Please read the following text carefully

An *ordinal number* (or *ordinal* in short) is an adjective that describes the numerical position or order of an object with respect to other objects. E.g. we say 1st (First), 2nd (Second), 3rd (Third), 4th (Fourth) and so on, to describe the position of different floors in a building. Thus, these are ordinal numbers. Moreover, it implies that all natural numbers (i.e., 0, 1, 2, 3, 4, ...) are ordinals. But what happens when we reach infinity ∞?

Ordinals allow us to continue the process of 'counting' beyond the natural numbers; i.e., they allow us to describe and compare positions of infinite entities. One can guess that this property suggests that there is no singular infinity, rather there are different sizes of infinities, such that we can count and compare them to one another.

Infinite ordinals

The first infinite ordinal (i.e., the first number that comes **after** all natural numbers) is *omega*, denoted by the Greek letter $\boldsymbol{\omega}$, which is defined as the set (a collection of elements) $\mathbb{N}$ of all natural numbers; i.e., $\omega = \{0, 1, 2, 3, ...\}$

Successor and predecessor of ordinals

For an ordinal number α, a *successor* of α, denoted by $S(\alpha)$, is some ordinal β, such that $\alpha < \beta$ and there is no other ordinal number between α and β (i.e., $\beta = \alpha + 1$).
For example:

- 2 is the successor of 1, i.e., $S(1) = 2$
- 974 is the successor of 973, i.e., $S(973) = 974$
- $\omega+1$ is the successor of ω, i.e., $S(\omega) = \omega+1$

For an ordinal number α, a *predecessor* of α is some ordinal γ, such that $\gamma < \alpha$ and there is no other ordinal number between α and γ (i.e., $\alpha = \gamma +1$).

For example:

- 5 is the predecessor of 6, i.e., $S(5) = 6$
- 2965 is the predecessor of 2966, i.e., $S(2965) = 2966$
- ω is the predecessor of $\omega+1$, i.e., $S(\omega) = \omega+1$

In set theory, ordinals are described as sets (i.e., collections of elements, denoted by { }), such that **every ordinal number is a set containing all smaller ordinals** (e.g., $1 = \{0\}$, $5 = \{0, 1, 2, 3, 4\}$). Intuitively, this implies that, for any pair α and β of ordinal numbers, if $\alpha < \beta$, then α is 'contained' within β (i.e., $\alpha \in \beta$). More generally, every ordinal number contains all of its predecessors (i.e., $0 \in 1 \in 2 \in 3 \in ..... \in \omega \in \omega+1 \in ...$).

Addition of ordinal numbers

Addition of ordinals is a generalization of addition on $\mathbb{N}$ (the natural numbers), and likewise, addition of ordinals is associative (i.e., the outcome will not change if we rearrange the parenthesis in the argument); e.g., $(3 + 1) + 6 = 3 + (1 + 6) = 10$.

Similarly to the 'regular' addition rules, for any ordinal α, $\alpha + 0 = \alpha$.

For example:

- $5 + 0 = 5$
- $\omega + 0 = \omega$

When applying addition between two ordinals α, β (i.e., $\alpha + \beta$), it means we attach β 'on top' of α. For example:

- $2 + 3 = \{0, 1\} + \{0, 1, 2\} = \{0, 1, 0', 1', 2'\} = \{0, 1, 2, 3, 4\} = 5$. We can consider the operation *2 + 3* as locating the second element (3) completely to the right of the first (2), i.e., $0 < 1 < 0' < 1' < 2'$ (where ' denotes the elements of the second ordinal, which is 3 in this example). By relabeling the elements of the second ordinal (e.g., we denote 0' as 2, to describe the 3rd element within the new set 2 + 3), we obtain the ordinal number 5.
- $\omega + \omega$ is the ordinal number obtained by 2 copies of $\mathbb{N}$ (recall $\mathbb{N}$ denotes the set of all natural numbers and $\omega = \{0, 1, 2, 3, ...\}$), ordered one after the other. Just like in the previous example, the second copy of $\mathbb{N}$ is located completely to the right of the first; i.e., $0 < 1 < 2 < 3 < ... < 0' < 1' < 2' < 3' < ...$ (where ' denotes the elements of the second ω). Hence, by relabeling the elements of the second ω, we get $\omega + \omega = \{0, 1, 2, 3, ...\} + \{0, 1, 2, 3, ...\} = \{0, 1, 2, 3, ..., 0', 1', 2', 3', ...\} = \{0, 1, 2, 3, ..., \omega, \omega+1, \omega+2, \omega+3, ...\} = \omega * 2$.

