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Ryszard Engelking

General Topology

Revised and completed edition

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Ryszard Engelking
Instytut Matematyki
Uniwersytet Warszawski
PKiN, 00-901 Waszawa
Poland

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All three stars in the maze of the cover, designed by Dave Phillips, can be joined by a closed line which never passes a piece of way twice.

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Contents

Preface to the first edition

This book contains a reasonably complete and up-to-date exposition of general topology. It is addressed primarily to the graduate student but it can also serve as a reference for the more advanced mathematician.

Chapter 1 provides fundamental definitions and basic results on general topological spaces and continuous mappings. Chapter 2 is devoted to operations on topological spaces, i.e., to standard methods of obtaining new spaces from old ones. In the next three chapters the classes of compact, metrizable and paracompact spaces are studied, as well as some related classes of spaces. Chapter 6 contains a discussion of connectedness. In Chapter 7 a condensed course in dimension theory of general topological spaces is presented. The last chapter is devoted to uniform spaces and proximity spaces.

The first five sections of Chapter 1 together with Sections 2.1 and 2.3 are intended to be an introduction to the book. After having read those seven sections, the reader should be able to continue the reading according to his own interests or needs. If, however, the reader is not familiar with the subject, he is advised to continue with Sections 4.1–4.3 which are relatively easy and can help in developing topological intuition.

The arrangement of the material follows the author's earlier book *Outline of general topology* published jointly by North-Holland Publishing Company and Polish Scientific Publishers in 1968. However, the present book is completely rewritten and much more exhaustive. About 40% of the text is devoted to the most recent developments in general topology which were not discussed in the *Outline*.

Each section ends with historical and bibliographic notes. Those are followed by exercises which are primarily intended to test the reader's comprehension of the material. The exercises beginning with the words "check", "verify", "note" or "observe" are rather easy. The exercises beginning with the words "show" or "give an example" are usually a little more difficult, and the exercises beginning with "prove" may be quite difficult.

The final section of each chapter contains problems which are supposed to be an integral part of the book. They usually contain detailed hints that in fact are sketches of proofs. The problems can also be just read, it is not absolutely necessary to solve them. There are a few series of problems discussed through several chapters; they are devoted to such topics as linearly ordered spaces, cardinal functions, spaces of closed subsets, semicontinuous functions and set-valued mappings.

A developed subject index (where reappearances of the most important items are indicated), a table of relations between various classes of spaces and the tables of invariance of topological properties under operations and mappings will give the reader better orientation in the material discussed in the book.

The mark ∎ indicates the end of a proof or of an example. If it appears just after a statement of a theorem, a proposition or a corollary, it means that the statement is obviously valid.

Numbers in square brackets refer to the bibliography at the end of the book. Papers of each author are numbered separately, the number being the year of publication.

It is a pleasure to acknowledge my indebtedness to several colleagues. The debts start with the *Outline*: while writing that book I was helped and encouraged by Professors A. Białynicki-Birula, J. Browkin, M. Karłowicz, A. Lelek, K. Maurin, J. Mycielski, C. Ryll-Nardzewski, R. Sikorski and M. Stark. The text of the present book was discussed in 1968–73

with students attending my topology classes and my seminars at Warsaw University. Their observations permitted some simplifications of proofs and hints. Thanks are due to K. Alster, J. Chaber, J. Kaniewski, P. Minc, K. Nowiński, J. Przytycki, E. Pol, Z. Słodkowski and K. Wojtkowska.

I am particularly obliged to two of my students, the first readers of this book: R. Pol and T. Przymusiński. Their valuable comments and suggestions led to many important improvements.

When preparing the English translation, during my stay at the University of Pittsburgh, I was helped by Professors D. Lutzer and E. Michael who verified my English and suggested further improvements.

Let me also mention the help I received from several readers of the *Outline* who, although not personally known to me, were kind enough to point out some mistakes and inaccuracies in my earlier book.

Ryszard Engelking.

Warsaw, June 1976

Preface to the revised edition

The purpose of this new edition of *General Topology* is to update the original edition in the light of current research. Important new results related to the topics discussed in the first edition were added, as were some older results which have recently proved to be important. Some errors and inaccuracies were corrected, several proofs simplified, historical and bibliographic notes were completed and references to the recent literature were added. As the result of all these changes, the bibliography was enlarged by some 130 items.

It is a real pleasure to acknowledge my indebtedness to the readers of the first edition who communicated their comments and called my attention to possible improvements. My thanks are primarily due to A. Bešlagić, A. Császár, E. van Douwen, K. P. Hart, A. M. Maurice and A. Mysior. I am also most obliged to J. Chaber and R. Pol for their reading of the new material included in this edition and for suggesting several improvements. I wish to thank as well W. Olszewski for his help in reading the proofs.

I was working on this new edition during my stay at Miami University in Oxford, Ohio, where I received all possible help from the Department of Mathematics and Statistics and greatly profited from conversations with D. Burke, S. Davis and D. Lutzer.

Finally, let me also express my thanks to N. Heldermann for his valuable contributions to this edition, both as a topologist and a publisher.

Ryszard Engelking.

Warsaw, September 1988

Introduction

The only prerequisite for understanding this book is a knowledge of fundamental facts of set theory and of some properties of real numbers. The purpose of this introduction is to list those facts and to make the reader familiar with our terminology and notations. With the exception of the equivalence of the axiom of choice, two maximal principles, and the well-ordering theorem, no proofs are given here, and the introduction cannot replace a course in set theory.

I.1. Algebra of sets. Functions

The union, intersection and difference of sets A and B are denoted by $A \cup B$, $A \cap B$ and $A \setminus B$ respectively; the union $A_1 \cup A_2 \cup \ldots \cup A_k$ is also denoted by $\bigcup_{i=1}^{k} A_i$ and the intersection $A_1 \cap A_2 \cap \ldots \cap A_k$ by $\bigcap_{i=1}^{k} A_i$. The empty set is denoted by $\emptyset$. Throughout the book we freely use all rules of the algebra of sets; in particular, *De Morgan's laws*

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C) \text{ and } A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

are often applied. We write $x \in A$ when x is an element of the set A and $x \notin A$ when x does not belong to A. The notation $A \subset B$ or $B \supset A$ means that A is contained in B, i.e., that every element of A belongs to B. When $A \subset B$ we say that A is a *subset* of B, and when $A \subset B$ and $A \neq B$ we say that A is a *proper subset* of B.

The set of all elements of X satisfying the condition $\varphi(x)$ is denoted by

$$\{x \in X : \varphi(x)\} \text{ or by } \{x : \varphi(x)\}$$

when it is clear from the context which set X is being considered.

The set consisting of finitely many elements $x_1, x_2, \ldots, x_k$ is denoted by $\{x_1, x_2, \ldots, x_k\}$. Sometimes we do not distinguish between the set $\{x\}$ and the element x.

The *ordered pair* (x, y) is the set $\{\{x\}, \{x, y\}\}$. Two ordered pairs (x_1, y_1) and (x_2, y_2) are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.

The *Cartesian product* $X \times Y$ of the sets X and Y is the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$. Finite Cartesian products are defined by the inductive formula $X_1 \times X_2 \times \ldots \times X_k = (X_1 \times X_2 \times \ldots \times X_{k-1}) \times X_k$.

Any subset of the Cartesian product $X \times Y$ is a *relation*. The relation $f \subset X \times Y$ is called a *function* from X to Y, or a *mapping* of X to Y, if for every $x \in X$ there exists a $y \in Y$ such that $(x, y) \in f$ and y is uniquely determined by x, i.e., $(x, y) \in f$ and $(x, y') \in f$ imply $y = y'$; the set X is the *domain* and the set Y is the *range* of the function f.

If f is a function from X to Y and if $x \in X$, then the unique y satisfying the condition $(x, y) \in f$ is denoted by $f(x)$; it is called the *value* of f at x. The *image* of the set $A \subset X$ under f is the set

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\},$$

and the *inverse image* of the set $B \subset Y$ under f is the set

$$f^{-1}(B) = \{x \in X : f(x) \in B\};$$

inverse images of one-point sets under f are called *fibers* of f.

Elementary formulas of the algebra of sets concerning images and inverse images are often used in the book; among the most important are

$$ff^{-1}(B) = B \cap f(X) \subset B \text{ and } f^{-1}f(A) \supset A.$$

If f is a function from X to Y and g a function from Y to Z, then the equality $(gf)(x) = g(f(x))$ defines a function gf from X to Z, the *composition* of f and g. One easily sees that $(gf)^{-1}(B) = f^{-1}(g^{-1}(B))$ for every $B \subset Z$.

A mapping f of X to Y is called *one-to-one* if for any $x_1, x_2 \in X$

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2.$$

If a mapping f of X to Y satisfies the condition $f(X) = Y$, we say that f is a mapping of X *onto* Y, or that f is a *mapping onto*. For a one-to-one mapping f of X onto Y there exists the *inverse mapping* f^{-1} which is a one-to-one mapping of Y onto X; the inverse mapping f^{-1} is defined by the condition

$$f^{-1}(y) = x \text{ whenever } f(x) = y.$$

The *identity mapping* of X onto itself, denoted by id_X, is defined by the formula $\mathrm{id}_X(x) = x$ for every $x \in X$.

A function defined on the set of all positive integers is a *sequence*; the value of a sequence at i usually is denoted by x_i, and the sequence itself is denoted by $x_1, x_2, \ldots$ or $(x_1, x_2, \ldots)$. The set consisting of all elements of the sequence $x_1, x_2, \ldots$ is denoted by $\{x_1, x_2, \ldots\}$. A sequence of sets $A_1, A_2, \ldots$ is called *increasing* if $A_i \subset A_{i+1}$ for $i = 1, 2, \ldots$ and *decreasing* if $A_{i+1} \subset A_i$ for $i = 1, 2, \ldots$

Sets whose elements are sets are called *families* or *classes* of sets and their elements are called *members*; families of families of sets are called *collections*. Throughout the book indexed as well as non-indexed families of sets will be discussed. An *indexed family* $\{A_s\}_{s\in S}$ is – strictly speaking – a function assigning to every $s \in S$ the set A_s, a *non-indexed family* is a simple set of sets. Every non-indexed family $\mathcal{A}$ can be regarded as an indexed one: we have only to take every member of the family as its own index, i.e., to assume that $\mathcal{A} = \{A\}_{A\in\mathcal{A}}$. In this way all notions defined in the text for indexed families (such as local finiteness or point-finiteness for example) pertain to non-indexed families as well.

The union and the intersection of a family $\{A_s\}_{s\in S}$ of sets are denoted by $\bigcup_{s\in S} A_s$ and $\bigcap_{s\in S} A_s$ respectively; in the case of a sequence $A_1, A_2, \ldots$ of sets we use the symbols $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$, and in the case of a non-indexed family $\mathcal{A}$ of sets we write $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$. The union and the intersection of all sets satisfying the condition $\varphi(A)$ are denoted by

$$\bigcup\{A : \varphi(A)\} \quad\text{and}\quad \bigcap\{A : \varphi(A)\}$$

respectively.

The *Cartesian product* of a family $\{X_s\}_{s\in S}$ of sets, i.e., the set of all functions f from S to $\bigcup_{s\in S} X_s$ such that $f(s) \in X_s$ for every $s \in S$, is denoted by $\prod_{s\in S} X_s$, or by $\prod_{i=1}^{\infty} X_i$ in the case of a sequence $X_1, X_2, \ldots$ of sets. For an $f \in \prod_{s\in S} X_s$, the point $f(s) \in X_s$ is called the *sth coordinate* of f. The element of the Cartesian product $\prod_{s\in S} X_s$, whose sth coordinate is the point $x_s \in X_s$ will be denoted in the sequel by the symbol $\{x_s\}$. In particular the sequence $x_1, x_2, \ldots$ of elements of X, which is an element of $\prod_{i=1}^{\infty} X_i$, where $X_i = X$ for $i = 1, 2, \ldots$, will be sometimes denoted by $\{x_i\}$.

Let us observe that the Cartesian product $\prod_{s\in S} X_s$, where $S = \{1,2,\ldots,k\}$, is not exactly the same set as the Cartesian product $X_1 \times X_2 \times \ldots \times X_k$; however, there is an obvious one-to-one correspondence between the elements of these two sets and we shall consider them as the same set, its elements being denoted by $(x_1, x_2, \ldots, x_k)$.

A relation E on the set X, i.e., a subset of $X \times X$, is called an *equivalence relation* on X when it has the following properties (we write xEy instead of $(x,y) \in E$):

(E1) *For every $x \in X$, xEx.*
(E2) *If xEy, then yEx.*
(E3) *If xEy and yEz, then xEz.*

Any equivalence relation E on the set X determines a decomposition of X into disjoint sets, the *equivalence classes* of E; two elements of X are in the same equivalence class of E if and only if they are E-related. Thus

$$X = \bigcup_{s\in S} A_s \text{ and } A_s \cap A_{s'} = \emptyset \text{ whenever } s \neq s',$$

where

$$x, y \in A_s \text{ for an } s \in S \text{ if and only if } xEy.$$

The equivalence class that contains the element x is denoted by $[x]$.

Conversely, any decomposition of X into disjoint sets $\{A_s\}_{s\in S}$ determines an equivalence relation E on X defined by the condition

$$xEy \text{ if and only if } x, y \in A_s \text{ for some } s \in S.$$

I.2. Cardinal numbers

The sets X and Y are called *equipotent* if there exists a one-to-one mapping of X onto Y. To every set X a cardinal number is assigned, it is called the *cardinality* of X and is denoted by $|X|$; the equality $|X| = |Y|$ holds if and only if X and Y are equipotent. For a finite set X the cardinality of X is equal to the number of elements of X. The cardinal number assigned to the set of all positive integers is denoted by the symbol $\aleph_0$ (*aleph zero*), and the cardinal number assigned to the set of all real numbers is denoted by $\mathfrak{c}$ (*continuum*). A set is *countable* when it is finite or has cardinality $\aleph_0$. Infinite countable sets are called *countably infinite*.

For cardinal numbers the operations of addition and of multiplication are defined. The *sum* of cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ is the cardinality of the set $X \cup Y$ where $|X| = \mathfrak{m}$, $|Y| = \mathfrak{n}$, and $X \cap Y = \emptyset$. The *product* of cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ is the cardinality of the set $X \times Y$, where $|X| = \mathfrak{m}$ and $|Y| = \mathfrak{n}$. The sum of cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ is denoted by $\mathfrak{m} + \mathfrak{n}$, and the product of $\mathfrak{m}$ and $\mathfrak{n}$ is denoted by $\mathfrak{m} \cdot \mathfrak{n}$ or $\mathfrak{mn}$. For every cardinal number $\mathfrak{m}$, the number $2^{\mathfrak{m}}$, also denoted by $\exp \mathfrak{m}$, is defined as the cardinality of the family of all subsets of a set X satisfying $|X| = \mathfrak{m}$. One proves that $2^{\aleph_0} = \mathfrak{c}$. More generally, we define $\mathfrak{n}^{\mathfrak{m}}$ as the cardinality of the set of all functions from X to Y, where $|X| = \mathfrak{m}$ and $|Y| = \mathfrak{n}$. One proves that

$$\mathfrak{n}^{\mathfrak{m}_1+\mathfrak{m}_2} = \mathfrak{n}^{\mathfrak{m}_1}\mathfrak{n}^{\mathfrak{m}_2}, \quad (\mathfrak{n}_1\mathfrak{n}_2)^{\mathfrak{m}} = \mathfrak{n}_1^{\mathfrak{m}}\mathfrak{n}_2^{\mathfrak{m}}, \quad (\mathfrak{n}^{\mathfrak{m}_1})^{\mathfrak{m}_2} = \mathfrak{n}^{\mathfrak{m}_1\mathfrak{m}_2}.$$

Let $\mathfrak{m}$ and $\mathfrak{n}$ be two cardinal numbers and let $|X| = \mathfrak{m}$ and $|Y| = \mathfrak{n}$. We say that $\mathfrak{m}$ is *not larger* than $\mathfrak{n}$, or that $\mathfrak{n}$ is *not smaller* than $\mathfrak{m}$, and we write $\mathfrak{m} \leq \mathfrak{n}$ or $\mathfrak{n} \geq \mathfrak{m}$, if there exists a one-to-one mapping of X to Y. The fundamental fact about inequalities for cardinal numbers is the following *Cantor-Bernstein theorem:*

$$\text{if } \mathfrak{m} \leq \mathfrak{n} \text{ and } \mathfrak{n} \leq \mathfrak{m}, \text{ then } \mathfrak{m} = \mathfrak{n}.$$

One proves also that $|f(X)| \leq |X|$ for any mapping f defined on X. This implies in particular that the family of all subsets of cardinality $\leq \mathfrak{m}$ in a set of cardinality $\mathfrak{n} \geq \mathfrak{m}$, has cardinality $\leq \mathfrak{n}^{\mathfrak{m}}$.

The sum of two cardinal numbers, at least one of which is infinite, is equal to the non-smaller of the two. The same holds for the product of two cardinal numbers which are distinct from zero. In particular

$$\mathfrak{m} + \mathfrak{m} = \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m} \text{ for } \mathfrak{m} \geq \aleph_0.$$

If $\mathfrak{m} \leq \mathfrak{n}$ and $\mathfrak{m} \neq \mathfrak{n}$, we say that $\mathfrak{m}$ is *smaller* than $\mathfrak{n}$, or that $\mathfrak{n}$ is *larger* than $\mathfrak{m}$, and we write $\mathfrak{m} < \mathfrak{n}$ or $\mathfrak{n} > \mathfrak{m}$. One proves that

$$\mathfrak{m} < 2^{\mathfrak{m}} \text{ for every cardinal number } \mathfrak{m};$$

in particular, $\aleph_0 < \mathfrak{c}$.

The *least upper bound* of a set $\{\mathfrak{m}_s\}_{s\in S}$ of cardinal numbers is defined as the smallest cardinal number $\mathfrak{m}$ such that $\mathfrak{m} \geq \mathfrak{m}_s$ for every $s \in S$ and is denoted by $\sup_{s\in S} \mathfrak{m}_s$; one can show that such a number always exists.

I.3. Order relations. Ordinal numbers

Let X be a set and $<$ a relation on X. We say that $<$ *linearly orders* X, or that $<$ is a *linear order* in X, if $<$ has the following properties:

(LO1) *If $x < y$ and $y < z$, then $x < z$.*
(LO2) *If $x < y$, then the relation $y < x$ does not hold.*
(LO3) *If $x \neq y$, then $x < y$ or $y < x$.*

A set X together with a linear order in X is called a *linearly ordered set.*

Let the set X be linearly ordered by $<$ and let the set Y be linearly ordered by $<'$; we say that a mapping f of X to Y is *order preserving* if $f(x) <' f(y)$ for every pair x, y of elements of X satisfying $x < y$. If there exists an order preserving mapping of a linearly ordered set X onto a linearly ordered set Y, then we say that X and Y are *similar.*

We say that the element x_0 of a linearly ordered set X is the *smallest element* of X if $x_0 < x$ for every $x \in X \setminus \{x_0\}$. The *largest element* of a linearly ordered set is defined analogously. As every subset of a linearly ordered set is linearly ordered itself, the smallest and the largest elements of a subset of a linearly ordered set are well-defined. Clearly, the smallest and the largest elements need not exist.

Any pair (D, E) of subsets of a set X linearly ordered by $<$ such that $D \cup E = X$, $D \neq \emptyset \neq E$ and if $x \in D$ and $y \in E$, then $x < y$, is called a *cut* of X. The set D is called the *lower section* and the set E the *upper section* of the cut; clearly, the sections are disjoint. For every cut of a linearly ordered set exactly one of the following four conditions is satisfied:

(1) The lower section has a largest element and the upper section has a smallest element.
(2) The lower section has a largest element but the upper section has no smallest element.
(3) The lower section has no largest element but the upper section has a smallest element.

(4) The lower section has no largest element and the upper section has no smallest element.

When the condition (1) is satisfied, we say that the cut is a *jump*, and when (4) holds, we say that the cut is a *gap*.

A linearly ordered set X is *densely ordered* if no cut of X is a jump; if, moreover, no cut of X is a gap, X is *continuously ordered*. A set X is densely ordered if and only if for every pair x, y of elements of X satisfying $x < y$, there exists a $z \in X$ such that $x < z < y$, i.e., if X does not contain any pair of consecutive elements. The set X is continuously ordered if and only if, besides the above, for every non-empty subset $X_0 \subset X$, the set

$$\{x \in X : a < x \text{ for every } a \in X_0 \setminus \{x\}\}$$

is empty or has a smallest element.

Every linearly ordered set X is similar to a subset of the set of all cuts of X which satisfy (1), (2) or (4) linearly ordered by defining that

$$(D_1, E_1) < (D_2, E_2) \text{ if and only if } D_1 \subset D_2 \text{ and } D_1 \neq D_2;$$

the latter set has no gaps, and is continuously ordered if X is densely ordered.

A linear order $<$ in a set X is called a *well-order*, and the set X together with $<$ is called a *well-ordered set*, if $<$ has the following additional property:

(WO) *Every non-empty subset of X has a smallest element.*

One can prove that every set of cardinal numbers is well-ordered by the relation $<$ defined in the last section. Thus, for each cardinal number $\mathfrak{m}$ there exists a smallest cardinal number larger than $\mathfrak{m}$; it is denoted by $\mathfrak{m}^+$. Cardinal numbers $\aleph_1, \aleph_2, \ldots$ are defined recursively by the condition $\aleph_{i+1} = \aleph_i^+$ for $i = 0, 1, \ldots$ The equality $\mathfrak{c} = \aleph_1$ is called the *continuum hypothesis*; the continuum hypothesis is independent of the axioms of set theory.

To every well-ordered set X an ordinal number α is assigned, it is called the *order type* of X; the order types of well-ordered sets X and Y are equal if and only if X and Y are similar.

Let α and β be two ordinal numbers, which are order types of X and Y respectively. We say that α is *smaller* than β, or that β is *larger* than α, and we write $\alpha < \beta$ or $\beta > \alpha$ if there exists a $y_0 \in Y$ such that the sets X and $\{y \in Y : y < y_0\}$ are similar. One can prove that every set of ordinal numbers is well-ordered by the relation $<$. Any well-ordered set of type α is similar to the set of all ordinal numbers smaller than α linearly ordered by the relation $<$.

An ordinal number λ is a *limit number* if there is no ordinal number immediately preceding λ, i.e., if for every $\xi < \lambda$ there exists an ordinal number α such that $\xi < \alpha < \lambda$. If the ordinal number ξ immediately precedes α, then we say that ξ is the *predecessor* of α and α the *successor* of ξ, and we write $\alpha = \xi + 1$.

Every ordinal number has a successor; for an ordinal number α and an integer $n \geq 0$ we define inductively $\alpha + n$ by letting $\alpha + 0 = \alpha$ and $\alpha + n = [\alpha + (n-1)] + 1$ for $n \geq 1$. It turns out that any ordinal number can be uniquely represented as $\lambda + n$, where λ is a limit number and n is a non-negative integer. The number $\lambda + n$ is *even (odd)* if n is even (odd).

If the set of all ordinal numbers smaller than a limit number λ contains a subset A of type α such that for every $\xi < \lambda$ there exists a $\xi' \in A$ satisfying $\xi < \xi' < \lambda$, then we say that the ordinal number α is *cofinal* with λ.

Since every order preserving mapping is one-to-one, for every pair X, Y of similar sets we have $|X| = |Y|$. Hence, to every ordinal number α corresponds a cardinal number, the cardinality of any well-ordered set of type α; this cardinal number is called the *cardinality* of α and is denoted by $|\alpha|$. If $|\alpha| \leq \aleph_0$ we say that the ordinal number α is *countable.*

An infinite ordinal number λ (i.e., the order type of an infinite well-ordered set) is an *initial number* if λ is the smallest among all ordinal numbers α satisfying $|\alpha| = |\lambda|$, i.e., if $|\xi| < |\lambda|$ for every $\xi < \lambda$. An initial ordinal number λ is *regular* if there is no $\alpha < \lambda$ which is cofinal with λ.

For every cardinal number $\mathfrak{m}$ there exists an initial ordinal number λ such that $|\lambda| = \mathfrak{m}$ and this λ is unique (cf. the Zermelo theorem in the next section). A cardinal number $\mathfrak{m}$ is *regular* if the initial ordinal number λ satisfying $|\lambda| = \mathfrak{m}$ is regular. One easily checks that for every cardinal number $\mathfrak{m}$, the cardinal number $\mathfrak{m}^+$ is regular. The initial number of cardinality $\aleph_0$ is denoted by ω_0; this is the order type of the set of all positive integers with the natural order. The initial number of cardinality $\aleph_i$, for $i = 1, 2, \ldots$, is denoted by ω_i. Thus ω_1 is the smallest uncountable ordinal number. For every sequence $\alpha_1, \alpha_2, \ldots$ of ordinal numbers smaller than ω_1, there exists an ordinal number $\alpha < \omega_1$ such that $\alpha_i < \alpha$ for $i = 1, 2, \ldots$ More generally, to any ordinal number α correspond a cardinal number $\aleph_\alpha$ and an ordinal number ω_α which is the initial number of cardinality $\aleph_\alpha$; one proves that every cardinal number is equal to $\aleph_\alpha$ for some α.

Let α be an ordinal number, and X an arbitrary set; by a *transfinite sequence of type* α *with values in* X we understand any mapping f of the set $W(\alpha)$ of all ordinal numbers smaller than α, to X. The element of X assigned to the ordinal number $\xi < \alpha$ is denoted by x_ξ rather than $f(\xi)$, and the transfinite sequence itself is denoted by $x_0, x_1, \ldots, x_\xi, \ldots,\ \xi < \alpha$; sometimes it is more convenient to begin transfinite sequences with the term x_1 and not x_0. A transfinite sequence $A_0, A_1, \ldots, A_\xi, \ldots,\ \xi < \alpha$ of sets is called *increasing* if $A_\eta \subset A_\xi$ for $\eta < \xi < \alpha$ and *decreasing* if $A_\xi \subset A_\eta$ for $\eta < \xi < \alpha$.

To define a transfinite sequence we usually apply

THE THEOREM ON DEFINITIONS BY TRANSFINITE INDUCTION. *Suppose we are given a set Z and an ordinal number α. Let G denote the set of all transfinite sequences of types smaller than α with values in Z. For each function h assigning to every $g \in G$ an element of Z, there exists exactly one transfinite sequence f of type α such that*

$$f(\xi) = h(f|W(\xi)) \text{ for every } \xi < \alpha,$$

where $f|W(\xi)$ is the transfinite sequence of type ξ obtained by restricting f to the set $W(\xi)$ of all ordinal numbers smaller than ξ.

The theorem on definitions by transfinite induction is often applied in the case when Z is the family of subsets of a set X. The function h is then usually defined by three separate formulas. The first formula gives the value of h for the sequence g of type 0 (the order type of the empty set), which is itself the empty sequence, i.e., the value $h(\emptyset)$. The second, gives the value $h(g)$ for all sequences g whose type is of the form $\xi + 1$, and the third gives the value $h(g)$ for all sequences g whose type λ is a limit number. For instance, the first formula being

$$h(\emptyset) = A,$$

the second may be of the form

$$h(g) = F(g(\xi)),$$

and the third of the form

$$h(g) = G\Big(\bigcup_{\xi<\lambda} g(\xi)\Big) \quad \text{or} \quad h(g) = G\Big(\bigcap_{\xi<\lambda} g(\xi)\Big),$$

where F and G are given functions and A is a fixed set. Then the sequence $A_0, A_1, \ldots, A_\xi, \ldots$, $\xi < \alpha$, which exists by virtue of the theorem on definitions by transfinite induction, satisfies the following conditions:

$$A_0 = A,$$

$$A_{\xi+1} = F(A_\xi),$$

$$A_\lambda = G(\bigcup_{\xi<\lambda} A_\xi) \quad \text{or} \quad A_\lambda = G(\bigcap_{\xi<\lambda} A_\xi).$$

Hence, in order to define a transfinite sequence $A_0, A_1, \ldots, A_\xi, \ldots$, $\xi < \alpha$ it suffices to determine A_0 and to describe how $A_{\xi+1}$ depends on A_ξ and how A_λ depends either on $\bigcup_{\xi<\lambda} A_\xi$ or on $\bigcap_{\xi<\lambda} A_\xi$.

If a set X is well-ordered by $<$, then each subset A of X which satisfies for every $x_0 \in X$ the condition

$$\text{if } \{x \in X : x < x_0\} \subset A, \text{ then } x_0 \in A,$$

is equal to the set X itself. That fact is the base for inductive proofs. We shall apply it in the case when X is the set of all natural numbers (proofs by mathematical induction) as well as in the case when X is the set of all ordinal numbers smaller than a given ordinal number α (proofs by transfinite induction).

Let X be a set and $\leq$ a relation on X. We say that $\leq$ *orders* X, or that $\leq$ is an *order* in X, if $\leq$ has the following properties:

(OR1) *If $x \leq y$ and $y \leq z$, then $x \leq z$.*
(OR2) *For every $x \in X$, $x \leq x$.*
(OR3) *If $x \leq y$ and $y \leq x$, then $x = y$.*

A set X together with an order $\leq$ in X is called an *ordered set.** Two elements x and y of an ordered set X can be *incomparable*, i.e., it can happen that neither $x \leq y$ nor $y \leq x$ holds.

Every family of sets is ordered by the relation $\subset$.

If X is linearly ordered by $<$, then defining for $x, y \in X$ that

$$x \leq y \quad \text{whenever} \quad x < y \quad \text{or} \quad x = y,$$

we obtain an order in X; hence, every linearly ordered set can be regarded as an ordered set. If for each pair x, y of elements of a subset A of an ordered set X we have $x \leq y$ or $y \leq x$, then defining that

$$x < y \quad \text{whenever} \quad x \leq y \quad \text{and} \quad x \neq y,$$

we obtain a linear order in A; we then say that A is a *linearly ordered subset* of the set X ordered by $\leq$.

An element u of an ordered set X is called the *least upper bound* of a subset $A \subset X$, if $x \leq u$ for every $x \in A$ and if any $v \in X$ satisfying $x \leq v$ for every $x \in A$ also satisfies the inequality $u \leq v$. The *greatest lower bound* of a subset $A \subset X$ is defined analogously. Let us note that the least upper bound of X, if it exists, is the largest element of X, and that the least upper bound of $\emptyset$, if it exists, is the smallest element of X.

Let X be a set and $\leq$ a relation on X. We say that $\leq$ *directs* X, or that X is *directed* by $\leq$, if $\leq$ has the following properties:

* The terms *partially orders*, *partial order* and *partially ordered* are also used.

(D1) *If* $x \le y$ *and* $y \le z$, *then* $x \le z$.
(D2) *For every* $x \in X$, $x \le x$.
(D3) *For any* $x, y \in X$ *there exists a* $z \in X$ *such that* $x \le z$ *and* $y \le z$.

A subset A of a set X directed by $\le$ is *cofinal* in X if for every $x \in X$ there exists an $a \in A$ such that $x \le a$. Cofinal subsets of linearly ordered sets and of ordered sets are defined similarly.

Let the sets X and Y be ordered (directed) by $\le$ and $\le'$ respectively; we say that a function f from X to Y is *nondecreasing* if $f(x) \le' f(y)$ for every pair x, y of elements of X satisfying $x \le y$. *Nonincreasing* functions are defined analogously.

I.4. The axiom of choice

The axiom of choice is often used in the book, and its applications are not specially mentioned. Sometimes however, it is more convenient not to use the axiom of choice in its original form but in one of its alternative forms, which are important theorems of set theory. We shall state the well-ordering theorem and two maximal principles, that are alternative forms of the axiom of choice, and we shall prove the equivalence of all four statements. First we must, however, give two definitions.

We say that an element x_0 of an ordered set X is a *maximal element* of X if $x_0 \le x \in X$ implies that $x_0 = x$. Suppose we are given a set X and a property $\mathcal{P}$ pertaining to subsets of X; we say that $\mathcal{P}$ is a property of *finite character* if the empty set has this property and a set $A \subset X$ has property $\mathcal{P}$ if and only if all finite subsets of A have this property.

THE AXIOM OF CHOICE. *For every family* $\{X_s\}_{s \in S}$ *of non-empty sets there exists a function* f *from* S *to* $\bigcup_{s \in S} X_s$ *such that* $f(s) \in X_s$ *for every* $s \in S$.

THE ZERMELO THEOREM ON WELL-ORDERING. *On every set* X *there exists a relation* $<$ *which well-orders* X.

THE TEICHMÜLLER-TUKEY LEMMA. *If* $\mathcal{P}$ *is a property of finite character pertaining to subsets of a set* X, *then every set* $A \subset X$ *which has property* $\mathcal{P}$ *is contained in a set* $B \subset X$ *which has property* $\mathcal{P}$ *and is maximal in the family of all subsets of* X *that have* $\mathcal{P}$ *ordered by the relation* $\subset$.

THE KURATOWSKI-ZORN LEMMA. *If for each linearly ordered subset* A *of a set* X *ordered by* $\le$ *there exists an* $x_0 \in X$ *such that* $x \le x_0$ *for every* $x \in A$, *then* X *has a maximal element.*

The Zermelo theorem follows from the axiom of choice:

Let f be a function assigning to every non-empty subset A of X an element $f(A) \in A$; let us assume, moreover, that $f(\emptyset) = x \notin X$. Denote by $\mathcal{W}$ the family of all well-ordering relations defined on subsets of X and let α be the smallest ordinal number larger than all order types of subsets of X ordered by members of $\mathcal{W}$.

According to the theorem on definitions by transfinite induction, there exists a transfinite sequence $x_0, x_1, \ldots, x_\xi, \ldots,\ \xi < \alpha$ such that

$$x_\xi = f(X \setminus \{x_\gamma :\ \gamma < \xi\}) \text{ for every } \xi < \alpha.$$

If $x_\xi \ne x$, then $x_\xi \in X \setminus \{x_\gamma :\ \gamma < \xi\}$ and $x_\xi \ne x_\gamma$ for $\gamma < \xi$. Hence, if for all $\xi < \alpha$ we had $x_\xi \ne x$, there would exist a transfinite sequence of type α with distinct terms belonging to X, which is impossible by definition of α. Therefore there exists the smallest ordinal ξ such that $x_\xi = x$; we then have $X = \{x_\gamma : \gamma < \xi\}$, i.e., all elements of X are arranged into a transfinite sequence $x_0, x_1, \ldots, x_\gamma, \ldots,\ \gamma < \xi$ with $x_\gamma \neq x_{\gamma'}$ for $\gamma \neq \gamma'$, and this means that X can be well-ordered. ■

The Teichmüller-Tukey lemma follows from the Zermelo theorem:

Suppose we are given a set X, a property $\mathcal{P}$ of finite character pertaining to subsets of X and a set $A \subset X$ which has property $\mathcal{P}$. According to the Zermelo theorem the set $X \setminus A$ can be well-ordered. Since every well-ordered set is similar to the set of all ordinal numbers smaller than an ordinal number α, this means that all elements of $X \setminus A$ can be arranged into a transfinite sequence $x_1, x_2, \ldots, x_\xi, \ldots,\ \xi < \alpha$.

According to the theorem on definitions by transfinite induction, the proof of which does not require the application of the axiom of choice, there exists a transfinite sequence $A_0, A_1, \ldots, A_\xi, \ldots,\ \xi < \alpha$ such that $A_0 = A$,

$$A_{\xi+1} = \begin{cases} A_\xi \cup \{x_{\xi+1}\}, & \text{if } A_\xi \cup \{x_{\xi+1}\} \text{ has property } \mathcal{P} \\ A_\xi & \text{otherwise} \end{cases}$$

and $A_\lambda = \bigcup_{\xi<\lambda} A_\xi$, when λ is a limit number.

One readily sees that the elements of that sequence all have property $\mathcal{P}$ (in the case of elements with limit indices this follows from the finite character of $\mathcal{P}$), and that $B = \bigcup_{\xi<\alpha} A_\xi$ contains A and is a maximal subset of X having property $\mathcal{P}$. ■

The Kuratowski-Zorn lemma follows from the Teichmüller-Tukey lemma:

The property "to be a linearly ordered subset of the set X ordered by $\leq$" is a property of finite character. According to the Teichmüller-Tukey lemma, where we take $A = \emptyset$, there exists a maximal linearly ordered subset B of the set X ordered by $\leq$. Take an $x_0 \in X$ such that $x \leq x_0$ for every $x \in B$. It turns out that x_0 is a maximal element of X. Indeed, for any $x \in X$ such that $x_0 \leq x$ we have $x_0 = x$, since otherwise $B \cup \{x\}$ would be a linearly ordered subset of X larger than B. ■

The axiom of choice follows from the Kuratowski-Zorn lemma:

Let $\{X_s\}_{s\in S}$ be a family of non-empty sets. Denote by $\mathcal{X}$ the set of all pairs (T, f), where $T \subset S$ and f is a function from T to $\bigcup_{s\in S} X_s$ such that $f(s) \in X_s$ for every $s \in T$; defining that

$$(T_1, f_1) \leq (T_2, f_2) \text{ whenever } T_1 \subset T_2 \text{ and } f_2(s) = f_1(s) \text{ for } s \in T_1,$$

we order the set $\mathcal{X}$. One readily observes that for every linearly ordered subset $\mathcal{A} = \{(T_w, f_w)\}_{w\in W}$ of the set $\mathcal{X}$, the formulas

$$T_0 = \bigcup_{w\in W} T_w \text{ and } f_0(s) = f_w(s) \text{ for } s \in T_w$$

define an element (T_0, f_0) of $\mathcal{X}$ and that $(T_w, f_w) \leq (T_0, f_0)$ for every $w \in W$. According to the Kuratowski-Zorn lemma there exists a maximal element (T, f) in $\mathcal{X}$; we shall show that $T = S$, which will complete the proof. Indeed, assuming that there exists an $s_0 \in S \setminus T$, taking an $x_0 \in X_{s_0}$ and letting

$$T' = T \cup \{s_0\},\ f'(s) = f(s) \text{ for } s \in T \text{ and } f'(s_0) = x_0,$$

we define a pair $(T', f') \in \mathcal{X}$ such that $(T, f) \leq (T', f')$ and $(T, f) \neq (T', f')$, and thus obtain a contradiction. ■

I.5. Real numbers

In the book we use the properties of arithmetic operations on real numbers, the properties of the relation $<$ linearly ordering the set of real numbers, and the properties of the absolute value function. We also assume familiarity with the notion of the limit of a sequence of real numbers, denoted by lim, and the notion of a continuous function. The fact that every bounded above non-empy set of real numbers has a least upper bound, i.e., the continuity of the real line, is also used; the least upper bound is denoted by sup and the greatest lower bound by inf (in the case of finite sets we use the symbols max and min). When describing some examples of topological spaces we apply trigonometric functions and use a system of coordinates in the plane.

The open interval with end-points a and b, is denoted by (a, b), and the closed interval with end-points a and b is denoted by $[a, b]$; half-open intervals are denoted by $(a, b]$ and $[a, b)$. The set of all real numbers is denoted by R, and N denotes the set of all natural numbers, i.e., positive integers. The closed unit interval $[0, 1]$ is denoted by I.

Historical and bibliographic notes

An exhaustive course in set theory is given in the excellent book by Kuratowski and Mostowski [1976]. When mentioning here, in a few instances, that some statement is independent of or consistent with the axioms of set theory, we refer to the system of axioms adopted in that book. Our exposition of general topology using naive set theory, could easily be based on that system of axioms. To readers interested in independence proofs we recommend Kunen's book [1980].

The Teichmüller-Tukey lemma was proved by Teichmüller in [1939] and Tukey in [1940], and the Kuratowski-Zorn lemma by Kuratowski in [1922] and Zorn in [1935].

Chapter 1
Topological spaces

In this chapter we introduce the basic notions of general topology.

In Section 1.1 we define what a topological space is and introduce open sets, neighbourhoods, bases, subbases and bases at a point. We also define the weight and the character of a topological space and formulate the axioms of countability. Next, we introduce closed sets and consider the closure and the interior operators. At the end of the section the notion of a locally finite family of sets appears.

Section 1.2 is devoted to different methods of generating a topology on a set.

Section 1.3 is a continuation of Section 1.1. We consider the boundary and the derived set operators and introduce further important classes of sets in topological spaces: dense sets, co-dense sets, nowhere dense sets and Borel sets (in particular F_σ-sets and G_δ-sets).

In Section 1.4 we introduce the notion of a continuous mapping which turns out to be as important in the study of topological spaces as the notion of a topological space itself. We also discuss closed mappings, open mappings and homeomorphisms; this last class of mappings leads to the notion of homeomorphic spaces. Next, we define invariants and inverse invariants of a class of mappings and conclude with some remarks about the subject of general topology.

In Section 1.5 we discuss the axioms of separation, i.e., different kinds of restrictions concerning separation of points and closed sets in topological spaces. In this section we prove Urysohn's lemma, one of the most important theorems of general topology.

The last section is devoted to nets and filters, which lead to two different ways of describing convergence in general topological spaces. We discuss also two classes of topological spaces in which sequences suffice to describe convergence and topology.

1.1. Topological spaces. Open and closed sets. Bases. Closure and interior of a set

A *topological space* is a pair $(X, \mathcal{O})$ consisting of a set X and a family $\mathcal{O}$ of subsets of X satisfying the following conditions:

(O1) $\emptyset \in \mathcal{O}$ *and* $X \in \mathcal{O}$.

(O2) *If* $U_1 \in \mathcal{O}$ *and* $U_2 \in \mathcal{O}$, *then* $U_1 \cap U_2 \in \mathcal{O}$.

(O3) *If* $\mathcal{A} \subset \mathcal{O}$, *then* $\bigcup \mathcal{A} \in \mathcal{O}$.

The set X is called a *space*, the elements of X are called *points* of the space, and the subsets of X belonging to $\mathcal{O}$ are called *open* in the space; the family $\mathcal{O}$ of open subsets of X is also called a *topology* on X. The properties (O1)–(O3) of the family of open sets can be restated as follows:

(O1′) *The empty set and the whole space are open sets.*

(O2′) *The intersection of two open sets is an open set.*

(O3′) *The union of any family of open sets is an open set.*

From (O2) if follows immediately that the intersection of any finite family of open sets is an open set.

If for some $x \in X$ and an open set $U \subset X$ we have $x \in U$, we say that U is a *neighbourhood* of x. A set $V \subset X$ is open if and only if for every $x \in V$ there exists a neighbourhood U_x of the point x contained in V. Indeed, if V is an open set then we can take $U_x = V$ for every $x \in V$, and if the condition is satisfied, then $V = \bigcup_{x\in V} U_x$ is open by virtue of (O3).

A family $\mathcal{B} \subset \mathcal{O}$ is called a *base for a topological space* $(X, \mathcal{O})$ if every non-empty open subset of X can be represented as the union of a subfamily of $\mathcal{B}$. One can easily check that a family $\mathcal{B}$ of subsets of X is a base for the topological space $(X, \mathcal{O})$ if and only if $\mathcal{B} \subset \mathcal{O}$ and for every point $x \in X$ and any neighbourhood V of x there exists a $U \in \mathcal{B}$ such that $x \in U \subset V$. Obviously, a topological space can have many bases. Any base $\mathcal{B}$ has the following properties:

(B1) *For any $U_1, U_2 \in \mathcal{B}$ and every point $x \in U_1 \cap U_2$ there exists a $U \in \mathcal{B}$ such that $x \in U \subset U_1 \cap U_2$.*

(B2) *For every $x \in X$ there exists a $U \in \mathcal{B}$ such that $x \in U$.*

Indeed, the first property follows from the fact that $U_1 \cap U_2$ is an open set and the second one from the fact that X is an open set.

Every set of cardinal numbers being well-ordered by $<$, the set of all cardinal numbers of the form $|\mathcal{B}|$, where $\mathcal{B}$ is a base for a topological space $(X, \mathcal{O})$, has a smallest element; this cardinal number is called the *weight of the topological space* $(X, \mathcal{O})$ and is denoted by $w((X, \mathcal{O}))$.

A family $\mathcal{P} \subset \mathcal{O}$ is called a *subbase for a topological space* $(X, \mathcal{O})$ if the family of all finite intersections $U_1 \cap U_2 \cap \ldots \cap U_k$, where $U_i \in \mathcal{P}$ for $i = 1, 2, \ldots, k$ is a base for $(X, \mathcal{O})$.

A family $\mathcal{B}(x)$ of neighbourhoods of x is called a *base for a topological space* $(X, \mathcal{O})$ *at the point* x if for any neighbourhood V of x there exists a $U \in \mathcal{B}(x)$ such that $x \in U \subset V$. One can easily check that if $\mathcal{B}$ is a base for $(X, \mathcal{O})$, the family $\mathcal{B}(x)$ consisting of all elements of $\mathcal{B}$ that contain x is a base for $(X, \mathcal{O})$ at the point x. On the other hand, if for every $x \in X$ a base $\mathcal{B}(x)$ for $(X, \mathcal{O})$ at the point x is given, then the union $\mathcal{B} = \bigcup_{x\in X} \mathcal{B}(x)$ is a base for the space $(X, \mathcal{O})$.

The *character of a point* x in a topological space $(X, \mathcal{O})$ is defined as the smallest cardinal number of the form $|\mathcal{B}(x)|$, where $\mathcal{B}(x)$ is a base for $(X, \mathcal{O})$ at the point x; this cardinal number is denoted by $\chi(x, (X, \mathcal{O}))$. The *character of a topological space* $(X, \mathcal{O})$ is defined as the supremum of all numbers $\chi(x, (X, \mathcal{O}))$ for $x \in X$; this cardinal number is denoted by $\chi((X, \mathcal{O}))$.

If $\chi((X, \mathcal{O})) \le \aleph_0$, then we say that the space $(X, \mathcal{O})$ satisfies the *first axiom of countability* or is *first-countable*; this means that at every point x of X there exists a countable base. If $w((X, \mathcal{O})) \le \aleph_0$, then we say that the space $(X, \mathcal{O})$ satisfies the *second axiom of countability* or is *second-countable*; this means that $(X, \mathcal{O})$ has a countable base.

Let $(X, \mathcal{O})$ be a topological space and suppose that for every $x \in X$ a base $\mathcal{B}(x)$ for $(X, \mathcal{O})$ at x is given; the collection $\{\mathcal{B}(x)\}_{x\in X}$ is called a *neighbourhood system for the* *topological space* $(X, \mathcal{O})$. We shall show that any neighbourhood system $\{\mathcal{B}(x)\}_{x \in X}$ has the following properties:

(BP1) *For every $x \in X$, $\mathcal{B}(x) \neq \emptyset$ and for every $U \in \mathcal{B}(x)$, $x \in U$.*
(BP2) *If $x \in U \in \mathcal{B}(y)$, then there exists a $V \in \mathcal{B}(x)$ such that $V \subset U$.*
(BP3) *For any $U_1, U_2 \in \mathcal{B}(x)$ there exists a $U \in \mathcal{B}(x)$ such that $U \subset U_1 \cap U_2$.*

Property (BP1) follows directly from the definition of a base at x. Properties (BP2) and (BP3) also follow from the definition of a base at x, because $U \in \mathcal{B}(y)$ and $U_1 \cap U_2$ are open sets containing x.

Let $(X, \mathcal{O})$ be a topological space; a set $F \subset X$ is called *closed* in the space if its complement $X \setminus F$ is an open set. From De Morgan's laws and properties (O1)–(O3) of open sets we infer that the family $\mathcal{C}$ of closed sets has the following properties dual to (O1)–(O3):

(C1) *$X \in \mathcal{C}$ and $\emptyset \in \mathcal{C}$.*
(C2) *If $F_1 \in \mathcal{C}$ and $F_2 \in \mathcal{C}$, then $F_1 \cup F_2 \in \mathcal{C}$.*
(C3) *If $\mathcal{A} \subset \mathcal{C}$, then $\bigcap \mathcal{A} \in \mathcal{C}$.*

The above properties can be restated as follows:

(C1′) *The whole space and the empty set are closed sets.*
(C2′) *The union of two closed sets is a closed set.*
(C3′) *The intersection of any family of closed sets is a closed set.*

By way of example, we shall prove that (C3) holds. Let $\{F_s\}_{s \in S}$ be a family of closed sets. By definition, the complement $U_s = X \setminus F_s$ is open for every $s \in S$. As

$$\bigcap_{s \in S} F_s = \bigcap_{s \in S} (X \setminus U_s) = X \setminus \bigcup_{s \in S} U_s,$$

and the union $\bigcup_{s \in S} U_s$ is open by (O3), the intersection $\bigcap_{s \in S} F_s$ is closed.

Sets which are simultaneously open and closed in a topological space are called *open-and-closed* or *closed-and-open* sets.

For any $A \subset X$ consider the family $\mathcal{C}_A$ of all closed sets containing A. By (C1) we have $\mathcal{C}_A \neq \emptyset$, and from (C3) it follows that the intersection $\overline{A} = \bigcap \mathcal{C}_A$ is closed. It is easily seen that $\overline{A}$ is the smallest closed set containing A; this set is called the *closure* of A. Obviously, a set is closed if and only if it is equal to its own closure.

For any subsets A, B of a space X we have

$$\text{(1)} \qquad \textit{if} \quad A \subset B, \quad \textit{then} \quad \overline{A} \subset \overline{B}.$$

Indeed, from the inclusion $A \subset B$ it follows that $\mathcal{C}_B \subset \mathcal{C}_A$, and due to the definition of closures this implies that $\overline{A} \subset \overline{B}$.

We shall prove now

1.1.1. PROPOSITION. *For every $A \subset X$ the following conditions are equivalent:*
(i) *The point x belongs to $\overline{A}$.*
(ii) *For every neighbourhood U of x we have $U \cap A \neq \emptyset$.*
(iii) *There exists a base $\mathcal{B}(x)$ at x such that for every $U \in \mathcal{B}(x)$ we have $U \cap A \neq \emptyset$.*

PROOF. To prove that (i)$\Rightarrow$(ii) suppose that (ii) does not hold, i.e., that for a neighbourhood U of x we have $U \cap A = \emptyset$. We then have $A \subset X \setminus U$ and $X \setminus U \in \mathcal{C}_A$; hence $\overline{A} \subset X \setminus U$ and $x \notin \overline{A}$, i.e., (i) does not hold.

The implication (ii)$\Rightarrow$(iii) is obvious; it remains to show that (iii)$\Rightarrow$(i). Suppose that (i) does not hold, i.e., that $x \notin \overline{A}$. There exists then a closed set $F \in \mathcal{C}_A$ such that $x \notin F$. For the open set $V = X \setminus F$ we have

$$x \in V \quad \text{and} \quad V \cap A = \emptyset. \tag{2}$$

Now, for every base $\mathcal{B}(x)$ at x there exists a $U \in \mathcal{B}(x)$ such that $x \in U \subset V$ and from (2) it follows that $U \cap A = \emptyset$, i.e., (iii) does not hold.■

1.1.2. COROLLARY. *If U is an open set and $U \cap A = \emptyset$, then also $U \cap \overline{A} = \emptyset$.*

In particular, *if U and V are disjoint open sets, then $U \cap \overline{V} = \emptyset = \overline{U} \cap V$.*

PROOF. Suppose that there exists an $x \in U \cap \overline{A}$. Thus $x \in \overline{A}$ and U is a neighbourhood of x. From Proposition 1.1.1 it follows that $U \cap A \neq \emptyset$ in contradiction to the assumption, so that we have $U \cap \overline{A} = \emptyset$.■

The most important properties of the closure operator are listed in the next theorem.

1.1.3. THEOREM. *The closure operator has the following properties:*

(CO1) $\overline{\emptyset} = \emptyset$.
(CO2) $A \subset \overline{A}$.
(CO3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(CO4) $\overline{(\overline{A})} = \overline{A}$.

PROOF. Properties (CO1) and (CO2) follow directly from the definition of closure, and property (CO4) follows from the fact that $\overline{A}$ is a closed set.

From (1) it follows that $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$; we then have

$$\overline{A} \cup \overline{B} \subset \overline{A \cup B}. \tag{3}$$

By (CO2), $A \subset \overline{A}$ and $B \subset \overline{B}$, so that $A \cup B \subset \overline{A} \cup \overline{B}$. Since the last set is closed, being the union of two closed sets, from the definition of closure it follows that

$$\overline{A \cup B} \subset \overline{A} \cup \overline{B}. \tag{4}$$

Formulas (3) and (4) establish the equality (CO3).■

The *interior* of a set $A \subset X$ is the union of all open sets contained in A or – equivalently – the largest open set contained in A; this set is denoted by $\operatorname{Int} A$. Obviously, a set is open if and only if it is equal to its own interior.

The following proposition is an immediate consequence of the definition of interior.

1.1.4. PROPOSITION. *The point x belongs to* $\operatorname{Int} A$ *if and only if there exists a neighbourhood U of x such that $U \subset A$.*■

As shown by the following theorem, the interior operator is closely related to the closure operator.

1.1.5. THEOREM. *For every* $A \subset X$ *we have* $\operatorname{Int} A = X \setminus \overline{X \setminus A}$.

PROOF. From (CO2) it follows that $X \setminus A \subset \overline{X \setminus A}$, so that $X \setminus \overline{X \setminus A} \subset X \setminus (X \setminus A) = A$. Since the set $X \setminus \overline{X \setminus A}$ is open, we have

(5) $$X \setminus \overline{X \setminus A} \subset \operatorname{Int} A.$$

For any open set U contained in A we have $X \setminus A \subset X \setminus U = \overline{X \setminus U}$, so that by (1) $\overline{X \setminus A} \subset X \setminus U$ or, equivalently, $U \subset X \setminus \overline{X \setminus A}$; in particular

$$\operatorname{Int} A \subset X \setminus \overline{X \setminus A}.$$

The last inclusion, together with (5), establishes our theorem. ∎

The properties of the interior operator dual to (CO1)–(CO4) are listed in the following theorem, which is a consequence of 1.1.3, 1.1.5 and De Morgan's laws.

1.1.6. THEOREM. *The interior operator has the following properties:*

(IO1) $\operatorname{Int} X = X$.

(IO2) $\operatorname{Int} A \subset A$.

(IO3) $\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B$.

(IO4) $\operatorname{Int}(\operatorname{Int} A) = \operatorname{Int} A$. ∎

1.1.7. EXAMPLE. Let X be an arbitrary set and $\mathcal{O}$ the family of all subsets of X. Clearly $(X, \mathcal{O})$ is a topological space. Every set $A \subset X$ is open-and-closed. Any set containing x is a neighbourhood of x. The family of all one-point subsets of X is a base for $(X, \mathcal{O})$. This is a base of minimal cardinality; therefore the weight of $(X, \mathcal{O})$ is equal to the cardinality of X. For every $x \in X$ the family consisting of the single set $\{x\}$ is a base for $(X, \mathcal{O})$ at x; hence $(X, \mathcal{O})$ satisfies the first axiom of countability. Every set $A \subset X$ is equal to its own closure and to its own interior.

The topological space described above is called the *discrete space*, and $\mathcal{O}$ is called the *discrete topology* on X. ∎

1.1.8. EXAMPLE. Let X be an arbitrary infinite set, x_0 a point in X and $\mathcal{O}$ the family consisting of all subsets of X that do not contain x_0 and of all subsets of X that have finite complement. One verifies readily that $(X, \mathcal{O})$ is a topological space. All one-point subsets of X, except for the set $\{x_0\}$, are open-and-closed; the set $\{x_0\}$ is closed but is not open. The family consisting of all one-point sets $\{x\}$ with $x \neq x_0$ and of all sets of the form $X \setminus F$ with F finite is a base for $(X, \mathcal{O})$. This is a base of minimal cardinality; therefore the weight of $(X, \mathcal{O})$ is equal to the cardinality of X. The family consisting of all one-point sets $\{x\}$ with $x \neq x_0$ and of all sets of the form $X \setminus \{x\}$ is a subbase for $(X, \mathcal{O})$. For every $A \subset X$ we have

$$\overline{A} = \begin{cases} A, & \text{if } A \text{ is finite,} \\ A \cup \{x_0\}, & \text{if } A \text{ is infinite} \end{cases}$$

and

$$\operatorname{Int} A = \begin{cases} A, & \text{if } X \setminus A \text{ is finite,} \\ A \setminus \{x_0\}, & \text{if } X \setminus A \text{ is infinite.} \end{cases}$$

This implies that any two infinite closed subsets of X have non-empty intersection. ∎

The topological spaces defined in the next two examples are of utmost importance for general topology.

1.1.9. EXAMPLE. Let R be the set of real numbers and $\mathcal{O}$ the family consisting of all sets $U \subset R$ with the property that for every $x \in U$ there exists an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset U$. Obviously, the family $\mathcal{O}$ has properties (O1)–(O3). From the definition of the limit of a sequence it follows that a set $A \subset R$ is closed if and only if together with any convergent sequence it contains its limit. The family of all open intervals with rational end-points is a base for $(R, \mathcal{O})$. This is a base of minimal cardinality; therefore $(R, \mathcal{O})$ is second-countable and, *a fortiori*, first-countable.

The topology $\mathcal{O}$ is called the *natural topology on the real line.* ■

1.1.10. EXAMPLE. Let $I = [0,1]$ be the closed unit interval and $\mathcal{O}$ the family consisting of all sets of the form $I \cap U$, where $U \subset R$ is open with respect to the natural topology on R. Clearly $(I, \mathcal{O})$ is a topological space. The family of all intervals of the form (q,r), $[0,r)$, $(q,1]$, where q,r are rational numbers and $0 < q < r < 1$, is a base for $(I, \mathcal{O})$. All intervals of the last two types form a subbase. The space $(I, \mathcal{O})$ is both first- and second-countable. A set $A \subset I$ is closed in I if and only if it is closed in R.

The topology $\mathcal{O}$ is called the *natural topology on the interval* I. ■

From the above examples it follows that for a set X one can select in many different ways a family $\mathcal{O}$ such that $(X, \mathcal{O})$ is a topological space. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two topologies on X and $\mathcal{O}_2 \subset \mathcal{O}_1$, then we say that the topology $\mathcal{O}_1$ is *finer* than the topology $\mathcal{O}_2$, or that $\mathcal{O}_2$ is *coarser* than $\mathcal{O}_1$. The discrete topology on X is the finest one. The coarsest topology on X consists of $\emptyset$ and X only; it is called the *anti-discrete* topology on X and the set X equipped with this topology is called the *anti-discrete space.* Clearly the family of all topologies on a set X is ordered by $\subset$.

Let X be an arbitrary infinite set, x_0 and x_0' two distinct points in X, $\mathcal{O}$ the topology defined in 1.1.8 and $\mathcal{O}'$ the topology defined in a similar way for x_0'. One can readily verify that the topologies $\mathcal{O}$ and $\mathcal{O}'$ are incomparable: neither of them is finer than the other.

In the sequel we shall usually consider only one topology $\mathcal{O}$ on a set X at a time. To simplify notations, we shall use the symbol X instead of $(X, \mathcal{O})$ for a topological space. Consequently, we shall write $w(X)$, $\chi(x, X)$ and $\chi(X)$. This notation is not quite precise, but it will always follow from the context which topology on X is being considered. Often, we shall simply say "a space" instead of "a topological space".

From the property (CO3) of the closure operator it follows that the closure of a finite union of sets is equal to the union of closures of those sets, i.e., that the closure operator is finitely additive. Simple examples show that the closure operator is not countably additive (cf. Exercise 1.1.B). We shall now define an important class of families of sets for which the closure operator is additive.

A family $\{A_s\}_{s \in S}$ of subsets of a topological space X is *locally finite* if for every point $x \in X$ there exists a neighbourhood U such that the set $\{s \in S : U \cap A_s \neq \emptyset\}$ is finite. If every point $x \in X$ has a neighbourhood that intersects at most one set of a given family, than we say that the family is *discrete.* Clearly any discrete family, as well as any finite family, is locally finite.

1.1.11. THEOREM. *For every locally finite family $\{A_s\}_{s\in S}$ we have the equality $\overline{\bigcup_{s\in S} A_s} = \bigcup_{s\in S} \overline{A}_s$.*

PROOF. From (1) it follows that $\overline{A}_s \subset \overline{\bigcup_{s\in S} A_s}$ for every $s \in S$; therefore we have $\bigcup_{s\in S} \overline{A}_s \subset \overline{\bigcup_{s\in S} A_s}$. To prove the reverse inclusion, let us note that, by local finiteness of $\{A_s\}_{s\in S}$, for every $x \in \overline{\bigcup_{s\in S} A_s}$ there exists a neighbourhood U such that the set $S_0 = \{s \in S : U \cap A_s \neq \emptyset\}$ is finite. From Proposition 1.1.1 it follows that $x \notin \overline{\bigcup_{s\in S\setminus S_0} A_s}$; since

$$x \in \overline{\bigcup_{s\in S} A_s} = \overline{\bigcup_{s\in S_0} A_s} \cup \overline{\bigcup_{s\in S\setminus S_0} A_s},$$

we have $x \in \overline{\bigcup_{s\in S_0} A_s} = \bigcup_{s\in S_0} \overline{A}_s \subset \bigcup_{s\in S} \overline{A}_s$.■

1.1.12. COROLLARY. *Let $\mathcal{F}$ be a locally finite family and $F = \bigcup \mathcal{F}$. If all members of $\mathcal{F}$ are closed, then F is a closed set and if all members of $\mathcal{F}$ are closed-and-open, then F is a closed-and-open set.*■

1.1.13. THEOREM. *If $\{A_s\}_{s\in S}$ is a locally finite (discrete) family, then the family $\{\overline{A}_s\}_{s\in S}$ also is locally finite (discrete)*.*■

We conclude this section with two theorems about families of open sets in a space of weight $\mathfrak{m}$. Both these theorems will be used later in the book.

1.1.14. THEOREM. *If $w(X) \leq \mathfrak{m}$, then for every family $\{U_s\}_{s\in S}$ of open subsets of X there exists a set $S_0 \subset S$ such that $|S_0| \leq \mathfrak{m}$ and $\bigcup_{s\in S_0} U_s = \bigcup_{s\in S} U_s$.*

PROOF. Take a base $\mathcal{B}$ for the space X satisfying $|\mathcal{B}| \leq \mathfrak{m}$ and denote by $\mathcal{B}_0$ the collection of all $U \in \mathcal{B}$ such that for some $s \in S$ we have $U \subset U_s$. For every $U \in \mathcal{B}_0$ choose an $s(U) \in S$ such that

$$U \subset U_{s(U)}. \tag{6}$$

In this way a function s from $\mathcal{B}_0$ into S is defined; we shall show that the set $S_0 = s(\mathcal{B}_0) \subset S$ satisfies the theorem.

First of all $|S_0| = |s(\mathcal{B}_0)| \leq |\mathcal{B}| \leq \mathfrak{m}$. Let us now take a point $x \in \bigcup_{s\in S} U_s$. There exists an $s \in S$ such that $x \in U_s$ and a $U \in \mathcal{B}$ such that $x \in U \subset U_s$; clearly, $U \in \mathcal{B}_0$ and $s(U) \in S_0$. From (6) it follows that

$$x \in U \subset U_{s(U)} \subset \bigcup_{s\in S_0} U_s,$$

therefore $\bigcup_{s\in S} U_s \subset \bigcup_{s\in S_0} U_s$. The reverse inclusion is obvious.■

1.1.15. THEOREM. *If $w(X) \leq \mathfrak{m}$ then for every base $\mathcal{B}$ for X there exists a base $\mathcal{B}_0$ such that $|\mathcal{B}_0| \leq \mathfrak{m}$ and $\mathcal{B}_0 \subset \mathcal{B}$.*

PROOF. Suppose that $\mathfrak{m} \geq \aleph_0$ and take a base $\mathcal{B}_1 = \{W_t\}_{t\in T}$ for X such that $|T| \leq \mathfrak{m}$. Let $\mathcal{B} = \{U_s\}_{s\in S}$ and for every $t \in T$ let

$$S(t) = \{s \in S : U_s \subset W_t\}.$$

* The way parentheses are used here allows us to write down two parallel statements simultaneously. To read the first statement one omits the words in parentheses and to read the second statement one omits the words that immediately precede parentheses.

Since $\mathcal{B}$ is a base for X we have $\bigcup_{s\in S(t)} U_s = W_t$ and by 1.1.14 there exists a set $S_0(t) \subset S(t)$ such that

$$|S_0(t)| \le \mathfrak{m} \tag{7}$$

and

$$W_t = \bigcup_{s\in S(t)} U_s = \bigcup_{s\in S_0(t)} U_s. \tag{8}$$

Let $\mathcal{B}_0 = \{U_s\}_{s\in S_0(t), t\in T}$. Since $|T| \le \mathfrak{m}$, from (7) and the equality $\mathfrak{m}^2 = \mathfrak{m}$ it follows that $|\mathcal{B}_0| \le \mathfrak{m}$. We shall show now that $\mathcal{B}_0$ is a base. Take an arbitrary point $x \in X$ and its neighbourhood V. As $\mathcal{B}_1$ is a base, for some $t \in T$ we have $x \in W_t \subset V$ and by (8) there exists an $s \in S_0(t)$ such that $x \in U_s \subset W_t \subset V$. Clearly $U_s \in \mathcal{B}_0$, and this proves that $\mathcal{B}_0$ is a base for X.

The proof in the case of a finite $\mathfrak{m}$ consists in showing that if $\mathcal{B}_1$ is a base and $|\mathcal{B}_1| = w(X) \le \mathfrak{m}$, then $\mathcal{B}_1 \subset \mathcal{B}$; this is left to the reader as an exercise. ■

1.1.16. REMARK. Let us note that in the proof of Theorem 1.1.14 we did not use the fact that members of $\mathcal{B}$ are open: all we used was that $|\mathcal{B}| \le \mathfrak{m}$ and that for every point $x \in X$ and its neighbourhood V there exists a $U \in \mathcal{B}$ such that $x \in U \subset V$ (cf. the notion of a network – defined in Section 3.1 – and Theorem 3.8.12).

Historical and bibliographic notes

The emergence of general topology is a consequence of the rebuilding of the foundations of calculus achieved during the 19th century. Endeavours at making analysis independent of naive geometric intuition and mechanical arguments, to which the inventors of calculus I. Newton (1642–1727) and G. Leibniz (1646–1716) referred, led to a precise definition of limit (J. d'Alembert (1717–1783) and A. L. Cauchy (1784–1857)), to formulation of tests for convergence of infinite series (C. F. Gauss (1777–1855)) and to clarifying the notion of a continuous function (B. Bolzano (1781–1848) and Cauchy). The necessity of resting calculus on a firmer base was generally recognized when various pathological phenomena in convergence of trigonometric series were discovered (N. H. Abel (1802–1829), P. G. Lejeune-Dirichlet (1805–1859), P. du Bois-Reymond (1831–1889)) and the first examples of nowhere differentiable continuous functions were described (Bolzano, B. Riemann (1826–1866) and K. Weierstrass (1815–1897) in 1830, 1854 and 1861, respectively). The latter examples unsettled common outlooks and led to a revision of the notion of a number and to the rise of rigorous theories of real numbers. The most important ones were: the theory proposed independently by Ch. Méray (1835–1911) and by G. Cantor (1845–1918), where real numbers were defined as equivalence classes of Cauchy sequences of rationals, and the theory due to R. Dedekind (1831–1916), where real numbers were defined as cuts in the set of rationals. Both theories gave a description of the topological structure of the real line.

General topology owes its beginnings to a sequel of papers by Cantor published in 1879–1884. Discussing the uniqueness problems for trigonometric series, Cantor concentrated on the study of sets of "exceptional points", where one could drop some hypotheses of a theorem without damaging the theorem itself. Later he devoted himself to an investigation of sets, originating in this way both set theory and topology. Cantor defined and studied, in the realm of subsets of Euclidean spaces, some of the fundamental notions of topology. Further important notions, also restricted to Euclidean spaces, were introduced in 1893–1905 by C. Jordan (1838–1922), H. Poincaré (1854–1912), E. Borel (1871–1956), R. Baire (1874–1932) and H. Lebesgue (1875–1941).

The decisive step forward was the move from Euclidean spaces to abstract spaces. Here, Riemann was the precursor; in 1854 he introduced and studied the notion of a two dimensional manifold and pointed out the possibility of studying higher dimensional manifolds as well as function spaces. Around 1900, when fundamental topological notions were already introduced, a few papers appeared exhibiting the existence of natural topological structures on some special sets, such as: the set of curves (G. Ascoli (1843–1896)), the set of functions (C. Arzelà (1847–1912), V. Volterra (1860–1940), D. Hilbert (1862–1943) and I. Fredholm (1866–1927)) and the set of lines and planes in the three dimensional space (Borel). In this way the ground was prepared for an axiomatic treatment of the notion of a limit and, more generally, of the notion of proximity of a point to a set.

The genesis of general topology that we have summarized here very briefly is exhaustively discussed in Manheim's book [1964]. The history of the first years of general topology is carefully described by Rosenthal and Zoretti in [1924] and by Tietze and Vietoris in [1930]. Rich historical information can also be found in Kuratowski's two-volume monograph [1966] and [1968].

Abstract spaces with a topological structure were first introduced by Fréchet in [1906] and by Riesz in [1907] and [1908]. Fréchet defined his spaces in terms of convergent sequences (see Problem 1.7.18), Riesz – in terms of accumulation points. A serious drawback of the Fréchet approach consisted in the restriction to countable sequences, which made the class of spaces under consideration decidedly too narrow. Nevertheless, a theory based on Fréchet's definitions has been developed (see his book [1926]). On the other hand, the Riesz definition was too general (the closure operator defined by adjoining to a set all its accumulation points did not satisfy (CO4)) and too complicated; a theory based on that definition of a space was only sketched by Riesz and was never developed in detail. The first satisfactory definition of topological spaces was proposed by Hausdorff in [1914]; he defined a topological space as an abstract set provided with a neighbourhood system satisfying conditions (BP1)–(BP3) and condition (BP4) in Section 1.5 (a narrower class, consisting of spaces which in addition satisfy the first axiom of countability, was discussed by Root in [1914] (announcement in [1911])). In his definition of a topological space, Hausdorff developed an idea appearing in papers of Hilbert [1902] and Weyl [1913], who gave an axiomatic description, in terms of neighbourhoods, of the plane and of a Riemann surface, respectively. Hausdorff's contribution was to give suitable generality to notions introduced by his predecessors and to develop a systematic and exhaustive theory. Hausdorff's book [1914] is still worth glancing through, even if only to see that due to a limpid, precise and elegant exposition it is perfectly readable after 70 years. Another set of axioms was proposed by R. L. Moore. In his book [1932] (revised edition [1962]) he gave an axiomatic characterization of the plane and discussed in detail various classes of abstract spaces obtained by assuming that only some of these axioms hold; the original version of this approach can be found in R. L. Moore's paper [1916].

The definition of a topological space given here, and now generally adopted, was first formulated by Kuratowski in [1922a] in terms of a closure operator satisfying (CO1)–(CO4). The notions of open and closed set as well as those of closure and interior were introduced and studied by Cantor in the class of subsets of Euclidean spaces. Hausdorff generalized them to abstract spaces in [1914], where both countability axioms can also be found. The notion of a locally finite family of sets was introduced by Alexandroff in [1924].

Exercises

1.1.A. Verify that, for all subsets A and B of a topological space, we have $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$; check that the inclusions cannot be replaced by equalities.

1.1.B. Show that for any sequence $A_1, A_2, \ldots$ of subsets of a topological space we have

$$\overline{\bigcup_{i=1}^{\infty} A_i} = \bigcup_{i=1}^{\infty} \overline{A_i} \cup \bigcap_{i=1}^{\infty} \overline{\bigcup_{j=0}^{\infty} A_{i+j}}.$$

Show by an example that the above equality does not hold when the second term on the right side is omitted.

1.1.C (Kuratowski [1922a]). A subset U of a topological space satisfying the condition $U = \operatorname{Int} \overline{U}$ is called an *open domain.*

(a) Verify that the interior of a closed set is an open domain.

(b) Show that the intersection of two open domains is an open domain. Note that the union of two open domains need not be an open domain.

(c) Verify that for open domains U and V the inclusion $U \subset V$ holds if and only if $\overline{U} \subset \overline{V}$.

(d) Prove that for any family $\{U_s\}_{s\in S}$ of open domains in a topological space X the set $\operatorname{Int}(\overline{\bigcup_{s\in S} U_s})$ is the least upper bound, and the set $\operatorname{Int}(\bigcap_{s\in S} U_s)$ is the greatest lower bound, of $\{U_s\}_{s\in S}$ in the family of all open domains in X ordered by inclusion.

(e) A subset A of a topological space satisfying the condition $A = \overline{\operatorname{Int} A}$ is called a *closed domain.* Show that A is a closed domain if and only if its complement is an open domain. Formulate and prove properties of closed domains which are dual to the properties of open domains stated in (a)–(d).

1.1.D. Show that for every family $\{\mathcal{O}_s\}_{s\in S}$ of topologies on a set X there exists a topology on X which is the least upper bound of $\{\mathcal{O}_s\}_{s\in S}$ (i.e., the coarsest topology in the family of all topologies that are finer than every $\mathcal{O}_s$) and that there exists also a topology on X which is the greatest lower bound of $\{\mathcal{O}_s\}_{s\in S}$ (i.e., the finest topology in the family of all topologies that are coarser than every $\mathcal{O}_s$).

1.2. Methods of generating topologies

Let X be an arbitrary set; by *generating a topology* on X we mean selecting a family $\mathcal{O}$ of subsets of X that satisfies conditions (O1)–(O3), i.e., a family $\mathcal{O}$ such that the pair $(X, \mathcal{O})$ is a topological space. Often it is more convenient not to describe the family $\mathcal{O}$ of open sets directly. We shall now give some other methods of generating topologies; they consist in the definition of a base, of a neighbourhood system, of the family of closed sets, of a closure operator, or of an interior operator.

1.2.1. PROPOSITION. *Suppose we are given a set X and a family $\mathcal{B}$ of subsets of X which has properties* (B1)–(B2). *Let $\mathcal{O}$ be the family of all subsets of X that are unions of subfamilies of $\mathcal{B}$, i.e., let $U \in \mathcal{O}$ if and only if $U = \bigcup \mathcal{B}_0$ for a subfamily $\mathcal{B}_0$ of $\mathcal{B}$. The family $\mathcal{O}$ satisfies conditions* (O1)–(O3) *and the family $\mathcal{B}$ is a base for the topological space $(X, \mathcal{O})$.*

The topology $\mathcal{O}$ is called the *topology generated by the base* $\mathcal{B}$.

PROOF. Condition (O1) is satisfied because $\emptyset = \bigcup \mathcal{B}_0$ for $\mathcal{B}_0 = \emptyset$ and, by (B2), $X = \bigcup \mathcal{B}_0$ for $\mathcal{B}_0 = \mathcal{B}$.

Take $U_1, U_2 \in \mathcal{O}$; we then have $U_1 = \bigcup_{s\in S} U_s$ and $U_2 = \bigcup_{t\in T} U_t$, where $U_s, U_t \in \mathcal{B}$ for $s \in S$ and $t \in T$. Since

$$U_1 \cap U_2 = \bigcup_{s\in S, t\in T} U_s \cap U_t,$$

to show that condition (O2) is satisfied, it is enough to prove that $U_s \cap U_t$ is the union of a subfamily of $\mathcal{B}$. By (B1), for every $x \in U_s \cap U_t$ there exists a $U(x) \in \mathcal{B}$ such that

$$x \in U(x) \subset U_s \cap U_t,$$

and this implies that

$$U_s \cap U_t = \bigcup \mathcal{B}_0 \quad \text{for} \quad \mathcal{B}_0 = \{U(x) : x \in U_s \cap U_t\}.$$

Condition (O3) is satisfied by the definition of the family $\mathcal{O}$.

Clearly $\mathcal{B}$ is a base for the space $(X, \mathcal{O})$. ■

1.2.2. EXAMPLE. Let K be the set of all real numbers and let $\mathcal{B}$ be the family of all intervals $[x, r)$, where $x, r \in K$, $x < r$ and r is a rational number. One can readily check that the family $\mathcal{B}$ has properties (B1)–(B2).

Members of $\mathcal{B}$ are open-and-closed with respect to the topology generated by the base $\mathcal{B}$. Obviously, $|\mathcal{B}| = \mathfrak{c}$; we shall show that $w(K) = \mathfrak{c}$. Let $\mathcal{R}$ be a family of open subsets of K such that $|\mathcal{R}| < \mathfrak{c}$. There exists a point $x_0 \in K$ which is not the greatest lower bound of any member of $\mathcal{R}$; hence the open set $[x_0, x_0 + 1)$ cannot be represented as the union of a subfamily of $\mathcal{R}$ and therefore $\mathcal{R}$ is not a base for K.

The space K is called the *Sorgenfrey line*. ■

1.2.3. PROPOSITION. *Suppose we are given a set X and a collection $\{\mathcal{B}(x)\}_{x\in X}$ of families of subsets of X which has properties* (BP1)–(BP3). *Let $\mathcal{O}$ be the family of all subsets of X that are unions of subfamilies of $\bigcup_{x\in X} \mathcal{B}(x)$. The family $\mathcal{O}$ satisfies conditions* (O1)–(O3) *and the collection $\{\mathcal{B}(x)\}_{x\in X}$ is a neighbourhood system for the topological space $(X, \mathcal{O})$.*

The topology $\mathcal{O}$ is called the *topology generated by the neighbourhood system* $\{\mathcal{B}(x)\}_{x\in X}$. ■

1.2.4. EXAMPLE. Let L be the subset of the plane defined by the condition $y \geq 0$, i.e., the closed upper half-plane; denote by L_1 the line $y = 0$ and let $L_2 = L \setminus L_1$. For every $x \in L_1$ and $r > 0$ let $U(x, r)$ be the set of all points of L inside the circle of radius r tangent to L_1 at x and let $U_i(x) = U(x, 1/i) \cup \{x\}$ for $i = 1, 2, \ldots$ For every $x \in L_2$ and $r > 0$ let $U(x, r)$ be the set of all points of L inside the circle of radius r and centre at x and let $U_i(x) = U(x, 1/i)$ for $i = 1, 2, \ldots$ One can readily check that the collection $\{\mathcal{B}(x)\}_{x \in L}$, where $\mathcal{B}(x) = \{U_i(x)\}_{i=1}^{\infty}$, has properties (BP1)–(BP3).

The set L_1 is closed with respect to the topology generated by the neighbourhood system $\{\mathcal{B}(x)\}_{x \in X}$.

The space L is called the *Niemytzki plane.* ■

1.2.5. PROPOSITION. *Suppose we are given a set X and a family $\mathcal{C}$ of subsets of X which has properties* (C1)–(C3). *The family $\mathcal{O} = \{X \setminus F : F \in \mathcal{C}\}$ satisfies conditions* (O1)–(O3) *and $\mathcal{C}$ is the family of all closed sets in the topological space* $(X, \mathcal{O})$.

The topology $\mathcal{O}$ is called the *topology generated by the family of closed sets* $\mathcal{C}$. ■

1.2.6. EXAMPLE. Let X be an arbitrary infinite set and $\mathcal{C}$ the family consisting of all finite subsets of X and X itself. One can readily check that the family $\mathcal{C}$ has properties (C1)–(C3).

All complements of finite sets and the empty set are the only subsets of X which are open with respect to the topology generated by the family $\mathcal{C}$ of closed sets. Any two non-empty open subsets of X have non-empty intersection. ■

1.2.7. PROPOSITION. *Suppose we are given a set X and an operator assigning to every set $A \subset X$ a set $\overline{A} \subset X$ in such a way that conditions* (CO1)–(CO4) *are satisfied. The family $\mathcal{O} \doteq \{X \setminus A : A = \overline{A}\}$ satisfies conditions* (O1)–(O3) *and for every $A \subset X$ the set $\overline{A}$ is the closure of A in the topological space* $(X, \mathcal{O})$.

The topology $\mathcal{O}$ is called the *topology generated by the closure operator* $^{-}$.

PROOF. To prove the first part of the proposition it suffices to show that the family $\mathcal{C} = \{A : A = \overline{A}\}$ has properties (C1)–(C3). Since $\overline{A} \subset X$ for every $A \subset X$ we have $\overline{X} \subset X$, and this together with (CO2) show that $\overline{X} = X$. By (CO1) we have $\overline{\emptyset} = \emptyset$. Thus the family $\mathcal{C}$ has property (C1).

Let us take $F_1, F_2 \in \mathcal{C}$, i.e., let $F_1 = \overline{F}_1$ and $F_2 = \overline{F}_2$. By (CO3) $\overline{F_1 \cup F_2} = \overline{F}_1 \cup \overline{F}_2 = F_1 \cup F_2$, and this implies that $F_1 \cup F_2 \in \mathcal{C}$. Thus the family $\mathcal{C}$ has property (C2).

Let us note that (CO3) implies that

$$\text{(1)} \qquad \text{if} \quad A \subset B, \quad \text{then} \quad \overline{A} \subset \overline{B}.$$

Indeed, if $A \subset B$, then $A \cup B = B$ and $\overline{A} \cup \overline{B} = \overline{A \cup B} = \overline{B}$. The last equality gives $\overline{A} \subset \overline{B}$.

Let us take now a family $\{F_s\}_{s \in S}$ of members of $\mathcal{C}$, i.e., let $F_s = \overline{F}_s$ for $s \in S$. As $\bigcap_{s \in S} F_s \subset F_s$, by (1) we have $\overline{\bigcap_{s \in S} F_s} \subset \overline{F}_s = F_s$ and this implies that $\overline{\bigcap_{s \in S} F_s} \subset \bigcap_{s \in S} F_s$. The last inclusion together with (CO2) show that $\overline{\bigcap_{s \in S} F_s} = \bigcap_{s \in S} F_s$. Thus the family $\mathcal{C}$ has property (C3).

Let the symbol $\tilde{A}$ denote the closure of the set A in the topological space $(X, \mathcal{O})$. We have to show that $\tilde{A} = \overline{A}$ for every $A \subset X$. By (CO4) for every $A \subset X$ we have $\overline{A} \in \mathcal{C}$, therefore $\tilde{A} \subset \overline{A}$. For every closed subset F of X that contains A, i.e., for every $F \subset X$ satisfying $F = \overline{F}$ and $A \subset F$, we have $\overline{A} \subset \overline{F} = F$ by virtue of (1). Hence $\overline{A} \subset \tilde{A} = \bigcap\{F : F = \overline{F}$ and $A \subset F\}$, and this proves that $\tilde{A} = \overline{A}$. ■

1.2.8. EXAMPLE. Let X be an arbitrary set containing more than one point and let x_0 be a point in X. Define $\overline{A} = A \cup \{x_0\}$ for every non-empty $A \subset X$ and let $\overline{\emptyset} = \emptyset$. The closure operator defined in this way satisfies conditions (CO1)–(CO4).

The set $\{x_0\}$ is the only one-point set in X which is closed with respect to the topology generated by this closure operator; other one-point sets are open but are not closed.■

1.2.9. PROPOSITION. *Suppose we are given a set X and an operator assigning to every set $A \subset X$ a set $\operatorname{Int} A \subset X$ in such a way that conditions* (IO1)-(IO4) *are satisfied. The family $\mathcal{O} = \{A : A = \operatorname{Int} A\}$ satisfies conditions* (O1)–(O3) *and for every $A \subset X$ the set $\operatorname{Int} A$ is the interior of A in the topological space $(X, \mathcal{O})$.*

The topology $\mathcal{O}$ is called the *topology generated by the interior operator* Int .■

1.2.10. EXAMPLE. Let X be an arbitrary set containing more than one point and let X_0 be a subset of X such that $|X \setminus X_0| > 1$. Define $\operatorname{Int} A = A \cap X_0$ for every proper subset $A \subset X$ and let $\operatorname{Int} X = X$. The interior operator defined in this way satisfies conditions (IO1)–(IO4).

All subsets of X_0 and the whole space are the only subsets of X which are open with respect to the topology generated by this interior operator. If X_0 is the empty set, X is the anti-discrete space.■

Historical and bibliographic notes

Various methods of generating topologies were first systematically described by Hausdorff in [1927]. The Sorgenfrey line appeared in Alexandroff and Urysohn [1929], but only after Sorgenfrey's paper [1947] did it become one of the "universal counterexamples" in general topology (see Examples 2.3.12, 3.8.15 and 5.1.31). The Niemytzki plane was defined (and attributed to Niemytzki) by Alexandroff and Hopf in [1935]; this is another "universal counterexample" (see Example 1.5.10 and Exercises 5.2.C(b) and 5.3.B(a)). Modifying the Sorgenfrey line and the Niemytzki plane is still a source of important and interesting examples (see Exercises 3.1.I and 5.4.B).

Exercises

1.2.A. State and prove a proposition, analogous to those of this section, on generating a topology by a subbase.

1.2.B. Let X be an arbitrary set and $\mathcal{B}_1$, $\mathcal{B}_2$ two collections of families of subsets of X which have properties (B1)–(B2). Show that the topologies generated by the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ coincide if and only if for every $x \in X$ and any $U \in \mathcal{B}_1$ containing x there exists a $V \in \mathcal{B}_2$ such that $x \in V \subset U$, and for any $V \in \mathcal{B}_2$ containing x there exists a $U \in \mathcal{B}_1$ such that $x \in U \subset V$.

State and prove the counterpart of the above for topologies generated by neighbourhood systems.

1.2.C. Verify that for a topological space $(X, \mathcal{O})$ and for a base for this space (a neighbourhood system, the family of all closed sets, the closure operator, the interior operator) the topology generated on X by this base (neighbourhood system, etc.) coincides with the original topology of X.

1.3. Boundary of a set and derived set. Dense and nowhere dense sets. Borel sets

In Section 1.1 we defined and studied the families of open sets and closed sets, the closure operator and the interior operator. In this section we define two more operators and other families of subsets in a topological space.

For any subset A of a topological space X we define the *boundary* of A to be the set

$$\operatorname{Fr} A = \overline{A} \cap \overline{X \setminus A} = \overline{A} \setminus \operatorname{Int} A.$$

Proposition 1.1.1 gives

1.3.1. PROPOSITION. *The point x belongs to $\operatorname{Fr} A$ if and only if for every neighbourhood U of x (or – equivalently – for every member U of a base $\mathcal{B}(x)$ at the point x) we have $U \cap A \neq \emptyset \neq U \setminus A$.* ■

The most important properties of the boundary operator are listed in the following theorem.

1.3.2. THEOREM. *The boundary operator has the following properties:*

(i) $\operatorname{Int} A = A \setminus \operatorname{Fr} A$. (ii) $\overline{A} = A \cup \operatorname{Fr} A$.
(iii) $\operatorname{Fr}(A \cup B) \subset \operatorname{Fr} A \cup \operatorname{Fr} B$. (iv) $\operatorname{Fr}(A \cap B) \subset (\overline{A} \cap \operatorname{Fr} B) \cup (\operatorname{Fr} A \cap \overline{B})$.
(v) $\operatorname{Fr}(X \setminus A) = \operatorname{Fr} A$. (vi) $X = \operatorname{Int} A \cup \operatorname{Fr} A \cup \operatorname{Int}(X \setminus A)$.
(vii) $\operatorname{Fr} \overline{A} \subset \operatorname{Fr} A$. (viii) $\operatorname{Fr} \operatorname{Int} A \subset \operatorname{Fr} A$.
(ix) *A is open if and only if* $\operatorname{Fr} A = \overline{A} \setminus A$.
(x) *A is closed if and only if* $\operatorname{Fr} A = A \setminus \operatorname{Int} A$.
(xi) *A is open-and-closed if and only if* $\operatorname{Fr} A = \emptyset$.

