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On extensions of the domain invariance theorem

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Abstract

If f maps continuously a compact subset X of R^n into R^n and x is a point whose distance from the boundary ∂X is greater than double diameter of the fibres of the points in $f(\partial X)$ then $f(x)$ is in the interior of $f(X)$. This theorem extends some results due to Borsuk and Sitnikov.

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Let us establish some terminology and notation. Throughout this paper all spaces are subspaces of the Euclidean n-space R^n with the metric ρ induced by the Euclidean norm $\|\cdot\|$. Fix a space $X \subset R^n$. For each point $x \in X$ and a real number $\varepsilon > 0$ the set $B(x, \varepsilon) := \{y \in X\colon\ \rho(x, y) < \varepsilon\}$ means the ε-ball. We denote by $d(x, A) := \inf\{\rho(x, a)\colon\ a \in A\}$ the distance of a point x from a set A, $\operatorname{diam} A := \sup\{\rho(x, y)\colon\ x, y \in A\}$—the diameter of A, and $B(A, \varepsilon) := \{x \in X\colon\ d(x, A) < \varepsilon\}$—$\varepsilon$-ball around the set A.

Symbols $\operatorname{Int} A$, $\overline{A}$, ∂A, $\operatorname{conv} A$ stand for interior, closure, boundary and convex hull of a set A, respectively.

The main result of this paper is Theorem 1 which implies some results of Borsuk and Sitnikov, cf. Theorems 2, 3. The proof is based on Lemma 5, preceded by the following four lemmas that are either easy to prove or known. In particular, Lemma 2 is an elementary instant of a technique used extensively in the context of refinable mappings; it can be derived from a result of Lončar and Mardešić [7] or Ancel [1] concerning cell-like relations (by taking convex hulls of the fibres).

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Lemma 1. *Let $f : X \to Y$ be a continuous map between compact. Then there exist a point $z \in Y$ such that*

$$\operatorname{diam} f^{-1}(z) = \sup\{\operatorname{diam} f^{-1}(y) \colon\ y \in Y\}.$$

Lemma 2. *Let $f : X \to Y$ be a continuous map from a compact space $X \subset R^n$ onto a metric space Y and let $a > 0$ be a real number such that for each $y \in Y$;* $\operatorname{diam} f^{-1}(y) < a$*. Then there exists a continuous map $g : Y \to R^n$ such that $\rho(x, g(f(x))) < a$ for each $x \in X$.*

Lemma 3. *Let $U \subset R^n$ be an open, connected and bounded set. Then for each number $\delta > 0$ there is an open and connected set W, $\overline{W} \subset U$, such that $d(x, \partial U) < \delta$ for each $x \in U \setminus W$.*

Lemma 4. *Let $Y \subset R^n$ be a compact set, $G \subset R^n$ an open set and $A \subset Y \cap G$ a compact connected set such that $A \cap \partial Y \neq \emptyset$. Then there exist an open set $V \subset R^n$ with $A \subset V \subset \overline{V} \subset G$, a compact boundary set $L \subset R^n$ and a continuous map $r : Y \to R^n$ such that $r(Y) \subset L \cup (Y \setminus V)$ with $r(y) = y$ for each $y \in Y \setminus V$.*

Proof. Fix a point $a \in A \cap \partial Y$. Compactness of A implies that there is a finite family $\{B_i \colon\ i \leqslant m\}$ of open balls such that $A \subset \bigcup\{B_i \colon\ i \leqslant m\}$, $\overline{B}_i \subset G$ and $B_i \cap A \neq \emptyset$ for each i. Let us put $V := \bigcup\{B_i \colon\ i \leqslant m\}$ and $L := \bigcup\{\partial B_i \colon\ i \leqslant m\}$. Since A is a connected set there is a chain $V_0, \ldots, V_s$ of sets consisting of the balls B_i's such that $a \in V_0$ and $V_{i-1} \cap V_i \neq \emptyset$ for each $i = 1, \ldots, s$ (in this chain each ball B_i may appear many times).

Define $r : Y \to R^n$ to be a composition of maps obtained by induction (by a standard “sweeping out” procedure) which are identity outside of the ball V_i and map each of them into the union of the boundaries ∂B_i's. □

Lemma 5. *Let x_0 be a point belonging to an open bounded subset U of R^n and let $f : \overline{U} \to R^n$ be a continuous map such that $f^{-1}(f(\partial U)) = \partial U$ and* $\operatorname{diam} f^{-1}(f(x)) < d(x_0, \partial U)$ *for each $x \in \partial U$. Then $f(x_0) \in \operatorname{Int} f(\overline{U})$.*

Proof. Let $C \subset U$ be a connected component such that $x_0 \in C$. Then $\partial C \subset \partial U$ and $\operatorname{diam} f^{-1}(f(x)) < d(x_0, \partial C)$ for each $x \in \partial C$. Thus, without loss of generality let us additionally assume that U is connected.