However, let's consider the ordinal numbers $\omega+3$ and $3+\omega$. Following the same idea, we can order these two ordinals as follows:

- $\omega+3 := \{0, 1, 2, 3, ...\} + \{0, 1, 2\}$ => $0 < 1 < 2 < 3 < ... < 0' < 1' < 2'$ => $\omega+3 = \{0, 1, 2, 3, ..., 0', 1', 2'\} = \{0, 1, 2, 3, ..., \omega, \omega+1, \omega+2\}$
- $3+\omega := \{0, 1, 2\} + \{0, 1, 2, 3, ...\}$ => $0 < 1 < 2 < 0' < 1' < 2' < 3' < ...$ => $3+\omega = \{0, 1, 2, 0', 1', 2', 3', ...\} = \{0, 1, 2, 3, 4, ...\} = \omega$

We can notice that $\omega+3$ is clearly different from ω (i.e., $\omega+3 \neq \omega$), because, $\omega+2$, for example, is included in $\omega+3$, but not in ω.
However, $3+\omega$ is just ω (i.e., $3+\omega = \omega$). This is because we can relabel the elements of $3+\omega$, such that it will just look like ω itself.
Therefore, we can conclude that $\omega+3 \neq 3+\omega$

In general, for any ordinal number β, addition of ordinals is defined as follows:
1. $\beta + 0 = \beta$
2. $\beta + (\alpha + \gamma) = (\beta + \alpha) + \gamma$
3. $\beta + \alpha = \sup \{\beta + \gamma \mid \gamma < \alpha\}$

<u>Multiplication of ordinal numbers</u>
Similarly to addition, multiplication of ordinals is a generalization of multiplication on $\mathbb{N}$, and likewise, multiplication of ordinals is associative (i.e., the outcome will not change if we rearrange the parenthesis in the argument); e.g., $(3 * 2) * 4 = 3 * (2 * 4) = 24$.

Similarly to the 'regular' multiplication rules, for any ordinal α, $\alpha * 0 = 0$.

For example:

- $5 * 0 = 0$
- $\omega * 0 = 0$

When applying multiplication between two ordinals α, β (i.e., $\alpha * \beta$), it means we attach α 'on top' of α 'on top' of α... β times (i.e., we add together β number of copies of α)

For example:

- $2 * 3 = 2 + 2 + 2 = \{0, 1\} + \{0, 1\} + \{0, 1\} = \{0, 1, 0', 1', 0'', 1''\} = 6$. Meaning, *2 * 3* is the ordinal obtained by adding 3 copies of 2 one after the other, i.e., $0 < 1 < 0' < 1' < 0'' < 1''$ (where ' denotes the elements of the second copy of 2 and " denotes the elements of the third copy). Now we can apply the rules of addition of ordinals and we can consider this process as locating the second copy of 2 completely to the right of the first, and locating the third copy completely to the right of the second. Where again, by relabeling the elements of the second and third copies (e.g., we denote 0' as 2 and 0" as 4, to describe the 3rd and 5th elements, respectively, within the new set $2 * 3 = 2 + 2 + 2$), we obtain the ordinal number 6.
- $\omega * 2$ is the ordinal number obtained by 2 copies of $\mathbb{N}$, ordered one after the other (i.e., $\omega + \omega$). Just as before, the second copy of $\mathbb{N}$ is located completely to the right of the first; i.e., $0 < 1 < 2 < 3 < ... < 0' < 1' < 2' < 3' < ...$ (where ' denotes the elements of the second ω). Hence, $\omega * 2 = \{0, 1, 2, 3, ...\} * 2 = \{0, 1, 2, 3, ...\} + \{0, 1, 2, 3, ...\} = \{0, 1, 2, 3, ..., 0', 1', 2', 3', ...\} = \{0, 1, 2, 3, ..., \omega, \omega+1, \omega+2, \omega+3, ...\} = \omega + \omega$

However, let's consider the ordinal numbers $2*\omega$. Following the same idea, we can order this ordinal as follows:

- $2 * \omega := \{0, 1\} + \{0, 1\} + \{0, 1\} + ...$ ω many times $\Rightarrow$ $0 < 1 < 0' < 1' < 0'' < 1'' < 0''' < 1''' < 0'''' < 1'''' < ...$ $\Rightarrow$ $2*\omega = \{0, 1, 0', 1', 0'', 1'', 0''', 1''', 0'''', 1'''', ...\} = \{0, 1, 2, 3, 4, 5, 6, ...\} = \omega$

Similarly to the ordinal $3+\omega$, after relabeling the elements of $2*\omega$, we obtain ω. Hence, $2*\omega$ is just ω itself.
Therefore, we can conclude that $\omega*2 \neq 2*\omega$.

In general, for any ordinal number β, multiplication of ordinals is defined as follows:
1. $\beta * 0 = 0$
2. $\beta * (\alpha + 1) = \beta * \alpha + \beta$
3. $\beta * \alpha = \sup \{\beta * \gamma \mid \gamma < \alpha\}$