PROOF. Proofs of properties (i)–(xi) consist of simple calculations; as a sample we shall prove (i) and (iii):

$$\begin{aligned} A \setminus \operatorname{Fr} A &= A \setminus (\overline{A} \cap \overline{X \setminus A}) = (A \setminus \overline{A}) \cup (A \setminus \overline{X \setminus A}) = A \setminus \overline{X \setminus A} \\ &= A \cap \operatorname{Int} A = \operatorname{Int} A. \\ Fr(A \cup B) &= \overline{A \cup B} \cap \overline{X \setminus (A \cup B)} = (\overline{A} \cup \overline{B}) \cap \overline{(X \setminus A) \cap (X \setminus B)} \\ &\subset (\overline{A} \cup \overline{B}) \cap \overline{X \setminus A} \cap \overline{X \setminus B} \subset (\overline{A} \cap \overline{X \setminus A}) \cup (\overline{B} \cap \overline{X \setminus B}) \\ &= \operatorname{Fr} A \cup \operatorname{Fr} B. \blacksquare \end{aligned}$$

A point x in a topological space X is called an *accumulation point of a set $A \subset X$* if $x \in \overline{A \setminus \{x\}}$; the set of all accumulation points of A is called the *derived set* of A and is denoted by A^{d}. Points from $A \setminus A^{\mathrm{d}}$ are called *isolated points of the set A*. A point x is an isolated point of the space X if and only if the one-point set $\{x\}$ is open. Indeed, the set $\{x\}$ is open if and only if $\{x\} = X \setminus \overline{X \setminus \{x\}}$, i.e. if $x \notin \overline{X \setminus \{x\}}$.

1.3.3. PROPOSITION. *The point x belongs to A^{d} if and only if every neighbourhood U of x (or – equivalently – every member U of a base $\mathcal{B}(x)$ at the point x) contains at least one point of the set A distinct from the point x.*

PROOF. Suppose that x is an accumulation point of A. By definition we have $x \in \overline{A \setminus \{x\}}$; hence for every neighbourhood U of the point x the intersection $U \cap (A \setminus \{x\})$ is non-empty, i.e., U contains a point of the set A distinct from x.

If every member U of a base $\mathcal{B}(x)$ at the point x contains a point of A distinct from x, i.e., if $U \cap (A \setminus \{x\}) \neq \emptyset$ for every $U \in \mathcal{B}(x)$, then $x \in \overline{A \setminus \{x\}}$ by 1.1.1, that is $x \in A^{\mathrm{d}}$.■

The proof of the following theorem is left to the reader.

1.3.4. THEOREM. *The derived set has the following properties:*

(i) $\overline{A} = A \cup A^{\mathrm{d}}$.

(ii) *If* $A \subset B$, *then* $A^{\mathrm{d}} \subset B^{\mathrm{d}}$.

(iii) $(A \cup B)^{\mathrm{d}} = A^{\mathrm{d}} \cup B^{\mathrm{d}}$.

(iv) $\bigcup_{s \in S} A_s^{\mathrm{d}} \subset (\bigcup_{s \in S} A_s)^{\mathrm{d}}$.■

We shall define now some further families of sets in topological spaces.

A set $A \subset X$ is called *dense* in X if $\overline{A} = X$.

A set $A \subset X$ is called *co-dense* in X if $X \setminus A$ is dense.

A set $A \subset X$ is called *nowhere dense* in X if $\overline{A}$ is co-dense.

A set $A \subset X$ is called *dense in itself* if $A \subset A^{\mathrm{d}}$.

1.3.5. PROPOSITION. *The set A is dense in X if and only if every non-empty open subset of X contains points of A.*

The set A is co-dense in X if and only if every non-empty open subset of X contains points of the complement of A.

The set A is nowhere dense in X if and only if every non-empty open subset of X contains a non-empty open set disjoint from A.■

The next two theorems will be often used.

1.3.6. THEOREM. *If A is dense in X, then for every open $U \subset X$ we have $\overline{U} = \overline{U \cap A}$.*

PROOF. For every $x \in \overline{U}$ and any neighbourhood W of x the intersection $W \cap U$ is open and non-empty. By 1.3.5 we then have $W \cap U \cap A \neq \emptyset$, and from 1.1.1 it follows that $x \in \overline{U \cap A}$. Thus the inclusion $\overline{U} \subset \overline{U \cap A}$ holds. The reverse inclusion is obvious.■

The *density of a space X* is defined as the smallest cardinal number of the form $|A|$, where A is a dense subset of X; this cardinal number is denoted by $d(X)$. If $d(X) \leq \aleph_0$, then we say that the space X is *separable*.

1.3.7. THEOREM. *For every topological space X we have $d(X) \leq w(X)$.*

PROOF. Let $\mathcal{B} = \{U_s\}_{s \in S}$ be a base for the space X consisting of non-empty sets such that $|S| = \mathfrak{m} = w(X)$. Let us choose for every $s \in S$ a point $a_s \in U_s$; obviously, the set $A = \{a_s : s \in S\}$ is dense in X. Since $|A| \leq |S| = \mathfrak{m}$, we have $d(X) \leq w(X)$.■

1.3.8. COROLLARY. *Every second-countable space is separable.*■

1.3.9. EXAMPLES. In a discrete space the boundary and the derived set of any set are empty and the only dense set is the space itself. The set of all rationals, as well as the set of all irrationals, are both dense and co-dense in the real line and in the Sorgenfrey line. The set

L_2 is dense in the Niemytzki plane L and the set L_1 is nowhere dense in L. The derived set of L_1 is empty and L_2 is dense in itself. The point x_0 is the only accumulation point of the space X defined in Example 1.1.8; other points are isolated. The real line R, the interval I, the Sorgenfrey line K and the Niemytzki plane L are all separable. The discrete space X is separable if and only if $|X| \le \aleph_0$. ■

As shown by simple examples, the intersection of countably many open sets need not be open and the union of countably many closed sets need not be closed. All subsets of a topological space X that can be obtained from open subsets of X, or – what is the same – from closed subsets of X, by taking countable unions and intersections as well as complements, are – from the topological point of view – of a particularly regular form, and the family of those sets is worth studying. By the family of *Borel sets* in a topological space X we mean the smallest family $\mathcal{S}$ of subsets of X satisfying the following conditions:

(BS1) *The family $\mathcal{S}$ contains all open subsets of X.*
(BS2) *If $A \in \mathcal{S}$, then $X \setminus A \in \mathcal{S}$.*
(BS3) *If $A_i \in \mathcal{S}$ for $i = 1, 2, \ldots$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$.*

The fact that there exists a smallest family $\mathcal{S}$ satisfying the above conditions follows from the simple observation that the family of all subsets of X satisfies (BS1)–(BS3) and that for any collection of families satisfying (BS1)–(BS3), the common part of all families in the collection is a family satisfying those conditions. Hence, the family of Borel sets in X could be as well defined to be the common part of all families $\mathcal{S}$ satisfying (BS1)–(BS3).

Let us note that in the definition of Borel sets condition (BS1) can be replaced by

(BS1′) *The family $\mathcal{S}$ contains all closed subsets of X,*

and condition (BS3) can be replaced by

(BS3′) *If $A_i \in \mathcal{S}$ for $i = 1, 2, \ldots$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{S}$.*

Indeed, the conjunction of (BS1′) and (BS2) is equivalent to the conjunction of (BS1) and (BS2), and the conjunction of (BS2) and (BS3′) is equivalent to the conjunction of (BS2) and (BS3).

The question arises whether there exist subsets of X which are not Borel sets. Clearly this depends on the space X, e.g., all subsets of a discrete space are Borel sets, but in general there are non-Borel sets in topological spaces (see Exercise 1.3.G).

The theory of Borel sets is a highly developed part of general topology; however to obtain interesting and deeper results on Borel sets one has to restrict the class of spaces under consideration (see Problems 1.7.5, 4.5.7, 4.5.8 and 7.4.22).

The most common Borel sets, besides open sets and closed sets, are countable unions of closed sets and countable intersections of open sets – the former are called *F_σ-sets* and the latter *G_δ-sets*. Clearly the complement of an F_σ-set is a G_δ-set and vice versa.

The intersection of two F_σ-sets is again an F_σ-set. Indeed, if $E = \bigcup_{i=1}^{\infty} E_i$ and $F = \bigcup_{j=1}^{\infty} F_j$, where E_i and F_j are closed sets, then $E \cap F = \bigcup_{i,j=1}^{\infty} (E_i \cap F_j)$, so that $E \cap F$ is an F_σ-set. Similarly, the union of two G_δ-sets is again a G_δ-set. Clearly a countable union (intersection) of F_σ-sets (G_δ-sets) is again an F_σ-set (a G_δ-set). The set of all rationals is an F_σ-set in R; the fact that it is not a G_δ-set is not so obvious (see Exercise 3.9.B(a)).

Historical and bibliographic notes

The notions of the boundary of a set, of the derived set and of dense and nowhere dense sets were introduced by Cantor, who also discovered their basic properties. Separable spaces were defined in Fréchet [1906]. A definition of Borel sets was sketched by Borel for subsets of the real line; the theory of Borel sets was originated by Lebesgue in [1905] (for Euclidean spaces) and by Hausdorff in [1914] (for metric spaces).

Exercises

1.3.A. Verify that if the sets A and B satisfy the condition $A \cap \overline{B} = \emptyset = \overline{A} \cap B$, then $\operatorname{Fr}(A \cup B) = \operatorname{Fr} A \cup \operatorname{Fr} B$.

1.3.B. Let $\{A_s\}_{s\in S}$ be a locally finite family of subsets of a topological space X.

(a) Show that $\operatorname{Fr}(\bigcup_{s\in S} A_s) \subset \bigcup_{s\in S} \operatorname{Fr} A_s$.

(b) Verify that if all members of the family $\{A_s\}_{s\in S}$ are nowhere dense in X, then the union $\bigcup_{s\in S} A_s$ also is nowhere dense in X.

1.3.C. For every positive integer n the *n-th derived set* $A^{(n)}$ of a subset A of a topological space X is defined inductively by the formulas:

$$A^{(1)} = A^{\mathrm{d}} \quad \text{and} \quad A^{(n)} = (A^{(n-1)})^{\mathrm{d}}.$$

(a) Give an example of a set of real numbers that has three consecutive derived sets distinct from each other.

(b) Give an example of a set of real numbers that has infinitely many derived sets distinct from each other.

1.3.D. (a) Generalize Theorem 1.3.6 by proving that for every open set U in a topological space X and every $A \subset X$ we have $\overline{U \cap \overline{A}} = \overline{U \cap A}$.

Hint. Apply the second inclusion in Exercise 1.1.A and the equality $U = X \setminus \overline{X \setminus U}$.

(b) Prove that for every closed set F in a topological space X and every $A \subset X$ we have $\operatorname{Int}(F \cup \operatorname{Int} A) = \operatorname{Int}(F \cup A)$.

1.3.E. Verify that the union of a co-dense set and a nowhere dense set is a co-dense set. Note that the union of two co-dense sets if not necessarily a co-dense set.

1.3.F. Show that any open subset of a dense in itself space is dense in itself.

1.3.G. Note that open subsets of the real line are F_σ-sets. Show that the family of all Borel sets in the real line can be represented as the union $\bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$, where $\mathcal{F}_0$ is the family of all closed sets, and $\mathcal{F}_\alpha$ consists of all countable unions of sets from $\bigcup_{\xi<\alpha} \mathcal{F}_\xi$ for an odd ordinal number α, and of all countable intersections of sets from $\bigcup_{\xi<\alpha} \mathcal{F}_\xi$ for an even ordinal number α. Note that all families $\mathcal{F}_\alpha$ are of cardinality $\mathfrak{c}$ and deduce that the real line has subsets which are non-Borel sets.

1.4. Continuous mappings. Closed and open mappings. Homeomorphisms

Let $(X, \mathcal{O})$ and $(Y, \mathcal{O}')$ be two topological spaces; a mapping f of X to Y is called *continuous* if $f^{-1}(U) \in \mathcal{O}$ for any $U \in \mathcal{O}'$, i.e., if the inverse image of any open subset of Y

is open in X. The fact that f is a continuous mapping of X to Y will be often written in symbols as $f: X \to Y$.

Our definition of a continuous mapping depends on the notion of an open set. Since we shall frequently use one of the methods described in Section 1.2 to generate a topology, it will be convenient to have some criteria for continuity of mappings formulated in corresponding terms. Criteria of this kind are listed in the following proposition.

1.4.1. PROPOSITION. *For a mapping f of a topological space X to a topological space Y the following conditions are equivalent:*

(i) *The mapping f is continuous.*

(ii) *Inverse images of all members of a subbase $\mathcal{P}$ for Y are open in X.*

(ii′) *Inverse images of all members of a base $\mathcal{B}$ for Y are open in X.*

(iii) *There are neighbourhood systems $\{\mathcal{B}(x)\}_{x \in X}$ and $\{\mathcal{D}(y)\}_{y \in Y}$ for X and Y respectively, such that for every $x \in X$ and a $V \in \mathcal{D}(f(x))$ there exists a $U \in \mathcal{B}(x)$ satisfying $f(U) \subset V$.*

(iv) *Inverse images of all closed subsets of Y are closed in X.*

(v) *For every $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$.*

(v′) *For every $B \subset Y$ we have $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.*

(vi) *For every $B \subset Y$ we have $f^{-1}(\operatorname{Int} B) \subset \operatorname{Int} f^{-1}(B)$.*

PROOF. The implication (i)⇒(ii) is obvious.

We shall prove that (ii)⇒(ii′). Let $\mathcal{P}$ be a subbase for Y such that $f^{-1}(V)$ is open in X for every $V \in \mathcal{P}$. Consider the base $\mathcal{B}$ for Y consisting of all finite intersections $V_1 \cap V_2 \cap \ldots \cap V_k$ of members of $\mathcal{P}$; since

$$f^{-1}(V_1 \cap V_2 \cap \ldots \cap V_k) = f^{-1}(V_1) \cap f^{-1}(V_2) \cap \ldots \cap f^{-1}(V_k),$$

inverse images of all members of $\mathcal{B}$ are open in X.

Now we show that (ii′)⇒(iii). For any $V \in \mathcal{D}(f(x))$ there exists a $W \in \mathcal{B}$ such that $f(x) \in W \subset V$. As $f^{-1}(W)$ is open in X and $x \in f^{-1}(W)$, there exists a $U \in \mathcal{B}(x)$ satisfying $U \subset f^{-1}(W)$. We then have $f(U) \subset ff^{-1}(W) \subset W \subset V$.

We shall prove the implication (iii)⇒(iv). Let $B = \overline{B}$ be a closed subset of Y. As $f^{-1}(B) = X \setminus f^{-1}(Y \setminus B)$, it suffices to show that the inverse image of $Y \setminus B$ is open in X. To achieve this we shall show that each point $x \in f^{-1}(Y \setminus B)$ has a neighbourhood U which is contained in $f^{-1}(Y \setminus B)$. For every $x \in f^{-1}(Y \setminus B)$ we have $f(x) \in Y \setminus B$; thus there exists a $V \in \mathcal{D}(f(x))$ such that $V \subset Y \setminus B$. By (iii) there exists a $U \in \mathcal{B}(x)$ satisfying $f(U) \subset V$; clearly

$$x \in U \subset f^{-1}f(U) \subset f^{-1}(V) \subset f^{-1}(Y \setminus B).$$

To prove that (iv)⇒(v) let us observe that $f^{-1}(\overline{f(A)})$ is a closed set containing A, and thus $\overline{A} \subset f^{-1}(\overline{f(A)})$, which gives

$$f(\overline{A}) \subset ff^{-1}(\overline{f(A)}) \subset \overline{f(A)}.$$

To prove that (v)⇒(v′) we apply (v) to $A = f^{-1}(B)$ and we obtain the inclusion

$$f(\overline{f^{-1}(B)}) \subset \overline{ff^{-1}(B)} \subset \overline{B},$$

which gives $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

To prove that (v′)⇒(vi) we apply (v′) to $Y \setminus B$ and we obtain the inclusion $\overline{f^{-1}(Y \setminus B)} \subset f^{-1}(\overline{Y \setminus B})$, which gives

$$\begin{aligned} f^{-1}(\operatorname{Int} B) &= f^{-1}(Y \setminus \overline{Y \setminus B}) = X \setminus f^{-1}(\overline{Y \setminus B}) \\ &\subset X \setminus \overline{f^{-1}(Y \setminus B)} = X \setminus \overline{X \setminus f^{-1}(B)} \\ &= \operatorname{Int} f^{-1}(B). \end{aligned}$$

To complete the proof of the proposition it remains to show that (vi)⇒(i). For every open $U \subset Y$ we have $U = \operatorname{Int} U$, and it follows from (vi) that $f^{-1}(U) \subset \operatorname{Int} f^{-1}(U)$. Thus we have $f^{-1}(U) = \operatorname{Int} f^{-1}(U)$, i.e., $f^{-1}(U)$ is open in X. ■

The characterization of continuity given in (iii) allows us to define continuity at a point. We shall say that a mapping f of X to Y is *continuous at a point* $x \in X$ if for every neighbourhood $V \subset Y$ of $f(x)$, there exists a neighbourhood $U \subset X$ of x such that $f(U) \subset V$. Clearly a mapping f of X to Y is continuous if and only if it is continuous at each point of the space X.

Let us observe, in connection with the above proposition, that if $f: X \to Y$ is continuous, then for any F_σ-set (G_δ-set) $B \subset Y$ the inverse image $f^{-1}(B)$ is an F_σ-set (a G_δ-set) in X.

Using the equality $(gf)^{-1}(U) = f^{-1}(g^{-1}(U))$, one can easily verify that if f is a continuous mapping of X to Y and g a continuous mapping of Y to Z, then the composition gf is a continuous mapping of X to Z.

1.4.2. EXAMPLE. If X is a discrete space, then any mapping of X to a topological space Y is continuous. Similarly, any mapping of a topological space X to an anti-discrete space Y is continuous. ■

1.4.3. EXAMPLE. If on a set X two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ are defined, then the identity mapping id_X is a continuous mapping of $(X, \mathcal{O}_1)$ to $(X, \mathcal{O}_2)$ if and only if the topology $\mathcal{O}_1$ is finer than the topology $\mathcal{O}_2$. ■

Let X be a topological space, R the real line with the natural topology, and I the closed unit interval with the natural topology. From the equivalence of conditions (i) and (iii) in 1.4.1, it follows that a mapping f of X to R or I is continuous if and only if for every $x \in X$ and any $\epsilon > 0$ there exists a neighbourhood U of x such that $|f(x) - f(x')| < \epsilon$ for every $x' \in U$. In particular, f is a continuous mapping of R to R if for every $x \in R$ and any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x')| < \epsilon$ for every x' satisfying $|x - x'| < \delta$. Continuous mappings to R or I will be called *continuous functions*. This terminology is consistent with the terminology of real analysis.

From what we observed above, it follows that for any continuous function $f: X \to R$, the function $|f|$, where $|f|(x) = |f(x)|$, is continuous. Indeed, $|f|$ is the composition of f and the absolute value function which is a continuous function from R to R. Similarly, one can verify that for continuous functions $f, g: X \to R$, the functions $f \pm g$, $f \cdot g$, $\min(f, g)$ and $\max(f, g)$, where

$$(f \pm g)(x) = f(x) \pm g(x), \quad (f \cdot g)(x) = f(x) \cdot g(x),$$

$$[\min(f, g)](x) = \min(f(x), g(x)), \quad [\max(f, g)](x) = \max(f(x), g(x)),$$

are continuous; in particular, for a real number t and a continuous function $f: X \to R$ the function tf, where $(tf)(x) = t \cdot f(x)$, is continuous. If a continuous function $f: X \to R$ vanishes at no point of X, then the function $1/f$, where $(1/f)(x) = 1/f(x)$, is continuous. The same holds for functions to the interval I, provided that the corresponding operations are performable.

1.4.4. EXAMPLE. Let K be the Sorgenfrey line defined in 1.2.2 and $[x, r)$ an element of the base $\mathcal{B}$ for that space. As the set $[x, r)$ is open-and-closed in X, the formula

$$f(y) = \begin{cases} 0 & \text{for } x \le y < r, \\ 1 & \text{otherwise,} \end{cases}$$

defines a continuous function $f: K \to I$.■

1.4.5. EXAMPLE. Let L be the Niemytzki plane defined in 1.2.4 and $U_i(x)$ an element of the base $\mathcal{B}(x)$ at the point $x \in L$. For every $y \in U_i(x) \setminus \{x\}$ denote by y' the point at which the ray starting at x and passing through y intersects the circle bounding $U_i(x)$. The reader can verify that the formula

$$f(y) = \begin{cases} 0 & \text{for } y = x, \\ 1 & \text{for } y \in L \setminus U_i(x), \\ \dfrac{|xy|}{|xy'|} & \text{for } y \in U_i(x) \setminus \{x\}, \end{cases}$$

where $|ab|$ denotes the length of the segment joining the points a and b, defines a continuous function $f: L \to I$.■

1.4.6. EXAMPLE. Let X be the space defined in 1.1.8. We shall show that for every continuous function $f: X \to R$ there exists a countable set $X_0 \subset X$ such that $f(x) = f(x_0)$ for every $x \in X \setminus X_0$.

The set $X_i = X \setminus f^{-1}((f(x_0) - 1/i, f(x_0) + 1/i))$ is closed and does not contain x_0; thus it is a finite set. One can readily verify that the set $X_0 = \bigcup_{i=1}^{\infty} X_i$ has the required property.■

Let X be a topological space and $\{f_i\}$ a sequence of functions from X to R or I. We say that the sequence $\{f_i\}$ is *uniformly convergent* to a real-valued function f if for every $\epsilon > 0$ there exists a k such that we have $|f(x) - f_i(x)| < \epsilon$ for every $x \in X$ and $i \ge k$; we write this in symbols as $f = \lim f_i$.

1.4.7. THEOREM. *If a sequence $\{f_i\}$ of continuous functions from X to R (from X to I) is uniformly convergent to a real-valued function f, then f is a continuous function from X to R (from X to I).*

PROOF. We shall show that for every $x_0 \in X$ and any $\epsilon > 0$ there exists a neighbourhood U of x_0 such that $|f(x_0) - f(x')| < \epsilon$ for every $x' \in U$.

Let us take an integer k such that

$$(1) \qquad |f(x) - f_i(x)| < \epsilon/3 \quad \text{for every} \quad x \in X \quad \text{and} \quad i \ge k.$$

As the function f_k is continuous there exists a neighbourhood U of x_0 such that

$$(2) \qquad |f_k(x_0) - f_k(x')| < \epsilon/3 \quad \text{for every} \quad x' \in U;$$

we shall show that the neighbourhood U has the required property. Let us take a point $x' \in U$. By virtue of (1) and (2) we have

$$|f(x_0) - f(x')| \leq |f(x_0) - f_k(x_0)| + |f_k(x_0) - f_k(x')| + |f_k(x') - f(x')|$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Obviously, if $f_i: X \to I$ for $i = 1, 2, \ldots$, then $f: X \to I$. ■

We shall describe now a method of generating topologies which is related to the notion of a continuous mapping.

1.4.8. PROPOSITION. *Suppose we are given a set X, a family $\{Y_s\}_{s \in S}$ of topological spaces and a family of mappings $\{f_s\}_{s \in S}$, where f_s is a mapping of X to Y_s. In the class of all topologies on X that make all the f_s's continuous there exists a coarsest topology; this is the topology $\mathcal{O}$ generated by the base consisting of all sets of the form $\bigcap_{i=1}^{k} f_{s_i}^{-1}(V_i)$, where $s_1, s_2, \ldots, s_k \in S$ and V_i is an open subset of Y_{s_i} for $i = 1, 2, \ldots, k$.*

The topology $\mathcal{O}$ is called the *topology generated by the family of mappings* $\{f_s\}_{s \in S}$.

PROOF. The family $\mathcal{B}$ consisting of all sets of the form $\bigcap_{i=1}^{k} f_{s_i}^{-1}(V_i)$ has properties (B1)–(B2), and all the f_s's are continuous with respect to the topology $\mathcal{O}$ generated by the base $\mathcal{B}$ (see Proposition 1.2.1). On the other hand, one can readily verify that, if all the f_s's are continuous with respect to a topology $\mathcal{O}'$ on X, then $\mathcal{B} \subset \mathcal{O}'$; this implies that $\mathcal{O} \subset \mathcal{O}'$, i.e., that the topology $\mathcal{O}$ is coarser than $\mathcal{O}'$. ■

1.4.9. PROPOSITION. *A mapping f of a topological space X to a topological space Y whose topology is generated by a family of mappings $\{f_s\}_{s \in S}$, where f_s is a mapping of Y to Y_s, is continuous if and only if the composition $f_s f$ is continuous for every $s \in S$.*

PROOF. If $f: X \to Y$ is continuous, then $f_s f$ is continuous as the composition of two continuous mappings. Let us suppose that $f_s f: X \to Y_s$ is continuous for every $s \in S$ and let us denote by $\mathcal{P}$ the subbase for the space Y consisting of all sets of the form $f_s^{-1}(V_s)$, where V_s is open in Y_s. By Proposition 1.4.1, it suffices to show that inverse images of members of $\mathcal{P}$ under the mapping f are open in X. This follows however from the equality

$$f^{-1} f_s^{-1}(V_s) = (f_s f)^{-1}(V_s). \blacksquare$$

If there exists a continuous mapping f of a space X onto a space Y, i.e., a mapping $f: X \to Y$ such that $f(X) = Y$, then we say that X can be *mapped onto* Y or that Y is *a continuous image of* X.

1.4.10. THEOREM. *If there exists a continuous mapping $f: X \to Y$ of X onto Y, then $d(Y) \leq d(X)$.*

PROOF. Let us take a dense subset A of X such that $|A| = d(X)$. By the equivalence of conditions (i) and (v) in 1.4.1 we have $Y = f(X) = f(\overline{A}) \subset \overline{f(A)}$. Thus the set $f(A)$ is dense in Y; since obviously $|f(A)| \leq |A| = d(X)$, we have $d(Y) \leq d(X)$. ■

1.4.11. COROLLARY. *A continuous image of a separable space is separable.* ■

We turn now to a study of two important classes of continuous mappings: closed mappings and open mappings. A continuous mapping $f: X \to Y$ is called *closed* (*open*) if for every closed (open) set $A \subset X$ the image $f(A)$ is closed (open) in Y. Mappings which are simultaneously closed and open are called *closed-and-open* mappings.

Clearly, the composition of two closed (open) mappings is closed (open).

1.4.12. THEOREM. *A continuous mapping $f: X \to Y$ is closed (open) if and only if for every $B \subset Y$ and every open (closed) set $A \subset X$ which contains $f^{-1}(B)$, there exists an open (a closed) set $C \subset Y$ containing B such that $f^{-1}(C) \subset A$.*

PROOF. We shall consider the case of a closed f; the argument for an open f is parallel.

Let us suppose that $f: X \to Y$ is closed, B is a subset of Y and A is an open subset of X which contains $f^{-1}(B)$. The set $C = Y \setminus f(X \setminus A)$ is open in Y and contains B. Moreover,

$$f^{-1}(C) = f^{-1}(Y \setminus f(X \setminus A)) = X \setminus f^{-1}f(X \setminus A) \subset X \setminus (X \setminus A) = A.$$

Let us suppose now that f satisfies the condition in the theorem and let us take a closed set $F \subset X$. The set $A = X \setminus F$ is open, and for $B = Y \setminus f(F)$ we have

$$f^{-1}(B) = X \setminus f^{-1}f(F) \subset X \setminus F = A;$$

thus there exists an open subset C of Y such that $Y \setminus f(F) \subset C$ and $f^{-1}(C) \subset A$, i.e., $f^{-1}(C) \cap F = \emptyset$. The last equality implies that $C \cap f(F) = \emptyset$, i.e., that $C \subset Y \setminus f(F)$. We then have $f(F) = Y \setminus C$, and this shows that the set $f(F)$ is closed.■

In the case of a closed mapping the condition in Theorem 1.4.12 can be replaced by a simpler one:

1.4.13. THEOREM. *A continuous mapping $f: X \to Y$ is closed if and only if for every point $y \in Y$ and every open set $U \subset X$ which contains $f^{-1}(y)$, there exists in Y a neighbourhood V of the point y such that $f^{-1}(V) \subset U$.*

PROOF. By 1.4.12, it suffices to show that if f satisfies the condition in the theorem, then f is closed. Let us take a set $B \subset Y$ and an open set $A \subset X$ such that $f^{-1}(B) \subset A$. For every $y \in B$ choose a neighbourhood $V_y \subset Y$ of the point y such that $f^{-1}(V_y) \subset A$. The open set $C = \bigcup_{y \in B} V_y$ satisfies the inclusions $B \subset C$ and $f^{-1}(C) \subset A$, thus f is closed by 1.4.12.■

1.4.14. THEOREM. *A continuous mapping $f: X \to Y$ is open if and only if there exists a base $\mathcal{B}$ for X such that $f(U)$ is open in Y for every $U \in \mathcal{B}$.*■

1.4.15. EXAMPLES. The mapping $r: R \to I$ defined by the formula

$$r(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x & \text{for } 0 \leq x \leq 1, \\ 1 & \text{for } x \geq 1, \end{cases}$$

is closed but not open.

The mapping $f: L \to R$, assigning to the point (x, y) of the Niemytzki plane L its first coordinate $x \in R$, is open but not closed.

The mapping $g: K \to D$ of the Sorgenfrey line K to the two-point discrete space $D = \{0, 1\}$ defined by the formula

$$g(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0, \end{cases}$$

is closed-and-open. ■

1.4.16. THEOREM. *If $f: X \to Y$ is an open mapping, then for every $x \in X$ we have $\chi(f(x), Y) \leq \chi(x, X)$. If, moreover, $f(X) = Y$, then $w(Y) \leq w(X)$ and $\chi(Y) \leq \chi(X)$.*

PROOF. The mapping f transforms any base at x to a base at $f(x)$ and any base for X to a base for $f(X)$. ■

In contrast to the above result, closed mappings may increase weight and character (cf. Theorem 3.7.19):

1.4.17. EXAMPLE. Let $X = R$ be the real line and $Y = (R \setminus N) \cup \{y_0\}$, where N denotes the set of positive integers and $y_0 \notin R$. Assign to any point $x \in X$ the point

$$f(x) = \begin{cases} x, & \text{if } x \in X \setminus N, \\ y_0, & \text{if } x \in N, \end{cases}$$

and consider on Y the topology generated by the family of closed sets $\mathcal{C} = \{A \subset Y : f^{-1}(A)$ is closed in $X\}$. One can easily check that $f: X \to Y$ is closed and that neighbourhoods of y_0 in Y are of the form $(U \setminus N) \cup \{y_0\}$, where U is open in X and contains the set N.

Let $(U_1 \setminus N) \cup \{y_0\}$, $(U_2 \setminus N) \cup \{y_0\}, \ldots$ be an arbitrary sequence of neighbourhoods of y_0. Let us choose for $i = 1, 2, \ldots$ a point $x_i \in U_i \setminus N$ such that $x_i > i$. The set $U = X \setminus \{x_1, x_2, \ldots\}$ is open in X and contains the set N; thus $V = (U \setminus N) \cup \{y_0\}$ is a neighbourhood of y_0. Since no element of our sequence is contained in V, it follows that the space Y has no countable base at y_0. Thus $\chi(Y) > \aleph_0$ and $w(Y) > \aleph_0$, while $w(X) = \chi(X) = \aleph_0$.

The mapping $f: X \to Y$ consists in "identifying the closed subset $N \subset X$ to a point"; by this procedure one also obtains other interesting examples. This is a particular case of the quotient space operation that will be discussed in Section 2.4. ■

A continuous mapping $f: X \to Y$ is called a *homeomorphism* if f maps X onto Y in a one-to-one way and the inverse mapping f^{-1} of Y to X is continuous. We say that two topological spaces X and Y are *homeomorphic* if there exists a homeomorphism of X onto Y.

For every space X the identity mapping $\mathrm{id}_X: X \to X$ is a homeomorphism. One can readily verify that if f is a homeomorphism, then the inverse mapping f^{-1} is a homeomorphism as well and that the composition gf of two homeomorphisms f and g is again a homeomorphism. Thus the relation "X and Y are homeomorphic" is an equivalence relation.

1.4.18. PROPOSITION. *For a one-to-one mapping f of a topological space X onto a topological space Y the following conditions are equivalent:*

(i) *The mapping f is a homeomorphism.*
(ii) *The mapping f is closed.*
(iii) *The mapping f is open.*
(iv) *The set $f(A)$ is closed in Y if and only if A is closed in X.*
(iv′) *The set $f^{-1}(B)$ is closed in X if and only if B is closed in Y.*
(v) *The set $f(A)$ is open in Y if and only if A is open in X.*
(v′) *The set $f^{-1}(B)$ is open in X if and only if B is open in Y.*

PROOF. The equivalence of (i) and (ii), as well as of (i) and (iii), follows from the fact that $(f^{-1})^{-1}(A) = f(A)$ for every $A \subset X$. The equivalence of (ii) and (iv), as well as of (iii) and (v), follows from the equality $A = f^{-1}f(A)$. From the equivalence of (i) and (iv) and of (i) and (v) it follows that both (iv′) and (v′) are equivalent to f^{-1} being a homeomorphism, and this is equivalent to (i).■

1.4.19. EXAMPLES. Let X be the set of real numbers equipped with one of the following topologies: (a) the discrete topology; (b) the natural topology; (c) the topology defined in 1.1.8 with $x_0 = 0$; (d) the Sorgenfrey line topology; (e) the topology defined in 1.2.6; (f) the topology defined in 1.2.8 with $x_0 = 0$; (g) the anti-discrete topology.

For every real number $a > 0$ the mapping $f_a: X \to X$ defined by $f_a(x) = ax$ is a homeomorphism. For $a < 0$ the mapping f_a is not continuous with respect to the topology (d), but it is a homeomorphism with respect to the other topologies considered here.■

Let $\mathcal{M}$ be a class of continuous mappings and $\mathcal{P}$ a property of topological spaces. We shall say that $\mathcal{P}$ is an *invariant of the class* $\mathcal{M}$, or that $\mathcal{P}$ is *invariant under mappings from* $\mathcal{M}$, if $\mathcal{P}$ is preserved by mappings in the class $\mathcal{M}$, i.e., if for each $f \in \mathcal{M}$, where $f: X \to Y$ and $f(X) = Y$, the space Y has the property $\mathcal{P}$, provided that X has $\mathcal{P}$. Using this terminology one can restate Theorem 1.4.10 saying that the property "density is $\leq \mathfrak{m}$" is an invariant of continuous mappings; similarly, one can restate Theorem 1.4.16 saying that the properties "weight is $\leq \mathfrak{m}$" and "character is $\leq \mathfrak{m}$" are invariants of open mappings.

We shall say that a property $\mathcal{P}$ is an *inverse invariant of a class* $\mathcal{M}$ of continuous mappings, if for each $f \in \mathcal{M}$, where $f: X \to Y$ and $f(X) = Y$, the space X has the property $\mathcal{P}$, provided that Y has $\mathcal{P}$. Let us note that $\mathcal{P}$ is an inverse invariant of the class $\mathcal{M}$ if and only if the property non-$\mathcal{P}$ is an invariant of the class $\mathcal{M}$. Thus the notion of an inverse invariant can be reduced to the notion of an invariant; it is introduced in order to simplify some statements. Clearly, if $\mathcal{M}_1 \subset \mathcal{M}_2$, then every invariant (inverse invariant) of the class $\mathcal{M}_2$ is an invariant (inverse invariant) of the class $\mathcal{M}_1$.

Invariants of homeomorphisms are of particular importance; they are also called *topological properties*. As the inverse mapping of a homeomorphism is a homeomorphism, the notions of an invariant and of an inverse invariant coincide for the class of homeomorphisms. Thus a space X has a topological property $\mathcal{P}$ if and only if every space homeomorphic to X has the property $\mathcal{P}$. Since a homeomorphism $f: X \to Y$ establishes a one-to-one correspondence between points of X and those of Y and also between open sets in both spaces, every property defined in terms of open sets and in terms of set theory is a topological property. We have already encountered several topological properties here; the most important are: "weight is $\leq \mathfrak{m}$", "character is $\leq \mathfrak{m}$" and "density is $\leq \mathfrak{m}$".

Any property $\mathcal{P}$ determines the class of all spaces that have this property. If $\mathcal{P}$ is a topological property, then the class determined by $\mathcal{P}$ is topologically invariant, i.e., together with a space X it contains all spaces homeomorphic to X. The topological properties listed at the end of the preceding paragraph determine for $\mathfrak{m} = \aleph_0$ the classes of second-countable, first-countable and separable spaces respectively; all these classes are topologically invariant. In the sequel we shall define a great number of topological properties, i.e., of topologically invariant classes of topological spaces. When defining a new class of spaces, we will usually not point out explicitly that the class is topologically invariant; this will follow, however, from the form of the definitions in which only set-theoretic notions, and notions reducible to the notion of an open set, will appear. Classes of topological spaces that are not topologically invariant will not be considered here. Also the classes of mappings we shall discuss will be topologically invariant, i.e., composing those mappings with homeomorphisms (on either side) will not take them out of the discussed class.

The object of topology is to study topological properties. When considering a particular space X, we try to determine which topological properties X has. When developing a general theory, we usually study a particular topological property $\mathcal{P}$, its relationship to other topological properties, and we try to determine which operations on topological spaces preserve $\mathcal{P}$ and what are the classes of mappings of which $\mathcal{P}$ is an invariant. Thus, from the topological point of view, two homeomorphic spaces may be considered as the same object.

1.4.20. EXAMPLES. Let X and Y be any two sets of the same cardinality and let us consider in both X and Y the discrete topology. Obviously, every one-to-one mapping of X onto Y is a homeomorphism. On the other hand, if the discrete spaces X and Y have distinct cardinalities, then they cannot be homeomorphic. Thus, a discrete space X depends – up to a homeomorphism – only on the cardinality of the set X. The discrete space of cardinality $\mathfrak{m}$ will be denoted by $D(\mathfrak{m})$.

The same holds for infinite sets X and Y with topology defined as in 1.1.8; here however – in the case of sets of the same cardinality – in order to obtain homeomorphisms we have to consider one-to-one mappings of X onto Y that map x_0 to y_0, the accumulation point of Y. The space obtained as in 1.1.8 from a set of cardinality $\mathfrak{m} \geq \aleph_0$ will be denoted by $A(\mathfrak{m})$. ■

Historical and bibliographic notes

Continuous mappings and homeomorphisms of abstract spaces were first considered by Fréchet in [1910]; in a narrower sense the notion of a homeomorphism was introduced earlier by Poincaré. The first exhaustive and systematic exposition of the matter was given by Hausdorff in [1914]. Topologies generated by families of mappings were considered by Bourbaki in [1951]. The notion of a closed mapping was introduced by Hurewicz in [1926] and by Alexandroff in [1927], and the notion of an open mapping of the plane into itself – by Weyl in [1913] and by Stoïlow in [1928] (the latter assumed moreover that fibers of the mappings did not contain non-trivial continua); open mappings of topological spaces were defined by Aronszajn in [1931] (see also Sierpiński [1930]).

Exercises

1.4.A. Show that the topology of the Sorgenfrey line can be generated by a family of mappings into a two-point discrete space.

1.4.B. Verify that the Sorgenfrey line can be mapped onto $D(\aleph_0)$, but cannot be mapped onto $D(\mathfrak{c})$.

1.4.C. Verify that a mapping f of X to Y is closed if and only if $\overline{f(A)} = f(\overline{A})$ for every $A \subset X$ and that f is an open mapping if and only if f is continuous and $f(\operatorname{Int} A) \subset \operatorname{Int} f(A)$ for every $A \subset X$. Give an example to show that in the above characterization of open mappings the inclusion cannot be replaced by an equality.

Show that a mapping f of X to Y is open if and only if $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$ or –

equivalently – $\operatorname{Int} f^{-1}(B) = f^{-1}(\operatorname{Int} B)$ for every $B \subset Y$.

1.4.D. Verify that

(a) The image of a closed domain under a closed-and-open mapping is a closed domain.

(b) The image of an open domain under a closed-and-open mapping is not necessarily an open domain.

(c) The image of a closed domain under a closed (open) mapping is not necessarily a closed domain.

(d) The inverse image of a closed (an open) domain under an open mapping is a closed (an open) domain.

(e) The inverse image of a closed (an open) domain under a closed mapping is not necessarily a closed (an open) domain.

1.4.E. Show that the Sorgenfrey line and the Niemytzki plane are not homeomorphic.

Hint. Apply Example 1.4.4.

1.4.F. (a) Prove that if an infinite space Y is a continuous image of the space $A(\mathfrak{m})$ under a closed mapping, then $Y = A(\mathfrak{n})$ for some $\mathfrak{n} \leq \mathfrak{m}$.

(b) Show that a continuous mapping of $A(\mathfrak{m})$ onto a topological space Y is closed if and only if for every pair y_1, y_2 of distinct points of Y there exist open sets $U_1, U_2 \subset Y$ such that $y_1 \in U_1, y_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$, i.e., if Y is a Hausdorff space (see the next section).

1.4.G. Verify that for every continuous mapping $f: X \to Y$, inverse images of Borel sets in Y are Borel sets in X.

1.5. Axioms of separation

The definition of a topological space is very general; not many interesting theorems can be proved about all topological spaces. Throughout the book we will study various classes of topological spaces, ranging from fairly general to more and more special. Clearly, the narrower the considered class is, the more theorems hold about spaces in this class.

The restrictions we put on topological spaces are of various kinds. We already discussed the axioms of countability which stipulate the existence of "small" bases. In this section we shall discuss axioms of separation which concern the ways of separating points and closed sets in topological spaces.

A topological space X is called a *T_0-space* if for every pair of distinct points $x_1, x_2 \in X$ there exists an open set containing exactly one of these points. The anti-discrete space and – more generally – the space described in Example 1.2.10 are examples of topological spaces that are not T_0-spaces; all other spaces defined above are T_0-spaces.

1.5.1. THEOREM. *For every T_0-space we have* $|X| \leq \exp w(X)$.

PROOF. Let $\mathcal{B}$ be a base for X such that $|\mathcal{B}| = w(X)$ and for every $x \in X$ let $\mathcal{B}(x) = \{U \in \mathcal{B}: x \in U\}$. From the definition of a T_0-space it follows that $\mathcal{B}(x) \neq \mathcal{B}(y)$ for $x \neq y$. Since the number of all distinct families $\mathcal{B}(x)$ is not larger than $\exp|\mathcal{B}|$, we have $|X| \leq \exp w(X)$. ■

A topological space X is called a *T_1-space* if for every pair of distinct points $x_1, x_2 \in X$ there exists an open set $U \subset X$ such that $x_1 \in U$ and $x_2 \notin U$. Let us observe that there also exists an open set $V \subset X$ such that $x_2 \in V$ and $x_1 \notin V$; it suffices to consider the pair x_2, x_1. This contrasts with the situation in T_0-spaces, where for a pair x_1, x_2 of distinct points there may exist either an open set U such that $x_1 \in U$ and $x_2 \notin U$ or an open set V such that $x_2 \in V$ and $x_1 \notin V$, but not necessarily both.

Clearly, every T_1-space is a T_0-space. The space described in Example 1.2.8 is a T_0-space but not a T_1-space; all other spaces defined above, except for the space described in Example 1.2.10, are T_1-spaces.

Let us note that X is a T_1-space if and only if every point $x \in X$ is the intersection of all its neighbourhoods. This implies in particular that every one-point subset of a first-countable T_1-space is a G_δ-set.

It turns out that X is a T_1-space if and only if for every $x \in X$ the set $\{x\}$ is closed. Indeed, if X is a T_1-space then for every $x \in X$ we have $\{x\} = \bigcap\{X \setminus U : x \notin U \in \mathcal{O}\}$, where $\mathcal{O}$ is the topology of X; hence the set $\{x\}$ is closed. On the other hand, if for every $x \in X$ the set $\{x\}$ is closed, then X is a T_1-space, because for every pair of distinct points $x_1, x_2 \in X$ the open set $U = X \setminus \{x_2\}$ contains x_1 and does not contain x_2.

A topological space X is called a *T_2-space*, or a *Hausdorff space*, if for every pair of distinct points $x_1, x_2 \in X$ there exist open sets U_1, U_2 such that $x_1 \in U_1, x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Clearly, every T_2-space is a T_1-space. The space described in Example 1.2.6 is a T_1-space but not a T_2-space; all spaces defined in Section 1.1, as well as the Sorgenfrey line and the Niemytzki plane, are Hausdorff spaces.

Let us note that X is a T_2-space if and only if every point $x \in X$ is the intersection of the closures of all its neighbourhoods.

The reader can easily prove the following

1.5.2. PROPOSITION. *Suppose we are given a set X and a collection $\{\mathcal{B}(x)\}_{x\in X}$ of families of subsets of X which has properties* (BP1)–(BP3). *If in addition the collection $\{\mathcal{B}(x)\}_{x\in X}$ has the following property*

(BP4) *For every pair of distinct points $x, y \in X$ there exist $U \in \mathcal{B}(x)$ and $V \in \mathcal{B}(y)$ such that $U \cap V = \emptyset$,*

then the space X with the topology generated by the neighbourhood system $\{\mathcal{B}(x)\}_{x\in X}$ is a Hausdorff space. ∎

1.5.3. THEOREM. *For every Hausdorff space X we have $|X| \leq \exp\exp d(X)$ and $|X| \leq [d(X)]^{\chi(X)}$.*

PROOF. Let A be a dense subset of X such that $|A| = d(X)$ and $\{\mathcal{B}(x)\}_{x\in X}$ a neighbourhood system for X.

To prove the first inequality assign to every $x \in X$ the family $\mathcal{A}(x) = \{U \cap A : U \in \mathcal{B}(x)\}$ of subsets of A. From the equality $\overline{U \cap A} = \overline{U}$ it follows that the intersection of the closures of all members of $\mathcal{A}(x)$ equals $\{x\}$; hence $\mathcal{A}(x) \neq \mathcal{A}(y)$ for $x \neq y$. Since the number of all distinct families $\mathcal{A}(x)$ is not larger than $\exp\exp|A|$, we have $|X| \leq \exp\exp d(X)$.

Suppose now that $|\mathcal{B}(x)| \leq \chi(X) \geq \aleph_0$ for every $x \in X$ and denote by $\mathcal{A}_0$ the family of all subsets of A whose cardinality is $\leq \chi(X)$; clearly $|\mathcal{A}_0| \leq [d(X)]^{\chi(X)}$. For every U in $\mathcal{B}(x)$ pick a point $a(x, U) \in U \cap A$ and take the set $A(x) = \{a(x, U) : U \in \mathcal{B}(x)\} \in \mathcal{A}_0$.

To prove the second inequality in the theorem let us assign to every $x \in X$ the family $\mathcal{A}_0(x) = \{U \cap A(x): U \in \mathcal{B}(x)\} \subset \mathcal{A}_0$; clearly $|\mathcal{A}_0(x)| \leq \chi(X)$. Since $x \in \overline{U \cap A(x)} \subset \overline{U}$ for every $U \in \mathcal{B}(x)$, the intersection of the closures of all members of $\mathcal{A}_0(x)$ equals $\{x\}$; hence $\mathcal{A}_0(x) \neq \mathcal{A}_0(y)$ for $x \neq y$. Since the number of all distinct families $\mathcal{A}_0(x)$ is not larger than $|\mathcal{A}_0|^{\chi(X)}$, we have

$$|X| \leq [[d(X)]^{\chi(X)}]^{\chi(X)} = [d(X)]^{\chi(X)}.$$

When $\chi(X)$ is finite, the space X is discrete and the second inequality in the theorem is obvious. ∎

In the sequel we shall frequently use the following theorem.

1.5.4. THEOREM. *For any pair f, g of continuous mappings of a space X into a Hausdorff space Y, the set $\{x \in X: f(x) = g(x)\}$ is closed in X.*

PROOF. We shall show that the set $A = \{x \in X: f(x) \neq g(x)\}$ is open. For every $x \in A$ we have $f(x) \neq g(x)$; hence there exist in Y open sets U_1, U_2 such that $f(x) \in U_1, g(x) \in U_2$ and $U_1 \cap U_2 = \emptyset$. The set $f^{-1}(U_1) \cap g^{-1}(U_2)$ is a neighbourhood of x contained in A. ∎

A topological space X is called a T_3*-space*, or a *regular space*, if X is a T_1-space and for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exist open sets U_1, U_2 such that $x \in U_1, F \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

1.5.5. PROPOSITION. *A T_1-space X is a regular space if and only if for every $x \in X$ and every neighbourhood V of x in a fixed subbase $\mathcal{P}$ there exists a neighbourhood U of x such that $\overline{U} \subset V$.*

In particular, *a T_1-space X is a regular space if and only if for every $x \in X$ and every neighbourhood V of x there exists a neighbourhood U of x such that $\overline{U} \subset V$.*

PROOF. Suppose that X is a regular space and $\mathcal{P}$ a subbase for X; take an $x \in X$ and a neighbourhoood $V \in \mathcal{P}$ of x. By the definition of a regular space, there exist open sets U_1, U_2 such that $x \in U_1, X \setminus V \subset U_2$ and $U_1 \cap U_2 = \emptyset$. We then have $U_1 \subset X \setminus U_2 \subset V$ which implies that $\overline{U}_1 \subset V$, since $X \setminus U_2$ is closed.

Suppose now that the condition is satisfied. Take an $x \in X$ and a closed set F such that $x \notin F$. By the definition of a subbase there exist $V_1, V_2, \ldots, V_k \in \mathcal{P}$ satisfying $x \in \bigcap_{i=1}^k V_i \subset X \setminus F$. For $i = 1, 2, \ldots, k$ take a neighbourhood W_i of x such that $\overline{W}_i \subset V_i$. The open sets $U_1 = \bigcap_{i=1}^k W_i$ and $U_2 = X \setminus \bigcap_{i=1}^k \overline{W}_i$ are disjoint, $x \in U_1$ and

$$F \subset X \setminus \bigcap_{i=1}^{k} V_i \subset X \setminus \bigcap_{i=1}^{k} \overline{W}_i = U_2. \blacksquare$$

Clearly, every regular space is a Hausdorff space. It is in order to guarantee this implication that we assume – besides the possibility of separation of points and closed sets – that a regular space is a T_1-space. The anti-discrete space has the required property of separation but it is not a T_1-space.*

All Hausdorff spaces described above are regular; we shall now give an example of a Hausdorff space that is not regular.

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* The reader should be warned that some authors do not include the assumption that X is a T_1-space in the definition of regular, completely regular and normal spaces.

1.5.6. EXAMPLE. Let X be the set of real numbers and $Z \subset X$ the set of reciprocals of all integers different from zero. For every $x \in X$ let $U_i(x) = (x - 1/i, x + 1/i)$ and

$$B(x) = \begin{cases} \{U_i(x)\}_{i=1}^{\infty}, & \text{if } x \neq 0, \\ \{U_i(x) \setminus Z\}_{i=1}^{\infty}, & \text{if } x = 0. \end{cases}$$

One can easily verify that the collection $\{\mathcal{B}(x)\}_{x \in X}$ has properties (BP1)–(BP4); hence the space X with the topology generated by the neighbourhood system $\{\mathcal{B}(x)\}_{x \in X}$ is a Hausdorff space. The set Z is closed in X and $0 \notin Z$; and yet for any open sets U_1, U_2 such that $0 \in U_1$ and $Z \subset U_2$, we have $U_1 \cap U_2 \neq \emptyset$, so that X is not regular.■

1.5.7. THEOREM. *For every regular space X we have $w(X) \leq \exp d(X)$.*

PROOF. Let A be a dense subset of X such that $|A| = d(X)$ and let $\mathcal{B} = \{\operatorname{Int} \overline{B} : B \subset A\}$. Since by Theorem 1.3.6 for every open set $U \subset X$ we have $\overline{U \cap A} = \overline{U} \supset U$, from the regularity of X it follows that $\mathcal{B}$ is a base for X, and clearly $|\mathcal{B}| \leq \exp|A|$.■

Tychonoff spaces are a still narrower class of spaces. A topological space X is called a *$T_{3\frac{1}{2}}$-space*, or a *Tychonoff space*, or a *completely regular space*, if X is a T_1-space and for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a continuous function $f: X \to I$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$. Since for the open sets $U_1 = f^{-1}([0, 1/2))$ and $U_2 = f^{-1}((1/2, 1])$ we have $x \in U_1, F \subset U_2$ and $U_1 \cap U_2 = \emptyset$, every Tychonoff space is a regular space.

In contrast to the definition of T_i-spaces for $i \leq 3$, the definition of a $T_{3\frac{1}{2}}$-space involves – besides set-theoretic notions and notions reducible to the notion of an open set – the notion of a continuous real-valued function. Hence, topological invariance of the class of $T_{3\frac{1}{2}}$-spaces requires a proof, while it followed directly from the definition of T_i-spaces with $i \leq 3$. This proof however consists in the simple observation that the composition fh of a homeomorphism h and a continuous function f is a continuous function.

1.5.8. PROPOSITION. *A T_1-space X is a Tychonoff space if and only if for every $x \in X$ and every neighbourhood V of x in a fixed subbase $\mathcal{P}$ there exists a continuous function $f: X \to I$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in X \setminus V$.*

In particular, *a T_1-space X is a Tychonoff space if and only if for every $x \in X$ and every neighbourhood V of x there exists a continuous function $f: X \to I$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in X \setminus V$.*

PROOF. Necessity of the condition follows from that fact that $X \setminus V$ is closed and does not contain x.

To prove sufficiency take an $x \in X$ and a closed set F such that $x \notin F$. By the definition of a subbase there exist $V_1, V_2, \ldots, V_k \in \mathcal{P}$ satisfying $x \in \bigcap_{i=1}^{k} V_i \subset X \setminus F$. For $i = 1, 2, \ldots, k$ take a function $f_i: X \to I$ such that $f_i(x) = 0$ and $f(y) = 1$ for $y \in X \setminus V_i$. One can readily verify that for $f = \max(f_1, f_2, \ldots, f_k)$ we have $f(x) = 0$ and $f(y) = 1$ for $y \in F$.■

In Examples 1.4.4 and 1.4.5 we proved that the Sorgenfrey line and the Niemytzki plane are Tychonoff spaces; all spaces defined in Section 1.1 are Tychonoff spaces as well.

We shall now give an example of a regular space that is not completely regular. Let us note that there even exist regular spaces on which every continuous real-valued function is constant; they are, however, fairly complicated (see Problem 2.7.17).

1.5.9. EXAMPLE. Let M_0 be the subset of the plane defined by the condition $y \geq 0$, i.e., the closed upper half-plane, let z_0 be the point $(0, -1)$ and let $M = M_0 \cup \{z_0\}$. Denote by L the line $y = 0$ and by L_i, where $i = 1, 2, \ldots$, the segment consisting of all points $(x, 0) \in L$ with $i - 1 \leq x \leq i$. For each point $z = (x, 0) \in L$ denote by $A_1(z)$ the set of all points $(x, y) \in M_0$, where $0 \leq y \leq 2$, by $A_2(z)$ the set of all points $(x + y, y) \in M_0$, where $0 \leq y \leq 2$, and let $\mathcal{B}(z)$ be the family of all sets of the form $(A_1(z) \cup A_2(z)) \setminus B$, where B is a finite set such that $z \notin B$. Furthermore, for each point $z \in M_0 \setminus L$ let $\mathcal{B}(z) = \{\{z\}\}$ and, finally, let $\mathcal{B}(z_0) = \{U_i(z_0)\}_{i=1}^{\infty}$, where $U_i(z_0)$ consists of z_0 and all points $(x, y) \in M_0$ with $x \geq i$. One can easily verify that the collection $\{\mathcal{B}(z)\}_{z \in M}$ has properties (BP1)–(BP4); hence the space M with the topology generated by the neighbourhood system $\{\mathcal{B}(z)\}_{z \in M}$ is a Hausdorff space.

We shall show that M is regular. First of all, let us observe that for each $z \in M_0$ the family $\mathcal{B}(z)$ consists of open-and-closed subsets of M. Thus, to establish the regularity of M, it suffices to show that for every closed set $F \subset M$ such that $z_0 \notin F$ there exist open sets U_1, U_2 such that $z_0 \in U_1, F \subset U_2$ and $U_1 \cap U_2 = \emptyset$; one readily checks that the sets $U_1 = U_{i_0+2}(z_0)$ and $U_2 = M \setminus (U_{i_0+2}(z_0) \cup L_{i_0} \cup L_{i_0+1})$, where $F \cap U_{i_0}(z_0) = \emptyset$, have the required properties.

Now, let us consider a continuous function $f: M \to I$ such that $f(L_1) = \{0\}$. To prove that the space M is not completely regular it suffices to show that $f(z_0) = 0$. The last equality follows directly from the continuity of f and the fact that the set $K_i = \{z \in L_i: f(z) = 0\}$ is infinite for $i = 1, 2, \ldots$, which we are going to establish by induction. Obviously, the set $K_1 = L_1$ is infinite. Assume now that $|K_n| \geq \aleph_0$ and consider a countably infinite set $K'_n \subset K_n$. Arguing as in Example 1.4.6, one readily sees that for every $z \in K'_n$ there exists a countable set $A_0(z) \subset A_2(z)$ such that $f(A_2(z) \setminus A_0(z)) = \{0\}$. The projection A of the union $\bigcup\{A_0(z): z \in K'_n\}$ onto L is countable. Now, for every $t \in L_{n+1} \setminus A$ the set $A_1(t)$ meets each of the sets $A_2(z) \setminus A_0(z)$ with $z \in K'_n$, so that by the continuity of f we have $f(t) = 0$; it follows that $L_{n+1} \setminus A \subset K_{n+1}$ which implies that $|K_{n+1}| \geq \aleph_0$. Thus K_i is infinite for $i = 1, 2, \ldots$, and the proof is concluded. ■

A topological space X is called a *T_4-space*, or a *normal space*, if X is a T_1-space and for every pair of disjoint closed subsets $A, B \subset X$ there exist open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. Let us note that a T_1-space X is normal if and only if for every closed set $F \subset X$ and every open set $V \subset X$ that contains F there exists an open set $U \subset X$ such that $A \subset U \subset \overline{U} \subset V$. Clearly, every T_4-space is a T_3-space; from Theorem 1.5.11 below it follows that every T_4-space is a $T_{3\frac{1}{2}}$-space as well, but this fact is by no means obvious. All discrete spaces $D(\mathfrak{m})$ are normal. One can easily check that all spaces $A(\mathfrak{m})$ are normal as well. Theorem 1.5.16 below implies that the real line R and the interval I are normal. It turns out that the Sorgenfrey line K is normal, too.

Indeed, let A, B be disjoint closed subsets of K. For each $a \in A$ choose an interval $[a, x(a))$ disjoint from B, and for each $b \in B$ an interval $[b, x(b))$ disjoint from A. Letting

$$U = \bigcup_{a \in A} [a, x(a)) \quad \text{and} \quad V = \bigcup_{b \in B} [b, x(b))$$

we define open sets such that $A \subset U$ and $B \subset V$. For each $a \in A$ and $b \in B$ we have $[a, x(a)) \cap [b, x(b)) = \emptyset$, because otherwise we would have $b \in [a, x(a))$ or $a \in [b, x(b))$ depending on whether $a < b$ or $b < a$. Thus $U \cap V = \emptyset$.

We shall now give an example of a Tychonoff space that is not normal; it turns out that the Niemytzki plane is such a space.

1.5.10. EXAMPLE. We have to show that the Niemytzki plane L is not normal. We have already observed that the derived set of $L_1 \subset L$ is empty. From 1.3.4 it follows that $A^d = \emptyset$ for every $A \subset L_1$ and that every such A is closed in L. Let us take the set C consisting of all points in L_2 having both coordinates rational; clearly C is a dense subset of L.

Assume that L is a normal space. Thus for every $A \subset L_1$ there exist open sets $U_A, V_A \subset L$ such that $A \subset U_A, L_1 \setminus A \subset V_A$ and $U_A \cap V_A = \emptyset$. To every $A \subset L_1$ assign the set $C_A = C \cap U_A$. We shall show that $C_A \neq C_B$ for $A \neq B$, which will yield a contradiction, because L_1 contains $2^{\mathfrak{c}}$ distinct subsets, and C contains only $\mathfrak{c}$ distinct subsets. Take then $A, B \subset L_1$ such that $A \neq B$; by the symmetry of assumptions we can assume that $A \setminus B \neq \emptyset$. Since $A \setminus B \subset U_A \cap V_B$, we have $U_A \cap V_B \neq \emptyset$, and this implies, due to the density of C in L, that $\emptyset \neq C \cap U_A \cap V_B \subset C_A \setminus U_B \subset C_A \setminus C_B$; hence $C_A \neq C_B$.■

Our next theorem is a fundamental one; for historical reasons it is called *Urysohn's lemma*:

1.5.11. THEOREM (URYSOHN'S LEMMA). *For every pair A, B of disjoint closed subsets of a normal space X there exists a continuous function $f: X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

PROOF. For every rational number r in the interval $[0,1]$ we shall define an open set $V_r \subset X$ subject to the conditions:

$$\overline{V}_r \subset V_{r'} \quad \text{whenever} \quad r < r', \tag{1}$$

$$A \subset V_0, \quad B \subset X \setminus V_1. \tag{2}$$

The sets V_r will be defined inductively. Let us arrange into an infinite sequence $r_3, r_4, \ldots$ all rational numbers in the interval $(0,1)$ and let $r_1 = 0$ and $r_2 = 1$. Take $V_0 = U$ and $V_1 = X \setminus B$, where U is an open set satisfying $A \subset U \subset \overline{U} \subset X \setminus B$. Obviously, $\overline{V}_0 \subset V_1$. Condition (2) as well as the condition

$$\overline{V}_{r_i} \subset V_{r_j} \quad \text{whenever} \quad r_i < r_j \quad \text{and} \quad i, j \leq k \tag{3_k}$$

for $k = 2$ are thus satisfied.

Assume that the sets V_{r_i} are already defined for $i \leq n$, where $n \geq 2$, and satisfy (3_n). Let us denote by r_l and r_m respectively those of the numbers $r_1, r_2, \ldots, r_n$ that are closest to r_{n+1} from the left and from the right. Since $r_l < r_m$ it follows from (3_n) that $\overline{V}_{r_l} \subset V_{r_m}$. Let U be an open set satisfying $\overline{V}_{r_l} \subset U \subset \overline{U} \subset V_{r_m}$. Taking $V_{r_{n+1}} = U$ we obtain sets $V_{r_1}, V_{r_2}, \ldots, V_{r_{n+1}}$ that satisfy (3_{n+1}). The sequence $V_{r_1}, V_{r_2}, \ldots$ obtained in this way is subject to conditions (1) and (2).

Consider the function f from X to I defined by the formula

$$f(x) = \begin{cases} \inf\{r: x \in V_r\} & \text{for } x \in V_1, \\ 1 & \text{for } x \in X \setminus V_1. \end{cases}$$

Since (2) implies that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$, we have only to prove that f is continuous; by virtue of 1.4.1, it suffices to show that inverse images of intervals $[0,a)$ and $(b,1]$, where $a \leq 1$ and $b \geq 0$, are open. The inequality $f(x) < a$ holds if and only if there exists an $r < a$ such that $x \in V_r$, hence the set $f^{-1}([0,a)) = \bigcup\{V_r : r < a\}$ is open. And the inequality $f(x) > b$ holds if and only if there exists an $r' > b$ such that $x \notin V_{r'}$, which – by virtue of (1) – means that there exists an $r > b$ such that $x \notin \overline{V}_r$. Hence the set

$$f^{-1}((b,1]) = \bigcup\{X \setminus \overline{V}_r : r > b\} = X \setminus \bigcap\{\overline{V}_r : r > b\}$$

is open, too. ■

1.5.12. COROLLARY. *A subset A of a normal space X is a closed G_δ-set if and only if there exists a continuous function $f: X \to I$ such that $A = f^{-1}(0)$.*

PROOF. The one-point set $\{0\} \subset I$ is a closed G_δ-set, thus for every continuous $f: X \to I$, the inverse image $f^{-1}(0)$ is a closed G_δ-set.

Now let A be a closed G_δ-set in a normal space X. The complement of A is an F_σ-set, i.e., $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, where $\overline{F}_i = F_i$ for $i = 1,2,\ldots$ By Urysohn's lemma, for $i = 1,2,\ldots$ there exists a continuous function $f_i: X \to I$ such that $f_i(x) = 0$ for $x \in A$ and $f_i(x) = 1$ for $x \in F_i$. From 1.4.7 it follows that the formula

$$f(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x) \quad \text{for} \quad x \in X$$

defines a continuous function $f: X \to I$. For $x \in A$ we clearly have $f(x) = 0$; and when $x \notin A$ there exists an i such that $x \in F_i$, and we have $f(x) \geq \frac{1}{2^i} f_i(x) = \frac{1}{2^i} > 0$. Hence $A = f^{-1}(0)$. ■

Taking complements, we can restate 1.5.12 as follows:

1.5.13. COROLLARY. *A subset A of a normal space X is an open F_σ-set if and only if there exists a continuous function $f: X \to I$ such that $A = f^{-1}((0,1])$.* ■

Two subsets A and B of a topological space X are called *completely separated* if there exists a continuous function $f: X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$; we then say that f *separates* the sets A and B. Urysohn's lemma states that in a normal space any two disjoint closed sets are completely separated. One readily checks that if any two disjoint closed sets in a T_1-space are completely separated then the space is normal.

A subset A of a topological space X is called *functionally closed* * if $A = f^{-1}(0)$ for some continous function $f: X \to I$. Obviously, every functionally closed subset of X is closed in X. Let $f, g: X \to I$; since for $h_1, h_2: X \to I$ defined by the formulas

$$h_1(x) = f(x) \cdot g(x) \quad \text{and} \quad h_2(x) = \frac{1}{2}(f(x) + g(x))$$

we have

$$h_1^{-1}(0) = f^{-1}(0) \cup g^{-1}(0) \quad \text{and} \quad h_2^{-1}(0) = f^{-1}(0) \cap g^{-1}(0),$$

it follows that the union and the intersection of two (and of finitely many) functionally closed sets are functionally closed. A countable intersection of functionally closed sets is a

* The terms "functionally closed set" and "functionally open set" adopted here seem more suitable than the terms *zero-set* and *cozero-set* which are generally used.

functionally closed set as well. Indeed, if for $f_i: X \to I$, where $i = 1, 2, \ldots$, we define $f: X \to I$ by the formula

$$f(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x) \quad \text{for } x \in X,$$

we have $\bigcap_{i=1}^{\infty} f_i^{-1}(0) = f^{-1}(0)$.

The complement of a functionally closed subset of X is called *functionally open.* Obviously, every functionally open subset of X is open in X. Countable unions and finite intersections of functionally open sets are functionally open. One readily verifies that a T_1-space X is completely regular if and only if the family of all functionally open sets is a base for X. Closed-and-open sets are both functionally closed and functionally open. The inverse image of a functionally closed (open) set under a continuous mapping is a functionally closed (open) set. Corollaries 1.5.12 and 1.5.13 state that in a normal space functionally closed (open) sets coincide with closed G_δ-sets (open F_σ-sets).

1.5.14. THEOREM. *Any disjoint functionally closed sets A, B in a topological space X are completely separated; moreover, there exists a continuous function $f: X \to I$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.*

PROOF. Let $g, h: X \to I$ satisfy $A = g^{-1}(0)$ and $B = h^{-1}(0)$. Since $A \cap B = \emptyset$, the formula $f(x) = g(x)/(g(x)+h(x))$ defines a continuous function $f: X \to I$. One readily checks that $A = f^{-1}(0)$ and $B = f^{-1}(1)$. ■

We shall show now that every second-countable regular space is normal.

1.5.15. LEMMA. *If X is a T_1-space and for every closed set $F \subset X$ and every open $W \subset X$ that contains F there exists a sequence $W_1, W_2, \ldots$ of open subsets of X such that $F \subset \bigcup_{i=1}^{\infty} W_i$ and $\overline{W_i} \subset W$ for $i = 1, 2, \ldots$, then the space X is normal.*

PROOF. Let A and B be disjoint closed subsets of X. Letting $F = A$ and $W = X \setminus B$ we obtain a sequence $W_1, W_2, \ldots$ of open subsets of X such that

$$A \subset \bigcup_{i=1}^{\infty} W_i \quad \text{and} \quad B \cap \overline{W_i} = \emptyset \quad \text{for} \quad i = 1, 2, \ldots \tag{4}$$

Letting $F = B$ and $W = X \setminus A$ we obtain a sequence $V_1, V_2, \ldots$ of open subsets of X such that

$$B \subset \bigcup_{i=1}^{\infty} V_i \quad \text{and} \quad A \cap \overline{V_i} = \emptyset \quad \text{for} \quad i = 1, 2, \ldots \tag{5}$$

Let

$$G_i = W_i \setminus \bigcup_{j \le i} \overline{V_j} \quad \text{and} \quad H_i = V_i \setminus \bigcup_{j \le i} \overline{W_j}. \tag{6}$$

The sets G_i and H_i are open; moreover, (4) and (5) imply that

$$A \subset U = \bigcup_{i=1}^{\infty} G_i \quad \text{and} \quad B \subset V = \bigcup_{i=1}^{\infty} H_i.$$

To complete the proof we have to show that the open sets U and V are disjoint. Since (6) implies that $G_i \cap V_j = \emptyset$ for $j \leq i$, we have $G_i \cap H_j = \emptyset$ for $j \leq i$. Similarly, $H_j \cap W_i = \emptyset$ for $i \leq j$ and $G_i \cap H_j = \emptyset$ for $i \leq j$. Thus $G_i \cap H_j = \emptyset$ for $i, j = 1, 2, \ldots$ and therefore $U \cap V = \emptyset$.■

One can easily check that the condition in the above lemma is not only sufficient but also necessary for normality of a T_1-space X.

1.5.16. THEOREM. *Every second-countable regular space is normal.*

PROOF. Every regular space X with a countable base $\mathcal{B}$ satisfies the condition in Lemma 1.5.15, because for any $x \in F$ there is a $U_x \in \mathcal{B}$ such that $x \in U_x \subset \overline{U}_x \subset W$, the family of the all U_x's is countable and $F \subset \bigcup_{x \in F} U_x$.■

1.5.17. THEOREM. *Every countable regular space is normal.*

PROOF. Every countable regular space X satisfies the condition in Lemma 1.5.15, because for any $x \in F$ there is an open U_x such that $x \in U_x \subset \overline{U}_x \subset W$, the family of all the U_x's is countable and $F \subset \bigcup_{x \in F} U_x$.■

In connection with Theorem 1.5.17, it should be noted that there exist countable regular spaces that do not satisfy the first, and, *a fortiori*, the second axiom of countability (see Examples 1.6.19, 1.6.20 and 2.3.37).

Let us observe that from De Morgan's laws it follows that a T_1-space X is normal if and only if for every pair of open sets $U, V \subset X$ such that $X = U \cup V$ there exists a pair of closed sets $A, B \subset X$ such that $A \subset U$, $B \subset V$ and $X = A \cup B$. To sharpen this result we need a few simple notions.

A family $\{A_s\}_{s \in S}$ of subsets of a set X is called a *cover* of X if $\bigcup_{s \in S} A_s = X$. If X is a topological space and all the sets A_s are open (closed), we say that the cover $\{A_s\}_{s \in S}$ is *open (closed)*. A family $\{A_s\}_{s \in S}$ of subsets of a set X is called *point-finite (point-countable)* if for every $x \in X$ the set $\{s \in S : x \in A_s\}$ is finite (countable). Clearly every locally finite cover is point-finite. On the other hand, the open cover of the interval I consisting of I itself and of all intervals $(1/(i+1), 1/i)$, where $i = 1, 2, \ldots$, is point-finite and not locally finite.

1.5.18. THEOREM. *For every point-finite open cover $\{U_s\}_{s \in S}$ of a normal space X there exists an open cover $\{V_s\}_{s \in S}$ of X such that $\overline{V}_s \subset U_s$ for every $s \in S$.*

PROOF. Let $\mathcal{G}$ be the family of all functions G from the set S to the topology $\mathcal{O}$ of the space X subject to the conditions:

$$G(s) = U_s \quad \text{or} \quad \overline{G(s)} \subset U_s, \tag{7}$$

and

$$\bigcup_{s \in S} G(s) = X. \tag{8}$$

Let us order the family $\mathcal{G}$ by defining that $G_1 \leq G_2$ whenever $G_2(s) = G_1(s)$ for every $s \in S$ such that $G_1(s) \neq U_s$. We shall show that for each linearly ordered subfamily $\mathcal{G}_0 \subset \mathcal{G}$ the formula $G_0(s) = \bigcap_{G \in \mathcal{G}_0} G(s)$ for $s \in S$ defines a member of $\mathcal{G}$. Condition (7) is clearly satisfied for $G = G_0$; we shall verify condition (8). Take a point $x \in X$; as $\{U_s\}_{s\in S}$ is point-finite, there exists a finite set $S_0 = \{s_1, s_2, \ldots, s_k\} \subset S$ such that $x \in U_{s_i}$ for $i = 1, 2, \ldots, k$ and $x \notin U_s$ for $s \in S \setminus S_0$. If $G_0(s_i) = U_{s_i}$ for some $s_i \in S_0$, then $x \in G_0(s_i) \subset \bigcup_{s\in S} G_0(s)$. Assume now that for $i = 1, 2, \ldots, k$ there exists a $G_i \in \mathcal{G}_0$ such that $G_i(s_i) \neq U_{s_i}$. Since the family $\mathcal{G}_0$ is linearly ordered, there exists a $j \leq k$ such that $G_i \leq G_j$ for $i = 1, 2, \ldots, k$. Applying (8) to G_j we find an $i_0 \leq k$ such that $x \in G_j(s_{i_0}) = G_0(s_{i_0})$, so that also in this case $x \in \bigcup_{s\in S} G_0(s)$. One easily sees that $G \leq G_0$ for every $G \in \mathcal{G}_0$.

From the Kuratowski-Zorn lemma it follows that there exists a maximal element G in $\mathcal{G}$; to complete the proof it suffices to show that $\overline{G(s)} \subset U_s$ for every $s \in S$.

Let us suppose that $\overline{G(s_0)} \cap (X \setminus U_{s_0}) \neq \emptyset$. The set $A = X \setminus \bigcup\{G(s): s \in S \setminus \{s_0\}\} \subset G(s_0)$ is closed. By the normality of X there exists an open set U such that $A \subset U \subset \overline{U} \subset G(s_0)$. Since from (7) it follows that $G(s_0) = U_{s_0}$, the formula

$$G_0(s) = \begin{cases} U & \text{for } s = s_0, \\ G(s) & \text{for } s \neq s_0, \end{cases}$$

defines a function $G_0 \in \mathcal{G}$ such that $G \leq G_0$ and $G \neq G_0$. This contradiction to maximality of G shows that $\overline{G(s)} \subset U_s$ for every $s \in S$. ∎

A topological space X is called a *perfectly normal space* if X is a normal space and every closed subset of X is a G_δ-set. Clearly, a normal space X is perfectly normal if and only if every open subset of X is an F_σ-set.

Obviously, all discrete spaces $D(\mathfrak{m})$ and all countable normal spaces are perfectly normal. Every normal space X that has a countable base $\mathcal{B}$ also is perfectly normal. Indeed, for any open set $U \subset X$ we have $U = \bigcup\{\overline{V} \in \mathcal{B}: \overline{V} \subset U\}$ and thus U is an F_σ-set.

It turns out that the Sorgenfrey line K is perfectly normal, too. Indeed, for every open set $U \subset K$ the interior V of U with respect to the topology of the real line is the union of countably many open intervals (a, b) each of which is an F_σ-set in K, and thus V is an F_σ-set in K; now, for every point $a \in A = U \setminus V$ an interval $[a, x(a))$ is contained in U, and for distinct $a, b \in A$ we have $[a, x(a)) \cap [b, x(b)) = \emptyset$ (because otherwise a or b would belong to V), since every interval $[a, x(a))$ contains a rational number, $|A| \leq \aleph_0$ and $U = V \cup A$ is an F_σ-set in K.

On the other hand, the space $A(\mathfrak{m})$ is not perfectly normal for $\mathfrak{m} > \aleph_0$. Indeed, every F_σ-set in $A(\mathfrak{m})$ that does not contain x_0, the unique accumulation point of $A(\mathfrak{m})$, is countable, and this implies that the open set $A(\mathfrak{m}) \setminus \{x_0\}$ is not an F_σ-set.

Since the real line R is perfectly normal, by virtue of Corollary 1.5.12 functionally closed (open) subsets of a space X can be characterized as inverse images of closed (open) subsets of R under continuous functions from X to R.

Let us note that there exist normal spaces in which all one-point subsets are G_δ-sets (even normal first-countable spaces), that are not perfectly normal (see Example 3.1.26).

When looking at the definition of perfectly normal spaces, one could ask why perfect normality is ranged among the separation axioms; this becomes clear with the following theorem.

1.5.19. THE VEDENISSOFF THEOREM. *For every T_1-space X the following conditions are equivalent:*

(i) *The space X is perfectly normal.*

(ii) *Open subsets of X are functionally open.*
(iii) *Closed subsets of X are functionally closed.*
(iv) *For every pair of disjoint closed subsets $A, B \subset X$ there exists a continuous function $f: X \to I$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.*

PROOF. The implication (i)$\Rightarrow$(ii) follows from 1.5.13 and the implication (iii)$\Rightarrow$(iv) from 1.5.14. Implications (ii)$\Rightarrow$(iii) and (iv)$\Rightarrow$(i) are obvious. ■

Now we are going to discuss the invariance of axioms of separation under mappings. Let us start with a simple observation: since any space is the continuous image of a discrete space, none of the axioms of separation is invariant under continuous mappings. The situation improves when we pass to closed mappings:

1.5.20. THEOREM. *The class of T_i-spaces for $i = 1$ and 4 and the class of perfectly normal spaces are invariant under closed mappings.*

PROOF. The invariance of T_1-spaces follows from the fact that T_1-spaces are characterized as spaces in which one-point subsets are closed.

Now, let $f: X \to Y$ be a closed mapping of a normal space X onto a space Y. As we know, Y is a T_1-space. Consider a pair of open sets $U, V \subset Y$ such that $U \cup V = Y$; the sets $f^{-1}(U)$ and $f^{-1}(V)$ are open in X and cover the space X. By the normality of X there exist closed sets $A_0, B_0 \subset X$ such that $A_0 \subset f^{-1}(U)$, $B_0 \subset f^{-1}(V)$ and $A_0 \cup B_0 = X$. The sets $A = f(A_0)$ and $B = f(B_0)$ are closed in Y; moreover $A \subset ff^{-1}(U) = U, B \subset ff^{-1}(V) = V$ and $A \cup B = f(X) = Y$. Thus Y is a normal space.

If $f: X \to Y$ is a closed mapping of a perfectly normal space X onto a space Y, then Y is normal and for every open set $U \subset Y$ we have $f^{-1}(U) = \bigcup_{i=1}^{\infty} F_i$, where the F_i's are closed in X. Hence

$$U = ff^{-1}(U) = f(\bigcup_{i=1}^{\infty} F_i) = \bigcup_{i=1}^{\infty} f(F_i)$$

is an F_σ-set in Y, i.e., Y is perfectly normal. ■

It turns out that T_1, T_4 and perfect normality are the only axioms of separation invariant under closed mappings. To obtain a closed mapping of a $T_{3\frac{1}{2}}$-space onto a space that is not a T_2-space, we have to take any non-normal Tychonoff space X, a pair of disjoint closed subsets $A, B \subset X$ that cannot be separated by disjoint open sets, and to identify A to a point a and B to a point b, as described in Example 1.4.17 (cf. Example 2.4.12). A closed mapping of a T_0-space onto the two-point anti-discrete space is defined below.

1.5.21. EXAMPLE. Let X be the set of all integers with the topology consisting of the sets $\{x \in X: x \geq n\}$ for $n = 0, \pm 1, \pm 2, \ldots$; obviously X is a T_0-space. Let $Y = \{0, 1\}$ with the anti-discrete topology and let $f(x) = 0$ if x is even and $f(x) = 1$ if x is odd. One readily sees that f is a closed-and-open mapping of X onto Y. ■

The last example shows that the class of T_0-spaces is not invariant under open mappings. It turns out that none of the axioms of separation is invariant under open mappings:

1.5.22. EXAMPLE. Let us map the real line $R = X$ onto the two-point anti-discrete space $Y = \{0, 1\}$ by assigning 0 to all rationals and 1 to all irrationals. One readily checks that the mapping $f: X \to Y$ defined that way is open (cf. the remark to Exercise 4.2.D(b)). ■

Historical and bibliographic notes

T_0-spaces were introduced by Kolmogoroff (cited in Alexandroff and Hopf [1935]), T_1-spaces – by Riesz in [1907] and T_2-spaces – by Hausdorff in [1914]. Regular spaces were defined by Vietoris in [1921]; the $T_{3\frac{1}{2}}$property was formulated by Urysohn in [1925], the class of $T_{3\frac{1}{2}}$-spaces was studied by Tychonoff in [1930]. The class of normal spaces was defined by Tietze in [1923] and by Alexandroff and Urysohn in [1924]; the normality property appeared earlier in Vietoris [1921]. The property of perfect normality appeared in Urysohn [1925] and was studied by Alexandroff and Urysohn in [1929] in the realm of compact spaces; perfectly normal spaces were defined by Čech in [1932].

The definition of $T_{3\frac{1}{2}}$-spaces is significantly different from definitions of other classes of spaces studied in this section. It is an *external* definition, in which we assume the existence of some objects external to the space under consideration (in this case we assume the existence of the interval I), in distinction to *internal* definitions, in which only objects internal to the space under consideration are used. In the sequel a few classes of spaces will be introduced by external definitions, because sometimes such definitions are simpler and more natural, but we shall also give – conforming to well-established topological tradition – internal characterizations of those classes (obviously, the internal characterizations could also be taken as definitions). An internal characterization of $T_{3\frac{1}{2}}$-spaces is given in Exercise 1.5.G; the reader should note how complicated and less natural it is.

The first part of Theorem 1.5.3 was proved by Pospíšil in [1937a] (cf. Pospíšil [1937]). Example 1.5.9 was given by Mysior in [1981a]; the first example of a regular space that is not completely regular, much more complicated, can be found in Tychonoff's paper [1930]. The proof of non-normality of the Niemytzki plane given in 1.5.10 is taken from Jones [1937] (cf. Problem 1.7.12(c) and Corollary 2.1.10). Urysohn's lemma was established by Urysohn in [1925]; modifications of his argument are sometimes used to construct continuous real-valued functions (cf. Exercise 1.5.G and Problem 2.7.2(c)). The paper of Urysohn also contains Theorem 1.5.17. Theorem 1.5.16 was proved by Tychonoff in [1925]; the proof of Lemma 1.5.15 also is one of the standard topological arguments (cf. the proof of Lemma 7.2.5). Theorem 1.5.18 appeared in Lefschetz [1942]. The conditions for perfect normality in Theorem 1.5.19 were given in Vedenissoff's papers [1936] and [1940]. Hausdorff in [1935] was the first to observe that normality is an invariant of closed mappings.

Exercises

1.5.A. Verify that X is a T_0-space if and only if $\overline{\{x\}} \neq \overline{\{y\}}$ for every pair of distinct points $x, y \in X$.

1.5.B. Note that finite T_1-spaces are discrete.

Show that in T_1-spaces the derived set has the following properties:

$$(A^{\mathrm{d}})^{\mathrm{d}} \subset A^{\mathrm{d}}, \quad \overline{(A^{\mathrm{d}})} = A^{\mathrm{d}} = (\overline{A})^{\mathrm{d}}, \quad A^{\mathrm{d}} = \emptyset \text{ if } A \text{ is finite.}$$

Give examples to show that the above statements are not true in T_0-spaces.

1.5.C. A continuous mapping $f: X \to X$ is called a *retraction* of X, if $ff = f$; the set of all values of a retraction of X is called a *retract* of X.

Show that any retract of a Hausdorff space is closed.

1.5.D. (a) Note that in the definition of regular and completely regular spaces one could as well assume that the space under consideration is a T_0-space rather than a T_1-space. Verify that such a modification in the definition of normal spaces is impossible.

(b) Show that in 1.5.1 the assumption that X is a T_0-space cannot be omitted and that in 1.5.3 the assumption that X is a Hausdorff space cannot be weakened to the assumption that X is a T_1-space.

(c) Prove that in 1.5.7 the assumption that X is a regular space cannot be weakened to the assumption that X is a Hausdorff space.

Hint. Let $U(z,r)$ be the set of all points of the plane within the circle of radius r and centre at z. Generate a topology on the plane taking as the base the family consisting of all sets of the form $U(z,r) \setminus A$, where A is a set intersecting every line parallel to the x-axis in a finite number of points.

One can also use the fact that there exist separable Hausdorff spaces of cardinality $2^{\mathfrak{c}}$ (cf. Theorem 2.3.15 or Corollary 3.6.12) and apply the hint to Exercise 3.1.F(d).

1.5.E. (a) Show that the topology of a T_1-space X can be generated by a family of mappings into the real line if and only if X is a completely regular space.

(b) Verify that if on a set X two topologies $\mathcal{O}_1, \mathcal{O}_2$ are defined and both $(X, \mathcal{O}_1)$ and $(X, \mathcal{O}_2)$ are $T_{3\frac{1}{2}}$-spaces, then $\mathcal{O}_1 = \mathcal{O}_2$ if and only if the family of all real-valued functions on X continuous with respect to $\mathcal{O}_1$ coincides with the family of all real-valued functions on X continuous with respect to $\mathcal{O}_2$.

1.5.F. Show that for every finite family $\{F_i\}_{i=1}^k$ of pairwise disjoint closed subsets of a normal space X there exists a family $\{U_i\}_{i=1}^k$ of open subsets of X such that $F_i \subset U_i$ for $i = 1, 2, \ldots, k$ and $U_i \cap U_j = \emptyset$ for $i \neq j$ (cf. Theorem 2.1.14 and Exercise 2.1.G). Verify that if all the F_i's are finite it is sufficient to assume that X is a Hausdorff space.

1.5.G (O. Frink [1964], Zaĭcev [1967]). Prove that a T_1-space X is completely regular if and only if there exists a base $\mathcal{B}$ for X satisfying the following conditions:

(1) *For every $x \in X$ and every $U \in \mathcal{B}$ that contains x there exists a $V \in \mathcal{B}$ such that $x \notin V$ and $U \cup V = X$.*

(2) *For any $U, V \in \mathcal{B}$ satisfying $U \cup V = X$, there exist $U', V' \in \mathcal{B}$ such that $X \setminus V \subset U'$, $X \setminus U \subset V'$ and $U' \cap V' = \emptyset$.*

Hint. Modify the proof of Urysohn's lemma.

1.5.H. (a) Prove that every closed subset of the Niemytzki plane is a G_δ-set (spaces with this property are called *perfect spaces*).

(b) (Smirnov [1948]) Show that a T_1-space X is normal if and only if X satisfies the following conditions:

(1) *Every closed G_δ-set in X is functionally closed.*

(2) *For every closed $F \subset X$ and any open $G \subset X$ that contains F there exists in X a closed G_δ-set M such that $F \subset M \subset G$.*

(c) Verify that neither of the conditions in (b) by itself implies the normality of X.

1.5.I. Verify that if $(X, \mathcal{O}_1)$ is a T_i-space with $i \leq 2$, and $\mathcal{O}_2$ is a topology on X finer than $\mathcal{O}_1$, then $(X, \mathcal{O}_2)$ is a T_i-space. Show that this is not true for $i \geq 3$.

1.5.J. (a) Verify that countable unions of functionally closed sets are not necessarily functionally closed.

(b) Show that the union of a locally finite family of functionally closed sets is not necessarily functionally closed. Note that in a perfectly normal space the union of a locally finite family of functionally closed sets is functionally closed.

(c) Verify that the union of a family of functionally open sets is not necessarily functionally open.

(d) Note that the union of a locally finite family of functionally open sets is functionally open.

