Experimental Verification of Trapezoidal Convergence

Numerical Mathematics II

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1 Introduction

This report aims to experimentally verify the following claims through multiple documented test runs for various functions:

1. The summed trapezoidal rule generally converges slower than the adaptive trapezoidal rule.
2. However, if the function $f(x)$ is periodic and analytic on R and the interval $(b-a)$ equals the period, the summed trapezoidal rule is also very well-suited.

2 Methodology

To verify these claims, we implemented and compared the following numerical integration methods:

- **Adaptive Trapezoidal Rule**: Adjusts the interval sizes based on the function's behavior, providing higher accuracy with fewer function evaluations in regions with more variation.
- **Summed Trapezoidal Rule**: Uses a uniform grid over the interval, refining the intervals based on the specified tolerance.
- **Adaptive Simpson's Rule**: Adjusts the interval sizes using Simpson's method for better accuracy.

We tested these methods on the following functions over specified intervals and varying tolerances:

1. $f_1(x) = 2 + \sin(x^2)$ over $[0, 5]$
2. $f_2(x) = 4 + 2\sin(x) + \cos(3x)$ over $[0, 2\pi]$

The exact values of the integrals were:

1. $\int_0^5 (2 + \sin(x^2))\, dx \approx 10.5279$
2. $\int_0^{2\pi} (4 + 2\sin(x) + \cos(3x))\, dx \approx 25.1327$

3 Results

The following observations were made based on the plots and numerical results:

3.1 Function $f_1(x) = 2 + \sin(x^2)$ over $[0, 5]$

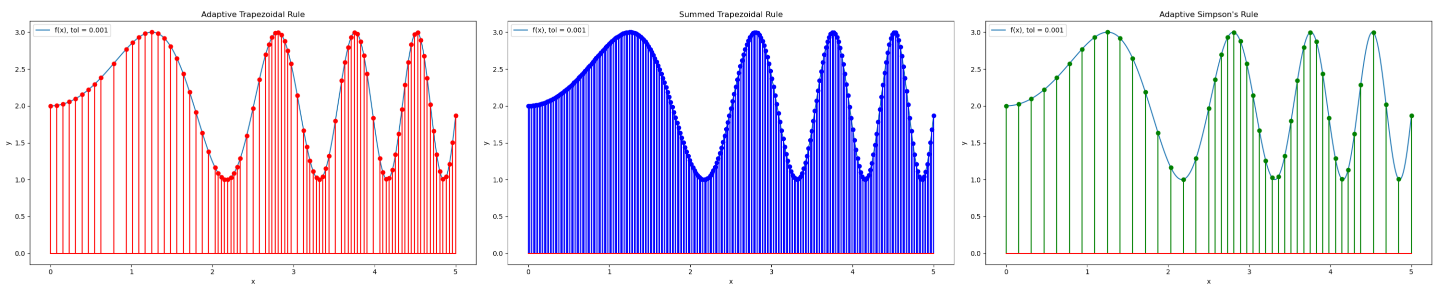

Figure 1: Function 1, Tolerance: 0.001

- **Adaptive Trapezoidal Rule**: Efficiently added more points in regions with higher variation, converging quickly to the exact value. Error = 5.4920759312793166e-05, Evaluations = 91
- **Summed Trapezoidal Rule**: Required significantly more points to achieve similar accuracy, especially for lower tolerances. Error = 0.00031529379112704703, Evaluations = 257
- **Adaptive Simpson's Rule**: Showed similar behavior to the adaptive trapezoidal rule, with even better accuracy due to the higher-order method. Error = 0.0024872495287091567, Evaluations = 45

3.2 Function $f_2(x) = 4 + 2\sin(x) + \cos(3x)$ over $[0, 2\pi]$

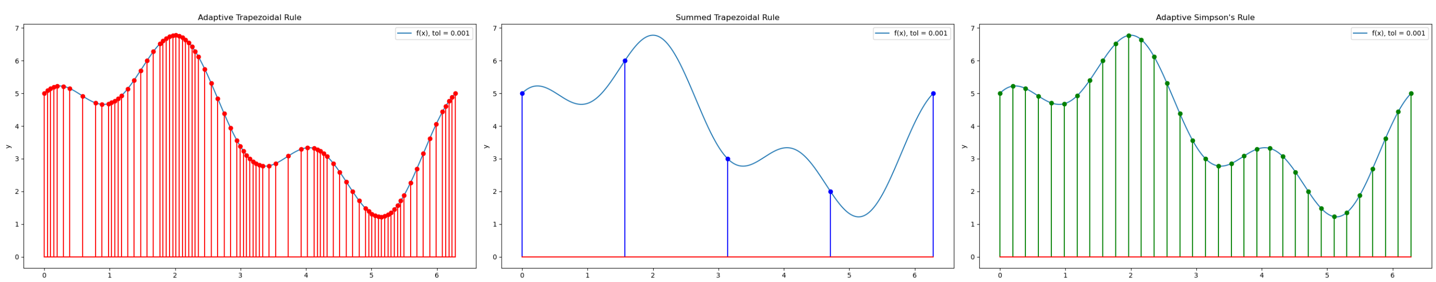

Figure 2: Function 2, Tolerance: 0.001

- **Adaptive Trapezoidal Rule**: Again showed efficient convergence by adapting to the function's oscillatory behavior. Error = 0.0, Evaluations = 85
- **Summed Trapezoidal Rule**: Provided one of the highest accuracy with fewer function evaluations compared to the other methods. Error = 0.0, Evaluations = 5
- **Adaptive Simpson's Rule**: Performed slightly worse than Adaptive Trapezoidal, here it wasn't as adaptive for higher oscilations. Error = 3.552713678800501e-15, Evaluations = 33

4 Discussion

The experimental results support the claims:

1. The **summed trapezoidal rule** converges slower than the **adaptive trapezoidal rule** for non-periodic functions, as seen with f_1 .
2. For periodic functions where the interval matches the period, such as f_2, the summed trapezoidal rule performs well, showing efficient convergence similar to the adaptive methods.

The adaptive methods, especially the adaptive Simpson's rule, provided higher accuracy with fewer function evaluations by focusing on regions with higher variation, demonstrating their efficiency for a wide range of functions.

5 Conclusion

The adaptive trapezoidal and Simpson's rules generally outperform the summed trapezoidal rule, particularly for non-periodic functions with varying behavior. However, for periodic functions with intervals matching the period, the summed trapezoidal rule is also very well-suited, providing efficient and accurate results. These findings are consistent with the theoretical expectations and demonstrate the practical advantages of adaptive numerical integration methods.