Let $a := d(x_0, \partial U)$ and $b := \sup\{\operatorname{diam} f^{-1}(f(x)) \colon\ x \in \partial U\}$. From Lemma 1 it follows that $b < a$. Fix an $\varepsilon > 0$ such that $a = b + 3\varepsilon$, Compactness of $\overline{U}$ and continuity of f imply that there exist a $\delta \in (0, \varepsilon)$ such that for each $x \in \partial U$, if $d(x, \partial U) < \delta$ then $\operatorname{diam} f^{-1}(f(x)) < b + \varepsilon$. According to Lemma 3 there is a connected open set W, $x_0 \in W \subset \overline{W} \subset U$, such that $d(x, \partial U) < \delta$ for each $x \in U \setminus W$.

Let $Y := f(\overline{U})$ and suppose that $f(x_0) \in \partial Y$. Applying Lemma 4 to the sets $A := f(\overline{W})$ and $G := R^n \setminus f(\partial U)$ we obtain an open set V such that $f(\overline{W}) \subset V \subset R^n \setminus f(\partial U)$, a compact boundary set $L \subset R^n$ and a continuous map $r : Y \to R^n$ such that $r(Y) \subset L \cup (Y \setminus V)$ and $r(y) = y$ for each $y \in Y \setminus V$.

Let $X := \overline{U} \setminus f^{-1}(V)$. Applying Lemma 2 to the map $f|X : X \to Y \setminus V$ we get a continuous map $g : Y \setminus V \to R^n$ such that $\rho(x, g(f(x)) < b + \varepsilon$ for each $x \in X$.

Let us verify that $x_0 \notin g(Y \setminus V)$. Suppose to the contrary that $x_0 \in g(Y \setminus V)$. Then there exists a point $x \in \overline{U} \setminus f^{-1}(V) \subset \overline{U} \setminus W$ such that $x_0 = g(f(x))$. Since $d(x, \partial U) < \delta < \varepsilon$ and $d(x_0, \partial U) = a$ we infer that $\rho(x, x_0) \geqslant a - \varepsilon = b + 2\varepsilon$. Thus, if we assume that for some $x \in \overline{U} \setminus f^{-1}(V)$, $x_0 = g(f(x))$, then we get that $b + 2\varepsilon \leqslant \rho(x, g(f(x)))$, contrary to $\rho(x, g(f(x))) < b + \varepsilon$.

In effect, $g : Y \setminus V \to R^n \setminus \{x_0\}$. Since L is a boundary subset of R^n hence $\dim L < n$ (cf. [4], p. 44) and therefore g has a continuous extension $g_1 : L \cup (Y \setminus V) \to R^n \setminus \{x_0\}$ (cf. [4, p. 84]).

Let $h : \overline{U} \to R^n \setminus \{x_0\}$ be the composition $h := g_1 \circ r \circ f$. Since $h|\partial U = (g \circ f)|\partial U$ hence $\rho(x, h(x)) < b + \varepsilon$ for each $x \in \partial U$.

Since for each $x \in \partial U$, $\rho(x, h(x)) < b + \varepsilon$ and $\rho(x, h(x)) \geqslant \rho(x, x_0) \geqslant a = b + 3\varepsilon$, the point x_0 is not in the segment with the points x and $h(x)$. Therefore the homotopy $F(x, t) := (1 - t)x + th(x)$ omits the point x_0.

The map $F : \partial U \times [0, 1] \to R^n \setminus \{x_0\}$ is a homotopy between the identity map Id and the map $h|\partial U$. Since $h : \overline{U} \to R^n \setminus \{x_0\}$ is an extension of the map $h|\partial U$ hence according to the Borsuk homotopy theorem (cf. [4, p. 86]) the identity map $Id|\partial U : \partial U \to \partial U \subset R^n \setminus \{x_0\}$ should have a continuous extension $Id^* : \overline{U} \to R^n \setminus \{x_0\}$. But it is impossible because it contradicts to non-retraction theorem which says that if $s : \overline{U} \to R^n$, $U \subset R^n$, is a continuous map from an open bounded subset of R^n such that $s(x) = x$ for each $x \in \partial U$ then $\overline{U} \subset s(\overline{U})$ (cf. [6]). □

Remark. For a given $a > 0$ let $I^n := [-a, a]^n$ means an n-dimensional cube with ith opposite faces $I_i^- := \{x \in I^n\colon x_i = -a_i\}$, $I_i^+ := \{x \in I^n\colon x_i = a\}$, $i = 1, \ldots, n$.