1.5.K. Show that a T_1-space X is perfectly normal if and only if for every open set $W \subset X$ there exists a sequence $W_1, W_2, \ldots$ of open subsets of X such that $W = \bigcup_{i=1}^{\infty} W_i$ and $\overline{W}_i \subset W$ for $i = 1, 2, \ldots$

1.5.L. (a) (Ponomarev [1959], Frolík [1961]) Prove that if $f: X \to Y$ is a closed-and-open mapping of X onto Y, then for every continuous function $g: X \to R$ that is bounded on all fibers of f, the formulas $g^*(y) = \sup\{g(x): x \in f^{-1}(y)\}$ and $g_*(y) = \inf\{g(x): x \in f^{-1}(y)\}$ define continuous functions $g^*, g_*: Y \to R$ (cf. Problem 1.7.16).

(b) (Chaber [1972]) Show that regularity and complete regularity are invariant under closed-and-open mappings.

Hint. In the case of complete regularity apply (a).

Remark. The class of Hausdorff spaces is not invariant under closed-and-open mappings (cf. an example of Alster's cited in Chaber [1972], where in lines 204_7, 205^3 and 205^5 one should read R instead of Q).

1.5.M. Give an example of an open mapping of a normal space onto a T_1-space that is not a T_2-space.

1.5.N (Fletcher and Lindgren [1973]). Let $\mathcal{U}$ be a point-finite open cover of a topological space X. Show that the set of all points $x \in X$ which have a neighbourhood that meets only finitely many members of $\mathcal{U}$ can be represented as the intersection of countably many dense open subsets of X.

Hint. For every $x \in X$ let $n(x) = |\{U \in \mathcal{U} : x \in U\}|$ and consider the open sets $G_i = \{x \in X : n(x) > i$ or x has a neighbourhood V such that $n(y) \leq i$ for each $y \in V\}$.

1.6. Convergence in topological spaces: Nets and filters. Sequential and Fréchet spaces

A *net in a topological space* X is an arbitrary function from a non-empty directed set to the space X. Nets will be denoted by the symbol $S = \{x_\sigma, \sigma \in \Sigma\}$, where x_σ is the point of X assigned to the element σ of the directed set Σ. The relation directing Σ will be denoted by $\leq$ and for $\sigma_1, \sigma_2 \in \Sigma$ we shall often write $\sigma_1 \geq \sigma_2$ instead of $\sigma_2 \leq \sigma_1$.

A point x is called a *limit of a net* $S = \{x_\sigma, \sigma \in \Sigma\}$ if for every neighbourhood U of x there exists a $\sigma_0 \in \Sigma$ such that $x_\sigma \in U$ for every $\sigma \geq \sigma_0$; we say then that the net S *converges* to x. A net can converge to many points; the set of all limits of the net $S = \{x_\sigma, \sigma \in \Sigma\}$ is denoted by $\lim S$ or $\lim_{\sigma\in\Sigma} x_\sigma$. When the net $S = \{x_\sigma,\ \sigma \in \Sigma\}$ has exactly one limit x, then we write $x = \lim S = \lim_{\sigma\in\Sigma} x_\sigma$.

A point x is called a *cluster point of a net* $S = \{x_\sigma,\ \sigma \in \Sigma\}$ if for every neighbourhood U of x and every $\sigma_0 \in \Sigma$ there exists a $\sigma \geq \sigma_0$ such that $x_\sigma \in U$.

We say that the net $S' = \{x_{\sigma'},\ \sigma' \in \Sigma'\}$ is *finer* than the net $S = \{x_\sigma,\ \sigma \in \Sigma\}$ if there exists a function ϕ from Σ' to Σ with the following properties:

(FN1) *For every $\sigma_0 \in \Sigma$ there exists a $\sigma'_0 \in \Sigma'$ such that $\phi(\sigma') \geq \sigma_0$ whenever $\sigma' \geq \sigma'_0$.*
(FN2) *$x_{\phi(\sigma')} = x_{\sigma'}$ for $\sigma' \in \Sigma'$.*

One can easily verify that every nondecreasing function ϕ from Σ' to Σ such that $\phi(\Sigma')$ is cofinal in Σ, has property (FN1).

1.6.1. PROPOSITION. *If x is a cluster point of the net S' that is finer than the net S, then x is a cluster point of S. If x is a limit of the net S, then x also is a limit of every net S' finer than S. If x is a cluster point of the net S, then x is a limit of a net S' that is finer than S.*

PROOF. Suppose that x is a cluster point of the net $S' = \{x_{\sigma'},\ \sigma' \in \Sigma'\}$ that is finer than the net $S = \{x_\sigma,\ \sigma \in \Sigma\}$, and let the function ϕ from Σ' to Σ have properties (FN1) and (FN2). Take an arbitrary neighbourhood U of x and a $\sigma_0 \in \Sigma$. There exists a $\sigma'_0 \in \Sigma$ such that $\phi(\sigma') \geq \sigma_0$ whenever $\sigma' \geq \sigma'_0$. By the definition of a cluster point, there exists a $\sigma'' \geq \sigma'_0$ such that $x_{\sigma''} \in U$. Thus we have $x_{\phi(\sigma'')} = x_{\sigma''} \in U$ and $\phi(\sigma'') \geq \sigma_0$, i.e., x is a cluster point of the net S.

Suppose now that x is a limit of the net $S = \{x_\sigma,\ \sigma \in \Sigma\}$. Let the net $S' = \{x_{\sigma'},\ \sigma' \in \Sigma'\}$ be finer than S, and let the function ϕ from Σ' to Σ have properties (FN1) and (FN2). Take an arbitrary neighbourhood U of x. There exist a $\sigma_0 \in \Sigma$ such that $x_\sigma \in U$ for every $\sigma \geq \sigma_0$ and a $\sigma'_0 \in \Sigma'$ such that $\phi(\sigma') \geq \sigma_0$ whenever $\sigma' \geq \sigma'_0$. Clearly for $\sigma' \geq \sigma'_0$ we have $x_{\sigma'} \in U$, i.e., x is a limit of the net S'.

Suppose that x is a cluster point of the net $S = \{x_\sigma,\ \sigma \in \Sigma\}$. Let us consider the set Σ' consisting of all ordered pairs (σ, U), where $\sigma \in \Sigma$, U is a neighbourhood of x and $x_\sigma \in U$; let us define that $(\sigma_1, U_1) \leq (\sigma_2, U_2)$ whenever $\sigma_1 \leq \sigma_2$ and $U_2 \subset U_1$. One can readily verify that Σ' is directed by $\leq$. The net $S' = \{x_{\sigma'},\ \sigma' \in \Sigma'\}$, where $x_{\sigma'} = x_\sigma$ for $\sigma' = (\sigma, U)$ is finer than Σ, because the function ϕ defined by $\phi((\sigma, U)) = \sigma$ is nondecreasing and maps Σ' onto Σ. For every neighbourhood U of x there exists a $\sigma \in \Sigma$ such that $x_\sigma \in U$; since for $\sigma' \geq (\sigma, U) \in \Sigma'$ we have $x_{\sigma'} \in U$, it follows that x is a limit of S'. ■

1.6.2. EXAMPLE. Let Σ be the set of all negative rational numbers directed by $\leq$ and let $x_r = r$ for every $r \in \Sigma$. Clearly $S = \{x_r,\ r \in \Sigma\}$ is a net in the real line R that converges to 0. Observe that 0 is the unique limit of S and that the set consisting of all elements of Σ and of the limit of Σ is not closed in R.

Let Σ' be the set of all positive integers directed by $\leq$ and let $\phi(n) = -1/n \in \Sigma$ for $n \in \Sigma'$. The function ϕ is nondecreasing and the set $\phi(\Sigma')$ is cofinal in Σ. The net $S' = \{x_n,\ n \in \Sigma'\}$, where $x_n = -1/n$, is finer than the net S; in conformity with 1.6.1, S' converges to 0. ■

1.6.3. PROPOSITION. *The point x belongs to $\overline{A}$ if and only if there exists a net consisting of elements of A converging to x.*

PROOF. The existence of such a net clearly implies that $x \in \overline{A}$. Let us suppose now that $x \in \overline{A}$ and let us denote by $\mathcal{U}$ the set of all neighbourhoods of x directed by the relation $\supset$, i.e., let us define that $U_1 \leq U_2$ if $U_1 \supset U_2$. One can easily check that $x \in \lim_{U \in \mathcal{U}} x_U$, where x_U is an arbitrary point of $A \cap U$.■

1.6.4. COROLLARY. *A set A is closed if and only if together with any net it contains all its limits.*■

1.6.5. COROLLARY. *The point x belongs to A^{d} if and only if there exists a net $S = \{x_\sigma,\ \sigma \in \Sigma\}$ converging to x, such that $x_\sigma \in A$ and $x_\sigma \neq x$ for every $\sigma \in \Sigma$.*■

Nets have many properties analogous to properties of simple sequences; this is the reason why some topologists consider nets to be a good tool to study properties of general topological spaces. The reader certainly noted that we have established for nets counterparts of fundamental properties of sequences; in those counterparts the role of subsequences is assumed by finer nets. Various topological notions can be characterized in terms of nets. One can also generate a topology on a set by specifying the class of convergent nets (see Problem 1.7.21). We shall show now the way in which continuous mappings and Hausdorff spaces can be characterized in terms of nets.

1.6.6. PROPOSITION. *A mapping f of a topological space X to a topological space Y is continuous if and only if*

$$f(\lim_{\sigma \in \Sigma} x_\sigma) \subset \lim_{\sigma \in \Sigma} f(x_\sigma)$$

for every net $\{x_\sigma,\ \sigma \in \Sigma\}$ in the space X.

PROOF. Suppose that $f: X \to Y$ is continuous and $x \in \lim_{\sigma \in \Sigma} x_\sigma$. By 1.4.1, for an arbitrary neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(U) \subset V$. Since $x \in \lim_{\sigma \in \Sigma} x_\sigma$, there exists a $\sigma_0 \in \Sigma$ such that $x_\sigma \in U$ for every $\sigma \geq \sigma_0$. This implies that $f(x_\sigma) \in V$ for $\sigma \geq \sigma_0$. Thus we have $f(x) \in \lim_{\sigma \in \Sigma} f(x_\sigma)$, which proves that $f(\lim_{\sigma \in \Sigma} x_\sigma) \subset \lim_{\sigma \in \Sigma} f(x_\sigma)$.

Conversely, suppose that the mapping f satisfies the condition in our proposition. To prove that f is continuous it suffices, again by 1.4.1, to show that $f(\overline{A}) \subset \overline{f(A)}$ for every $A \subset X$; this follows, however, from Proposition 1.6.3.■

1.6.7. PROPOSITION. *A topological space X is a Hausdorff space if and only if every net in X has at most one limit.*

PROOF. Suppose that X is a Hausdorff space. Let $S = \{x_\sigma,\ \sigma \in \Sigma\}$ be a net in X and let $x_1, x_2 \in \lim_{\sigma \in \Sigma} x_\sigma$. Take arbitrary neighbourhoods U_1, U_2 of points x_1, x_2 respectively. For $i = 1, 2$ there exists a $\sigma_i \in \Sigma$ such that $x_\sigma \in U_i$ for every $\sigma \geq \sigma_i$; the set Σ being directed this implies that $U_1 \cap U_2 \neq \emptyset$. Thus $x_1 = x_2$, i.e., S has at most one limit.

Conversely, suppose that X is not a Hausdorff space. This means that there are two distinct points $x_1, x_2 \in X$ such that for any neighbourhood U_1 of x_1 and any neighbourhood U_2 of x_2, we have $U_1 \cap U_2 \neq \emptyset$. The set Σ consisting of all intersections $U_1 \cap U_2$ is directed by $\supset$. One can easily check that $x_1, x_2 \in \lim_{\sigma \in \Sigma} x_\sigma$, where x_σ is an arbitrary point of $\sigma = U_1 \cap U_2$.■

Convergence in general topological spaces can be also described in terms of filters. We shall restrict our exposition of the theory of filters to formulations of fundamental definitions and counterparts of the above propositions on nets; we shall also sketch briefly how an equivalence can be established between the two ways of describing convergence in topological spaces.

Let $\mathcal{R}$ be a family of sets that together with A and B contains the intersection $A \cap B$. By a *filter in* $\mathcal{R}$ we mean a non-empty subfamily $\mathcal{F} \subset \mathcal{R}$ satisfying the following conditions:

(F1) $\emptyset \notin \mathcal{F}$.

(F2) *If* $A_1, A_2 \in \mathcal{F}$, *then* $A_1 \cap A_2 \in \mathcal{F}$.

(F3) *If* $A \in \mathcal{F}$ *and* $A \subset A_1 \in \mathcal{R}$, *then* $A_1 \in \mathcal{F}$.

A filter $\mathcal{F}$ in $\mathcal{R}$ is a *maximal filter* or an *ultrafilter in* $\mathcal{R}$, if for every filter $\mathcal{F}'$ in $\mathcal{R}$ that contains $\mathcal{F}$ we have $\mathcal{F}' = \mathcal{F}$.

A *filter-base in* $\mathcal{R}$ is a non-empty family $\mathcal{G} \subset \mathcal{R}$ such that $\emptyset \notin \mathcal{G}$ and

(FB) *If* $A_1, A_2 \in \mathcal{G}$, *then there exists an* $A_3 \in \mathcal{G}$ *such that* $A_3 \subset A_1 \cap A_2$.

One readily sees that for any filter-base $\mathcal{G}$ in $\mathcal{R}$, the family

$$\mathcal{F}_{\mathcal{G}} = \{A \in \mathcal{R} : \text{ there exists a } B \in \mathcal{G} \text{ such that } B \subset A\}$$

is a filter in $\mathcal{R}$.

By a *filter* (a *filter-base*) *in a topological space* X we mean a filter (a filter-base) in the family of all subsets of X. In this section we shall discuss only filters and filter-bases in topological spaces; we shall call them simply filters and filter-bases.

A point x is called a *limit of a filter* $\mathcal{F}$ if every neighbourhood of x belongs to $\mathcal{F}$; we then say that the filter $\mathcal{F}$ *converges* to x and we write $x \in \lim \mathcal{F}$. A point x is called a *limit of a filter-base* $\mathcal{G}$ if $x \in \lim \mathcal{F}_{\mathcal{G}}$; we then say that the filter-base $\mathcal{G}$ *converges* to x and we write $x \in \lim \mathcal{G}$. Clearly, $x \in \lim \mathcal{G}$ if and only if every neighbourhood of x contains a member of $\mathcal{G}$.

A point x is called a *cluster point of a filter* $\mathcal{F}$ *(of a filter-base* $\mathcal{G}$) if x belongs to the closure of every member of $\mathcal{F}$ (of $\mathcal{G}$). Clearly, x is a cluster point of a filter $\mathcal{F}$ (of a filter-base $\mathcal{G}$) if and only if every neighbourhood of x intersects all members of $\mathcal{F}$ (of $\mathcal{G}$). This implies in particular that every cluster point of an ultrafilter is a limit of this ultrafilter.

We say that the filter $\mathcal{F}'$ is *finer* than the filter $\mathcal{F}$ if $\mathcal{F}' \supset \mathcal{F}$.

1.6.8. PROPOSITION. *If x is a cluster point of the filter $\mathcal{F}'$ that is finer than the filter $\mathcal{F}$, then x is a cluster point of $\mathcal{F}$. If x is a limit of the filter $\mathcal{F}$, then x also is a limit of every filter $\mathcal{F}'$ finer than $\mathcal{F}$. If x is a cluster point of the filter $\mathcal{F}$, then x is a limit of a filter $\mathcal{F}'$ that is finer than $\mathcal{F}$.* ■

1.6.9. PROPOSITION. *The point x belongs to $\overline{A}$ if and only if there exists a filter-base consisting of subsets of A converging to x.* ■

1.6.10. PROPOSITION. *A mapping f of a topological space X to a topological space Y is continuous if and only if for every filter-base $\mathcal{G}$ in the space X and the filter-base $f(\mathcal{G}) = \{f(A) \colon A \in \mathcal{G}\}$ in the space Y we have*

$$f(\lim \mathcal{G}) \subset \lim f(\mathcal{G}).\ ■$$

1.6.11. PROPOSITION. *A topological space X is a Hausdorff space if and only if every filter in X has at most one limit.*■

We shall establish now a one-to-one correspondence between nets and filters in a topological space.

1.6.12. THEOREM. *For every net $S = \{x_\sigma,\ \sigma \in \Sigma\}$ in a topological space X, the family $\mathcal{F}(S)$, consisting of all sets $A \subset X$ with the property that there exists a $\sigma_0 \in \Sigma$ such that $x_\sigma \in A$ whenever $\sigma \geq \sigma_0$, is a filter in the space X and $\lim \mathcal{F}(S) = \lim S$. If the net S' is finer than the net S, then the filter $\mathcal{F}(S')$ is finer than the filter $\mathcal{F}(S)$.*■

1.6.13. THEOREM. *Let $\mathcal{F}$ be a filter in a topological space X; let us denote by Σ the set of all pairs (x, A), where $x \in A \in \mathcal{F}$ and let us define that $(x_1, A_1) \leq (x_2, A_2)$ if $A_2 \subset A_1$. The set Σ is directed by $\leq$, and for the net $S(\mathcal{F}) = \{x_\sigma,\ \sigma \in \Sigma\}$, where $x_\sigma = x$ for $\sigma = (x, A) \in \Sigma$, we have $\mathcal{F} = \mathcal{F}(S(\mathcal{F}))$ and $\lim S(\mathcal{F}) = \lim \mathcal{F}$.*■

As we observed above, the notion of a net is a modification of the notion of a sequence, adapted to handle convergence problems in general topological spaces. It is necessary to modify the notion of a sequence in such a way, because there exist topological spaces in which the topology cannot be described in terms of sequences, as opposed, for example, to the case of the real line, where closed subsets can be described as the sets that together with any convergent sequence contain its limit. The question arises in which topological spaces sequences suffice to describe topology. The question is not quite a precise one, and we shall see that it can be answered in two different ways – we shall define two classes of spaces that are natural in this context; both of them are larger than the class of first-countable spaces.

To begin, we make a few simple observations. A sequence $\{x_i\}$ in a topological space X is clearly a net defined on the set N of positive integers directed by $\leq$; thus the notions of a limit and of a cluster point are defined for sequences, and we know when a sequence converges to a point. The set of all limits of a sequence $\{x_i\}$ is denoted by $\lim x_i$, and when the sequence $\{x_i\}$ has exactly one limit x, then we write $x = \lim x_i$. Every subsequence $\{x_{k_i}\}$ of a sequence $\{x_i\}$ is a net finer than $\{x_i\}$; indeed, the function ϕ defined by $\phi(i) = k_i$ is nondecreasing and the set $\phi(N)$ is cofinal in N. It follows then from Proposition 1.6.1 that for every subsequence $\{x_{k_i}\}$ of $\{x_i\}$ if $x \in \lim x_i$, then $x \in \lim x_{k_i}$, and that every cluster point of $\{x_{k_i}\}$ is a cluster point of $\{x_i\}$.

Now, let us define the two classes of spaces we mentioned above. A topological space X is called a *sequential space* if a set $A \subset X$ is closed if and only if together with any sequence it contains all its limits. A topological space X is called a *Fréchet space* if for every $A \subset X$ and every $x \in \overline{A}$ there exists a sequence $x_1, x_2, \ldots$ of points of A converging to x.

1.6.14. THEOREM. *Every first-countable space is a Fréchet space and every Fréchet space is a sequential space.*

PROOF. If the space X has a countable base $\{U_i\}_{i=1}^\infty$ at the point x and $x \in \overline{A}$, then taking $x_i \in A \cap U_1 \cap U_2 \cap \ldots \cap U_i$ for $i = 1, 2, \ldots$, we define a sequence $\{x_i\}$ of points of A converging to x. Hence every first-countable space is a Fréchet space. The second part of the theorem is obvious.■

1.6.15. PROPOSITION. *A mapping f of a sequential space X to a topological space Y is* *continuous if and only if* $f(\lim x_i) \subset \lim f(x_i)$ *for every sequence* $\{x_i\}$ *in the space* X.

PROOF. Necessity of the condition follows from Proposition 1.6.6.

We shall show that if f satisfies the condition, then f is continuous. Let B be an arbitrary closed subset of Y; take a sequence $x_1, x_2, \ldots$ of points of $f^{-1}(B)$ and a point $x \in \lim x_i$. We have $f(x) \in \lim f(x_i) \subset B$; hence $x \in f^{-1}(B)$ and $f^{-1}(B)$ is closed.■

1.6.16. PROPOSITION. *If every sequence in a topological space* X *has at most one limit, then* X *is a* T_1*-space. If, moreover,* X *is first-countable, then* X *is a Hausdorff space.*

PROOF. If $y \in \overline{\{x\}}$, then every neighbourhood of y contains x, and $y \in \lim x_i$, where $x_i = x$ for $i = 1, 2, \ldots$ Since $x \in \lim x_i$ as well, and since every sequence has at most one limit, we have $y = x$, and thus $\{x\}$ is a closed set. The proof of the second part of the proposition is parallel to the proof of the second part of Proposition 1.6.7.■

From 1.6.16 and 1.6.7 we obtain

1.6.17. PROPOSITION. *A first-countable space* X *is a Hausdorff space if and only if every sequence in the space* X *has at most one limit.*■

1.6.18. EXAMPLE. The space Y defined in Example 1.4.17 is a Fréchet space that is not first-countable. Indeed, if $y \in \overline{A}$ for an $A \subset Y$, and $y \neq y_0$ or else $y = y_0$ and $y_0 \in A$, then there exists a sequence $y_1, y_2, \ldots$ of points of A converging to y. Suppose then that $y_0 \in \overline{A} \setminus A$. Observe that there exists a positive integer k such that A contains a sequence $\{y_i\}$ converging to k in R; the contrary means that every positive integer i has a neighbourhood $U_i \subset R$ disjoint from A, and this would imply that $[(\bigcup_{i=1}^{\infty} U_i) \setminus N] \cup \{y_0\}$ is a neighbourhood of y_0 in Y which is disjoint from A, a contradiction to the assumption that $y_0 \in \overline{A}$. From the definition of the topology on Y it follows that the sequence $\{y_i\}$, which converges to $k \in N$ in R, converges to y_0 in Y.■

1.6.19. EXAMPLE. We shall now define a sequential space that is not a Fréchet space. Let $X = \{0\} \cup \bigcup_{i=1}^{\infty} X_i$, where $X_i = \{1/i\} \cup \{1/i + 1/i^2, 1/i + 1/(i^2+1), \ldots\}$; one can easily verify that $X_i \cap X_k = \emptyset$ whenever $i \neq k$. The topology on X will be generated by a neighbourhood system. All points of the form $1/i + 1/j$ will be isolated points of X, i.e., $\mathcal{B}(x) = \{\{x\}\}$ for every x of that form. For an x of the form $1/i$ we take as $\mathcal{B}(x)$ the family of all sets $\{1/i\} \cup \{1/i + 1/k, 1/i + 1/(k+1), \ldots\}$ for $k = i^2, i^2+1, \ldots$ Finally, as members of $\mathcal{B}(0)$ we take all sets obtained from X by removing a finite number of X_i's and a finite number of points of the form $1/i + 1/j$ in all the remaining X_i's. One can readily show that the collection $\{\mathcal{B}(x)\}_{x\in X}$ has properties (BP1)–(BP4), hence X is a Hausdorff space. Since all members of the base $\bigcup_{x\in X} \mathcal{B}(x)$ are open-and-closed in X, the space X is regular, and from 1.5.17 it follows that X is perfectly normal.

One easily observes that the point 0 belongs to the closure of the set $X \setminus \{0, 1, 1/2, \ldots\}$, but there is no sequence in that set which converges to 0, so that X is not a Fréchet space. Let us note that thus X is a countable space that is not first-countable.

We shall now show that X is a sequential space, i.e., that every set $A \subset X$ which together with any convergent sequence contains its limit, is closed in X. Let us take an $x \in \overline{A}$. If $x \neq 0$, then $\mathcal{B}(x)$ is a countable base at x, and A contains a sequence converging to x (cf. the proof of Theorem 1.6.14). Now, let us consider the case when $x = 0$ and let us assume that $0 \in \overline{A} \setminus A$. There exists a subsequence $x_1, x_2, \ldots$ of the sequence $1, 1/2, 1/3, \ldots$ such that every neighbourhood of each x_i meets A, because the contrary would provide a neighbourhood of 0 disjoint from A. It follows that A contains all terms of the sequence $x_1, x_2, \ldots$, and as that sequence converges to 0, we have $0 \in A$, contrary to the assumption. ∎

1.6.20. EXAMPLE. By a modification of the space X in the above example we can define a countable perfectly normal space that is not a sequential space. In fact, it suffices to consider $Y = X \setminus \{1, 1/2, 1/3, \ldots\}$ and take as open sets all the intersections $Y \cap U$, where U is open in X. One readily observes that convergent sequences in Y are eventually constant, thus the set $Y \setminus \{0\}$ together with any convergent sequence contains its limit. Since $Y \setminus \{0\}$ is not closed, Y is not a sequential space. To verify that Y is perfectly normal, one applies the same argument as in Example 1.6.19. ∎

Historical and bibliographic notes

Nets were introduced by E. H. Moore and H. L. Smith in [1922]. Convergence in general topological spaces was described in terms of nets by Birkhoff in [1937]; his theory, however, was limited and complicated because an unproper counterpart of a subsequence was used in it. The notion of a finer net was defind in E. H. Moore's book [1939], and the proper description of convergence in general topological spaces in terms of nets was given by Kelley in [1950]. The setting of convergence in terms of filters was sketched by Cartan in [1937] and [1937a], and was developed by Bourbaki in [1940] (filters first appeared in Riesz [1908]). The equivalence of both theories was noted by Bartle in [1955] as well as by Bruns and Schmidt in [1955]. Sequential spaces and Fréchet spaces belonged to the folklore almost since the origin of general topology, but they were first thouroughly examined by Franklin in [1965] and [1967]; Example 6.1.19 is taken from Franklin [1965], the space in Example 6.1.20 was defined by Arens in [1950]. The first example of a normal countable space with no countable base at one of its points was given by Urysohn in [1925]. A normal countable space with no countable base at any of its points was defined by Novák in [1937] (for an example of such a space see Example 2.3.37 or Exercise 2.3.M).

Exercises

1.6.A. Show that the point x is a cluster point of the net $S = \{x_\sigma, \ \sigma \in \Sigma\}$ if and only if $x \in \bigcap_{\sigma_0 \in \Sigma} \overline{\{x_\sigma : \sigma \geq \sigma_0\}}$.

1.6.B. Let a net $S_\xi = \{x_\sigma^{(\xi)}, \ \sigma \in \Sigma_\xi\}$ in a space X be given for every ξ in a directed set Ξ and let $x \in \lim_{\xi \in \Xi} x_\xi$, where $x_\xi \in \lim S_\xi$. Verify that by defining $(\xi_0, f_0) \leq (\xi_1, f_1)$ whenever $\xi_0 \leq \xi_1$ and $f_0(\xi) \leq f_1(\xi)$ for every $\xi \in \Xi$, the product $\Sigma = \Xi \times \prod_{\xi \in \Xi} \Sigma_\xi$ becomes a directed set, and that $x \in \lim_{\sigma \in \Sigma} x_\sigma$, where $x_\sigma = x_{f(\xi)}^{(\xi)}$ for $\sigma = (\xi, f) \in \Sigma$.

1.6.C. Verify that for every net $S = \{x_\sigma, \ \sigma \in \Sigma\}$ with $|\Sigma| \leq \aleph_0$ there exists a sequence $\{x_i\}$ finer than S.

1.6.D. Prove that if X is a Fréchet space, then for every cluster point x of a sequence $\{x_i\}$ in X there exists a subsequence of $\{x_i\}$ that converges to x. Show that the assumption that X is a Fréchet space cannot be weakened to the assumption that X is a sequential space.

1.6.E. Give an example of a non-Hausdorff Fréchet space in which every sequence has at most one limit.

Hint. Adjoin a point to the space described in 1.6.18.

1.6.F. Show that a sequential space X is a Fréchet space if and only if the sequential closure operator which to every set $A \subset X$ assigns the set $\overline{A}$ consisting of all limits of sequences contained in A has properties (CO1)–(CO4). Observe that the assumption that X is a sequential space cannot be omitted.

1.7. Problems

Closure and complement yield only 14 different sets

1.7.1 (Kuratowski [1922a]). Prove that by applying alternatively closure and complement operators to a subset A of a topological space X one obtains at most 14 different sets. Find a subset of the real line from which one obtains in this way exactly 14 different sets.

Hint. Prove first that $A^{-\prime-\prime-\prime-} = A^{-\prime-}$, where $A^{-} = \overline{A}$ and $A' = X \setminus A$.

Left and right topology on an ordered set

1.7.2. Let X be a set ordered by $\leq$ and let $L(x) = \{y \in X : y \leq x\}$ for every $x \in X$. The *left topology induced on X by the order $\leq$* is the topology generated on X by the neighbourhood system $\{\{L(x)\}\}_{x \in X}$.

(a) Show that the intersection of every family of open sets in X with the left topology induced by $\leq$ is an open set.

(b) Verify that X is a T_0-space.

(c) Characterize points $x \in X$ such that the set $\{x\}$ is closed, open and closed-and-open.

(d) Describe the closure $\overline{\{x\}}$.

(e) Define in an analogous way the *right topology induced on X by the order $\leq$*.

1.7.3. Let X be a T_0-space with the property that the intersection of every family of open sets in X is an open set. Prove that there exists in X an order $\leq$ such that the original topology on X coincides with the left topology induced by $\leq$.

Linearly ordered spaces I (see Problems 2.7.5, 3.12.3, 3.12.4, 3.12.12(f), 5.5.22, 6.3.2 and 8.5.13(j))

1.7.4. Let X be a set linearly ordered by $<$ and containing at least two elements. For $a, b \in X$ satisfying $a < b$ let

$$(a, b) = \{x \in X : a < x < b\}, \quad (\leftarrow, a) = \{x \in X : x < a\}, \quad (a, \rightarrow) = \{x \in X : a < x\};$$

the sets defined above will be called *intervals* in X.

(a) Verify that the family $\mathcal{B}$ of all intervals in a linearly ordered set X has properties (B1)–(B2); the *topology induced on X by the linear order $<$* is the topology generated on X

by the base $\mathcal{B}$. A *linearly ordered space* is a space whose topology can be induced by a linear order. Note that every linearly ordered space is a T_1-space.

(b) Show that for every discrete family $\{\{x_s\}\}_{s\in S}$ of one-point subsets of a linearly ordered space X, there exists a family $\{U_s\}_{s\in S}$ of pairwise disjoint open subsets of X such that $x_s \in U_s$ for $s \in S$.

(c) (Mansfield [1957a]) Prove that for every discrete family $\{F_s\}_{s\in S}$ of closed subsets of a linearly ordered space X, there exists a family $\{U_s\}_{s\in S}$ of pairwise disjoint open subsets of X such that $F_s \subset U_s$ for $s \in S$.

Hint (Steen [1970]). Show that the open sets

$$W_s = \bigcup\{(x,y) : x, y \in F_s \text{ and } (x,y) \cap \bigcup_{s'\neq s} F_{s'} = \emptyset\}$$

form a discrete family and that $F_s \cap \overline{\bigcup_{s'\neq s} W_{s'}} = \emptyset$; observe that the family of all one-point subsets of the union $\bigcup_{s\in S}(F_s \setminus W_s)$ is discrete and apply (b).

(d) (Birkhoff [1940]) Prove that every linearly ordered space is normal.

Hint. Apply (c).

(e) (Meyer [1969]) Note that every linearly ordered sequential space is first-countable.

Borel sets I (see Problems 4.5.7, 4.5.8 and 7.4.22)

1.7.5. (a) Observe that in a perfect space X (cf. Exercise 1.5.H(a)) the family of Borel sets can be defined as the smallest family $\mathcal{S}$ of subsets of X satisfying conditions (BS1), (BS3) and (BS3′), or else, conditions (BS1′), (BS3) and (BS3′). Deduce from this observation that the family of Borel sets in a perfect space X can be represented as the union $\bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$ (the union $\bigcup_{\alpha<\omega_1} \mathcal{G}_\alpha$), where the family $\mathcal{F}_0$ (the family $\mathcal{G}_0$) consists of all closed (open) sets, and the family $\mathcal{F}_\alpha$ (the family $\mathcal{G}_\alpha$) consists of all countable unions (intersections) of sets from $\bigcup_{\xi<\alpha} \mathcal{F}_\xi$ (from $\bigcup_{\xi<\alpha} \mathcal{G}_\xi$) for an odd ordinal number α and of all countable intersections (unions) of sets from $\bigcup_{\xi<\alpha} \mathcal{F}_\xi$ (from $\bigcup_{\xi<\alpha} \mathcal{G}_\xi$) for an even ordinal number α. Verify that the families $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ are closed with respect to finite unions and intersections, that $\mathcal{F}_\beta \cup \mathcal{G}_\beta \subset \mathcal{F}_\alpha \cap \mathcal{G}_\alpha$ whenever $\beta < \alpha$, and that $A \in \mathcal{F}_\alpha$ if and only if $X \setminus A \in \mathcal{G}_\alpha$. Prove that the family $\mathcal{F}_\alpha$ (the family $\mathcal{G}_\alpha$) is closed with respect to countable unions if α is odd (even) – members of this family are called the *sets of the additive class* α in X – and is closed with respect to countable intersections if α is even (odd) – members of this family are called the *sets of the multiplicative class* α in X. Members of the family $\mathcal{F}_1$ (the family $\mathcal{G}_1$) are F_σ-sets (G_δ-sets); members of $\mathcal{F}_2, \mathcal{F}_3, \ldots$ are called $F_{\sigma\delta}$-sets, $F_{\sigma\delta\sigma}$-sets, ... and members of $\mathcal{G}_2, \mathcal{G}_3, \ldots$ are called $G_{\delta\sigma}$-sets, $G_{\delta\sigma\delta}$-sets, ... Show that if $f: X \to Y$ is a continuous mapping of a perfect space X to a perfect space Y, then the inverse image of any set of the additive or the multiplicative class α in Y is a set of the same class in X.

(b) (Lebesgue [1905]) A mapping f of a perfect space X to a perfect space Y is called *measurable of class* α if the inverse image of any closed subset of Y is a set of the multiplicative class α in X.

Observe that f is measurable of class α if and only if the inverse image of any open subset of Y is a set of the additive class α in X, and – in the case of a second-countable Y – if and only if inverse images of members of a fixed countable base for Y are sets of the additive class α in X. Verify that if X, Y and Z are perfect spaces, f is a mapping of X to

Y measurable of class α, and g is a continuous mapping of Y to Z, then the composition gf is measurable of class α.

Normally placed sets I (see Problems 2.7.3 and 3.12.25)

1.7.6 (Smirnov [1951c]). We say that a set A is *normally placed* in a space X if for every open $U \subset X$ that contains A there exists in X an F_σ-set H such that $A \subset H \subset U$.

(a) Show that if X is a normal space, then in the definition of a normally placed set one can assume that H is an open F_σ-set.

(b) Observe that a space X is perfect if and only if all subsets of X are normally placed.

Urysohn spaces and semiregular spaces I (see Problems 2.7.6 and 6.3.17)

1.7.7 (Urysohn [1925]). A topological space X is called a *Urysohn space* if for every pair of distinct points $x_1, x_2 \in X$ there exist open sets U_1, U_2 such that $x_1 \in U_1, x_2 \in U_2$ and $\overline{U}_1 \cap \overline{U}_2 = \emptyset$.

Note that every regular space is a Urysohn space and that every Urysohn space is a Hausdorff space. Give an example of a Hausdorff space that is not a Urysohn space and an example of a Urysohn space that is not regular.

1.7.8. (a) (M. H. Stone [1937]) A topological space X is called a *semiregular space* if X is a T_2-space and the family of all open domains is a base for X.

Show that every regular space is a semiregular space. Give an example of a Hausdorff space that is not semiregular and an example of a semiregular Urysohn space that is not regular. Note that there exist T_1-spaces in which open domains form a base but which are not T_2-spaces.

(b) (M. H. Stone [1937], Katětov [1947]) Let $(X, \mathcal{O})$ be a Hausdorff space. Generate on X a topology $\mathcal{O}' \subset \mathcal{O}$ by the base consisting of all open domains of $(X, \mathcal{O})$ and show that the space $(X, \mathcal{O}')$ is semiregular and has the same open domains as the space $(X, \mathcal{O})$.

(c) (Urysohn [1925]) Give an example of a Urysohn space that is not semiregular and an example of a semiregular space that is not a Urysohn space.

(d) Give an example of a countable Urysohn space X such that at no point of X there exists a base consisting of open domains (cf. Problem 6.3.17).

Hint. Generate a suitable topology on the subset of the plane consisting of all points having both coordinates rational.

(e) Give an example of a countable semiregular Urysohn space X such that every point $x \in X$ has a neighbourhood which does not contain the closure of any neighbourhood of x.

Hint. Apply the hint to part (d).

1.7.9. (a) Show that neither the property of being a Urysohn space nor semiregularity is invariant under closed mappings with finite fibers.

(b) Show that semiregularity is not invariant under closed-and-open mappings.

Remark. The example cited in the remark to Exercise 1.5.L(b) proves that the property of being a Urysohn space is not invariant under closed-and-open mappings.

The Cantor-Bendixson theorem

1.7.10. Show that if each member of a family $\mathcal{A}$ of subsets of a space X is dense in itself, then the union $\bigcup \mathcal{A}$ is dense in itself. Deduce that every topological space can be represented as the union of two disjoint sets, of which one is dense in itself and closed (such sets are called *perfect sets*) and the other contains no non-empty dense in itself subset (such sets are called *scattered sets*).

Remark. The reader must be careful not to confuse this notion of a perfect set with the notion of a perfect space defined in Exercise 1.5.H(a).

1.7.11 (Hausdorff [1914]; for subspaces of Euclidean spaces, Lindelöf [1903]). A point x of a topological space X is called a *condensation point* of a set $A \subset X$ if every neighbourhood of x contains uncountably many points of A; the set of all condensation points of a set A is denoted by A^0.

Verify that

$$A^0 \subset A^{\mathrm{d}}, \quad A^0 = \overline{A^0} \quad \text{and} \quad (A \cup B)^0 = A^0 \cup B^0.$$

Show that for every subset A of a second-countable space, the difference $A \setminus A^0$ is countable and $(A^0)^0 = A^0$.

Deduce from the above that every second-countable space can be represented as the union of two disjoint sets, of which one is perfect and the other countable (this is the *Cantor-Bendixson theorem*).

Remark. Cantor and I. Bendixson proved this fact independently in 1883 for subsets of the real line.

Cardinal functions I (see Problems 2.7.9–2.7.11, 3.12.4, 3.12.7–3.12.11, 3.12.12(h), 3.12.12(j) and 8.5.17)

1.7.12. A *cardinal function* is a function f assigning to every topological space X a cardinal number $f(X)$ such that $f(X) = f(Y)$ for any pair X, Y of homeomorphic spaces.

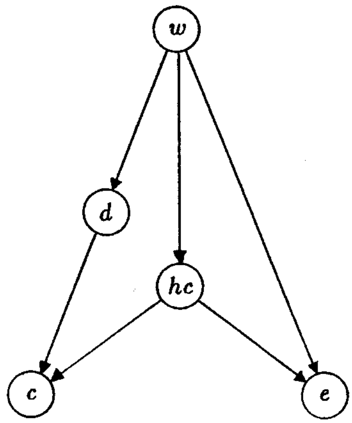
The smallest cardinal number $\mathfrak{m} \geq \aleph_0$ such that every family of pairwise disjoint non-empty open subsets of X has cardinality $\leq \mathfrak{m}$, is called the *Souslin number*, or *cellularity*, of the space X and is denoted by $c(X)$. If $c(X) = \aleph_0$, we say that the space X has the *Souslin property*.

The smallest cardinal number $\mathfrak{m} \geq \aleph_0$ such that every subset of X consisting exclusively of isolated points (i.e., every set $A \subset X$ satisfying the equality $A = A \setminus A^{\mathrm{d}}$) has cardinality $\leq \mathfrak{m}$, is denoted by $hc(X)$; for an explanation of the symbol hc and the name of the function $hc(X)$ see Problem 2.7.9(b) (some authors use the symbol $s(X)$ and the name *spread* of the space X).

The smallest cardinal number $\mathfrak{m} \geq \aleph_0$ such that every closed subset of X consisting exclusively of isolated points has cardinality $\leq \mathfrak{m}$, is called the *extent* of the space X and is denoted by $e(X)$.

For sake of simplicity, in all problems about cardinal functions, the cardinal functions defined in the main body of the book (weight, character and density, as yet) will be re-defined to assume only infinite values: the new value of $f(X)$ is defined to be $\aleph_0$ if the old value is finite, and to be equal to the old value if this is an infinite cardinal number. (Sometimes topologists say that "there are no finite cardinal numbers in general topology").

(a) Prove that the following diagram, where an arrow from f to g means that $g(X) \leq f(X)$ for every topological space X, contains all inequalities that hold between the cardinal functions that appear in it:



(b) Verify that if Y is a continuous image of X, then $c(Y) \leq c(X)$ and $hc(Y) \leq hc(X)$; note that if, moreover, X is a T_1-space, then also $e(Y) \leq e(X)$.

(c) (Jones [1937]) Prove that for every normal space X and every closed subset A of X consisting exclusively of isolated points we have $|A| < \exp|A| \leq \exp d(X)$. Deduce that for every normal space X we have $e(X) \leq \exp d(X)$.

Hint. Apply the argument in Example 1.5.10.

Remark. Juhász's books [1971] and [1980] are devoted to cardinal functions; see also Hodel [1984] and Juhász [1984].

1.7.13 (Arhangel'skiĭ and Ponomarev [1968], Arhangel'skiĭ [1970]). The *tightness of a point* x in a topological space X is the smallest cardinal number $\mathfrak{m} \geq \aleph_0$ with the property that if $x \in \overline{C}$, then there exists a $C_0 \subset C$ such that $|C_0| \leq \mathfrak{m}$ and $x \in \overline{C_0}$; this cardinal number is denoted by $\tau(x, X)$. The *tightness of a topological space* X is the supremum of all numbers $\tau(x, X)$ for $x \in X$; this cardinal number is denoted by $\tau(X)$.

(a) Note that for every topological space X and every $x \in X$ we have $\tau(x, X) \leq \chi(x, X)$ and $\tau(X) \leq \chi(X)$. Give an example of a perfectly normal space X such that $\tau(X) < \chi(X)$.

(b) Prove that for every topological space X, the tightness $\tau(X)$ is equal to the smallest cardinal number $\mathfrak{m} \geq \aleph_0$ with the property that for any $C \subset X$ which is not closed there exists a $C_0 \subset C$ such that $|C_0| \leq \mathfrak{m}$ and $\overline{C_0} \setminus C \neq \emptyset$.

Hint. Observe that $\tau(X) \leq \mathfrak{m}$ if and only if for every $C \subset X$ we have $\overline{C} = [C]_{\mathfrak{m}}$, where $[C]_{\mathfrak{m}} = \bigcup\{\overline{M} : M \subset C \text{ and } |M| \leq \mathfrak{m}\}$.

(c) Note that for every sequential space X we have $\tau(X) = \aleph_0$.

Remark. Spaces of countable tightness were introduced by R. C. Moore and Mrówka in [1964]; they appear also in Corson [1961].

Semicontinuous functions I (see Problems 2.7.4, 3.12.23(g) and 5.5.20)

1.7.14. A real-valued function f defined on a topological space X is *lower (upper) semicontinuous* if for every $x \in X$ and every real number r satisfying the inequality $f(x) > r$ (the inequality $f(x) < r$) there exists a neighbourhood $U \subset X$ of x such that $f(x') > r$ (that $f(x') < r$) for every $x' \in U$.

(a) Verify that a function f is lower (upper) semicontinuous if and only if for every real number r the set $\{x: f(x) \leq r\}$ (the set $\{x: f(x) \geq r\}$) is closed. Note that a function is both lower and upper semicontinuous if and only if it is continuous. Show that if f and g are lower (upper) semicontinuous, then $\min(f,g)$, $\max(f,g)$ and $f+g$ are lower (upper) semicontinuous, as is $f \cdot g$ provided that $f(x) \geq 0$ and $g(x) \geq 0$ for $x \in X$. Note that the function $-f$ is lower (upper) semicontinuous if and only if f is upper (lower) semicontinuous. Prove that for any family $\{f_s\}_{s\in S}$ of lower (upper) semicontinuous functions with the property that the set $\{f_s(x): s \in S\}$ is bounded above (below) for every $x \in X$, the function $\sup_{s\in S} f_s$ (the function $\inf_{s\in S} f_s$) defined by the formula $[\sup_{s\in S} f_s](x) = \sup_{s\in S} f_s(x)$ (by the formula $[\inf_{s\in S} f_s](x) = \inf_{s\in S} f_s(x)$) is lower (upper) semicontinuous. Give an example of a sequence $\{f_i\}$ of continuous functions from I to itself such that $\inf f_i$ is not lower semicontinuous and $\sup f_i$ is not upper semicontinuous.

(b) (Fort [1955]) Prove that for every lower or upper semicontinuous function f defined on a topological space X there exists a set $G \subset X$, which can be represented as the intersection of countably many dense open subsets of X, such that f is continuous at every point of G.

Hint. For a lower semicontinuous f let $A_r = \{x: f(x) > r\}$ and $G_r = A_r \cup (X \setminus \overline{A_r})$ for every rational number r; define G as the intersection of all the G_r's.

Remark. For functions defined on the real line the notion of semicontinuity was introduced by Baire in 1899, who also established (a) and (b) in this particular case.

1.7.15. (a) (Bourbaki [1948]) Prove that a T_1-space X is completely regular if and only if for every lower (upper) semicontinuous function f defined on X such that there exists a continuous function $g: X \to R$ satisfying $f(x) \geq g(x)$ (satisfying $f(x) \leq g(x)$) for every $x \in X$, there exists a family of continuous functions $\{f_s\}_{s\in S}$ such that $f = \sup_{s\in S} f_s$ (that $f = \inf_{s\in S} f_s$).

(b) (Tong [1952] (announcement [1948]), Katětov [1951a] (correction [1953]); for paracompact spaces, Dieudonné [1944]; for metric spaces, Hahn [1917]) Prove that a T_1-space X is normal if and only if for every pair f, g of real-valued functions defined on X, where f is upper semicontinuous, g is lower semicontinuous and $f(x) \leq g(x)$ for every $x \in X$, there exists a continuous function $h: X \to R$ such that $f(x) \leq h(x) \leq g(x)$ for every $x \in X$.

Hint (Tong [1952]; cf. Problem 2.7.2(c)). To begin, observe that if $f_i, g_i: X \to I$ are continuous for $i = 1, 2, \ldots$ and if $f_0 = \inf f_i$ together with $g_0 = \sup g_i$ satisfy $f_0(x) \leq g_0(x)$ for every $x \in X$, then there exists a continuous function $h: X \to I$ satisfying $f_0(x) \leq h(x) \leq g_0(x)$ for every $x \in X$. For this purpose, assuming – with no loss of generality – that $f_{i+1}(x) \leq f_i(x)$ and $g_{i+1}(x) \geq g_i(x)$ for $i = 1, 2, \ldots$ and every $x \in X$, let $k_i = \max_{j\leq i}[\min(f_j, g_j)]$, $l_i = \max(k_{i-1}, f_i)$ and show that $\sup k_i = \inf l_i$.

Returning to the general case, observe that it suffices to consider functions f, g with values in the interval $(0,1)$. Then for every pair of positive integers j, i with $j < i$ let

$$A_{j,i} = \{x : g(x) \leq j/i\} \quad \text{and} \quad B_{i,j} = \{x : f(x) \geq j/i + 1/2i\}$$

and take a continuous function $f_{j,i}$ equal to $j/i + 1/2i$ on $A_{j,i}$ and to 1 on $B_{j,i}$ such that $f_{j,i}(x) \geq j/i + 1/2i$ for every $x \in X$. Verify that for the function $f_i = \min_{j<i} f_{j,i}$ and an $x \in X$ we have either $f_i(x) < g(x)$ or $|f_i(x) - g(x)| < 3/2i$ and that $f_i(x) \geq f(x)$; deduce that the function $f_0 = \inf f_i$ satisfies $f(x) \leq f_0(x) \leq g(x)$ for every $x \in X$. Applying this result to functions $-g$ and $-f_0$, obtain $g_0 = \sup g_n$ such that $f_0(x) \leq g_0(x) \leq g(x)$ for every $x \in X$, and apply the first part of the hint.

(c) (Tong [1952]; for metric spaces, Hahn [1917]; for the real line, Baire [1904]) Prove that a T_1-space X is perfectly normal if and only if for every lower (upper) semicontinuous function f defined on X there exists a sequence $\{f_i\}$ of continuous real-valued functions on X such that $f_i(x) \leq f_{i+1}(x)$ (that $f_i(x) \geq f_{i+1}(x)$) for $i = 1, 2, \ldots$ and $x \in X$, and that $f(x) = \lim f_i(x)$ for every $x \in X$.

Hint (Tong [1952]). Prove that a space X with the above property satisfies condition (iii) in 1.5.19. When constructing the sequence $\{f_i\}$ for a lower semicontinuous f discuss first the case of an f taking only finitely many values. Then, assuming that $0 < f(x) < 1$, for every pair of integers j, i with $0 \leq j < i$, let

$$C_{j,i} = \{x : j/i < f(x) \leq (j+1)/i\}$$

and for $i = 1, 2, \ldots$ take a lower semicontinuous function g_i equal to j/i on $C_{j,i}$; note that $g_i(x) \leq f(x)$ for $x \in X$. Choose for every i a sequence $\{g_k^i\}$ of continuous functions such that $g_k^i(x) \leq g_{k+1}^i(x)$, $\lim g_k^i(x) = g_i(x)$ and $g_k^i(x) \geq 0$ for every $x \in X$. Rearrange all functions g_k^i into a sequence $h_1, h_2, \ldots$, let

$$k_i = \max_{j \leq i} h_j \quad \text{and} \quad k(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} k_i(x) \quad \text{for} \quad x \in X.$$

Observe that $k(x) > 0$ for every $x \in X$ and prove that the functions $f_i = \max(k, k_i)$ have the required properties and take only positive values.

(d) (Michael [1956]) Prove that a T_1-space X is perfectly normal if and only if for every pair f, g of real-valued functions defined on X, where f is upper semi-continuous, g is lower semi-continuous and $f(x) \leq g(x)$ for every $x \in X$, there exists a continuous function $h: X \to R$ such that $f(x) \leq h(x) \leq g(x)$ for every $x \in X$ and $f(x) < h(x) < g(x)$ whenever $f(x) < g(x)$.

Hint (cf. Problem 5.5.20(b)). It suffices to consider functions f, g with values in the interval $(0, 1)$. Apply (b) and (c) to obtain a sequence $\{f_i\}$ of continuous real-valued functions on X such that $f(x) \leq f_i(x) \leq g(x)$ for $i = 1, 2, \ldots$ and $x \in X$ and that $f(x) = \lim f_i(x)$ for every $x \in X$. Show that the formula

$$h_1(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x) \quad \text{for } x \in X$$

defines a continuous function $h_1: X \to R$ such that $f(x) \leq h_1(x) \leq g(x)$ for every $x \in X$ and $h_1(x) < g(x)$ whenever $f(x) < g(x)$. Define a continuous function $h_2: X \to R$ such that $f(x) \leq h_2(x) \leq g(x)$ for every $x \in X$ and $f(x) < h_2(x)$ whenever $f(x) < g(x)$, and let $h(x) = (1/2)(h_1(x) + h_2(x))$ for $x \in X$.

1.7.16 (Isiwata [1967]; cf. Exercise 1.5.L(a)). Prove that if $f: X \to Y$ is an open (a closed) mapping of X onto Y, then for every continuous function $g: X \to R$ that is bounded on all fibers of f, the formulas $g^*(y) = \sup\{g(x) : x \in f^{-1}(y)\}$ and $g_*(y) = \inf\{g(x) : x \in f^{-1}(y)\}$ define, respectively, a lower and an upper (an upper and a lower) semicontinuous function on the space Y.

Set-valued mappings I (see Problems 2.7.21 and 3.12.28)

1.7.17 (Kuratowski [1932] and [1963]). A mapping F assigning to each point y of a topological space Y a closed subset $F(y)$ of a topological space X is *lower (upper) semicontinuous* if for every open set $U \subset X$ the set $\{y: F(y) \cap U \neq \emptyset\}$ (the set $\{y: F(y) \subset U\}$) is open in Y.

(a) Note that F is lower (upper) semicontinuous if and only if for every closed set $K \subset X$ the set $\{y: F(y) \subset K\}$ (the set $\{y: F(y) \cap K \neq \emptyset\}$) is closed in Y. Verify that a mapping f of a topological space Y into a T_1-space X is continuous if and only if the set-valued mapping $F(y) = \{f(y)\}$ is lower, or – equivalently – upper, semicontinuous. Prove that a mapping $f: X \to Y$ onto a T_1-space Y is closed (open) if and only if the set-valued mapping $F(y) = f^{-1}(y)$ is upper (lower) semicontinuous. Verify that a real-valued function f defined on a topological space Y is lower (upper) semicontinuous (see Problem 1.7.14) if and only if the set-valued mapping F assigning to $y \in Y$ the closed set $F(y) = \{r \in R: r \leq f(y)\} \subset R$ is lower (upper) semicontinuous.

(b) Show that for every family $\{F_s\}_{s\in S}$ of lower semicontinuous set-valued mappings, the set-valued mapping $F = \overline{\bigcup_{s\in S} F_s}$, defined by the formula $F(y) = \overline{\bigcup_{s\in S} F_s(y)}$ is lower semicontinuous. Verify that if F_1, F_2 are upper semicontinuous set-valued mappings, then their union $F = F_1 \cup F_2$, defined by the formula $F(y) = F_1(y) \cup F_2(y)$, is upper semicontinuous. Show that the last result cannot be generalized as in the case of lower semicontinuous mappings.

(c) Prove that if F_1, F_2 are upper semicontinuous set-valued mappings assigning to points of Y closed subsets of a normal space X, then their intersection $F = F_1 \cap F_2$, defined by the formula $F(y) = F_1(y) \cap F_2(y)$, is upper semicontinuous. Show that the last result cannot be generalized to countable intersections and that the counterpart for lower semicontinuous set-valued mappings does not hold.

Hint. Observe that $\{y : F(y) \subset U\} = \bigcup[\{y : F_1(y) \subset U_1\} \cap \{y : F_2(y) \subset U_2\}]$, where the union is taken over all pairs U_1, U_2 of open subsets of X such that $U_1 \cap U_2 = U$.

Topologies described by sequences

1.7.18 (Fréchet [1906] and [1918], Urysohn [1926a]). An *$\mathcal{L}^*$-space* is a pair (X, λ), where X is a set and λ a function (called the *limit operator*) assigning to some sequences of points of X an element of X (called the *limit* of the sequence) in such a way that the following conditions are satisfied (we write λx_i instead of $\lambda(\{x_i\})$ and say that $\{x_i\}$ *converges* to x if $\lambda x_i = x$):

(L1) *If $x_i = x$ for $i = 1, 2, \ldots$, then $\lambda x_i = x$.*

(L2) *If $\lambda x_i = x$, then $\lambda x_{k_i} = x$ for every subsequence $\{x_{k_i}\}$ of $\{x_i\}$.*

(L3) *If a sequence $\{x_i\}$ does not converge to x, then it contains a subsequence $\{x_{k_i}\}$ such that no subsequence of $\{x_{k_i}\}$ converges to x.*

(a) In an $\mathcal{L}^*$-space one defines the closure operator by letting $x \in \overline{A}$ whenever A contains a sequence converging to x. Verify that the closure operator has properties (CO1)–(CO3), but not necessarily property (CO4).

(b) Show that the closure operator defined in (a) has property (CO4) if and only if the following condition is satisfied:

(L4) *If* $\lambda x_i = x$ *and* $\lambda x^i_j = x_i$ *for* $i = 1, 2, \ldots$, *then there exist sequences of positive integers* $i_1, i_2, \ldots$ *and* $j_1, j_2, \ldots$ *such that* $\lambda x^{i_k}_{j_k} = x$.

An $\mathcal{L}^*$-space X satisfying (L4) is called an *S^*-space* and the topology generated on X by the closure operator defined in (a) is called the *Fréchet topology* induced by the limit operator λ. Verify that every S^*-space with the Fréchet topology is a T_1-space.

(c) Prove that for a sequence $x_1, x_2, \ldots$ of points of an S^*-space with the Fréchet topology, $x = \lim x_i$ if and only if $\lambda x_i = x$, i.e., that the convergence *a posteriori* is equivalent to the convergence *a priori*.

1.7.19 (Kisyński [1960]). Show that in any $\mathcal{L}^*$-space one can generate a topology by taking as the family of closed sets all sets containing together with any convergent sequence the limit of this sequence; this topology is called the *sequential topology* induced by the limit operator λ. Verify that every $\mathcal{L}^*$-space with the sequential topology is a T_1-space. Show that in an S^*-space the sequential topology and the Fréchet topology coincide. Prove that for a sequence $x_1, x_2, \ldots$ of points of an $\mathcal{L}^*$-space with the sequential topology, $x = \lim x_i$ if and only if $\lambda x_i = x$, i.e., that the convergence *a posteriori* is equivalent to the convergence *a priori*.

1.7.20. Prove that in a topological space X a limit operator λ, such that (X, λ) is an S^*-space (an $\mathcal{L}^*$-space) and the Fréchet topology (the sequential topology) induced by λ coincides with the original topology of X, can be defined if and only if X is a Fréchet space (a sequential space) in which every sequence has at most one limit.

1.7.21 (Kelley [1950]). State and prove counterparts of conditions (L1)–(L3) in Problem 1.7.18 for nets. Define a closure operator in a set X, where the class of convergent nets and their limits are specified which satisfy those counterparts as well as the iterated limits condition in Exercise 1.6.B. Prove that every topology can be *generated by a class of convergent nets*, i.e., obtained in the way described above.

Chapter 2

Operations on topological spaces

It is the purpose of this chapter to investigate operations on topological spaces, i.e., methods of constructing new topological spaces from old ones. Six sections correspond to six operations investigated. The theorems on universality of the Tychonoff cube for all $T_{3\frac{1}{2}}$-spaces and of the Alexandroff cube for all T_0-spaces, to be proved in Section 2.3, show that all spaces in the above classes can be obtained by applying just two operations to very simple spaces. Chapter 2 is, in a sense, the heart of this book; in subsequent chapters we shall, when defining various classes of spaces, always test their behaviour under the operations we are going to discuss.

Section 2.1 is devoted to subspaces. Having defined the notion of a subspace, we study restrictions and extensions of continuous mappings and functions; the most important result in this context is the Tietze-Urysohn theorem. Next, we examine the notion of a hereditary topological property. The section is closed with some observations on combinations of mappings.

In Section 2.2 we discuss the operation of the sum of topological spaces and the additivity of topological properties.

Section 2.3, the longest in this chapter, is devoted to Cartesian products. After defining the Tychonoff topology on Cartesian products and proving a few elementary propositions, we discuss the multiplicativity of topological properties. Then we pass to a study of the embedding of spaces in Cartesian products and we prove the two theorems on universal spaces mentioned above. The final part of the section is devoted to Cartesian products and diagonals of mappings.

Quotient spaces and quotient mappings are discussed in Section 2.4.

In Section 2.5 we study inverse systems of spaces, their limit spaces and mappings.

The last section is devoted to function spaces. We introduce the topology of uniform convergence on the set of continuous real-valued functions defined on a topological space and the topology of pointwise convergence on the set of continuous mappings of X to Y. The section is closed with a discussion of acceptable topologies on mapping spaces. The same topic will be treated in Section 3.4, where we shall introduce another topology on mapping spaces and obtain deeper results.

2.1. Subspaces

Suppose we are given a topological space X and a set $M \subset X$. It is easy to see that the family $\mathcal{O}$ of all sets $M \cap U$, where U is open in X, satisfies conditions (O1)–(O3). Indeed, condition (O1) is satisfied since $\emptyset = M \cap \emptyset$ and $M = M \cap X$, while from the equalities

$$(M \cap U_1) \cap (M \cap U_2) = M \cap (U_1 \cap U_2) \quad \text{and} \quad \bigcup_{s \in S} (M \cap U_s) = M \cap \bigcup_{s \in S} U_s$$

it follows that (O2) and (O3) are satisfied, too.

Taking the family $\{M \cap U : U \text{ is open in } X\}$ as the family of open sets in M, we define a topology on M; the set M with this topology is called a *subspace of the space* X, and the topology itself is called the *induced topology* or the *subspace topology.*

2.1.1. PROPOSITION. *Let X be a topological space and M a subspace of X. The set $A \subset M$ is closed in M if and only if $A = M \cap F$, where F is closed in X. The closure $\tilde{A}$ of a set $A \subset M$ in the subspace M and the closure $\overline{A}$ of A in the space X are related by the equality $\tilde{A} = \overline{A} \cap M$.*

PROOF. If $A = M \cap F$, where $F = \overline{F} \subset X$, then $M \setminus A = M \cap (X \setminus F)$ and A is closed in M as the complement of an open set. Conversely, if A is a closed subset of M, then $M \setminus A = M \cap U$, where U is open in X. Thus

$$A = M \setminus (M \setminus A) = M \setminus (M \cap U) = M \cap (X \setminus U),$$

and $A = M \cap F$, where $F = X \setminus U$ is closed in X.

By definition of the closure operator, $\tilde{A}$ is equal to the intersection of all closed subsets of M that contain A, i.e., of all sets $M \cap F$, where $F = \overline{F}$ and $A \subset F$. This gives the equality $\tilde{A} = M \cap \overline{A}$. ∎

2.1.2. PROPOSITION. *If M is a subspace of a space X, and L a subset of M, then the two topologies defined on L, viz., the topology of a subspace of M and the topology of a subspace of X, coincide.* ∎

We say that a subspace M of X is a *closed subspace* of X if M is a closed subset of X. If M is a closed subspace of X, then a set $A \subset M$ is closed in M if and only if it is closed in X and, consequently, $\tilde{A} = \overline{A}$ for every $A \subset M$. *Open subspaces* and *dense subspaces* are defined analogously. Clearly, if M is an open (a dense) subspace of X, then a set $A \subset M$ is open (dense) in M if and only if it is open (dense) in X. In the sequel the words "subspace" and "subset" will be used interchangeably; e.g., we shall say "separable subset", "compact subset" etc. and we shall mean by that a subset which with regard to the subspace topology is a separable, compact etc. space.

For every topological space X and any subspace M of X, the formula $i_M(x) = x$ defines a mapping of M to X; since $i_M^{-1}(U) = M \cap U$, this mapping is continuous. The mapping $i_M: M \to X$ is called the *embedding of the subspace M in the space X*. One can readily show that the subspace topology coincides with the topology generated by the mapping i_M of the set M to the topological space X. The embedding i_M is closed (open) if and only if the subspace M is closed (open).

For a continuous mapping $f: X \to Y$ and a subspace M of X, the composition $f i_M: M \to Y$ is a continuous mapping; it is called the *restriction of f to M* and is denoted by $f|M$. Since the composition of closed (open) mappings is a closed (an open) mapping, the restriction of a closed (an open) mapping to a closed (an open) set $M \subset X$ is closed (open).

2.1.3. PROPOSITION. *If the composition gf of continuous mappings $f: X \to Y$ and $g: Y \to Z$ is closed (open), then the restriction $g|f(X): f(X) \to Z$ is closed (open).*

PROOF. Any closed (open) subset of $f(X)$ is of the form $A \cap f(X)$, where A is closed (open) in Y. As the inverse image $f^{-1}(A)$ is closed (open) in X and gf is a closed (an open) mapping, the set

$$[g|f(X)](A \cap f(X)) = g(A \cap f(X)) = gf(f^{-1}(A))$$

is closed (open) in Z.∎

One can easily check that even if gf is closed (open), neither g nor f is necessarily closed (open).

For a continuous mapping $f: X \to Y$ and a subspace M of X, the *restriction of f to M and $f(M)$* is defined as the mapping of M to $f(M)$ that assigns to every $x \in M$ the point $f(x)$ of the subspace $f(M) \subset Y$; this restriction is denoted by $f\|M$, obviously $f\|M$ is continuous. The difference between $f|M$ and $f\|M$ consists in different ranges of the mappings: we have $f|M: M \to Y$ and $f\|M: M \to f(M)$; in particular $f\|X: X \to f(X)$. One easily sees that if f and M are closed (open), then $f\|M$ also is closed (open).

For a continuous mapping $f: X \to Y$ and a subspace L of Y, the *restriction of f to L* is defined as the mapping of the subspace $f^{-1}(L) \subset X$ to the subspace $L \subset Y$ that assigns $f(x) \in L$ to $x \in f^{-1}(L)$; this restriction is denoted by f_L, obviously $f_L: f^{-1}(L) \to L$ is continuous.

The reader can readily verify the following simple formulas for images and inverse images under restrictions:

$$\begin{aligned} &(f|M)(A) = f(A) \text{ for } A \subset M, && (f|M)^{-1}(B) = M \cap f^{-1}(B) \text{ for } B \subset Y, \\ &(f\|M)(A) = f(A) \text{ for } A \subset M, && (f\|M)^{-1}(B) = M \cap f^{-1}(B) \text{ for } B \subset f(M), \\ &f_L(A) = f(A) \text{ for } A \subset f^{-1}(L), && f_L^{-1}(B) = f^{-1}(B) \text{ for } B \subset L. \end{aligned}$$

2.1.4. PROPOSITION. *If $f: X \to Y$ is a closed (an open) mapping, then for any subspace $L \subset Y$ the restriction $f_L: f^{-1}(L) \to L$ is closed (open).*

PROOF. For $A \subset X$ we have

$$f_L(A \cap f^{-1}(L)) = f(A \cap f^{-1}(L)) = f(A) \cap L$$

so that the proposition follows from the definition of the topologies on the subspaces $f^{-1}(L)$ and L.∎

A mapping $f: X \to Y$ is called a *homeomorphic embedding* if it is a composition of a homeomorphism and an embedding, i.e., if there exist a subspace L of Y and a homeomorphism $f': X \to L$ such that $f = i_L f'$; clearly, we then have $L = f(X)$ and $f' = f\|X$. If for a space X there exists a homeomorphic embedding $f: X \to Y$ in a space Y, we say that X is *embeddable in Y*.

Proposition 2.1.2 implies that the composition of homeomorphic embeddings and restrictions of a homeomorphic embedding are homeomorphic embeddings. One readily sees that a homeomorphic embedding $f: X \to Y$ is closed (open) if and only if $f(X)$ is closed (open) in Y.

2.1.5. EXAMPLES. The interval I with its natural topology is a closed subspace of the real line R with its natural topology. The *natural topology* of any interval of R is the induced topology. In the sequel by the real line or an interval we shall always mean those sets together with their natural topology. It is easy to verify that any two closed intervals containing more than one point are homeomorphic. The same holds for any two open intervals and any two half-open intervals.

The discrete space of cardinality $\mathbf{c}$ is embeddable in the Niemytzki plane L: it is homeomorphic to the closed subspace L_1 of L. The discrete space of cardinality $\aleph_0$ is embeddable, also as a closed subspace, in the real line: it is homeomorphic to the set N of all positive integers with the induced topology; in the sequel, we shall often assume that $D(\aleph_0) = N$. The real line is embeddable in the interval $J = [-1, 1]$: it is homeomorphic to the interval $(-1, 1)$, a homeomorphic embedding $i: R \to J$ can be defined by the formula $i(x) = x/(1 + |x|)$.■

We say that a topological property $\mathcal{P}$ is *hereditary (hereditary with respect to closed subsets, open subsets* etc.) if for any space X that has the property $\mathcal{P}$, every subspace (every closed subspace, open subspace etc.) of X also has the property $\mathcal{P}$. Having weight $\leq \mathfrak{m}$ or having character $\leq \mathfrak{m}$ – and, in particular, satisfying the second or the first axiom of countability – are examples of hereditary properties. As shown in 2.1.5, separability is not a hereditary property; one can easily verify, however, that separability is hereditary with respect to open subsets. If the property $\mathcal{P}$ is not a hereditary property but every subspace of a space X has $\mathcal{P}$, then we say that X has $\mathcal{P}$ *hereditarily*; in this sense the terms such as *"hereditarily separable space"*, *"hereditarily normal space"* etc. will be used in the sequel.

2.1.6. THEOREM. *Any subspace of a T_i-space is a T_i-space for $i \leq 3\frac{1}{2}$. Normality is hereditary with respect to closed subsets. Perfect normality is a hereditary property.*

PROOF. In the case of T_1-spaces our theorem follows immediately from 2.1.1. The fact that perfect normality is a hereditary property follows from 2.1.1 and the equivalence of conditions (i) and (iii) in 1.5.19. The proofs of the remaining five cases are similar to each other; as a sample we shall prove that a subspace M of a regular space X is regular.

As we already know, M is a T_1-space. Let us take a point $x \in M$ and a closed set $A \subset M$ such that $x \notin A$. By 2.1.1 we have $A = M \cap \overline{A}$, so that $x \notin \overline{A}$. Then there exist sets U_1, V_1, open in X, such that $x \in U_1$, $\overline{A} \subset V_1$ and $U_1 \cap V_1 = \emptyset$. Defining $U = U_1 \cap M$ and $V = V_1 \cap M$, we obtain open subsets of M such that $x \in U$, $A \subset V$ and $U \cap V = \emptyset$. Hence the space M is regular.■

Example 2.3.36 below shows that normality is not hereditary (cf. Exercise 2.1.E(b)).

From the last theorem it follows that every perfectly normal space is hereditarily normal. The reverse implication does not hold: the space $A(\mathfrak{m})$ is not perfectly normal for $\mathfrak{m} > \aleph_0$, and yet is hereditarily normal because all its subspaces are of the form $A(\mathfrak{n})$ or $D(\mathfrak{n})$ with $\mathfrak{n} \leq \mathfrak{m}$. Let us also observe that from 2.1.4 and 1.5.20 it follows that the class of hereditarily normal spaces is invariant under closed mappings.

We shall now give two characterizations of hereditarily normal spaces. In the second, the notion of separated sets is used; two subsets A and B of a topological space X are called *separated* if $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. Two disjoint sets are separated if and only if neither of them contains accumulation points of the other. In particular, any two disjoint closed sets are separated. Also any two disjoint open sets are separated. If $A, B \subset X$ are separated and $A_1 \subset A, B_1 \subset B$, then the sets A_1, B_1 also are separated.

2.1.7. THEOREM. *For every T_1-space X the following conditions are equivalent:*

(i) *The space X is hereditarily normal.*

(ii) *Every open subspace of X is normal.*

(iii) *For every pair of separated sets $A, B \subset X$ there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

PROOF. The implication (i)$\Rightarrow$(ii) is obvious. We shall show that (ii)$\Rightarrow$(iii). Let us consider a pair of separated sets $A, B \subset X$ and the open subspace $M = X \setminus (\overline{A} \cap \overline{B}) \subset X$; obviously $A, B \subset M$. The closures of A and B in M are disjoint, thus – by normality of M – there exist sets U, V open in M such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. As M is an open subspace of X, the sets U and V are open in X.

To complete the proof it remains to show that (iii)$\Rightarrow$(i). Let M be an arbitrary subspace of X and $A, B \subset M$ a pair of disjoint closed subsets of M. Clearly A and B are separated in X, so that there exist open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. The intersections $M \cap U$ and $M \cap V$ are open in M, disjoint and contain A and B respectively. ∎

Condition (iii) in the last theorem shows that hereditary normality can be ranged among the separation axioms; hereditarily normal spaces are sometimes called *T_5-spaces*, and members of the narrower class of perfectly normal spaces are called *T_6-spaces*.

If, for a continuous mapping $f: M \to Y$ defined on a subspace M of a space X, there exists a continuous mapping $F: X \to Y$ such that $F|M = f$, then we say that f is *continuously extendable*, or – briefly – *extendable*, over X and we call F an *extension* of f over X. Not every continuous mapping defined on a subspace has a continuous extension over the space; the existence of a continuous extension is rather an exception. Theorems that give sufficient conditions for extendability of continuous mappings or continuous real-valued functions are among the most important ones in topology, and usually are rather difficult. Let us note that Urysohn's lemma can be restated as a theorem of that type. Indeed, Urysohn's lemma says that if a subspace M of a normal space X can be represented as the union of two disjoint subsets A, B closed in X, then the function $f: M \to I$, defined by $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$, is continuously extendable over X.

It turns out that a much more general theorem holds:

2.1.8. THE TIETZE-URYSOHN THEOREM. *Every continuous function from a closed subspace M of a normal space X to I or R is continuously extendable over X.*

PROOF. First we shall prove our theorem for functions from X to I. For simplicity of notation, we shall discuss the case of a function $f: M \to J$, where J is the interval $[-1, 1]$ homeomorphic to I.

Let us begin with the remark that for any $f_0: M \to R$ satisfying $|f_0(x)| \leq c$ for all $x \in M$, there exists a $g: X \to R$ such that

(1) $$|g(x)| \leq \frac{1}{3}c \quad \text{for} \quad x \in X$$

and

(2) $$|f_0(x) - g(x)| \leq \frac{2}{3}c \quad \text{for} \quad x \in M.$$

Indeed, the sets $A = f_0^{-1}([-c, -\frac{1}{3}c])$ and $B = f_0^{-1}([\frac{1}{3}c, c])$ are disjoint and closed in M; thus they are closed in X, and by Urysohn's lemma there exists a function $k: X \to I$ such that $k(A) \subset \{0\}$ and $k(B) \subset \{1\}$. One readily verifies that by letting $g(x) = \frac{2}{3}c(k(x) - \frac{1}{2})$ we obtain $g: X \to R$ satisfying (1) and (2).

We are now going to define, by induction, a sequence $g_1, g_2, \ldots$ of continuous functions from X to R such that

$$|g_i(x)| \leq \frac{1}{3}(\frac{2}{3})^{i-1} \quad \text{for} \quad x \in X \tag{3}$$

and

$$|f(x) - \sum_{j=1}^{i} g_j(x)| \leq (\frac{2}{3})^i \quad \text{for} \quad x \in M. \tag{4}$$

To obtain g_1 we apply the above remark to $f_0 = i_J f$ and $c = 1$, where i_J is the embedding of J in R. Assume that $g_1, g_2, \ldots, g_i$ are already defined; applying the same remark to $f_0 = i_J f - (\sum_{j=1}^{i} g_j)|M$ and $c = (\frac{2}{3})^i$ we obtain a function g_{i+1} satisfying (3) and (4) with i replaced by $i+1$.

From (3) and 1.4.7 it follows that the formula $F(x) = \sum_{i=1}^{\infty} g_i(x)$ defines a continuous function $F: X \to J$, and (4) implies that $F(x) = f(x)$ for $x \in M$, so that F is an extension of f over X.

Now, let us consider a function $f: M \to R$. By the part of the theorem which is already proved, the function $if: M \to J$, where $i: R \to J$ is the homeomorphic embedding defined in 2.1.5, is extendable over X to $F_1: X \to J$. Clearly the set $L = F_1^{-1}(\{-1, 1\})$ is a closed subset of X disjoint from M, so that there exists a continuous $k: X \to I$ such that $k(L) \subset \{0\}$ and $k(M) \subset \{1\}$. One readily sees that $F_2: X \to J$, defined by $F_2(x) = F_1(x) \cdot k(x)$, also is an extension of if over X, and that $F_2(X) \subset i(R)$. The function $F: X \to R$ defined by $F(x) = i^{-1}F_2(x)$ is the required extension of f over X. ∎

Let us note that the extension property established in the Tietze-Urysohn theorem characterizes normal spaces in the class of T_1-spaces. Indeed, if a T_1-space X is not normal, then it contains two disjoint closed subsets A, B that cannot be separated by open sets, and this implies that the function $f: A \cup B \to I$ defined by $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$ cannot be continuously extended over X.

We shall prove one more theorem about extensions of mappings.

2.1.9. THEOREM. *If a continuous mapping f of a dense subset of a topological space X to a Hausdorff space Y is continuously extendable over X, then the extension is uniquely determined by f.*

PROOF. Suppose that $F_1: X \to Y$ and $F_2: X \to Y$ are both extensions of f. By 1.5.4 the set

$$B = \{x \in X : F_1(x) = F_2(x)\}$$

is closed; since B contains a dense subset of X, we have $B = X$. ∎

2.1.10. COROLLARY. *No separable normal space contains a closed discrete subspace of cardinality continuum.*

PROOF. Assume that a normal space X contains a countable dense set C and a closed discrete subspace A of cardinality $\mathfrak{c}$. From 2.1.9 it follows that every continuous real-valued function on X is determined by its restriction to C, so that there are at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ such functions. On the other hand, by the Tietze-Urysohn theorem, each of the $2^{\mathfrak{c}}$ continuous real-valued functions defined on A is continuously extendable over X, and we have a contradiction. ■

From Corollary 2.1.10 it follows immediately that the Niemytzki plane is not normal (cf. Example 1.5.10); this is a standard way of proving non-normality of certain spaces (cf. Examples 2.3.12 and 3.6.19, Exercise 3.1.H(a) and Problem 2.7.20(f)). Let us observe that the argument in Example 1.5.10 yields another proof of Corollary 2.1.10 (cf. Problem 1.7.12(c)).

To define a mapping f from a space X to a space Y, it is sometimes convenient to split X into subspaces and define f separately (e.g., by different formulas) on each of these subspaces (cf. Examples 1.4.4 and 1.4.5). We shall now prove two propositions giving conditions for continuity of a mapping defined in that way.

Suppose we are given a topological space X, a cover $\{A_s\}_{s\in S}$ of the space X and a family of continuous mappings $\{f_s\}_{s\in S}$, where $f_s\colon A_s \to Y$. We say that the mappings f_s are *compatible* if for every pair s_1, s_2 of elements of S we have

$$f_{s_1}|A_{s_1} \cap A_{s_2} = f_{s_2}|A_{s_1} \cap A_{s_2};$$

letting

$$f(x) = f_s(x) \quad \text{for} \quad x \in A_s$$

we define then a mapping f of X to Y, which is called the *combination of the mappings* $\{f_s\}_{s\in S}$ and is denoted by the symbol $\triangledown_{s\in S} f_s$, or by $f_1 \triangledown f_2 \triangledown \ldots \triangledown f_k$ if $S = \{1, 2, \ldots, k\}$. Let us note that if, according to the definition given in our Introduction, one views functions as sets, the combination $\triangledown_{s\in S} f_s$ is just the union $\bigcup_{s\in S} f_s$.

2.1.11. PROPOSITION. *If $\{U_s\}_{s\in S}$ is an open cover of a space X and $\{f_s\}_{s\in S}$, where $f_s\colon U_s \to Y$, is a family of compatible continuous mappings, then the combination $f = \triangledown_{s\in S} f_s$ is a continuous mapping of X to Y.*

PROOF. For every open subset U of Y we have

$$f^{-1}(U) = \bigcup_{s\in S} f_s^{-1}(U).$$

The set $f_s^{-1}(U)$ is open in U_s, and thus in X. This implies that the inverse image $f^{-1}(U)$ is open in X. ■

2.1.12. COROLLARY. *A mapping f of a topological space X to a topological space Y is continuous if and only if every point $x \in X$ has a neighbourhood U such that $f|U$ is continuous.* ■

2.1.13. PROPOSITION. *If $\{F_s\}_{s\in S}$ is a locally finite closed cover of a space X and $\{f_s\}_{s\in S}$, where $f_s\colon F_s \to Y$, is a family of compatible continuous mappings, then the combination $f = \triangledown_{s\in S} f_s$ is a continuous mapping of X to Y.*

PROOF. For every closed subset F of Y we have

$$f^{-1}(F) = \bigcup_{s\in S} f_s^{-1}(F).$$

The set $f_s^{-1}(F)$ is closed in F_s, and thus in X. As the family $\{f_s^{-1}(F)\}_{s\in S}$ is locally finite, Corollary 1.1.12 implies that the inverse image $f^{-1}(F)$ is closed in X. ■

2.1.14. THEOREM. *For every countable discrete family $\{F_i\}_{i=1}^{\infty}$ of closed subsets of a normal space X there exists a family $\{U_i\}_{i=1}^{\infty}$ of open subsets of X such that $F_i \subset U_i$ for $i = 1, 2, \ldots$ and $\overline{U}_i \cap \overline{U}_j = \emptyset$ for $i \neq j$.*

PROOF. The union $M = \bigcup_{i=1}^{\infty} F_i$ is a closed subspace of X, and $\{f_i\}_{i=1}^{\infty}$, where $f_i: F_i \to R$ is defined by $f_i(x) = i$, is a family of compatible continuous mappings, so that the combination f of the mappings $\{f_i\}_{i=1}^{\infty}$ is a continuous mapping of M to R. By the Tietze-Urysohn theorem, f is extendable to a mapping $F: X \to R$; it is easy to verify that the sets $U_i = F^{-1}((i - \frac{1}{3}, i + \frac{1}{3}))$ have the required properties. ■

We close this section with a proposition giving conditions for the combination of mappings to be closed or open.

2.1.15. PROPOSITION. *Suppose we are given a topological space X, a cover $\{A_s\}_{s\in S}$ of the space X and a family $\{f_s\}_{s\in S}$ of compatible mappings, where $f_s: A_s \to Y$, such that the combination $f = \nabla_{s\in S} f_s: X \to Y$ is continuous. If all mappings f_s are open (closed and the family $\{f_s(A_s)\}_{s\in S}$ is locally finite), then the combination f is open (closed).*

PROOF. It suffices to apply the equality

$$f(A) = f(A \cap \bigcup_{s\in S} A_s) = \bigcup_{s\in S} f_s(A \cap A_s). \blacksquare$$

Historical and bibliographic notes

The systematic study of subspaces started with Hausdorff's book [1914]. Theorem 2.1.7, the observation that perfectly normal spaces are hereditarily normal and the Tietze-Urysohn theorem appeared in Urysohn [1925]. Particular cases of the latter theorem were obtained by Lebesgue in [1907] (for the plane) and by Tietze in [1915] (for metric spaces). Corollary 2.1.10 was proved by Jones in [1937], the argument in its proof was given by Katětov in [1950]. Theorem 2.1.14 is a particular case of a result in Kuratowski [1935].

Exercises

2.1.A. Verify that the interior and the boundary of a set A in a subspace M of a topological space X are equal respectively to

$$M \setminus \overline{M \setminus A} \quad \text{and} \quad M \cap \overline{A} \cap \overline{M \setminus A}$$

where the bar denotes the closure operator in X. Deduce from the above that for every $B \subset X$ the boundary in M of the intersection $M \cap B$ is contained in the intersection of M with the boundary of B in X; observe that this is not true for interiors.

2.1.B. (a) Let M be a subspace of X; verify that a set $A \subset M$ is an F_σ-set (a G_δ-set) in M if and only if $A = M \cap B$, where B is an F_σ-set (a G_δ-set) in X. Deduce that if the subspace

M of X is an F_σ-set (a G_δ-set) in X, then a set $A \subset M$ is an F_σ-set (a G_δ-set) in M if and only if it is an F_σ-set (a G_δ-set) in X.

(b) Prove that if the subspace M of X is an open (a closed) domain, then a set $A \subset M$ is an open (a closed) domain in M if and only if it is an open (a closed) domain in X. Show that an open domain in a subspace M of X is not necessarily an intersection of an open domain in X with the subspace M. Give an example of an open domain in a topological space X whose intersection with a subspace M of X is not an open domain in M.

(c) Prove that if the subspace M of X is a functionally open subset of X, then a set $A \subset M$ is functionally open in M if and only if it is functionally open in X. Give an example of a functionally closed set M in a Tychonoff space X such that there exists a functionally closed set in the subspace M which is not functionally closed in X. Observe that the latter set is not an intersection of M with a functionally closed set in X. Note that the intersection of a functionally open (closed) set in a topological space X with a subspace M of X is functionally open (closed) in M.

2.1.C. (a) Prove that if M is a dense subspace of a regular space X, then $\chi(x, X) = \chi(x, M)$ for every $x \in M$. Show that the assumption of regularity cannot be weakened to the assumption that X is a Hausdorff space.

(b) A family $\mathcal{B}(A)$ of open subsets of a space X is called a *base for the space X at a set $A \subset X$* if all members of $\mathcal{B}(A)$ contain A and for any open set V that contains A there exists a $U \in \mathcal{B}(A)$ such that $A \subset U \subset V$. The *character of a set A* in a topological space X is defined as the smallest cardinal number of the form $|\mathcal{B}(A)|$, where $\mathcal{B}(A)$ is a base for X at the set A; this cardinal number is denoted by $\chi(A, X)$.

Prove that if M is a dense subspace of a space X and $A \subset M$ has the property that for every closed set $B \subset X$ disjoint from A there exist open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$, then $\chi(A, X) = \chi(A, M)$ (cf. Theorem 3.1.6).

Deduce from the above that if M is a dense subspace of a normal space X, then $\chi(F, X) = \chi(F, M)$ for every closed subset F of the space X that is contained in M.

(c) Show that if M is a closed subspace of X, then $\chi(A \cap M, M) \le \chi(A, X)$ for every $A \subset X$. Verify that the assumption of M being closed cannot be omitted.

(d) Note that if $f: X \to Y$ is a closed mapping, then $\chi(f^{-1}(B), X) \le \chi(B, Y)$ for every $B \subset Y$, and that $\chi(f^{-1}(B), X) = \chi(B, Y)$ if $f(X) = Y$. Note that if $f: X \to Y$ is an open mapping, then $\chi(f(A), Y) \le \chi(A, X)$ for every $A \subset X$.

2.1.D. Verify that a subspace M of a topological space X is a retract of X if and only if every continuous mapping defined on M is extendable over X or – equivalently – if and only if there exists a continuous mapping $r: X \to M$ such that $r|M = \mathrm{id}_M$.

2.1.E. (a) (Urysohn [1925]) Prove that normality is hereditary with respect to F_σ-sets.

Hint. Apply Lemma 1.5.15.

(b) Prove that by adjoining one point to the Niemytzki plane K one can obtain a normal space of which K is a subspace.

2.1.F. Verify that if $x_1, x_2, \ldots$ is a sequence of points in a T_2-space X and x_0 is the limit of this sequence, then the subspace $M = \{x_0, x_1, x_2, \ldots\}$ of X is either finite or is homeomorphic to the space $A(\aleph_0)$. Show that the assumption that X is a T_2-space cannot be weakened to the assumption that X is a T_1-space.

2.1.G. Show that for every countable discrete family $\{F_i\}_{i=1}^{\infty}$ of finite subsets of a regular space X there exists a family $\{U_i\}_{i=1}^{\infty}$ of open subsets of X such that $F_i \subset U_i$ for $i = 1, 2, \ldots$ and $\overline{U}_i \cap \overline{U}_j = \emptyset$ for $i \neq j$.

2.1.H. (a) Verify that a subspace of a sequential space is not necessarily a sequential space (cf. Examples 1.6.19 and 1.6.20), but a closed subspace or an open subspace is a sequential space (cf. Exercise 2.4.G(b)).

(b) Verify that every subspace of a Fréchet space is a Fréchet space.

2.1.I. Prove that the Sorgenfrey line is hereditarily separable.

2.1.J (Gillman and Jerison [1960]). Show that every continuous function from a subspace M of a topological space X to I is continuously extendable over X if and only if any disjoint sets $A, B \subset M$ functionally closed in M are completely separated in X.

2.1.K. Show that Proposition 2.1.13 and the parenthetical part of Proposition 2.1.15 do not hold without the assumption that the considered families of sets are locally finite.

2.2. Sums

Suppose we are given a family $\{X_s\}_{s\in S}$ of pairwise disjoint topological spaces, i.e., that $X_s \cap X_{s'} = \emptyset$ for $s \neq s'$; consider the set $X = \bigcup_{s\in S} X_s$ and the family $\mathcal{O}$ of all sets $U \subset X$ such that $U \cap X_s$ is open in X_s for every $s \in S$. It is easily seen that the family $\mathcal{O}$ satisfies conditions (O1)–(O3), so that $\mathcal{O}$ is a topology on the set X. The set X with this topology is called the *sum of the spaces* $\{X_s\}_{s\in S}$ and is denoted by $\bigoplus_{s\in S} X_s$, or by $X_1 \oplus X_2 \oplus \ldots \oplus X_k$ if $S = \{1, 2, \ldots, k\}$.

2.2.1. PROPOSITION. *A set $A \subset \bigoplus_{s\in S} X_s$ is closed if and only if the intersection $A \cap X_s$ is closed in X_s for every $s \in S$.*

PROOF. The set A is closed if and only if its complement $\bigoplus_{s\in S} X_s \setminus A$ is open. Hence the proposition follows from the equality

$$(\bigoplus_{s\in S} X_s \setminus A) \cap X_{s_0} = X_{s_0} \setminus (A \cap X_{s_0}). \blacksquare$$

2.2.2. COROLLARY. *All sets X_s are open-and-closed in $\bigoplus_{s\in S} X_s$.* ■

Clearly, every X_s is a subspace of the sum $\bigoplus_{s\in S} X_s$; the embedding of X_s in $\bigoplus_{s\in S} X_s$ is denoted by i_s.

2.2.3. PROPOSITION. *If $\{X_s\}_{s\in S}$ is a family of pairwise disjoint topological spaces and A_s is a subspace of X_s for every $s \in S$, then the two topologies defined on the set $\bigcup_{s\in S} A_s$, viz., the topology of the sum of subspaces $\{A_s\}_{s\in S}$ and the topology of the subspace of the sum $\bigoplus_{s\in S} X_s$, coincide.* ■

2.2.4. PROPOSITION. *If a topological space X can be represented as the union of a family $\{X_s\}_{s\in S}$ of pairwise disjoint open subsets, then $X = \bigoplus_{s\in S} X_s$.*

PROOF. The sets X and $\bigoplus_{s\in S} X_s$ coincide, so we have only to check that families of open sets in X and $\bigoplus_{s\in S} X_s$ coincide, too. If U is open in X, then the intersection $U\cap X_s$ is open in X_s for every $s\in S$, so that U is open in $\bigoplus_{s\in S} X_s$. Conversely, if U is open in $\bigoplus_{s\in S} X_s$, then the intersection $U\cap X_s$ is open in X_s, and thus in X, for every $s\in S$, which implies that $U=\bigcup_{s\in S}(U\cap X_s)$ is open in X. ∎

2.2.5. COROLLARY. *Let $\{X_s\}_{s\in S}$ be a family of pairwise disjoint topological spaces. If $S=\bigcup_{t\in T} S_t$, where $S_t\cap S_{t'}=\emptyset$ for $t\neq t'$, then $\bigoplus_{s\in S} X_s=\bigoplus_{t\in T}(\bigoplus_{s\in S_t} X_s)$, i.e., the sum of spaces is associative.* ∎

2.2.6. PROPOSITION. *A mapping f of the sum $\bigoplus_{s\in S} X_s$ to a topological space Y is continuous if and only if the composition fi_s is continuous for every $s\in S$.*

PROOF. If $f\colon \bigoplus_{s\in S} X_s\to Y$ is continuous, then every fi_s is continuous, being the composition of two continuous mappings. Conversely, if for a mapping f of $\bigoplus_{s\in S} X_s$ to Y every fi_s is continuous, then $f=\nabla_{s\in S} fi_s$ is continuous by Proposition 2.1.11. ∎

The sum $\bigoplus_{s\in S} X_s$ can also be defined for a family of topological spaces $\{X_s\}_{s\in S}$ which are not pairwise disjoint. To do that, one has simply to take a family $\{X'_s\}_{s\in S}$ of pairwise disjoint spaces such that X'_s is homeomorphic to X_s for every $s\in S$ and define $\bigoplus_{s\in S} X_s=\bigoplus_{s\in S} X'_s$; e.g., one can take $X'_s=X_s\times\{s\}$ with the topology generated by the mapping $p_s\colon X'_s\to X_s$, where $p_s(x,s)=x$. The reader can easily verify that the spaces $\bigoplus_{s\in S} X'_s$ obtained in that way are all homeomorphic to each other. In the sequel we shall assume that any family of spaces has a sum (determined up to a homeomorphism), but in the proofs we shall tacitly assume that the discussed family consists of pairwise disjoint spaces.

We say that a topological property $\mathcal{P}$ is *additive* ($\mathfrak{m}$-*additive, finitely additive*) if for any family $\{X_s\}_{s\in S}$ (such that $|S|\le\mathfrak{m}$, $|S|<\aleph_0$) of spaces with property $\mathcal{P}$, the sum $\bigoplus_{s\in S} X_s$ also has property $\mathcal{P}$.

2.2.7. THEOREM. *Any sum of T_i-spaces is a T_i-space for $i\le 6$.*

PROOF. In the case of T_1-spaces our theorem follows immediately from 2.2.1. The proofs of all the remaining cases are similar to each other; as a sample we shall show that normality is additive. Let $\{X_s\}_{s\in S}$ be a family of normal spaces and A,B two disjoint closed subsets of the sum $\bigoplus_{s\in S} X_s$. By Proposition 2.2.1 the intersections $A\cap X_s$ and $B\cap X_s$ are closed in X_s for every $s\in S$. From the normality of X_s it follows that there exist sets U_s,V_s open in X_s and such that

$$A\cap X_s\subset U_s,\quad B\cap X_s\subset V_s\quad\text{and}\quad U_s\cap V_s=\emptyset.$$

Clearly

$$A\subset U=\bigcup_{s\in S} U_s,\quad B\subset V=\bigcup_{s\in S} V_s\quad\text{and}\quad U\cap V=\emptyset;$$

since U and V are open in $\bigoplus_{s\in S} X_s$, the sum $\bigoplus_{s\in S} X_s$, being a T_1-space, is normal. ∎

One readily sees that the properties "weight is $\le\mathfrak{m}$" and "density is $\le\mathfrak{m}$" are both $\mathfrak{m}$-additive for $\mathfrak{m}\ge\aleph_0$, but are not additive properties; the property "character is $\le\mathfrak{m}$" is an additive property.

2.2.8. EXAMPLES. The discrete space $D(\mathfrak{m})$ is the sum of $\mathfrak{m}$ one-point spaces.

For every point x of the Sorgenfrey line K and for every neighbourhood U of x, the Sorgenfrey line can be represented as a sum $X_1 \oplus X_2$, where $x \in X_1 \subset U$. Indeed, as X_1 one can take any interval $[x, r)$ contained in U and as X_2 its complement $K \setminus X_1$. Since X_1 is open-and-closed in K, the equality $K = X_1 \oplus X_2$ follows from Proposition 2.2.4.

The real line R cannot be represented as the sum $X_1 \oplus X_2$ of non-empty subspaces X_1, X_2. Suppose that this is possible, i.e., that $R = X_1 \oplus X_2$ and $X_1 \neq \emptyset \neq X_2$. Take arbitrary points $x_1 \in X_1$ and $x_2 \in X_2$; with no loss of generality one can assume that $x_1 < x_2$. The set $X_1 \cap [x_1, x_2]$ is bounded; let x_0 be its least upper bound. Since X_1 is closed, we have $x_0 \in X_1$, which implies that $x_0 < x_2$. Since X_1 is open, there exists an $\epsilon > 0$ such that $(x_0 - \epsilon, x_0 + \epsilon) \subset X_1$ and thus we have $X_1 \cap (x_0, x_2] \neq \emptyset$, contrary to the definition of the least upper bound. ∎

Historical and bibliographic notes

Sums of topological spaces appeared for the first time in Tietze's paper [1923]. Theorems concerning this operation are usually very easy and belong to topological folklore. However, application of the sum operation sometimes simplifies proofs and description of examples.

Exercises

2.2.A. Show that no continuous image of the real line can be represented as the sum $X_1 \oplus X_2$ with $X_1 \neq \emptyset \neq X_2$. How can this be strengthened?

2.2.B. Show that the property of being a discrete space is an additive property.

2.2.C. For $X = \bigoplus_{s \in S} X_s$ express $w(X)$, $d(X)$, $\chi(X)$ and $\chi(x, X)$, in terms of $w(X_s)$, $d(X_s)$, $\chi(X_s)$, $\chi(x, X_s)$ and $|S|$.

2.2.D (Kuratowski [1921]). Let $X_0, X_1, \ldots$ be the subspaces of R defined by

$$X_0 = [0, 1) \quad \text{and} \quad X_i = (i, i+1) \quad \text{for} \quad i = 1, 2, \ldots$$

Prove that the spaces $Y_1 = (\bigoplus_{i=1}^{\infty} X_i) \oplus D(\aleph_0)$ and $Y_2 = X_0 \oplus Y_1$ are not homeomorphic, but either of them can be mapped by a continuous one-to-one mapping onto the other.

Hint. When proving that Y_1 and Y_2 are not homeomorphic apply Exercise 2.2.A.

2.2.E. (a) Suppose we are given two families $\{X_s\}_{s \in S}$ and $\{Y_s\}_{s \in S}$ of pairwise disjoint topological spaces and a family of continuous mappings $\{f_s\}_{s \in S}$, where $f_s: X_s \to Y_s$. Letting $f(x) = f_s(x)$ for $x \in X_s$ we define a mapping f of the sum $\bigoplus_{s \in S} X_s$ to the sum $\bigoplus_{s \in S} Y_s$ which is called the *sum of the mappings* $\{f_s\}_{s \in S}$ and is denoted by $\bigoplus_{s \in S} f_s$, or by $f_1 \oplus f_2 \oplus \ldots \oplus f_k$ if $S = \{1, 2, \ldots, k\}$.

Verify that f is continuous and that f is closed (open, is a homeomorphic embedding) if and only if all mappings f_s are closed (open, are homeomorphic embeddings).

(b) Suppose we are given a family $\{X_s\}_{s \in S}$ of pairwise disjoint topological spaces and a family of continuous mappings $\{f_s\}_{s \in S}$, where $f_s: X_s \to Y$. Formulate conditions for the combination $\nabla_{s \in S} f_s: \bigoplus_{s \in S} X_s \to Y$ to be closed or open (cf. Proposition 2.1.15).

2.2.F. Show that the space X is homeomorphic to the sum $\bigoplus_{s\in S} X_s$ if and only if there exists a family of continuous mappings $\{i_s\}_{s\in S}$, where $i_s: X_s \to X$, satisfying the following conditions:

(1) *For every space Y and a pair f, g of continuous mappings of X to Y, if $fi_s = gi_s$ for every $s \in S$, then $f = g$.*

(2) *For every space Y and a family of continuous mappings $\{f_s\}_{s\in S}$, where $f_s: X_s \to Y$, there exists a continuous mapping $f: X \to Y$ such that $fi_s = f_s$ for every $s \in S$.*

2.3. Cartesian products

Suppose we are given a family $\{X_s\}_{s\in S}$ of topological spaces; consider the Cartesian product $X = \prod_{s\in S} X_s$ and the family of mappings $\{p_s\}_{s\in S}$, where p_s assigns to the point $x = \{x_s\} \in \prod_{s\in S} X_s$ its sth coordinate $x_s \in X_s$. The set $X = \prod_{s\in S} X_s$ with the topology generated by the family of mappings $\{p_s\}_{s\in S}$ is called the *Cartesian product of the spaces* $\{X_s\}_{s\in S}$ and the topology itself is called the *Tychonoff topology* on $\prod_{s\in S} X_s$; the mapping $p_s: \prod_{s\in S} X_s \to X_s$ are called the *projections of* $\prod_{s\in S} X_s$ *onto* X_s. For any family $\{X_s\}_{s\in S}$ of topological spaces, the symbol $\prod_{s\in S} X_s$ will from now on denote not a set, the Cartesian product of the sets $\{X_s\}_{s\in S}$, but rather this set with the Tychonoff topology, i.e., a topological space. The Cartesian product of a finite family of spaces $\{X_i\}_{i=1}^k$ is also denoted by $X_1 \times X_2 \times \ldots \times X_k$. If all spaces in the family $\{X_s\}_{s\in S}$ are equal to each other, i.e., if $X_s = X$ for $s \in S$, then the Cartesian product $\prod_{s\in S} X_s$ is also denoted by $X^{\mathfrak{m}}$, where $\mathfrak{m} = |S|$; one can readily verify that the space $X^{\mathfrak{m}}$ does not depend (up to a homeomorphism) on the set S but only on its cardinality $\mathfrak{m}$. The Cartesian product $X^{\mathfrak{m}}$ is called the $\mathfrak{m}$*th power* of the space X; the product $X \times X$ is also called the *square* of X.

2.3.1. PROPOSITION. *The family of all sets $\prod_{s\in S} W_s$, where W_s is an open subset of X_s and $W_s \neq X_s$ only for finitely many $s \in S$, is a base for the Cartesian product $\prod_{s\in S} X_s$.*

Moreover, *if for every $s \in S$ a base $\mathcal{B}_s$ for X_s is fixed, then the subfamily consisting of those $\prod_{s\in S} W_s$ in which $W_s \in \mathcal{B}_s$ whenever $W_s \neq X_s$, also is a base.*

PROOF. By 1.4.8, the family of all sets of the form $\bigcap_{i=1}^k p_{s_i}^{-1}(W_i)$ where $s_1, s_2, \ldots, s_k \in S$ and W_i is open in X_{s_i}, is a base for $\prod_{s\in S} X_s$. Hence to prove the first part of the proposition it suffices to observe that

$$p_{s_0}^{-1}(W_{s_0}) = \prod_{s\in S} W_s, \quad \text{where } W_s = X_s \text{ for } s \neq s_0$$

and that

$$\Big(\prod_{s\in S} W_s\Big) \cap \Big(\prod_{s\in S} W'_s\Big) = \prod_{s\in S} (W_s \cap W'_s).$$

The second part is an immediate consequence of the first part and of the definition of a base. ■

The base for $\prod_{s\in S} X_s$ described in the first part of the above proposition is called the *canonical base* for the Cartesian product. Clearly, the family of all sets $\prod_{s\in S} W_s$, where W_s is an open subset of X_s and $W_s \neq X_s$ only for one $s \in S$, is a subbase for the Cartesian product $\prod_{s\in S} X_s$.

2.3.2. PROPOSITION. *If $\{X_s\}_{s\in S}$ is a family of topological spaces and A_s is for every $s \in S$ a subspace of X_s, then the two topologies defined on the set $A = \prod_{s\in S} A_s$, viz., the topology of the Cartesian product of subspaces $\{A_s\}_{s\in S}$ and the topology of a subspace of the Cartesian product $\prod_{s\in S} X_s$, coincide.*

PROOF. One can assume that $A_s \neq \emptyset$ for $s \in S$. Since the restrictions $p_s|A: A \to A_s$ of the projections p_s are continuous, the subspace topology on A is finer than the topology of the Cartesian product by the definition of the latter. Any open subset of the subspace A is the intersection of A with the union of a family of members of the canonical base for $\prod_{s\in S} X_s$, i.e., the union of intersections of A with members of that family. Since every such intersection is a member of the canonical base for $\prod_{s\in S} A_s$, the topology of the Cartesian product on A is finer than the subspace topology. ■

2.3.3. PROPOSITION. *For every family of sets $\{A_s\}_{s\in S}$, where $A_s \subset X_s$, in the Cartesian product $\prod_{s\in S} X_s$ we have*

$$\overline{\prod_{s\in S} A_s} = \prod_{s\in S} \overline{A_s}. \tag{1}$$

PROOF. From 1.1.1 it follows that $x \in \overline{\prod_{s\in S} A_s}$ if and only if for every member $\prod_{s\in S} W_s$ of the canonical base for $\prod_{s\in S} X_s$ that contains x we have $\prod_{s\in S} W_s \cap \prod_{s\in S} A_s = \prod_{s\in S} W_s \cap A_s \neq \emptyset$, i.e., if for every $s \in S$ and any neighbourhood W_s of the sth coordinate of x we have $W_s \cap A_s \neq \emptyset$. The last condition is satisfied if and only if $x \in \prod_{s\in S} \overline{A_s}$. ■

2.3.4. COROLLARY. *The set $\prod_{s\in S} A_s$, where $\emptyset \neq A_s \subset X_s$, is closed in the Cartesian product $\prod_{s\in S} X_s$ if and only if A_s is closed in X_s for every $s \in S$.*

PROOF. From the proposition it follows that if all the A_s's are closed, then $\prod_{s\in S} A_s$ also is closed. Conversely, if $\prod_{s\in S} A_s$ is closed then $\prod_{s\in S} A_s = \overline{\prod_{s\in S} A_s} = \prod_{s\in S} \overline{A_s}$ and, since the sets A_s are non-empty, $A_s = \overline{A_s}$ for every $s \in S$. ■

2.3.5. COROLLARY. *The set $\prod_{s\in S} A_s$, where $A_s \subset X_s \neq \emptyset$, is dense in the Cartesian product $\prod_{s\in S} X_s$ if and only if A_s is dense in X_s for every $s \in S$.* ■

Let us note that since not every subset of $\prod_{s\in S} X_s$ can be represented in the form $\prod_{s\in S} A_s$, the Tychonoff topology on the Cartesian product $\prod_{s\in S} X_s$ cannot be generated by a closure operator defined by the formula (1).

From Proposition 1.4.9 we immediately obtain

2.3.6. PROPOSITION. *A mapping f of a topological space X to a Cartesian product $\prod_{s\in S} Y_s$ is continuous if and only if the composition $p_s f$ is continuous for every $s \in S$.* ■

2.3.7. PROPOSITION. *Let $\{X_s\}_{s\in S}$ be a family of topological spaces. If $S = \bigcup_{t\in T} S_t$, where $S_t \cap S_{t'} = \emptyset$ for $t \neq t'$. Then the spaces $\prod_{s\in S} X_s$ and $\prod_{t\in T}(\prod_{s\in S_t} X_s)$ are homeomorphic, i.e., the Cartesian product is associative.*

PROOF. To the point $x = \{x_s\} \in \prod_{s\in S} X_s$ we assign the point

$$f(x) = \{x_t\} \in \prod_{t\in T}\Big(\prod_{s\in S_t} X_s\Big),$$

where $x_t = \{x_s\} \in \prod_{s\in S_t} X_s$. The mapping f defined in this way is a one-to-one mapping of $\prod_{s\in S} X_s$ onto $\prod_{t\in T}(\prod_{s\in S_t} X_s)$; applying 2.3.6 one can readily verify that f and f^{-1} are continuous. ■

Let us note that from the last proposition it follows that the spaces $(X^{\mathfrak{m}})^{\mathfrak{m}}$ and $X^{\mathfrak{m}}$ are homeomorphic for every space X and every cardinal number $\mathfrak{m} \geq \aleph_0$.

2.3.8. PROPOSITION. *Let $\{X_s\}_{s\in S}$ be a family of topological spaces and ϕ a one-to-one mapping of S onto itself. Then the spaces $\prod_{s\in S} X_s$ and $\prod_{s\in S} X_{\phi(s)}$ are homeomorphic, i.e., the Cartesian product is commutative.* ■

2.3.9. EXAMPLES. For every positive integer n the space R^n, the Cartesian product of n copies of the real line, is called *Euclidean n-space*, and the space I^n, the Cartesian product of n copies of the closed unit interval, is called the *unit n-cube*. If $m > n$, then the subspace of R^m consisting of all points with last $m - n$ coordinates equal to zero is homeomorphic to R^n; hence R^n is embeddable in R^m for $m > n$. The subspace of R^{n+1} consisting of all points $(x_1, x_2, \ldots, x_{n+1})$ such that $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1$ is called the *unit n-sphere* and is denoted by S^n. Replacing $=$ by $\leq$ in the condition defining the $(n-1)$-sphere, we obtain a subset of R^n which is called the *unit n-ball* and is denoted by B^n. The 1-sphere S^1 is a *circle* and the Cartesian product $S^1 \times S^1$ is a *torus*. ■

For every non-empty set $\prod_{s\in S} W_s \subset \prod_{s\in S} X_s$ we have $p_{s_0}(\prod_{s\in S} W_s) = W_{s_0}$ so that the projections $p_s: \prod_{s\in S} X_s \to X_s$ are open mappings. The example below shows that generally the projections are not closed (cf. Theorem 3.1.16).

2.3.10. EXAMPLE. The projection $p: R^2 \to R$ of the plane R^2 onto the x-axis is not closed. Indeed, the set $F = \{(x,y) \in R^2 : xy = 1\}$ is closed in R^2 and yet its image $p(F) = R \setminus \{0\}$ is not closed in R. ■

Suppose we are given two families $\{X_s\}_{s\in S}$ and $\{Y_s\}_{s\in S}$ of topological spaces and a family of continuous mappings $\{f_s\}_{s\in S}$, where $f_s: X_s \to Y_s$. By Proposition 2.3.6, the mapping assigning to the point $x = \{x_s\} \in \prod_{s\in S} X_s$ the point $\{f_s(x_s)\} \in \prod_{s\in S} Y_s$ is continuous; it is called the *Cartesian product of the mappings* $\{f_s\}_{s\in S}$ and is denoted by $\prod_{s\in S} f_s$, or by $f_1 \times f_2 \times \ldots \times f_k$ if $S = \{1, 2, \ldots, k\}$. If all mappings in the family $\{f_s\}_{s\in S}$ are equal to each other, i.e., if $f_s = f$ for $s \in S$, then the Cartesian product $\prod_{s\in S} f_s$ is also denoted by $f^{\mathfrak{m}}$, where $\mathfrak{m} = |S|$.

Let us observe that for the Cartesian product $f = \prod_{s\in S} f_s$ we have

$$f(\prod_{s\in S} A_s) = \prod_{s\in S} f_s(A_s) \quad \text{and} \quad f^{-1}(\prod_{s\in S} B_s) = \prod_{s\in S} f_s^{-1}(B_s),$$

where $f_s: X_s \to Y_s$, $A_s \subset X_s$ and $B_s \subset Y_s$.

Suppose we are given a topological space X, a family $\{Y_s\}_{s\in S}$ of topological spaces and a family of continuous mappings $\{f_s\}_{s\in S}$, where $f_s: X \to Y_s$. By Proposition 2.3.6, the mapping assigning to the point $x \in X$ the point $\{f_s(x)\} \in \prod_{s\in S} Y_s$ is continuous; it is called the *diagonal of the mappings* $\{f_s\}_{s\in S}$ and is denoted by $\triangle_{s\in S} f_s$, or by $f_1 \triangle f_2 \triangle \ldots \triangle f_k$ if $S = \{1, 2, \ldots, k\}$.

Let us observe that for the diagonal $f = \triangle_{s\in S} f_s$ we have

$$f(A) \subset \prod_{s\in S} f_s(A) \quad \text{and} \quad f^{-1}(\prod_{s\in S} B_s) = \bigcap_{s\in S} f_s^{-1}(B_s),$$

where $f_s : X \to Y_s$, $A \subset X$ and $B_s \subset Y_s$.

Note that the diagonal $\triangle_{s\in S} f_s$ is the composition of the diagonal $i = \triangle_{s\in S} \mathrm{id}_{X_s} : X \to \prod_{s\in S} X_s$, where $X_s = X$ for $s \in S$, and the Cartesian product $\prod_{s\in S} f_s : \prod_{s\in S} X_s \to \prod_{s\in S} Y_s$. The image $\Delta = i(X) \subset X^{\mathfrak{m}} = \prod_{s\in S} X_s$, where $\mathfrak{m} = |S|$, is called the *diagonal of the Cartesian product* $X^{\mathfrak{m}}$; from 1.5.4 it follows that if X is a Hausdorff space, then the diagonal $\Delta = \bigcap_{s',s''\in S}\{x \in \prod_{s\in S} X_s : p_{s'}(x) = p_{s''}(x)\}$ is closed in $X^{\mathfrak{m}}$.

We say that a topological property $\mathcal{P}$ is *multiplicative* ($\mathfrak{m}$-*multiplicative, finitely multiplicative*) if for any family $\{X_s\}_{s\in S}$ (such that $|S| \le \mathfrak{m}$, $|S| < \aleph_0$) of spaces with property $\mathcal{P}$, the Cartesian product $\prod_{s\in S} X_s$ also has property $\mathcal{P}$.

2.3.11. THEOREM. *Any Cartesian product of T_i-spaces is a T_i-space for $i \le 3\frac{1}{2}$. If the Cartesian product $\prod_{s\in S} X_s$ is a non-empty T_i-space, then all the X_s's are T_i-spaces for $i \le 6$.*

PROOF. In the case of T_1-spaces our theorem follows immediately from 2.3.4. The proofs of multiplicativity of T_i-spaces for $i = 0, 2, 3$, and $3^1/_2$ are similar to each other; as a sample we shall prove that the Cartesian product of Tychonoff spaces $\{X_s\}_{s\in S}$ is a Tychonoff space. By Proposition 1.5.8 it suffices to show that for every $x = \{x_s\} \in \prod_{s\in S} X_s$ and every neighbourhood V of x of the form $p_{s_0}^{-1}(W_{s_0})$, where W_{s_0} is an open subset of X_{s_0}, there exists a continuous function $f : \prod_{s\in S} X_s \to I$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in \prod_{s\in S} X_s \setminus V$. One can readily verify that the composition $f_0 p_{s_0}$, where $f_0 : X_{s_0} \to I$ satisfies $f_0(x_{s_0}) = 0$ and $f_0(X_{s_0} \setminus W_{s_0}) \subset \{1\}$, has the required properties.

Now let us consider the Cartesian product $\prod_{s\in S} X_s$ of non-empty spaces $\{X_s\}_{s\in S}$, let us choose a point $x_s^* \in X_s$ for every $s \in S$, and let us define $X_{s_0}^* = \prod_{s\in S} A_s$, where $A_s = \{x_s^*\}$ for $s \ne s_0$ and $A_{s_0} = X_{s_0}$. Clearly, the restriction $i_{s_0} = p_{s_0}|X_{s_0}^* \to X_{s_0}$ is a homeomorphism, so that X_{s_0} is embeddable in the Cartesian product $\prod_{s\in S} X_s$. Moreover, if the product $\prod_{s\in S} X_s$ is a T_i-space with $i \ge 1$, then $X_{s_0}^*$ is its closed subset and X_{s_0} is embeddable in $\prod_{s\in S} X_s$ as a closed subspace. By virtue of Theorem 2.1.6 it follows that if $\prod_{s\in S} X_s$ is a non-empty T_i-space, then all the X_s's are T_i-spaces. ■

Now we are going to give an example to show that normality, hereditary normality and perfect normality are not finitely multiplicative.

2.3.12. EXAMPLE. In Section 1.5 we proved that the Sorgenfrey line K is perfectly normal, so that K is hereditarily normal as well. It turns out that the Cartesian product $K \times K$ is not normal. Indeed, $K \times K$ contains a closed subset homeomorphic to $D(\mathfrak{c})$ – the set $\{(x, y) : y = -x\}$ – and a countable dense subset – the set of all points with both coordinates rational. Thus $K \times K$ is not normal by Corollary 2.1.10. ■

Let us note that there exist perfectly normal spaces with normal, but not hereditarily normal, squares (see Exercise 3.10.C(c)). In Example 2.3.36 below we define two hereditarily normal spaces with normal, but non hereditarily normal, product. Further information about normality of Cartesian products can be found in Exercise 2.3.E and Problems 2.7.16, 3.12.5, 3.12.20, 4.5.15 and 4.5.16; the fact that $N^{\aleph_1}$ is not normal deserves special mention (see Exercise 2.3.E(a)).

2.3.13. THEOREM. *If $w(X_s) \le \mathfrak{m} \ge \aleph_0$ for every $s \in S$ and $|S| \le \mathfrak{m}$, then $w(\prod_{s\in S} X_s) \le \mathfrak{m}$. If $\chi(X_s) \le \mathfrak{m} \ge \aleph_0$ for every $s \in S$ and $|S| \le \mathfrak{m}$, then $\chi(\prod_{s\in S} X_s) \le \mathfrak{m}$.*

PROOF. The first part follows from Proposition 2.3.1, where as $\mathcal{B}_s$ one takes any base for X_s such that $|\mathcal{B}_s| \le \mathfrak{m}$. The proof of the second part is left to the reader.■

2.3.14. COROLLARY. *First-countability and second-countability are $\aleph_0$-multiplicative properties.*■

2.3.15. THE HEWITT-MARCZEWSKI-PONDICZERY THEOREM. *If $d(X_s) \le \mathfrak{m} \ge \aleph_0$ for every $s \in S$ and $|S| \le 2^{\mathfrak{m}}$, then $d(\prod_{s\in S} X_s) \le \mathfrak{m}$.*

PROOF. Suppose that the spaces X_s are non-empty and that $|S| = 2^{\mathfrak{m}}$; let A_s be a dense subspace of X_s such that $|A_s| \le \mathfrak{m}$. By virtue of 2.3.5 it suffices to prove that the Cartesian product $\prod_{s\in S} A_s$ contains a dense subset of cardinality $\le \mathfrak{m}$. Since the Cartesian product $f = \prod_{s\in S} f_s$, where f_s is an arbitrary mapping of the discrete space $D(\mathfrak{m})$ onto A_s, is a continuous mapping of $[D(\mathfrak{m})]^{2^{\mathfrak{m}}}$ onto $\prod_{s\in S} A_s$, Theorem 1.4.10 reduces the proof to the verification that $d([D(\mathfrak{m})]^{2^{\mathfrak{m}}}) \le \mathfrak{m}$.

Let us denote by T the $\mathfrak{m}$th power of the two-point discrete space; we then have $|T| = 2^{\mathfrak{m}}$ and $w(T) \le \mathfrak{m}$. Let us take a base $\mathcal{B}$ for T such that $|\mathcal{B}| \le \mathfrak{m}$ and let us denote by $\mathcal{T}$ the collection of all finite families $\{U_1, U_2, \ldots, U_k\}$ of pairwise disjoint members of $\mathcal{B}$; clearly $|\mathcal{T}| \le \mathfrak{m}$.

In the Cartesian product $[D(\mathfrak{m})]^{2^{\mathfrak{m}}} = \prod_{t\in T} Y_t$, where $Y_t = D(\mathfrak{m})$ for every $t \in T$, take the subset A consisting of all functions f from T to $D(\mathfrak{m})$ such that there exists $\{U_1, U_2, \ldots, U_k\} \in \mathcal{T}$ with the property that f is constant on every set U_i and on the set $T \setminus (U_1 \cup U_2 \cup \ldots \cup U_k)$. Since $|\mathcal{T}| \le \mathfrak{m}$, we also have $|A| \le \mathfrak{m}$.

We shall show that the set A is dense in $[D(\mathfrak{m})]^{2^{\mathfrak{m}}}$, i.e., that for every non-empty open subset $V \subset \prod_{t\in T} Y_t$ we have $A \cap V \ne \emptyset$. Let us take points $t_1, t_2, \ldots, t_k \in T$, where $t_i \ne t_j$ for $i \ne j$, and points $y_1, y_2, \ldots, y_k \in D(\mathfrak{m})$ such that $\bigcap_{i=1}^{k} p_{t_i}^{-1}(y_i) \subset V$. As T is a Hausdorff space, there exists a family $\{U_1, U_2, \ldots, U_k\} \in \mathcal{T}$ such that $t_i \in U_i$ for $i = 1, 2, \ldots, k$. The function f from T to $D(\mathfrak{m})$ defined by

$$f(t) = \begin{cases} y_i, & \text{if } t \in U_i \text{ for } i = 1, 2, \ldots, k, \\ y_1, & \text{if } t \in T \setminus (U_1 \cup U_2 \cup \ldots \cup U_k), \end{cases}$$

belongs to both A and V, so that $A \cap V \ne \emptyset$.■

2.3.16. COROLLARY. *Separability is a $\mathfrak{c}$-multiplicative property.*■

2.3.17. THEOREM. *If $d(X_s) \le \mathfrak{m} \ge \aleph_0$ for every $s \in S$, then any family of pairwise disjoint non-empty open subsets of the Cartesian product $\prod_{s\in S} X_s$ has cardinality $\le \mathfrak{m}$.*

PROOF. Let $\{U_t\}_{t\in T}$ be a family of pairwise disjoint non-empty open subsets of $\prod_{s\in S} X_s$. Without loss of generality we can assume that $\{U_t\}_{t\in T}$ consists of members of the canonical base for $\prod_{s\in S} X_s$, i.e., that for every $t \in T$ there exist a finite set $S_t \subset S$ and a family $\{W_s^t\}_{s\in S}$, where W_s^t is an open subset of X_s and $W_s^t = X_s$ if $s \notin S_t$, such that $U_t = \prod_{s\in S} W_s^t$.

Assume that $|T| > \mathfrak{m}$; obviously, we can suppose that $|T| \le 2^{\mathfrak{m}}$. The set $S_0 = \bigcup_{t\in T} S_t$ also has cardinality $\le 2^{\mathfrak{m}}$, so that the Cartesian product $\prod_{s\in S_0} X_s$ contains, by the Hewitt-Marczewski-Pondiczery theorem, a dense subset A of cardinality $\leq \mathfrak{m}$. The family $\{\prod_{s\in S_0} W_s^t\}_{t\in T}$ consists of non-empty open subsets of $\prod_{s\in S_0} X_s$. Since $U_t = \prod_{s\in S_0} W_s^t \times \prod_{s\in S\setminus S_0} X_s$ for every $t \in T$, the members of $\{\prod_{s\in S_0} W_s^t\}_{t\in T}$ are pairwise disjoint; since every member contains an element of A, it follows that $|T| \leq |A| \leq \mathfrak{m}$, a contradiction.■

2.3.18. COROLLARY. *In the Cartesian product of separable spaces any family of pairwise disjoint non-empty open sets is countable.*■

Let us observe that, in contrast to the case of Theorem 2.3.17, the restriction of number of factors in the Hewitt-Marczewski-Pondiczery theorem is essential (see the first part of Theorem 1.5.3 and Exercise 2.3.F(c)).

Now, we shall discuss embedding of topological spaces in Cartesian products. We begin with two definitions and an important technical theorem, called the diagonal theorem.

Suppose we are given a topological space X, a family $\{Y_s\}_{s\in S}$ of topological spaces and a family of continuous mappings $\mathcal{F} = \{f_s\}_{s\in S}$, where $f_s: X \to Y_s$. We say that the family $\mathcal{F}$ *separates points* if for every pair of distinct points $x, y \in X$ there exists a mapping $f_s \in \mathcal{F}$ such that $f_s(x) \neq f_s(y)$. If for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a mapping $f_s \in \mathcal{F}$ such that $f_s(x) \notin \overline{f_s(F)}$, then we say that the family $\mathcal{F}$ *separates points and closed sets*. Let us note that if X is a T_0-space, then every family $\mathcal{F}$ separating points and closed sets separates points as well. Indeed, if $x \neq y$, then there exists an open set U containing exactly one of these points; denoting by z that one of the points x, y which does not belong to the closed set $F = X \setminus U$ we have $f_s(z) \notin \overline{f_s(F)}$ for some $f_s \in \mathcal{F}$, which obviously implies that $f_s(x) \neq f_s(y)$.

2.3.19. LEMMA. *If the continuous mapping $f: X \to Y$ is one-to-one and the one-element family $\{f\}$ separates points and closed sets, then f is a homeomorphic embedding.*

PROOF. It suffices to show that for every closed $F \subset X$ we have

$$f(F) = f(X) \cap \overline{f(F)}. \tag{2}$$

If $y = f(x) \in f(X) \setminus f(F)$, then $x \notin F$ and $y = f(x) \notin \overline{f(F)}$. Hence the right-hand side of (2) is contained in the left-hand side. The reverse inclusion is obvious.■

2.3.20. THE DIAGONAL THEOREM. *If the family $\mathcal{F} = \{f_s\}_{s\in S}$ of continuous mappings, where $f_s: X \to Y_s$, separates points, then the diagonal $f = \triangle_{s\in S} f_s: X \to \prod_{s\in S} Y_s$ is a one-to-one mapping. If, moreover, the family $\mathcal{F}$ separates points and closed sets, then f is a homeomorphic embedding.*

In particular, *if there exists an $s \in S$ such that f_s is a homeomorphic embedding, then f is a homeomorphic embedding.*

PROOF. If the family $\mathcal{F}$ separates points, then for every pair of distinct points $x, y \in X$ there exists an $f_s \in \mathcal{F}$ such that $f_s(x) \neq f_s(y)$, so that we have $f(x) \neq f(y)$ which means that f is a one-to-one mapping.

If the family $\mathcal{F}$ separates points and closed sets, then the family $\{f\}$ also does, because if $f(x) \in \overline{f(F)}$ for an $F = \overline{F} \subset X$, then $f_s(x) = p_s f(x) \in p_s(\overline{f(F)}) \subset \overline{p_s f(F)} = \overline{f_s(F)}$ for every $s \in S$. To complete the proof it suffices to apply Lemma 2.3.19.■

If $X_s = X$ for every $s \in S$, then the diagonal $i = \Delta_{s\in S}\,\mathrm{id}_{X_s}: X \to \prod_{s\in S} X_s$ is a homeomorphic embedding, and thus we have

2.3.21. COROLLARY. *The diagonal Δ of the Cartesian product $X^{\mathfrak{m}}$ is homeomorphic to X.* ∎

By the *graph of a mapping* f of a space X to a space Y, we mean the subset of the Cartesian product $X \times Y$ defined by

$$G(f) = \{(x,y) \in X \times Y : y = f(x)\}.$$

Let us observe that, according to the definition of a function given in our Introduction, $G(f)$ is the same as f. We introduce the notion of the graph and the symbol $G(f)$ for the convenience of those readers who, like most topologists, view functions as being transformations.

2.3.22. COROLLARY. *For every continuous mapping $f: X \to Y$ the graph $G(f)$ is the image of X under the homeomorphic embedding $\mathrm{id}_X \Delta f: X \to X \times Y$. The restriction $p|G(f)$ of the projection $p: X \times Y \to X$ is a homeomorphism. If Y is a Hausdorff space, then $G(f)$ is a closed subset of $X \times Y$.*

PROOF. The first part of the corollary is obvious. The second part follows from the fact that the restriction $p|G(f)$ is the inverse mapping of the homeomorphism $(\mathrm{id}_X\Delta f)|X$. The third part is a consequence of the equality $G(f) = (f \times \mathrm{id}_Y)^{-1}(\Delta)$, where Δ is the diagonal of $Y \times Y$, and of the fact that Δ is a closed subset of $Y \times Y$. ∎

We say that the space X is *universal* for all spaces having a topological property $\mathcal{P}$ if X has the property $\mathcal{P}$ and every space that has the property $\mathcal{P}$ is embeddable in X.

Let us call the reader's attention to the fact that theorems on the existence of universal spaces are interesting and useful. Indeed, they allow us to reduce the study of a class of spaces with a property $\mathcal{P}$ to the study of subspaces of a fixed topological space (cf., however, Exercise 2.3.I); this is a consequence of the fact that, as explained at the end of Section 1.4, we do not distinguish between homeomorphic spaces.

The *Tychonoff cube of weight* $\mathfrak{m} \geq \aleph_0$ is the space $I^{\mathfrak{m}}$, i.e., the Cartesian product $\prod_{s\in S} I_s$, where $I_s = I$ for every $s \in S$ and $|S| = \mathfrak{m}$. The Tychonoff cube $I^{\aleph_0}$ is called the *Hilbert cube*. Let us note that if $\mathfrak{n} \leq \mathfrak{m}$, then the cube $I^{\mathfrak{n}}$ is embeddable in $I^{\mathfrak{m}}$.

2.3.23. THEOREM. *The Tychonoff cube $I^{\mathfrak{m}}$ is universal for all Tychonoff spaces of weight $\mathfrak{m} \geq \aleph_0$.*

PROOF. In Section 1.5 we observed that the interval I is a Tychonoff space, thus by Theorem 2.3.11 the cube $I^{\mathfrak{m}}$ also is a Tychonoff space. From Theorem 2.3.13 it follows that we have $w(I^{\mathfrak{m}}) \leq \mathfrak{m}$.

Now, we shall show that every Tychonoff space X of weight $\mathfrak{m}$ is embeddable in $I^{\mathfrak{m}}$. As the discrete space $D(\mathfrak{m})$ is a Tychonoff space of weight $\mathfrak{m}$, this will prove in particular that $w(I^{\mathfrak{m}}) \geq \mathfrak{m}$, which will complete the proof of the equality $w(I^{\mathfrak{m}}) = \mathfrak{m}$.

Since X is a Tychonoff space, the family of all functionally open sets is a base for X. From Theorem 1.1.15 it follows that there exists a base $\{U_s\}_{s\in S}$ for the space X consisting of functionally open sets and such that $|S| = \mathfrak{m}$. For every $s \in S$ consider a continuous mapping $f: X \to I$ such that $U_s = f_s^{-1}((0,1])$. The family $\mathcal{F} = \{f_s\}_{s\in S}$ separates points and closed sets, and as X is a T_0-space, it follows from the diagonal theorem that the diagonal $\triangle_{s\in S} f_s$ is a homeomorphic embedding of X in $I^{\mathfrak{m}}$.■

From now on, the two-point discrete space $D(2)$ will be denoted simply by D and will be identified with the subspace $\{0,1\}$ of the real line.

The *Cantor cube of weight* $\mathfrak{m} \geq \aleph_0$ is the space $D^{\mathfrak{m}}$, i.e., the Cartesian product $\prod_{s\in S} D_s$, where $D_s = D$ for every $s \in S$ and $|S| = \mathfrak{m}$. The Cantor cube $D^{\aleph_0}$ is called the *Cantor set.* From Theorem 2.3.24 below it follows that the weight of $D^{\mathfrak{m}}$ is equal to $\mathfrak{m}$; in Section 6.2 we shall prove that the Cantor cube $D^{\mathfrak{m}}$ also is a universal space, namely for all zero-dimensional spaces of weight $\mathfrak{m} \geq \aleph_0$ (see Theoem 6.2.16).

2.3.24. THEOREM. *For every* $\mathfrak{m} \geq \aleph_0$ *and every* $x \in D^{\mathfrak{m}}$, *we have* $\chi(x, D^{\mathfrak{m}}) = \mathfrak{m}$.

PROOF. Suppose that $\chi(x, D^{\mathfrak{m}}) = \mathfrak{n} < \mathfrak{m}$ and denote by $\mathcal{B}(x)$ a base for $D^{\mathfrak{m}}$ at x such that $|\mathcal{B}(x)| = \mathfrak{n}$; as $D^{\mathfrak{m}}$ is dense in itself we have $\mathfrak{n} \geq \aleph_0$. Let us shrink every member U of $\mathcal{B}(x)$ to a neighbourhood of x having the form $\prod_{s\in S} W_s$, where $W_s \neq D_s$ only for s in a finite subset $S(U) \subset S$. The family $\mathcal{B}_0(x)$ obtained in this way also is a base at x and has cardinality $\mathfrak{n}$. The set $S_0 = \bigcup_{U\in\mathcal{B}(x)} S(U)$ has cardinality $\leq \mathfrak{n} \cdot \aleph_0 = \mathfrak{n} < \mathfrak{m}$, so that there exists an $s_0 \in S \setminus S_0$. Clearly, the neighbourhood $p_{s_0}^{-1} p_{s_0}(x)$ of x does not contain any element of $\mathcal{B}_0(x)$, which shows that $\mathcal{B}_0(x)$ is not a base at x.■

2.3.25. COROLLARY. *For every* $\mathfrak{m} \geq \aleph_0$ *and every* $x \in I^{\mathfrak{m}}$, *we have* $\chi(x, I^{\mathfrak{m}}) = \mathfrak{m}$.■

It turns out that there exists also a universal space for all T_0-spaces of weight $\mathfrak{m} \geq \aleph_0$. Let F be the topological space defined in Example 1.2.8, where $X = \{0,1\}$ and $x_0 = 1$, i.e., let $F = \{0,1\}$ with the topology consisting of the empty set, the set $\{0\}$ and the whole space; clearly F is a T_0-space.

The *Alexandroff cube of weight* $\mathfrak{m} \geq \aleph_0$ is the space $F^{\mathfrak{m}}$, i.e., the Cartesian product $\prod_{s\in S} F_s$, where $F_s = F$ for every $s \in S$ and $|S| = \mathfrak{m}$.

Applying the diagonal theorem and the fact that for every open subset U of a T_0-space X the formula

$$f(x) = \begin{cases} 0 & \text{for } x \in U, \\ 1 & \text{for } x \in X \setminus U, \end{cases}$$

defines a continuous mapping $f: X \to F$, one easily obtains the following

2.3.26. THEOREM. *The Alexandroff cube* $F^{\mathfrak{m}}$ *is universal for all* T_0*-spaces of weight* $\mathfrak{m} \geq \aleph_0$.■

The next four propositions describe relations between closedness and openess of mappings $\{f_s\}_{s\in S}$, of their Cartesian product $\prod_{s\in S} f_s$ and of their diagonal $\triangle_{s\in S} f_s$.

2.3.27. PROPOSITION. *If the Cartesian product* $f = \prod_{s\in S} f_s$, *where* $f_s: X_s \to Y_s$ *and* $X_s \neq \emptyset$ *for* $s \in S$, *is closed, then all mappings* f_s *are closed.*

PROOF. Take an $s_0 \in S$ and a non-empty closed set $F \subset X_{s_0}$. The set $p_{s_0}^{-1}(F)$ is closed in $\prod_{s\in S} X_s$ and so is its image under f in $\prod_{s\in S} Y_s$. Since $f(p_{s_0}^{-1}(F)) = \prod_{s\in S} A_s$, where $A_{s_0} = f_{s_0}(F)$ and $A_s = f_s(X_s)$ for $s \neq s_0$, it follows from 2.3.4 that $f_{s_0}(F)$ is closed in Y_{s_0}.■

2.3.28. EXAMPLE. Let $f_1 = \mathrm{id}_R$ be the identity mapping of the real line onto itself and let f_2 be the mapping of R into the one-point discrete space $\{0\}$. Both f_1 and f_2 are closed, however, the Cartesian product $f_1 \times f_2: R^2 \to R \times \{0\}$ is not closed (cf. Example 2.3.10).■

2.3.29. PROPOSITION. *The Cartesian product $f = \prod_{s\in S} f_s$, where $f_s: X_s \to Y_s$ and $X_s \neq \emptyset$ for $s \in S$, is open if and only if all mappings f_s are open and there exists a finite set $S_0 \subset S$ such that $f_s(X_s) = Y_s$ for $s \in S \setminus S_0$.*

PROOF. From 1.4.14 and the equality $f(\prod_{s\in S} W_s) = \prod_{s\in S} f_s(W_s)$ it follows that if the mappings f_s satisfy the above conditions, then f is open.

Conversely, suppose that f is an open mapping. Take an $s_0 \in S$ and a non-empty open set $U \subset X_{s_0}$. The set $U \times \prod_{s\in S\setminus\{s_0\}} X_s$ is non-empty and open in $\prod_{s\in S} X_s$, so that the set

$$p_{s_0} f(U \times \prod_{s\in S\setminus\{s_0\}} X_s) = f_{s_0}(U)$$

is open in Y_{s_0}, because the projection p_{s_0} is an open mapping. This implies that f_{s_0} is open. As $\prod_{s\in S} X_s \neq \emptyset$, the set $f(\prod_{s\in S} X_s) = \prod_{s\in S} f_s(X_s)$ is a non-empty open subset of $\prod_{s\in S} Y_s$, so that it contains a set of the form $\prod_{s\in S} W_s$, where $W_s \neq Y_s$ only for s in a finite set $S_0 \subset S$; then for $s \in S \setminus S_0$ we have $f_s(X_s) = Y_s$. ■

2.3.30. PROPOSITION. *If the mappings $f_1, f_2, \ldots, f_k$, where $f_i: X \to Y_i$, are closed, Y_1 is a T_1-space and $Y_2, Y_3, \ldots, Y_k$ are T_3-spaces, then the diagonal $f = f_1 \triangle f_2 \triangle \ldots \triangle f_k$ is closed.*

PROOF. It suffices to show that if $f_i: X \to Y_i$ is closed for $i = 1, 2$, where Y_1 is a T_1-space and Y_2 is a T_3-space, then $f = f_1 \triangle f_2$ is closed. Let us take a closed set $A \subset X$ and a point $(y_1, y_2) \in Y_1 \times Y_2 \setminus f(A)$. We have $A \cap f_1^{-1}(y_1) \cap f_2^{-1}(y_2) = \emptyset$, which implies that

$$f_2^{-1}(y_2) \subset X \setminus A \cap f_1^{-1}(y_1);$$

so that – by 1.4.13 and regularity of Y_2 – there exists a neighbourhood $V_2 \subset Y_2$ of y_2 satisfying

$$f_2^{-1}(\overline{V}_2) \subset X \setminus A \cap f_1^{-1}(y_1).$$

We then have

$$f_1^{-1}(y_1) \subset X \setminus A \cap f_2^{-1}(\overline{V}_2),$$

which implies – again by 1.4.13 – that there exists a neighbourhood $V_1 \subset Y_1$ of y_1 such that

$$f_1^{-1}(V_1) \subset X \setminus A \cap f_2^{-1}(\overline{V}_2), \quad \text{i.e., such that} \quad A \cap f_1^{-1}(V_1) \cap f_2^{-1}(\overline{V}_2) = \emptyset.$$

The last equality shows that the neighbourhood $V_1 \times V_2$ of (y_1, y_2) is disjoint from $f(A)$, and this proves that f is closed. ■

2.3.31. EXAMPLES. For $i = 1, 2$ let f_i denote the projection of the plane R^2 onto the ith axis. The diagonal $f_1 \triangle f_2$ is the identity mapping of R^2 onto itself, so it is closed, but neither f_1 nor f_2 is a closed mapping (cf. Example 2.3.10). Hence closedness of the diagonal $f_1 \triangle f_2$ does not imply that f_1 and f_2 are closed.

We shall show that Proposition 2.3.30 cannot be generalized to infinite diagonals. Let $X = N$ and $Y_i = D = \{0, 1\}$ for $i = 1, 2, \ldots$; the mappings $f_i: X \to Y_i$, where

$$f_i(j) = 1 \text{ for } j \leq i \quad \text{and} \quad f_i(j) = 0 \text{ for } j > i,$$

are closed, and yet the diagonal $f = \triangle_{i=1}^{\infty} f_i: N \to D^{\aleph_0}$ is not closed. Indeed, the set $f(N)$ contains all points of the form $(0, 0, \ldots, 0, 1, 1, \ldots)$ but does not contain the point $(0, 0, \ldots)$, so that $f(N)$ is not closed in $D^{\aleph_0}$. Further examples related to Proposition 2.3.30 are suggested in Exercise 2.3.J. ∎

2.3.32. PROPOSITION. *If the diagonal $f = \triangle_{s\in S} f_s$, where $f_s: X \to Y_s$, is open, then all mappings f_s are open.*

PROOF. This follows from the equality $f_s = p_s f$, the projections p_s being open. ∎

2.3.33. EXAMPLE. Let $f_1 = f_2 = \mathrm{id}_R: R \to R$. The image of R under the diagonal $f = f_1 \triangle f_2$ is not open in R^2, so that the diagonal of two open mappings is not necessarily an open mapping. In the above example, the restriction $f|R: R \to f(R)$ is a homeomorphism and, *a fortiori*, is open. The reader should easily find two open mappings f_1, f_2 of the interval I into itself such that the restriction $f|I$ of the diagonal $f = f_1 \triangle f_2$ is not open. ∎

From Proposition 2.3.2 it follows that the Cartesian product of homeomorphic embeddings is a homeomorphic embedding. One can readily verify that if the Cartesian product $\prod_{s\in S} f_s$, where $f_s: X_s \to Y_s$ and $X_s \neq \emptyset$ for $s \in S$, is a homeomorphic embedding, then all mappings f_s are homeomorphic embeddings. Theorem 2.3.20 gives a sufficient condition for the diagonal to be a homeomorphic embedding; a little more sophisticated necessary and sufficient condition can be found in Exercise 2.3.D.

2.3.34. PROPOSITION. *A net $T = \{x_\sigma, \sigma \in \Sigma\}$ in the Cartesian product $\prod_{s\in S} X_s$ converges to $x \in \prod_{s\in S} X_s$ if and only if the net $T_s = \{p_s(x_\sigma), \sigma \in \Sigma\}$ converges to $p_s(x)$ for every $s \in S$.*

PROOF. If $x \in \lim T$, then – by Proposition 1.6.6 – $p_s(x) \in \lim T_s$ for every $s \in S$.

Conversely, assume that for a point $x \in \prod_{s\in S} X_s$ we have $p_s(x) \in \lim T_s$ for every $s \in S$. For every neighbourhood U of x there exist elements $s_1, s_2, \ldots, s_k \in S$ and open sets $U_i \subset X_{s_i}$, where $i = 1, 2, \ldots, k$, such that $x \in p_{s_1}^{-1}(U_1) \cap p_{s_2}^{-1}(U_2) \cap \ldots \cap p_{s_k}^{-1}(U_k) \subset U$. By the assumption, there exists $\sigma_1, \sigma_2, \ldots, \sigma_k \in \Sigma$ such that $p_{s_i}(x_\sigma) \in U_i$ for every $\sigma \geq \sigma_i$ and $i = 1, 2, \ldots, k$. As the set Σ is directed, there exists a $\sigma_0 \in \Sigma$ such that $\sigma_i \leq \sigma_0$ for $i = 1, 2, \ldots, k$; for every $\sigma \geq \sigma_0$ we have

$$x_\sigma \in p_{s_1}^{-1}(U_1) \cap p_{s_2}^{-1}(U_2) \cap \ldots \cap p_{s_k}^{-1}(U_k) \subset U,$$

so that $x \in \lim T$. ∎

The next proposition is the filter counterpart of the above.

2.3.35. PROPOSITION. *If $\mathcal{F}$ is a filter in the Cartesian product $\prod_{s\in S} X_s$, then for every $s \in S$ the family $\mathcal{F}_s = \{p_s(F): F \in \mathcal{F}\}$ is a filter in X_s; the filter $\mathcal{F}$ converges to $x \in \prod_{s\in S} X_s$ if and only if the filter $\mathcal{F}_s$ converges to $p_s(x)$ for every $s \in S$.* ∎

We close this section with two important examples in the constructions of which Cartesian products are applied in a crucial way.

2.3.36. EXAMPLE. We shall show that normality is not a hereditary property (cf. Exercise 2.1.E(b)). Let $X = A(\aleph_0)$, $Y = A(\mathfrak{c})$, and let x_0 and y_0 be the accumulation points of X and Y respectively (see Example 1.4.20).

The Cartesian product $X \times Y$ is a normal space. This follows from the fact that for every pair of disjoint closed subsets of $X \times Y$ there exists an open-and-closed set $V \times W \subset X \times Y$ containing (x_0, y_0) and disjoint from at least one of the closed sets, and from normality of the subspace $X \times Y \setminus (V \times W)$, which in its turn follows from 2.2.4 and 2.2.7. Let us note that the normality of $X \times Y$ follows also from Theorems 3.1.9 and 3.2.4 below (cf. Example 3.2.7).

We shall show that the subspace $Z = X \times Y \setminus \{(x_0, y_0)\}$ of the space $X \times Y$ is not normal. Since the sets $A = (X \setminus \{x_0\}) \times \{y_0\}$ and $B = \{x_0\} \times (Y \setminus \{y_0\})$ are disjoint and closed in Z, it suffices to show that for each pair of open sets $U, V \subset Z$ such that $A \subset U$ and $B \subset V$ we have $U \cap V \neq \emptyset$. To begin, let us note that for every $x \in X \setminus \{x_0\}$ the point (x, y_0) belongs to U, so that there exists a finite set $F(x) \subset Y \setminus \{y_0\}$ such that $\{x\} \times (Y \setminus F(x)) \subset U$. The set $C = \bigcup\{F(x) : x \in X \setminus \{x_0\}\}$ is countable, and thus there exists a point $y \in Y \setminus (C \cup \{y_0\})$; clearly $(X \setminus \{x_0\}) \times \{y\} \subset U$. Now, the point $(x_0, y) \in B$ belongs both to the closure of the set $(X \setminus \{x_0\}) \times \{y\}$ and to V, so that $U \cap V \neq \emptyset$. ■

2.3.37. EXAMPLE. In Section 1.6 we described two countable spaces that are not first-countable; both of them have character $\leq \aleph_0$ at all but one point. Now we are going to define a countable space with character $> \aleph_0$ at all points (cf. Exercise 2.3.M).

From 2.3.16 it follows that the Cantor cube $D^{\mathfrak{c}} = \prod_{s \in S} D_s$, where $D_s = D$ for every $s \in S$ and $|S| = \mathfrak{c}$, contains a dense countable subspace X; we shall prove that $\chi(x, X) > \aleph_0$ for every $x \in X$ (let us note that Theorem 2.3.24 and Exercise 2.1.C(a) yield even more, viz., that $\chi(x, X) = \mathfrak{c}$). If suffices to show that for every sequence $U_1, U_2, \ldots$ of neighbourhoods of x in X there exists a neighbourhood $U \subset X$ of x such that

$$U_i \setminus U \neq \emptyset \quad \text{for} \quad i = 1, 2, \ldots \tag{3}$$

For $i = 1, 2, \ldots$ there exists a finite set $S_i \subset S$ and a family $\{W_s^i\}_{s \in S}$ of non-empty subsets of D such that $W_s^i = D$ for $s \in S \setminus S_i$ and

$$X \cap \prod_{s \in S} W_s^i \subset U_i. \tag{4}$$

Let us take an $s_0 \in S \setminus \bigcup_{i=1}^{\infty} S_i$ and consider the neighbourhood $U = X \cap p_{s_0}^{-1} p_{s_0}(x)$ of x. The set $(D^{\mathfrak{c}} \setminus p_{s_0}^{-1} p_{s_0}(x)) \cap \prod_{s \in S} W_s^i$ is non-empty and open in $D^{\mathfrak{c}}$ for $i = 1, 2, \ldots$, so that – by the density of X in $D^{\mathfrak{c}}$ – we have $(X \setminus U) \cap \prod_{s \in S} W_s^i \neq \emptyset$ and this together with (4) gives (3). ■

Historical and bibliographic notes

Compared to other operations on topological spaces, the operation of the Cartesian product leads to the most interesting theorems, examples and problems. Cartesian products (of finitely many manifolds) appeared for the first time in Steinitz's paper [1908]. Fréchet in [1910] was the first to discuss (finite) Cartesian products of abstract spaces. Finite and countably infinite Cartesian products of metric spaces belonged to the topological folklore of the twenties. The Cartesian product of arbitrarily many topological spaces was defined by Tychonoff in [1930]. Theorem 2.3.15 was proved by Pondiczery in [1944], Hewitt in [1946], and for $\mathfrak{m} = \aleph_0$ also by Marczewski in [1947]. Corollary 2.3.18 was established by

Marczewski in [1947] (under the stronger assumption that the spaces are second-countable in [1941]). Theorem 2.3.17 is a consequence of a stronger result of Šanin's formulated here in Problem 2.7.11(b). Theorem 2.3.23 was proved by Tychonoff in [1930] and Theorem 2.3.26 by Alexandroff in [1936]; related results are to be found in Exercise 2.3.I and Problems 2.7.7, 2.7.8 and 2.7.18(b). Example 2.3.12 was given by Sorgenfrey in [1947]. Example 2.3.36 combines ideas of Tychonoff [1930] and Novák [1948].

Exercises

2.3.A. Show that for any two families $\{X_s\}_{s\in S}$ and $\{Y_t\}_{t\in T}$ of topological spaces, the spaces $(\bigoplus_{s\in S} X_s) \times (\bigoplus_{t\in T} Y_t)$ and $\bigoplus_{s\in S, t\in T}(X_s \times Y_t)$ are homeomorphic. Generalize this result to an arbitrary collection of families of topological spaces.

2.3.B. (a) Verify that for any $A \subset X$ and $B \subset Y$ in the Cartesian product $X \times Y$ we have

$$\operatorname{Int}(A \times B) = \operatorname{Int} A \times \operatorname{Int} B \quad \text{and} \quad \operatorname{Fr}(A \times B) = (\overline{A} \times \operatorname{Fr} B) \cup (\operatorname{Fr} A \times \overline{B}).$$

(b) Prove that if A_s is an F_σ-set (a G_δ-set) in X_s for every $s \in S$ and $|S| < \aleph_0$ (and $|S| \leq \aleph_0$), then $\prod_{s\in S} A_s$ is an F_σ-set (a G_δ-set) in the Cartesian product $\prod_{s\in S} X_s$. Note that in both cases the assumption about the cardinality of S is essential.

(c) Prove that if A_i is an open (a closed) domain in X_i for $i = 1, 2, \ldots, k$, then $A_1 \times A_2 \times \ldots \times A_k$ is an open (a closed) domain in the Cartesian product $X_1 \times X_2 \times \ldots \times X_k$. Note that the assumption of finiteness of the Cartesian product is essential.

2.3.C. (a) Show that X is a Hausdorff space if and only if the diagonal Δ of the Cartesian product $X \times X$ is closed in $X \times X$ (cf. Exercise 2.4.C(c)).

Verify that the diagonal Δ of the Cartesian product $X \times X$ is open in $X \times X$ if and only if the space X is discrete (cf. Exercise 2.4.C(b)).

(b) Give an example of a discontinuous function f of R into itself such that the graph $G(f)$ is a closed subspace of the plane R^2 (cf. Corollary 2.3.22 and Exercise 3.1.D(a)).

2.3.D. Prove that the diagonal $f = \triangle_{s\in S} f_s$ of the family $\{f_s\}_{s\in S}$ of continuous mappings, where $f_s\colon X \to Y_s$, is a homeomorphic embedding if and only if the family $\{f_s\}_{s\in S}$ separates points and the family $\{f_T\}_{T\in\mathcal{T}}$, where $\mathcal{T}$ is the family of all finite subsets of S and $f_T = \triangle_{s\in T} f_s$, separates points and closed sets.

2.3.E. (a) (A. H. Stone [1948]) Prove that the Cartesian product $N^{\aleph_1}$ is not normal (cf. Exercise 3.1.H(a) and Problem 2.7.12(c)).

Hint. Let $N^{\aleph_1} = \prod_{s\in S} N_s$, where $N_s = N$ and $|S| = \aleph_1$; for each $S_0 \subset S$ denote by p_{S_0} the projection of $\prod_{s\in S} N_s$ onto $\prod_{s\in S_0} N_s$. Show that the sets $A_1, A_2 \subset \prod_{s\in S} N_s$, where A_i consists of all points $\{j_s\}$ such that for every $j \neq i$ the equality $j_s = j$ holds for at most one $s \in S$, are closed and disjoint. Consider an open set U containing A_1, the point $x_1 \in A_1$ all of whose coordinates are equal to 1, and find a finite set $S_1 = \{s_1, s_2, \ldots, s_{n_1}\} \subset S$ such that $p_{S_1}^{-1}p_{S_1}(x_1) \subset U$; define a point $x_2 \in A_1$ by letting $p_{s_n}(x_2) = n$ for $n = 1, 2, \ldots, n_1$ and $p_s(x_2) = 1$ for $s \notin S_1$, and enlarge S_1 to a finite set $S_2 = \{s_1, s_2, \ldots, s_{n_1}, s_{n_1+1}, \ldots, s_{n_2}\} \subset S$ such that $p_{S_2}^{-1}p_{S_2}(x_2) \subset U$; apply induction to obtain points $x_1, x_2, x_3, \ldots$ and finite sets $S_1 \subset S_2 \subset S_3 \subset \ldots$ Then show that the set $\{y_1, y_2, y_3, \ldots\}$, where $p_s(y_i) = p_s(x_i)$ for $s \in S_i$

and $p_s(y_i) = 2$ for $s \notin S_i$, is contained in U and has an accumulation point in A_2, so that $\overline{U} \cap A_2 \neq \emptyset$.

(b) (Pospíšil [1937b]) Show that no Cartesian product of uncountably many spaces of cardinality > 1 is hereditarily normal.

2.3.F. (a) Prove that the weight of an infinite Cartesian product $X = \prod_{s\in S} X_s$, where $w(X_s) > 1$, is equal to the larger of the two cardinal numbers $|S|$ and $\sup_{s\in S} w(X_s)$.

(b) Prove that for a point $x = \{x_s\}$ of an infinite Cartesian product $X = \prod_{s\in S} X_s$, where X_s is a T_1-space and $|X_s| > 1$, the character $\chi(x, X)$ is equal to the larger of the two cardinal numbers $|S|$ and $\sup_{s\in S} \chi(x_s, X_s)$.

(c) (Pondiczery [1944], Marczewski [1947]) Prove that the Cartesian product $X = \prod_{s\in S} X_s$, where X_s is a T_2-space and $|X_s| > 1$, is not separable if $|S| > \mathfrak{c}$.

(d) Give an example of a T_1-space X with $|X| > 1$ and $X^{\mathfrak{m}}$ separable for every $\mathfrak{m}$.

2.3.G. Show that if $d(X_s) \le \mathfrak{m} \ge \aleph_0$ for every $s \in S$, then each dense open subset U of the Cartesian product $\prod_{s\in S} X_s$ contains a set of the form $U_0 \times \prod_{s\in S\setminus S_0} X_s$, where $|S_0| \le \mathfrak{m}$ and U_0 is a dense open subset of $\prod_{s\in S_0} X_s$.

Hint. Consider a maximal family of disjoint members of the canonical base for $\prod_{s\in S} X_s$ that are contained in U and apply Theorem 2.3.17.

2.3.H. Show that the space X is homeomorphic to the Cartesian product $\prod_{s\in S} X_s$ if and only if there exists a family of continuous mappings $\{p_s\}_{s\in S}$, where $p_s\colon X \to X_s$, satisfying the following conditions:

(1) *For every space Y and a pair f, g of continuous mappings of Y to X, if $p_s f = p_s g$ for every $s \in S$, then $f = g$.*

(2) *For every space Y and a family of continuous mappings $\{f_s\}_{s\in S}$, where $f_s\colon Y \to X_s$, there exists a continuous mappings $f\colon Y \to X$ such that $p_s f = f_s$ for every $s \in S$.*

2.3.I. Let E be the topological space defined in Example 1.2.10, where $X = \{0, 1, 2\}$ and $X_0 = \{0\}$, i.e., let $E = \{0, 1, 2\}$ with the topology consisting of the empty set, the set $\{0\}$ and the whole space. Show that the space $E^{\mathfrak{m}}$ is universal for all topological spaces of weight $\mathfrak{m} \ge \aleph_0$ and cardinality $\le 2^{\mathfrak{m}}$. Note that every topological space is homeomorphic to a subspace of a power of E.

2.3.J. (a) Let X be a non-normal regular space and let A, B be a pair of disjoint closed subsets of X that cannot be separated by disjoint open sets; denote by X_1 and X_2 the spaces obtained from X by identifying A and B, respectively, to a point (see Examples 1.4.17 and 2.4.12) and let f_i be the natural mapping of X onto X_i for $i = 1, 2$. Verify that X_1 and X_2 are T_2-spaces and that f_1 and f_2 are closed mappings, but the diagonal $f_1 \triangle f_2$ is not closed; conclude that even for $k = 2$ the assumption of regularity in Proposition 2.3.30 cannot be weakened to the assumption that $Y_2, Y_3, \ldots, Y_k$ are Hausdorff spaces.

(b) Let $Y_1 = F = \{0, 1\}$ with the topology consisting of the empty set, the set $\{0\}$ and the whole space, and let $Y_2 = A(\aleph_0)$ with the unique accumulation point y_0. Consider $X = Y_1 \times Y_2 \setminus \{(0, y_0)\}$ and $f_i = p_i | X\colon X \to Y_i$, where p_i is the projection of $Y_1 \times Y_2$ onto Y_i for $i = 1, 2$, and verify that both f_1 and f_2 are closed mappings, but the diagonal $f_1 \triangle f_2$ is not closed; conclude that in Proposition 2.3.30 the assumption that Y_1 is a T_1-space cannot be weakened to the assumption that Y_1 is a T_0-space.

(c) Modifying the example in (a), define closed mappings f_1 and f_2 of a Tychonoff space X to Hausdorff spaces Y_1 and Y_2, respectively, such that the restriction $f|X: X \to f(X)$ of the diagonal $f = f_1 \triangle f_2$ is not closed. Improve (b) and the second part of 2.3.31 in a similar way.

2.3.K (Franklin [1965] and [1967]). (a) Verify that the Cartesian product $X \times X$, where X is the Fréchet space in Exercise 1.6.E, is not a sequential space (cf. Exercise 2.4.G(c)).

Hint. Apply Exercise 2.3.C(a).

(b) Verify that the Cartesian product $X \times Y$, where $X = I$ and Y is the Fréchet space in Example 1.6.18, is not a Fréchet space (from Exercise 3.3.J it follows that this Cartesian product is a sequential space; cf. Exercise 3.10.J(d) and the remark to Exercise 3.3.J).

2.3.L. (a) Show that if a topological property $\mathcal{P}$ is hereditary with respect to both closed subsets and open subsets and is countably multiplicative, then, in the class of Hausdorff spaces, $\mathcal{P}$ is hereditary with respect to G_δ-sets.

(b) (van der Slot [1966]) Show that if a topological property $\mathcal{P}$ is hereditary with respect to both closed subsets and open subsets and is multiplicative, then if the closed interval I has $\mathcal{P}$, all Tychonoff spaces have $\mathcal{P}$.

2.3.M. Let X be a countable space with no countable base at the point x_0. Show that the subspace of $X^{\aleph_0}$ consisting of all points $\{x_i\}$ such that $x_i = x_0$ for every i larger than some integer j is countable and yet has no countable base at any of its points.

2.4. Quotient spaces and quotient mappings

Suppose we are given a topological space X and an equivalence relation E on the set X. Let us denote by X/E the set of all equivalence classes of E and by q the mapping of X to X/E assigning to the point $x \in X$ the equivalence class $[x] \in X/E$. Now, in looking for a good topology on X/E, it is reasonable to require q to be continuous. It turns out that in the class of all topologies on X/E that make q continuous there exists the finest one: this is the family of all sets U such that $q^{-1}(U)$ is open in X. This topology is called the *quotient topology*, the set X/E equipped with it is called the *quotient space*, and $q: X \to X/E$ is called the *natural quotient mapping*, or briefly, the *natural mapping*.

2.4.1. PROPOSITION. *A set $A \subset X/E$ is closed in the quotient space if and only if $q^{-1}(A)$ is a closed subset of X.*

PROOF. The proposition follows from the equality $q^{-1}(X/E \setminus A) = X \setminus q^{-1}(A)$. ■

2.4.2. PROPOSITION. *A mapping f of a quotient space X/E to a topological space Y is continuous if and only if the composition fq is continuous.*

PROOF. If f is continuous, then fq is continuous, too. Conversely, if fq is continuous, then for every open set $U \subset Y$ the set $(fq)^{-1}(U) = q^{-1}f^{-1}(U)$ is open in X, which means that $f^{-1}(U)$ is open in X/E. ■

Suppose we are given topological spaces X and Y and a continuous mapping f of X onto Y. Consider the equivalence relation $E(f)$ on the set X determined by the decomposition

$\{f^{-1}(y)\}_{y\in Y}$ of X into the fibers of f. The mapping $f: X \to Y$ can be represented as the composition $\overline{f}q$, where $q: X \to X/E(f)$ is the natural mapping and $\overline{f}$ is the mapping of the quotient space $X/E(f)$ onto Y defined by letting $\overline{f}(f^{-1}(y)) = y$; by 2.4.2 the mapping $\overline{f}$ is continuous. The following diagram illustrates the situation

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & X/E(f) \arrow[ur, leftarrow, "\overline{f}"'] & \end{tikzcd}$$

Clearly, $\overline{f}$ is a one-to-one continuous mapping of $X/E(f)$ onto Y, but generally it need not be a homeomorphism. Indeed, if f is a one-to-one mapping of the discrete space X of cardinality $\mathfrak{c}$ onto the interval I, the quotient space $X/E(f)$ is discrete, so that $\overline{f}$ cannot be a homeomorphism.

Now, we are going to study the class of all mappings f such that $\overline{f}$ is a homeomorphism. It turns out that those mappings, as far as mappings onto are considered, constitute a common generalization of closed and open mappings.

A continuous mapping $f: X \to Y$ of X onto Y is called a *quotient mapping* if it is a composition of a natural quotient mapping and a homeomorphism, i.e., if there exist an equivalence relation E on the set X and a homeomorphism $f': X/E \to Y$ such that $f = f'q$, where $q: X \to X/E$ is the natural mapping.

2.4.3. PROPOSITION. *For a mapping f of a topological space X onto a topological space Y the following conditions are equivalent:*

(i) *The mapping f is quotient.*

(ii) *The set $f^{-1}(U)$ is open in X if and only if U is open in Y.*

(iii) *The set $f^{-1}(F)$ is closed in X if and only if F is closed in Y.*

(iv) *The mapping $\overline{f}: X/E(f) \to Y$ is a homeomorphism.*

PROOF. Suppose that f is a quotient mapping, i.e., that $f = f'q$, where $f': X/E \to Y$ is a homeomorphism and $q: X \to X/E$ the natural mapping. By the definition of the quotient topology, the set $f^{-1}(U) = q^{-1}f'^{-1}(U)$ is open in X if and only if $f'^{-1}(U)$ is open in X/E; as f' is a homeomorphism, the latter holds if and only if U is open in Y. Hence we have proved that (i)$\Rightarrow$(ii).

The implication (ii)$\Rightarrow$(iii) follows directly from the equality $f^{-1}(F) = X \setminus f^{-1}(Y \setminus F)$.

Now assume that f satisfies (iii). As the mapping $\overline{f}: X/E(f) \to Y$ is one-to-one and onto, to prove (iv) it suffices to show (see Proposition 1.4.18) that for every closed $F \subset X/E(f)$, the set $\overline{f}(F)$ is closed in Y. However, since $f^{-1}\overline{f}(F) = q^{-1}\overline{f}^{-1}\overline{f}(F) = q^{-1}(F)$ is closed in X, the set $\overline{f}(F)$ is closed in Y by virtue of (iii).

The implication (iv)$\Rightarrow$(i) is obvious.■

2.4.4. COROLLARY. *The composition of two quotient mappings is a quotient mapping.*■

2.4.5. COROLLARY. *If the composition gf of continuous mappings $f: X \to Y$ and $g: Y \to Z$ is quotient, then $g: Y \to Z$ is a quotient mapping.*

PROOF. Clearly $g(Y) = Z$, because $(gf)(X) = Z$. If the inverse image $g^{-1}(U)$ of $U \subset Z$ is open in Y, then $f^{-1}(g^{-1}(U)) = (gf)^{-1}(U)$ is open in X, and U is open in Z, because gf is a quotient mapping. ■

One can easily check that if the composition gf is quotient, the mapping f – even if it is an onto mapping – need not be quotient.

2.4.6. COROLLARY. *If for a continuous mapping $f: X \to Y$ there exists a set $A \subset X$ such that $f(A) = Y$ and the restriction $f|A$ is quotient, then f is a quotient mapping.*

PROOF. This follows from 2.4.5 and the equality $f|A = fi_A$. ■

2.4.7. COROLLARY. *Every one-to-one quotient mapping is a homeomorphism.* ■

2.4.8. COROLLARY. *Closed mappings onto and open mappings onto are quotient mappings.*

PROOF. If $f: X \to Y$ is an onto mapping, then $ff^{-1}(B) = B$ for every $B \subset Y$. ■

In connection with the last corollary the question arises how to characterize those equivalence relations for which the natural quotient mapping is closed or open. The next proposition gives the answer.

2.4.9. PROPOSITION. *For an equivalence relation E on a topological space X the following conditions are equivalent:*

(i) *The natural mapping $q: X \to X/E$ is closed (open).*

(ii) *For every closed (open) set $A \subset X$ the union of all equivalence classes that meet A is closed (open) in X.*

(iii) *For every open (closed) set $A \subset X$ the union of all equivalence classes that are contained in A is open (closed) in X.*

PROOF. The equivalence of (i) and (ii) follows from Proposition 2.4.1 and from the definition of the quotient topology; the equivalence of (ii) and (iii) is an immediate consequence of De Morgan's laws. ■

2.4.10. COROLLARY. *The quotient mapping $f: X \to Y$ is closed (open) if and only if the set $f^{-1}f(A) \subset X$ is closed (open) for every closed (open) $A \subset X$.* ■

We say that an equivalence relation E on a space X is *closed (open)* if the natural mapping $q: X \to X/E$ is closed (open). Conditions (ii) and (iii) in Proposition 2.4.9 internally characterize closed and open equivalence relations. From condition (ii) it follows that the equivalence classes of a closed equivalence relation on a T_1-space are closed.

As we know, there is one-to-one correspondence between equivalence relations on a set and decompositions of this set into disjoint sets. Sometimes it is more convenient to use decompositions rather than equivalence relations; decompositions of a topological space that correspond to closed (open) equivalence relations are called *upper (lower) semicontinuous* (cf. Problem 1.7.17(a) for the origin of this terminology). In this context the word *identification* also is often used, mainly with the respect to upper semicotinuous decompositions: we say that the quotient space X/E, where E is the equivalence relation corresponding to the decomposition $\mathcal{E}$, is obtained by identifying each element of $\mathcal{E}$ to a point.

2.4.11. EXAMPLE. Let the mapping $f: R \to S^1$ be defined by $f(x) = (\cos 2\pi x, \sin 2\pi x)$; obviously for $x, y \in R$ we have $xE(f)y$ if and only if the difference $x - y$ is an integer. The effect of the mapping f is to wrap the real line around the circle in such a manner that every interval $(x, y]$ of length 1 is wrapped around the whole circle in a one-to-one way. Clearly, f transforms open intervals (x, y) with $y - x < 1$ onto open arcs of S^1, so that f is an open mapping and, *a fortiori*, a quotient mapping. It follows that the quotient space $R/E(f)$ is homeomorphic to S^1. One easily verifies that f is not closed.

The restriction $g = f|I: I \to S^1$ is – as can be readily checked – a closed mapping (but is not open); in particular, g is a quotient mapping and the quotient space $I/E(g)$ is homeomorphic to S^1. The decomposition of I corresponding to $E(g)$ consists of all one-point sets x for $0 < x < 1$ and the set $\{0, 1\}$; thus the quotient space S^1 is obtained by identifying the end-points of the interval I to a point. The reader can easily verify that the Cartesian product $g \times g: I^2 \to S^1 \times S^1$ also is a quotient mapping, and can determine which identifications one has to make in the square to obtain the torus.■

Using the language of quotient spaces, we shall now describe precisely the operation of identifying closed sets to points which was applied above in construction of a few examples (cf. Example 1.4.17, the remark preceding Example 1.5.21 and Exercise 2.3.J(a)).

2.4.12. EXAMPLE. Suppose we are given a topological space X and a sequence $A_1, A_2, \ldots, A_k$ of pairwise disjoint closed subsets of X. Denote by E the equivalence relation on X corresponding to the decomposition of X into the sets $A_1, A_2, \ldots, A_k$ and one-point sets $\{x\}$, where $x \in X \setminus (A_1 \cup A_2 \cup \ldots \cup A_k)$. For every closed set $A \subset X$ the union of all equivalence classes that meet A is equal to the union of A and those A_i's that satisfy $A_i \cap A \neq \emptyset$; this is a closed set, and the natural mapping $q: X \to X/E$ is closed by 2.4.9. The quotient space X/E is obtained from X by identifying each of the sets A_i to the point $q(A_i)$; one can readily verify that the subspaces $X \setminus (A_1 \cup A_2 \cup \ldots \cup A_k)$ and $X/E \setminus \{q(A_1), q(A_2), \ldots, q(A_k)\}$ are homeomorphic (cf. Proposition 2.1.4). The space obtained by identifying to a point a single closed subset A of a space X is denoted by X/A.■

Now we are going to define an important particular case of the quotient space operation, viz., the adjunction space. Suppose we are given two disjoint topological spaces X and Y and a continuous mapping $f: M \to Y$ defined on a closed subset M of the space X. Denote by E the equivalence relation on the sum $X \oplus Y$ corresponding to the decomposition of $X \oplus Y$ into one-point sets $\{x\}$, where $x \in X \setminus M$, and sets of the form $\{y\} \cup f^{-1}(y)$, where $y \in Y$ (if $y \in Y \setminus f(M)$ the latter set is the one-point set $\{y\}$). The quotient space $(X \oplus Y)/E$ is called the *adjunction space* determined by X, Y and f and is denoted by $Y \cup_f Y$. One readily sees that if Y is a one-point space, then $X \cup_f Y$ is homeomorphic to X/M.

Let $i_X: X \to X \oplus Y$ and $i_Y: Y \to X \oplus Y$ be the embeddings of the subspaces X and Y in $X \oplus Y$ and let $g: X \oplus Y \to X \cup_f Y$ be the natural quotient mapping. The compositions

$$j = qi_X: X \to X \cup_f Y \quad \text{and} \quad k = qi_Y: Y \to X \cup_f Y$$

are continuous and a set $C \subset X \cup_f Y$ is open (closed) if and only if $j^{-1}(C)$ and $k^{-1}(C)$ are open (closed) in X and Y respectively. One can easily verify that

(1) $$j^{-1}k(B) = f^{-1}(B) \quad \text{and} \quad k^{-1}k(B) = B \quad \text{for} \quad B \subset Y,$$

(2) $$j^{-1}j(A) = A \cup f^{-1}f(A \cap M) \quad \text{and} \quad k^{-1}j(A) = f(A \cap M) \quad \text{for} \quad A \subset X.$$

Formulas (1) imply that $k: Y \to X \cup_f Y$ is a closed mapping. Since k is a one-to-one mapping it is a homeomorphic embedding; the image $k(Y)$ is closed in $X \cup_f Y$. Similarly, formulas (2) imply that $j|X \setminus M$ is an open mapping. Since $j|X \setminus M$ is a one-to-one mapping it is a homeomorphic embedding, the image $j(X \setminus M)$ is open in $X \cup_f Y$ and is the complement of $k(Y)$. From formulas (2) it follows also that j is a closed mapping if and only if the mapping f is closed. Thus, the quotient mapping $q: X \oplus Y \to X \cup_f Y$ is closed if and only if the mapping f is closed.

If in the situation described above $f(M) = Y$, then $j(X) = X \cup_f Y$, so that – since the closedness of f implies that of j – we have

2.4.13. THEOREM. *If M is a closed subspace of X and $\mathcal{E}$ is an upper semicontinuous decomposition of M, then the decomposition of X into elements of $\mathcal{E}$ and one-point sets $\{x\}$ with $x \in X \setminus M$ is upper semicontinuous.* ∎

In contrast to the operations discussed in Sections 2.1–2.3, the operation of forming a quotient space preserves rather few topological properties. Clearly, the topological properties that are preserved by the operation of forming a quotient space are exactly those which are invariant under quotient mappings. As both closed mappings onto and open mappings onto are quotient, invariants of quotient mappings are invariants of both closed mappings and open mappings. Hence, among topological properties whose invariance was discussed in Sections 1.4 and 1.5, only the property of having density $\leq \mathfrak{m}$ could be an invariant of quotient mappings; as we know (see Theorem 1.4.10) this property is even an invariant of all continuous mappings. In order to obtain a quotient space X/E with nice topological properties, one usually makes additional assumptions about the equivalence classes of E or about their mutual relationship; such assumptions often can be expressed as additional properties of the natural quotient mapping corresponding to E. For instance, one readily sees that the quotient space X/E is a T_1-space if and only if the equivalence classes are closed in X, and from Theorem 1.5.20 it follows that if X is normal and the equivalence relation E on X is closed, then the quotient space X/E also is normal.

2.4.14. PROPOSITION. *A quotient space of a quotient space of X is a quotient space of X.*

More precisely, *if E is an equivalence relation on the space X and E' is an equivalence relation on the quotient space X/E, then the mapping $\overline{q'q}: X/E(q'q) \to (X/E)/E'$, where $q: X \to X/E$ and $q': X/E \to (X/E)/E'$ are natural quotient mappings, is a homeomorphism.*

PROOF. By 2.4.4 the composition $q'q$ is a quotient mapping, so that $\overline{q'q}$ is a homeomorphism by the equivalence of (i) and (iv) in Proposition 2.4.3. ∎

Suppose we are given a topological space X and an equivalence relation E on X; on any subspace A of the space X the equivalence relation $E|A = (A \times A) \cap E$, the *restriction* of E to A, is defined and $E|A = E(q|A)$, where $q: X \to X/E$ is the natural mapping. The question arises if the mapping $\overline{q|A}: A/(E|A) \to q(A) \subset X/E$ is a homeomorphism. Clearly, this question is equivalent to the question whether the restriction $f|A: A \to f(A)$ of a quotient mapping f is a quotient mapping. As shown in Examples 2.4.16 and 2.4.17 below, the answer is generally negative; let us note however that from Proposition 2.4.3 we immediately obtain the following positive result (cf. Exercise 2.4.F):

2.4.15. PROPOSITION. *If $f: X \to Y$ is a quotient mapping, then for any set $B \subset Y$ which is either closed or open, the restriction $f_B: f^{-1}(B) \to B$ is a quotient mapping.*

In other words, *if E is an equivalence relation on a space X, then for any $A \subset X$ which is either open or closed and satisfies the condition $q^{-1}q(A) = A$, where $q: X \to X/E$ is the natural mapping, the mapping $\overline{q|A}: A/(E|A) \to q(A) \subset X/E$ is a homeomorphism.* ∎

2.4.16. EXAMPLE. Let $g: I \to S^1$ be the quotient mapping defined in 2.4.11; the restriction $g|A: A \to g(A) = S^1$ to the open subset $A = (0,1]$ of I is one-to-one but is not a homeomorphism, so that $g|A$ is not a quotient mapping. Moreover, the combination $h = \mathrm{id}_{S^1} \nabla (g|A): S^1 \oplus A \to S^1$ is a quotient mapping by Corollary 2.4.6, because $h|S^1 = \mathrm{id}_{S^1}$, and yet the restriction $h|A = g|A$ to the open-and-closed subset A of $S^1 \oplus A$ is not a quotient mapping. ∎

2.4.17. EXAMPLE. Let $X = (0,1/2] \cup \{1, 1+1/2, 1+1/3, \ldots\}$ with the topology of a subspace of R and let E be the equivalence relation on X defined by letting xEy whenever $|x-y| = 1$ or $x = y$. One easily observes that the set $A = \{1\} \cup [(0,1/2] \setminus \{1/2, 1/3, \ldots\}] \subset X$ is of the form $q^{-1}(B)$ for $B = q(A) \subset X/E$ and that the restriction $q_B: A \to q(A)$ is a one-to-one mapping. Since the point 1 is isolated in A and $q_B(1)$ is not isolated in $q(A)$, the restriction q_B is not a homeomorphism, so that q_B is not a quotient mapping.