In the case when $\overline{U}$ is a cube I^n and $x_0 = \mathbf{0}$, in the final part of the proof Borsuk's theorem can be replaced by Poincaré theorem (announced in 1883) (cf. [6]):

Let $h : I^n \to R^n$, $h = (h_1, \ldots, h_n)$, be a continuous map such that for each $i \leqslant n$, $h_i(I_i^-) \subset (-\infty, 0]$ and $h_i(I_i^+) \subset [0, \infty)$. Then there exists a point $c \in I^n$ such that $h(c) = \mathbf{0}$.

The map h which appears in the proof of Lemma 5 satisfies the assumptions of the Poincaré theorem whenever we assume that $x_0 = \mathbf{0}$. From theorem of Poincaré we obtain $x_0 \in h(I^n)$, contrary to $x_0 \notin h(I^n)$. Poincaré's theorem gives a weak form of domain invariance:

If $h = (h_1, \ldots, h_n) : I^n \to R^n$ is a continuous map such $h_i(I_i^-) \subset (-\infty, 0)$ and $h_i(I_i^+) \subset (0, \infty)$ for each $i \leqslant n$, then $\mathbf{0} \in \operatorname{Int} h(I^n)$

To prove this, note that by compactness of I^n and from the assumptions it follows that there exists $\delta > 0$ such that $h_i(I_i^-) \subset (-\infty, \delta)$ and $h_i(I_i^+) \subset (\delta, \infty)$ for each $i \leqslant n$. Now observe that for each $b \in J^n := [-\delta, \delta]^n$ the map $h_b(x) := h(x) - b$, $x \in I^n$, also satisfies the assumptions of the Poincaré theorem and therefore there is $c \in I^n$ such that $h_b(c) = \mathbf{0}$, i.e., $h(c) = b$. Thus we have proved that $J^n \subset h(I^n)$.

A simple proof of the Brouwer domain invariance theorem based on the Poincaré theorem is given in [5].

Theorem 1. *Let X be a compact subset of R^n. Fix $x_0 \in \operatorname{Int} X$ and let $a := \frac{1}{2} d(x_0, \partial X)$. Then for any continuous map $f : X \to R^n$ such that $\operatorname{diam} f^{-1}(f(x)) < a$ for each $x \in \partial X$, the point $f(x_0)$ belongs to the interior of $f(X)$, $f(x_0) \in \operatorname{Int} f(X)$.*

Proof. Let $f : X \to R^n$ be a continuous map such that $\operatorname{diam} f^{-1}(f(x)) < a$. Define $U := X \setminus f^{-1}(f(\partial X))$. Then for each $x \in \partial U$, $\operatorname{diam} f^{-1}(f(x)) < a \leqslant d(x_0, \partial U)$. The map $f|\overline{U}$ satisfies the assumptions of Lemma 5 and therefore $f(x_0) \in \operatorname{Int} f(\overline{U}) \subset \operatorname{Int} f(X)$. □

From Theorem 1 as a corollary we immediately obtain some extension of Borsuk's result as well Sitnikov's theorem;

Theorem 2 (Borsuk [2]). *If $f : R^n \to R^n$ is a continuous map such that*

$$\limsup_{\|x\| \to \infty} \frac{\operatorname{diam} f^{-1}(f(x))}{\|x\|} < \frac{1}{2}$$

then $f(R^n)$ is an open set.

Theorem 3 (Sitnikov [8]). *If $f : U \to R^n$ is a continuous map from an open bounded subset $U \subset R^n$ such that*

$$\lim_{d(x,\partial U) \to 0} \operatorname{diam} f^{-1}\big(f(x)\big) = 0$$

then $f(U)$ is an open subset of R^n.

Proof. Fix a point $x_0 \in U$. Replacing U by its connected component containing x_0 without loss of generality we may assume that U is connected open set. Let $a := \frac{1}{3} d(x_0, \partial U)$. From the assumptions of Sitnikov's theorem it follows that there exists δ, $0 < \delta < a$, such that for each point $x \in U$; if $d(x, \partial U) < \delta$ then $\operatorname{diam} f^{-1}(f(x)) < a$. According to Lemma 3 there is an open set $W \subset U$ such that for each $x \in U \setminus W$, $\operatorname{diam} f^{-1}(f(x)) < a$. Now, applying Theorem 1 to the map $f|\overline{W}$ we infer that $x_0 \in \operatorname{Int} f(\overline{W}) \subset f(U)$. □

For each compact set $X \subset R^n$ let $a_X := \sup\{\rho(x, \partial X)\colon x \in X\}$. The set $M_X := \{x \in X\colon d(x, \partial X) = a_X\}$ is said to be a set of middle points. For each compact subset of R^n with nonempty interior the set of middle points is nonempty. From Theorem 3 it follows that if $f : X \to R^n$ is a continuous map from a compact subset $X \subset R^n$ such that $\operatorname{diam} f^{-1}(f(x)) < \frac{1}{2} a_X$ for each $x \in \partial X$ then $f(M_X) \subset \operatorname{Int} f(X)$.

The following problems are related to Theorem 1.

Problem 1. Let $f : X \to R^n$ be a continuous map from a compact subset $X \subset R^n$ with nonempty interior, $\operatorname{Int} X \neq \emptyset$, such that $\operatorname{diam} f^{-1}(f(x)) < 2a_X$. Is the interior of the image $f(X)$ nonempty, $\operatorname{Int} f(X) \neq \emptyset$?

Problem 2. Let x_0 be a point belonging to an open bounded subset U of R^n and let $f : \overline{U} \to R^n$ be a continuous map such that for each $x \in \partial U$, $\operatorname{diam} f^{-1}(f(x)) < a$, where $a := d(x_0, \partial U)$. Is it true that $f(x_0) \in \operatorname{Int} f(\overline{U})$?

Comments. The following examples of two maps $f_i : I^2 \to R^2$, $i = 1, 2$, show that in Problems 1 and 2 the strong inequalities "$<$" in the assumptions about $\operatorname{diam} f^{-1}(f(x))$ cannot be replaced by weak inequalities "$\leqslant$". To see this consider $f_1(x, y) := (0, y)$ and $f_2(x, y) := (0, y)$ if $x \leqslant 0$, $f_2(x, y) := (x, y)$ if $x > 0$.

Problem 1 is related to the results of the author who in [6] proved the following non-squeezing theorem:

Let $f : I^n \to Y$ be a continuous map onto Hausdorff space Y such that $f(I_i^-) \cap f(I_i^+) = \emptyset$ for each $i \leqslant n$. Then $\dim f(I^n) \geqslant n$.

In the case $Y \subset R^n$ from the above theorem and the fact that for each subset $A \subset R^n$, $\dim A = n$ iff $\operatorname{Int} A \neq \emptyset$ (see [4, p. 44]) we get

Theorem 4. *If $f : I^n \to R^n$, $I^n = [-a, a]^n$, is a continuous map such that* $\operatorname{diam} f^{-1}(f(x)) < 2a$ *for each $x \in \partial I^n$, then* $\operatorname{Int} f(I^n) \neq \emptyset$

The first theorem of this type having great significance for topology was proved in 1911 by Brouwer [3].

If $f : I^n \to R^n$, $I^n = [-a, a]^n$, is a continuous map such that $\rho(x, f(x)) < a$ for each $x \in \partial I^n$, then $\operatorname{Int} f(I^n) \neq \emptyset$

It is clear that Theorem 4 strengthens Brouwer's result because if $f(x_1) = f(x_2)$ and $\rho(x_i, f(x_i)) < a$ for $i = 1, 2$ then $\rho(x_1, x_2) \leqslant \rho(x_1, f(x_1)) + \rho(f(x_2), x_2) < 2a$.

Theorem 4 is not sufficient to prove the Brouwer domain invariance theorem.

If $f : U \to R^n$ is one-to-one continuous map from an open set $U \subset R^n$, then $f(U)$ is an open set.

But the following slight modification of this theorem which is a consequence of Theorem 1 yields immediately the Brouwer theorem.

If $f : I^n \to R^n$, $I^n = [-a, a]^n$, is a continuous map such that $\operatorname{diam} f^{-1}(f(x)) < \frac{a}{2}$, *then* $f(\mathbf{0}) \in \operatorname{Int} f(I^n)$.

From this theorem we also obtain Borsuk's theorem [2].

If $f : R^n \to R^n$ is a continuous map such that for some $a > 0$, $\operatorname{diam} f^{-1}(f(x)) < a$ *for each $x \in R^n$, then $f(R^n)$ is an open subset of R^n.*

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