The reader can easily verify that if one restricts oneself to a suitable countable subspace of X, one obtains as the quotient space the space in Example 1.6.19 (cf. Exercise 2.4.G(a)). ∎

2.4.18. PROPOSITION. *Suppose we are given a topological space X, a cover $\{A_s\}_{s\in S}$ of the space X and a family $\{f_s\}_{s\in S}$ of compatible mappings, where $f_s: A_s \to Y$, such that the combination $f = \nabla_{s\in S} f_s: X \to Y$ is continuous. If there exists a set $S_0 \subset S$ such that the restrictions $f_s|A_s: A_s \to f_s(A_s)$ are quotient for $s \in S_0$ and $\{f_s(A_s)\}_{s\in S_0}$ is either an open cover of Y or a locally finite closed cover of Y, then the combination f is a quotient mapping.*

PROOF. Suppose that $\{f_s(A_s)\}_{s\in S_0}$ is an open cover of Y and that the inverse image $f^{-1}(U)$ of a set $U \subset Y$ is open in X. As the set $A_s \cap f^{-1}(U) = (f_s|A_s)^{-1}(U \cap f_s(A_s))$ is open in A_s, for every $s \in S_0$ the set $U \cap f_s(A_s)$ is open in $f_s(A_s)$ and also in Y. Hence the union $\bigcup_{s\in S_0} U \cap f_s(A_s) = U \cap \bigcup_{s\in S_0} f_s(A_s) = U \cap Y = U$ is open in Y. Similarly one proves that if $\{f_s(A_s)\}_{s\in S_0}$ is a locally finite closed cover of Y, then from the closedness of $f^{-1}(F)$ in X the closedness of F in Y follows. In both cases the fact that f is a quotient mapping is a consequence of Proposition 2.4.3. ∎

Now, suppose we are given a family $\{X_s\}_{s\in S}$ of topological spaces and for every $s \in S$ an equivalence relation E_s on X_s. Letting $\{x_s\}E\{y_s\}$ if and only if $x_s E_s y_s$ for every $s \in S$ we define an equivalence relation E on the Cartesian product $\prod_{s\in S} X_s$; this relation is called the *Cartesian product of the relations* $\{E_s\}_{s\in S}$ and is denoted by the symbol $\prod_{s\in S} E_s$, or by $E_1 \times E_2 \times \ldots \times E_k$ if $S = \{1, 2, \ldots, k\}$. One easily sees that $\prod_{s\in S} E_s = E(\prod_{s\in S} q_s)$, where $q_s: X_s \to X_s/E_s$ is the natural mapping. The question arises whether the mapping

$$\overline{\prod_{s\in S} q_s} : \prod_{s\in S} X_s / \prod_{s\in S} E_s \to \prod_{s\in S} (X_s/E_s)$$

is a homeomorphism. Clearly, this question is equivalent to the question whether the Cartesian product of quotient mappings is a quotient mapping. As shown in Example 2.4.20 below, the answer is generally negative, even in the case of two factors, one of which is an identity mapping and the other is a closed mapping (cf. Theorem 3.3.17 and Problem 3.12.14(b)). Let us note that for open equivalence relations we have the following positive result which is a consequence of Proposition 2.3.29:

2.4.19. PROPOSITION. *If E_s is for every $s \in S$ an open equivalence relation on a space X_s and $q_s: X_s \to X_s/E_s$ is the natural mapping, then the mapping*

$$\overline{\prod_{s\in S} q_s}: \prod_{s\in S} X_s / \prod_{s\in S} E_s \to \prod_{s\in S}(X_s/E_s)$$

is a homeomorphism. ■

2.4.20. EXAMPLE. Let $X_1 = Y_1 = R\setminus\{1/2, 1/3, \ldots\}$ with the topology of a subspace of R and let $f_1 = \mathrm{id}_{X_1}: X_1 \to Y_1$. Let Y_2 be the quotient space obtained from $X_2 = R$ by identifying the set of positive integers to a point (cf. Examples 1.4.17 and 2.4.12) and let $f_2: X_2 \to Y_2$ be the natural mapping; as we know the mapping f_2 is closed. We shall show that the Cartesian product $f = f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is not a quotient mapping.

The set $F = \{(1/i + \pi/j, i + 1/j : i, j = 2, 3, \ldots\}$ is closed in $X_1 \times X_2$. Since $(0, f_2(1))$ belongs to $\overline{f(F)} \setminus f(F)$, the set $f(F)$ is not closed in $Y_1 \times Y_2$, and from the obvious equality $F = f^{-1}f(F)$ it follows that f is not a quotient mapping. ■

Historical and bibliographic notes

Quotient spaces appeared for the first time in R. L. Moore's paper [1925] and in Alexandroff's paper [1927] (announcement [1925]); both authors discussed only a particular case, that of a quotient space determined by an upper semicontinuous decomposition (Moore studied only decompositions of the plane into continua). The notion of the quotient space in full generality, as well as the notion of a quotient mapping, were introduced by Baer and Levi in [1932], and were systematically discussed for the first time in Bourbaki's books [1940] and [1951]. Adjunction spaces were defined by Borsuk in [1935] (for compact metric spaces).

Exercises

2.4.A (M. H. Stone [1936]). Let X be an arbitrary topological space; verify that by letting xE_0y whenever $\overline{\{x\}} = \overline{\{y\}}$, we define an equivalence relation E_0 on X and that X/E_0 is a T_0-space. Show that if for an equivalence relation E on X the quotient space X/E is a T_0-space, than $E_0 \subset E$.

Hint. The set $\overline{\{x\}}$ is a union of equivalence classes of E_0.

2.4.B. Suppose we are given an equivalence relation E on a space X, an equivalence relation E' on a space Y, and a continuous mapping $f: X \to Y$ such that xEy implies $f(x)E'f(y)$. Define a mapping $f^*: X/E \to Y/E'$ satisfying $f^*q = q'f$, where $q: X \to X/E$ and $q': Y \to Y/E'$ are natural mappings, and prove that if f is quotient, then f^* also is quotient, and that if both f and q' are closed (open), then f^* also is closed (open).

Observe that if we are given a closed subset A of X, a closed subset B of Y, and a continuous mapping $f: X \to Y$ such that $f(A) \subset B$, then $f^*: X/A \to Y/B$ is closed if f is closed.

2.4.C. (a) Give an example of a closed (open) equivalence relation E on a space X such that the set E is not closed (open) in the Cartesian product $X \times X$. Give an example of an equivalence relation E on a space X such that E is not a closed relation, but the set E is closed in the Cartesian product $X \times X$.

(b) Note that if E is an equivalence relation on a space X and the set E is open in the Cartesian product $X \times X$, then the quotient space X/E is discrete and E is an open relation.

(c) Suppose we are given an equivalence relation E on a topological space X. Show that if the quotient space X/E is a Hausdorff space, then E is a closed subset of the Cartesian product $X \times X$. Verify that if E is a closed subset of the Cartesian product $X \times X$ and the relation E is open, then X/E is a Hausdorff space.

2.4.D. (a) Suppose we are given two families $\{X_s\}_{s\in S}$ and $\{Y_s\}_{s\in S}$ of topological spaces such that $X_s \cap Y_s = \emptyset$ for $s \in S$ and a family of continuous mappings $\{f_s\}_{s\in S}$, where $f_s: M_s \to Y_s$ and M_s is a closed subspace of X_s. Verify that the adjunction space $(\bigoplus_{s\in S} X_s) \cup_f (\bigoplus_{s\in S} Y_s)$, where $f = \bigoplus_{s\in S} f_s$, is homeomorphic to the sum $\bigoplus_{s\in S}(X_s \cup_{f_s} Y_s)$.

(b) Define for $i = 1,2$ disjoint topological spaces X_i, Y_i and a continuous mapping $f_i: M_i \to Y_i$, where M_i is a closed subspace of X_i, such that the adjunction space $(X_1 \times X_2) \cup_{(f_1\times f_2)} (Y_1 \times Y_2)$ is not homeomorphic to the Cartesian product $(X_1 \cup_{f_1} Y_1) \times (X_2 \cup_{f_2} Y_2)$.

Hint. See Example 2.4.20.

2.4.E. (a) Verify that the sum $\bigoplus_{s\in S} f_s$ is a quotient mapping if and only if all mappings f_s are quotient.

(b) Give an example of two open mappings $f_1: X \to Y_1$ and $f_2: X \to Y_2$ such that the restriction $f|X: X \to f(X)$ of the diagonal $f = f_1 \triangle f_2$ is not a quotient mapping.

Hint. Take $X = [0,1)$.

(c) Observe that for every retraction $f: X \to X$ the restriction $f|X: X \to f(X)$ is a quotient mapping.

2.4.F (McDougle [1958] and [1959], Arhangel'skiĭ [1963], Din' N'e T'ong [1963], Filippov [1969a]). A continuous mapping $f: X \to Y$ of X onto Y is called *hereditarily quotient* if for every $B \subset Y$ the restriction $f_B: f^{-1}(B) \to B$ is a quotient mapping.

(a) Prove that a continuous mapping $f: X \to Y$ of X onto Y is hereditarily quotient if and only if the set $f(\overline{f^{-1}(B)}) \subset Y$ is closed for every $B \subset Y$ or – equivalently – if and only if for every $y \in Y$ and any open $U \subset X$ that contains $f^{-1}(y)$, we have $y \in \operatorname{Int} f(U)$.

(b) Verify that the composition of two hereditarily quotient mappings is a hereditarily quotient mapping, that the sum of hereditarily quotient mappings is hereditarily quotient and that Proposition 2.4.18 holds also for hereditarily quotient mappings.

(c) Show that any quotient mapping $f: X \to Y$ onto a Fréchet space Y in which every sequence has at most one limit (hence, in particular, onto a Fréchet T_2-space) is hereditarily quotient.

(d) Applying (c) to corresponding examples for quotient mappings, observe that restrictions of hereditarily quotient mappings to open-and-closed subsets of the domain, Cartesian products of two hereditarily quotient mappings and restrictions of diagonals (see Exercise 2.4.E(b)) of two hereditarily quotient mappings need not be quotient mappings.

2.4.G (Arhangel'skiĭ [1963a], Franklin [1965] and [1967]). (a) Show that the image of a sequential space under a quotient mapping is a sequential space and the image of a Fréchet space under a hereditarily quotient mapping is a Fréchet space.

Note that the image of a sequential space under a continuous mapping is not necessarily a sequential space and that the image of a Fréchet space under a quotient mapping is not necessarily a Fréchet space.

(b) Let X be a sequential space and C the family of all sequences $x_0, x_1, x_2, \ldots$ of points of X such that $x_0 \in \lim x_i$. For every $c = \{x_i\} \in C$ let $X_c = \{c\} \times \{0, 1, 1/2, \ldots\}$, where $\{c\}$ is the one-point discrete space and $\{0, 1, 1/2, \ldots\}$ has the topology of a subspace of R, and let $f_c: X_c \to X$ be defined by the formulas

$$f_c((c,0)) = x_0 \quad \text{and} \quad f_c((c,1/i)) = x_i.$$

Show that the combination $f_X = \nabla_{c\in C} f_c : \bigoplus_{c\in C} X_c \to X$ is a quotient mapping and note that sequential spaces can be characterized as the images of first-countable spaces under quotient mappings (cf. Exercise 4.2.D(c)). Verify that a subspace M of a sequential space X is sequential if and only if the restriction $(f_X)_M: f_X^{-1}(M) \to M$ is a quotient mapping. Deduce that a space X is hereditarily sequential if and only if it is a Fréchet space (cf. Exercise 3.3.I) and note that Fréchet spaces can be characterized as the images of first-countable spaces under hereditarily quotient mappings (cf. Exercise 4.2.D(c)).

(c) Apply Example 2.4.20 to show that the Cartesian product of a normal second-countable space and a normal Fréchet space is not necessarily a sequential space (cf. Exercise 2.3.K and Example 3.3.29).

2.5. Limits of inverse systems

Suppose that to every σ in a set Σ directed by the relation $\leq$ corresponds a topological space X_σ, and that for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$ a continuous mapping $\pi^\sigma_\rho: X_\sigma \to X_\rho$ is defined; suppose further that $\pi^\rho_\tau \pi^\sigma_\rho = \pi^\sigma_\tau$ for any $\sigma, \rho, \tau \in \Sigma$ satisfying $\tau \leq \rho \leq \sigma$ and that $\pi^\sigma_\sigma = \mathrm{id}_{X_\sigma}$ for every $\sigma \in \Sigma$. In this situation we say that the family $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ is an *inverse system of the spaces* X_σ; the mappings π^σ_ρ are called *bonding mappings* of the inverse system $\mathbf{S}$.

An inverse system $\mathbf{S} = \{X_i, \pi^i_j, N\}$, where N is the set of all positive integers directed by its natural order, is called an *inverse sequence*; to denote an inverse sequence we shall write briefly $\{X_i, \pi^i_j\}$.

Let $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ be an inverse system; an element $\{x_\sigma\}$ of the Cartesian product $\prod_{\sigma\in\Sigma} X_\sigma$ is called a *thread* of $\mathbf{S}$ if $\pi^\sigma_\rho(x_\sigma) = x_\rho$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$, and the subspace of $\prod_{\sigma\in\Sigma} X_\sigma$ consisting of all threads of $\mathbf{S}$ is called the *limit of the inverse system* $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ and is denoted by $\varprojlim \mathbf{S}$ or by $\varprojlim \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$.

2.5.1. PROPOSITION. *The limit of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of Hausdorff spaces X_σ is a closed subset of the Cartesian product $\prod_{\sigma\in\Sigma} X_\sigma$.*

PROOF. For all $\rho, \tau \in \Sigma$ satisfying $\tau \leq \rho$ let

$$M_{\rho\tau} = \{\{x_\sigma\} \in \prod_{\sigma\in\Sigma} X_\sigma : \pi^\rho_\tau(x_\rho) = x_\tau\};$$

as the sets $M_{\rho\tau}$ are closed in $\prod_{\sigma\in\Sigma} X_\sigma$ by Theorem 1.5.4, the set $\varprojlim \mathrm{S} = \bigcap_{\tau\leq\rho} M_{\rho\tau}$ also is closed in $\prod_{\sigma\in\Sigma} X_\sigma$. ■

Theorems 2.1.6 and 2.3.11 imply

2.5.2. THEOREM. *The limit of an inverse system of T_i-spaces is a T_i-space for $i \leq 3\frac{1}{2}$.* ■

From Example 2.5.3 below and Exercise 2.3.E(a) (or Exercise 3.1.H(a)) it follows that the limit of an inverse system of perfectly normal spaces need not be normal. On the other hand, the limit of an inverse sequence of perfectly normal spaces is perfectly normal (see Problem 2.7.16(b)); this is not true for normal spaces or hereditarily normal spaces (see the remark to Problem 2.7.16).

2.5.3. EXAMPLE. Suppose we are given a family $\{X_s\}_{s\in S}$ of topological spaces where $|S| \geq \aleph_0$. Observe that the family Σ of all finite subsets of S is directed by inclusion, i.e., by the relation $\leq$ defined by letting $\rho \leq \sigma$ if and only if $\rho \subset \sigma$. Letting $X_\sigma = \prod_{s\in\sigma} X_s$ for every $\sigma \in \Sigma$, for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$ we have a continuous mapping $\pi^\sigma_\rho: X_\sigma \to X_\rho$ defined – the restriction of elements of X_σ to the subset ρ of the set σ. One can easily verify that $\mathrm{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ is an inverse system of topological spaces.

For every $s \in S$ let $\sigma_s = \{s\} \in \Sigma$; the reader can readily check that by assigning to the point $\{x_\sigma\} \in \varprojlim \mathrm{S}$, the point $\{x_{\sigma_s}\} \in \prod_{s\in S} X_s$, we define a homeomorphism of the space $\varprojlim \mathrm{S}$ onto the Cartesian product $\prod_{s\in S} X_s$. Hence, applying the operation of the limit of an inverse system, one can express infinite Cartesian products in terms of the finite Cartesian products. ■

2.5.4. EXAMPLE. Let Σ be a family of subspaces of a topological space X which is directed by the relation $\supset$, i.e., has the property that for every $L_1, L_2 \in \Sigma$ there exists an $M \in \Sigma$ such that $M \subset L_1 \cap L_2$. For any $M, L \in \Sigma$ satisfying $M \subset L$ let $\pi^M_L: M \to L$ be the embedding of M in L. One can easily verify that $\mathrm{S} = \{M, \pi^M_L, \Sigma\}$ is an inverse system (where the space corresponding to an $M \in \Sigma$ is M itself) and that $\varprojlim \mathrm{S}$ is homeomorphic to the subspace $\bigcap \Sigma$ of X. In particular, every subspace A of a T_1-space X can be represented as the limit of an inverse system of open subspaces of X, because $A = \bigcap \Sigma$, where Σ is the directed family of all open subsets of X that contain A and have finite complement. ■

Let $\mathrm{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ be an inverse system of topological spaces and let $X = \varprojlim \mathrm{S}$. For every $\sigma \in \Sigma$ a continuous mapping $\pi_\sigma = p_\sigma|X: X \to X_\sigma$, where $p_\sigma : \prod_{\sigma\in\Sigma} X_\sigma \to X_\sigma$ is the projection, is defined; it is called the *projection of the limit of* S *to* X_σ. Clearly, for any $\sigma, \rho \in \Sigma$ such that $\rho \leq \sigma$, the projections π_σ and π_ρ satisfy the equality $\pi_\rho = \pi^\sigma_\rho \pi_\sigma$.

2.5.5. PROPOSITION. *The family of all sets $\pi^{-1}_\sigma(U_\sigma)$, where U_σ is an open subset of X_σ and σ runs over a subset Σ' cofinal in Σ, is a base for the limit of the inverse system* $\mathrm{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$.

Moreover, *if for every $\sigma \in \Sigma$ a base $\mathcal{B}_\sigma$ for X_σ is fixed, then the subfamily consisting of those $\pi^{-1}_\sigma(U_\sigma)$ in which $U_\sigma \in \mathcal{B}_\sigma$, also is a base.*

PROOF. Denote by $\mathcal{B}$ the family described in the first part of the proposition and let $X = \varprojlim \mathrm{S}$. As the projections π_σ are continuous, all members of $\mathcal{B}$ are open in X, and it remains to show that any non-empty open set $U \subset X$ can be represented as the union of a subfamily of $\mathcal{B}$. Clearly, if suffices to show that for every $x = \{x_\sigma\} \in U$ there exist a $\sigma \in \Sigma'$ and an open subset U_σ of X_σ such that $x \in \pi_\sigma^{-1}(U_\sigma) \subset U$.

By the definition of the induced topology there exists an open set $V \subset \prod_{\sigma\in\Sigma} X_\sigma$ such that $U = X \cap V$ and by Proposition 2.3.1 there exist $\sigma_1, \sigma_2, \ldots, \sigma_k \in \Sigma$ and $U_1, U_2, \ldots, U_k$, open in $X_{\sigma_1}, X_{\sigma_2}, \ldots, X_{\sigma_k}$ respectively such that

$$x \in p_{\sigma_1}^{-1}(U_1) \cap p_{\sigma_2}^{-1}(U_2) \cap \ldots \cap p_{\sigma_k}^{-1}(U_k) \subset V. \tag{1}$$

Since the set Σ is directed and Σ' is cofinal in Σ, there exists a $\sigma \in \Sigma'$ such that $\sigma_i \leq \sigma$ for $i = 1, 2, \ldots, k$. All the sets $(\pi_{\sigma_i}^\sigma)^{-1}(U_i)$ and their intersection $U_\sigma = \bigcap_{i=1}^k (\pi_{\sigma_i}^\sigma)^{-1}(U_i)$ are open in X_σ; as $\pi_{\sigma_i}^\sigma(x_\sigma) = x_{\sigma_i}$, we have

$$x_\sigma \in U_\sigma. \tag{2}$$

Clearly

$$\pi_\sigma^{-1}(\pi_{\sigma_i}^\sigma)^{-1}(U_i) = \pi_{\sigma_i}^{-1}(U_i) = X \cap p_{\sigma_i}^{-1}(U_i); \tag{3}$$

applying (2), (3) and (1) we obtain

$$\begin{aligned} x \in \pi_\sigma^{-1}(U_\sigma) &= \pi_\sigma^{-1}\Big(\bigcap_{i=1}^k (\pi_{\sigma_i}^\sigma)^{-1}(U_i)\Big) = \bigcap_{i=1}^k \pi_\sigma^{-1}(\pi_{\sigma_i}^\sigma)^{-1}(U_i) \\ &= X \cap \bigcap_{i=1}^k p_{\sigma_i}^{-1}(U_i) \subset X \cap V = U \end{aligned}$$

which completes the proof that $\mathcal{B}$ is a base for X.

The second part of the proposition is an immediate consequence of the first part and of the definition of a base. ∎

2.5.6. PROPOSITION. *For every subspace A of the limit of an inverse system $\mathbf{S} = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$, the family $\mathbf{S}_A = \{\overline{A}_\sigma, \tilde{\pi}_\rho^\sigma, \Sigma\}$, where $A_\sigma = \pi_\sigma(A)$ and $\tilde{\pi}_\rho^\sigma(x) = \pi_\rho^\sigma(x)$ for $x \in \overline{A}_\sigma$, is an inverse system and $\varprojlim \mathbf{S}_A = \overline{A} \subset X$.*

PROOF. As $\pi_\rho(x) = \tilde{\pi}_\rho^\sigma \pi_\sigma(x)$ for $x \in A$ and $\rho \leq \sigma$, we have

$$\tilde{\pi}_\rho^\sigma(\overline{A}_\sigma) = \tilde{\pi}_\rho^\sigma(\overline{\pi_\sigma(A)}) \subset \overline{\tilde{\pi}_\rho^\sigma \pi_\sigma(A)} = \overline{\pi_\rho(A)} = \overline{A}_\rho,$$

which proves that $\mathbf{S}_A$ is an inverse system.

Clearly $\varprojlim \mathbf{S}_A \subset X$, and from Propositions 2.3.2 and 2.1.2 it follows that $\varprojlim \mathbf{S}_A$ is a subspace of X; moreover, it is a closed subspace of X. Indeed, for every $x = \{x_\sigma\} \in X \setminus \varprojlim \mathbf{S}_A$ there exists a $\sigma_0 \in \Sigma$ such that $x_{\sigma_0} \in X_{\sigma_0} \setminus \overline{A}_{\sigma_0}$, so that $\pi_{\sigma_0}^{-1}(X_{\sigma_0} \setminus \overline{A}_{\sigma_0})$ is a neighbourhood of x disjoint from $\varprojlim \mathbf{S}_A$. As we clearly have $A \subset \varprojlim \mathbf{S}_A$, we then have $\overline{A} \subset \varprojlim \mathbf{S}_A$.

Now let us take any point $x = \{x_\sigma\} \in \varprojlim \mathbf{S}_A$. By Proposition 2.5.5, the family of all sets $\pi_\sigma^{-1}(U_\sigma)$, where U_σ is a neighbourhood of x_σ in the space X_σ, is a base for X at the point x. For every member $\pi_\sigma^{-1}(U_\sigma)$ of that base we have $x_\sigma \in \overline{A}_\sigma \cap U_\sigma$, so that $A_\sigma \cap U_\sigma \neq \emptyset$ or – equivalently – $A \cap \pi_\sigma^{-1}(U_\sigma) \neq \emptyset$. This implies that $x \in \overline{A}$, proving that $\varprojlim \mathbf{S}_A \subset \overline{A}$. ∎

2.5.7. COROLLARY. *Every closed subspace A of the limit of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ is the limit of the inverse system $\mathbf{S}_A = \{\overline{A}_\sigma, \tilde{\pi}^\sigma_\rho, \Sigma\}$ of closed subspaces $\overline{A}_\sigma$ of the spaces X_σ.* ∎

From 2.5.1, 2.5.3 and 2.5.7 we obtain

2.5.8. THEOREM. *Let $\mathcal{P}$ be a topological property that is hereditary with respect to closed subsets and finitely multiplicative. A topological space X is homeomorphic to the limit of an inverse system of T_2-spaces with the property $\mathcal{P}$ if and only if X is homeomorphic to a closed subspace of a Cartesian product of T_2-spaces with the property $\mathcal{P}$.* ∎

Suppose we are given two inverse systems $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ and $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$; a *mapping of the system* $\mathbf{S}$ *to the system* $\mathbf{S}'$ is a family $\{\phi, f_{\sigma'}\}$ consisting of a nondecreasing function ϕ from Σ' to Σ such that the set $\phi(\Sigma')$ is cofinal in Σ, and of continuous mappings $f_{\sigma'}: X_{\phi(\sigma')} \to Y_{\sigma'}$ defined for all $\sigma' \in \Sigma'$ and such that

$$\pi^{\sigma'}_{\rho'} f_{\sigma'} = f_{\rho'} \pi^{\phi(\sigma')}_{\phi(\rho')}, \tag{4}$$

i.e., such that the diagram

$$\begin{tikzcd} X_{\phi(\sigma')} \arrow[r, "f_{\sigma'}"] \arrow[d, "\pi^{\phi(\sigma')}_{\phi(\rho')}"] & Y_{\sigma'} \arrow[d, "\pi^{\sigma'}_{\rho'}"] \\ X_{\phi(\rho')} \arrow[r, "f_{\rho'}"] & Y_{\rho'} \end{tikzcd}$$

is commutative for any $\sigma', \rho' \in \Sigma'$ satisfying $\rho' \le \sigma'$.

Any mapping of an inverse system $\mathbf{S}$ to an inverse system $\mathbf{S}'$ induces a continuous mapping of $\varprojlim \mathbf{S}$ to $\varprojlim \mathbf{S}'$. To show this, let us consider a mapping $\{\phi, f_{\sigma'}\}$ of $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ to $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$; for a thread $x = \{x_\sigma\} \in X = \varprojlim \mathbf{S}$ and every $\sigma' \in \Sigma'$ let us define

$$y_{\sigma'} = f_{\sigma'}(x_{\phi(\sigma')}); \tag{5}$$

the point $\{y_{\sigma'}\} \in \prod_{\sigma' \in \Sigma'} Y_{\sigma'}$ thus obtained is a thread, i.e., $\{y_{\sigma'}\} \in Y = \varprojlim \mathbf{S}'$. Indeed, for any $\sigma', \rho' \in \Sigma'$ satisfying $\rho' \le \sigma'$ we have by virtue of (4) and (5)

$$\pi^{\sigma'}_{\rho'}(y_{\sigma'}) = \pi^{\sigma'}_{\rho'} f_{\sigma'}(x_{\phi(\sigma')}) = f_{\rho'} \pi^{\phi(\sigma')}_{\phi(\rho')}(x_{\phi(\sigma')}) = f_{\rho'}(x_{\phi(\rho')}) = y_{\rho'}.$$

Assigning to $x = \{x_\sigma\} \in X$ the point $y = \{y_{\sigma'}\} \in Y$ we define a mapping f of X to Y; we shall show that f is a continuous mapping. By Proposition 2.5.5 it suffices to show that inverse images under f of all sets $\pi^{-1}_{\sigma'}(U_{\sigma'})$, where $U_{\sigma'}$ is an open subset of $Y_{\sigma'}$, are open in X. From (5) it follows that for $x = \{x_\sigma\} \in X$ we have

$$\pi_{\sigma'} f(x) = f_{\sigma'}(x_{\phi(\sigma')}) = f_{\sigma'} \pi_{\phi(\sigma')}(x), \tag{6}$$

so that the inverse image $f^{-1}\pi^{-1}_{\sigma'}(U_{\sigma'}) = \pi^{-1}_{\phi(\sigma')} f^{-1}_{\sigma'}(U_{\sigma'})$ is open, $f_{\sigma'}$ and $\pi_{\phi(\sigma')}$ being continuous. The mapping $f: X \to Y$ is called the *limit mapping induced by* $\{\phi, f_{\sigma'}\}$ and is denoted by $\varprojlim \{\phi, f_{\sigma'}\}$.

2.5.9. LEMMA. *Let $\{\phi, f_{\sigma'}\}$ be a mapping of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ to an inverse system $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$. If all mappings $f_{\sigma'}$ are one-to-one, the limit mapping $f = \varprojlim \{\phi, f_{\sigma'}\}$ also is one-to-one. If, moreover, all mappings $f_{\sigma'}$ are onto, f also is a mapping onto.*

PROOF. Let $x = \{x_\sigma\}$ and $z = \{z_\sigma\}$ be two distinct points of $\varprojlim \mathbf{S}$. Take a $\sigma_0 \in \Sigma$ such that $x_{\sigma_0} \neq z_{\sigma_0}$ and a $\sigma' \in \Sigma'$ satisfying $\sigma_0 \leq \phi(\sigma')$. Clearly $x_{\phi(\sigma')} \neq z_{\phi(\sigma')}$, so that – $f_{\sigma'}$ being one-to-one – $f_{\sigma'}(x_{\phi(\sigma')}) \neq f_{\sigma'}(z_{\phi(\sigma')})$, which shows that $f(x) \neq f(z)$.

Now suppose that $f_{\sigma'}$ is for every $\sigma' \in \Sigma'$ a one-to-one mapping of $X_{\phi(\sigma')}$ onto $Y_{\sigma'}$, and take a point $y = \{y_{\sigma'}\} \in \varprojlim \mathbf{S}'$. From (4) it follows that $\phi(\sigma') = \phi(\rho')$ implies $f^{-1}_{\sigma'}(y_{\sigma'}) = f^{-1}_{\rho'}(y_{\rho'})$, so that for every $\phi(\sigma') \in \phi(\Sigma')$ the point

$$z_{\phi(\sigma')} = f^{-1}_{\sigma'}(y_{\sigma'}) \in X_{\phi(\sigma')}$$

is well-defined. For every $\sigma \in \Sigma$ choose an element of $\sigma' \in \Sigma'$ such that $\sigma \leq \phi(\sigma')$ and define

$$x_\sigma = \pi^{\phi(\sigma')}_\sigma(z_{\phi(\sigma')}) \in X_\sigma;$$

one can readily verify that x_σ does not depend on the choice of σ', that $x = \{x_\sigma\}$ is a thread of $\mathbf{S}$ and that $f(x) = y$. ■

2.5.10. PROPOSITION. *Let $\{\phi, f_{\sigma'}\}$ be a mapping of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ to an inverse system $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$. If all mappings $f_{\sigma'}$ are homeomorphisms, the limit mapping $f = \varprojlim \{\phi, f_{\sigma'}\}$ also is a homeomorphism.*

PROOF. By the lemma and Proposition 2.5.5 it suffices to show that for any $\sigma' \in \Sigma'$ and any open $U \subset X_{\phi(\sigma')}$ the image under f of the set $\pi^{-1}_{\phi(\sigma')}(U)$ is open in $\varprojlim \mathbf{S}'$. From (6) it follows that for $A \subset Y_{\sigma'}$, we have $\pi^{-1}_{\phi(\sigma')} f^{-1}_{\sigma'}(A) = f^{-1}\pi^{-1}_{\sigma'}(A)$; letting $A = f_{\sigma'}(U)$ and applying the fact that $f_{\sigma'}$ is a one-to-one mapping of $X_{\phi(\sigma')}$ onto $Y_{\sigma'}$, we obtain

$$\pi^{-1}_{\phi(\sigma')}(U) = \pi^{-1}_{\phi(\sigma')} f^{-1}_{\sigma'} f_{\sigma'}(U) = f^{-1}\pi^{-1}_{\sigma'} f_{\sigma'}(U).$$

Therefore the set

$$f\pi^{-1}_{\phi(\sigma')}(U) = ff^{-1}\pi^{-1}_{\sigma'} f_{\sigma'}(U) = \pi^{-1}_{\sigma'} f_{\sigma'}(U)$$

is open in $\varprojlim \mathbf{S}'$. ■

2.5.11. COROLLARY. *Let $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ be an inverse system and Σ' a subset cofinal in Σ. The mapping consisting in restricting all threads from $X = \varprojlim \mathbf{S}$ to Σ' is a homeomorphism of X onto the space $X' = \varprojlim \mathbf{S}'$, where $\mathbf{S}' = \{X_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$.* ■

2.5.12. COROLLARY. *Let $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ be an inverse system; if in the directed set Σ there exists an element σ_0 such that $\sigma \leq \sigma_0$ for every $\sigma \in \Sigma$, then the limit of $\mathbf{S}$ is homeomorphic to the space X_{σ_0}.* ■

From Corollary 2.5.11 it follows that if a mapping $\{\phi, f_{\sigma'}\}$ of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ to an inverse system $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$ is being considered, one can always assume that $\phi(\Sigma') = \Sigma$. Indeed, when one identifies the limit of $\mathbf{S}$ with the limit of the inverse system

$\mathbf{S}'' = \{X_\sigma, \pi^\sigma_\rho, \phi(\Sigma')\}$ as described in Corollary 2.5.11, the limit mappings of $\varprojlim \mathbf{S}$ to $\varprojlim \mathbf{S}'$ and of $\varprojlim \mathbf{S}''$ to $\varprojlim \mathbf{S}'$ coincide.

One can readily verify that any countable directed set contains a cofinal subset linearly ordered by $\le$ and similar to the set N of all positive integers. Hence, from 2.5.11 it follows that instead of inverse systems $\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ with $|\Sigma| = \aleph_0$ one can consider inverse sequences. Considering a mapping $\{\phi, f_i\}$ of an inverse sequence to another inverse sequence, one can always assume that $\phi = \mathrm{id}_N$.

Now we are going to prove two theorems showing that limit mappings can be expressed in terms of Cartesian products and that projections of limits of inverse systems can be expressed in terms of limit mappings. These theorems will be applied later, when proving that limit mappings and projections belong to some classes of mappings (see Corollary 3.2.15, Theorems 3.7.12 and 3.7.13, and Problem 6.3.16(a)).

2.5.13. THEOREM. *For every mapping $\{\phi, f_{\sigma'}\}$ of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ to an inverse system $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$ there exists a homeomorphic embedding $h: \varprojlim \mathbf{S} \to \prod_{\sigma' \in \Sigma'} Z_{\sigma'}$, where $Z_{\sigma'} = X_{\phi(\sigma')}$, such that $\varprojlim \{\phi, f_{\sigma'}\} = (\prod_{\sigma' \in \Sigma'} f_{\sigma'})h$. If all $X_{\phi(\sigma')}$ are Hausdorff spaces, then $h(\varprojlim \mathbf{S})$ is a closed subset of $\prod_{\sigma' \in \Sigma'} Z_{\sigma'}$.*

PROOF. For every $\sigma \in \phi(\Sigma')$ the diagonal $i_\sigma = \triangle_{\sigma' \in \phi^{-1}(\sigma)} \mathrm{id}_{X_\sigma}: X_\sigma \to \prod_{\sigma' \in \phi^{-1}(\sigma)} Z_{\sigma'}$ is a homeomorphic embedding and if X_σ is a Hausdorff space, then $i_\sigma(X_\sigma)$ is closed. The composition h of the homeomorphism of $\varprojlim \mathbf{S}$ onto $\varprojlim \mathbf{S}''$, where $\mathbf{S}'' = \{X_\sigma, \pi^\sigma_\rho, \phi(\Sigma')\}$, described in 2.5.11 and the restriction $(\prod_{\sigma \in \phi(\Sigma')} i_\sigma)|\varprojlim \mathbf{S}'' : \varprojlim \mathbf{S}'' \to \prod_{\sigma' \in \Sigma'} Z_{\sigma'}$ is also a homeomorphic embedding and if all $X_{\phi(\sigma')}$ are Hausdorff spaces, then $h(\varprojlim \mathbf{S})$ is a closed subset of $\prod_{\sigma' \in \Sigma'} Z_{\sigma'}$. The equality $\varprojlim \{\phi, f_{\sigma'}\} = (\prod_{\sigma' \in \Sigma'} f_{\sigma'})h$ is obvious. ∎

2.5.14. THEOREM. *For every inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ and any $\sigma_0 \in \Sigma$ there exist an inverse system $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$, where $Y_{\sigma'} = X_{\sigma_0}$ for all $\sigma' \in \Sigma'$, a homeomorphism $h : \varprojlim \mathbf{S}' \to X_{\sigma_0}$, and a mapping $\{\phi, f_{\sigma'}\}$ of $\mathbf{S}$ to $\mathbf{S}'$, where $f_{\sigma'}$ are bonding mappings of $\mathbf{S}$, such that $\pi_{\sigma_0} = h\varprojlim \{\phi, f_{\sigma'}\}$.*

PROOF. Letting $\Sigma' = \{\sigma' \in \Sigma : \sigma_0 \le \sigma'\}$, $Y_{\sigma'} = X_{\sigma_0}$ for $\sigma' \in \Sigma'$ and $\pi^{\sigma'}_{\rho'} = \mathrm{id}_{X_{\sigma_0}}$ for any $\sigma', \rho' \in \Sigma'$ satisfying $\rho' \le \sigma'$, we define an inverse system $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$. The limit of $\mathbf{S}'$ coincides with the diagonal Δ of the Cartesian product $\prod_{\sigma' \in \Sigma'} Y_{\sigma'} = X^{\mathfrak{m}}_{\sigma_0}$, where $\mathfrak{m} = |\Sigma'|$; denote by h the homeomorphism of Δ onto X_{σ_0} which is inverse to the homeomorphism $(\triangle_{\sigma' \in \Sigma'} \mathrm{id}_{X_{\sigma_0}})|X_{\sigma_0}: X_{\sigma_0} \to \Delta$. The family $\{\phi, f_{\sigma'}\}$, where $\phi(\sigma') = \sigma'$ and $f_{\sigma'} = \pi^{\sigma'}_{\sigma_0}$ for $\sigma' \in \Sigma'$, is a mapping of the inverse system $\mathbf{S}$ to the inverse system $\mathbf{S}'$. The equality $\pi_{\sigma_0} = h\varprojlim \{\phi, f_{\sigma'}\}$ is obvious. ∎

We close this section by considering two important particular cases of mappings of inverse systems.

For every topological space X and any non-empty directed set Σ, the inverse system $\mathbf{S}(X, \Sigma) = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$, where $X_\sigma = X$ for $\sigma \in \Sigma$ and $\pi^\sigma_\rho = \mathrm{id}_X$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \le \sigma$, is called the *constant inverse system* of the space X on the set Σ. Clearly, the mapping $h: X \to \varprojlim \mathbf{S}(X, \Sigma)$ assigning to $x \in X$ the thread $\{x_\sigma\}$, where $x_\sigma = x$ for $\sigma \in \Sigma$, is a homeomorphism of the space X onto the space $\varprojlim \mathbf{S}(X, \Sigma)$ which coincides with the diagonal of the Cartesian product $\prod_{\sigma\in\Sigma} X_\sigma = X^{\mathfrak{m}}$, where $|\Sigma| = \mathfrak{m}$.

For an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ with $\Sigma \neq \emptyset$, a topological space X and a family of continuous mappings $\{f_\sigma\}_{\sigma\in\Sigma}$, where $f_\sigma: X \to X_\sigma$, such that $\pi^\sigma_\rho f_\sigma = f_\rho$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \le \sigma$, the family $\{\mathrm{id}_\Sigma, f_\sigma\}$ is a mapping of the constant inverse system $\mathbf{S}(X, \Sigma)$ to the inverse system $\mathbf{S}$, so that the limit mapping $f = \varprojlim\{\mathrm{id}_\Sigma, f_\sigma\}$ of $\varprojlim \mathbf{S}(X,\Sigma)$ to $\varprojlim \mathbf{S}$ is defined. The composition $fh: X \to \varprojlim \mathbf{S}$, where h is the above defined homeomorphism of X onto $\varprojlim \mathbf{S}(X,\Sigma)$, is called the *limit mapping induced by* $\{f_\sigma\}_{\sigma\in\Sigma}$ and is denoted by $\varprojlim f_\sigma$.

Similarly, for an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ with $\Sigma \neq \emptyset$, a topological space X and a family of continuous mapping $\{f_\sigma\}_{\sigma\in\Sigma}$, where $f_\sigma: X_\sigma \to X$, such that $f_\rho \pi^\sigma_\rho = f_\sigma$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \le \sigma$, the family $\{\mathrm{id}_\Sigma, f_\sigma\}$ is a mapping of the inverse system $\mathbf{S}$ to the constant inverse system $\mathbf{S}(X,\Sigma)$, so that the limit mapping $f = \varprojlim\{\mathrm{id}_\Sigma, f_\sigma\}$ of $\varprojlim \mathbf{S}$ to $\varprojlim \mathbf{S}(X,\Sigma)$ is defined. The composition $h^{-1}f : \varprojlim \mathbf{S} \to X$, where h is the above defined homeomorphism of X onto $\varprojlim \mathbf{S}(X,\Sigma)$, is called the *limit mapping induced by* $\{f_\sigma\}_{\sigma\in\Sigma}$ and is denoted by $\varprojlim f_\sigma$.

Historical and bibliographic notes

The notion of the limit of an inverse sequence appeared, in a slightly different form, in Alexandroff's paper [1929]; the present definition was first stated by Lefschetz in [1931]. Mappings of inverse sequences and induced limit mappings were first studied in Freudenthal's paper [1937]. In [1936] Steenrod discussed the particular case of inverse systems, where Σ is a set of ordinal numbers directed by its natural order; in full generality inverse systems were defined by Lefschetz in [1942]. An exhaustive discussion of inverse systems was presented by Eilenberg and Steenrod in their book [1952]; after that book was published, inverse systems began to be widely studied and applied. Let us note that the notions of an inverse system and of the limit of an inverse system could be defined without a topology on the X_σ's; these notions are also studied in algebra and analysis.

Exercises

2.5.A.(a) Show that if all bonding mappings of an inverse sequence $\mathbf{S} = \{X_i, \pi^i_j\}$ of non-empty spaces are onto, then $\varprojlim \mathbf{S} \neq \emptyset$.

(b) (Waterhouse [1972]) Let S be an arbitrary uncountable set and Σ the family of all finite subsets of S directed by inclusion. Consider the inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$, where X_σ is the discrete space consisting of all one-to-one mappings of $\sigma \subset S$ into N and $\pi^\sigma_\rho(f) = f|\rho$ for every $f \in X_\sigma$ and any $\sigma, \rho \in \Sigma$ satisfying $\rho \subset \sigma$. Note that all bonding mappings of the inverse system $\mathbf{S}$ are onto, and yet $\varprojlim \mathbf{S} = \emptyset$ (cf. Exercise 3.1.K(b) and Theorem 3.2.13).

Remark. The first example of an inverse system with similar properties was given by Henkin in [1950].

(c) Give an example of an inverse sequence $\mathbf{S} = \{X_i, \pi^i_j\}$ of non-empty spaces such that all bonding mappings of $\mathbf{S}$ are one-to-one and $\varprojlim \mathbf{S} = \emptyset$.

2.5.B. (a) Show that if in an inverse sequence $\mathbf{S} = \{X_i, \pi^i_j\}$ all bonding mappings are onto, then all projections also are onto. Note that this is not true for inverse systems (see Exercise 2.5.A(b)).

(b) Show that if in an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ all bonding mappings are one-to-one, then all projections also are one-to-one. Verify that if, moreover, all bonding mappings are onto, then all projections also are onto.

2.5.C. Give an example of inverse sequences $\mathbf{S} = \{X_i, \pi^i_j\}$ and $\mathbf{S}' = \{Y_i, \tilde{\pi}^i_j\}$ and of a mapping $\{\mathrm{id}_N, f_i\}$ of $\mathbf{S}$ to $\mathbf{S}'$ such that all $\pi^i_j, \tilde{\pi}^i_j$ and f_i are onto, but the limit mapping $\varprojlim\{\mathrm{id}_N, f_i\}$ is not onto.

2.5.D. (a) Verify that if for every $s \in S$ an inverse system $\mathbf{S}(s) = \{X(s)_\sigma, \pi(s)^\sigma_\rho, \Sigma\}$ is given and $X(s)_\sigma \cap X(s')_\sigma = \emptyset$ for $s \neq s'$ and $\sigma \in \Sigma$, then the family $\bigoplus_{s\in S} \mathbf{S}(s) = \{\bigoplus_{s\in S} X(s)_\sigma, \bigoplus_{s\in S} \pi(s)^\sigma_\rho, \Sigma\}$ is an inverse system and $\varprojlim(\bigoplus_{s\in S} \mathbf{S}(s)) = \bigoplus_{s\in S} \varprojlim \mathbf{S}(s)$.

(b) Verify that, if for every $s \in S$ an inverse system $\mathbf{S}(s) = \{X(s)_\sigma, \pi(s)^\sigma_\rho, \Sigma\}$ is given, then the family $\prod_{s\in S} \mathbf{S}(s) = \{\prod_{s\in S} X(s)_\sigma, \prod_{s\in S} \pi(s)^\sigma_\rho, \Sigma\}$ is an inverse system and $\varprojlim(\prod_{s\in S} \mathbf{S}(s))$ is homeomorphic to $\prod_{s\in S} \varprojlim \mathbf{S}(s)$.

2.5.E. (a) Note that the limit of an inverse sequence of first-countable (second-countable) spaces is first-countable (second-countable) and observe that the limit of an inverse system of second-countable spaces need not be first-countable.

(b) Observe that the limit of an inverse sequence of separable spaces need not be separable.

Hint. Apply Example 2.5.4 to the Niemytzki plane.

Remark. The limit of an inverse sequence of Fréchet spaces need not be a sequential space (see Exercise 3.3.E(b)).

2.5.F. Show that the space X is homeomorphic to the limit of the inverse system $\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ if and only if there exists a family of continuous mappings $\{\pi_\sigma\}_{\sigma\in\Sigma}$, where $\pi_\sigma: X \to X_\sigma$, satisfying the following conditions:

(1) *For any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$ we have $\pi^\sigma_\rho \pi^\sigma = \pi_\rho$.*

(2) *For every space Y and a pair f, g of continuous mappings of Y to X, if $\pi_\sigma f = \pi_\sigma g$ for every $\sigma \in \Sigma$, then $f = g$.*

(3) *For every space Y and a family of continuous mappings $\{f_\sigma\}_{\sigma\in\Sigma}$ where $f_\sigma: Y \to X_\sigma$ and $\pi^\sigma_\rho f_\sigma = f_\rho$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$, there exists a continuous mapping $f: Y \to X$ such that $\pi_\sigma f = f_\sigma$ for every $\sigma \in \Sigma$.*

2.6. Function spaces I: The topology of uniform convergence on R^X and the topology of pointwise convergence

The last operation we shall discuss here is the operation of forming function spaces. To every pair X, Y of topological spaces corresponds the set Y^X of all continuous mappings of X to Y; we shall try to define a natural topology on that set. It turns out that, in contrast to the previously discussed operations, the set Y^X has several rather natural topologies. We shall begin with defining a topology on the set R^X of all continuous real-valued functions on the space X. Then we shall define a topology on Y^X for arbitrary X and Y. The section will close with a discussion of conditions that an "acceptable topology" on Y^X should satisfy.

Still another topology on function spaces, satisfying those conditions for a fairly large class of spaces, will be defined and studied in Section 3.4, when the notion of compactness will be at our disposal.

For $A \subset R^X$ and $f \in R^X$, define $\overline{A}$ by letting

$$(1) \qquad f \in \overline{A} \quad \text{whenever} \quad f = \lim f_i, \quad \text{where} \quad f_i \in A \quad \text{for} \quad i = 1, 2, \ldots;$$

the equality $f = \lim f_i$ means, as defined in Section 1.4, that the sequence $\{f_i\}$ is uniformly convergent to f.

2.6.1. PROPOSITION. *The closure operator defined in* R^X *by formula* (1) *satisfies conditions* (CO1)–(CO4).

PROOF. Condition (CO1) is clearly satisfied. As $\lim f_i = f$, if $f_i = f$ for $i = 1, 2, \ldots$, condition (CO2) also is satisfied.

It follows immediately from (1) that

$$(2) \qquad \text{if} \quad A \subset B, \quad \text{then} \quad \overline{A} \subset \overline{B};$$

hence, in order to prove that condition (CO3) is satisfied it suffices to show that

$$(3) \qquad \overline{A \cup B} \subset \overline{A} \cup \overline{B}.$$

Take an $f \in \overline{A \cup B}$ and a sequence $\{f_i\}$ of functions belonging to $A \cup B$ such that $f = \lim f_i$. In one of the sets A, B, say in A, there is a subsequence $\{f_{k_i}\}$ of $\{f_i\}$; by definition of uniform convergence, we have $f = \lim f_{k_i}$ and this proves that $f \in \overline{A}$, so that (3) holds.

As from (2) it follows that $\overline{A} \subset \overline{(\overline{A})}$, in order to prove that condition (CO4) is satisfied it suffices to show that

$$(4) \qquad \overline{(\overline{A})} \subset \overline{A}.$$

Take an $f \in \overline{(\overline{A})}$ and a sequence $\{f_i\}$ of functions belonging to $\overline{A}$ such that $f = \lim f_i$. For every positive integer k there exists an $i(k)$ such that

$$(5) \qquad |f(x) - f_{i(k)}(x)| \leq 1/2k \quad \text{for} \quad x \in X.$$

Since $f_{i(k)} \in \overline{A}$, we have $f_{i(k)} = \lim g_j^k$, where $g_j^k \in A$ for $j = 1, 2, \ldots$, and there exists a positive integer $j(k)$ such that

$$(6) \qquad |f_{i(k)}(x) - g_{j(k)}^k(x)| \leq 1/2k \quad \text{for} \quad x \in X.$$

The function $g_k = g_{j(k)}^k$ belongs to A for $k = 1, 2, \ldots$; as from (5) and (6) it follows that $f = \lim g_k$, we have $f \in \overline{A}$, which completes the proof of (4). ∎

The topology generated on R^X – according to Proposition 1.2.7 – by the closure operator defined by formula (1) is called the *topology of uniform convergence* on R^X. The reader can easily verify that for any $f \in R^X$, the family $\{U_i(f)\}_{i=1}^{\infty}$, where

$$U_i(f) = \{g \in R^X \ : \ \text{there exists an } a < 1/i \text{ such that } |f(x) - g(x)| < a \text{ for } x \in X\},$$

is a base at the point f for the space R^X with the topology of uniform convergence.

We are not going to study the topology of uniform convergence now; its nature will be clarified in Sections 4.2 and 8.2. Let us observe, however, that the topology of uniform convergence on R^X induces the topology of a subspace on I^X and that Theorem 1.4.7 implies

2.6.2. PROPOSITION. *For every topological space X the set I^X is closed in the space R^X with the topology of uniform convergence.* ■

Now let X and Y be arbitrary topological spaces; for $A \subset X$ and $B \subset Y$ define

$$M(A,B) = \{f \in Y^X : f(A) \subset B\}. \tag{7}$$

Denote by $\mathcal{F}$ the family of all finite subsets of X and let $\mathcal{O}$ be the topology of Y. The family $\mathcal{B}$ of all sets $\bigcap_{i=1}^k M(A_i, U_i)$, where $A_i \in \mathcal{F}$ and $U_i \in \mathcal{O}$ for $i = 1, 2, \ldots, k$, generates – according to Proposition 1.2.1 – a topology on Y^X; this topology is called the *topology of pointwise convergence* on Y^X. The family $\mathcal{B}$ is a base for the space Y^X with the topology of pointwise convergence.

2.6.3. PROPOSITION. *The topology of pointwise convergence on Y^X coincides with the topology of a subspace of the Cartesian product $\prod_{x \in X} Y_x$, where $Y_x = Y$ for every $x \in X$.*

PROOF. Every open set in Y^X with the topology of a subspace of the Cartesian product $\prod_{x \in X} Y_x$ is a union of sets of the form

$$Y^X \cap p_{x_1}^{-1}(U_1) \cap p_{x_2}^{-1}(U_2) \cap \ldots \cap p_{x_k}^{-1}(U_k), \tag{8}$$

where $x_i \in X$ and $U_i \in \mathcal{O}$ for $i = 1, 2, \ldots, k$. However,

$$Y^X \cap p_x^{-1}(U) = M(\{x\}, U), \tag{9}$$

so that all sets of the form (8), and all sets that are open with respect to the topology of a subspace of the Cartesian product, are open with respect to the topology of pointwise convergence.

Conversely, from (9) it follows that for $A = \{x_1, x_2, \ldots, x_k\} \in \mathcal{F}$ and $U \in \mathcal{O}$, we have

$$M(A,U) = Y^X \cap p_{x_1}^{-1}(U) \cap p_{x_2}^{-1}(U) \cap \ldots \cap p_{x_k}^{-1}(U),$$

so that all sets that are open with respect to the topology of pointwise convergence are open with respect to the topology of a subspace of the Cartesian product. ■

The last proposition together with Theorems 2.1.6 and 2.3.11 imply (cf. Example 2.6.7):

2.6.4. THEOREM. *If Y is a T_i-space, then the space Y^X with the topology of pointwise convergence also is a T_i-space for $i \leq 3\frac{1}{2}$.* ■

From Propositions 2.6.3 and 2.3.34 we obtain

2.6.5. PROPOSITION. *A net $\{f_\sigma, \sigma \in \Sigma\}$ in the space Y^X with the topology of pointwise convergence converges to $f \in Y^X$ if and only if the net $\{f_\sigma(x), \sigma \in \Sigma\}$ converges to $f(x)$ for every $x \in X$.* ■

The topology of pointwise convergence on Y^X might as well be defined as the topology of a subspace of a Cartesian product or as the topology generated by a family of mappings. The present definition – although it may seem unnecessarily complicated – is chosen for its similarity with the definition of compact-open topology on Y^X, where instead of the family $\mathcal{F}$ of finite subsets of X, the family $\mathcal{Z}(X)$ of all compact subsets of X is being considered (cf. Section 3.4).

2.6.6. PROPOSITION. *For every topological space X the topology of uniform convergence on R^X is finer than the topology of pointwise convergence.*

PROOF. The equivalence of conditions (i) and (v) in Proposition 1.4.1 shows that it suffices to prove that if $f \in R^X$ is in the closure of a set $A \subset R^X$ with respect to the topology of uniform convergence, then f is in the closure of A with respect to the topology of pointwise convergence. Let $U = R^X \cap \bigcap_{i=1}^k p_{x_i}^{-1}(U_i)$ be a neighbourhood of f in the topology of pointwise convergence; since the sets U_i are open in R, there exists an $\epsilon > 0$ such that $(f(x_i) - \epsilon, f(x_i) + \epsilon) \subset U_i$ for $i = 1, 2, \ldots, k$. As $f = \lim f_j$, where $f_j \in A$ for $j = 1, 2, \ldots$, there exists a j such that $|f(x) - f_j(x)| < \epsilon$ for every $x \in X$, in particular, $f_j(x_i) \in U_i$ for $i = 1, 2, \ldots, k$, and this shows that $U \cap A \neq \emptyset$. ∎

Let us observe that the topologies on R^X and Y^X defined above do not depend on the topology of the space X, although the sets R^X and Y^X obviously depend on it.

2.6.7. EXAMPLE. Taking $Y = K$ and $X = D$ we infer from Proposition 2.6.3 and Example 2.3.12 that the space Y^X with the topology of pointwise convergence need not be normal even for perfectly normal Y. ∎

2.6.8. EXAMPLE. On R^N the topology of uniform convergence differs from the topology of pointwise convergence. Indeed, one easily checks that the function $f_0 \in R^N$ defined by $f_0(x) = 0$ for $x \in N$ belongs to the closure of the set $A = \{f \in R^N : f(N) \subset \{0,1\}$ and $|f^{-1}(0)| < \aleph_0\} \subset R^N$ with respect to the topology of pointwise convergence but f_0 does not belong to the closure of A with respect to the topology of uniform convergence. ∎

For every topological space Y and a one-point space $\{p\}$ assigning to the point $y \in Y$ the element $i_Y(y)$ of $Y^{\{p\}}$, where $[i_Y(y)](p) = y$, defines a one-to-one mapping of Y onto $Y^{\{p\}}$. The mapping $i_Y : Y \to Y^{\{p\}}$ is a homeomorphism with respect to the topology of pointwise convergence on $Y^{\{p\}}$ and, for $Y = R$, also with respect to the topology of uniform convergence on $R^{\{p\}}$.

From Propositions 2.1.11 and 2.2.6 it follows that if $X_s \neq \emptyset$ for $s \in S$, then the combination ∇ is a one-to-one mapping of the Cartesian product $\prod_{s\in S}(Y^{X_s})$ onto the space $Y^{(\bigoplus_{s\in S} X_s)}$. Similarly, from Proposition 2.3.6 it follows that the diagonal $\triangle$ is a one-to-one mapping of the Cartesian product $\prod_{s\in S}(Y_s^X)$ onto the space $(\prod_{s\in S} Y_s)^X$. The reader can easily prove the following two propositions:

2.6.9. PROPOSITION. *For every family $\{X_s\}_{s\in S}$ of non-empty topological spaces and a topological space Y, the combination $\nabla : \prod_{s\in S}(Y^{X_s}) \to Y^{(\bigoplus_{s\in S} X_s)}$ is a homeomorphism with respect to the topology of pointwise convergence on function spaces.* ∎

2.6.10. PROPOSITION. *For every topological space X and a family $\{Y_s\}_{s\in S}$ of topological* *spaces, the diagonal* $\triangle : \prod_{s\in S}(Y_s^X) \to (\prod_{s\in S} Y_s)^X$ *is a homeomorphism with respect to the topology of pointwise convergence on function spaces.* ■

One readily verifies that for $Y = R$ the counterpart of Proposition 2.6.9 for the topology of uniform convergence holds if and only if the family $\{X_s\}_{s\in S}$ is finite (cf. Example 2.6.8).

Let us observe that any continuous mappings $g: Y \to Z$ and $h: T \to X$ induce mappings Φ_g of Y^X to Z^X and Ψ_h of Y^X to Y^T defined by letting

(10) $$\Phi_g(f) = gf \quad \text{for} \quad f \in Y^X \quad \text{and} \quad \Psi_h(f) = fh \quad \text{for} \quad f \in Y^X.$$

Since

(11) $$\Phi_g^{-1}(M(A,B)) = M(A, g^{-1}(B)) \quad \text{and} \quad \Psi_h^{-1}(M(A,B)) = M(h(A), B),$$

both Φ_g and Ψ_h are continuous with respect to the topology of pointwise convergence on function spaces. One can readily check that for any continuous mapping $h: T \to X$ the mapping $\Psi_h: R^X \to R^T$ is also continuous with respect to the topology of uniform convergence on R^X and R^T. On the other hand, the mapping Φ_g of R^X into itself need not be continuous even for a homeomorphism $g: R \to R$ (cf. Example 4.2.14). This shows that the relation between the topology on R and the topology of uniform convergence on R^X is not quite satisfactory. We shall show later that the topology of uniform convergence on R^X is associated with a particular metric on R, or – more exactly – with a particular uniform structure on R, rather than with the topology on R (see Theorem 4.2.20 and Exercises 4.2.A(c) and 8.1.A(b)).

If $i: Y \to Z$ is a homeomorphic embedding, then $\Phi_i: Y^X \to Z^X$ also is a homeomorphic embedding with respect to the topology of pointwise convergence on function spaces. If $i: T \to X$ is an embedding, then $\Psi_i: Y^X \to Y^T$ is the restriction; generally Ψ_i is not one-to-one and is not a mapping onto, because some elements of Y^T may be not extendable over X. On the other hand, if $h: T \to X$ is a continuous mapping onto then – as one readily checks – $\Psi_h: Y^X \to Y^T$ is a homeomorphic embedding both with respect to the topology of pointwise convergence and, when $Y = R$, with respect to the topology of uniform convergence.

The mappings Φ_g and Ψ_h are connected with the operation Σ of composition of mappings; in fact, (10) can be restated as

$$\Phi_g(f) = \Sigma(g, f) \quad \text{and} \quad \Psi_h(f) = \Sigma(f, h).$$

The question arises whether one can define a topology on function spaces Z^Y, Y^X and Z^X in such a way that Σ would be a continuous mapping of the Cartesian product $Z^Y \times Y^X$ to Z^X. It turns out that under natural supplementary assumptions, to exclude for example the discrete topology, in order to define such a topology on function spaces we must restrict the class of spaces under consideration (see Theorem 3.4.2 and Exercise 3.4.A).

The mapping Ω of $Y^X \times X$ to Y defined by $\Omega(f, x) = f(x)$ is called the *evaluation mapping* of Y^X. It is also connected with the operation Σ; namely, Ω is the composition of the mappings

(12) $$Y^X \times X \xrightarrow{\mathrm{id}_{Y^X} \times i_X} Y^X \times X^{\{p\}} \xrightarrow{\Sigma} Y^{\{p\}} \xrightarrow{i_Y^{-1}} Y, \quad \text{i.e.,} \quad \Omega = i_Y^{-1}\Sigma(\mathrm{id}_{Y^X} \times i_X).$$

One easily sees that the formula

(13) $$\{[\Lambda(f)](z)\}(x) = f(z, x),$$

where f is a mapping of $Z \times X$ to Y, defines a one-to-one correspondence Λ between the set of all (not necessarily continuous) mappings of $Z \times X$ to Y and the set of all mappings of Z to the set of all mappings of X to Y; this correspondence is called the *exponential mapping.* It is natural to study the behaviour of Λ and of Λ^{-1} on the sets of continuous mappings $Y^{(Z\times X)}$ and $(Y^X)^Z$, the last set being well-defined if there is a topology on Y^X.

First of all the question arises whether $\Lambda(f)$ belongs to $(Y^X)^Z$ for an $f \in Y^{(Z\times X)}$ – this will happen if the topology of Y^X is not too fine. Another natural question is whether $\Lambda^{-1}(g)$ belongs to $Y^{(Z\times X)}$ for a $g \in (Y^X)^Z$ – this will happen if the topology of Y^X is not too coarse. Finally, one can ask when the exponential mapping is continuous and under what circumstances it is a homeomorphism.

We say that a topology on Y^X is *proper* if for every space Z and any $f \in Y^{(Z\times X)}$ the mapping $\Lambda(f)$ belongs to $(Y^X)^Z$. Similarly, we say that a topology on Y^X is *admissible* if for every space Z and any $g \in (Y^X)^Z$ the mapping $\Lambda^{-1}(g)$ belongs to $Y^{(Z\times X)}$. A topology on Y^X that is both proper and admissible is called an *acceptable* topology.

2.6.11. PROPOSITION. *A topology on Y^X is admissible if and only if the evaluation mapping of Y^X is continuous, i.e., if $\Omega: Y^X \times X \to Y$.*

PROOF. If the topology on Y^X is admissible, then for $Z = Y^X$ and $g = \mathrm{id}_{Y^X}$ the mapping $\Lambda^{-1}(g): Y^X \times X \to Y$ is continuous. Since

$$\{[\Lambda(\Omega)](f)\}(x) = \Omega(f, x) = f(x), \quad \text{i.e.,} \quad \Lambda(\Omega) = \mathrm{id}_{Y^X}, \tag{14}$$

it follows that $\Omega = \Lambda^{-1}(g)$, so that Ω is continuous.

Conversely, if the evaluation mapping of Y^X is continuous, then for every space Z and any $g \in (Y^X)^Z$ the mapping $\Lambda^{-1}(g)$ is continuous because $\Lambda^{-1}(g) = \Omega(g \times \mathrm{id}_X): Z \times X \to Y$. Indeed, for any $(z, x) \in Z \times X$ we have

$$\{[\Lambda(\Omega(g \times \mathrm{id}_X))](z)\}(x) = [\Omega(g \times \mathrm{id}_X)](z, x) = \Omega((g(z), x)) = [g(z)](x),$$

so that $\Lambda(\Omega(g \times \mathrm{id}_X)) = g$ and $\Lambda^{-1}(g) = \Omega(g \times \mathrm{id}_X)$. ■

2.6.12. PROPOSITION. *For every pair X, Y of topological spaces and any two topologies $\mathcal{O}, \mathcal{O}'$ on the function space Y^X we have:*

(i) *If the topology $\mathcal{O}$ is proper and $\mathcal{O}' \subset \mathcal{O}$, then the topology $\mathcal{O}'$ is proper.*

(ii) *If the topology $\mathcal{O}$ is admissible and $\mathcal{O} \subset \mathcal{O}'$, then the topology $\mathcal{O}'$ is admissible.*

(iii) *If the topology $\mathcal{O}$ is proper and the topology $\mathcal{O}'$ is admissible, then $\mathcal{O} \subset \mathcal{O}'$.*

(iv) *On Y^X there exists at most one acceptable topology.*

PROOF. Both (i) and (ii) follow directly from the definitions of proper and admissible topology and (iv) is a consequence of (iii). To prove (iii) let us take a proper topology $\mathcal{O}$ and an admissible topology $\mathcal{O}'$ on Y^X and let us denote the space $(Y^X, \mathcal{O})$ by Y_p^X and the space $(Y^X, \mathcal{O}')$ by Y_a^X.

The definitions of an admissible and a proper topology give inclusions

$$\Lambda^{-1}[(Y_a^X)^{Y_a^X}] \subset Y^{(Y_a^X \times X)} \quad \text{and} \quad \Lambda[Y^{(Y_a^X \times X)}] \subset (Y_p^X)^{(Y_a^X)},$$

which imply that $\mathrm{id}_{Y^X} = \Lambda\Lambda^{-1}(\mathrm{id}_{Y^X}) \in (Y_p^X)^{(Y_a^X)}$, i.e., that $\mathcal{O} \subset \mathcal{O}'$. ■

From Proposition 2.3.6 it follows that the topology of pointwise convergence is proper; indeed, $\Lambda(f)$ is continuous if and only if $p_{x_0}\Lambda(f)$ is continuous for every $x_0 \in X$ and from (13) it follows that $[p_{x_0}\Lambda(f)](z) = f(z, x_0)$, so that if $f \in Y^{(Z\times X)}$, then $\Lambda(f) \in (Y^X)^Z$. On the other hand, the topology of pointwise convergence is generally not admissible; indeed for this topology the fact that g is in $(Y^X)^Z$ means that for all $z_0 \in Z$ and $x_0 \in X$ the mappings $[g(z_0)](x)$ and $[g(z)](x_0)$ are continuous, while the fact that $\Lambda^{-1}(g)$ is in $Y^{(Z\times X)}$ means that g is continuous with respect to both coordinates.

From Proposition 2.6.11 it follows immediately that the topology of uniform convergence is admissible. On the other hand, the topology of uniform convergence is generally not proper. We leave it to the reader to define a function $f: R \times R \to R$ such that $\Lambda(f)$ does not belong to $(R^R)^R$, where R^R has the topology of uniform convergence.

Let us note that part (iii) of Proposition 2.6.12 and the facts established in last two paragraphs give immediately Proposition 2.6.6.

Historical and bibliographic notes

The topology of uniform convergence stems from the classical notion of a uniformly convergent sequence of functions, and the topology of pointwise convergence stems from the notion of a convergent sequence of functions. Admissible topologies were first discussed by Arens in [1946]; they were defined via continuity of the evaluation mapping. In [1951] Arens and Dugundji defined proper topologies and proved Propositions 2.6.11 and 2.6.12.

Exercises

2.6.A. Note that the sets $M(\{x\}, U)$, where $x \in X$ and U belongs to a fixed subbase $\mathcal{P}$ for Y, form a subbase for the space Y^X with the topology of pointwise convergence.

2.6.B. Verify that for any two families $\{X_s\}_{s\in S}$ and $\{Y_s\}_{s\in S}$ of topological spaces the Cartesian product $\prod : \prod_{s\in S}(Y_s^{X_s}) \to (\prod_{s\in S} Y_s)^{(\prod_{s\in S} X_s)}$ is a homeomorphic embedding with respect to the topology of pointwise convergence on function spaces.

2.6.C (Arens and Dugundji [1951]). We say that a net $\{f_\sigma,\ \sigma \in \Sigma\}$ in the function space Y^X *converges continuously* to a mapping $f \in Y^X$ if for every net $\{x_{\sigma'},\ \sigma' \in \Sigma'\}$ in the space X and any $x \in \lim_{\sigma'\in\Sigma'} x_{\sigma'}$, the net $\{f_\sigma(x_{\sigma'}),\ (\sigma,\sigma') \in \Sigma \times \Sigma'\}$ in Y, where the set $\Sigma \times \Sigma'$ is directed by letting $(\sigma_1, \sigma_1') \le (\sigma_2, \sigma_2')$ whenever $\sigma_1 \le \sigma_2$ and $\sigma_1' \le \sigma_2'$, converges to the point $f(x)$ (cf. Exercise 4.2.E).

(a) Show that a topology on Y^X is proper if and only if for any net $\{f_\sigma,\ \sigma \in \Sigma\}$ in Y^X, continuous convergence of $\{f_\sigma,\ \sigma \in \Sigma\}$ to a mapping $f \in Y^X$ implies that $f \in \lim_{\sigma\in\Sigma} f_\sigma$ with respect to this topology.

(b) Show that a topology on Y^X is admissible if and only if for any net $\{f_\sigma, \sigma \in \Sigma\}$ in Y^X and any $f \in \lim_{\sigma\in\Sigma} f_\sigma$ with respect to this topology, the net $\{f_\sigma,\ \sigma \in \Sigma\}$ converges continuously to f.

2.6.D. (a) Note that the composition operation Σ generally is not continuous with respect to the topology of pointwise convergence.

(b) Verify that the composition operation Σ generally is not continuous with respect to the topology of uniform convergence.

(c) Prove that if topologies on Z^Y and Y^X are admissible, and topology on Z^X is proper, then the composition operation Σ is a continuous mapping of $Z^Y \times Y^X$ to Z^X.

Hint. Express Σ in terms of Ω and Λ.

2.6.E (Fox [1945]). Prove that there is no acceptable topology on the space R^Q, where Q is the space of rational numbers (cf. Theorem 3.4.3 and Exercise 3.4.A).

Hint. Show that if a topology on R^Q is proper then $\operatorname{Int} M(\overline{U},(0,1)) = \emptyset$ for every open subset $U \subset Q$. To that end, observe that $\overline{U}$ contains $D(\aleph_0)$ as a closed subspace and applying the Tietze-Urysohn theorem define for any $f \in M(\overline{U},(0,1))$ a function $F\colon [0,1] \times Q \to R$ such that $F(0,x) = f(x)$ for $x \in Q$ and $[\Lambda(F)](z) \notin M(\overline{U},(0,1))$ for $z > 0$.

2.6.F. Prove that for topological spaces X, Y and Z the exponential mapping $\Lambda\colon Y^{(Z\times X)} \to (Y^X)^Z$ is a homeomorphic embedding with respect to the topology of pointwise convergence on function spaces.

Hint. Apply Exercise 2.6.A.

2.6.G. Show that addition and subtraction of functions are continuous mappings of the Cartesian product $R^X \times R^X$ to R^X both with respect to the topology of uniform convergence and with respect to the topology of pointwise convergence on R^X. Verify that multiplication of functions is a continuous mapping of the Cartesian product $R^X \times R^X$ to R^X with respect to the topology of pointwise convergence. Observe that multiplication of functions by real numbers is not a continuous mapping of $R^X \times R$ to R^X with respect to the topology of uniform convergence.

2.7. Problems

Locally closed sets

2.7.1 (Kuratowski and Sierpiński [1921a]). A subset A of a topological space X is *locally closed* if every point $x \in A$ has a neighbourhood U in the space X such that the intersection $A \cap U$ is closed in the subspace U of X.

Show that for a subset A of a topological space X the following conditions are equivalent:

(1) *The set A is locally closed.*
(2) *The difference $\overline{A} \setminus A$ is closed.*
(3) *The set A is a difference of two closed sets.*

Separated F_σ-sets in normal spaces

2.7.2. (a) (Urysohn [1925]) Prove that, for every pair A, B of separated F_σ-sets in a normal space X, there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Hint. Define sequences $W_1, W_2, \ldots$ and $V_1, V_2, \ldots$ of open subsets of X satisfying (4) and (5) in the proof of Lemma 1.5.15.

(b) (Urysohn [1925]) Deduce from part (a) that normality is hereditary with respect to F_σ-sets (cf. Exercise 2.1.E(a)).

(c) (Bonan [1970]) Apply (a) to give a solution of Problem 1.7.15(b).

Hint (Katětov [1951a]). Modifying the proof of Urysohn's lemma, define for every rational number r an open set $V_r \subset X$ such that

$$g^{-1}((-\infty, r)) \subset V_r \subset \overline{V}_r \subset f^{-1}((-\infty, r])$$

and $\overline{V}_r \subset V_{r'}$ whenever $r < r'$.

Normally placed sets II (see Problems 1.7.6 and 3.12.25)

2.7.3 (Smirnov [1951c]). Prove the following properties of normally placed sets:

(a) If A_i is a normally placed set in a space X for $i = 1, 2, \ldots$, then the union $\bigcup_{i=1}^{\infty} A_i$ is normally placed in X.

(b) If A is a normally placed set in a space X and B is an F_σ-set in the subspace A, then B is normally placed in X.

(c) If A is a normally placed set in a space X and B is normally placed in the subspace A, then B is normally placed in X.

(d) If X is a normal space, then every normally placed subset of X with the subspace topology is normal.

Semicontinuous functions II (see Problems 1.7.14–1.7.16, 3.12.23(g) and 5.5.20)

2.7.4. (a) (Hausdorff [1919]) Note that the Tietze-Urysohn theorem is an easy consequence of the characterization of normality given in Problem 1.7.15(b).

(b) (Tong [1952]; for metric spaces, Tietze [1915]) Prove that a T_1-space X is normal if and only if, for every closed subset $A \subset X$ and any real-valued function k which is defined and bounded below on X and continuous at all points of A, there exists a continuous function $h: X \to R$ such that $h|A = k|A$ and $h(x) \le k(x)$ for every $x \in X$.

Hint. Show that the function g defined by letting $g(x) = \sup_U(\inf_{z\in U} k(z))$, where sup is taken over all neighbourhoods U of x, is lower semicontinuous and apply the characterization of normality given in Problem 1.7.15(b).

(c) (Tong [1952]; for metric spaces, Tietze [1915]) Prove that a T_1-space X is perfectly normal if and only if for every closed subset $A \subset X$ and any lower (upper) semicontinuous function f defined on X such that $f|A$ is continuous, there exists a sequence $\{f_i\}$ of continuous functions on X such that $f(x) = \lim f_i(x)$ for every $x \in X$, $f_i|A = f|A$ for $i = 1, 2, \ldots$, and $f_i(x) \le f_{i+1}(x)$ (and $f_i(x) \ge f_{i+1}(x)$) for $i = 1, 2, \ldots$ and $x \in X$.

Hint. Apply parts (c) and (b) of Problem 1.7.15.

Linearly ordered spaces II (see Problems 1.7.4, 3.12.3, 3.12.4, 3.12.12(f), 5.5.22, 6.3.2 and 8.5.13(j))

2.7.5. (a) Let X be a space with topology induced by the linear order $<$. Observe that for any subset M of X containing at least two elements, the topology of a subspace of X on M is finer than the topology induced on M by the restriction of the linear order $<$ to M; give an example of a closed-and-open subset M of a linearly ordered space X such that the two topologies on M are distinct. Verify that if M is dense in X in the sense of order, i.e., if for every $x, y \in X$ satisfying $x < y$ there exists a $z \in M$ such that $x < z < y$, then the two topologies on M coincide; show that this may be false if M is only assumed to be topologically dense in X, i.e., if $\overline{M} = X$.

(b) A subset C of a linearly ordered set X is *convex* if $(x, y) \subset C$ for any $x, y \in C$.

Observe that if a family $\mathcal{C}$ consists of convex subsets of X and $\bigcap \mathcal{C} \neq \emptyset$, then $\bigcup \mathcal{C}$ is a convex set. Show that any subset $M \subset X$ can be represented as the union of disjoint maximal convex sets, i.e., convex sets that cannot be enlarged to convex subsets of X contained in M; these sets are called the *convex components* of M. Verify that in every convex subset C of X containing at least two elements the topology of a subspace of X and the topology induced by the restriction of the linear order in X to C coincide. Show that the convex components of an open subset $M \subset X$ are open in X.

(c) (Bourbaki [1948]) Prove that every linearly ordered space is hereditarily normal.

Hint. Apply (b) and Problem 1.7.4(d).

Urysohn spaces and semiregular spaces II (see Problems 1.7.7–1.7.9 and 6.3.17)

2.7.6. (a) Verify that the property of being a Urysohn space is hereditary and multiplicative.

(b) Show that semiregularity is multiplicative and is hereditary with respect to open subsets and to dense subsets. Give an example to show that semiregularity is not hereditary with respect to closed subsets.

Embedding in Cartesian products

2.7.7 (Šanin [1944], Mrówka [1956]). Show that for a T_0-space X and a topological space Y the following conditions are equivalent:

(1) *The space X is embeddable in a power of Y.*

(2) *The family of all sets of the form $f^{-1}(U)$, where $f: X \to Y$ and U is an open subset of Y, is a subbase for X.*

(3) *The topology of X can be generated by a family of mappings into Y.*

(4) *For every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a continuous mapping f of X into a finite power of Y such that $f(x) \notin \overline{f(F)}$.*

2.7.8 (Mrówka [1956]). (a) Prove that there is no T_1-space Y such that every T_1-space is embeddable in a power of Y (cf. Problem 2.7.18(b)).

Hint. Apply Problem 2.7.7.

(b) For every $\mathfrak{m} \geq \aleph_0$ let $L(\mathfrak{m})$ be the space defined in Example 1.2.6, where $|X| = \mathfrak{m}$. Show that every T_1-space X such that $|X| \leq \mathfrak{m} \geq \aleph_0$ and $w(X) \leq \mathfrak{m}$ is embeddable in the $\mathfrak{m}$th power of $L(\mathfrak{m})$.

Cardinal functions II (see Problems 1.7.12, 1.7.13, 3.12.4, 3.12.7–3.12.11, 3.12.12(h), 3.12.12(j) and 8.5.17)

2.7.9. For a cardinal function f we denote by hf the cardinal function whose value on a space X is equal to $\sup f(M)$, where the sup is taken over all subspaces M of the space X; the function hf is called *hereditary f* – in this sense the terms such as *hereditary density,* *hereditary Souslin number* etc. will be used in the sequel.

(a) Verify that

$$hw(X) = w(X), \quad h\chi(X) = \chi(X) \quad \text{and} \quad h\tau(X) = \tau(X).$$

(b) Verify that the function hc defined in Problem 1.7.12 is the hereditary Souslin number and that $hc(X) = he(X)$ for every topological space X.

(c) Show that $hd(X) \geq \tau(X)$ for every topological space X and give an example of a space X such that $hd(X) > d(X)$ and $hd(X) > \tau(X)$.

(d) Note that if A is a dense subspace of X, then $c(A) = c(X)$, but not necessarily $d(A) \leq d(X)$.

(e) (Smirnov [1951d]; for $\aleph_0$, Sierpiński [1921a]) Prove that a topological space X satisfies $hd(X) \leq \aleph_\alpha$ if and only if for every increasing transfinite sequence $F_0 \subset F_1 \subset \ldots \subset F_\xi \subset \ldots$, $\xi < \omega_{\alpha+1}$ of closed subsets of X there exists a $\xi_0 < \omega_{\alpha+1}$ such that $F_\xi = F_{\xi_0}$ for every $\xi \geq \xi_0$ (cf. Problem 3.12.7(b)).

(f) (Sierpiński [1921a]) Give an example of a Hausdorff space X such that $hd(X) > hc(X) = \aleph_0$.

Hint. Take R with the topology generated by the base consisting of all sets of the form $(a,b) \setminus A$, where $|A| \leq \aleph_0$.

Remark. Todorčević described in [1986] a completely regular space X satisfying $hd(X) > hc(X) > \aleph_0$. No example of a regular space X such that $hd(X) > hc(X) = \aleph_0$ is known within the realm of classical set theory. The existence of such a space is connected with *Souslin's problem* – the question, raised by M. Souslin in 1920, whether there exists a linearly ordered space X such that $c(X) = \aleph_0$ and $d(X) > \aleph_0$ (a *Souslin space*); indeed, one proves that the inequality $hd(X) > hc(X) = \aleph_0$ holds for any Souslin space X (see Problem 3.12.4(e)). As shown by Jech in [1967] and by Tennenbaum in [1968], the existence of a Souslin space is independent of the axioms of set theory (the history and connections of Souslin's problem are thoroughly discussed by Kurepa in [1968] and by M. E. Rudin in [1969]).

A hereditarily normal space X such that $hd(X) > hc(X) = \aleph_0$ was constructed in [1974] by Hajnal and Juhász under the assumption of the continuum hypothesis; a much simpler construction was described by van Douwen, Tall and Weiss in [1977] (cf. the remark to Problem 3.12.7(c)).

(g) (Zenor [1980]) Show that if a family $\{X_s\}_{s\in S}$ of topological spaces has the property that $hd(\prod_{s\in S_0} X_s) \leq \mathfrak{m}$ for every finite $S_0 \subset S$ and if $|S| \leq \mathfrak{m}$, then $hd(\prod_{s\in S} X_s) \leq \mathfrak{m}$.

2.7.10. (a) Give an example of a perfectly normal space X such that $hd(X) = hc(X) = e(X) = \aleph_0$ and $hd(X \times X) = hc(X \times X) = e(X \times X) = \mathfrak{c}$.

Hint. See Exercise 2.1.I.

(b) (Kurepa [1950]) Prove that if X is a Souslin space, then $c(X \times X) > \aleph_0$.

Hint. For every $\alpha < \omega_1$ define by transfinite induction points $a_\alpha < b_\alpha < c_\alpha$ in X in such a way that $b_\beta \notin (a_\alpha, c_\alpha)$ whenever $\beta < \alpha$. To that end, choose arbitrarily $a_0 < b_0 < c_0$ and assuming that $a_\beta, b_\beta, c_\beta$ are already defined for $\beta < \alpha$ find a non-empty interval (a_α, c_α) disjoint from $A \cup \{b_\beta : \beta < \alpha\}$, where A is the set of isolated points of X. Observe that $[(a_\beta, b_\beta) \times (b_\beta, c_\beta)] \cap [(a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha)] = \emptyset$ for $\beta < \alpha$.

Remark. From (b) and the remark in the preceding problem it follows that one cannot prove that the Souslin property is finitely multiplicative. It turns out that finite multiplicativity of the Souslin property is independent of the axioms of set theory (see Arhangel'skiĭ

[1971] or Juhász [1971]). Todorčević described in [1986] a compact space X satisfying $c(X \times X) > c(X) > \aleph_0$.

(c) (Šanin [1948] (announcement [1946])) Show that for every family $\{S_t\}_{t\in T}$ of finite sets, where $|T| = \mathfrak{m} > \aleph_0$ is a regular cardinal number, there exists a set $T_0 \subset T$ of cardinality $\mathfrak{m}$ and a set Z such that $S_t \cap S_{t'} = Z$ for any pair t, t' of distinct elements of T_0.

Hint (Juhász [1971]). One can assume that $|S_t| = n < \aleph_0$. Apply induction with respect to n. Consider a maximal subset T_1 of T such that $S_t \cap S_{t'} = \emptyset$ for any pair t, t' of distinct elements of T_1.

(d) Prove that the Cartesian product $\prod_{s\in S} X_s$ has the Souslin property if and only if for every finite $S_0 \subset S$ the Cartesian product $\prod_{s\in S_0} X_s$ has the Souslin property.

Hint. Apply (c) for $\mathfrak{m} = \aleph_1$.

Remark. Kurepa proved in [1962] that if $c(X_s) \leq \mathfrak{m}$ for every $s \in S$, then $c(\prod_{s\in S} X_s) \leq 2^{\mathfrak{m}}$; a proof of this fact can be also found in Juhász [1971] or [1980].

2.7.11 (Šanin [1948] (announcement [1946a])). We say that a cardinal number $\mathfrak{m} > \aleph_0$ is a *caliber* of a space X if every family of cardinality $\mathfrak{m}$ consisting of non-empty open subsets of X contains a subfamily of cardinality $\mathfrak{m}$ with non-empty intersection. The smallest cardinal number $\mathfrak{m} \geq \aleph_0$ such that every family of cardinality $> \mathfrak{m}$ consisting of non-empty open subsets of X contains a subfamily of cardinality $> \mathfrak{m}$ with non-empty intersection is called the *Šanin number* of the space X and is denoted by $\check{s}(X)$; clearly, $\check{s}(X)$ is the smallest cardinal number $\mathfrak{m} \geq \aleph_0$ with the property that the cardinal number $\mathfrak{m}^+$ is a caliber of the space X.

(a) Note that $c(X) \leq \check{s}(X) \leq d(X)$ for every topological space X and give examples of Hausdorff spaces X and Y such that $\check{s}(X) > hc(X)$ and $d(Y) > \check{s}(Y)$.

(b) Prove that if a regular cardinal number is a caliber of X_s for every $s \in S$, then it is also a caliber of the Cartesian product $\prod_{s\in S} X_s$. Deduce that if $\check{s}(X_s) = \mathfrak{m}$ for every $s \in S$, then $\check{s}(\prod_{s\in S} X_s) = \mathfrak{m}$.

Hint. First consider finite products and then apply Problem 2.7.10(c).

(c) Give examples of completely regular spaces X and Y such that $\check{s}(X) > c(X)$ and $d(Y) > \check{s}(Y)$.

Remark. The existence of normal spaces satisfying the above inequalities follows from Problem 2.7.15 and Theorems 2.3.17 and 3.1.9.

Functions on Cartesian products

2.7.12. (a) (Ross and Stone [1964]) Let $\{X_s\}_{s\in S}$ be a family of separable spaces. Show that the closure of any open set $U \subset \prod_{s\in S} X_s$ (i.e., every closed domain; cf. Exercise 1.1.C(e)) depends on countably many coordinates, i.e., that there exists a countable set $S_0 \subset S$ such that if $\{x_s\} \in \overline{U}$, $\{y_s\} \in \prod_{s\in S} X_s$ and $y_s = x_s$ for $s \in S_0$, then $\{y_s\} \in \overline{U}$.

Hint. Consider a maximal family of disjoint members of the canonical base for $\prod_{s\in S} X_s$ that are contained in U and apply Corollary 2.3.18.

(b) (Bockstein [1948], Ross and Stone [1964]) Applying (a) show that for every pair of disjoint open sets $U, V \subset \prod_{s\in S} X_s$, where all X_s are separable spaces, there exists a countable set $S_0 \subset S$ such that the projections of U and V onto $\prod_{s\in S_0} X_s$ are disjoint. Verify that if all X_s are second-countable, then there exist open sets $U_1, V_1 \subset \prod_{s\in S} X_s$ which are countable unions of members of the canonical base for $\prod_{s\in S} X_s$ such that $U \subset U_1, V \subset V_1$

and $U_1 \cap V_1 = \emptyset$.

(c) Apply (b) to solve Exercise 2.3.E(a).

(d) (Mazur [1952], Corson [1959], Corson and Isbell [1960], Ross and Stone [1964]) Let $\{X_s\}_{s\in S}$ be a family of topological spaces; we say that a continuous mapping $f: A \to Y$ of a subspace A of the Cartesian product $\prod_{s\in S} X_s$ to a space Y *depends on countably many coordinates* if there exists a countable set $S_0 \subset S$ such that $f(x) = f(y)$ for every pair $x = \{x_s\}$, $y = \{y_s\}$ of points of A satisfying $x_s = y_s$ for $s \in S_0$; otherwise, we say that f *depends on uncountably many coordinates.*

Prove that if $\{X_s\}_{s\in S}$ is a family of separable spaces and Y a second-countable Hausdorff space, then every continuous mapping $f : \prod_{s\in S} X_s \to Y$ depends on countably many coordinates and that there exist a countable set $S_0 \subset S$ and a continuous mapping $f_0 : \prod_{s\in S_0} X_s \to Y$ such that f coincides with the composition $f_0 p_{S_0}$ of the projection $p_{S_0} : \prod_{s\in S} X_s \to \prod_{s\in S_0} X_s$ and the mapping f_0 (cf. Exercises 3.2.H and 4.1.G and Problem 2.7.14).

2.7.13. (a) (Noble and Ulmer [1972]) Let $\{X_s\}_{s\in S}$ be a family of completely regular spaces, where $|S| > \aleph_0$ and $|X_s| > 1$ for $s \in S$. Prove that every continuous real-valued function on the Cartesian product $X = \prod_{s\in S} X_s$ depends on countably many coordinates if and only if every locally finite family of non-empty open subsets of X is countable.

Hint. Assuming that there exists a continuous real-valued function $f: X \to R$ which depends on uncountably many coordinates, observe that there exists an $\epsilon > 0$ with the property that for every countable $S' \subset S$ one can find $x', y' \in X$ such that $p_{S'}(x') = p_{S'}(y')$ and $|f(x') - f(y')| \geq \epsilon$. Then consider a maximal family $\{S_t\}_{t\in T}$ of pairwise disjoint finite subsets of S with the property that there exist points $x_t, y_t \in X$ such that $|f(x_t) - f(y_t)| \geq \epsilon/2$ and $p_s(x_t) = p_s(y_t)$ for $s \notin S_t$; show that $|T| > \aleph_0$. Find neighbourhoods $U_t = \prod_{s\in S} U_s^t$ and $V_t = \prod_{s\in S} V_s^t$ of x_t and y_t such that $U_s^t = V_s^t$ for $s \notin S_t$ and $|f(x) - f(y)| \geq \epsilon/4$ whenever $x \in U_t$ and $y \in V_t$ and show that the family $\{U_t\}_{t\in T}$ is locally finite.

Assuming that X contains a locally finite family $\{U_t\}_{t\in T}$ of non-empty open sets with $|T| = \aleph_1$ and that $U_t = \prod_{s\in S} U_s^t$, where $U_s^t = X_s$ for all s in the complement of a finite set $S_t \subset S$, for each $t \in T$ find an $s_t \in S \setminus S_t$ in such a way that $s_t \neq s_{t'}$, for $t \neq t'$ and define continuous functions $f_t : \prod_{s\in S_t} X_s \to I$ and $g_t: X_{s_t} \to I$ such that $f_t p_{S_t}(x) = 0$ for $x \notin U_t$, $f_t^{-1}(1) \neq \emptyset$ and $g_t^{-1}(0) \neq \emptyset \neq g_t^{-1}(1)$; then show that the function $f: X \to R$, where $f(x)$ is the sum of the numbers $(f_t p_{S_t}(x)) \cdot (g_t p_{s_t}(x))$ (of whose only finitely many differs from 0 for each $x \in X$), depends on uncountably many coordinates.

(b) (Noble and Ulmer [1972], Ščepin [1976]) Let $\{X_s\}_{s\in S}$ be a family of completely regular spaces, where $|S| > \aleph_0$ and $|X_s| > 1$ for $s \in S$, and let $X = \prod_{s\in S} X_s$. Prove that the following conditions are equivalent:

(1) *Every continuous real-valued function defined on an open subspace of X depends on countably many coordinates.*

(2) *The closure of any open subset of X depends on countably many coordinates.*

(3) *The space X has the Souslin property.*

Hint. To prove the implication (2)⇒(3), assume that X contains a family $\{U_t\}_{t\in T}$ of pairwise disjoint non-empty members of the canonical base for X with $|T| = \aleph_1$, for each $t \in T$ find an $s_t \in S$ in such a way that $\overline{p_{s_t}(U_t)} = X_{s_t}$ and $s_t \neq s_{t'}$, for $t \neq t'$, and define a non-empty open set $V_t \subset U_t$ such that $\overline{p_{s_t}(V_t)} \neq X_{s_t}$; then show that the set $\overline{\bigcup_{t\in T} V_t}$ depends on uncountably many coordinates. To prove that (3)⇒(1) consider a continuous real-valued function f defined on an open subspace U of X and use the hint to Problem 2.7.12(a) to obtain for $i = 1, 2, \ldots$ a countable family $\mathcal{V}_i$ of members of the canonical base for X such that $U \subset \overline{\bigcup \mathcal{V}_i}$ and $\delta(f(V)) < 1/i$ for $V \in \mathcal{V}_i$.

Remark. Further information about functions on Cartesian products can be found in Engelking [1966] and Hušek [1978].

Σ-products I (see Problems 3.12.24, 4.5.12 and Exercise 3.10.D)

2.7.14 (Corson [1959], A. M. Gleason cited in Isbell [1964]). Let $\{X_s\}_{s\in S}$ be a family of topological spaces and let $a = \{a_s\}$ be a point of the Cartesian product $\prod_{s\in S} X_s$; by $\Sigma(a)$ we denote the subspace of $\prod_{s\in S} X_s$ consisting of all points $\{x_s\}$ such that $x_s \neq a_s$ only for countably many $s \in S$. All subspaces of $\prod_{s\in S} X_s$ that are of the form $\Sigma(a)$ for an $a \in \prod_{s\in S} X_s$ are called *Σ-products* of the spaces $\{X_s\}_{s\in S}$.

(a) Prove that if $\{X_s\}_{s\in S}$ is a family of separable spaces and Y is a Hausdorff space whose one-point subsets are G_δ-sets, then for every $a \in \prod_{s\in S} X_s$ and every continuous mapping $f : \Sigma(a) \to Y$ there exist a countable set $S_0 \subset S$ and a continuous mapping $f_0 : \prod_{s\in S_0} X_s \to Y$ such that f coincides with the composition $f_0(p_{S_0}|\Sigma(a))$ of the restriction of the projection $p_{S_0} : \prod_{s\in S} X_s \to \prod_{s\in S_0} X_s$ to $\Sigma(a)$ and the mapping f_0; observe that, in particular, f depends on countably many coordinates.

Hint. First of all show that for every $x = \{x_s\} \in \Sigma(a)$ there exists a countable set $S(x) \subset S$ such that, if $x' = \{x'_s\} \in \Sigma(a)$ and $x'_s = x_s$ for $s \in S(x)$, then $f(x') = f(x)$. Then define inductively countable subsets $\Sigma_1, \Sigma_2, \ldots$ of $\Sigma(a)$, where $\Sigma_1 = \{a\}$, and countable subsets $S_1, S_2, \ldots$ of S such that $S_i = \bigcup_{j=1}^{i} \bigcup_{x\in\Sigma_j} S(x)$ and the projection of Σ_{i+1} onto $\prod_{s\in S_i} X_s$ is dense in $\prod_{s\in S_i} X_s$. Finally consider the set $S_0 = \bigcup_{i=1}^{\infty} S_i$.

(b) Deduce from the above that under the assumptions of (a) every continuous mapping $f : \Sigma(a) \to Y$ is continuously extendable over $\prod_{s\in S} X_s$.

(c) Show that (a) remains valid when $\Sigma(a)$ is replaced by any open subset of $\prod_{s\in S} X_s$, i.e., that under the assumptions of (a) for every open set $U \subset \prod_{s\in S} X_s$ and every continuous mapping $f : U \to Y$ there exist a countable set $S_0 \subset S$ and a continuous mapping $f_0 : p_{S_0}(U) \to Y$ such that $f = f_0(p_{S_0}|U)$; note that this gives a solution of Problem 2.7.12(b).

2.7.15 (Kombarov and Malyhin [1973]). Prove that if $\{X_s\}_{s\in S}$ is a family of topological spaces with the property that for every countable set $S_0 \subset S$ the Cartesian product $\prod_{s\in S_0} X_s$ is normal and hereditarily separable, then any Σ-product of spaces $\{X_s\}_{s\in S}$ is normal (cf. Problem 2.7.9(g)).

Hint. Consider the Σ-product $\Sigma(a)$, where $a = \{a_s\} \in \prod_{s\in S} X_s$; for any $x = \{x_s\}$ in $\Sigma(a)$ let $S(x) = \{s \in S : x_s \neq a_s\}$ and let $S(M) = \bigcup_{x\in M} S(x)$ for any $M \subset \Sigma(a)$. For a pair A, B of disjoint closed subsets of $\Sigma(a)$ choose an $s_0 \in S$ and define inductively increasing sequences $S_1 \subset S_2 \subset \ldots$, where $S_1 = \{s_0\}$, $A_1 \subset A_2 \subset \ldots$ and $B_1 \subset B_2 \subset \ldots$ of countable subsets of S, A and B, respectively, such that for $i = 1, 2, \ldots$.

$$p_{S_i}(A) \subset \overline{p_{S_i}(A_i)}, \quad p_{S_i}(B) \subset \overline{p_{S_i}(B_i)} \quad \text{and} \quad S(A_i) \cup S(B_i) \subset S_{i+1}.$$

Observe that for $S_0 = \bigcup_{i=1}^{\infty} S_i$, $A_0 = \bigcup_{i=1}^{\infty} A_i$ and $B_0 = \bigcup_{i=1}^{\infty} B_i$ the inclusions $p_{S_0}(A) \subset \overline{p_{S_0}(A_0)}$ and $p_{S_0}(B) \subset \overline{p_{S_0}(B_0)}$ hold and that $\overline{p_{S_0}(A_0)} \cap \overline{p_{S_0}(B_0)} = \emptyset$.

Normality and related properties in Cartesian products I (see Problems 3.12.15, 3.12.20, 4.5.15, 4.5.16, 5.5.5, 5.5.6, 5.5.18, 5.5.19 and Exercise 2.3.E)

2.7.16. (a) (Katětov [1948]) Show that if the Cartesian product $X \times Y$ is hereditarily normal, then X is perfectly normal or all countable subsets of Y are closed.

Hint. Assume that there exists a closed set $F \subset X$ that is not a G_δ-set and a countable subset M of Y such that $\overline{M} \setminus M \neq \emptyset$; take a $y \in \overline{M} \setminus M$ and consider the subsets $A = F \times M$ and $B = \{X \setminus F) \times \{y\}$ of the Cartesian product $X \times Y$.

(b) (Cook and Fitzpatrick [1968]) Prove that the limit of an inverse sequence of perfectly normal spaces is perfectly normal.

Hint. Apply condition (iii) in Theorem 1.5.19.

(c) (Katětov [1948]) Show that a countable Cartesian product $\prod_{i=1}^{\infty} X_i$ is perfectly normal if and only if all finite Cartesian products $X_1 \times X_2 \times \ldots \times X_i$ are perfectly normal.

(d) (Katětov [1948]) Prove that a countable Cartesian product $\prod_{i=1}^{\infty} X_i$, where $|X_i| > 1$ for $i = 1, 2, \ldots$, is perfectly normal if and only if it is hereditarily normal.

(e) Apply (a) to solve Exercise 2.3.E(b).

Remark. As observed by Michael in [1971], there exists a non-normal Cartesian product $\prod_{i=1}^{\infty} X_i$ such that all finite Cartesian products $X_1 \times X_2 \times \ldots \times X_i$ are hereditarily normal (in fact, hereditarily paracompact). Indeed, it suffices to take the space X in Example 5.1.32 as X_1, and the countably infinite discrete space N as X_i for $i > 1$ (see Example 5.1.32 and Exercise 4.3.G). Przymusiński's paper [1984] is a survey of recent results on normality of Cartesian products.

A regular space on which every continuous real-valued function is constant

2.7.17 (Hewitt [1946], Novák [1948]). (a) Define a regular space Z containing points z_0 and z_0' with the property that $f(z_0) = f(z_0')$ for every continuous function $f: Z \to R$.

Hint (Mysior [1981a]). Adjoin the point $z_0' = (0, -2)$ to the space M in Example 1.5.9 and let $\mathcal{B}(z_0') = \{U_i(z_0')\}_{i=1}^{\infty}$, where $U_i(z_0')$ consists of z_0' and all points $(x, y) \in M_0$ with $x \leq -i$, except for the points $(x, 0)$ with $-i - 2 \leq x \leq -i$.

(b) Define a regular space X such that $|X| > 1$ and every continuous function $f: X \to R$ is constant.

Hint (van Douwen [1972]). Let the space Z and the points z_0 and z_0' have the property stated in (a). For an arbitrary set S of cardinality $|Z|$ let $Y = \bigoplus_{s \in S}(Z \times \{s\})$, $Z_0 = \{(z_0, s) : s \in S\}$ and $Z_0' = \{(z_0', s) : s \in S\}$. Take an arbitrary one-to-one function g of Z_0 onto $Y \setminus (Z_0 \cup Z_0')$ and consider the quotient space $X = Y/E$, where E is the equivalence relation corresponding to the decomposition of Y consisting of the set Z_0' and the two-point sets $\{(z_0, s), g((z_0, s))\}$ for $s \in S$.

2.7.18 (Herrlich [1965a]). (a) For every T_1-space Y define a regular space X such that $|X| > 1$ and every continuous function $f: X \to Y$ is constant.

Hint (Brandenburg and Mysior [1984]). Define first a regular space Z containing points z_0 and z_0' with the property that $f(z_0) = f(z_0')$ for every continuous function $f: Z \to Y$ and then proceed as in the hint to Problem 2.7.17(b).

(b) Deduce from (a) that there is no T_1-space Y with the property that every regular space is embeddable in a power of Y.

Remark. Part (b) was obtained independently by Ramer in [1965].

Inverse systems I (see Problem 3.12.13 and 6.3.16)

2.7.19. (a) (Gentry [1969], E. Pol [1973]) Give an example of inverse sequences $\mathbf{S} = \{X_i, \pi^i_j\}$ and $\mathbf{S}' = \{Y_i, \tilde{\pi}^i_j\}$ of subspaces of the real line and of a mapping $\{\mathrm{id}_N, f_i\}$ of $\mathbf{S}$ to $\mathbf{S}'$, where π^i_j, $\tilde{\pi}^i_j$ and f_i are closed-and-open mappings onto, such that $\varprojlim\{\mathrm{id}_N, f_i\}$ is an onto mapping but is not quotient.

(b) (Lokucievskiĭ [1954]) Prove that if all bonding mappings π^i_j of an inverse sequence $\{X_i, \pi^i_j\}$ are open and onto then the projections π_i are open.

Remark. This is not true for arbitrary inverse systems (see E. Pol [1973]).

(c) Give an example of an inverse sequence $\{X_i, \pi^i_j\}$, where all bonding mappings π^i_j are open, such that the projections π_i are not open.

(d) (Zenor [1969]) Prove that if all bonding mappings π^i_j of an inverse sequence $\{X_i, \pi^i_j\}$ are closed, then the projections π_i also are closed.

Remark. This is not true for arbitrary inverse systems (see E. Pol [1973]).

(e) (E. Pol [1973]) Prove that if all bonding mappings π^i_j of an inverse sequence $\{X_i, \pi^i_j\}$ are hereditarily quotient, then the projections π_i also are hereditarily quotient.

Remark. This is not true for arbitrary inverse systems, even if the projections π_σ are mappings onto. If one assumes only that all bonding mappings π^i_j of an inverse sequence $\{X_i, \pi^i_j\}$ are quotient, then the projections π_i are not necessarily quotient (see E. Pol [1973]).

Spaces of closed subsets I (see Problems 3.12.27, 4.5.23, 6.3.22, 8.5.13(i) and 8.5.16)

2.7.20. (a) (Vietoris [1922]) For any topological space X we denote by 2^X the family of all non-empty closed subsets of X. Verify that the family $\mathcal{B}$ of all the sets of the form

$$\mathcal{V}(U_1, U_2, \ldots, U_k) = \{B \in 2^X : B \subset \bigcup_{i=1}^{k} U_i \text{ and } B \cap U_i \neq \emptyset \text{ for } i = 1, 2, \ldots, k\},$$

where $U_1, U_2, \ldots, U_k$ is a finite sequence of open subsets of X, satisfies conditions (B1) and (B2), so that it generates a topology on 2^X; this topology is called the *Vietoris topology on* 2^X and the set 2^X with the Vietoris topology is called the *exponential space* of X or the *hyperspace* of X.

(b) (Michael [1951]) For $i = 1, 2, \ldots$ let $J_i(X)$ be the subspace of 2^X consisting of all sets of cardinality $\leq i$ and let $J(X) = \bigcup_{i=1}^{\infty} J_i(X)$. Verify that if X is a T_1-space then by assigning to $(x_1, x_2, \ldots, x_i) \in X^i$ the point $\{x_1, x_2, \ldots, x_i\} \in J_i(X)$ we define a continuous mapping $j_i\colon X^i \to J_i(X) \subset 2^X$.

Note that if X is a T_1-space, then $J_1(X)$ is homeomorphic to X. Show that if X is a T_2-space, then all sets $J_i(X)$ are closed in 2^X and that if X is a T_1-space and $J_1(X)$ is closed in 2^X, then X is a T_2-space.

Verify that if X is a T_1-space, then $J(X)$ is dense in 2^X and deduce that $d(2^X) = d(X)$ for every infinite T_1-space X. Note that if X is a T_1-space then 2^X is dense in itself if and only if X is dense in itself.

(c) Show that if X is a T_2-space, then the mapping $j_i: X^i \to J_i(X)$ is closed for $i = 1, 2, \ldots$ Note that generally the mappings j_i are not open.

(d) (Michael [1951]) Verify that a mapping F assigning to each point y of a topological space Y a non-empty closed subset $F(y)$ of a topological space X is a continuous mapping of Y to the exponential space 2^X if and only if F is both lower and upper semicontinuous (see Problem 1.7.17).

(e) (Michael [1951]) Show that 2^X is always a T_0-space and that 2^X is a T_1-space provided that X is a T_1-space, but not vice versa. Prove that for a T_1-space X the exponential space 2^X is a Hausdorff (a regular or – equivalently – a completely regular) space if and only if X is a regular (a normal) space (cf. the remark to Problem 3.12.27(a)).

(f) (Ivanova [1955]) Prove that the exponential space 2^N is not normal.

Hint (Keesling [1970]). Decompose N into a union $N_1 \cup N_2$, where $N_1 \cap N_2 = \emptyset$ and $|N_1| = |N_2| = \aleph_0$. For $i = 1, 2$ take a one-to-one function f_i from N onto N_i and verify that $\{f_1(A) \cup f_2(N \setminus A) : A \subset N\}$ is a closed subspace of 2^N that is homeomorphic to $D(\mathfrak{c})$.

Set-valued mappings II (see Problems 1.7.17 and 3.12.28)

2.7.21 (Kuratowski [1963]). Prove that a T_1-space X is normal if and only if by assigning the intersection $A \cap B \subset X$ to each (A, B) in $2^X \times 2^X$, where 2^X is taken with the Vietoris topology, we define an upper semicontinuous set-valued mapping.

Chapter 3
Compact spaces

Compact spaces, the study of which is the main object of this chapter, are one of the most important classes of topological spaces. They are defined as Hausdorff spaces with the property that every cover by open sets contains a finite subcover. The class of compact spaces contains all bounded closed subsets of Euclidean spaces, and it turns out that many well-known properties of the latter are in fact properties of all compact spaces. In Section 3.10, three classes of spaces related to the class of compact spaces are studied; they all coincide with compact spaces in the realm of subspaces of Euclidean spaces, but in general they do not behave as well as the class of compact spaces. An investigation of those classes, as well as of the classes of Lindelöf spaces and realcompact spaces, gives a better insight into the role and place of compactness in general topology.

In Section 3.1 we state the definition of compact spaces, prove several rather simple theorems about compactness, and give a few examples. The notion of a network is also introduced in this section; it turns out to be a good tool in proving theorems on weight of compact spaces. We conclude by showing that the cardinality of any first-countable compact space does not exceed $\mathfrak{c}$.

Section 3.2 is devoted to an investigation of the behaviour of compact spaces under different operations defined in the preceding chapter. The Tychonoff theorem, asserting that any Cartesian product of compact spaces is a compact space (one of the most useful results of general topology), and the Stone-Weierstrass theorem are both proved in that section.

In Section 3.3 we discuss locally compact spaces and their quotient spaces, the k-spaces.

The compact-open topology on function spaces, already mentioned in Section 2.6, is discussed in Section 3.4. The last part of this section deals with Ascoli-type theorems that give necessary and sufficient conditions for compactness of sets in function spaces.

Section 3.5 is devoted to compactifications. In the family $\mathcal{C}(X)$ of all compactifications of a Tychonoff space X an order is defined and it is proved that in $\mathcal{C}(X)$ there exist all least upper bounds with respect to this order; it turns out that existence of greatest lower bounds is equivalent to local compactness of X.

In Section 3.6 we study the Čech-Stone compactification βX, the largest element of $\mathcal{C}(X)$, as well as the Wallman extension wX, a substitute for βX defined for all T_1-spaces.

Perfect mappings are defined and studied in Section 3.7. We show that this important class of mappings behaves well under operations on mappings and has numerous invariants and inverse invariants.

The last four sections are of a slightly more special character. Section 3.8 is devoted to Lindelöf spaces. In Section 3.9 we study Čech-complete spaces; among other things, we show that in this class of spaces – and, in particular, in the class of locally compact spaces – the Baire category theorem holds. Section 3.10 deals with three classes of spaces related to the class of compact spaces, namely countably compact, pseudo-compact and sequentially compact spaces. We give a few examples (involving the Čech-Stone compactification) to show that none of these three classes behaves as well with respect to operations as the class of compact spaces does. In particular, the Cartesian product of two countably compact (pseudocompact) spaces is not necessarily countably compact (pseudocompact). Realcompact spaces, which have some applications in functional analysis, are studied in Section 3.11.

3.1. Compact spaces

Let us recall that a cover of a set X is a family $\{A_s\}_{s\in S}$ of subsets of X such that $\bigcup_{s\in S} A_s = X$, and that – if X is a topological space – $\{A_s\}_{s\in S}$ is an open (a closed) cover of X if all sets A_s are open (closed). We say that a cover $\mathcal{B} = \{B_t\}_{t\in T}$ is a *refinement* of another cover $\mathcal{A} = \{A_s\}_{s\in S}$ of the same set X if for every $t \in T$ there exists an $s \in S$ such that $B_t \subset A_s$; in this situation we say also that $\mathcal{B}$ *refines* $\mathcal{A}$. A cover $\mathcal{A}' = \{A_{s'}\}_{s\in S'}$ of X is a *subcover* of another cover $\mathcal{A} = \{A_s\}_{s\in S}$ of X if $S' \subset S$ and $A'_s = A_s$ for every $s \in S'$. In particular, any subcover is a refinement.

A topological space X is called a *compact space* if X is a Hausdorff space and every open cover of X has a finite subcover, i.e., if for every open cover $\{U_s\}_{s\in S}$ of the space X there exists a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ such that $X = U_{s_1} \cup U_{s_2} \cup \ldots \cup U_{s_k}$. *

One readily sees that a Hausdorff space X is compact if and only if every open cover of X has a finite refinement. Since every open cover of a topological space has a refinement consisting of elements of any fixed base for this space, it turns out that a Hausdorff space X is compact if and only if every open cover of X by elements of a fixed base has a finite subcover (cf. Problem 3.12.2).

We say that a family $\mathcal{F} = \{F_s\}_{s\in S}$ of subsets of a set X has the *finite intersection property* if $\mathcal{F} \neq \emptyset$ and $F_{s_1} \cap F_{s_2} \cap \ldots \cap F_{s_k} \neq \emptyset$ for every finite set $\{s_1, s_2, \ldots, s_k\} \subset S$.

3.1.1. THEOREM. *A Hausdorff space X is compact if and only if every family of closed subsets of X which has the finite intersection property has non-empty intersection.*

PROOF. Let X be a compact space and $\{F_s\}_{s\in S}$ a family of closed subsets of X such that the intersection $\bigcap_{s\in S} F_s$ is empty. Consider open sets $U_s = X \setminus F_s$; as

$$\bigcup_{s\in S} U_s = \bigcup_{s\in S} (X \setminus F_s) = X \setminus \bigcap_{s\in S} F_s = X,$$

the family $\{U_s\}_{s\in S}$ is an open cover of X. The space X being compact, the cover $\{U_s\}_{s\in S}$ has a finite subcover $\{U_{s_1}, U_{s_2}, \ldots, U_{s_k}\}$. Hence

$$X = \bigcup_{i=1}^{k} U_{s_i} = \bigcup_{i=1}^{k} (X \setminus F_{s_i}) = X \setminus \bigcap_{i=1}^{k} F_{s_i},$$

which implies that $\bigcap_{i=1}^{k} F_{s_i} = \emptyset$; thus, if a family $\{F_s\}_{s\in S}$ of closed subsets of X has the finite intersection property, then $\bigcap_{s\in S} F_s \neq \emptyset$.

* The reader should be warned that some authors do not include the assumption that X is a Hausdorff space in the definition of compactness (cf. our notion of quasi-compactness defined at the end of this section).

The proof of compactness of a Hausdorff space in which families of closed subsets with the finite intersection property have non-empty intersections is left to the reader.■

The next theorem is a consequence of Theorem 3.1.1.

3.1.2. THEOREM. *Every closed subspace of a compact space is compact.*■

We shall now prove several theorems on compact subspaces of arbitrary topological spaces.

From the definition of the subspace topology, we immediately obtain

3.1.3. THEOREM. *If a subspace A of a topological space X is compact, then for every family $\{U_s\}_{s\in S}$ of open subsets of X such that $A \subset \bigcup_{s\in S} U_s$ there exists a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ such that $A \subset \bigcup_{i=1}^{k} U_{s_i}$.*■

3.1.4. COROLLARY. *Let X be a Hausdorff space and $\{F_1, F_2, \ldots, F_k\}$ a family of closed subsets of X. The subspace $F = \bigcup_{i=1}^{k} F_i$ of X is compact if and only if all subspaces F_i are compact.*■

3.1.5. COROLLARY. *Let U be an open subset of a topological space X. If a family $\{F_s\}_{s\in S}$ of closed subsets of X contains at least one compact set – in particular, if X is compact – and if $\bigcap_{s\in S} F_s \subset U$, then there exists a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ such that $\bigcap_{i=1}^{k} F_{s_i} \subset U$.*

PROOF. Let the set F_{s_0} be compact. Replacing the space X by F_{s_0}, the set U by $U \cap F_{s_0}$ and the family $\{F_s\}_{s\in S}$ by $\{F_{s_0} \cap F_s\}_{s\in S}$ we reduce the problem to the case of a compact space, so that we can assume that the space X is compact. Applying Theorem 3.1.3 to the sets $A = X \setminus U$ and $U_s = X \setminus F_s$ we obtain a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ with the required property.■

3.1.6. THEOREM. *If A is a compact subspace of a regular space X, then for every closed set B disjoint from A there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

If, moreover, B is a compact subspace of X, then it suffices to assume that X is a Hausdorff space.

PROOF. The space X being regular, for every $x \in A$ there exist open sets $U_x, V_x \subset X$ such that

$$(1) \qquad x \in U_x, \quad B \subset V_x \quad \text{and} \quad U_x \cap V_x = \emptyset.$$

Clearly $A \subset \bigcup_{x\in A} U_x$, so that by 3.1.3 there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset A$ such that $A \subset \bigcup_{i=1}^{k} U_{x_i}$. One readily verifies that the sets $U = \bigcup_{i=1}^{k} U_{x_i}$ and $V = \bigcap_{i=1}^{k} V_{x_i}$ have the required properties.

Let us observe that if B is a one-point set, then only the fact that X is a Hausdorff space is used in the proof of the first part of the theorem. Now if B is a compact subspace of X, then for every $x \in A$ we obtain open sets $U_x, V_x \subset X$ satisfying (1) by applying the above observation to the compact subspace B and the one-point set $\{x\}$.■

3.1.7. THEOREM. *If A is a compact subspace of a Tychonoff space X, then for every closed set B disjoint from A there exists a continuous function $f: X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

PROOF. For every $x \in A$ there exists a function $f_x: X \to I$ such that $f_x(x) = 0$ and $f_x(B) \subset \{1\}$. Since $A \subset \bigcup_{x \in A} f_x^{-1}([0, 1/2))$, by 3.1.3 there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset A$ such that $A \subset \bigcup_{i=1}^{k} f_{x_i}^{-1}([0, 1/2))$. The function $g = \min(f_{x_1}, f_{x_2}, \ldots, f_{x_k})$ satisfies the inclusions

$$A \subset g^{-1}([0, 1/2)) \text{ and } g(B) \subset \{1\}.$$

One readily verifies that the continuous function $f: X \to I$, defined by the formula $f(x) = 2\max(g(x) - 1/2, 0)$, has the required properties. ■

3.1.8. THEOREM. *Every compact subspace of a Hausdorff space X is a closed subspace of X.*

PROOF. Let A be a compact subspace of X. By the second part of 3.1.6, for every $x \in X \setminus A$ there exists an open set $V \subset X$ such that $x \in V$ and $A \cap V = \emptyset$, so that $X \setminus A$ is an open subset of X. ■

The second part of 3.1.6 together with 3.1.2 give

3.1.9. THEOREM. *Every compact space is normal.* ■

In the next three theorems we discuss properties of continuous mappings defined on compact spaces.

3.1.10. THEOREM. *If there exists a continuous mapping $f: X \to Y$ of a compact space X onto a Hausdorff space Y, then Y is a compact space.*

In other words, *a continuous image of a compact space is compact, provided it is a Hausdorff space.*

PROOF. Let $\{U_s\}_{s \in S}$ be an open cover of the space Y. The family $\{f^{-1}(U_s)\}_{s \in S}$ is an open cover of X; thus there exists a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ such that

$$f^{-1}(U_{s_1}) \cup f^{-1}(U_{s_2}) \cup \ldots \cup f^{-1}(U_{s_k}) = X,$$

and this implies that $U_{s_1} \cup U_{s_2} \cup \ldots \cup U_{s_k} = Y$. ■

3.1.11. COROLLARY. *If $f: X \to Y$ is a continuous mapping of a compact space X to a Hausdorff space Y, then the equality $\overline{f(A)} = f(\overline{A})$ holds for every $A \subset X$.*

PROOF. As f is continuous, we have $f(\overline{A}) \subset \overline{f(A)}$ by Proposition 1.4.1; the reverse inclusion follows from the definition of the closure and from Theorems 3.1.2, 3.1.10 and 3.1.8. ■

The last corollary gives immediately

3.1.12. THEOREM. *Every continuous mapping of a compact space to a Hausdorff space is closed.* ■

From 3.1.12 and 1.4.18 we obtain the following important theorem (see also Problem 3.12.5(e)):

3.1.13. THEOREM. *Every continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism.* ■

3.1.14. COROLLARY. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two topologies defined on a set X and let $\mathcal{O}_1$ be finer than $\mathcal{O}_2$. If the space $(X, \mathcal{O}_1)$ is compact and $(X, \mathcal{O}_2)$ is a Hausdorff space, then $\mathcal{O}_1 = \mathcal{O}_2$.*

In other words, *among all Hausdorff topologies, compact topologies are minimal.*

PROOF. The identity of X is a one-to-one continuous mapping of $(X, \mathcal{O}_1)$ onto $(X, \mathcal{O}_2)$, and by Theorem 3.1.13 this is a homeomorphism.∎

We shall now establish an interesting characterization of compact spaces in terms of Cartesian products.

3.1.15. LEMMA. *If A is a compact subspace of a space X and y a point of a space Y, then for every open set $W \subset X \times Y$ containing $A \times \{y\}$ there exist open sets $U \subset X$ and $V \subset Y$ such that $A \times \{y\} \subset U \times V \subset W$.*

PROOF. For every $x \in A$ the point (x, y) has a neighbourhood of the form $U_x \times V_x$ contained in W. Clearly $A \times \{y\} \subset \bigcup_{x\in A} U_x \times V_x$, so that by Theorem 3.1.3 there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset A$ such that $A \times \{y\} \subset \bigcup_{i=1}^{k} U_{x_i} \times V_{x_i}$. One readily verifies that the sets $U = \bigcup_{i=1}^{k} U_{x_i}$ and $V = \bigcap_{i=1}^{k} V_{x_i}$ have the required properties.∎

3.1.16. THE KURATOWSKI THEOREM. *For a Hausdorff space X the following conditions are equivalent:*

(i) *The space X is compact.*

(ii) *For every topological space Y the projection $p: X \times Y \to Y$ is closed.*

(iii) *For every normal space Y the projection $p: X \times Y \to Y$ is closed.*

PROOF. Let X be a compact space and $F = \overline{F} \subset X \times Y$. Take a point $y \notin p(F)$; as $X \times \{y\} \subset (X \times Y) \setminus F$, it follows from the lemma that y has a neighbourhood V such that $(X \times V) \cap F = \emptyset$. We then have $p(F) \cap V = \emptyset$, which shows that $p(F)$ is a closed subset of Y and yields the implication (i)⇒(ii).

The implication (ii)⇒(iii) being obvious, we pass to the proof that (iii)⇒(i). Let X be a Hausdorff space with property (iii); suppose that there exists a family $\{F_s\}_{s\in S}$ of closed subsets of X with the finite intersection property such that $\bigcap_{s\in S} F_s = \emptyset$. Take a point $y_0 \notin X$ and on the set $Y = X \cup \{y_0\}$ consider the topology consisting of all subsets of X and of all sets of the form

$$\{y_0\} \cup (F_{s_1} \cap F_{s_2} \cap \ldots \cap F_{s_k}) \cup K, \text{ where } s_1, s_2, \ldots, s_k \in S \text{ and } K \subset X.$$

From the equality $\bigcap_{s\in S} F_s = \emptyset$ it follows that Y is a T_1-space; since every subset of Y that does not contain y_0 is open, Y is a normal space.

As X has property (iii), the projection $p(F)$ of the set $F = \overline{\{(x, x) : x \in X\}} \subset X \times Y$ is closed in Y. From the inclusion $X \subset p(F)$ it follows that $y_0 \in p(F)$, because $y_0 \in \overline{X} = Y$; hence there exists a point $x_0 \in X$ such that $(x_0, y_0) \in F$. For every neighbourhood $U \subset X$ of x_0 and every $s \in S$ we have $[U \times (\{y_0\} \cup F_s)] \cap \{(x, x) : x \in X\} \neq \emptyset$, i.e., $U \cap F_s \neq \emptyset$, which shows that $x_0 \in F_s$ for every $s \in S$. This implies that $\bigcap_{s\in S} F_s \neq \emptyset$, and we have a contradiction.∎

3.1.17. REMARK. A slight change in the above argument (see Problem 3.12.14(a)) provides a compact space Y such that $w(Y) \leq w(X)$; hence, a Hausdorff space X is compact if and only if the projection $p: X \times Y \to Y$ is closed for every compact space Y such that $w(Y) \leq w(X)$.

We now introduce the concept of a network and the related concept of the network weight; both concepts are defined for arbitrary topological spaces, but they prove particularly useful when studying classes of topological spaces more or less related to the class of compact spaces.

A family $\mathcal{N}$ of subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighbourhood U of x there exists an $M \in \mathcal{N}$ such that $x \in M \subset U$. * Clearly, any base for X is a network for X: it is a network of a special kind, namely, all members of which are open. The family of all one-point subsets of a space is another example of a network. The *network weight* of a space X is defined as the smallest cardinal number of the form $|\mathcal{N}|$, where $\mathcal{N}$ is a network for X; this cardinal number is denoted by $nw(X)$. Clearly, for every topological space X we have $nw(X) \leq w(X)$ and $nw(X) \leq |X|$.

Let us note that if there exists a network $\mathcal{N}$ for X such that $|\mathcal{N}| \leq \mathfrak{m}$, then X has a dense subset of cardinality $\leq \mathfrak{m}$ (see the proof of Theorem 1.3.7), so that for every topological space X we have $d(X) \leq nw(X)$. The reverse inequality generally is not true: arguing as in Example 1.2.2 one easily shows that for the Sorgenfrey line K we have $nw(K) = \mathfrak{c}$, so that $nw(K) > d(K) = \aleph_0$. For every T_0-space X we have $|X| \leq \exp nw(X)$; this can be proved exactly as the weaker inequality in 1.5.1.

3.1.18. LEMMA. *For every Hausdorff space X there exists a continuous one-to-one mapping of X onto a Hausdorff space Y such that $w(Y) \leq nw(X)$.*

PROOF. Let $nw(X) = \mathfrak{m}$ and let $\mathcal{N}$ be a network for X satisfying $|\mathcal{N}| = \mathfrak{m}$. One can readily verify that, if $\mathfrak{m} < \aleph_0$, then X is a discrete space of cardinality $\mathfrak{m}$, so that $w(X) = \mathfrak{m}$ and one can take $Y = X$.

Suppose that $\mathfrak{m} \geq \aleph_0$. Denote by $\mathcal{O}_1$ the topology of the space X. Consider those pairs M_1, M_2 of members of $\mathcal{N}$ for which there exist disjoint open sets $U_1, U_2 \in \mathcal{O}_1$ containing M_1 and M_2 respectively, and choose for every such pair some open sets U_1, U_2; let $\mathcal{B}_0$ be the family of sets obtained in this way. The family $\mathcal{B}$ of all finite intersections of members of $\mathcal{B}_0$ has properties (B1)–(B2). From the definition of a network and the fact that $(X; \mathcal{O}_1)$ is a Hausdorff space it follows that the set X with the topology $\mathcal{O}_2$ generated by the base $\mathcal{B}$ is a Hausdorff space – denote this space by Y. As $|\mathcal{B}| \leq \mathfrak{m}$, we have $w(Y) \leq nw(X)$, and from the inclusion $\mathcal{O}_2 \subset \mathcal{O}_1$ it follows that the identity of X is a continuous one-to-one mapping of X onto Y.∎

The preceding lemma and Theorem 3.1.13 give

3.1.19. THEOREM. *For every compact space X we have $nw(X) = w(X)$.*∎

3.1.20. COROLLARY. *If a compact space X has a cover $\{A_s\}_{s \in S}$ such that $w(A_s) \leq \mathfrak{m} \geq \aleph_0$ for $s \in S$ and $|S| \leq \mathfrak{m}$, then $w(X) \leq \mathfrak{m}$.*

PROOF. The family $\bigcup_{s \in S} \mathcal{B}_s$, where $\mathcal{B}_s$ is a base of cardinality $\leq \mathfrak{m}$ for the subspace A_s, is a network for X and has cardinality $\leq \mathfrak{m}$.∎

Let us note that in the above corollary it suffices to assume that $nw(A_s) \leq \mathfrak{m}$.

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* Networks are sometimes called *nets* or *grids*; obviously they have nothing to do with the nets introduced in Section 1.6.

From Theorem 3.1.19 two further important theorems follow.

3.1.21. THEOREM. *For every compact space X we have $w(X) \leq |X|$.* ■

3.1.22. THEOREM. *If a compact space Y is a continuous image of a topological space X, then $w(Y) \leq w(X)$.*

PROOF. It follows from condition (iii) in Theorem 1.4.1 that if $f: X \to Y$ maps X onto Y, then the family $\{f(U) : U \in \mathcal{B}\}$, where $\mathcal{B}$ is a base for X, is a network for Y. ■

Clearly, under the assumption of the above theorem the stronger inequality $w(Y) \leq nw(X)$ holds.

As there exists countable spaces without a countable base (see Examples 1.6.19, 1.6.20 or 2.3.37), the compactness assumption in Theorems 3.1.19 and 3.1.21 cannot be omitted. Considering any mapping of the space N onto a countable space without a countable base, we see that the assumption of compactness cannot be omitted in Theorem 3.1.22, either. It turns out, however, that in all three of those theorems compactness can be replaced by weaker assumptions (see Theorems 3.3.5, 3.7.19 and Exercise 3.9.E).

The next theorem characterizes compactness in terms of nets.

3.1.23. THEOREM. *A Hausdorff space X is compact if and only if every net in X has a cluster point.*

PROOF. Let X be a compact space and $S = \{x_\sigma, \sigma \in \Sigma\}$ a net in X. The family $\{F_\sigma\}_{\sigma\in\Sigma}$, where

$$F_\sigma = \overline{\{x_{\sigma'} : \sigma \leq \sigma'\}},$$

consists of closed sets and has the finite intersection property, because $F_{\sigma_1} \subset F_{\sigma_2}$ whenever $\sigma_2 \leq \sigma_1$. By theorem 3.1.1 there exists an $x \in \bigcap_{\sigma\in\Sigma} F_\sigma$; the point x is a cluster point of S. Indeed, if x were not a cluster point of S, then there would exist a neighbourhood U of x and a $\sigma_0 \in \Sigma$ such that $U \cap \{x_{\sigma'} : \sigma_0 \leq \sigma'\} = \emptyset$, and we would have $x \notin F_{\sigma_0}$.

Now, let X be a Hausdorff space such that every net in X has a cluster point. Consider a family $\mathcal{F}$ of closed subsets of X that has the finite intersection property. Denote by Σ the set consisting of all finite subfamilies $\{F_1, F_2, \ldots, F_k\}$ of $\mathcal{F}$ and for $\sigma = \{F_1, F_2, \ldots, F_k\}$, $\sigma' = \{F_1', F_2', \ldots, F_l'\} \in \Sigma$ let $\sigma \leq \sigma'$ if $F_1 \cap F_2 \cap \ldots \cap F_k \supset F_1' \cap F_2' \cap \ldots \cap F_l'$. The set Σ is directed by $\leq$ and $S = \{x_\sigma, \sigma \in \Sigma\}$, where for every $\sigma = \{F_1, F_2, \ldots, F_k\} \in \Sigma$ the point x_σ is an arbitrary element of the intersection $F_1 \cap F_2 \cap \ldots \cap F_k$, is a net in X. To complete the proof it suffices to show that if x is a cluster point of S, then x belongs to all members of $\mathcal{F}$.

Let F_0 be a member of $\mathcal{F}$. For every neighbourhood U of x there exists a $\sigma = \{F_1, F_2, \ldots, F_k\} \geq \sigma_0 = \{F_0\}$ such that $x_\sigma \in U$. Since $x_\sigma \in F_1 \cap F_2 \cap \ldots \cap F_k \subset F_0$, we have $F_0 \cap U \neq \emptyset$ and, the set F_0 being closed, this proves that $x \in F_0$. ■

The filter counterpart of the above theorem reads as follows:

3.1.24. THEOREM. *A Hausdorff space X is compact if and only if every filter in X has a cluster point.* ■

3.1.25. EXAMPLES. The discrete space $D(\mathfrak{m})$ is compact if and only if $\mathfrak{m}$ is finite.

The real line and the Sorgenfrey line are not compact: the open cover $\{(-i,i)\}_{i=1}^{\infty}$ has no finite subcover.

We shall now show that the space $A(\mathfrak{m})$ defined in 1.4.20 is compact for every $\mathfrak{m} \geq \aleph_0$. Let $\{U_s\}_{s\in S}$ be an open cover of the space $A(\mathfrak{m})$. There exists an $s_0 \in S$ such that the unique accumulation point x_0 of $A(\mathfrak{m})$ belongs to U_{s_0}; by the definition of the topology on $A(\mathfrak{m})$, the set $A(\mathfrak{m}) \setminus U_{s_0}$ is finite. Let $A(\mathfrak{m}) \setminus U_{s_0} = \{x_1, x_2, \ldots, x_k\}$ and let $x_i \in U_{s_i}$ for $i = 1,2,\ldots,k$. Clearly $\{U_{s_i}\}_{i=0}^{k}$ is a finite subcover of $\{U_s\}_{s\in S}$; being a Hausdorff space, $A(\mathfrak{m})$ is compact.

Every open cover of the space X defined in 1.2.6 has a finite subcover, and yet the space X is not compact because it is not a Hausdorff space.

We shall prove that every closed interval $J = [a,b] \subset R$ is a compact space. Let $\{U_s\}_{s\in S}$ be an open cover of the space J and let A be the set of all $x \in J$ such that the interval $[a,x]$ is contained in the union of finitely many members of $\{U_s\}_{s\in S}$. It suffices to show that the set $J \setminus A$ is empty.

Assume that $J \setminus A \neq \emptyset$ and denote by x_0 the greatest lower bound of this set; clearly $x_0 \in J \setminus A$ and $x_0 \neq a$. Let $x_0 \in U_{s_0}$; as $x_0 > a$, there exists a $y < x_0$ such that $(y, x_0] \subset U_{s_0}$. By the definition of x_0 we have $y \in A$, so that $[a,y] \subset \bigcup_{i=1}^{k} U_{s_i}$ for some $s_1, s_2, \ldots, s_k \in S$. It follows that $[a, x_0] \subset \bigcup_{i=0}^{k} U_{s_i}$, and we have a contradiction. ■

3.1.26. EXAMPLE. Consider in the plane R^2 two concentric circles $C_i = \{(x,y) \in R^2 : x^2 + y^2 = i\}$, where $i = 1,2$, and their union $X = C_1 \cup C_2$; the projection of C_1 onto C_2 from the point $(0,0)$ will be denoted by p. On the set X we shall generate a topology by defining a neighbourhood system $\{\mathcal{B}(z)\}_{z\in X}$; namely let $\mathcal{B}(z) = \{\{z\}\}$ for $z \in C_2$ and for $z \in C_1$ let $\mathcal{B}(z) = \{U_j(z)\}_{j=1}^{\infty}$, where $U_j = V_j \cup p(V_j \setminus \{z\})$ and V_j is the arc of C_1 with centre at z and of length $1/j$. One easily verifies that the collection $\{\mathcal{B}(z)\}_{z\in X}$ has properties (BP1)–(BP4), so that – by Proposition 1.5.2 – the set X with the topology generated by the neighbourhood system $\{\mathcal{B}(z)\}_{z\in X}$ is a Hausdorff space.

The space X is called the *Alexandroff double circle.*

The subspace $C_2 \subset X$ is a discrete space of cardinality $\mathfrak{c}$; it is an open and dense subset of X. The subspace $C_1 \subset X$ is the unit circle S^1 with its usual topology; it is a compact subspace of X, because S^1, being a continuous image of I, is a compact space.

We shall now prove that X is a compact space. Let $\{U_s\}_{s\in S}$ be an open cover of the space X. Without loss of generality we can assume that the sets U_s are members of the neighbourhood system defined above. The subspace C_1 being compact, there exists a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ such that

$$C_1 \subset U_{s_1} \cup U_{s_2} \cup \ldots \cup U_{s_k}. \tag{2}$$

If we omit all one-point sets on the right-hand side of (2), the inclusion remains still valid; hence, we can assume that $U_{s_i} = U_{j_i}(z_i)$, where $z_i \in C_1$ for $i = 1,2,\ldots,k$. Thus we have

$$X \setminus \{p(z_1), p(z_2), \ldots, p(z_k)\} \subset U_{s_1} \cup U_{s_2} \cup \ldots \cup U_{s_k}.$$

Let us choose for $i = 1,2,\ldots,k$ an $s_i' \in S$ such that $p(z_i) \in U_{s_i'}$. Clearly the family $\{U_{s_i}\}_{i=1}^{k} \cup \{U_{s_i'}\}_{i=1}^{k}$ is a finite subcover of $\{U_s\}_{s\in S}$, and this proves that X is compact.

One easily verifies that C_2 is not an F_σ-set in X, so that X is not a perfectly normal space. On the other hand, it can be readily checked that X is a hereditarily normal space. The space X is first-countable but not second-countable; it is not even separable.

One can easily define a continuous mapping of the space X onto $A(\mathfrak{c})$: it suffices to map C_2 in a one-to-one way onto the set of all isolated points of $A(\mathfrak{c})$ and the whole of C_1 to the accumulation point of $A(\mathfrak{c})$; this shows that there is no counterpart of Theorem 3.1.22 for the character of spaces.■

3.1.27. EXAMPLE. Let W be the set of all ordinal numbers less than or equal to the first uncountable ordinal number ω_1. The set W is well-ordered by the natural order $<$. Consider on W the topology generated by the base $\mathcal{B}$ consisting of all segments $(y,x] = \{z \in W : y < z \leq x\}$, where $y < x \leq \omega_1$, and the one-point set $\{0\}$, where 0 is the order type of the empy set. One easily sees that W is a Hausdorff space.

We shall prove that W is a compact space. Let $\{U_s\}_{s\in S}$ be an open cover of the space W and let A be the set of all $x \in W$ such that the interval $[0,x]$ is contained in the union of finitely many members of $\{U_s\}_{s\in S}$. It suffices to show that the set $W \setminus A$ is empty.

Assume that $W \setminus A \neq \emptyset$ and denote by x_0 the smallest element of this set; clearly $x_0 \neq 0$. Let $x_0 \in U_{s_0}$; as $x_0 > 0$ there exists a $y < x_0$ such that $(y, x_0] \subset U_{s_0}$. By the definition of x_0 we have $y \in A$, so that $[0,y] \subset \bigcup_{i=1}^k U_{s_i}$ for some $s_1, s_2, \ldots, s_k \in S$. It follows that $[0,x_0] \subset \bigcup_{i=0}^k U_{s_i}$, and we have a contradiction.

Let us now consider the subspace $W_0 = W \setminus \{\omega_1\}$ of W. The space W_0 is normal; moreover, for every pair A, B of disjoint closed subsets of W_0, the closures $\overline{A}$ and $\overline{B}$ of A and B in the space W are disjoint. This follows from the fact that only one of the sets A and B can be cofinal in W_0. Indeed, if both A and B were cofinal in W_0, we could define inductively two sequences $a_1, a_2, \ldots.$ and $b_1, b_2, \ldots$ of countable ordinal numbers satisfying

$$a_i < b_i < a_{i+1} \quad \text{and} \quad a_i \in A, \quad b_i \in B \quad \text{for} \quad i = 1,2,\ldots$$

Since no countable set is cofinal in W_0, the set C consisting of all elements of W_0 that are larger than all the a_i's and b_i's would be non-empty, and this is impossible, because – as one easily verifies – the smallest element of C would then belong to $A \cap B$.

From the above one can deduce (see Theorem 3.2.1 or Corollary 3.6.4) that every continuous function $f: W_0 \to I$ is extendable over W. It turns out that even more is true: for every continuous function $f: W_0 \to I$ there exists an ordinal number $x_0 < \omega_1$ such that $f(x) = f(x_0)$ for every $x \geq x_0$, i.e., f is eventually constant.

It suffices to show that for every positive integer i there exists an $x_i \in W_0$ such that $|f(x) - f(x_i)| < 1/i$ for every $x \geq x_i$; indeed, one easily verifies that any x_0 larger than all the x_i's has the required property.

Let us assume that for some i and every $x \in W_0$ one can find an $x' \in W_0$ such that $|f(x) - f(x')| \geq 1/i$ and $x' \geq x$. This enables us to define inductively two sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ of countable ordinal numbers satisfying

$$a_j < b_j < a_{j+1} \quad \text{and} \quad |f(a_j) - f(b_j)| \geq 1/i \quad \text{for} \quad j = 1,2,\ldots$$

It turns out, however, that the existence of such sequences contradicts the continuity of f. Indeed, let us denote by c the smallest ordinal number larger than all the a_i's and b_i's; we have $c \in W_0$ and – as one readily verifies – every neighbourhood of c contains almost all elements of both sequences; since the set $f^{-1}((f(c) - 1/2i, f(c) + 1/2i))$ contains at most one of the points a_j, b_j for $j = 1,2,\ldots$ it is not a neighbourhood of the point c.

The set F of all countable limit ordinal numbers is closed in W_0; as there is no continuous function $f: W_0 \to I$ such that $F = f^{-1}(0)$, from the Vedenissoff theorem it follows that W_0 is not a perfectly normal space. Hence W is not a perfectly normal space either. One the other hand, one can prove (see Problems 3.12.3(c) and 2.7.5(c)) that W is a hereditarily normal space. The space W_0 is first-countable; the space W has no countable base at ω_1, and, in fact, W is not even a sequential space.

As the reader certainly observed, the topology on W and W_0, as well as the topology on R and I, are closely connected with the natural linear order relations defined in those sets. In a similar way one can define a topology on every linearly ordered set; topological spaces thus obtained form the interesting class of linearly ordered spaces (see Problems 1.7.4, 2.7.5, 3.12.3, 3.12.4, 3.12.12(f), 5.5.22, 6.3.2 and 8.5.13(j)).■

3.1.28. EXAMPLE. We shall now show that the Cantor set $D^{\aleph_0}$ is homeomorphic to a subspace of the real line. Let us consider the set C of all real numbers of the segment I that have a tryadic expansion in which the digit 1 does not occur, i.e., the set of all numbers of the form

(3) $$x = \sum_{i=1}^{\infty} \frac{2x_i}{3^i}, \quad \text{where} \quad x_i \in \{0,1\} \quad \text{for} \quad i = 1,2,\ldots$$

Any element x of C is representable in the form (3) in a unique way, so that letting $f(x) = \{x_i\}$ we define a one-to-one mapping f of C onto $D^{\aleph_0}$. From 2.3.6 it follows that f is continuous with respect to the topology on C as a subspace of R. By Theorem 3.1.13 to show that f is a homeomorphism it suffices to prove that C is a compact space. This follows, however, from the equality $C = \bigcap_{i=1}^{\infty} F_i$, where F_i is the subset of I consisting of all numbers that have a tryadic expansion in which 1 does not occur as the jth digit for $j \le i$, because all the F_i's are closed. One easily sees that the set F_1 is obtained from I by removing the "middle" interval $(1/3, 2/3)$, the set F_2 is obtained from F_1 by removing the "middle" intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ of both segments in F_1, and so on.

Let us observe that it follows from the above that the Cantor set $D^{\aleph_0}$ is compact. In the next section, we shall show that the Cantor cube $D^{\mathfrak{m}}$ is compact for every $\mathfrak{m} \ge \aleph_0$ (see Theorem 3.2.4).■

We close this section with an important theorem which implies, in particular, that every first-countable compact space has cardinality $\le \mathfrak{c}$ (cf. Problems 3.12.11(d) and 3.12.10(a)).

3.1.29. THEOREM. *For every infinite compact space X we have $|X| \le \exp \chi(X)$.*

PROOF. Let X be an infinite compact space and let $\chi(X) = \mathfrak{m}$; clearly $\mathfrak{m} \ge \aleph_0$. For every $x \in X$ choose a base $\mathcal{B}(x)$ for X at the point x satisfying $|\mathcal{B}(x)| \le \mathfrak{m}$.

Let τ be the initial number of cardinality $\mathfrak{m}^+$. Applying transfinite induction we shall define a transfinite sequence $F_0, F_1, \ldots, F_\alpha, \ldots,\ \alpha < \tau$ of closed subsets of X such that for every $\alpha < \tau$

(4) $$|F_\alpha| \le 2^{\mathfrak{m}}, \quad F_\beta \subset F_\alpha \quad \text{for} \quad \beta < \alpha$$

and

(5) for any finite $\mathcal{U} \subset \bigcup\{\mathcal{B}(x) : x \in \bigcup_{\beta<\alpha} F_\beta\}$ if $X \setminus \bigcup \mathcal{U} \ne \emptyset$, then $F_\alpha \setminus \bigcup \mathcal{U} \ne \emptyset$.

Assume that $\alpha_0 = 0$ or that $\alpha_0 > 0$ and the sets F_α satisfying (4) and (5) are defined for all $\alpha < \alpha_0$. Let

$$\mathcal{B} = \bigcup\{\mathcal{B}(x) : x \in \bigcup_{\alpha<\alpha_0} F_\alpha\}$$

and

$$\mathbf{B} = \{\mathcal{U} \subset \mathcal{B} : \mathcal{U} \text{ is finite and } X \setminus \bigcup \mathcal{U} \neq \emptyset\}.$$

We clearly have $|\mathcal{B}| \leq 2^{\mathfrak{m}}$ and $|\mathbf{B}| \leq 2^{\mathfrak{m}}$. Denote by B the set obtained by choosing for every $\mathcal{U} \in \mathbf{B}$ a point in the complement of $\bigcup\mathcal{U}$; obviously $|B| \leq 2^{\mathfrak{m}}$. By the second inequality in Theorem 1.5.3, the set $F_{\alpha_0} = \overline{B \cup \bigcup_{\alpha<\alpha_0} F_\alpha}$ has cardinality $\leq 2^{\mathfrak{m}}$, so that conditions (4) and (5) are satisfied for $\alpha = \alpha_0$.

To conclude the proof it suffices to show that the union $F = \bigcup_{\alpha<\tau} F_\alpha$ equals X. To begin, let us observe that, τ being regular, by virtue of the second part of (4) for every set $A \subset F$ of cardinality $\leq \mathfrak{m}$ there exists an $\alpha < \tau$ such that $A \subset F_\alpha$. Hence $\overline{A} \subset F_\alpha \subset F$ and this, together with the equality $\chi(X) = \mathfrak{m}$, implies that F is a closed subset of X; in particular, F is a compact space.

Now, suppose that there exists a point $y \in X \setminus F$ and for every $x \in F$ take a $U_x \in \mathcal{B}(x)$ such that $y \notin U_x$. There exists a finite subfamily $\mathcal{U}$ of $\{U_x\}_{x\in F}$ such that $F \subset \bigcup\mathcal{U}$ and an $\alpha < \tau$ such that $\mathcal{U} \subset \bigcup\{\mathcal{B}(x) : x \in \bigcup_{\beta<\alpha} F_\beta\}$. Thus we have $X \setminus \bigcup\mathcal{U} \neq \emptyset$ and $F_\alpha \setminus \bigcup\mathcal{U} = \emptyset$ which contradicts (5). ∎

3.1.30. COROLLARY. *Every first-countable compact space has cardinality* $\leq \mathfrak{c}$. ∎

Besides compact spaces, a larger class of quasi-compact spaces is sometimes considered. A topological space X is called a *quasi-compact space* if every open cover of X has a finite subcover. Hence, compact spaces are Hausdorff quasi-compact spaces. The reader can easily verify that Theorems 3.1.1–3.1.3, Corollaries 3.1.4 and 3.1.5, Theorems 3.1.10, 3.1.16, 3.1.23 and 3.1.24 of this section, as well as Theorems 3.2.3, 3.2.4, and 3.2.10 of the next section, remain valid, along with their proofs, when one replaces "compact" by "quasi-compact" and "Hausdorff space" by "topological space". One proves also that for every infinite quasi-compact T_1-space X we have $|X| \leq \exp\psi(X)$ (see Exercise 3.1.F(a) for the definition of $\psi(X)$).

Historical and bibliographic notes

The genesis of the notion of compactness is connected with the Borel theorem (proved in 1894) stating that every countable open cover of a closed interval has a finite subcover, and with the Lebesgue observation that the same holds for every open cover of a closed interval (in [1903] Borel generalized this result, in Lebesgue's setting, to all bounded closed subsets of Euclidean spaces). When general topology was in its infancy, defining new classes of spaces often consisted in taking a property of the closed interval I or of the real line R and considering the class of all spaces that have this property; classes of separable, compact, complete and connected spaces were defined following this pattern. At first this method was used to define some classes of metric spaces, later definitions were extended to topological spaces. Sometimes properties equivalent in the class of metric spaces, when extended to topological spaces, led to different classes of topological spaces (see, e.g., Theorem 4.1.15) and it was not immediately clear which class was the proper generalization. This happened with compactness, and for some time there was doubt whether the proper extension of the class of compact metric spaces is the class of compact spaces, the class of countably compact spaces, or the class of sequentially compact spaces (cf. Section 3.10). By now, it is quite clear that it is the class of compact spaces; this class behaves best with respect to operations on topological spaces, is most often met in applications and leads to the most interesting problems.

The concept of a (regular) compact space was introduced by Vietoris in [1921]; the definition was similar to the condition in Theorem 3.1.23. In that paper, Vietoris proved our Theorems 3.1.1, 3.1.8 and 3.1.9 (Riesz states in [1908] that every family of bounded closed subsets of a Euclidean space which has the finite intersection property has non-empty intersection). An attempt at the definition of compactness appeared earlier in Janiszewski's paper [1912] (see also Janiszewski [1913]). The equivalence of some topological properties with the Borel-Lebesgue condition defining compact spaces was proved by Kuratowski and Sierpiński in [1921] and by Saks in [1921], but they did not consider the class of compact spaces. The definition of compactness adopted here was given by Alexandroff and Urysohn in [1923]. They defined the class of compact spaces independently of Vietoris, and gave a deep analysis of the concept of compactness in their important paper [1929] (main definitions and results were announced in [1923] and [1924]); in particular, this paper contains our Theorems 3.1.1, 3.1.2, 3.1.3, 3.1.8 and 3.1.9; Theorems 3.1.10, 3.1.12 and 3.1.13 were proved by Alexandroff in [1927] (announcement [1925]).

Theorems 3.1.6 and 3.1.7 are symptomatic of a fairly general but rather dim regularity usually expressed by the phrase that "compact sets behave like points" (cf. Theorems 3.2.10 and 3.3.2). Kuratowski proved in [1931] that in the class of metric spaces projections parallel to a compact space are closed mappings; this was generalized to topological spaces by Bourbaki in [1940], and Mrówka observed in [1959] that the property characterizes compact spaces.

Networks were defined by Arhangel'skiĭ in [1959], where Theorem 3.1.19 was established; our proof is based on Holsztyński's paper [1966a]. Theorem 3.1.21 follows easily from early results of Alexandroff (see Exercise 3.1.F(a)); Theorem 3.1.22 was proved by Alexandroff in [1939]. The Alexandroff double circle was described in Alexandroff and Urysohn's paper [1929]. Theorem 3.1.29 was proved by Arhangel'skiĭ in [1969a]; it solves an outstanding problem of general topology that had been attacked for about 30 years. Our proof is taken from R. Pol's paper [1974] (it is closely related to a proof given by Ponomarev in [1971]), a similar proof is contained in Šapirovskiĭ's paper [1974]; it appeared in retrospect that the idea of Pol and Šapirovskiĭ's proof was implicit in Arhangel'skiĭ's original argument. The counterpart of Theorem 3.1.29 for quasi-compact spaces quoted at the end of this section was proved by Gryzlov in [1980].

Chapter 3, besides many important theorems, contains some examples fundamental for general topology. Not all of them are discussed in the main body of this book; examples sketched in Exercises 3.1.I, 3.2.E, 3.6.I(a) and 3.10.C, as well as examples in Problems 3.12.19 and 3.12.20 are particularly recommended to the reader's attention.

Exercises

3.1.A. (a) Give examples to show that Theorems 3.1.8, 3.1.9, 3.1.12, 3.1.13, 3.1.19, 3.1.21 and 3.1.22 do not hold if one replaces the term "compact" by "quasi-compact" and the term "Hausdorff space" by "T_1-space".

(b) Verify that in Corollary 3.1.4 the assumption that X is a Hausdorff space can be omitted (cf. Theorem 3.7.22).

3.1.B. Prove that every compact subspace of the Sorgenfrey line is countable.

Hint. Apply Theorem 3.1.13.

3.1.C. A continuous mapping $f: X \to Y$ of X onto Y is *irreducible* if $f(A) \neq Y$ for every proper closed subset $A \subset X$.

(a) Show that for every continuous mapping with compact fibers $f: X \to Y$ of X onto Y there exists a closed subspace $X_0 \subset X$ such that $f(X_0) = Y$ and $f|X_0: X_0 \to Y$ is irreducible (cf. Problem 5.5.12).

Hint. Apply the Kuratowski-Zorn lemma.

(b) Verify that every irreducible open mapping defined on a Hausdorff space is a homeomorphism, and note that this is not true for mappings defined on a T_1-space.

(c) Observe that if the set of all one-point fibers of a continuous mapping $f: X \to Y$ of X onto Y is dense in the space X, then f is irreducible. Give an example of an irreducible mapping of a compact space X onto a compact space Y which has no one-point fiber.

3.1.D. (a) Show applying the Kuratowski theorem, that a mapping f of a space X to a compact space Y is continuous if and only if the graph of f is a closed subset of $X \times Y$ (cf. Exercise 2.3.C(b)).

(b) Prove that if X is a compact space, then for every topological space Y the projection $p: X \times Y \to Y$ maps functionally closed sets to functionally closed sets.

Hint. Show that for every continuous function $f: X \times Y \to I$ the formula $F(y) = \sup\{f(x, y) : x \in X\}$ defines a continuous function $F: Y \to I$ (cf. Exercise 1.5.L(a)).

3.1.E (Arhangel'skiĭ [1965], Čoban [1967]). (a) Prove that if A_1 and A_2 are subsets of a space X such that $A_1 \subset A_2$ and for every set $F \subset A_2 \setminus A_1$ that is closed in A_2 there exist disjoint open subsets of X containing A_1 and F respectively, then $\chi(A_1, X) \leq \chi(A_1, A_2)\chi(A_2, X)$ (cf. Theorem 3.1.6 and Exercise 3.8.B).

Deduce that for compact subsets F_1, F_2 of a Hausdorff space X such that $F_1 \subset F_2$ we have $\chi(F_1, X) \leq \chi(F_1, F_2)\chi(F_2, X)$.

Hint. Let $\mathfrak{m} = \chi(A_1, A_2)\chi(A_2, X)$ and let $\{W_s\}_{s \in S}$ and $\{V_t\}_{t \in T}$ be bases for X at A_2 and for A_2 at A_1 respectively, satisfying $|S| \leq \mathfrak{m}$ and $|T| \leq \mathfrak{m}$. For every $t \in T$ take disjoint open sets $G_t, H_t \subset X$ such that $A_1 \subset G_t$ and $A_2 \setminus V_t \subset H_t$, and verify that the sets $U_{t,s} = G_t \cap W_s$ form a base for X at A_1.

(b) A Hausdorff space X is of *pointwise countable type* if for every point $x \in X$ there exists a compact set $F \subset X$ such that $x \in F$ and $\chi(F, X) \leq \aleph_0$.

Note that compact spaces and first-countable Hausdorff spaces are of pointwise countable type. Prove that closed subspaces and G_δ-subspaces of a space of pointwise countable type are spaces of pointwise countable type. Give an example of a normal space that is not of pointwise countable type, and observe that the property of being of pointwise countable type is not a hereditary property (Theorem 3.2.4 can be applied).

3.1.F. (a) (Alexandroff [1924a]) The *pseudocharacter of a point* x in a T_1-space X is defined as the smallest cardinal number of the form $|\mathcal{U}|$, where $\mathcal{U}$ is a family of open subsets of X such that $\bigcap \mathcal{U} = \{x\}$; this cardinal number is denoted by $\psi(x, X)$. The *pseudocharacter of a T_1-space* X is defined as the supremum of all numbers $\psi(x, X)$ for $x \in X$; this cardinal number is denoted by $\psi(X)$.

Note that for every T_1-space X and any $x \in X$ we have $\psi(x, X) \leq \chi(x, X)$ and $\psi(X) \leq \chi(X)$. Prove that if X is a compact space, then $\psi(x, X) = \chi(x, X)$, for every $x \in X$, and $\psi(X) = \chi(X)$; observe that this implies Theorem 3.1.21. Show that for every Hausdorff space X we have $\psi(X) \leq \exp d(X)$ (cf. Theorem 1.5.7).

(b) For a Hausdorff space X we denote by $h(X)$ the smallest cardinal number $\mathfrak{m}$ with the property that for every point $x \in X$ there exists a compact set $F \subset X$ such that $x \in F$ and $\chi(F, X) \leq \mathfrak{m}$, hence a Hausdorff space X is of pointwise countable type (cf. Exercise 3.1.E(b)) if and only if $h(X) \leq \aleph_0$.

Prove that for every Hausdorff space X we have $\chi(X) = \psi(X)h(X)$.

(c) Observe that if there exists an open mapping of a Hausdorff space X onto a Hausdorff space Y, then $h(Y) \leq h(X)$, and that it is not necessarily so if there exists a closed mapping of X onto Y (cf. Exercise 3.7.G(b)).

(d) Show that for every regular space X we have $|X| \leq \exp[d(X)\psi(X)]$. Verify that regularity cannot be weakened to the assumption that X is a Hausdorff space.

Hint. Applying the equality $d(I^{\mathfrak{c}}) = \aleph_0$, define a Hausdorff space X of cardinality $2^{\mathfrak{c}}$ that contains a countable dense subset A consisting of isolated points of X such that the subspace $X \setminus A$ is discrete.

3.1.G. Let X be a compact space, $X_i = X \times \{i\}$ for $i = 1, 2$ and $A(X) = X_1 \cup X_2$. Generalizing Example 3.1.26 generate on $A(X)$ a topology of a compact space in such a way that X_1 is homeomorphic to X and X_2 is a discrete subspace. Verify that for every set $M \subset X$ the subspace $X_1 \cup M_2 \subset A(X)$, where $M_2 = M \times \{2\} \subset X_2$, is compact and that, under the additional assumption that M is dense in X and X is dense in itself, M_2 is dense in $X_1 \cup M_2$.

3.1.H. (a) (Engelking [1968]) Prove that the discrete space $D(\mathfrak{c})$ is embeddable as a closed subspace in $[D(\aleph_0)]^{\mathfrak{c}}$; deduce that the Cartesian product $[D(\aleph_0)]^{\mathfrak{c}}$ is not normal (cf. Exercise 2.3.E(a)).

Hint. Let X_1 be the closed interval I with its natural topology and let X_2 be the interval I with the discrete topology. For every $t \in I$ define a function f_t from I to the discrete space $D(\aleph_0) = N \oplus \{0\}$ letting $f_t(t) = 0$ and $f_t(x) = i$ whenever $1/(i+1) < |x - t| \leq 1/i$. Verify that $f = \triangle_{t \in I} f_t$ is a homeomorphic embedding of X_2 in $Y = [D(\aleph_0)]^{\mathfrak{c}}$ and note that if $y = \{y_t\} \notin f(I)$ and $y_t = 0$ for some $t \in I$, then y has in Y a neighbourhood disjoint from $f(I)$. Show that if $y = \{y_t\} \in \overline{f(I)}$, then $y_t = 0$ for some $t \in I$; to that end, consider the family $\{\overline{f^{-1}(U)}\}_{U \in \mathcal{B}}$ of closed subsets of X_1, where $\mathcal{B}$ is a base at $y \in Y$.

Remark. As proved by Mycielski in [1964], the discrete space $D(\mathfrak{m})$ is embeddable as a closed subspace in $[D(\aleph_0)]^{\mathfrak{m}}$ if $\mathfrak{m}$ is less than the first weakly inaccessible cardinal number (a cardinal number $\aleph_\alpha$ is *weakly inaccessible* if it is a regular number and α is a limit number larger than 0; the assumption that there exists a weakly inaccessible cardinal number implies the consistency of the axioms of set theory and – under this assumption – the statement that the first weakly inaccessible cardinal number is less than $\mathfrak{c}$ is independent of the axioms; it is consistent with the axioms of set theory that no cardinal number is weakly inaccessible;

proofs are given in Kunen [1980]).

(b) (Juhász [1969]) Using the fact that there exists a compact space X such that $|X| = 2^{\mathfrak{m}}$ and $\chi(x,X) = \mathfrak{m}$ for every $x \in X$ (see Theorems 2.3.24 and 3.2.4) show that for every $\mathfrak{m} \geq \aleph_0$ the discrete space $D(2^{\mathfrak{m}})$ is embeddable as a closed subspace in $[D(\mathfrak{m})]^{2^{\mathfrak{m}}}$.

Hint. For every $t \in X$ choose a base $\{U_s(t)\}_{s\in S}$ at the point t, where S is a well-ordered set of cardinality $\mathfrak{m}$ such that $0 \notin S$, and define a function f_t from X to the discrete space $D(\mathfrak{m}) = S \oplus \{0\}$ letting $f_t(t) = 0$ and $f_t(x) = s$, where s is the smallest element in S such that $x \notin U_s(t)$ whenever $x \neq t$. Then follow the hint in (a).

3.1.I (Alexandroff and Niemytzki [1938], McAuley [1956]). Generate a topology on the plane leaving neighbourhoods of all points (x,y) with $y \neq 0$ unchanged and taking as a base at each point $(x,0)$ the family $\{\{(x,0)\} \cup U_i(x)\}_{i=1}^{\infty}$, where $U_i(x)$ is the set of all points inside the circle of radius $1/i$ and centre at $(x,0)$ but outside the two circles of radius i tangent to the x-axis at $(x,0)$.

Show that the space X obtained in this way is perfectly normal and that on the sets $X_1 = \{(x,0) : x \in R\}$ and $X_2 = X \setminus X_1$ the topology induced by the topology of X coincides with the topology induced by the topology of R^2. Verify that $d(X) = \chi(X) = nw(X) = \aleph_0$ and that $w(X) = \mathfrak{c}$.

The space described above is sometimes called the *butterfly space* or the *bow-tie space.*

3.1.J. (a) Note that for every subspace M of a space X we have $nw(M) \leq nw(X)$.

(b) Prove that the network weight of an infinite Cartesian product $X = \prod_{s\in S} X_s$, where $nw(X_s) > 1$, is equal to the larger of the two cardinal numbers $|S|$ and $\sup_{s\in S} nw(X_s)$.

3.1.K. (a) Applying the diagonal theorem and Example 3.1.28, prove that for every countable ordinal number α the subspace $Y_\alpha = \{\gamma : \gamma \leq \alpha\}$ of the space W defined in 3.1.27 is embeddable in the real line.

Observe that the subspace W_0 of W is not embeddable in the real line.

(b) (Higman and Stone [1954], Isbell [1964]) Let Y_α, where $\alpha \in W_0$, be the spaces defined in (a); consider the inverse system $\mathbf{S} = \{X_\alpha, \pi^\alpha_\beta, W_0\}$, where X_α is the discrete space consisting of all homeomorphic embeddings of the space Y_α in the real line and $\pi^\alpha_\beta(f) = f|Y_\beta$ for every $f \in X_\alpha$ and $\alpha, \beta \in W_0$ satisfying $\beta \leq \alpha$. Note that all bonding mappings of the inverse system $\mathbf{S}$ are onto and yet $\varprojlim \mathbf{S} = \emptyset$ (cf. Exercise 2.5.A(b) and Theorem 3.2.13).

3.2. Operations on compact spaces

Let us begin with a discussion of problems related to the subspace operation. First of all note that, as follows immediately from Theorems 3.1.2 and 3.1.8, compactness is hereditary with respect to closed subspaces and only with respect to closed subspaces.

The next theorem gives a criterion for extendability of mappings into compact spaces.

3.2.1. THEOREM. *Let A be a dense subspace of a topological space X and f a continuous mapping of A to a compact space Y. The mapping f has a continuous extension over X if and only if for every pair B_1, B_2 of disjoint closed subsets of Y the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have disjoint closures in the space X.*

PROOF. Let $F: X \to Y$ be an extension of f. If $B_i = \overline{B_i} \subset Y$ for $i = 1,2$ and $B_1 \cap B_2 = \emptyset$, then

$$F^{-1}(B_i) = \overline{F^{-1}(B_i)} \quad \text{for} \quad i = 1,2 \quad \text{and} \quad F^{-1}(B_1) \cap F^{-1}(B_2) = \emptyset,$$

so that

$$\overline{f^{-1}(B_1)} \cap \overline{f^{-1}(B_2)} \subset F^{-1}(B_1) \cap F^{-1}(B_2) = \emptyset.$$

Thus the condition in the theorem is necessary for the extendability of f.

We shall now show that the condition is also sufficient. For every $x \in X$ denote by $\mathcal{B}(x)$ the family of all neighbourhoods of x in the space X and consider the family $\mathcal{F}(x) = \{\overline{f(A \cap U)}\}_{U \in \mathcal{B}(x)}$ of closed subsets of Y.

Since for $U_1, \ldots, U_k \in \mathcal{B}(x)$ we have

$$\overline{f(A \cap U_1 \cap U_2 \cap \ldots \cap U_k)} \subset \overline{f(A \cap U_1)} \cap \overline{f(A \cap U_2)} \cap \ldots \cap \overline{f(A \cap U_k)}, \tag{1}$$

the family $\mathcal{F}(x)$ has the finite intersection property. By Theorem 3.1.1, the intersection $\bigcap \mathcal{F}(x)$ is non-empty for every $x \in X$.

We shall prove that $\bigcap \mathcal{F}(x)$ consists of a single point; this will imply in particular that $\bigcap \mathcal{F}(x) = \{f(x)\}$ for $x \in A$. Suppose that $y_1, y_2 \in \bigcap \mathcal{F}(x)$ and that $y_1 \neq y_2$. There exist neighbourhoods V_1 and V_2 of y_1 and y_2 respectively, such that $\overline{V}_1 \cap \overline{V}_2 = \emptyset$, and from the condition under consideration it follows that $\overline{f^{-1}(V_1)} \cap \overline{f^{-1}(V_2)} = \emptyset$. Without loss of generality one can assume that $x \notin \overline{f^{-1}(V_1)}$; thus $X \setminus \overline{f^{-1}(V_1)} \in \mathcal{B}(x)$ and $y_1 \in \bigcap \mathcal{F}(x) \subset \overline{f(A \setminus \overline{f^{-1}(V_1)})}$. On the other hand, since $V_1 \cap f(A \setminus \overline{f^{-1}(V_1)}) = \emptyset$, we have $y_1 \notin \overline{f(A \setminus \overline{f^{-1}(V_1)})}$, a contradiction.

Assigning to $x \in X$ the unique point in $\bigcap \mathcal{F}(x)$ we define a mapping F of X to Y which is an extension of f; it remains to prove that F is continuous. We shall show that F satisfies condition (iii) in Proposition 1.4.1.

Let V be a neighbourhood of $F(x)$ in the space Y. Since $\{F(x)\} = \bigcap_{U \in \mathcal{B}(x)} \overline{f(A \cap U)} \subset V$, from 3.1.5 it follows that there exists a finite family $\{U_1, U_2, \ldots, U_k\} \subset \mathcal{B}(x)$ such that

$$\overline{f(A \cap U_1)} \cap \overline{f(A \cap U_2)} \cap \ldots \cap \overline{f(A \cap U_k)} \subset V. \tag{2}$$

Clearly $U_1 \cap U_2 \cap \ldots \cap U_k = U \in \mathcal{B}(x)$, and by (1) and (2) we have $F(x') \in \overline{f(A \cap U)} \subset V$ for every $x' \in U$, i.e., $F(U) \subset V$. ■

The next theorem is a consequence of Theorem 3.2.1.

3.2.2. THEOREM. *Every compact space of weight $\mathfrak{m} \geq \aleph_0$ is a continuous image of a closed subspace of the Cantor cube $D^{\mathfrak{m}}$.*

PROOF. Let Y be a compact space of weight $\mathfrak{m}$. From Theorem 2.3.26 it follows that Y is homeomorphic to a subspace of the Alexandroff cube $F^{\mathfrak{m}}$. For sake of simplicity we shall assume that $Y \subset F^{\mathfrak{m}}$. The spaces $F^{\mathfrak{m}}$ and $D^{\mathfrak{m}}$ have the same underlying set and every member of the canonical base $\mathcal{B}$ for $F^{\mathfrak{m}}$ is open-and-closed in $D^{\mathfrak{m}}$, so that the identity mapping h of $D^{\mathfrak{m}}$ onto $F^{\mathfrak{m}}$ is continuous.

We shall show that the assumptions of Theorem 3.2.1 are satisfied if we let $A = h^{-1}(Y)$, $X = \overline{A} \subset D^{\mathfrak{m}}$ and $f = h|A: A \to Y$.

Let B_1, B_2 be a pair of disjoint closed subsets of Y. There exist closed subsets K_1, K_2 of $F^{\mathfrak{m}}$ such that $B_i = Y \cap K_i$ for $i = 1,2$. Since $B_1 \cap B_2 = \emptyset$, we have $Y \subset F^{\mathfrak{m}} \setminus (K_1 \cap K_2)$.

The last set being open, for every $x \in Y$ there exists a $U_x \in \mathcal{B}$ satisfying the condition $x \in U_x \subset F^{\mathfrak{m}} \setminus (K_1 \cap K_2)$. By Theorem 3.1.3 there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset Y$ such that

$$Y \subset U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_k} \subset F^{\mathfrak{m}} \setminus (K_1 \cap K_2).$$

The union $U = U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_k}$ is an open-and-closed subset of $D^{\mathfrak{m}}$ containing A, so that

$$X = \overline{A} \subset U \subset D^{\mathfrak{m}} \setminus (K_1 \cap K_2).$$

Since $\overline{f^{-1}(B_i)} = \overline{A \cap K_i} \subset \overline{A} \cap K_i$ for $i = 1, 2$, we have

$$\overline{f^{-1}(B_1)} \cap \overline{f^{-1}(B_2)} \subset \overline{A} \cap K_1 \cap K_2 \subset [D^{\mathfrak{m}} \setminus (K_1 \cap K_2)] \cap K_1 \cap K_2 = \emptyset,$$

and by Theorem 3.2.1 there exists a continuous extension $F: X \to Y$ of the mapping f. As $Y = f(A) \subset F(X)$, the space Y is a continuous image of the closed subspace X of $D^{\mathfrak{m}}$.■

Our only theorem connected with the sum operation is the following:

3.2.3. THEOREM. *The sum $\bigoplus_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is compact if and only if all spaces X_s are compact and the set S is finite.*

PROOF. If the sum $X = \bigoplus_{s \in S} X_s$ is compact, then all spaces X_s are compact as closed subspaces of X and the set S is finite, for otherwise the open cover $\{X_s\}_{s \in S}$ of X would not have a finite subcover.

Conversely, if $\{X_i\}_{i=1}^k$ is a family of compact spaces, then the sum $X = X_1 \oplus X_2 \oplus \ldots \oplus X_k$ is compact by Theorem 2.2.7 and Corollary 3.1.4.■

We shall now consider the Cartesian product operation. The next theorem is fundamental in this context, and is also one of the most important theorems of general topology.

3.2.4. THE TYCHONOFF THEOREM. *The Cartesian product $\prod_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is compact if and only if all spaces X_s are compact.*

PROOF. If the Cartesian product $X = \prod_{s \in S} X_s$ is compact and non-empty, then all the X_s's are Hausdorff spaces by Theorem 2.3.11 and are compact by Theorem 3.1.10, because the projection $p_s: X \to X_s$ is a continuous mapping of X onto X_s.

Let us now consider a family $\{X_s\}_{s \in S}$ of compact spaces. By Theorem 2.3.11 the Cartesian product $X = \prod_{s \in S} X_s$ is a Hausdorff space. Consider a family $\mathcal{F}_0$ of closed subsets of X which has the finite intersection property. Since the finite intersection property is a property of finite character, it follows from the Teichmüller-Tukey lemma (see the Introduction) that the family $\mathcal{F}_0$ is contained in a maximal family $\mathcal{F}$ of subsets of X which has the finite intersection property.

In order to prove that $\bigcap \mathcal{F}_0 \neq \emptyset$ it suffices to show that there exists a point $x \in X$ such that

$$x \in \overline{A} \quad \text{for every} \quad A \in \mathcal{F}. \tag{3}$$

From the maximality of $\mathcal{F}$ we have:

$$\text{if} \quad A_1, A_2, \ldots, A_k \in \mathcal{F}, \quad \text{then} \quad A_1 \cap A_2 \cap \ldots \cap A_k \in \mathcal{F} \tag{4}$$

and

(5) if $A_0 \subset X$ and $A_0 \cap A \neq \emptyset$ for every $A \in \mathcal{F}$, then $A_0 \in \mathcal{F}$.

As $\mathcal{F}$ has the finite intersection property, the family $\mathcal{F}_s = \{\overline{p_s(A)}\}_{A \in \mathcal{F}}$ of closed subsets of X_s also has this property for every $s \in S$. Hence, for every $s \in S$ there exists a point

$$x_s \in \bigcap_{A \in \mathcal{F}} \overline{p_s(A)} \subset X_s.$$

Let W_s be a neighbourhood of x_s in X_s. By the above formula, $W_s \cap p_s(A) \neq \emptyset$ for every $A \in \mathcal{F}$, i.e.,

$$p_s^{-1}(W_s) \cap A \neq \emptyset \text{ for every } A \in \mathcal{F}.$$

By virtue of (5) we infer that $p_s^{-1}(W_s) \in \mathcal{F}$, and from (4) it follows that all members of the canonical base for X that contain the point $x = \{x_s\}$ belong to the family $\mathcal{F}$. As $\mathcal{F}$ has the finite intersection property, every $A \in \mathcal{F}$ intersects all members of the canonical base for X that contain the point x, and this gives (3). ■

Note that compactness of a finite Cartesian product $X_1 \times X_2 \times \ldots \times X_k$ of compact spaces can be proved directly in a simpler way; it also follows from the Kuratowski theorem, because for every Y the projection $p: X_1 \times X_2 \times \ldots \times X_k \times Y \to Y$ is closed as the composition of the closed mappings

$$\begin{aligned}
&p_1: X_1 \times X_2 \times \ldots \times X_k \times Y \to X_2 \times X_3 \times \ldots \times X_k \times Y,\\
&p_2: X_2 \times X_3 \times \ldots \times X_k \times Y \to X_3 \times X_4 \times \ldots \times X_k \times Y,\\
&\ldots\ldots\ldots\\
&p_k: X_k \times Y \to Y.
\end{aligned}$$

The Tychonoff theorem, along with Theorem 2.3.23, gives the next two theorems.

3.2.5. THEOREM. *The Tychonoff cube $I^{\mathfrak{m}}$ is universal for all compact spaces of weight $\mathfrak{m} \geq \aleph_0$.* ■

3.2.6. THEOREM. *A topological space is a Tychonoff space if and only if it is embeddable in a compact space.* ■

3.2.7. EXAMPLE. Let us note that from the Tychonoff theorem and Theorem 3.1.9 it follows that the Cartesian product $X \times Y$ considered in Example 2.3.36 is a normal space. However, when we have the Tychonoff theorem at our disposal, we can easily produce another example of a non-normal subspace of a normal space: by Theorem 2.3.23, the Tychonoff cube $I^{\mathfrak{c}}$ contains a subspace homeomorphic to the Niemytzki plane. ■

A subset A of Euclidean n-space R^n is *bounded* if there exists a closed interval $J = [a, b] \subset R$ such that $A \subset J^n \subset R^n$. A real-valued function f defined on a topological space X is *bounded* if the image $f(X)$ is a bounded subset of R.

The Tychonoff theorem yields a characterization of compact subspaces of Euclidean spaces.

3.2.8. THEOREM. *A subspace A of Euclidean n-space R^n is compact if and only if the set A is closed and bounded.*

PROOF. Suppose that A is a compact subspace of R^n. Theorem 3.1.8 implies that the set A is closed. Since $A \subset \bigcup_{i=1}^{\infty} K_i^n$, where $K_i = (-i, i)$, and since $K_i^n \subset K_j^n$ for $i \leq j$, there exists an integer i_0 such that $A \subset K_{i_0}^n$ and this shows that A is bounded.

Conversely, since for every $J = [a, b]$ the space J^n is compact by the Tychonoff theorem, every closed and bounded subspace of R^n is compact by Theorem 3.1.2. ■

3.2.9. COROLLARY. *Every continuous real-valued function defined on a compact space is bounded and attains its bounds.* ■

We shall prove one more theorem connected with the Cartesian product operation.

3.2.10. THE WALLACE THEOREM. *If A_s is a compact subspace of a topological space X_s for $s \in S$, then for every open subset W of the Cartesian product $\prod_{s\in S} X_s$ which contains the set $\prod_{s\in S} A_s$ there exist open sets $U_s \subset X_s$ such that $U_s \neq X_s$ only for finitely many $s \in S$ and $\prod_{s\in S} A_s \subset \prod_{s\in S} U_s \subset W$.*

PROOF. We first consider the case of a Cartesian product of two spaces, i.e., we assume that $S = \{1, 2\}$. From Lemma 3.1.15 it follows that for every $y \in A_2$ there exist open sets $U_1(y) \subset X_1$ and $U_2(y) \subset X_2$ such that $A_1 \times \{y\} \subset U_1(y) \times U_2(y) \subset W$. Since A_2 is a compact space, there exists a finite set $\{y_1, y_2, \ldots, y_k\} \subset A_2$ such that $A_2 \subset \bigcup_{i=1}^{k} U_2(y_i)$. One easily verifies that the sets $U_1 = \bigcap_{i=1}^{k} U_1(y_i)$ and $U_2 = \bigcup_{i=1}^{k} U_2(y_i)$ have all required properties.

Assume now that the theorem holds for Cartesian products of k-1 spaces, where $k \geq 3$, and consider an open subset W of the Cartesian product $X_1 \times X_2 \times \ldots \times X_k$ that contains the set $A_1 \times A_2 \times \ldots \times A_k$. Since the product $A_2 \times A_3 \times \ldots \times A_k$ is compact, by the special case of the theorem proved above, there exist open sets $U_1 \subset X_1$ and $U_2' \subset X_2 \times X_3 \times \ldots \times X_k$ such that

$$A_1 \times (A_2 \times A_3 \times \ldots \times A_k) \subset U_1 \times U_2' \subset W \subset X_1 \times (X_2 \times X_3 \times \ldots \times X_k).$$

By the inductive hypothesis there exist open sets $U_2 \subset X_2, U_3 \subset X_3, \ldots, U_k \subset X_k$ such that

$$A_2 \times A_3 \times \ldots \times A_k \subset U_2 \times U_3 \times \ldots \times U_k \subset U_2'.$$

One readily sees that the sets $U_1, U_2, \ldots, U_k$ have all required properties; thus the theorem holds for finite Cartesian products.

Finally we consider an arbitrary Cartesian product and a set $A = \prod_{s\in S} A_s \subset W \subset \prod_{s\in S} X_s$. For every $a \in A$ take a member of the canonical base for $\prod_{s\in S} X_s$ that contains a and is contained in W; by the Tychonoff theorem, A is a subset of a finite union of those sets:

$$A \subset \prod_{s\in S} W_s^1 \cup \prod_{s\in S} W_s^2 \cup \ldots \cup \prod_{s\in S} W_s^k \subset W.$$

There exists a finite set $S_0 = \{s_1, s_2, \ldots, s_l\}$ such that $W_s^i = X_s$ for $s \in S \setminus S_0$ and $i = 1, 2, \ldots, k$. Letting

$$W_1 = \bigcup_{i=1}^{k} \prod_{s\in S_0} W_s^i, \quad W_2 = \prod_{s\in S\setminus S_0} X_s \quad \text{and} \quad A_2 = \prod_{s\in S\setminus S_0} A_s$$

we have

$$(A_{s_1} \times A_{s_2} \times \ldots \times A_{s_l}) \times A_2 \subset W_1 \times W_2 \subset W,$$

and – our theorem being proved for finite Cartesian products – there exist $U_{s_1}, U_{s_2}, \ldots, U_{s_l}$, open in $X_{s_1}, X_{s_2}, \ldots, X_{s_l}$ respectively, such that

$$A_{s_1} \times A_{s_2} \times \ldots \times A_{s_l} \subset U_{s_1} \times U_{s_2} \times \ldots \times U_{s_l} \subset W_1.$$

Taking $U_s = X_s$ for each $s \in S \setminus S_0$ we have an open set $U_s \subset X_s$ defined for every $s \in S$; it is easy to check that those sets have all required properties. ∎

On the subject of the quotient space operation we have the following simple theorem.

3.2.11. THE ALEXANDROFF THEOREM. *For every closed equivalence relation E on a compact space X there exists exactly one (up to a homeomorphism) Hausdorff space Y and a continuous mapping $f: X \to Y$ of X onto Y such that $E = E(f)$, viz., the quotient space X/E and the natural quotient mapping $q: X \to X/E$; moreover, Y is a compact space.*

Conversely, *for every continuous mapping $f: X \to Y$ of a compact space X onto a Hausdorff space Y the equivalence relation $E(f)$ is closed.*

PROOF. If E is a closed equivalence relation on a compact space X, then the natural mapping $q: X \to X/E$ is closed, so that, by Theorems 3.1.9 and 1.5.20, the quotient space X/E is normal; moreover, by Theorem 3.1.10, the space X/E is compact.

If for a Hausdorff space Y there exists a mapping $f: X \to Y$ of X onto Y such that $E = E(f)$, then f is a closed mapping and $\overline{f}: X/E \to Y$ is a homeomorphism by 2.4.3 and 2.4.8. Clearly, when one identifies X/E with Y applying the homeomorphism $\overline{f}$, the mapping f coincides with the natural mapping $q: X \to X/E$.

The second part of the theorem is a consequence of Theorem 3.1.12. ∎

3.2.12. EXAMPLES. We shall show that the assumption of compactness is essential in the first part of the above theorem. Let $X = D(\mathfrak{c})$ and let $f_1: X \to I$ and $f_2: X \to I \oplus \{2\}$ be arbitrary one-to-one mappings. Clearly, $E(f_1) = E(f_2)$ is the equality relation in X, so that the same closed equivalence relation in X is determined by mappings of X onto distinct compact spaces.

Similarly, the assumption that Y is a Hausdorff space cannot be omitted. Indeed, let $X = I$ and let Y be the closed interval I with the topology described in Example 1.2.6; the continuous mapping $f: X \to Y$ defined by letting $f(x) = x$ and the identity $\mathrm{id}_X: X \to X$ determine the same closed equivalence relation in X, viz., the equality relation, and yet X and Y are not homeomorphic. ∎

We now pass to a discussion of inverse systems of compact spaces.

3.2.13. THEOREM. *The limit of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of non-empty compact spaces is compact and non-empty.*

PROOF. For each $\rho \in \Sigma$ let

$$Z_\rho = \{\{x_\sigma\} \in \prod_{\sigma \in \Sigma} X_\sigma : \pi^\rho_\tau(x_\rho) = x_\tau \text{ for } \tau \le \rho\};$$

consider a point $z_\rho \in X_\rho$, let $z_\tau = \pi^\rho_\tau(z_\rho)$ for $\tau \le \rho$, and pick arbitrarily a $z_\sigma \in X_\sigma$ for all remaining $\sigma \in \Sigma$; as $z = \{z_\sigma\} \in Z_\rho$, the set Z_ρ is non-empty. From Theorem 1.5.4 it follows that Z_ρ is a closed subset of $\prod_{\sigma \in \Sigma} X_\sigma$. Since $Z_{\rho_1} \subset Z_{\rho_2}$ whenever $\rho_2 \le \rho_1$ and since Σ is a directed set, the family $\{Z_\rho\}_{\rho\in\Sigma}$ of closed subsets of the Cartesian product $\prod_{\sigma\in\Sigma} X_\sigma$ has the finite intersection property. The Tychonoff theorem yields the inequality $\bigcap_{\rho\in\Sigma} Z_\rho \neq \emptyset$, which – along with the obvious relation $\underleftarrow{\lim} \mathrm{S} = \bigcap_{\rho\in\Sigma} Z_\rho$ – concludes the proof. ■

3.2.14. THEOREM. *Let $\{\varphi, f_{\sigma'}\}$ be a mapping of an inverse system $\mathrm{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of compact spaces to an inverse system $\mathrm{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$ of T_1-spaces. If all mappings $f_{\sigma'}$ are onto, the limit mapping $f = \underleftarrow{\lim}\{\varphi, f_{\sigma'}\}$ also is a mapping onto.*

PROOF. By Corollary 2.5.11 we can assume that $\varphi(\Sigma') = \Sigma$. Let us consider an auxiliary inverse system $\mathrm{S}'' = \{X_{\varphi(\sigma')}, \pi^{\varphi(\sigma')}_{\varphi(\rho')}, \Sigma'\}$, where every space X_σ occurs $|\varphi^{-1}(\sigma)|$ times and where the bonding mappings are those of S; furthermore, let us consider the mapping $\{\mathrm{id}_{\Sigma'}, f_{\sigma'}\}$ of the inverse system S'' to the inverse system S', and let us denote by f' the limit mapping $\underleftarrow{\lim}\{\mathrm{id}_{\Sigma'}, f_{\sigma'}\}$.

Let $y = \{y_{\sigma'}\}$ be an arbitrary point of $\underleftarrow{\lim}\mathrm{S}'$. For every $\sigma' \in \Sigma'$ the set $Z_{\sigma'} = f^{-1}_{\sigma'}(y_{\sigma'})$ is closed in $X_{\varphi(\sigma')}$, and thus is compact. For any $\sigma', \rho' \in \Sigma'$ satisfying $\rho' \leq \sigma'$ we have

$$\pi^{\varphi(\sigma')}_{\varphi(\rho')}(Z_{\sigma'}) = \pi^{\varphi(\sigma')}_{\varphi(\rho')} f^{-1}_{\sigma'}(y_{\sigma'}) \subset f^{-1}_{\rho'}\pi^{\sigma'}_{\rho'}(y_{\sigma'}) = f^{-1}_{\rho'}(y_{\rho'}) = Z_{\rho'},$$

so that $\mathrm{S}''' = \{Z_{\sigma'}, \tilde{\pi}^{\varphi(\sigma')}_{\varphi(\rho')}, \Sigma'\}$, where $\tilde{\pi}^{\varphi(\sigma')}_{\varphi(\rho')}(x) = \pi^{\varphi(\sigma')}_{\varphi(\rho')}(x)$ for $x \in Z_{\sigma'}$, is an inverse system of non-empty compact spaces. By Theorem 3.2.13 there exists a point $x' = \{x_{\sigma'}\} \in \underleftarrow{\lim}\mathrm{S}'''$, clearly $f'(x') = y$.

Consider now the mapping $\{\varphi, \mathrm{id}_{X_{\varphi(\sigma')}}\}$ of the inverse system S to the inverse system S''; from Proposition 2.5.10 it follows that the limit mapping $f'' = \underleftarrow{\lim}\{\varphi, \mathrm{id}_{X_{\varphi(\sigma')}}\}$ is a homeomorphism, so that there exists an $x \in \underleftarrow{\lim}\mathrm{S}$ such that $f''(x) = x'$. Since – as one can easily check – $f = f'f''$, we have $f(x) = y$. ■

Theorems 3.2.14 and 2.5.14 yield

3.2.15. COROLLARY. *If in an inverse system $\mathrm{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of compact spaces all bonding mappings π^σ_ρ are onto, then the projections $\pi_\sigma: \underleftarrow{\lim}\mathrm{S} \to X_\sigma$ are also mappings onto.* ■

We have also two further corollaries concerning the two particular cases of mappings of inverse systems discussed at the end of Section 2.5.

3.2.16. COROLLARY. *If $\mathrm{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$, where $\Sigma \neq \emptyset$, is an inverse system of T_1-spaces, X is a compact space, and $\{f_\sigma\}_{\sigma\in\Sigma}$ where $f_\sigma: X \to X_\sigma$, is a family of mappings onto such that $\pi^\sigma_\rho f_\sigma = f_\rho$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$, then the limit mapping $\underleftarrow{\lim} f_\sigma$ also is a mapping onto.* ■

3.2.17. COROLLARY. *If $\mathrm{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$, where $\Sigma \neq \emptyset$, is an inverse system of compact spaces, X is a T_1-space and $\{f_\sigma\}_{\sigma\in\Sigma}$, where $f_\sigma: X_\sigma \to X$, is a family of mappings onto such that $f_\rho\pi^\sigma_\rho = f_\sigma$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$, then the limit mapping $\underleftarrow{\lim} f_\sigma$ also is a mapping onto.* ■

We shall conclude this section with an important theorem about the space of continuous real-valued functions defined on a compact space. General function spaces will be discussed in Section 3.4.

A family $P \subset R^X$ of continuous real-valued functions defined on a topological space X is a *ring of functions* provided that for all $f, g \in P$ the functions $f+g$, $f-g$ and $f \cdot g$ also belong to P.

As proved in Section 1.4, the whole space R^X is a ring of functions that contains all constant functions and is closed with respect to uniform convergence; moreover, if X is a Tychonoff space, then the family R^X separates points, i.e., for every pair of distinct points $x, y \in X$ there exists a function $f \in R^X$ such that $f(x) \neq f(y)$. For compact spaces the converse is also true: every ring of continuous real-valued functions defined on a compact space X and satisfying the above conditions coincides with the whole of R^X. The proof of this theorem will be preceded by three lemmas. The second lemma is a particular case of a well-known theorem of analysis and is included here to make the book selfcontained. The first one is used only in the proof of the second.

3.2.18. LEMMA (THE DINI THEOREM). *Let X be a compact space and $\{f_i\}$ a sequence of continuous real-valued functions defined on X and satisfying $f_i(x) \leq f_{i+1}(x)$ for all $x \in X$ and $i = 1, 2, \ldots$ If there exists a function $f \in R^X$ such that $f(x) = \lim f_i(x)$ for every $x \in X$, then $f = \lim f_i$, i.e., the sequence $\{f_i\}$ is uniformly convergent to f.*

PROOF. Let ϵ be an arbitrary positive number. The sets $F_i = \{x : f(x) - f_i(x) \geq \epsilon\}$ are closed and form a decreasing sequence $F_1 \supset F_2 \supset \ldots$ Since $\bigcap_{i=1}^{\infty} F_i = \emptyset$, the family $\{F_i\}_{i=1}^{\infty}$ cannot have the finite intersection property, so that there exists an i_0 such that $F_{i_0} = \emptyset$ and this proves the lemma. ■

3.2.19. LEMMA. *There exists a sequence $\{w_i\}$ of polynomials which is uniformly convergent to the function $\sqrt{t}$ on the closed interval I.*

PROOF. The sequence $\{w_i\}$ is defined recursively by the formulas

$$(6) \qquad w_1(t) = 0 \quad \text{and} \quad w_{i+1}(t) = w_i(t) + \frac{1}{2}(t - w_i^2(t)) \quad \text{for } i = 1, 2, \ldots$$

We shall prove by induction that

$$(7) \qquad w_i(t) \leq \sqrt{t} \quad \text{for } t \in I \text{ and } i = 1, 2, \ldots$$

The last inequality is valid for $i = 1$, suppose that $w_i(t) \leq \sqrt{t}$. Since

$$\sqrt{t} - w_{i+1}(t) = \sqrt{t} - w_i(t) - \frac{1}{2}(t - w_i^2(t)) = (\sqrt{t} - w_i(t))[1 - \frac{1}{2}(\sqrt{t} + w_i(t))],$$

the inductive hypothesis and the inequality $t \leq 1$ yield

$$\sqrt{t} - w_{i+1}(t) \geq (\sqrt{t} - w_i(t))(1 - \frac{1}{2}2\sqrt{t}) \geq 0,$$

which completes the proof of (7).

From (6) and (7) it follows that $w_i(t) \leq w_{i+1}(t)$ for $t \in I$ and $i = 1, 2, \ldots$, which – along with (7) – shows that for every $t \in I$ there exists the limit $f(t)$ of the sequence $\{w_i(t)\}$. A passage to the limit in (6) gives the equality $f(t) = \sqrt{t}$ for every $t \in I$; from the preceding lemma it follows that $f = \lim w_i$. ■

3.2.20. LEMMA. *Let P be a ring of continuous and bounded real-valued functions defined on a topological space X. If the ring P contains all constant functions and is closed with respect to uniform convergence, then for $f, g \in P$ the functions $\max(f,g)$ and $\min(f,g)$ belong to P.*

PROOF. Since

$$\min(f,g) = \frac{1}{2}(f+g-|f-g|) \quad \text{and} \quad \max(f,g) = \frac{1}{2}(f+g+|f-g|),$$

it suffices to show that if $f \in P$, then $|f| \in P$. Take an $f \in P$ and a positive number c such that $|f(x)| \le c$ for $x \in X$. As it is sufficient to prove that $(1/c)|f| \in P$, one can assume without loss of generality that $|f(x)| \le 1$ for $x \in X$. Now, by the preceding lemma the function $|f| = \sqrt{f^2}$ is the limit of a uniformly convergent sequence of functions in P, viz., of the sequence $\{f_i\}$, where $f_i(x) = w_i[(f(x))^2]$. ■

3.2.21. THE STONE-WEIERSTRASS THEOREM. *If a ring P of continuous real-valued functions defined on a compact space X contains all constant functions, separates points and is closed with respect to uniform convergence (i.e., is a closed subset of the space R^X with the topology of uniform convergence), then P coincides with the ring of all continuous real-valued functions defined on X.*

PROOF. It suffices to prove that for every $f \in R^X$ and any $\epsilon > 0$ there exists an $f_\epsilon \in P$ satisfying $|f(x) - f_\epsilon(x)| < \epsilon$ for every $x \in X$.

For every pair of distinct points $a, b \in X$ there exists a function $h \in P$ such that $h(a) \neq h(b)$. The function g defined by letting $g(x) = (h(x) - h(a)) \cdot (h(b) - h(a))^{-1}$ belongs to P and has the property that $g(a) = 0$ and $g(b) = 1$. For the function $f_{a,b} \in P$ defined by the formula $f_{a,b}(x) = (f(b) - f(a))g(x) + f(a)$ we have

$$f_{a,b}(a) = f(a) \quad \text{and} \quad f_{a,b}(b) = f(b).$$

The sets

$$U_{a,b} = \{x : f_{a,b}(x) < f(x) + \epsilon\} \quad \text{and} \quad V_{a,b} = \{x : f_{a,b}(x) > f(x) - \epsilon\}$$

are neighbourhoods of points a and b respectively. Let us fix the point b and take a finite subcover $\{U_{a_i,b}\}_{i=1}^k$ of the open cover $\{U_{a,b}\}_{a \in X}$ of X. By Lemma 3.2.20, the function $f_b = \min\{f_{a_1,b}, f_{a_2,b}, \ldots, f_{a_k,b}\}$ belongs to P and we clearly have

$$f_b(x) < f(x) + \epsilon \text{ for } x \in X \quad \text{and} \quad f_b(x) > f(x) - \epsilon \text{ for } x \in V_b = \bigcap_{i=1}^{k} V_{a_i,b}.$$

The set V_b is a neighbourhood of the point b. Let us take a finite subcover $\{V_{b_i}\}_{i=1}^l$ of the open cover $\{V_b\}_{b \in X}$ of X. By Lemma 3.2.20, the function $f_\epsilon = \max\{f_{b_1}, f_{b_2}, \ldots, f_{b_l}\}$ belongs to P and we clearly have

$$|f(x) - f_\epsilon(x)| < \epsilon \text{ for every } x \in X. ■$$

The importance of the Stone-Weierstrass theorem lies in the fact that it yields a method of uniform approximation of all continuous real-valued functions defined on a compact space X by special classes of functions. Indeed, every continuous real-valued function on X can be arbitrarily closely approximated by polynomials (in several variables) of elements of a fixed family of functions that separates points. Since for every closed interval $J \subset R$ the family $\{f\}$ consisting of the function $f: J \to R$, defined by $f(x) = x$, separates points, Theorem 3.2.21 implies the classical Weierstrass theorem which states that every continuous real-valued function on J is the limit of a uniformly convergent sequence of polynomials.

3.2.22. EXAMPLE. We shall show that in Theorem 3.2.21 the assumption of compactness is essential. Indeed, one easily observes that the ring P, defined as the closure in the space R^R with the topology of uniform convergence of the set of all continuous functions from R to itself that are constant beyond an interval, satisfies all the assumptions of the Stone-Weierstrass theorem, but is not equal to the whole of R^R because it does not contain the function $\sin x$. One can prove that if a completely regular space X satisfies the Stone-Weierstrass theorem, then X is compact (see Exercise 3.2.K). ■

As observed at the end of Section 3.1, Theorems 3.2.3, 3.2.4 and 3.2.10 hold also for quasi-compact spaces. The reader can easily verify that the counterparts of Theorems 3.2.1, 3.2.2, 3.2.11, 3.2.13, 3.2.14 and 3.2.21 for quasi-compact spaces do not hold. One proves, however, that the limit of an inverse system of non-empty quasi-compact T_0-spaces is quasi-compact and non-empty, provided that all bonding mappings are closed; this fact implies easily formulated counterparts of Theorem 3.2.14 and Corollaries 3.2.15–3.2.17 for quasi-compact spaces.

Historical and bibliographic notes

Theorem 3.2.1 was proved by Taĭmanov in [1952] and, in a dual statement (see Exercise 3.2.A(a)), by Eilenberg and Steenrod in [1952]. Theorem 3.2.2 was proved by Alexandroff in [1936]. Tychonoff proved Theorems 3.2.5 and 3.2.6 in [1930]; in particular, he showed that all cubes $I^{\mathfrak{m}}$ are compact which easily implies the validity of Theorem 3.2.4, but the theorem itself was formulated for the first time in Tychonoff's paper [1935a]. Our proof of the Tychonoff theorem follows Chevalley and O. Frink [1941]. The finite case of the Wallace theorem appears in Gottshalk and Hedlund [1955] and in Kelley [1955], where it is attributed to Wallace; the theorem in full generality was proved by Frolík in [1960] and by Lin in [1960]. The Alexandroff theorem was announced in [1925] and proved in [1927]. Theorem 3.2.13 was, in principle, proved by Steenrod in [1936] (its counterpart for quasi-compact spaces quoted at the end of this section was proved by A. H. Stone in [1979]); Corollary 3.2.15 can be found in Eilenberg and Steenrod [1952]. Lemma 3.2.18 was established by U. Dini in 1878 for functions defined on a closed interval. The Stone-Weierstrass theorem, generalizing the classical result of Weierstrass obtained in 1885 (and stated here just before Example 3.2.22), was proved by M. H. Stone in [1937]; the simple proof given here and an interesting discussion of the subject can be found in M. H. Stone's [1947] paper.

Exercises

3.2.A. (a) (Eilenberg and Steenrod [1952]) Let A be a dense subspace of a topological space X and f a continuous mapping of A to a compact space Y. Prove that the mapping f has a continuous extension over X if and only if for every finite open cover $\{V_i\}_{i=1}^k$ of the space Y there exists a finite open cover $\{U_i\}_{i=1}^l$ of the space X such that the open cover $\{U_i \cap A\}_{i=1}^l$ of the subspace A is a refinement of $\{f^{-1}(V_i)\}_{i=1}^k$.

(b) (Bourbaki and Dieudonné [1939]) Let A be a dense subspace of a topological space X and f a continuous mapping of A to a regular space Y. Prove that the mapping f has a continuous extension over X if and only if f has a continuous extension over $A \cup \{x\}$ for every $x \in X \setminus A$. Note that the assumption of regularity of Y cannot be weakened to the assumption that Y is a Hausdorff space.

3.2.B. Show that the formula

$$f(x) = \sum_{i=1}^{\infty} \frac{x_i}{2^i}, \quad \text{where} \quad x = \{x_i\},$$

defines a continuous mapping of the Cantor set $D^{\aleph_0}$ onto the interval I; apply this fact to prove Theorem 3.2.2. Verify that there exists a countable dense set $A \subset I$ such that $|f^{-1}(y)| = 2$ for every $y \in A$ and $|f^{-1}(y)| = 1$ for $y \in I \setminus A$.

3.2.C. The *segment* in R^n with *end-points* $x, y \in R^n$ is the set of all points of the form $(1-t)x + ty$, where $0 \leq t \leq 1$ and addition of points as well as multiplication of a point by a real number are defined as in the appendix to Section 7.3. A subset A of R^n is *convex* if for each pair x, y of points of A the segment with end-points x, y is contained in A.

Show that every convex compact set $A \subset R^n$ such that $\operatorname{Int} A \neq \emptyset$ is homeomorphic to the unit n-ball B^n and that the boundary $\operatorname{Fr} A$ is homeomorphic to the unit $(n-1)$-sphere S^{n-1}. Note that, in particular, the spaces I^n and B^n, as well as the spaces $\operatorname{Fr} I^n \subset R^n$ and S^{n-1}, are homeomorphic for $n = 1, 2, \ldots$

Hint. Take a point $x \in \operatorname{Int} A$ and prove that every ray starting from x meets $\operatorname{Fr} A$ at exactly one point.

3.2.D. Note that the Tychonoff theorem follows from the Kuratowski theorem and the Wallace theorem.

3.2.E (Kelley [1955]). The subspace X of the Tychonoff cube $I^{\mathfrak{c}} = \prod_{t \in I} I_t$, where $I_t = I$ for every $t \in I$, consisting of all nondecreasing functions from I to I is called the *Helly space.* Prove that

(a) The Helly space is compact.

(b) The Helly space contains a subspace homeomorphic to the discrete space $D(\mathfrak{c})$ and a subspace homeomorphic to the Sorgenfrey line K.

(c) The Helly space is not hereditarily normal.

Hint. Verify that X^2 is embeddable in X.

(d) The Helly space is first-countable.

Hint. The set of points of discontinuity of every function in X is countable.

(e) The Helly space is separable.

Hint. See the proof of the Hewitt-Marczewski-Pondiczery theorem.

3.2.F (Arhangel'skiĭ [1965]). Show that the Cartesian product $\prod_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is of pointwise countable type if and only if all spaces X_s are of pointwise countable type and there exists a countable set $S_0 \subset S$ such that X_s is compact for $s \in S \setminus S_0$ (see Exercise 3.1.E(b)).

3.2.G. Show that if X is a compact space, then an equivalence relation E on X is closed if and only if the set E is closed in the Cartesian product $X \times X$.

3.2.H. (a) (Mibu [1944]) Applying the Stone-Weierstrass theorem prove that every continuous real-valued function $f: \prod_{s\in S} X_s \to R$ defined on a Cartesian product of compact spaces depends on countably many coordinates (see the definition in Problem 2.7.12(d)).

(b) Show that there exists a continuous real-valued function (in fact, a continuous function assuming only values 0 and 1) defined on the Cartesian product $T \times \prod_{t\in T} D_t$, where $T = D(\mathfrak{c})$ and $D_t = D$ for every $t \in T$, that depends on uncountably many coordinates.

Hint. Consider the function $f: T \times \prod_{t\in T} D_t \to D$ defined by the formula $f(t,x) = p_t(x)$, where p_t is the projection of $\prod_{t\in T} D_t$ onto D_t.

(c) (Kellerer [1968]) Prove that if X_s is for every $s \in S$ a Hausdorff space that contains a dense subset which can be represented as a countable union of compact sets (in particular, if every X_s is separable), then every continuous real-valued function $f: \prod_{s\in S} X_s \to R$ depends on countably many coordinates (cf. Problem 2.7.12(d) and Exercise 4.1.G).

Hint. Let $X_{s,1}, X_{s,2}, \ldots$ be an increasing sequence of compact subsets of X_s such that the union $\bigcup_{i=1}^{\infty} X_{s,i}$ is dense in X_s; consider the Cartesian products $X_i = \prod_{s\in S} X_{s,i}$ and note that the union $\bigcup_{i=1}^{\infty} X_i$ is dense in $\prod_{s\in S} X_s$.

(d) Prove counterparts of (a) and (c) for continuous mappings of Cartesian products into a Tychonoff space of weight $\mathfrak{m}$.

(e) Prove that if an inverse system $\mathbf{S} = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$ of compact spaces has the property that for every set $\Sigma_0 \subset \Sigma$ of cardinality $\mathfrak{m} \geq \aleph_0$ there exists a $\sigma_0 \in \Sigma$ such that $\sigma \leq \sigma_0$ for $\sigma \in \Sigma_0$, then for every continuous mapping f of $\varprojlim \mathbf{S}$ onto a compact space Y of weight $\leq \mathfrak{m}$ there exists a $\sigma \in \Sigma$ and a continuous mapping $f_\sigma: X_\sigma \to Y$ such that $f = f_\sigma \pi_\sigma$.

3.2.I. (a) Show that for a Tychonoff space X and a cardinal number $\mathfrak{m} \geq \aleph_0$ each of the following conditions is a consequence of the preceding one and that if X is compact, then all the conditions are equivalent:

(1) *The space R^X with the topology of uniform convergence contains a dense subset of cardinality $\leq \mathfrak{m}$.*

(2) *The space X has a base of cardinality $\leq \mathfrak{m}$.*

(3) *The set R^X contains a subset of cardinality $\leq \mathfrak{m}$ that separates points.*

Apply this result to prove Theorem 3.1.21 and – under the additional assumption that X is compact – Theorem 3.1.22.

Verify that neither conditions (1) and (2) nor conditions (2) and (3) are equivalent.

Hint. When proving Theorem 3.1.22 note that if there exists a continuous mapping of X onto Y, then R^Y is embeddable in R^X and verify that if $d(R^X) \leq \mathfrak{m}$, then also $w(R^X) \leq \mathfrak{m}$.

(b) Prove that condition (3) in (a) is equivalent to

(4) *The space R^X with the topology of pointwise convergence contains a dense subset of cardinality $\leq \mathfrak{m}$.*

Deduce that $d(R^{2^{\mathfrak{m}}}) \leq \mathfrak{m}$ (cf. the Hewitt-Marczewski-Pondiczery theorem).

3.2.J (M. H. Stone [1947]). (a) Show that every continuous real-valued function defined on a compact subspace M of a Tychonoff space X is continuously extendable over X.

Hint. Embed X in a Tychonoff cube and apply the Tietze-Urysohn theorem.

(b) Observe that part (a) follows from the Stone-Weierstrass theorem without the use of the Tietze-Urysohn theorem (cf. Exercise 3.6.C).

3.2.K (Hewitt [1947]). Observe that if a completely regular space X satisfies the Stone-Weierstrass theorem, then X is compact.

Hint. Embed X in a Tychonoff cube.

3.3. Locally compact spaces and k-spaces

A topological space X is called a *locally compact space* if for every $x \in X$ there exists a neighbourhood U of the point x such that $\overline{U}$ is a compact subspace of X. Since the compact space $\overline{U}$ is a T_1-space, the set $\{x\}$ is closed in $\overline{U}$; this implies that $\{x\}$ is closed in X, i.e., that every locally compact space is a T_1-space. It turns out that the following stronger result holds:

3.3.1. THEOREM. *Every locally compact space is a Tychonoff space.*

PROOF. Let x be a point of a locally compact space X and let F be a closed subset of X such that $x \notin F$. Take a neighbourhood U of the point x such that $\overline{U}$ is compact. The set $F_0 = (\overline{U} \setminus U) \cup (\overline{U} \cap F)$ is a closed subset of the space $\overline{U}$; as $x \in \overline{U} \setminus F_0$ there exists a continuous function $f_1: \overline{U} \to I$ such that $f_1(x) = 0$ and $f_1(F_0) \subset \{1\}$. Since $\overline{U} \cap (X \setminus U) = \overline{U} \setminus U \subset F_0$, the combination f of f_1 and the constant function $f_2: X \setminus U \to I$, defined by letting $f_2(y) = 1$ for every $y \in X \setminus U$, is continuous by virtue of 2.1.13. One easily sees that $f(x) = 0$ and $f(F) \subset \{1\}$. ∎

3.3.2. THEOREM. *For every compact subspace A of a locally compact space X and every open set V that contains A there exists an open set U such that $A \subset U \subset \overline{U} \subset V$ and $\overline{U}$ is compact.*

PROOF. For every $x \in A$ take a neighbourhood V_x of the point x such that $\overline{V_x} \subset V$ and a neighbourhood W_x of x such that $\overline{W_x}$ is compact. The set $\overline{U_x}$, where $U_x = V_x \cap W_x$, is compact because it is a closed subset of the compact space $\overline{W_x}$. By Theorem 3.1.3 there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset A$ such that $A \subset U = U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_k}$. The set $\overline{U} = \overline{U_{x_1}} \cup \overline{U_{x_2}} \cup \ldots \cup \overline{U_{x_k}}$ is compact by virtue of 3.1.4 and we clearly have $\overline{U} \subset \overline{V_{x_1}} \cup \overline{V_{x_2}} \cup \ldots \cup \overline{V_{x_k}} \subset V$. ∎

From Theorems 3.3.1, 3.1.7 and 3.1.2 we obtain

3.3.3. COROLLARY. *For every compact subspace A of a locally compact space X and every open set V that contains A there exists a continuous function $f: X \to I$ such that $f(x) = 0$ for $x \in A$, $f(x) = 1$ for $x \in X \setminus V$ and the set $f^{-1}([0, a])$ is compact for every $a < 1$.* ∎

The next theorem gives an evaluation for characters of points in a locally compact space (cf. Exercise 3.1.F(a)).

3.3.4. THEOREM. *The character of a point x in a locally compact space X is equal to the smallest cardinal number of the form $|\mathcal{U}|$, where $\mathcal{U}$ is a family of open subsets of X such that $\bigcap \mathcal{U} = \{x\}$.*

PROOF. Let $\{x\} = \bigcap_{s\in S} V_s$, where V_s is open in X and $|S| \leq \mathfrak{m}$; it suffices to show that $\chi(x, X) \leq \mathfrak{m}$. One can assume that $\mathfrak{m} \geq \aleph_0$, because if $\mathfrak{m}$ is finite, then x is an isolated point of X and $\chi(x, X) = 1 \leq \mathfrak{m}$. From Theorem 3.3.2 it follows that $\{x\} = \bigcap_{s\in S} \overline{U}_s$, where U_s is a neighbourhood of x such that $\overline{U}_s$ is a compact subspace of X. By 3.1.5 for every neighbourhood U of x there exists a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ such that $U_{s_1} \cap U_{s_2} \cap \ldots \cap U_{s_k} \subset \overline{U}_{s_1} \cap \overline{U}_{s_2} \cap \ldots \cap \overline{U}_{s_k} \subset U$. Hence, all finite intersections of members of $\{U_s\}_{s\in S}$ form a base at the point x; as the cardinality of the family of all such intersections does not exceed $\mathfrak{m}$, we have $\chi(x, X) \leq \mathfrak{m}$. ■

It turns out that Theorem 3.1.19, as well as its consequences, can be generalized to locally compact spaces (cf. Exercise 3.9.E).

3.3.5. THEOREM. *For every locally compact space X we have $nw(X) = w(X)$.*

PROOF. It suffices to show that $w(X) \leq nw(X)$; obviously, one can assume that $nw(X) = \mathfrak{m} \geq \aleph_0$. Let $\mathcal{N}$ be a network for X satisfying $|\mathcal{N}| = \mathfrak{m}$. From the definition of local compactness it follows that the family $\{M_s\}_{s\in S}$ consisting of all members of $\mathcal{N}$ that have compact closures is a cover of X. By 3.3.2 for every $s \in S$ there exists an open set $U_s \subset X$ such that $\overline{M}_s \subset U_s$ and $\overline{U}_s$ is compact. Since $nw(\overline{U}_s) \leq nw(X) = \mathfrak{m}$ it follows from Theorem 3.1.19 that $w(U_s) \leq \mathfrak{m}$, i.e., that there exists a base $\mathcal{B}_s$ of cardinality $\leq \mathfrak{m}$ for the subspace U_s of X. One easily sees that the union $\bigcup_{s\in S} \mathcal{B}_s$ is a base of cardinality $\leq \mathfrak{m}$ for the space X; thus we have $w(X) \leq \mathfrak{m}$. ■

Let us note that the last theorem is an immediate consequence of Theorem 3.1.19 and Theorem 3.5.11 below.

3.3.6. COROLLARY. *For every locally compact space X we have $w(X) \leq |X|$.* ■

3.3.7. COROLLARY. *If a locally compact space Y is a continuous image of a topological space X, then $w(Y) \leq w(X)$.* ■

3.3.8. THEOREM. *If X is a locally compact space, then every subspace of X that can be repesented in the form $F \cap V$, where F is closed in X and V is open in X, also is locally compact.*

PROOF. It suffices to show that local compactness is hereditary both with respect to closed subsets and with respect to open subsets, because the intersection $F \cap V$ is an open subspace of the closed subspace F of X.

Let F be a closed subspace of a locally compact space X. For every $x \in F$ there exists a neighbourhood U of the point x in the space X such that $\overline{U}$ is a compact space. The intersection $F \cap U$ is a neighbourhood of the point x in the subspace F, and the closure $\overline{F \cap U} \cap F = \overline{F \cap U}$ of this neighbourhood in F is compact, being a closed subset of the compact space $\overline{U}$.

The fact that local compactness is hereditary with respect to open subsets follows directly from Theorem 3.3.2 applied to one-point subspaces. ■

3.3.9. THEOREM. *Every locally compact subspace M of a Hausdorff space X is an open subset of the closure $\overline{M}$ of the set M in the space X, i.e., it can be represented in the form $F \cap V$, where F is closed in X and V is open in X.*

PROOF. It suffices to show that a locally compact dense subspace M of a Hausdorff space X is open in X.

Every point $x \in M$ has a neighbourhood U in the subspace M such that the set $\overline{U} \cap M$ is compact and thus closed in X. Since $U \subset \overline{U} \cap M$, we have $\overline{U} \subset \overline{U} \cap M \subset M$. Let W be an open subset of X satisfying $U = M \cap W$; by virtue of Theorem 1.3.6

$$x \in W \subset \overline{W} = \overline{M \cap W} = \overline{U} \subset M,$$

which shows that every point $x \in M$ has a neighbourhood W in the space X contained in the subspace M, i.e., that M is open in X.■

The last two theorems give

3.3.10. COROLLARY. *A subspace M of a locally compact space X is locally compact if and only if it can be represented in the form $F \cap V$, where F is closed in X and V is open in X.*■

From 3.3.10, 3.3.1 and 3.2.6 we obtain

3.3.11. COROLLARY. *A topological space is locally compact if and only if it is homeomorphic to an open subspace of a compact space.*■

3.3.12. THEOREM. *The sum $\bigoplus_{s \in S} X_s$ is locally compact if and only if all spaces X_s are locally compact.*

PROOF. If the sum $\bigoplus_{s \in S} X_s$ is locally compact, then all the X_s's are locally compact by Theorem 3.3.8 and Corollary 2.2.2.

Conversely, if all the X_s's are locally compact, then for every $x \in X = \bigoplus_{s \in S} X_s$ there exists a neighbourhood U of x in the space X_{s_0} that contains x such that the closure of U in X_{s_0} is compact; clearly U is a neighbourhood of x in X and the closure of U in X, being identical with the closure of U in X_{s_0}, is compact.■

3.3.13. THEOREM. *The Cartesian product $\prod_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is locally compact if and only if all spaces X_s are locally compact and there exists a finite set $S_0 \subset S$ such that X_s is compact for $s \in S \setminus S_0$.*

PROOF. By the Tychonoff theorem and Proposition 2.3.7 to prove that the above condition is sufficient for local compactness of the Cartesian product it is enough to show that the Cartesian product of finitely many locally compact spaces $X_1, X_2, \ldots, X_k$ is locally compact. Take a point $x = \{x_1, x_2, \ldots, x_k\} \in X_1 \times X_2 \times \ldots \times X_k$; by local compactness of X_i, for $i = 1, 2, \ldots, k$ there exists a neighbourhood V_i of x_i in X_i such that $\overline{V}_i$ is compact. The set $V = V_1 \times V_2 \times \ldots \times V_k$ is a neighbourhood of x in $X_1 \times X_2 \times \ldots \times X_k$ and $\overline{V}$ is compact by 2.3.3 and the Tychonoff theorem.

Conversely, suppose that $\prod_{s \in S} X_s$ is a non-empty locally compact space. Take an $s_0 \in S$ and a point $x \in X_{s_0}$; we shall show that the point x has a neighbourhood $W \subset X_{s_0}$ such that $\overline{W}$ is a compact subspace of X_{s_0}. Let x_s be an arbitrary point of X_s for $s \neq s_0$ and let $x_{s_0} = x$. The point $\{x_s\} \in \prod_{s \in S} X_s$ has a neighbourhood U such that $\overline{U}$ is compact. Clearly, there exists a member $\prod_{s \in S} W_s$ of the canonical base for $\prod_{s \in S} X_s$ such that $\{x_s\} \in \prod_{s \in S} W_s \subset U$ and $W_s = X_s$ for $s \in S \setminus S_0$, where $|S_0| < \aleph_0$. From Theorem 3.1.2 it follows that the Cartesian product $\prod_{s \in S} \overline{W}_s = \overline{\prod_{s \in S} W_s} \subset \overline{U}$ is compact; this implies that $W = W_{s_0} \subset X_{s_0}$ is a neighbourhood of x which has a compact closure and that X_s is compact for $s \in S \setminus S_0$.■

Let us note that the last theorem and Example 2.5.3 imply that the limit of an inverse sequence of locally compact spaces need not be locally compact.

3.3.14. EXAMPLES. Every discrete space is locally compact. The real line is locally compact because it is homeomorphic to the open subspace $(-1,1)$ of the compact space $[-1,1]$. From the last theorem it follows that Euclidean n-space R^n also is locally compact. The space W_0 of all countable ordinal numbers is locally compact, too.

The space Z defined in Example 2.3.36 is locally compact as an open subspace of the compact space $X \times Y$; it is an example of a non-normal locally compact space. ∎

3.3.15. THEOREM. *If there exists an open mapping $f: X \to Y$ of a locally compact space X onto a Hausdorff space Y, then Y is a locally compact space.*

PROOF. Let y be a point of Y; take an arbitrary point $x \in f^{-1}(y)$ and a neighbourhood U of x such that $\overline{U}$ is a compact subspace of X. The image $f(U)$ is a neighbourhood of y in Y and – since the set $f(\overline{U})$ is compact and thus closed in Y – the closure $\overline{f(U)} \subset f(\overline{U})$ is compact. ∎

On the other hand, it turns out that local compactness is not an invariant of closed mappings:

3.3.16. EXAMPLES. In Example 1.4.17 we defined a closed mapping $f: X \to Y$ of the real line $R = X$ onto the quotient space $R/N = Y$ obtained by identifying the set N of positive integers to the point $y_0 \in Y$. We also observed there that the space Y has no countable base at y_0; by virtue of Corollary 3.3.7 this implies that the space Y is not locally compact.

Another example can be obtained by identifying to a point the closed subset A of the locally compact space Z described in Example 2.3.36; the quotient space Z/A is not regular, so it is not a locally compact space either. ∎

As shown in Example 2.4.20, the Cartesian product of a quotient mapping and an identity mapping need not be a quotient mapping; for identities on locally compact spaces we have however (cf. Problem 3.12.14(b)):

3.3.17. THE WHITEHEAD THEOREM. *For every locally compact space X and any quotient mapping $g: Y \to Z$, the Cartesian product $f = \mathrm{id}_X \times g: X \times Y \to X \times Z$ is a quotient mapping.*

PROOF. Let us suppose that the inverse image $f^{-1}(W) \subset X \times Y$ of a set $W \subset X \times Z$ is open and let us take an arbitrary point $(x_0, z_0) \in W$. Choose a point $y_0 \in g^{-1}(z_0)$ and a neighbourhood U of x_0 such that $\overline{U}$ is compact and $\overline{U} \times \{y_0\} \subset f^{-1}(W)$. Since for every $y \in Y$ we have

$$(1) \qquad \overline{U} \times g^{-1}g(y) \subset f^{-1}(W) \quad \text{whenever} \quad \overline{U} \times \{y\} \subset f^{-1}(W),$$

the inclusion $\overline{U} \times g^{-1}(z_0) \subset f^{-1}(W)$ holds. The set $V = \{z \in Z : \overline{U} \times g^{-1}(z) \subset f^{-1}(W)\}$ satisfies the condition $(x_0, z_0) \in U \times V \subset W$, so that it suffices to show that V is an open subset of Z. As g is a quotient mapping, this reduces to showing that

$$g^{-1}(V) = \{y \in Y : \overline{U} \times g^{-1}g(y) \subset f^{-1}(W)\}$$

is an open subset of Y.

From (1) it follows that $g^{-1}(V) = \{y \in Y : \overline{U} \times \{y\} \subset f^{-1}(W)\}$, and the Kuratowski theorem implies that the last set is open, because it is the complement of the projection of the closed set $(\overline{U} \times Y) \setminus f^{-1}(W)$ under the projection $p: \overline{U} \times Y \to Y$ parallel to the compact axis $\overline{U}$. ■

The second half of this section is devoted to a study of k-spaces, a class of spaces that is closely related to the class of locally compact spaces. A topological space X is called a *k-space* if X is a Hausdorff space and X is an image of a locally compact space under a quotient mapping. In other words, k-spaces are Hausdorff spaces that can be represented as quotient spaces of locally compact spaces. Clearly every locally compact space is a k-space.

3.3.18. THEOREM. *A Hausdorff space X is a k-space if and only if for each $A \subset X$, the set A is closed in X provided that the intersection of A with any compact subspace Z of the space X is closed in Z.*

PROOF. Let X be a k-space and let $f: Y \to X$ be a quotient mapping of a locally compact space Y onto X. Suppose that the intersection of a set $A \subset X$ with any compact subspace Z of X is closed in Z. Take a point $y \in \overline{f^{-1}(A)}$ and a neighbourhood $U \subset Y$ of the point y such that $\overline{U}$ is a compact subspace of Y. The set $f^{-1}(A \cap f(\overline{U})) \subset f^{-1}(A)$ is closed in Y and contains $f^{-1}(A) \cap \overline{U}$; we then have $y \in f^{-1}(A)$. This implies that $\overline{f^{-1}(A)} = f^{-1}(A)$ and – f being a quotient mapping – that $\overline{A} = A$.

Now, consider a Hausdorff space X and denote by $\mathcal{Z}(X)$ the family of all non-empty compact subspaces of X. The space $\tilde{X} = \bigoplus_{Z \in \mathcal{Z}(X)} Z$ is locally compact and the mapping $f = \nabla_{Z \in \mathcal{Z}(X)} i_Z : \tilde{X} \to X$, where i_Z is the embedding of the subspace Z in the space X, is continuous by virtue of Proposition 2.1.11. One easily sees that if a set $A \subset X$ is closed provided that the intersections $A \cap Z$ are closed in Z for all $Z \in \mathcal{Z}(X)$, then f is a quotient mapping. ■

Let us observe that in the above theorem we could as well assume that all intersections $A \cap Z$ are compact or closed in X.

3.3.19. COROLLARY. *A Hausdorff space X is a k-space if and only if for each $A \subset X$, the set A is open in X provided that the intersection of A with any compact subspace Z of the space X is open in Z.* ■

3.3.20. THEOREM. *Every sequential Hausdorff space – and, in particular, every first-countable Hausdorff space – is a k-space.*

PROOF. Suppose that a subset A of a sequential space X is not closed. There exists then a sequence $x_1, x_2, \ldots$ of points of A and a point $x_0 \in \lim x_i$ such that $x_o \notin A$. If, moreover, X is a Hausdorff space, then – as one can easily verify – the subspace $Z = \{x_0, x_1, x_2, \ldots\}$ of X is homeomorphic to the space $A(\aleph_0)$, the point x_0 being the unique accumulation point of Z. This implies that the intersection of A with the compact subspace Z of X is not closed. ■

3.3.21. THEOREM. *A mapping f of a k-space X to a topological space Y is continuous if and only if for every compact subspace $Z \subset X$ the restriction $f|Z: Z \to Y$ is continuous.*

PROOF. It suffices to show that continuity of all restrictions $f|Z$ implies continuity of

f. Take a closed set $A \subset Y$; for every compact $Z \subset X$ the set $f^{-1}(A) \cap Z = (f|Z)^{-1}(A)$ is closed, which shows that $f^{-1}(A)$ is closed. Hence f is continuous. ∎

3.3.22. THEOREM. *A continuous mapping $f: X \to Y$ of a topological space X to a k-space Y is closed (open, quotient) if and only if for every compact subspace $Z \subset Y$ the restriction $f_Z: f^{-1}(Z) \to Z$ is closed (open, quotient).*

PROOF. By Propositions 2.1.4 and 2.4.15 it suffices to show that if all restrictions f_Z are closed, open or quotient, then so is the mapping f.

First, suppose that the restriction $f_Z: f^{-1}(Z) \to Z$ is closed (open) for every compact subspace $Z \subset Y$ and consider a closed (an open) set $A \subset X$. The equalities

$$f(A) \cap Z = f(A \cap f^{-1}(Z)) = f_Z(A \cap f^{-1}(Z))$$

and the fact that f_Z is closed (open) imply that the set $f(A) \cap Z$ is closed (open) in Z for every compact subspace $Z \subset Y$. Since Y is a k-space it follows that $f(A)$ is closed (open), and this shows that the mapping f is closed (open).

Now, suppose that the restriction $f_Z: f^{-1}(Z) \to Z$ is quotient for every compact subspace $Z \subset Y$ and consider a set $B \subset Y$ such that $f^{-1}(B)$ is closed in X. The set $f_Z^{-1}(B \cap Z) = f^{-1}(B) \cap f^{-1}(Z)$ is closed in $f^{-1}(Z)$; hence the intersection $B \cap Z$ is closed in Z for every compact subspace $Z \subset Y$. Since Y is a k-space it follows that B is closed in Y, and this shows that the mapping f is quotient. ∎

From the definition of a k-space we readily obtain

3.3.23. THEOREM. *If there exists a quotient mapping $f: X \to Y$ of a k-space X onto a Hausdorff space Y, then Y is a k-space.* ∎

3.3.24. EXAMPLES. Let us note that there exist k-spaces which are not regular: such a space is defined in the second part of 3.3.16.

There also exist perfectly normal spaces which are not k-spaces: the space Y defined in Example 1.6.20 is such a space. To prove that Y is not a k-space it suffices to show that all compact subspaces of Y are finite, because the set $Y \setminus \{0\}$ is not closed in Y and yet its intersection with every finite subset of Y is closed. Assume that Z is an infinite compact subspace of Y. Since each of the intersections $Z \cap (1/n, 1/n + 1/n^2]$ must be finite, there exists a sequence $x_1, x_2, \ldots$ of points of Z converging to 0 with respect to the natural topology of the real line. The set $\{x_1, x_2, \ldots\} \subset Z$ with the topology of a subspace of Y is homeomorphic to the discrete space $D(\aleph_0)$ and is closed in Y, and this contradicts the assumption that Z is compact.

The space Y is a continuous image of $D(\aleph_0)$ which is a k-space, thus a continuous image of a k-space is not necessarily a k-space, even if it is a perfectly normal space.

Let us observe also that the space Y is a subspace of a sequential space (the space X described in Example 1.6.19); so that – in the light of Theorem 3.3.20 – the property of being a k-space is not hereditary (cf. Exercise 3.3.I). ∎

From 3.3.8 and 2.4.15 we obtain however

3.3.25. THEOREM. *The property of being a k-space is hereditary both with respect to closed subsets and with respect to open subsets.* ∎

Theorem 3.3.18 yields

3.3.26. THEOREM. *The sum $\bigoplus_{s\in S} X_s$ is a k-space if and only if all spaces X_s are k-spaces.* ■

The Whitehead theorem implies

3.3.27. THEOREM. *The Cartesian product $X \times Y$ of a locally compact space X and a k-space Y is a k-space.* ■

Let us observe that from Theorem 3.3.23 it follows that if a non-empty Cartesian product $\prod_{s\in S} X_s$ is a k-space, then all the X_s's are k-spaces.

We shall now prove a theorem on Cartesian products of quotient mappings which will yield as a corollary the important fact that the property of being a k-space is not multiplicative.

3.3.28. THEOREM. *If $f_i: X_i \to Y_i$ is a quotient mapping for $i = 1, 2$ and if X_1 and $Y_1 \times Y_2$ are k-spaces, then the Cartesian product $f = f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is a quotient mapping.*

PROOF. Let us first observe that if $X \times T$ is a k-space, then for every quotient mapping $g: Y \to T$ the Cartesian product $h = \mathrm{id}_X \times g: X \times Y \to X \times T$ is a quotient mapping. To that end, let us consider compact subspaces $Z_1 \subset X$ and $Z_2 \subset T$, their Cartesian product $Z = Z_1 \times Z_2$ and the restriction $h_Z: h^{-1}(Z) \to Z$; from the equality $h^{-1}(Z) = Z_1 \times g^{-1}(Z_2)$ it follows that $h_Z = \mathrm{id}_{Z_1} \times g_{Z_2}$, thus h_Z is a quotient mapping by the Whitehead theorem. Since every compact subspace of the Cartesian product $X \times T$ is contained in the Cartesian product of its projections onto X and T, i.e., in a compact set of the form $Z_1 \times Z_2$, it follows from Theorem 3.3.22 that h is a quotient mapping.

Let us now consider a special case of the theorem, viz., the case of a locally compact X_1. Since by 3.3.27 the Cartesian product $X_1 \times Y_2$ is a k-space, from the fact observed in the last paragraph it follows that the mappings

$$\mathrm{id}_{X_1} \times f_2: X_1 \times X_2 \to X_1 \times Y_2 \quad \text{and} \quad f_1 \times \mathrm{id}_{Y_2}: X_1 \times Y_2 \to Y_1 \times Y_2$$

are quotient. Hence f, being their composition, also is a quotient mapping.

Finally let us consider an arbitrary k-space X_1. There exists a locally compact space X' and a quotient mapping $f': X' \to X_1$ of X' onto X_1. By the special case of the theorem established above, the Cartesian product $(f_1 f') \times f_2: X' \times X_2 \to Y_1 \times Y_2$ is a quotient mapping. Since $(f_1 f') \times f_2 = (f_1 \times f_2)(f' \times \mathrm{id}_{X_2})$, the Cartesian product $f = f_1 \times f_2$ is a quotient mapping by 2.4.5. ■

3.3.29. EXAMPLE. In 2.4.20 we considered the subspace $X_1 = Y_1 = R \setminus \{1/2, 1/3, \ldots\}$ of the real line and the quotient space Y_2 obtained from $X_2 = R$ by identifying the set of positive integers to a point. Both Y_1 and Y_2 are k-spaces and yet the Cartesian product $Y_1 \times Y_2$ is not a k-space. Indeed, as shown in 2.4.20, the Cartesian product $f = f_1 \times f_2$, where $f_1 = \mathrm{id}_{X_1}$ and f_2 is the natural quotient mapping, is not a quotient mapping, thus from the last theorem it follows that $Y_1 \times Y_2$ cannot be a k-space. ■

We close this section with a construction leading from an arbitrary Hausdorff space to a k-space which has the same underlying set. Let X be a Hausdorff space; one readily checks that the family C consisting of those subsets of X that have closed intersections with all compact subspaces of X has properties (C1)–(C3) – the set X with the topology generated by the family $\mathcal{C}$ of closed subsets will be denoted by kX. Clearly, a subset of kX is open if and only if its intersection with any compact subspace Z of the space X is open in Z. The topology of kX is finer than the topology of X, so that kX is a Hausdorff space and the formula $\kappa_X(x) = x$ defines a continuous mapping $\kappa_X: kX \to X$. From Theorem 3.1.10 it follows that if Z is a compact subspace of kX, then Z is also a compact subspace of X; on the other hand, if Z is a compact subspace of X, then the two topologies on Z, viz., the topology induced by the topology of X and the topology induced by the topology of kX, coincide, so that Z is also a compact subspace of kX. Thus the spaces kX and X have the same compact subspaces and the same topology on those subspaces. This implies that kX is a k-space and – by virtue of Theorem 3.3.21 – that for every continuous mapping $f: X \to Y$, where X and Y are Hausdorff spaces, the mapping kf assigning to $x \in kX$ the point $f(x) \in kY$ is continuous, i.e., $kf: kX \to kY$; clearly the mapping kf satisfies the equality $f\kappa_X = \kappa_Y kf$.

Historical and bibliographic notes

The notion of a locally compact space was introduced by Alexandroff in [1923]; in this paper Theorem 3.3.4 and Corollary 3.3.11 were announced (proofs were given in [1924b]). Theorem 3.3.17 was proved by Whitehead in [1948]. The class of k-spaces was introduced in Gale's paper [1950] (where the notion is ascribed to Hurewicz); the characterization established in Theorem 3.3.18 is taken as the definition there and it is proved that locally compact spaces and first-countable spaces are k-spaces. D. E. Cohen's paper [1954] contains Theorem 3.3.27, the construction of kX and the second half of the proof of Theorem 3.3.18. Kelley proved in [1955] Theorems 3.3.21 and 3.3.23. Theorem 3.3.25 was proved in Arhangel'skiĭ's paper [1965] (announcement in [1963]), and Theorem 3.3.28 in Michael's paper [1968a]. The first example of two k-spaces, whose Cartesian product is not a k-space, was given by Dowker in [1952].

Exercises

3.3.A. Prove that every space X that can be represented as the union of a locally finite family of locally compact closed subspaces is itself locally compact (cf. Theorem 3.7.22).

3.3.B. (a) Verify that every Hausdorff space X that can be represented as the union of a family of locally compact open subspaces is itself locally compact.

(b) Give an example of a T_1-space that can be represented as the union of two open compact subspaces and yet is not a T_2-space.

3.3.C. Define a subspace of the real line that can be represented as the union of two locally compact subspaces, one of which is closed and the other open, and that is not a locally compact space.

3.3.D (Parhomenko [1941]). Show that for every locally compact space X there exists a one-to-one continuous mapping of X onto a compact space.

Hint. Apply Corollary 3.3.11 and Theorem 3.2.11.

3.3.E. (a) (Kelley [1955]) Verify that the Cartesian product $N^{\aleph_1}$ is not a k-space.

Hint. Consider the subset of $N^{\aleph_1}$ consisting of all points x such that for some integer $i > 1$ at most i coordinates of x are equal to 1 and the remaining coordinates are equal to i.

(b) Show that the limit of an inverse sequence of Fréchet spaces need not be a k-space.

Hint. Represent the space Y_1 in Example 3.3.29 as the limit of an inverse sequence of locally compact spaces, apply Theorem 3.3.17 and Exercise 2.5.D(b).

3.3.F. Prove Theorem 3.3.27 by applying the characterization of k-spaces given in Theorem 3.3.18.

3.3.G (Arhangel'skiĭ [1965]). (a) Give an example of a k-space X, of a subset $A \subset X$ and of a point $x \in \overline{A}$ such that $x \notin \overline{A \cap Z}$ for every compact subspace $Z \subset X$.

Hint. There exists even a sequential space with this property.

(b) Show that for every Hausdorff space there exists exactly one k-space that has the same underlying set and the same compact subspaces.

3.3.H (Arhangel'skiĭ [1965]). (a) Note that every locally compact space is of pointwise countable type.

(b) Prove that every space of pointwise countable type is a k-space.

Hint. Modify the proof of Theorem 3.9.5.

(c) Give an example of a k-space that is not a space of pointwise countable type.

3.3.I (Arhangel'skiĭ [1968]). Prove that a Hausdorff space X is a hereditarily k-space if and only if it is a Fréchet space.

Hint. To begin, observe that for every subspace A of a hereditarily k-space X and any point $x \in \overline{A} \setminus A$ there exists a compact set $Z \subset A \cup \{x\}$ such that $x \in \overline{Z \setminus \{x\}}$. Then take such a set of minimal cardinality $\mathfrak{m}$, show that $\chi(x, Z) = \mathfrak{m}$ and consider a base $V_1, V_2, \ldots, V_\alpha, \ldots,\ \alpha < \tau$ for the space Z at the point x, where τ is the initial ordinal number of cardinality $\mathfrak{m}$. Applying transfinite induction define a transfinite sequence $U_1, U_2, \ldots, U_\alpha, \ldots,\ \alpha < \tau$ of neighbourhoods of x in Z and a transfinite sequence $x_1, x_2, \ldots, x_\alpha, \ldots,\ \alpha < \tau$ of points of Z such that

$$\{x_\beta : \beta < \alpha\} \cap \overline{U}_\alpha = \emptyset, \quad U_\alpha \subset V_\alpha \quad \text{and} \quad x_\alpha \in \Big(\bigcap_{\beta \leq \alpha} U_\beta\Big) \setminus \{x\} \text{ for } \alpha < \tau;$$

show that $\{x\} \cup \{x_\alpha : \alpha < \tau\}$ contains $A(\mathfrak{m})$.

3.3.J (Boehme [1965]). Prove that the Cartesian product $X \times Y$ of sequential spaces X and Y is sequential provided that the space X is locally compact (cf. Exercise 3.10.J(c)).

Hint. Apply Theorem 3.3.17 and Exercise 2.4.G(b).

Remark. As shown by Simon in [1980], there exist compact Fréchet spaces X and Y such that the Cartesian product $X \times Y$ is not a Fréchet space (cf. Exercises 2.3.K and 2.4.G(c)); such spaces were defined earlier under additional set-theoretic assumptions by Malyhin (see Malyhin und Šapirovskiĭ [1973]) and by Boehme and Rosenfeld [1974] (cf. Olson [1974]).

3.4. Function spaces II: The compact-open topology

In Section 2.6 we defined the topology of pointwise convergence on the set Y^X of all continuous mappings of X to Y as the topology generated by the base consisting of all sets $\bigcap_{i=1}^k M(A_i, U_i)$, where A_i is a finite subset of X and U_i is an open subset of Y for $i = 1, 2, \ldots, k$ and where, for $A \subset X$ and $B \subset Y$,

$$M(A, B) = \{f \in Y^X : f(A) \subset B\}.$$

The *compact-open* topology on Y^X is the topology generated by the base consisting of all sets $\bigcap_{i=1}^k M(C_i, U_i)$, where C_i is a compact subset of X and U_i is an open subset of Y for $i = 1, 2, \ldots, k$.

As in the case of both topologies discussed in Section 2.6, for every topological space Y and a one-point space $\{p\}$, assigning to the point $y \in Y$ the element $i_Y(y)$ of $Y^{\{p\}}$, where $[i_Y(y)](p) = y$, defines a homeomorphism of Y onto the space $Y^{\{p\}}$ with the compact-open topology.

Formulas (11) in Section 2.6 imply that

(1) $\quad \Phi_g: Y^X \to Z^X$ is continuous for every continuous mapping $g: Y \to Z$

and

(2) $\quad \Psi_h: Y^X \to Y^T$ is continuous for every continuous mapping $h: T \to X$ to a T_2-space X,

where $\Phi_g(f) = gf$ for $f \in Y^X$, $\Psi_h(f) = fh$ for $f \in Y^X$ and the function spaces have the compact-open topology.

The reader can easily check that for a homeomorphic embedding $i: Y \to Z$ and for any continuous mapping $h: T \to X$ to a Hausdorff space X such that for every compact $C \subset X$ there exists a compact $C' \subset T$ satisfying $h(C') = C$ (cf. Problem 5.5.11 and Theorem 3.7.2) – in particular for every continuous mapping $h: T \to X$ of a compact space T onto a Hausdorff space X – the mappings $\Phi_i: Y^X \to Z^X$ and $\Psi_h: Y^X \to Y^T$ are homeomorphic embeddings.

Let us note that on R^X the compact-open topology generally differs from the topology of uniform convergence (cf. however Theorems 4.2.17 and 4.2.20); indeed, this follows from Example 2.6.8, since on R^N the compact-open topology coincides with the topology of pointwise convergence. Similarly, the compact-open topology generally differs from the topology of pointwise convergence; as an example the reader can consider the space I^I.

3.4.1. THEOREM. *For every pair X, Y of topological spaces the compact-open topology on Y^X is proper.*

PROOF. Let Z be a topological space and f a mapping in $Y^{(Z \times X)}$; take a set $M(C, U) \subset Y^X$, where C is a compact subset of X and U an open subset of Y. Equality (13) in Section 2.6 yields

$$\begin{aligned}[\Lambda(f)]^{-1}(M(C,U)) &= \{z \in Z : \{[\Lambda(f)](z)\}(x) \in U \text{ for } x \in C\} \\ &= \{z \in Z : f(z,x) \in U \text{ for } x \in C\} = \{z \in Z : f(\{z\} \times C) \subset U\} \\ &= \{z \in Z : \{z\} \times C \subset f^{-1}(U)\}.\end{aligned}$$

From Lemma 3.1.15 it follows that the last set is open, which, along with the fact that the sets $M(C, U)$ form a subbase for Y^X, implies that $\Lambda(f) \in (Y^X)^Z$. ■

The last theorem, part (iii) of Proposition 2.6.12 and the penultimate paragraph of Section 2.6 imply that on R^X the compact-open topology is coarser than the topology of uniform convergence. Since one-point sets are compact and for any finite $A = \{x_1, x_2, \dots, x_k\}$ we have $M(A, U) = \bigcap_{i=1}^{k} M(\{x_i\}, U)$, the compact-open topology is finer than the topology of pointwise convergence.

3.4.2. THEOREM. *For every pair X, Z of topological spaces and every locally compact space Y the operation Σ of composition of mappings is continuous with respect to the compact-open topology on the function spaces Y^X, Z^Y and Z^X, i.e., $\Sigma\colon Z^Y \times Y^X \to Z^X$.*

PROOF. We shall show that, for a compact set $C \subset X$ and an open set $U \subset Z$, the inverse image $\Sigma^{-1}(M(C, U))$ is open. For every pair $(g, f) \in \Sigma^{-1}(M(C, U))$ we have $gf(C) \subset U$, i.e., $f(C) \subset g^{-1}(U)$. Since the space Y is locally compact, it follows from Theorem 3.3.2 that there exists an open set $W \subset Y$ such that $f(C) \subset W \subset \overline{W} \subset g^{-1}(U)$ and $\overline{W}$ is compact. One readily checks that

$$(g, f) \in M(\overline{W}, U) \times M(C, W) \subset \Sigma^{-1}(M(C, U)),$$

which shows that $\Sigma^{-1}(M(C, U))$ is open. ■

3.4.3. THEOREM. *If X is locally compact, then for every topological space Y the compact-open topology on Y^X is acceptable.*

PROOF. Theorem 3.4.2, formula (12) in Section 2.6 and Proposition 2.6.11 imply that the compact-open topology on Y^X is admissible. To conclude the proof it suffices to apply Theorem 3.4.1. ■

It turns out that local compactness of X is crucial; indeed, one can prove (see Exercise 3.4.A) that if for a completely regular space X there exists an acceptable topology on the set R^X, then X is locally compact.

The reader can easily prove the following counterparts of Propositions 2.6.9 and 2.6.10:

3.4.4. PROPOSITION. *For every family $\{X_s\}_{s \in S}$ of non-empty topological spaces and a topological space Y, the combination $\nabla\colon \prod_{s\in S}(Y^{X_s}) \to Y^{(\bigoplus_{s\in S} X_s)}$ is a homeomorphism with respect to the compact-open topology on function spaces.* ■

3.4.5. PROPOSITION. *For every topological space X and a family $\{Y_s\}_{s\in S}$ of topological spaces, the diagonal $\triangle\colon \prod_{s\in S}(Y_s^X) \to (\prod_{s\in S} Y_s)^X$ is a homeomorphism with respect to the compact-open topology on function spaces.* ■

We shall now consider the restriction of the exponential mapping Λ to the set $Y^{(Z\times X)}$; for simplicity this restriction will also be denoted by Λ and will be called the *exponential mapping.*

3.4.6. LEMMA. *For every pair X, Y of topological spaces and every subbase $\mathcal{P}$ for the space Y, the sets $M(C, U)$, where C is a compact subset of X and $U \in \mathcal{P}$, form a subbase for the space Y^X with the compact-open topology.*

PROOF. It suffices to prove that for every compact set $C \subset X$, every open set $U \subset Y$ and any $f \in M(C, U)$ there exist compact subsets $C_1, C_2, \dots, C_k$ of X and members

$U_1^1, U_2^1, \ldots, U_{n_1}^1, U_1^2, U_2^2, \ldots U_{n_2}^2, \ldots, U_1^k, U_2^k, \ldots, U_{n_k}^k$ of $\mathcal{P}$ such that

$$f \in W = \bigcap_{i=1}^{k} \bigcap_{j=1}^{n_i} M(C_i, U_j^i) \subset M(C, U). \tag{3}$$

By the definition of a subbase for every $x \in C$ there exist sets $U_1^x, U_2^x, \ldots, U_{n_x}^x \in \mathcal{P}$ satisfying the conditions

$$x \in \bigcap_{j=1}^{n_x} f^{-1}(U_j^x) \quad \text{and} \quad \bigcap_{j=1}^{n_x} U_j^x \subset U. \tag{4}$$

From Theorem 3.1.3 it follows that there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset C$ such that $C \subset \bigcup_{i=1}^{k} \bigcap_{j=1}^{n_i} f^{-1}(U_j^i)$, where $U_j^i = U_j^{x_i}$ and $n_i = n_{x_i}$. Theorem 1.5.18 implies that there exists a closed cover $\{C_i\}_{i=1}^{k}$ of the space C such that

$$C_i \subset \bigcap_{j=1}^{n_i} f^{-1}(U_j^i) \quad \text{for} \quad i = 1, 2, \ldots, k \tag{5}$$

(this fact can also be proved directly by applying regularity and compactness of C). Clearly the sets C_i are compact and from (5) it follows that the mapping f belongs to the set W defined in (3); it remains to prove that $W \subset M(C, U)$. Take a mapping $g \in W$ and a point $x \in C$; there exists an $i \le k$ such that $x \in C_i$ and we have $g(x) \in \bigcap_{j=1}^{n_i} U_j^i$, which, along with the second part of (4), implies that $g(x) \in U$; i.e., $g \in M(C, U)$. ■

3.4.7. THEOREM. *For every pair X, Z of Hausdorff spaces and every topological space Y, the exponential mapping $\Lambda: Y^{(Z \times X)} \to (Y^X)^Z$ is a homeomorphic embedding with respect to the compact-open topology on function spaces.*

PROOF. For $C \subset X$, $C' \subset Z$ and $U \subset Y$ we clearly have

$$\Lambda^{-1}[M(C', M(C, U))] = M(C' \times C, U); \tag{6}$$

hence the last lemma implies that Λ is continuous. Since Λ is a one-to-one mapping it remains to prove that for every open set $W \subset Y^{(Z \times X)}$, the image $\Lambda(W)$ is open in the subspace $\Lambda(Y^{(Z \times X)})$ of $(Y^X)^Z$. Equality (6) implies that

$$\Lambda(M(C' \times C, U)) = \Lambda(Y^{(Z \times X)}) \cap M(C', M(C, U));$$

hence – Λ being a one-to-one mapping – it suffices to show that the sets

(7) $\qquad M(C' \times C, U)$, where $C \subset X$ and $C' \subset Z$ are compact and $U \subset Y$ is open,

form a subbase for $Y^{(Z \times X)}$.

Take a compact subset C'' of $Z \times X$, an open subset U of Y and a mapping $f \in M(C'', U)$. Since C'' is compact there exist open sets $V_1, V_2, \ldots, V_k \subset X$ and $V_1', V_2', \ldots, V_k' \subset Z$ satisfying

$$C'' \subset \bigcup_{i=1}^{k} (V_i' \times V_i) \subset f^{-1}(U). \tag{8}$$

As in the proof of Lemma 3.4.6 we find a cover $\{C_i\}_{i=1}^k$ of the space C'' consisting of compact sets such that

(9) $$C_i \subset V_i' \times V_i \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Formulas (8) and (9) imply that $f \in \bigcap_{i=1}^k M(p'(C_i) \times p(C_i), U) \subset M(C'', U)$, where $p': Z \times X \to Z$ and $p: Z \times X \to X$ are the projections. Since Z and X are Hausdorff spaces, the sets $p'(C_i)$ and $p(C_i)$ are compact, which proves that the sets in (7) form a subbase for $Y^{(Z \times X)}$. ■

Theorems 3.4.3 and 3.4.7 imply

3.4.8. THEOREM. *For every topological space Y, a Hausdorff space Z and a locally compact space X, the exponential mapping $\Lambda: Y^{(Z \times X)} \to (Y^X)^Z$ is a homeomorphism with respect to the compact-open topology on function spaces.* ■

3.4.9. THEOREM. *If $Z \times X$ is a k-space, then for every topological space Y the exponential mapping $\Lambda: Y^{(Z \times X)} \to (Y^X)^Z$ is a homeomorphism with respect to the compact-open topology on function spaces.*

PROOF. By Theorem 3.4.7 it suffices to show that if $Z \times X$ is a k-space, then for every $g \in (Y^X)^Z$ the mapping $\Lambda^{-1}(g)$ of $Z \times X$ to Y is continuous. From Theorem 3.3.21 it follows that it is enough to prove that the restriction $\Lambda^{-1}(g)|C: C \to Y$ is continuous for every compact subset C of $Z \times X$. Even more is true: the restriction of $\Lambda^{-1}(g)$ to a larger set $Z \times X_0$, whee X_0 is the projection of C onto X, is continuous. Indeed, X_0 being compact, we infer from 3.4.3 that the exponential mapping $\Lambda_0: Y^{(Z \times X_0)} \to (Y^{X_0})^Z$ is onto, and obviously

$$\Lambda^{-1}(g)|(Z \times X_0) = \Lambda_0^{-1}(g_0),$$

where $g_0 \in (Y^{X_0})^Z$ is defind by the formula $[g_0(z)](x) = [g(z)](x)$ for $x \in X_0$. ■

3.4.10. COROLLARY. *If X and Z are first-countable Hausdorff spaces, then for every topological space Y the exponential mapping $\Lambda: Y^{(Z \times X)} \to (Y^X)^Z$ is a homeomorphism with respect to the compact-open topology on function spaces.* ■

Next we shall show how the space of mappings of a k-space X into a topological space Y can be represented as the limit of an inverse system of the function spaces Y^C, where C is a compact subset of X.

Let $Z(X)$ denote the family of all non-empty compact subsets of a Hausdorff space X. The family $Z(X)$ is ordered by the relation $\leq$ defined by letting $C_2 \leq C_1$ whenever $C_2 \subset C_1$. Moreover, the family $Z(X)$ is directed by $\leq$, because in a Hausdorff space the union of two compact subsets is a compact subset. For any $C_1, C_2 \in Z(X)$ satisfying $C_2 \leq C_1$, and for an arbitrary topological space Y, a continuous mapping $\pi_{C_2}^{C_1}: Y^{C_1} \to Y^{C_2}$ is defined, viz., $\pi_{C_2}^{C_1} = \Psi_i$, where $i: C_2 \to C_1$ is the embedding; clearly $\pi_{C_2}^{C_1}(f) = f|C_2$ for every $f \in Y^{C_1}$.

3.4.11. THEOREM. *If X is a k-space, then for every topological space Y the space Y^X with the compact-open topology (with the topology of pointwise convergence) is homeomorphic to the limit of the inverse system $\mathrm{S}(X) = \{Y^C, \pi_{C_2}^{C_1}, Z(X)\}$ of the spaces Y^C with the compact-open topology (with the topology of pointwise convergence).*

PROOF. For every $f = \{f_C\} \in \varprojlim \mathbf{S}(X)$ the mappings $\{f_C\}_{C \in Z(X)}$ are compatible and from Theorem 3.3.21 it follows that $F(f) = \nabla_{C \in Z(X)} f_C: X \to Y$ is continuous, i.e., that $F(f) \in Y^X$. Clearly F is a one-to-one mapping of $\varprojlim \mathbf{S}(X)$ onto Y^X; since for every $f = \{f_C\}$, every $C \in Z(X)$ and any $A \subset C$ we have

$$F(f) \in M(A,U) \quad \text{if and only if} \quad f_C \in M(A,U),$$

Proposition 2.5.5 implies that F is a homeomorphism. ∎

We shall now examine how the topological properties of the space Y^X with the compact-open topology depend on properties of X and Y. We start with an investigation of the axioms of separation. As the compact-open topology is finer than the topology of pointwise convergence, from Theorem 2.6.4 it follows immediately that if Y is a T_i-space, then Y^X with the compact-open topology also is a T_i-space for $i \leq 2$. We shall show that the same holds for $i = 3$ and $i = 3\frac{1}{2}$.

3.4.12. LEMMA. *For every pair X, Y of topological spaces, any subset A of X and any closed subset B of Y, the set $M(A,B)$ is closed in the space Y^X with the topology of pointwise convergence and, a fortiori, in the space Y^X with the compact-open topology.*

PROOF. Clearly

$$M(A,B) = \bigcap_{x \in A} M(\{x\}, B)$$

and since the set $M(\{x\},B) = Y^X \setminus M(\{x\}, Y \setminus B)$ is closed for every $x \in A$, the set $M(A,B)$ is closed. ∎

3.4.13. THEOREM. *If Y is a regular space, then the space Y^X with the compact-open topology also is a regular space.*

PROOF. By virtue of Proposition 1.5.5 it suffices to show that for every $f \in Y^X$ and every neighbourhood $M(C,U)$ of f, where C is a compact subset of X and U is an open subset of Y, there exists an open set $V \subset Y$ such that

$$f \in M(C,V) \subset \overline{M(C,V)} \subset M(C,U). \tag{10}$$

From Theorems 3.1.10 and 3.1.6 it follows that there exists an open set $V \subset Y$ satisfying

$$f(C) \subset V \subset \overline{V} \subset U. \tag{11}$$

By the last lemma the set $M(C,\overline{V})$ is closed, so that $\overline{M(C,V)} \subset M(C,\overline{V})$ and (10) follows from (11). ∎

3.4.14. LEMMA. *Let X be a topological space and C a compact subspace of X. Assigning to any $f \in I^X$ the number $\Xi(f) = \sup_{x \in C} f(x)$ defines a function Ξ from I^X to I which is continuous with respect to the compact-open topology on I^X.*

PROOF. We shall show that for every open interval $(a,b) \subset R$ the inverse image $\Xi^{-1}(I \cap (a,b))$ is open in I^X. Corollary 3.2.9 implies that $\sup_{x \in C} f(x) < b$ if and only if $f(x) < b$ for every $x \in C$. Hence for $A = \{x \in I : x \leq a\}$ and $B = \{x \in I : x < b\}$ we have

$$\Xi^{-1}(I \cap (a,b)) = (I^X \setminus M(C,A)) \cap M(C,B),$$

which, along with Lemma 3.4.12, implies that $\Xi^{-1}(I \cap (a,b))$ is open in I^X.■

3.4.15. THEOREM. *If Y is a Tychonoff space, then the space Y^X with the compact-open topology also is a Tychonoff space.*

PROOF. By virtue of Proposition 1.5.8 it suffices to show that for every $f \in Y^X$ and every neighbourhood $M(C,U)$ of f, where C is a compact subset of X and U is an open subset of Y, there exists a continuous function $G: Y^X \to I$ such that $G(f) = 0$ and $G(h) = 1$ for $h \in Y^X \setminus M(C,U)$.

From Theorems 3.1.10 and 3.1.7 it follows that there exists a continuous function $g: Y \to I$ such that $g(f(C)) \subset \{0\}$ and $g(y) = 1$ for $y \in Y \setminus U$. Let $G(h) = \sup_{x \in C} gh(x)$ for every $h \in Y^X$; since $G = \Xi\Phi_g$, the last lemma implies that G is a continuous function from Y^X to I. We have clearly $G(f) = 0$; if $h \notin M(C,U)$, there exists a point $x \in C$ such that $h(x) \in Y \setminus U$ and we have $gh(x) = 1$ which implies that $G(h) = 1$.■

It is readily seen that if X is a discrete space, then the space Y^X with the compact-open topology is homeomorphic to the Cartesian product $Y^{|X|}$. Since there exist perfectly normal spaces with non-normal square (cf. Example 2.3.12), the space Y^X with the compact-open topology need not be normal even if Y is a perfectly normal space and X is a two-point discrete space.

3.4.16. THEOREM. *If the weight of both X and Y is not larger than $\mathfrak{m} \geq \aleph_0$ and X is a locally compact space, then the weight of the space Y^X with the compact-open topology is not larger than $\mathfrak{m}$.*

PROOF. Let $\mathcal{B}$ be a base for X such that $|\mathcal{B}| \leq \mathfrak{m}$, finite unions of members of $\mathcal{B}$ belong to $\mathcal{B}$ and $\overline{V}$ is compact for every $V \in \mathcal{B}$; let $\mathcal{D}$ be a base for Y such that $|\mathcal{D}| \leq \mathfrak{m}$ and finite unions of members of $\mathcal{D}$ belong to $\mathcal{D}$. The family $\mathcal{E}$ consisting of all sets $M(\overline{V}, W)$, where $V \in \mathcal{B}$ and $W \in \mathcal{D}$ has cardinality $\leq \mathfrak{m}$, so that to complete the proof it suffices to show that $\mathcal{E}$ is a subbase for the space Y^X. Indeed, for every $f \in Y^X$, a compact subset C of X and an open subset U of Y such that $f \in M(C,U)$, there exist a $V \in \mathcal{B}$ satisfying $C \subset V \subset \overline{V} \subset f^{-1}(U)$ and a $W \in \mathcal{D}$ satisfying $f(\overline{V}) \subset W \subset U$. Therefore, we have $f \in M(\overline{V}, W) \subset M(C,U)$ and $M(\overline{V}, W) \in \mathcal{E}$.■

One proves (see Exercise 3.4.E) that if the space R^X with the compact-open topology is first-countable and X is a first-countable Tychonoff space, then X is locally compact; similarly, one proves (see Exercise 5.1.H) that if the space R^X with the compact-open topology is first-countable and X is a metrizable space, then X is locally compact and second-countable.

We shall close this section with two characterizations of compact subspaces of function spaces with the compact-open topology. In these characterizations the concept of an evenly continuous family of mappings is applied; this is – as the reader will see in Chapter 8 – a topological counterpart of the concept of an equicontinuous family of mappings. We say that a family F of mappings of X to Y is *evenly continuous* if for every $x \in X$, every $y \in Y$ and any neighbourhood V of y there exist a neighbourhood U of x and a neighbourhood W of y such that $\Omega[(F \cap M(\{x\}, W)) \times U] \subset V$, i.e., such that the conditions $f \in F$ and $f(x) \in W$ imply the inclusion $f(U) \subset V$. It follows directly from the definition that if a family F of mappings of X to Y is evenly continuous, then all members of F are continuous, i.e., $F \subset Y^X$.

3.4.17. LEMMA. *If Y is a regular space, then for every evenly continuous family of mappings $F \subset Y^X$ the closure $\overline{F}$ of the set F in the Cartesian product $\prod_{x\in X} Y_x$, where $Y_x = Y$ for every $x \in X$, is an evenly continuous family of mappings and, in particular, $\overline{F} \subset Y^X$.*

PROOF. Take a point $x_0 \in X$ and a point $y_0 \in Y$; to every neighbourhood V of y_0 assign a neighbourhood V' of this point such that $\overline{V'} \subset V$. Since the family F is evenly continuous, for every V there exist a neighbourhood U_V of x_0 and a neighbourhood W_V of y_0 such that $\Omega[(F \cap M(\{x_0\}, W_V)) \times U_V] \subset V'$; this yields

$$\begin{aligned} F \subset F(x_0, y_0, V) &= \{f \in \prod_{x\in X} Y_x : f(x_0) \in W_V \text{ implies that } f(U_V) \subset \overline{V'}\} \\ &= p_{x_0}^{-1}(Y \setminus W_V) \cup \bigcap_{x\in U_V} p_x^{-1}(\overline{V'}). \end{aligned}$$

The last set being closed, it follows that $\overline{F} \subset F(x_0, y_0, V)$. Hence the set $\overline{F}$ is contained in the intersection of all sets $F(x, y, V)$, where $x \in X$, $y \in Y$ and V is a neighbourhood of y, and this implies that $\overline{F}$ is an evenly continuous family of mappings. ∎

3.4.18. LEMMA. *If $F \subset Y^X$ is an evenly continuous family of mappings, then the restriction $\Omega|F \times X$ of the evaluation mapping is continuous with respect to the topology of pointwise convergence on F.*

PROOF. For $f \in F$, $x \in X$, $y = f(x)$ and a neighbourhood V of y there exist a neighbourhood U of x and a neighbourhood W of y such that $\Omega[(F \cap M(\{x\}, W)) \times U] \subset V$; it suffices to note that $(F \cap M(\{x\}, W)) \times U$ is a neighbourhood of (f, x) in the space $F \times X$. ∎

3.4.19. LEMMA. *Let Y be a regular space, X an arbitrary topological space and Y^X the space of all continuous mappings of X to Y with the topology of pointwise convergence. If a set $F \subset Y^X$ is compact and the restriction $\Omega|F \times X$ of the evaluation mapping is continuous, then F is an evenly continuous family of mappings.*

PROOF. Let $x \in X$, $y \in Y$ and let V be a neighbourhood of y. The space Y being regular, there exists a neighbourhood W of y such that $\overline{W} \subset V$ and from Lemma 3.4.12 it follows that the set $F_0 = F \cap M(\{x\}, \overline{W})$ is compact. Since $\Omega(F_0 \times \{x\}) \subset V$, we have $F_0 \times \{x\} \subset (\Omega|F \times X)^{-1}(V)$ and by Lemma 3.1.15 there exists a neighbourhood U of the point x such that $\Omega(F_0 \times U) \subset V$. As we clearly have $\Omega[(F \cap M(\{x\}, W)) \times U] \subset \Omega(F_0 \times U)$, the family of mappings F is evenly continuous. ∎

3.4.20. THE ASCOLI THEOREM. *If X is a k-space and Y is a regular space, then a closed subset F of the space Y^X with the compact-open topology is compact if and only if F is an evenly continuous family of mappings and the set $\Omega(F \times \{x\}) = \{f(x) : f \in F\} \subset Y$ has a compact closure for every $x \in X$.*

PROOF. Suppose that $F \subset Y^X$ is an evenly continuous family of mappings and that the set $F_x = \overline{\Omega(F \times \{x\})} \subset Y$ is compact for every $x \in X$. The Cartesian product $\prod_{x\in X} F_x$ is compact, so that the closure $\overline{F}$ of the set F in the Cartesian product $\prod_{x\in X} Y_x$, where $Y_x = Y$ for every $x \in X$, is compact. Lemma 3.4.17 implies that $\overline{F} \subset Y^X$ is an evenly continuous family of mappings, and from Lemma 3.4.18 it follows that the restriction $\Omega_0 = \Omega|\overline{F} \times X$ is continuous, i.e., that $\Omega_0 \in Y^{\overline{F}\times X}$. By virtue of Theorem 3.4.1 we have $\Lambda(\Omega_0) \in (Y^X)^{\overline{F}}$, where Y^X has the compact-open topology. Since $[\Lambda(\Omega_0)](f) = f$ for every $f \in \overline{F}$, it follows from Theorem 3.1.13 that on $\overline{F}$ the compact-open topology and the topology of pointwise convergence coincide. Hence we have $F = \overline{F}$ and the set F is compact.

Conversely, suppose that F is a compact subset of Y^X. By virtue of Lemma 3.4.19 it suffices to show that the restriction $\Omega|F \times X$ of the evaluation mapping is continuous, because compactness of the sets $\Omega(F \times \{x\})$ follows immediately from compactness of F and continuity of $\Omega|F \times X$. Since – by Theorem 3.3.27 – $F \times X$ is a k-space, it is enough to check that for every compact subspace $C \subset X$ the restriction $\Omega_0 = \Omega|F \times C$ is continuous. From Theorem 3.4.3 it follows that the evaluation $\Omega_C: Y^C \times C \to Y$ is continuous and this implies that Ω_0 is continuous, because $\Omega_0 = \Omega_C[(\Psi_i|F) \times \mathrm{id}_C]$, where $i: C \to X$ is the embedding of C in X.■

The following theorem is a variant of the Ascoli theorem; the symbol $F|Z$ that appears in it denotes, for $F \subset Y^X$ and $Z \subset X$, the family of restrictions $\{f|Z : f \in F\} \subset Y^Z$.

3.4.21. THEOREM. *If X is a k-space and Y is a regular space, then a closed subset F of the space Y^X with the compact-open topology is compact if and only if $F|Z$ is an evenly continuous family of mappings for each compact $Z \subset X$ and the set $\Omega(F \times \{x\}) = \{f(x) : f \in F\} \subset Y$ has a compact closure for every $x \in X$.*

PROOF. By virtue of the Ascoli theorem it suffices to show that compactness of F follows from conditions in the theorem. This however, is a consequence of 2.5.7, 3.4.17, 3.4.20, 3.4.11 and 3.2.13.■

Historical and bibliographic notes

The compact-open topology was defined by Fox in [1945]. Fox proved Theorems 3.4.1 and 3.4.3 and showed that the exponential mapping Λ in Corollary 3.4.10 is one-to-one and onto; the fact that Λ is a homeomorphism was proved by Jackson in [1952a]. Jackson's paper also contains Lemma 3.4.6 and Theorems 3.4.7 and 3.4.8. Theorem 3.4.9 was proved by Morita in [1956a]. Theorems 3.4.13, 3.4.15 and 3.4.16 were proved by Arens in [1946]. The concept of an evenly continuous family of mappings was introduced by Kelley and A. P. Morse (see Kelley [1955]). They also proved Theorem 3.4.21 and Theorem 3.4.20, the latter under the more restrictive assumption that X is locally compact; the generalization to k-spaces was obtained by Bagley and Yang in [1966]. Different variants of the Ascoli theorem (generalizing a classical result of Ascoli obtained in 1883) are often applied in analysis, mainly in proofs of existence theorems (e.g. in the proof that there exists a solution of the differential equation $y' = u(x, y)$ under the sole assumption of continuity of the function $u(x, y)$).

Exercises

3.4.A (Fox [1945], Arens [1946]). Prove that if X is a completely regular space and there exists an acceptable topology on R^X, then X is locally compact (cf. Exercise 2.6.E).

Hint. Define $f: X \to R$ by letting $f(x) = 0$ for every $x \in X$; for a point $x_0 \in X$ take a neighbourhood V of x_0 and a neighbourhood W of f satisfying the inclusion $\Omega(W \times V) \subset (-1, 1)$ and show that $\overline{V}$ is compact. To that end, for any family $\{V_s\}_{s \in S}$ of open subsets of

X which cover $\overline{V}$, consider the topology on R^X generated by the base consisting of all sets $\bigcap_{i=1}^k M(A_i, U_i)$, where A_i is closed, and contained either in some V_s or in $X \setminus \overline{V}$, and U_i is open in R for $i = 1, 2, \ldots, k$.

3.4.B. Verify that for every family $\{X_s\}_{s\in S}$ of Hausdorff spaces and a family $\{Y_s\}_{s\in S}$ of arbitrary topological spaces, the Cartesian product $\prod: \prod_{s\in S} Y_s^{X_s} \to (\prod_{s\in S} Y_s)^{(\prod_{s\in S} X_s)}$ is a homeomorphic embedding with respect to the compact-open topology on function spaces.

3.4.C. (a) Show that if X is a compact space and Y a Hausdorff space then the set of all continuous mappings of X onto Y is closed in the space Y^X with the compact-open topology. Verify that the assumption of compactness of X cannot be omitted.

(b) For a compact space X consider the space X^X with the compact-open topology and its subspace H consisting of all homeomorphisms of X onto itself. Show that assigning to $h \in H$ its inverse $h^{-1} \in H$ defines a homeomorphism of H onto itself (cf. Theorem 3.4.2 and Example 8.1.17). Verify that the assumption of compactness of X is essential.

3.4.D (A. H. Stone [1963]). Verify that the space of all continuous mappings of the interval I into the Tychonofff cube $I^{\mathfrak{c}}$ with the compact-open topology is not normal (cf. Exercise 3.8.D).

Hint. Observe that the space I^I contains $D(\aleph_0)$ as a closed subspace, then apply Proposition 3.4.5 and Exercise 2.3.E(a) or 3.1.H(a).

3.4.E (Arens [1946]). A Hausdorff space X is *hemicompact* if in the family of all compact subspaces of X ordered by $\subset$ there exists a countable cofinal subfamily.

(a) Prove that every first-countable hemicompact space is locally compact.

(b) Give an example of a countable hemicompact space which is not a k-space.

(c) Show that in the realm of second-countable spaces hemicompactness is equivalent to local compactness.

(d) Prove that if the space R^X with the compact-open topology is first-countable and X is a Tychonoff space, then X is hemicompact.

3.4.F. Note that if X is a Tychonofff space and Y contains a subspace homeomorphic to R, then the space Y^X with the compact-open topology is compact if and only if Y is compact and X is discrete.

3.4.G (Kelley [1955]). Deduce from the Stone-Weierstrass theorem that if a ring P of continuous real-valued functions defined on a topological space X contains all constant functions, separates points and is a closed subset of the space R^X with the compact-open topology, then P coincides with the ring of all continuous real-valued functions defined on X.

Observe that the above proposition generalizes the Stone-Weierstrass theorem.

3.4.H. (a) (Michael [1961]) Show that $nw(Y^X) \le w(X)w(Y)$ with respect both to the compact-open topology and to the topology of pointwise convergence on Y^X. Deduce that if X and Y are second-countable, then Y^X is hereditarily separable with respect to both compact-open topology and the topology of pointwise convergence.

Hint. Let $\mathcal{B}$ and $\mathcal{D}$ be bases of cardinality $\le w(X)w(Y)$ for X and Y respectively. Consider the topology on Y^X generated by the base consisting of all sets $\bigcap_{i=1}^k M(U_i, V_i)$, where $U_i \in \mathcal{B}$ and $V_i \in \mathcal{D}$ for $i = 1, 2, \ldots, k$; apply Theorem 3.4.1 and Proposition 2.6.11.

(b) (Warner [1958]) Prove that if X is a Tychonoff space, then the space R^X with the compact-open topology contains a dense subset of cardinality $\leq \mathfrak{m} \geq \aleph_0$ if and only if there exists a one-to-one continuous mapping of X onto a Tychonoff space of weight $\leq \mathfrak{m}$.

Hint. Deduce from Exercise 3.2.J(a) that if $h: X \to Z$ is a one-to-one continuous mapping of X onto Z, then Ψ_h maps R^Z onto a dense subset of R^X, and apply (a).

(c) Show that for a Tychonoff space X and a cardinal number $\mathfrak{m} \geq \aleph_0$ the following conditions are equivalent:

(1) *The space R^X with the compact-open topology contains a dense subset of cardinality $\leq \mathfrak{m}$.*

(2) *The space R^X with the topology of pointwise convergence contains a dense subset of cardinality $\leq \mathfrak{m}$.*

Hint. Cf. Exercise 3.2.I.

3.4.I (Kaul [1969]). We say that a family F of mappings of X to Y is *uniformly regular* if for every open cover $\mathcal{V}$ of the space Y there exists an open cover $\mathcal{U}$ of the space X which is a refinement of every cover of the form $\{f^{-1}(V) : V \in \mathcal{V}\}$, where $f \in F$.

Prove that if X is a k-space and Y is a regular space, then a closed subset F of the space Y^X with the compact-open topology is compact if and only if F is a uniformly regular family of mappings and the set $\Omega(F \times \{x\}) = \{f(x) : f \in F\} \subset Y$ has a compact closure for every $x \in X$.

Hint. Show that if the set $\{f(x) : f \in F\}$ has a compact closure for every $x \in X$, then the family F is uniformly regular if and only if it is evenly continuous.

3.4.J. Show that addition, subtraction and multiplication of functions are continuous mappings of the Cartesian product $R^X \times R^X$ to R^X and that multiplication by real numbers is a continuous mapping of $R^X \times R$ to R^X with respect to the compact-open topology on R^X.

3.5. Compactifications

A pair (Y, c), where Y is a compact space and $c: X \to Y$ is a homeomorphic embedding of X in Y such that $\overline{c(X)} = Y$, is called a *compactification of the space* X. If a space X is embeddable in a compact space Y, i.e., if there exists a homeomorphism $f: X \to M$ onto a subspace $M = f(X)$ of Y, then clearly the pair $(\overline{f(X)}, if)$, where i denotes the embedding of M in $\overline{M}$, is a compactification of the space X. Hence every space which is embeddable in a compact space has a compactification; this fact, along with Theorems 3.2.6, 3.2.5 and 2.3.23, yields the following two theorems:

3.5.1. THEOREM. *A topological space X has a compactification if and only if X is a Tychonoff space.* ■

3.5.2. THEOREM. *Every Tychonoff space X has a compactification (Y, c) such that $w(Y) = w(X)$.* ■

In the sequel, by a compactification of X we shall mean not only a pair (Y, c) but also the compact space Y in which X can be embedded as a dense subspace. Compactifications of a space X will be usually denoted by symbols cX, c_iX, αX, etc., where c, c_i and α are the symbols for homeomorphic embedding of X in the corresponding compactification. Hence, when considering a compactification as a space, we shall always know which homeomorphic embedding is being used: for a compactification cX of a space X we have

$$c: X \to cX,\ c|X: X \to c(X) \text{ is a homeomorphism, and } \overline{c(X)} = cX.$$

We shall say that compactifications c_1X and c_2X of a space X are *equivalent* if there exists a homeomorphism $f: c_1X \to c_2X$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "c_1"] & X \arrow[d, "c_2"] \\ c_1X \arrow[r, "f"] & c_2X \end{tikzcd}$$

is commutative, i.e., $fc_1(x) = c_2(x)$ for every $x \in X$. Thus two compactifications of X are equivalent if they are homeomorphic and if the space X is embedded in both of them in the same way. One can readily verify that the equivalence of compactifications is an equivalence relation.

In the sequel we shall often identify equivalent compactifications; any class of equivalent compactifications will be considered as a single compactification and will be denoted by the symbol cX, where cX is an arbitrary compactification in this class.

If cX is a compactification of a compact space X, then clearly $c(X) = cX$ and c is a homeomorphism. Hence, for any two compactifications c_1X and c_2X of a compact space X, the mapping $f = c_2c_1^{-1}: c_1X \to c_2X$ is a homeomorphism and $fc_1(x) = c_2(x)$ for every $x \in X$, which means that any two compactifications of a compact space are equivalent. In particular, any compactification of a compact space X is equivalent to the compactification (X, id_X), which we identify with the space X itself.

Theorems 1.5.3 and 1.5.7 imply

3.5.3. THEOREM. *For every compactification Y of a space X we have $|Y| \le \exp\exp d(X)$ and $w(Y) \le \exp d(X)$.* ■

From the last theorem and Theorem 3.2.5 it follows that all compactifications of a space X (up to equivalence) are subspaces of the Tychonoff cube $I^{\exp d(X)}$. Thus, for any space X, one can consider the family $C(X)$ of all compactifications of X. Strictly speaking $C(X)$ is the family of all classes consisting of equivalent compactifications of X which are subspaces of the Tychonoff cube $I^{\exp d(X)}$.

We shall now define an order relation in the family $C(X)$. Let $c_2X \le c_1X$ if there exists a continuous mapping $f: c_1X \to c_2X$ such that $fc_1 = c_2$; thus the inequality $c_2X \le c_1X$ means that c_1X can be mapped onto c_2X in such a way that every point of the space X, considered as a subspace of both c_1X and c_2X, is mapped onto itself. One can readily see that if $c_1X \le c_2X$ and $c_2X \le c_3X$ then $c_1X \le c_3X$. Hence, to show that $\le$ is an order in the family $C(X)$ it suffices to prove the following theorem.

3.5.4. THEOREM. *Compactifications c_1X and c_2X of a space X are equivalent if and only if $c_1X \le c_2X$ and $c_2X \le c_1X$.*

PROOF. Clearly if c_1X and c_2X are equivalent, then $c_1X \le c_2X$ and $c_2X \le c_1X$.

Assume now that $c_1X \leq c_2X$ and $c_2X \leq c_1X$. Let $f_1: c_1X \to c_2X$ and $f_2: c_2X \to c_1X$ satisfy $f_1c_1 = c_2$ and $f_2c_2 = c_1$. Since $f_2f_1c_1 = f_2c_2 = c_1$, the restrictions of f_2f_1 and id_{c_1X} to the dense subspace $c_1(X)$ of c_1X coincide, and thus $f_2f_1 = \mathrm{id}_{c_1X}$ by Theorem 1.5.4. Similarly, $f_1f_2 = \mathrm{id}_{c_2X}$. Hence f_1 is a homeomorphism and the compactifications c_1X and c_2X are equivalent. ■

Another necessary and sufficient condition for the equivalence of two compactifications is contained in the next theorem.

3.5.5. THEOREM. *Compactifications c_1X and c_2X of a space X are equivalent if and only if for every pair A, B of closed subsets of X we have*

$$\overline{c_1(A)} \cap \overline{c_1(B)} = \emptyset \quad \textit{if and only if} \quad \overline{c_2(A)} \cap \overline{c_2(B)} = \emptyset. \tag{1}$$

PROOF. Clearly if c_1X and c_2X are equivalent then (1) holds for every pair A, B of closed subsets of X.

Assume now that for compactifications c_1X and c_2X we have (1) for every pair A, B of closed subsets of X. By virtue of Theorem 3.2.1 the mappings

$$c_1h_2: c_2(X) \to c_1X \quad \text{and} \quad c_2h_1: c_1(X) \to c_2X,$$

where $h_i: c_i(X) \to X$ is the inverse of the homeomorphism $c_i|X$ for $i = 1, 2$, are extendable to continuous mappings

$$C_2: c_2X \to c_1X \quad \text{and} \quad C_1: c_1X \to c_2X.$$

As we clearly have $C_2c_2 = c_1$ and $C_1c_1 = c_2$, it follows from Theorem 3.5.4 that c_1X and c_2X are equivalent. ■

Let cX be a compactification of a space X; the set $cX \setminus c(X)$, i.e., the set of points in which cX differs from $c(X)$, is called the *remainder* of the compactification cX. The crucial property of remainders is stated in Theorem 3.5.7 below; in the proof of this theorem we shall use the following lemma.

3.5.6. LEMMA. *Let A be a dense subspace of a Hausdorff space X and let $f: X \to Y$ be a continuous mapping of X to an arbitrary space Y. If $f|A: A \to f(A) \subset Y$ is a homeomorphism, then $f(X \setminus A) \cap f(A) = \emptyset$.*

PROOF. Assume that there exists a point $x \in X \setminus A$ such that $f(x) \in f(A)$. Without loss of generality we can suppose that $X = A \cup \{x\}$ and $Y = f(A)$. Let $f(x) = f(y)$, where $y \in A$, and let $U, V \subset X$ be disjoint neighbourhoods of x and y respectively. The set $f(A \setminus V) = (f|A)(A \setminus V)$ is closed in $Y = f(A)$, so that the set $f^{-1}f(A \setminus V) = A \setminus V$ is closed in X. Since $x \notin \overline{V}$, this implies that $x \notin \overline{A}$, a contradiction. ■

3.5.7. THEOREM. *If c_1X and c_2X are compactifications of a space X and a continuous mapping $f: c_1X \to c_2X$ satisfies the condition $fc_1 = c_2$, then*

$$f(c_1(X)) = c_2(X) \quad \textit{and} \quad f(c_1X \setminus c_1(X)) = c_2X \setminus c_2(X).$$

PROOF. The first of the above equalities follows from the condition $fc_1 = c_2$; the second equality follows from the first one, the last lemma and from the fact that $f(c_1X) = c_2X$.■

It turns out that some classes of Tychonoff spaces can be characterized by properties of remainders of compactifications. We shall now give such a characterization for locally compact spaces; another important class of spaces that has a characterization of this kind, the class of Čech complete spaces, will be discussed in Section 3.9 (see also Exercise 3.5.G, Theorem 3.11.10, Problems 3.12.25, 3.12.26 and 8.5.13(b)).

3.5.8. THEOREM. *For every Tychonoff space X the following conditions are equivalent:*
(i) *The space X is locally compact.*
(ii) *For every compactification cX of the space X the remainder $cX \setminus c(X)$ is closed in cX.*
(iii) *There exists a compactification cX of the space X such that the remainder $cX \setminus c(X)$ is closed in cX.*

PROOF. Implications (ii)⇒(iii) and (iii)⇒(i) are obvious; the implication (i)⇒(ii) follows from Theorem 3.3.9.■

The next theorem states an important property of the family $C(X)$ of all compactifications of X.

3.5.9. THEOREM. *Every non-empty subfamily $C_0 \subset C(X)$ has a least upper bound with respect to the order $\leq$ in $C(X)$.*

PROOF. Let $C_0 = \{c_sX\}_{s\in S}$ and $c_S = \triangle_{s\in S}\, c_s\colon X \to \prod_{s\in S} c_sX$. By the diagonal theorem c_S is a homeomorphic embedding. We shall show that $c_SX = \overline{c_S(X)} \subset \prod_{s\in S} c_sX$ is the least upper bound of the family C_0.

Since the projection $p_s\colon \prod_{s\in S} c_sX \to c_sX$ satisfies $p_sc_S = c_s$, we have $c_sX \leq c_SX$ for every $s \in S$. Assume that a compactification cX of X satisfies $c_sX \leq cX$ for every $s \in S$, i.e., that there exist mappings $f_s\colon cX \to c_sX$ such that $f_sc = c_s$ for every $s \in S$. One readily sees that the diagonal $F = \triangle_{s\in S} f_s$ satisfies $Fc = c_S$, so that we have $c_SX \leq cX$.■

3.5.10. COROLLARY. *For every Tychonoff space X there exists in $C(X)$ a largest element with respect to the order $\leq$.*■

The largest element in $C(X)$ is called the *Čech-Stone compactification* of X or the *maximal compactification* of X and is denoted by βX; this compactification will be studied in detail in the following section.

In connection with the last theorem the question arises for which spaces X every subfamily of $C(X)$ has a greatest lower bound. If for a subfamily C_0 of $C(X)$ the family $C_0' = \{c'X : c'X \leq cX \text{ for every } cX \in C_0\}$ is non-empty, then its least upper bound is the greatest lower bound of C_0; hence greatest lower bounds exist in $C(X)$ if and only if there exists a smallest element in $C(X)$. It turns out that this is equivalent to the local compactness of the space X.

3.5.11. THE ALEXANDROFF COMPACTIFICATION THEOREM. *Every non-compact locally compact space X has a compactification ωX with one-point remainder. This compactification is the smallest element in $C(X)$ with respect to the order $\leq$, its weight is equal to the weight of the space X.*

PROOF. Take a point $\Omega \notin X$ and let $\omega X = X \cup \{\Omega\}$. Define open sets in ωX as sets of the form $\{\Omega\} \cup (X \setminus F)$, where F is a compact subspace of X, together with all sets that are open in X. One readily sees that ωX with this topology is a Hausdorff space and that the mapping $\omega: X \to \omega X$, defined by letting $\omega(x) = x$, is a homeomorphic embedding whose image $\omega(X) = X$ is dense in ωX. We shall show that ωX is a compact space. Let $\{U_s\}_{s \in S}$ be an open cover of ωX. There exists an $s_0 \in S$ such that $\Omega \in U_{s_0}$ and from the definition of the topology on ωX it follows that the set $F = X \setminus U_{s_0}$ is compact. By Theorem 3.1.3 there exists a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ such that $F \subset \bigcup_{i=1}^k U_{s_i}$, so that the cover $\{U_s\}_{s \in S}$ contains the finite subcover $\{U_{s_i}\}_{i=0}^k$.

In order to prove that $cX \geq \omega X$ for every compactification cX of the space X it suffices to show that the mapping f of cX to ωX defined by letting

$$f(x) = \begin{cases} \omega(c^{-1}(x)) & \text{for } x \in c(X), \\ \Omega & \text{for } x \in cX \setminus c(X), \end{cases}$$

satisfies the equality $fc = \omega$ and is continuous. The equality $fc = \omega$ follows directly from the definition, and Theorem 3.5.8 implies that the inverse image of every open subset of ωX is open in cX, either as an open subset of the open subspace $c(X)$ or as the complement of a compact subset of cX.

The last part of the theorem follows from Corollary 3.1.20. ■

The compactification ωX of a locally compact non-compact space X is called the *Alexandroff compactification of* X, the *one-point compactification of* X, or the *minimal compactification of* X; one can say that it is obtained by adjoining to the space X a "point in infinity".

3.5.12. THEOREM. *If in the family $C(X)$ of all compactifications of a non-compact Tychonoff space X there exists an element cX which is smallest with respect to the order $\leq$, then X is locally compact and cX is equivalent to the Alexandroff compactification ωX of X.*

PROOF. By virtue of Theorems 3.5.11 and 3.5.4, it suffices to prove that X is locally compact. To that end we shall show that the remainder of cX is a one-point set.

Assume that the remainder $cX \setminus c(X)$ contains two distinct points x_1 and x_2. The space $X_1 = cX \setminus \{x_1, x_2\}$ is locally compact and the Alexandroff compactification of X_1 is a compactification of the space X, i.e., we have $\omega X_1 = c_1 X$. Since $cX \leq c_1 X$ there exists a continuous mapping $f: c_1 X \to cX$ such that $f|c(X) = \mathrm{id}_{c(X)}$. Theorem 1.5.4 implies that $f|X_1 = \mathrm{id}_{X_1}$ and from Theorem 3.5.7, applied to the compactifications $c_1 X$ and cX of the space X_1, it follows that $f(\{\Omega\}) = \{x_1, x_2\}$, which is impossible. ■

3.5.13. THEOREM. *If a compact space Y is a continuous image of the remainder $cX \setminus c(X)$ of a compactification cX of a locally compact space X, then the space X has a compactification $c'X \leq cX$ with the remainder homeomorphic to the space Y.*

PROOF. Let f be a continuous mapping of $cX \setminus c(X)$ onto Y. From Theorems 3.5.8 and 2.4.13 it follows that the decomposition of cX consisting of all fibers of f and all one-point subsets of $c(X)$ is upper semicontinuous. The equivalence relation E corresponding to this decomposition of cX is closed and the quotient space cX/E is compact. Since $c(X)$ is an open subspace of cX, it follows from Proposition 2.4.15 that the space $cX/E = c'X$, where $c' = qc$ and $q: cX \to cX/E$ is the natural quotient mapping, is a compactification of X with the remainder homeomorphic to Y. ■

3.5.14. EXAMPLES. The circle S^1 and the interval I are compactifications of the real line R; the circle is the Alexandroff compactification of R. More generally, the n-sphere S^n and the n-cube I^n are compactifications of the Euclidean n-space R^n; the n-sphere is the Alexandroff compactification of R^n.

The space W described in Example 3.1.27 is the Alexandroff compactification of its subspace W_0.

The space $A(\mathfrak{m})$ defined in 1.4.20 is the Alexandroff compactification of the discrete space $D(\mathfrak{m})$. Every finite sum $A(\mathfrak{m}) \oplus A(\mathfrak{m}) \oplus \ldots \oplus A(\mathfrak{m})$ is also a compactification of $D(\mathfrak{m})$. The Alexandroff double circle defined in 3.1.26 is a compactification of the discrete space $D(\mathfrak{c})$; this compactification is incomparable (with respect to the order $\leq$) to the compactification $A(\mathfrak{c}) \oplus A(\mathfrak{c})$ of $D(\mathfrak{c})$. ■

3.5.15. EXAMPLE. Consider the subspace $X = X_1 \cup X_2 \cup X_3$ of the plane R^2, where $X_1 = \{0\} \times [-1, 1]$, $X_2 = \{(x, \sin 1/x) : 0 < x \leq 2/3\pi\}$ and X_3 is an arc with end-points $(0, -1)$ and $(2/3\pi, -1)$ and with interior disjoint from $X_1 \cup X_2$. As X is a closed and bounded subset of R^2, the space X is compact. The subspace $X \setminus X_1$ of X is homeomorphic to R and is dense in X, so that X is a compactification of R; the remainder of this compactification is homeomorphic to I. Replacing $X \setminus X_1$ by a suitably defined discrete subspace Y of cardinality $\aleph_0$, we obtain a compactification $Y \cup X_1$ of the space $D(\aleph_0)$ that has the remainder homeomorphic to I.

From Theorem 3.5.13 it follows that every compact space which is a continuous image of I (see Problem 6.3.14) is the remainder of a compactification of R and also of a compactification of $D(\aleph_0)$; from Example 3.1.26 we infer that every such space is also the remainder of a compactification of $D(\mathfrak{c})$ (cf. Exercise 3.5.H). ■

Historical and bibliographic notes

Compactifications (of open subsets of the plane) were first studied by Carathéodory in [1913] in the context of analytic functions; earlier, similar constructions were used in different theories of real numbers. Theorems 3.5.1 and 3.5.2 can be found in Tychonoff's paper [1930]. The order $\leq$ in the family of all compactifications of a space was defined and Theorems 3.5.9 and 3.5.12 were proved by Lubben in [1941]. The existence of a largest compactification was established earlier by Čech in [1937] and by M. H. Stone in [1937]. Theorem 3.5.5 was proved by Smirnov in [1952] (our proof is due to Mrówka [1956a]). Theorem 3.5.11 was proved by Alexandroff in [1924b]. Theorem 3.5.13 – at least in the realm of metrizable spaces – was a part of topological folklore in the thirties (cf. the final remarks in Hausdorff [1938] and in Kuratowski [1938]); it was first stated explicitly in Magill [1966].

Exercises

3.5.A. Show that if $c_s X_s$ is a compactification of X_s for $s \in S$, then the Cartesian product $\prod_{s \in S} c_s X_s$ is a compactification of the Cartesian product $\prod_{s \in S} X_s$.

3.5.B. Let cX be a compactification of X and E a closed equivalence relation on cX. Show that if all one-point subsets of $c(X)$ are equivalence classes of E, then cX/E also is a compactification of X.

3.5.C. Show that the limit of an inverse system $\{c_\sigma X, \pi^\sigma_\rho, \Sigma\}$ of compactifications of X, where $c_\rho = \pi^\sigma_\rho c_\sigma$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$, is a compactification of X.

3.5.D. Observe that any two compactifications $c_1 N$ and $c_2 N$ of the space $N = D(\aleph_0)$ that have finite remainders of the same cardinality are homeomorphic, and yet can be incomparable with respect to the order $\leq$. Give an example of two compactifications of $D(\mathfrak{c})$ that have two-point remainders and are not homeomorphic.

3.5.E. Show that the maximal compactification of a Tychonoff space X can be obtained by taking the closure in $\prod_{f\in\mathcal{F}} I_f$ of the image of the space X under the mapping $\triangle_{f\in\mathcal{F}} f$, where $\mathcal{F}$ is the family of all continuous functions from X to I and $I_f = I$ for $f \in \mathcal{F}$.

3.5.F. Prove that for every compactification cX of X there exists a compactification $c'X \leq cX$ such that $w(c'X) = wX$.

3.5.G. (a) (Arhangel'skiĭ [1965]) Prove that for every Tychonoff space X the following conditions are equivalent:

(1) *The space X is of pointwise countable type.*

(2) *For every compactification cX of the space X the subspace $c(X)$ can be represented as the union of a family of G_δ-sets in cX.*

(3) *There exists a compactification cX of the space X such that the subspace $c(X)$ can be represented as the union of a family of G_δ-sets in cX.*

(b) Observe that for every Tychonoff space X the following conditions are equivalent:

(1) *The space X is hemicompact.*

(2) *For every compactification cX of the space X we have $\chi(cX \setminus c(X), cX) \leq \aleph_0$.*

(3) *There exists a compactification cX of the space X such that $\chi(cX \setminus c(X), cX) \leq \aleph_0$.*

3.5.H. Show that for every compact space X such that $d(X) \leq \mathfrak{m} \geq \aleph_0$ there exists a compactification of $D(\mathfrak{m})$ with the remainder homeomorphic to X (cf. Problem 3.12.18(c)).

Hint. Apply Exercise 3.1.G and Theorem 3.5.13.

3.5.I. Prove that a circle and a closed interval are the only compactifications of the real line with finite remainder.

Hint. Apply the final part of 2.2.8.

3.5.J (Levy and McDowell [1975], van Douwen [1977]). (a) Show that for every Tychonoff space X all compactifications of the space X have the same density.

Hint. Apply Exercise 3.1.C(c) and observe that a closed irreducible mapping cannot lower density.

(b) Give an example of a Tychonoff space X and of a compactification cX of X such that $d(cX) < d(X)$.

Hint. Apply Corollary 2.3.16.

3.6. The Čech-Stone compactification and the Wallman extension

Let us recall that the largest element in the family $\mathcal{C}(X)$ of all compactifications of a Tychonoff space X is called the Čech-Stone compactification of X and is denoted by βX.

For simplicity's sake, in this section we shall identify the space X with the subspace $c(X)$ of any compactification cX of X, i.e., we shall assume that X is a subspace of any compactification cX (this assumption is also made, without warning the reader, in a few instances further in the book); the embedding of X in cX will be denoted by c. Identifying X and $c(X)$ allows us to discuss extendability of mappings from a space X over its compactifications.

3.6.1. THEOREM. *Every continuous mapping $f: X \to Z$ of a Tychonoff space X to a compact space Z is extendable to a continuous mapping $F: \beta X \to Z$.*

If every continuous mapping of a Tychonoff space X to a compact space is continuously extendable over a compactification αX of X, then αX is equivalent to the Čech-Stone compactification of X.

PROOF. The diagonal theorem implies that $c = \beta \triangle f: X \to \beta X \times Z$ is a homeomorphic embedding, so that $cX = \overline{c(X)} \subset \beta X \times Z$ is a compactification of X. By the maximality of βX there exists a continuous mapping $g: \beta X \to cX$ such that $g\beta = c$. Let $p: cX \to Z$ be the restriction to cX of the projection of $\beta X \times Z$ onto Z and let $F = pg: \beta X \to Z$. Since $F\beta = pg\beta = pc = f$, the mapping F is an extension of f.

If a compactification αX of a Tychonoff space X has the property stated in the second part of the theorem, then there exists an extension $B: \alpha X \to \beta X$ of the embedding $\beta: X \to \beta X$. We then have $B\alpha = \beta$, i.e., $\beta X \leq \alpha X$, which implies that αX und βX are equivalent. ∎

From Theorem 3.6.1 a series of important corollaries follows.

3.6.2. COROLLARY. *Every pair of completely separated subsets of a Tychonoff space X has disjoint closures in βX.*

If a compactification αX of a Tychonoff space X has the property that every pair of completely separated subsets of the space X has disjoint closures in αX, then αX is equivalent to the Čech-Stone compactification of X.

PROOF. Let A, B be a pair of completely separated subsets of a Tychonoff space X and let $f: X \to I$ be a continuous function that separates A and B. By the last theorem, f is extendable to a continuous function $F: \beta X \to I$, so that we have $\overline{A} \cap \overline{B} = \emptyset$, because $\overline{A} \subset F^{-1}(0)$ and $\overline{B} \subset F^{-1}(1)$.

If a compactification αX of a Tychonoff space X has the property stated in the second part of the corollary, then completely separated subsets of X, and – by Urysohn's lemma and Theorem 3.1.9 – only such subsets, have disjoint closures in αX. The same holds for βX, so that αX and βX are equivalent by virtue of Theorem 3.5.5. ∎

3.6.3. COROLLARY. *Every continuous function $f: X \to I$ from a Tychonoff space X to the closed interval I is extendable to a continuous function $F: \beta X \to I$.*

If every continuous function from a Tychonoff space X to the closed interval I is continuously extendable over a compactification αX of X, then αX is equivalent to the Čech-Stone compactification of X. ∎

3.6.4. COROLLARY. *Every pair of disjoint closed subsets of a normal space X has disjoint closures in βX.*

If a compactification αX of a Tychonoff space X has the property that every pair of disjoint closed subsets of the space X has disjoint closures in αX, then X is normal and αX is equivalent to the Čech-Stone compactification of X. ∎

3.6.5. COROLLARY. *For every open-and-closed subset A of a Tychonoff space X the closure $\overline{A}$ of A in βX is open-and-closed.* ■

Let X and Y be Tychonoff spaces, cX and $c'Y$ compactifications of X and Y respectively, and let $f: X \to Y$ be a continuous mapping; if there exists a continuous mapping $F: cX \to c'Y$ such that $F(x) = f(x)$ for $x \in X$, then we say that f is *continuously extendable*, or – briefly – *extendable, over compactifications cX and $c'Y$* and we call F an *extension of f over cX and $c'Y$*. This modification of the notion of extendability allows a simple formulation of the following important property of the Čech-Stone compactification.

3.6.6. COROLLARY. *For every compactification αY of a Tychonoff space Y and every continuous mapping $f: X \to Y$ of a Tychonoff space X to the space Y there exists a continuous extension $F: \beta X \to \alpha Y$ over βX and αY.* ■

3.6.7. COROLLARY. *If a subspace M of a Tychonoff space X has the property that every continuous function $f: M \to I$ is continuously extendable over X, then the closure $\overline{M}$ of M in βX is a compactification of M equivalent to βM. If, moreover, M is dense in X, then $\beta X = \beta M$.* ■

The last corollary and the Tietze-Urysohn theorem imply

3.6.8. COROLLARY. *For every closed subspace M of a normal space X the closure $\overline{M}$ of M in βX is a compactification of M equivalent to βM.* ■

3.6.9. COROLLARY. *For every Tychonoff space X and a space T such that $X \subset T \subset \beta X$ we have $\beta T = \beta X$.* ■

3.6.10. EXAMPLE. As shown in Example 3.1.27, every continuous function $f: W_0 \to I$ defined on the space W_0 of all countable ordinal numbers is extendable over the space W of all ordinal numbers $\leq \omega_1$; therefore we have $W = \beta W_0$.

Let us observe that the space W_0 has only one compactification; indeed, the Alexandroff compactification ωW_0 is obtained by identifying the remainder $\beta W_0 \setminus W_0$ to a point, so that for our space we have $\omega W_0 = \beta W_0$. More generally, the equality $\omega X = \beta X$ holds for every space X of the form $\beta Y \setminus \{x\}$, where $x \in \beta Y \setminus Y$, because for such a space we have $\omega X = \beta Y$ and – by Corollary 3.6.9 – also $\beta X = \beta Y$. ■

We are now going to study in detail Čech-Stone compactifications of discrete spaces and, in particular, the compactification βN, where N is the space of positive integers with the discrete topology.

3.6.11. THEOREM. *For every $\mathfrak{m} \geq \aleph_0$ the Čech-Stone compactification of the space $D(\mathfrak{m})$ has cardinality $2^{2^{\mathfrak{m}}}$ and weight $2^{\mathfrak{m}}$.*

PROOF. The Tychonoff cube $I^{2^{\mathfrak{m}}}$ is a compact space of cardinality $2^{2^{\mathfrak{m}}}$ and of weight $2^{\mathfrak{m}}$. By virtue of the Hewitt-Marczewski-Pondiczery theorem , $I^{2^{\mathfrak{m}}}$ contains a dense subset A of cardinality $\mathfrak{m}$. Let $f: D(\mathfrak{m}) \to I^{2^{\mathfrak{m}}}$ be an arbitrary mapping of $D(\mathfrak{m})$ to $I^{2^{\mathfrak{m}}}$ such that $f(D(\mathfrak{m})) = A$. It follows from Theorem 3.6.1 that f is extendable to a continuous mapping $F: \beta D(\mathfrak{m}) \to I^{2^{\mathfrak{m}}}$; obviously $F(\beta D(\mathfrak{m})) = I^{2^{\mathfrak{m}}}$, because the set $F(\beta D(\mathfrak{m}))$ is closed in $I^{2^{\mathfrak{m}}}$ and contains the dense subset A. We then have $|\beta D(\mathfrak{m})| \geq |I^{2^{\mathfrak{m}}}| = 2^{2^{\mathfrak{m}}}$ and – by Theorem

3.1.22 – $w(\beta D(\mathfrak{m})) \geq w(I^{2^{\mathfrak{m}}}) = 2^{\mathfrak{m}}$. To complete the proof it suffices to apply Theorem 3.5.3 and the Cantor-Bernstein theorem. ∎

3.6.12. COROLLARY. *The space βN has cardinality $2^{\mathfrak{c}}$ and weight $\mathfrak{c}$.* ∎

3.6.13. THEOREM. *For every $\mathfrak{m} \geq \aleph_0$ the Čech-Stone compactification of the space $\beta D(\mathfrak{m})$ has a base consisting of open-and-closed sets.*

PROOF. It suffices to show that for every point $x \in \beta D(\mathfrak{m})$ and each neighbourhood V of x there exists an open-and-closed subset U of $\beta D(\mathfrak{m})$ such that $x \in U \subset V$. Let W be a neighbourhood of x satisfying $x \in W \subset \overline{W} \subset V$ and let $A = W \cap D(\mathfrak{m})$. From Corollary 3.6.5 it follows that the set $U = \overline{A}$ is open-and-closed in $\beta D(\mathfrak{m})$ and from Theorem 1.3.6 it follows that $U = \overline{W \cap D(\mathfrak{m})} = \overline{W}$, so that $x \in U \subset V$. ∎

The class of spaces that have a base consisting of open-and-closed sets is studied in Section 6.2 (such spaces are called zero-dimensional).

Theorem 3.6.11 can be significantly strengthened in the case of $\mathfrak{m} = \aleph_0$; we have (cf. Exercise 3.6.G(d)):

3.6.14. THEOREM. *Every infinite closed set $F \subset \beta N$ contains a subset homeomorphic to βN; in particular F has cardinality $2^{\mathfrak{c}}$ and weight $\mathfrak{c}$.*

PROOF. By induction one can readily define a sequence of points $a_1, a_2, \ldots$ and a sequence of open sets $V_1, V_2, \ldots$ such that $a_i \in V_i$, $V_i \cap V_j = \emptyset$ for $i \neq j$ and $A = \{a_1, a_2, \ldots\} \subset F$.

Let $g: A \to I$ be a continuous function. The function $g_0: N \to I$ defined by the formula

$$g_0(n) = \begin{cases} g(a_i), & \text{if } n \in N \cap V_i, \\ 0, & \text{if } n \in N \setminus \bigcup_{i=1}^{\infty} V_i, \end{cases}$$

has an extension $G_0: \beta N \to I$. Since N is dense in βN, for every $x \in \overline{V_i} = \overline{N \cap V_i}$ we have $G_0(x) = g(a_i)$, so that $G_0|A = g$. Hence for every function $g: A \to I$ there exists an extension $G = G_0|\overline{A}: \overline{A} \to I$ over the compactification $\overline{A}$ of the space A and by Corollary 3.6.3 we have $\overline{A} = \beta A$. The space A being homeomorphic to N, it follows that the space $\beta A = \overline{A} \subset F$ is homeomorphic to βN. ∎

3.6.15. COROLLARY. *The space βN does not contain any subspace homeomorphic to $A(\aleph_0)$, i.e., in βN there are no non-trivial convergent sequences.* ∎

3.6.16. COROLLARY. *No non-discrete subspace of βN is a sequential space.* ∎

3.6.17. COROLLARY. *No space $N \cup \{x\} \subset \beta N$, where $x \in \beta N \setminus N$, is first-countable.* ∎

3.6.18. EXAMPLE. The space $\beta N \setminus N$, the remainder of the Čech-Stone compactification of the discrete space of cardinality $\aleph_0$, contains a family of cardinality $\mathfrak{c}$ consisting of pairwise disjoint non-empty open sets.

Let us first note that in the set N there exists a family $\{N_t\}_{t \in I}$ of infinite subsets such that the intersection $N_t \cap N_{t'}$ is finite for every pair t, t' of distinct numbers in the interval I. Indeed, it suffices to arrange all rational numbers of the interval I into a sequence $q_1, q_2, \ldots$ and to define $N_t = \{n_1, n_2, \ldots\}$, where $q_{n_1}, q_{n_2}, \ldots$ is a subsequence of $q_1, q_2, \ldots$ that converges to t and whose elements are distinct from each other.

The family $\{U_t\}_{t \in I}$, where $U_t = (\beta N \setminus N) \cap \overline{N}_t$, has cardinality $\mathfrak{c}$ and consists of non-empty open subsets of $\beta N \setminus N$. For every pair t, t' of distinct numbers in the interval I we have $N_{t'} = F \cup M$, where $|F| < \aleph_0$ and $M \cap N_t = \emptyset$. Since $\overline{F} = F \subset N$ and by Corollary 3.6.4 we have $\overline{M} \cap \overline{N}_t = \emptyset$,

$$\begin{aligned} U_t \cap U_{t'} &= (\beta N \setminus N) \cap \overline{N}_t \cap \overline{N}_{t'} = (\beta N \setminus N) \cap \overline{N}_t \cap (\overline{F} \cup \overline{M}) \\ &= (\beta N \setminus N) \cap [(\overline{N}_t \cap \overline{F}) \cup (\overline{N}_t \cap \overline{M})] \subset (\beta N \setminus N) \cap N = \emptyset. \blacksquare \end{aligned}$$

3.6.19. EXAMPLE. We shall now use the family $\{U_t\}_{t \in I}$ constructed in the preceding example to define a non-normal Tychonoff space which will prove useful in Section 6.2.

Let us choose a point $x_t \in U_t$ for every $t \in I$ and let us define $X = N \cup \{x_t : t \in I\}$. Since N is locally compact and dense in X, it follows from Theorem 3.3.9 that the set $X \setminus N$ is closed in X; as the space $X \setminus N$ consists exclusively of isolated points, it follows that it is homeomorphic to $D(\mathfrak{c})$. Thus, by virtue of Corollary 2.1.10, the space X is not normal. $\blacksquare$

The space βN is used in the construction of many interesting examples; some of them will be discussed in later sections of this chapter.

From Theorems 3.6.11 and 2.3.23 it follows that the space $\beta D(\mathfrak{m})$ is embeddable in the Tychonoff cube $I^{2^{\mathfrak{m}}}$; we shall now describe a subspace of the Cantor cube $D^{2^{\mathfrak{m}}} \subset I^{2^{\mathfrak{m}}}$ which is homeomorphic to $\beta D(\mathfrak{m})$.

3.6.20. EXAMPLE. Let $\mathcal{F}$ be the family of all mappings of $D(\mathfrak{m})$, where $\mathfrak{m} \geq \aleph_0$, to the two-point discrete space $D = \{0, 1\}$; clearly $|\mathcal{F}| = 2^{\mathfrak{m}}$. By virtue of the diagonal theorem the mapping $F = \triangle_{f \in \mathcal{F}} f \colon D(\mathfrak{m}) \to D^{2^{\mathfrak{m}}} = \prod_{f \in \mathcal{F}} D_f$, where $D_f = D$ for every $f \in \mathcal{F}$, is a homeomorphic embedding, so that the closure $\overline{F(D(\mathfrak{m}))} \subset D^{2^{\mathfrak{m}}}$ is a compactification of $D(\mathfrak{m})$. For every pair A, B of disjoint closed subsets of $D(\mathfrak{m})$ the closures of $F(A)$ and $F(B)$ in this compactification are disjoint. Indeed, for the function $f \colon D(\mathfrak{m}) \to D$ defined by the formulas:

$$f(x) = \begin{cases} 0, & \text{if } x \in A, \\ 1, & \text{if } x \in D(\mathfrak{m}) \setminus A, \end{cases}$$

we have

$$F(A) \subset p_f^{-1}(0) \quad \text{and} \quad F(B) \subset p_f^{-1}(1),$$

and the sets $p_f^{-1}(0)$ and $p_f^{-1}(1)$ are disjoint and closed in $D^{2^{\mathfrak{m}}}$.

Corollary 3.6.4 implies that $\overline{F(D(\mathfrak{m}))}$ is a compactification of $D(\mathfrak{m})$ which is equivalent to $\beta D(\mathfrak{m})$. $\blacksquare$

We shall now define for every T_1-space X a quasi-compact T_1-space wX that contains X as a dense subspace and has the property that every continuous mapping $f \colon X \to Z$ of X to a compact space Z is extendable to a continuous mapping $F \colon wX \to Z$. Thus wX is a substitute for the Čech-Stone compactification which is defined for every T_1-space X. It turns out that wX is a Hausdorff space if and only if X is a normal space; obviously, in this case wX is a compactification of the space X equivalent to βX.

Let X be a T_1-space and let $\mathcal{D}(X)$ denote the family of all closed subsets of X. It follows readily from the Teichmüller-Tukey lemma that every family of closed subsets of X which has the finite intersection property is contained in an ultrafilter in $\mathcal{D}(X)$; generally this ultrafilter is not uniquely determined. The family of all ultrafilters in $\mathcal{D}(X)$ will be denoted by $F(X)$. Each ultrafilter $\mathcal{F} \in F(X)$ has the following properties:

(1) *$\emptyset \notin \mathcal{F}$.*
(2) *If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$.*
(3) *If $A_0 \in \mathcal{D}(X)$ and $A_0 \cap A \neq \emptyset$ for every $A \in \mathcal{F}$, then $A_0 \in \mathcal{F}$.*
(4) *If $A \in \mathcal{F}$ and $A \subset A_1 \in \mathcal{D}(X)$, then $A_1 \in \mathcal{F}$.*
(5) *If $A_1, A_2 \in \mathcal{D}(X)$ and $A_1 \cup A_2 \in \mathcal{F}$, then either $A_1 \in \mathcal{F}$ or $A_2 \in \mathcal{F}$.*
(6) *If $\mathcal{F} \neq \mathcal{F}' \in F(X)$, then there exist $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$ such that $A \cap A' = \emptyset$.*

Properties (1), (2) and (4) follow from the definition of a filter, property (3) follows from the fact that the family $\mathcal{F} \cup \{A_0\} \subset \mathcal{D}(X)$ has the finite intersection property and thus is contained in an ultrafilter $\mathcal{F}'$ in $\mathcal{D}(X)$ which must coincide with $\mathcal{F}$. To prove (5) note that if $A_1 \notin \mathcal{F}$ and $A_2 \notin \mathcal{F}$, then by (3) there exist $A_1', A_2' \in \mathcal{F}$ such that $A_1 \cap A_1' = \emptyset = A_2 \cap A_2'$; thus we have $(A_1 \cup A_2) \cap A_1' \cap A_2' = \emptyset$ which by (1) and (2) implies that $A_1 \cup A_2 \notin \mathcal{F}$. Finally property (6) follows from the fact that, if $\mathcal{F} \neq \mathcal{F}'$, then – by the maximality of $\mathcal{F}$ – there exists an $A \in \mathcal{F}$ such that $A \notin \mathcal{F}'$ and, by (3), there exists an $A' \in \mathcal{F}'$ such that $A \cap A' = \emptyset$.

One readily sees that properties (1)–(3) characterize ultrafilters in the class of all subfamilies of $\mathcal{D}(X)$.

To every $x \in X$ there corresponds an ultrafilter $\mathcal{F}(x) = \{A \in \mathcal{D}(X) : x \in A\}$; clearly $\bigcap \mathcal{F}(x) = \{x\}$. If $\mathcal{F} \in F(X)$ and $x \in \bigcap \mathcal{F}$, then $\mathcal{F} \subset \mathcal{F}(x)$, so that $\mathcal{F} = \mathcal{F}(x)$. Hence, for every ultrafiler $\mathcal{F} \in F(X)$ the intersection $\bigcap \mathcal{F}$ is either a one-point set or is empty. Ultrafilters that have an empty intersection are called *free ultrafilters*; they form a subfamily $F_0(X)$ of the family $F(X)$.

Let $wX = X \cup F_0(X)$; for every open set $U \subset X$ define

$$U^* = U \cup \{\mathcal{F} \in F_0(X) : A \subset U \quad \text{for some} \quad A \in \mathcal{F}\} \subset wX,$$

and for every closed set $A \subset X$ define

$$A_* = A \cup \{\mathcal{F} \in F_0(X) : A \in \mathcal{F}\} \subset wX.$$

Since, by virtue of (3), for every $\mathcal{F} \in F(X)$ and an open set $U \subset X$

$$X \setminus U \notin \mathcal{F} \quad \text{if and only if} \quad A \subset U \text{ for some } A \in \mathcal{F},$$

we have

$$U^* = wX \setminus (X \setminus U)_* \quad \text{and} \quad A_* = wX \setminus (X \setminus A)^*. \tag{7}$$

From (2), (5) and (4) it follows that for any closed sets $A_1, A_2 \subset X$ we have

$$(A_1 \cap A_2)_* = A_{1*} \cap A_{2*} \quad \text{and} \quad (A_1 \cup A_2)_* = A_{1*} \cup A_{2*}, \tag{8}$$

which by (7) and De Morgan's laws implies that for any open sets $U_1, U_2 \subset X$ we have

$$(U_1 \cup U_2)^* = U_1^* \cup U_2^* \quad \text{and} \quad (U_1 \cap U_2)^* = U_1^* \cap U_2^*. \tag{9}$$

The second part of (9) and the equality $X^* = wX$, imply that the family $\mathcal{B}$ of all sets U^*, where U is an open subset of X, has properties (B1)–(B2) in Section 1.1; the set wX with the topology generated by the base $\mathcal{B}$ is called the *Wallman extension of the space X*.

3.6.21. THEOREM. *For every T_1-space X the Wallman extension wX is a quasi-compact T_1-space that contains X as a dense subspace and has the property that every continuous mapping $f: X \to Z$ of X to a compact space Z is extendable to a continuous mapping $F: wX \to Z$.*

PROOF. The fact that the topology of X as a subspace of wX coincides with the original topology of X, and the fact that X is dense in wX, follow directly from the definition of U^*.

We shall show that wX is a T_1-space. First of all note that from the second part of (7) it follows that the set A_* is closed in wX for every $A \in \mathcal{D}(X)$. Since clearly $\{x\} = \{x\}_*$ for every $x \in X$ and, by (6), $\{\mathcal{F}\} = \bigcap_{A \in \mathcal{F}} A_*$ for every $\mathcal{F} \in F_0(X)$, we see that all one-point subsets of wX are closed.

To prove that wX is a quasi-compact space consider an arbitrary family $\{F_s\}_{s \in S}$ of closed subsets of wX that has the finite intersection property. By the definition of the topology on wX for every $s \in S$ we have $F_s = \bigcap_{t \in T_s} A_{t^*}$, where A_t is a closed subset of X. The family $\{A_{t^*}\}_{t \in T}$, where $T = \bigcup_{s \in S} T_s$, has the finite intersection property; the first part of (8) implies that the family $\{A_t\}_{t \in T}$ of closed subsets of X also has the finite intersection property, so that the family $\{A_t\}_{t \in T}$ is contained in an ultrafilter $\mathcal{F}$ in $\mathcal{D}(X)$. If $\bigcap \mathcal{F} \neq \emptyset$, then there exists an $x \in X$ such that $\mathcal{F} = \mathcal{F}(x)$ and we have $x \in \bigcap_{t \in T} A_t \subset \bigcap_{t \in T} A_{t^*} = \bigcap_{s \in S} F_s$; if $\bigcap \mathcal{F} = \emptyset$, then $\mathcal{F} \in F_0(X)$ and we have $\mathcal{F} \in \bigcap_{t \in T} A_{t^*} = \bigcap_{s \in S} F_s$. Hence in both cases the family $\{F_s\}_{s \in S}$ has a non-empty intersection.

Let us now consider a continuous mapping $f: X \to Z$ of X to a compact space Z. For every pair B_1, B_2 of disjoint closed subsets of Z the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are closed in X and disjoint; so that – by the fist part of (8) – the sets $[f^{-1}(B_1)]_*$ and $[f^{-1}(B_2)]_*$ are disjoint. Since the last two sets are closed in wX and contain $f^{-1}(B_1)$ and $f^{-1}(B_2)$ respectively, the closures of $f^{-1}(B_1)$ and $f^{-1}(B_2)$ in wX are disjoint. The existence of a continuous extension $F: wX \to Z$ of the mapping f now follows from Theorem 3.2.1. ■

3.6.22. THEOREM. *The Wallman extension wX of a T_1-space X is a Hausdorff space if and only if X is a normal space.*

PROOF. If wX is a Hausdorff space, then wX is a compactification of the space X; since for every pair A, B of disjoint closed subsets of X the closures of A and B in wX are disjoint, the space X is normal.

Assume now that the space X is normal. By (6) for every pair $\mathcal{F}, \mathcal{F}'$ of distinct elements of $F(X)$ there exist closed sets $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$ such that $A \cap A' = \emptyset$; furthermore, by normality of X, there exist open sets $U(\mathcal{F}, \mathcal{F}')$, $V(\mathcal{F}, \mathcal{F}') \subset X$ such that $A \subset U(\mathcal{F}, \mathcal{F}')$, $A' \subset V(\mathcal{F}, \mathcal{F}')$ and $U(\mathcal{F}, \mathcal{F}') \cap V(\mathcal{F}, \mathcal{F}') = \emptyset$. One readily sees that for every pair of distinct points $\mathcal{F}, \mathcal{F}' \in F_0(X) \subset wX$, the sets $U(\mathcal{F}, \mathcal{F}')^*$ and $V(\mathcal{F}, \mathcal{F}')^*$ are disjoint neighbourhoods of these points in the space wX. Similarly, for every pair of distinct poins $x, x' \in X \subset wX$, the sets $U(\mathcal{F}(x), \mathcal{F}(x'))^*$ and $V(\mathcal{F}(x), \mathcal{F}(x'))^*$ are disjoint neighbourhoods of these points in wX. Finally, for a point $\mathcal{F} \in F_0(X)$ and a point $x \in X$, the required neighbourhoods are $U(\mathcal{F}, \mathcal{F}(x))^*$ and $V(\mathcal{F}, \mathcal{F}(x))^*$. Hence wX is a Hausdorff space. ■

3.6.23. COROLLARY. *For every normal space X the Wallman extension wX is a compactification of the space X equivalent to the Čech-Stone compactification of X.* ■

Historical and bibliographic notes

The Čech-Stone compactification was introduced by Čech in [1937] and by M. H. Stone in [1937]; those papers, along with M. H. Stone's paper [1948], contain fundamental results on βX, i.e., Theorem 3.6.1 and Corollaries 3.6.2–3.6.9. The first of the two authors defined the maximal compactification using the method indicated in Exercise 3.5.E, thus developing an idea appearing in Tychonoff's paper [1930]; the second one used the construction described in Problem 3.12.22(c) below. Let us note that Tychonoff defined a space homeomorphic to βN in [1935] (see Exercise 3.6.H). Example 3.6.10 was given by Čech in [1937]. Theorem 3.6.11 was proved by Pospíšil [1937]; the proof given in this book is due to Mrówka [1959a] (the theorem can be also deduced from an earlier result in set theory (see Exercise 3.6.F)). Theorem 3.6.14 was proved by Čech in 1939 but has not been published (see Novák [1953a]); our proof is taken from Novák [1953a]. The fact that in the remainder $\beta N \setminus N$ there exists a family of cardinality $\mathfrak{c}$ consisting of pairwise disjoint non-empty open sets was observed by Nakamura and Kakutani in [1943]; the construction of such a family given in Example 3.6.18, as well as Example 3.6.19, are taken from Katětov's paper [1950] (the family $\{N_t\}_{t\in I}$ was, in principle, defined by Sierpiński in [1928a]). The Wallman extension was constructed and Theorems 3.6.21 and 3.6.22 were proved by Wallman in [1938]. A large number of results concerning the Čech-Stone compactification is collected in Walker's book [1974]. Van Mill's paper [1984] is a survey of recent results on βN and $\beta N \setminus N$.

Problems connected with the Čech-Stone compactification are among the most interesting problems of general topology. Some of them, in particular problems connected with Čech-Stone compactifications of discrete spaces, are very close to set theory. The compactification βX can be constructed in many ways (see Exercises 3.5.E, 3.6.K, Problems 3.12.22(c), 8.5.8(a), Examples 8.3.18 and 8.4.14), has many interesting properties (see for example Theorems 7.1.15 and 7.1.17), is applied in constructing many interesting examples (see Examples 3.10.19, 3.10.29, 5.1.23, Problems 3.12.17 and 4.5.20(c)) and also in proving some theorems (see hints to Problems 5.5.8(a), 7.4.15 and 7.4.16).

Exercises

3.6.A (Novák [1953], W. Rudin [1956]). (a) Note that every open-and-closed subset of the remainder $\beta N \setminus N$ is of the form $A^* = \overline{A} \cap (\beta N \setminus N)$, where $A \subset N$; check that $A^* \subset B^*$ if and only if the difference $A \setminus B$ is finite.

(b) Show that $\beta N\setminus N$ has no isolated points and that every non-empty G_δ-set in $\beta N\setminus N$ has a non-empty interior. Observe that the latter property is equivalent to the fact that in $\beta N\setminus N$ the closure of any union $\bigcup_{i=1}^{\infty} A_i^*$, where $A_1^* \subset A_2^* \subset \ldots$ and $A_i^* \neq A_{i+1}^*$ for $i = 1, 2, \ldots$, is not an open set.

Remark. It follows from (a) that properties of $\beta N \setminus N$ can be formulated in terms of sets of integers or infinite sequences consisting of zeros and ones; some of these properties were first proved in such a form by Hausdorff in [1936] (cf. Engelking [1972]).

3.6.B. (a) Define a function $f: D(\mathfrak{c}) \to I$ which is not continuously extendable over the compactifications of $D(\mathfrak{c})$ described in 3.5.14.

(b) Show that for every $\mathfrak{m} \geq \aleph_0$ the remainder $\beta D(\mathfrak{m}) \setminus D(\mathfrak{m})$ contains a subset homeomorphic to $\beta D(\mathfrak{m})$.

Hint. Apply the equality $\mathfrak{m}^2 = \mathfrak{m}$.

3.6.C (M. H. Stone [1948]). Deduce Corollary 3.6.8 from Corollary 3.6.4 and – applying Exercise 3.2.J(b) – obtain the Tietze-Urysohn Theorem.

3.6.D. (a) Deduce from Exercise 3.2.H(a) and (b) that the Cartesian product $\beta X \times \beta Y$ is not necessarily the Čech-Stone compactification of $X \times Y$ (even if Y is a compact space).

Remark. A necessary and sufficient condition for the equality $\prod_{s\in S} \beta X_s = \beta \prod_{s\in S} X_s$ is given in Problem 3.12.21(d).

(b) Show that the function $f: N \times N \to I$ defined by letting $f((m,n)) = n/(n+m)$ for $(n,m) \in N \times N$ is not continuously extendable over $\beta N \times \beta N$; deduce that the Cartesian product $\beta N \times \beta N$ is not the Čech-Stone compactification of $N \times N$ (cf. Problem 6.3.21(a)).

(c) Prove that for every separable compact space X the Cartesian product $W \times X$ is the Čech-Stone compactification of $W_0 \times X$.

Remark. The assumption of separability is not essential (see Problem 3.12.20(c) or 3.12.21(c)).

(d) Prove that for every completely regular space Y there exists a completely regular space Z such that $\beta Z \setminus Z = Y$.

Hint. If $w(Y) \leq \aleph_0$ the existence of Z follows from part (c), and if $w(Y) \leq \mathfrak{c}$ – from (c) and Corollary 2.3.16; to obtain the general statement, modify (c) by considering an appropriate space of ordinal numbers instead of W_0.

Remark. This follows directly from Problem 3.12.20(c) and from Problem 3.12.24(c).

(e) Observe that the Čech-Stone compactification of the limit of an inverse system $\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of Tychonoff spaces can differ from the limit of the inverse system $\{\beta X_\sigma, \tilde{\pi}^\sigma_\rho, \Sigma\}$, where $\tilde{\pi}^\sigma_\rho$ is the extension of π^σ_ρ over βX_σ and βX_ρ.

3.6.E. Prove that for a continuous mapping $f: X \to Y$ of a Tychonoff space X onto a Tychonoff space Y there exist compactifications cX and $c'Y$ such that $w(cX) = w(X)w(Y)$, $w(c'Y) = w(Y)$ and f is extendable to a continuous mapping $F: cX \to c'Y$.

3.6.F. We say that a family $\{A_s\}_{s\in S}$ of subsets of a set X consists of *independent sets* provided that for every finite sequence $s_1, s_2, \ldots, s_k$ of distinct elemens of S and any sequence $i_1, i_2, \ldots, i_k$ consisting of zeros and ones we have $A^{i_1}_{s_1} \cap A^{i_2}_{s_2} \cap \ldots \cap A^{i_k}_{s_k} \neq \emptyset$, where $A^0 = A$ and $A^1 = X \setminus A$.

(a) (Hausdorff [1936a]; for $\mathfrak{m} = \aleph_0$ and $\mathfrak{m} = 2^{\aleph_0}$, Fichtenholz and Kantorovitch [1934]) Show that the family of all subsets of a set X of cardinality $\mathfrak{m} \geq \aleph_0$ contains a subfamily of cardinality $2^{\mathfrak{m}}$ consising of independent sets.

Hint. Apply the Hewitt-Marczewski-Pondiczery theorem and take as X a dense subset of the Cantor cube $D^{2^{\mathfrak{m}}}$.

(b) Deduce from (a) that the Cantor cube $D^{2^{\mathfrak{m}}}$ has a dense subset of cardinality $\mathfrak{m}$.

Hint. Use a family of independent subsets of a set of cardinality $\mathfrak{m}$ to index $2^{\mathfrak{m}}$ factors of $D^{2^{\mathfrak{m}}}$.

(c) Observe that Theorem 3.6.11 follows from (a).

3.6.G. (a) (Čech [1959], Gillman and Jerison [1960]) Prove that for any Tychonoff space X every non-empty closed G_δ-set in βX which is contained in the remainder $\beta X \setminus X$ contains a subset homeomorphic to βN and thus has cardinality not less than $2^{\mathfrak{c}}$ (cf. part (d) below). Note that no point of the remainder $\beta X \setminus X$ is a G_δ-set in βX. Observe that if X can be represented as a countable union of compact subspaces, then the remainder $\beta X \setminus X$ has no isolated points.

Hint. Let $f: \beta X \to I$ be a function that vanishes only at points of the set under consideration. Define a sequence $a_1, a_2, \ldots$ of points of X such that $f(a_1) > f(a_2) > \ldots$ and $f(a_i) \leq 1/i$ for $i = 1, 2, \ldots$ Applying normality of the interval $(0, 1]$ observe that any two disjoint closed subsets of the set $A = \{a_1, a_2, \ldots\}$ are completely separated in X and deduce that the closure of A in βX is homeomorphic to βN.

(b) (Čech [1937]) Show that first-countable Tychonoff spaces X and Y are homeomorphic if and only if the compactifications βX and βY are homeomorphic.

(c) (Negrepontis [1967]) Let X be a locally compact space that can be represented as a countable union of compact subspaces. Prove that for each F_σ-set M in $\beta X \setminus X$ every continuous function $f: M \to I$ is continuously extendable over $\beta X \setminus X$, so that the closure of M in $\beta X \setminus Y$ is the Čech-Stone compactification of M.

Hint. Apply Exercise 2.1.E(a) to show that $X \cup M$ is a normal space and use Corollary 3.6.8.

Remark. It is enough to assume that M has the Lindelöf property (see Section 3.8); cf. Exercise 3.8.F(b).

(d) (Gillman and Henriksen [1956], Gillman and Jerison [1960]) A completely regular space X is an *F-space*, if for each functionally open set $M \subset X$ every continuous function $f: M \to I$ is continuously extendable over X.

Prove that a completely regular space X is an F-space if and only if any disjoint sets A, B functionally open in X are completely separated; deduce that a normal space X is an F-space if and only if every pair of disjoint open F_σ-sets in X has disjoint closures.

Show that in an F-space there are no non-trivial convergent sequences and deduce that every sequential F-space is discrete. Prove that every infinite closed subset of a compact F-space contains a subset homeomorphic to βN. Apply part (c) to show that for any Tychonoff space X every closed G_δ-set in βX which is contained in the remainder $\beta X \setminus X$ is an F-space.

Hint. In the proof of the first part apply Exercises 2.1.B(c) and 2.1.J.

3.6.H. For every $t \in I$ and $i = 1, 2, \ldots$ let $d_i(t)$ denote the ith digit of the dyadic expansion of the number t, with the additional assumption that if t has two expansions then we consider the one with infinitely many zeros; thereby for every i a point d_i of the Tychonoff cube $I^{\mathfrak{c}} = \prod_{t \in I} I_t$, where $I_t = I$ for $t \in I$, is defined. Show that the subspace $\{d_1, d_2, \ldots\}$ of $I^{\mathfrak{c}}$ is homeomorphic to N and its closure in $I^{\mathfrak{c}}$ is homeomorphic to βN.

3.6.I. (a) (Mrówka [1954], Franklin [1967]) Let $\{N_s\}_{s \in S}$, where $S \cap N = \emptyset$, be an infinite family of infinite subsets of N such that the intersection $N_s \cap N_{s'}$ is finite for every pair s, s' of distinct elements of S and that $\{N_s\}_{s \in S}$ is maximal with respect to the last property (cf. Example 3.6.18 and the Teichmüller-Tukey lemma). Generate a topology on the set $X = N \cup S$ by the neighbourhood system $\{\mathcal{B}(x)\}_{x \in X}$, where $\mathcal{B}(x) = \{\{n\}\}$ if $x = n \in N$ and $\mathcal{B}(x) = \{\{s\} \cup (N_s \setminus \{1, 2, \ldots, i\})\}_{i=1}^{\infty}$ if $x = s \in S$. Verify that X is a non-normal locally compact space and prove that ωX is a sequential space but is not a Fréchet space (cf. Exercise 3.10.E).

Remark. A similar construction (without the assumption of maximality) can be found in Alexandroff and Urysohn's paper [1929].

(b) Note that from the existence of a locally compact space X with a countable dense set A consisting of isolated points of X such that the subspace $X \setminus A$ contains $\mathfrak{c}$ isolated points it follows easily that the remainder $\beta N \setminus N$ contains a family of cardinality $\mathfrak{c}$ consisting of pairwise disjoint non-empty open sets.

3.6.J (Gillman and Jerison [1960]). Note that both Theorem 3.6.1 and Theorem 3.6.21 imply the Tychnoff theorem.

Hint. Consider the Čech-Stone compactification and the Wallman extension of the Cartesian product.

3.6.K (Gillman and Jerison [1960). (a) Verify that if X is a Tychonoff space, then modifying the construction of the Wallman extension of X by taking ultrafilters in the family $\mathcal{D}_0(X)$ of all functionally closed subsets of X instead of ultrafilters in $\mathcal{D}(X)$ one obtains the Čech-Stone compactification of X. Note that in this case all the sets U^*, where U is a functionally open subset of X, form a base for $X \cup \mathcal{F}_0(X)$.

(b) Observe that a Tychonoff space X is compact if and only if every family of functionally closed subsets of X which has the finite intersection property has non-empty intersection.

3.7. Perfect mappings

A continuous mapping $f: X \to Y$ is *perfect* if X is a Hausdorff space, f is a closed mapping and all fibers $f^{-1}(y)$ are compact subsets of X. A one-to-one mapping $f: X \to Y$ defined on a Hausdorff space X is perfect if and only if it is a closed mapping, i.e., if f is a homeomorphic embedding and the set $f(X)$ is closed in Y. In particular, an embedding $i_M: M \to X$ is a perfect mapping if and only if M is a Hausdorff space and $\overline{M} = M$. Theorems 3.1.2 and 3.1.12 imply that every continuous mapping of a compact space to a Hausdorff space is perfect. The Kuratowski theorem yields

3.7.1. THEOREM. *If X is a compact space and Y is a Hausdorff space, then the projection $p: X \times Y \to Y$ is perfect.* ∎

3.7.2. THEOREM. *If $f: X \to Y$ is a perfect mapping, then for every compact subspace $Z \subset Y$ the inverse image $f^{-1}(Z)$ is compact.*

PROOF. As $f^{-1}(Z)$ is clearly a Hausdorff space, it suffices to show that for any family $\{U_s\}_{s \in S}$ of open subsets of X whose union contains $f^{-1}(Z)$ there exists a finite set $S_0 \subset S$ such that $f^{-1}(Z) \subset \bigcup_{s \in S_0} U_s$. Let $\mathcal{T}$ be the family of all finite subsets of S and $U_T = \bigcup_{s \in T} U_s$ for $T \in \mathcal{T}$. For each $z \in Z$ the fiber $f^{-1}(z)$ is compact and thus is contained in the set U_T for some $T \in \mathcal{T}$; it follows that $z \in Y \setminus f(X \setminus U_T)$, and thus $Z \subset \bigcup_{T \in \mathcal{T}} (Y \setminus f(X \setminus U_T))$. The sets $Y \setminus f(X \setminus U_T)$ being open, there exist $T_1, T_2, \ldots, T_k \in \mathcal{T}$ such that $Z \subset \bigcup_{i=1}^{k} (Y \setminus f(X \setminus U_{T_i}))$.

Hence

$$f^{-1}(Z) \subset \bigcup_{i=1}^{k} f^{-1}(Y \setminus f(X \setminus U_{T_i})) = \bigcup_{i=1}^{k} (X \setminus f^{-1}f(X \setminus U_{T_i})) \subset$$
$$\subset \bigcup_{i=1}^{k} (X \setminus (X \setminus U_{T_i})) = \bigcup_{i=1}^{k} U_{T_i} = \bigcup_{s \in S_0} U_s,$$

where $S_0 = T_1 \cup T_2 \cup \ldots \cup T_k$.■

The last theorem yields

3.7.3. COROLLARY. *The composition of two perfect mappings is a perfect mapping.*■

In the proof of the next proposition we shall use a lemma that shall be also applied further in this section; let us note that the lemma generalizes Lemma 3.5.6.

3.7.4. LEMMA. *A perfect mapping $f: X \to Y$ cannot be continuously extended over any Hausdorff space Z that contains X as a dense proper subspace.*

PROOF. Assume that $F: Z \to Y$ is a continuous extension of f to a Hausdorff space Z that contains X as a proper subset. Without loss of generality we can suppose that $Z = X \cup \{x\}$. The point x does not belong to the compact set $f^{-1}(F(x))$, so that by the second part of Theorem 3.1.6 there exist open sets $U, V \subset Z$ such that $x \in U$, $f^{-1}(F(x)) \subset V$ and $U \cap V = \emptyset$. The set $f(X \setminus V)$ being closed in Y, its inverse image $F^{-1}(f(X \setminus V))$ is closed in Z. Thus we have

$$\overline{X \setminus V} \subset F^{-1}(f(X \setminus V)) = f^{-1}f(X \setminus V) \subset X,$$

where the bar denotes the closure operation in Z. Since $x \notin \overline{V}$, this implies that X is closed in Z.■

3.7.5. PROPOSITION. *If the composition gf of continuous mappings $f: X \to Y$ and $g: Y \to Z$, where Y is a Hausdorff space, is perfect, then the mappings $g|f(X)$ and f are perfect.*

PROOF. For every point $z \in Z$ the fiber $(g|f(X))^{-1}(z) = f(X) \cap g^{-1}(z) = f[(gf)^{-1}(z)]$ is compact, because the fiber $(gf)^{-1}(z)$ is compact. The fact that $g|f(X)$ is a closed mapping follows from Proposition 2.1.3 and thus the mapping $g|f(X)$ is perfect.

For every point $y \in Y$ the fiber $f^{-1}(y) = (gf)^{-1}(g(y)) \cap f^{-1}(y)$ is compact. To conclude the proof it suffices to show that f is closed. For every closed set $F \subset X$ the mapping $(gf)|F$ is perfect, so that – by the first part of our proof – the restriction $g|f(F)$ is perfect; since the latter mapping can be continuously extended over $\overline{f(F)}$, it follows from the lemma that $f(F) = \overline{f(F)}$, and thus f is a closed mapping.■

Applying Proposition 2.1.4, one easily obtains

3.7.6. PROPOSITION. *If $f: X \to Y$ is a pefect mapping, then for any closed $A \subset X$ and any $B \subset Y$ the restrictions $f|A: A \to Y$ and $f_B: f^{-1}(B) \to B$ are perfect.*■

Propositions 2.1.15, 2.1.11 and 2.1.13 imply the following two propositions.

3.7.7. PROPOSITION. *Suppose we are given a Hausdorff space X, a finite cover $\{A_i\}_{i=1}^k$ of the space X and a family $\{f_i\}_{i=1}^k$ of compatible mappings, where $f_i: A_i \to Y$, such that the combination $f = f_1 \triangledown f_2 \triangledown \ldots \triangledown f_k$ is continuous. If all mappings f_i are perfect, then the combination f is perfect.* ■

3.7.8. PROPOSITION. *If $\{A_i\}_{i=1}^k$ is a finite closed cover or a finite open cover of a Hausdorff space X and $\{f_i\}_{i=1}^k$, where $f_i: A_i \to Y$, is a family of compatible perfect mappings, then the combination $f = f_1 \triangledown f_2 \triangledown \ldots \triangledown f_k$ is a perfect mapping of X to Y.* ■

3.7.9. THEOREM. *The Cartesian product $f = \prod_{s\in S} f_s$, where $f_s: X_s \to Y_s$ and $X_s \neq \emptyset$ for $s \in S$, is perfect if and only if all mappings f_s are perfect.*

PROOF. From Proposition 3.7.6 it follows that if the Cartesian product f is perfect, then all mappings f_s are perfect.

Assume now that all mappings f_s are perfect. By Theorem 2.3.11 the Cartesian product $\prod_{s\in S} X_s$ is a Hausdorff space and by the Tychonoff theorem the fiber $f^{-1}(y) = \prod_{s\in S} f_s^{-1}(y_s)$ is compact for every $y = \{y_s\} \in \prod_{s\in S} Y_s$; hence, it remains to show that f is a closed mapping.

Take a point $y = \{y_s\} \in \prod_{s\in S} Y_s$ and an open set $U \subset \prod_{s\in S} X_s$ which contains $f^{-1}(y) = \prod_{s\in S} f_s^{-1}(y_s)$. By the Wallace theorem there exist open sets $U_s \subset X_s$ such that $U_s \neq X_s$ only for $s \in \{s_1, s_2, \ldots, s_k\} \subset S$ and that $\prod_{s\in S} f_s^{-1}(y_s) \subset \prod_{s\in S} U_s \subset U$. Since f_{s_i} is a closed mapping, it follows from Theorem 1.4.13 that for $i = 1, 2, \ldots, k$ there exists a neighbourhood $V_{s_i} \subset Y_{s_i}$ of y_{s_i} such that $f_{s_i}^{-1}(V_{s_i}) \subset U_{s_i}$. The neighbourhood $V = \prod_{s\in S} V_s$ of the point y, where $V_s = Y_s$ for $s \notin \{s_1, s_2, \ldots, s_k\}$ satisfies the condition

$$f^{-1}(V) = \prod_{s\in S} f_s^{-1}(V_s) \subset \prod_{s\in S} U_s \subset U,$$

so that f is a closed mapping by Theorem 1.4.13. ■

From the above theorem on the Cartesian product of perfect mappings we shall deduce a series of important properties of this class of mappings. However, to give the reader an idea how strong that theorem is, let us first note that the Tychonoff theorem and the Wallace theorem (the latter under the additional assumption that all the X_s's are Hausdorff spaces) are its immediate consequences. Indeed, if $\{X_s\}_{s\in S}$ is a family of compact spaces, then the mapping $f_s: X_s \to Y_s$ of X_s to a one-point space Y_s is perfect for every $s \in S$, so that the mapping $f = \prod_{s\in S} f_s$ also is perfect and the Cartesian product $\prod_{s\in S} X_s = f^{-1}(y)$, where $\{y\} = \prod_{s\in S} Y_s$, is compact. To deduce the Wallace theorem from 3.7.9 it suffices to consider the Cartesian product of the natural quotient mappings $f_s: X_s \to X_s/A_s$ and apply Theorem 1.4.13.

Let us pass to properties of perfect mappings that follow from Theorem 3.7.9.

3.7.10. THEOREM. *The diagonal of any family of perfect mappings is a perfect mapping.*

PROOF. The diagonal can be represented as the restriction of the Cartesian product of mappings to a closed set. ■

Let us note that the fact that the diagonal $f_1 \triangle f_2$ is a perfect mapping does not imply that f_1 and f_2 are perfect (see 2.3.31).

It turns out that the last theorem can be significantly strengthened.

3.7.11. THEOREM. *Suppose we are given a family of continuous mappings $\{f_s\}_{s\in S}$, where $f_s: X \to Y_s$. If there exists an $s_0 \in S$ such that f_{s_0} is a perfect mapping and Y_s is a Hausdorff space for every $s \in S \setminus \{s_0\}$, then the diagonal $\triangle_{s\in S} f_s$ is a perfect mapping.*

PROOF. It suffices to consider the diagonal $h = f\triangle g$ of a perfect mapping $f: X \to Y$ and a continuous mapping $g: X \to Z$ to a Hausdorff space Z. The diagonal h can be represented as the composition

$$X \xrightarrow{\mathrm{id}_X \triangle g} X \times Z \xrightarrow{f\times \mathrm{id}_Z} Y \times Z.$$

The mapping $\mathrm{id}_X\triangle g$ is perfect by virtue of Corollary 2.3.22 and $f\times\mathrm{id}_Z$ is perfect by Theorem 3.7.9; thus it follows from Corollary 3.7.3 that h is a perfect mapping.■

Observe that the last theorem is in fact a strengthening of Theorem 3.7.10; indeed, if $f: X \to Y$ is a perfect mapping, then – as shown below in Theorem 3.7.20 – $f(X)$ is a Hausdorff space.

Theorem 3.7.9, along with Theorems 2.5.13 and 2.5.14, yields the next two theorems.

3.7.12. THEOREM. *If $\{\varphi, f_{\sigma'}\}$ is a mapping of an inverse system $\mathrm{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ to an inverse system $\mathrm{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$ and all mappings $f_{\sigma'}$ are perfect, then the limit mapping $\varprojlim\{\varphi, f_{\sigma'}\}$ also is a perfect mapping.*■

3.7.13. THEOREM. *If all bonding mappings π^σ_ρ of an inverse system $\mathrm{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ are perfect, then all projections also are perfect mappings.*■

The following interesting characterization of perfect mappings also is connected with Theorem 3.7.9.

3.7.14. THEOREM. *For a continuous mapping $f: X \to Z$ defined on a Hausdorff space X the following conditions are equivalent:*

(i) *The mapping f is perfect.*

(ii) *For every Hausdorff space Y the Cartesian product $f \times \mathrm{id}_Y$ is perfect.*

(iii) *For every Hausdorff space Y the Cartesian product $f \times \mathrm{id}_Y$ is closed.*

PROOF. Implications (i)⇒(ii) and (ii)⇒(iii) are obvious. By Proposition 2.3.27 to conclude the proof it suffices to show that from (iii) it follows that all fibers of f are compact. Let us take a $z_0 \in Z$ and an arbitrary Hausdorff space Y; the restriction $g_0 = g_{(\{z_0\}\times Y)}: f^{-1}(z_0) \times Y \to \{z_0\} \times Y$ of the closed mapping $g = f \times \mathrm{id}_Y$ is closed. Thus the composition $p_0 g_0$, where $p_0: \{z_0\} \times Y \to Y$ is the projection, also is closed. The last composition coincides with the projection $p: f^{-1}(z_0) \times Y \to Y$, so that – $f^{-1}(z_0) \subset X$ being a Hausdorff space – the compactness of $f^{-1}(z_0)$ follows from the Kuratowski theorem.■

3.7.15. REMARK. Let us observe that from Remark 3.1.17 it follows that a continuous mapping $f: X \to Z$ defined on a Hausdorff space X is perfect if and only if the Cartesian product $f \times \mathrm{id}_Y$ is closed for every compact space Y such that $w(Y) \le w(X)$ or – by Theorems 3.2.5 and 3.1.8 – if and only if the Cartesian product $f \times \mathrm{id}_{I^{w(X)}}$ is closed.

Clearly, the Cartesian product $f \times g$ of a perfect mapping f and an arbitrary closed mapping g is not necessarily closed (cf. Example 2.3.28).

In the realm of Tychonoff spaces the class of perfect mappings can be characterized in terms of extensions. We shall now give two characterizations of this kind; in the first of these characterizations we assume that the spaces X and Y are subspaces of their compactifications considered in the theorem.

3.7.16. THEOREM. *For a continuous mapping $f: X \to Y$, where X and Y are Tychonoff spaces, the following conditions are equivalent:*

(i) *The mapping f is perfect.*

(ii) *For every compactification αY the extension $F_\alpha: \beta X \to \alpha Y$ of the mapping f satisfies the conditions $F_\alpha(\beta X \setminus X) \subset \alpha Y \setminus Y$.*

(iii) *The extension $F: \beta X \to \beta Y$ of the mapping f satisfies the condition $F(\beta X \setminus X) \subset \beta Y \setminus Y$.*

(iv) *There exists a compactification αY such that the extension $F_\alpha: \beta X \to \alpha Y$ of the mapping f satisfies the condition $F_\alpha(\beta X \setminus X) \subset \alpha Y \setminus Y$.*

PROOF. Assume that the mapping $f: X \to Y$ is perfect and consider its extension $F_\alpha: \beta X \to \alpha Y$. Since f can be extended over $Z = F_\alpha^{-1}(Y)$ without changing the range, it follows from lemma 3.7.4 that $Z = X$, i.e., that $F_\alpha^{-1}(Y) \subset X$ and $F_\alpha(\beta X \setminus X) \subset \alpha Y \setminus Y$. Thus the implication (i)$\Rightarrow$(ii) is proved.

Implications (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(iv) are obvious; the implication (iv)$\Rightarrow$(i) follows from Proposition 3.7.6.∎

3.7.17. THEOREM. *A continuous mapping $f: X \to Y$, where X and Y are Tychonoff spaces, is perfect if and only if it cannot be continuously extended over any Hausdorff space Z that contains X as a dense proper subspace.*

PROOF. It suffices to observe that if $f: X \to Y$ is not perfect, then by the last theorem for the extension $F: \beta X \to \beta Y$ of the mapping f we have $F(\beta X \setminus X) \cap Y \neq \emptyset$, so that f is extendable over the space $Z = F^{-1}(Y)$ that contains X as a dense proper subspace.∎

The next theorem characterizes perfect mappings which take values in k-spaces.

3.7.18. THEOREM. *For a continuous mapping $f: X \to Y$ of a Hausdorff space X to a k-space Y the following conditions are equivalent:*

(i) *The mapping f is perfect.*

(ii) *For every compact subspace $Z \subset Y$ the restriction $f_Z: f^{-1}(Z) \to Z$ is perfect.*

(iii) *For every compact subspace $Z \subset Y$ the inverse image $f^{-1}(Z)$ is compact.*

PROOF. The implication (i)$\Rightarrow$(ii) follows from Proposition 3.7.6 and the implication (ii)$\Rightarrow$(iii) from Theorem 3.7.2. The implication (iii)$\Rightarrow$(i) is a consequence of Theorems 3.3.22 and 3.1.12.∎

We shall now study invariance and inverse invariance of topological properties under perfect mappings.

3.7.19. THEOREM. *If there exists a perfect mapping $f: X \to Y$ onto a space Y, then $w(Y) \leq w(X)$.*

PROOF. Let $w(X) = \mathfrak{m}$. Since the validity of our theorem is obvious for $\mathfrak{m} < \aleph_0$, we can assume that $\mathfrak{m} \geq \aleph_0$. Let $\{U_s\}_{s\in S}$ be a base for X such that $|S| = \mathfrak{m}$ and let $\mathcal{T}$ be the family of all finite subsets of S. Since $|\mathcal{T}| = \mathfrak{m}$ it suffices to show that the family $\{W_T\}_{T\in\mathcal{T}}$, where $W_T = Y \setminus f(X \setminus \bigcup_{s\in T} U_s)$, is a base for Y. It follows from the definition that the sets W_T are open. Let us take a point $y \in Y$ and a neighbourhood $W \subset Y$ of y. The inverse image $f^{-1}(y)$ is a compact subset of $f^{-1}(W)$; thus there exists a $T \in \mathcal{T}$ such that $f^{-1}(y) \subset \bigcup_{s\in T} U_s \subset f^{-1}(W)$. Clearly $y \in W_T$, and since

$$Y \setminus W = f(X \setminus f^{-1}(W)) \subset f(X \setminus \bigcup_{s\in T} U_s),$$

we have $W_T \subset W$.■

Let us observe that, as noted in the final paragraph of Example 3.1.26, perfect mappings can raise the character of spaces.

The last theorem implies that by identifying a finite family of pairwise disjoint compact subspaces of a Hausdorff space to points (cf. Example 2.4.12), we do not raise the weight; Example 1.4.17 shows that the assumption of compactness is essential.

3.7.20. THEOREM. *The class of T_i-spaces is invariant under perfect mappings for $i = 2, 3, 4, 5$ and* 6.

PROOF. By Theorem 1.5.20 and the fact, noted in Section 2.1, that the class of hereditarily normal spaces is invariant under closed mappings it suffices to prove the theorem for $i = 2$ and $i = 3$. The cases are similar to each other, and we shall consider only the case of $i = 2$.

Let $f: X \to Y$ be a perfect mapping of a T_2-space X onto a space Y and let y_1, y_2 be a pair of distinct points of Y. The inverse images $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are compact and disjoint; hence by Theorem 3.1.6 there exist open sets $U, V \subset X$ such that $f^{-1}(y_1) \subset U, f^{-1}(y_2) \subset V$ and $U \cap V = \emptyset$. The sets $Y \setminus f(X \setminus U)$ and $Y \setminus f(X \setminus V)$ are open in Y, the former contains y_1 and the latter contains y_2. Moreover,

$$\begin{aligned}[Y \setminus f(X \setminus U)] \cap [Y \setminus f(X \setminus V)] &= Y \setminus [f(X \setminus U) \cup f(X \setminus V)] \\ &= Y \setminus f[(X \setminus U) \cup (X \setminus V)] = Y \setminus f[X \setminus (U \cap V)] = Y \setminus f(X) = \emptyset.\blacksquare\end{aligned}$$

Let us note that complete regularity is not an invariant of perfect mappings (see Exercise 3.7.D).

Theorems 3.7.20 and 3.1.10 imply that compactness is an invariant of perfect mappings; similarly, Theorems 3.7.20 and 3.3.23 imply that the property of being a k-space is an invariant of perfect mappings.

The same holds for local compactness:

3.7.21. THEOREM. *Local compactness is an invariant of perfect mappings.*

PROOF. Let $f: X \to Y$ be a perfect mapping of a locally compact space X onto a space Y. By Theorem 3.3.2 for every $y \in Y$ there exists an open set $V \subset X$ such that $f^{-1}(y) \subset V$ and $\overline{V}$ is compact. The set $W = Y \setminus f(X \setminus V)$ is a neighbourhood of y and since

$$W = Y \setminus f(X \setminus V) \subset Y \setminus f(X \setminus \overline{V}) \subset f(\overline{V}),$$

the closure $\overline{W}$ is a compact subspace of Y.■

3.7.22. THEOREM. *Let $\mathcal{P}$ be a (finitely) additive topological property which is an invariant of perfect mappings. If a space X can be represented as the union of a locally finite (finite) family $\{X_s\}_{s\in S}$ of closed subspaces each of which is a Hausdorff space and has the property $\mathcal{P}$, then X also has the property $\mathcal{P}$.*

PROOF. The mapping $\nabla_{s\in S}\, i_{X_s}\colon \bigoplus_{s\in S} X_s \to X$ is perfect. ■

Let us now pass to a discussion of inverse invariants of perfect mappings.

3.7.23. THEOREM. *Regularity is an inverse invariant of perfect mappings.*

PROOF. Let $f\colon X \to Y$ be a perfect mapping onto a regular space Y. Take a point $x \in X$ and a closed set $F \subset X$ such that $x \notin F$. The set $F \cap f^{-1}f(x)$ is compact and does not contain x, so that by Theorem 3.1.6 there exist disjoint open sets $U_1, V_1 \subset X$ such that $x \in U_1$ and $F \cap f^{-1}f(x) \subset V_1$. The set $f(F \setminus V_1)$ is closed in Y and does not contain $f(x)$, hence – by regularity of Y – there exist disjoint open sets $U_2, V_2 \subset Y$ such that $f(x) \in U_2$ and $f(F\setminus V_1) \subset V_2$. The sets $U = U_1 \cap f^{-1}(U_2)$ and $V = V_1 \cup f^{-1}(V_2)$ are open in X, disjoint, and contain the point x and the set F respectively. ■

The remaining axioms of separation are not inverse invariants of perfect mappings. To check this for hereditary and perfect normality it suffices to map $I^{\mathfrak{c}}$ onto a one-point space (cf. Exercise 3.10.C(c)); for normality, see Problem 3.12.20(e) or Example 5.1.40, and for complete regularity, see the notes to this section.

3.7.24. THEOREM. *Compactness and local compactness are inverse invariants of perfect mappings.*

PROOF. The inverse invariance of compactness follows directly from Theorem 3.7.2. If $f\colon X \to Y$ is a perfect mapping and Y is a locally compact space, then for every $x \in X$ there exists a neighbourhood $U \subset X$ such that $f(U)$ is contained in a compact subspace Z of the space Y; since $f(\overline{U}) \subset \overline{f(U)} \subset Z$, the set $\overline{U} \subset f^{-1}(Z)$ is compact. ■

3.7.25. THEOREM. *If there exists a perfect mapping $f\colon X \to Y$ of X onto a k-space Y, then X is a k-space.*

PROOF. Let us consider spaces kX and kY and the mapping $kf\colon kX \to kY$ defined at the end of Section 3.3. Since Y is a k-space, we have $Y = kY$ and $kf = f\kappa_X$. From Theorem 3.7.18 it follows that kf is a perfect mapping, and this – along with Proposition 3.7.5 – implies that κ_X is perfect. As κ_X is a one-to-one mapping, it is a homeomorphism; therefore X is a k-space. ■

From Theorem 3.7.1 it follows that if a topological property $\mathcal{P}$ is an inverse invariant of perfect mappings, then the Cartesian product $X \times Y$ of a compact space X and a T_2-space Y which has the property $\mathcal{P}$, also has the property $\mathcal{P}$. Hence, theorems on inverse invariance of topological properties under perfect mappings are generalizations of corresponding theorems on Cartesian products. As shown in the next theorem, in the realm of Tychonoff spaces inverse invariance under perfect mappings and invariance under Cartesian multiplication by a compact space are equivalent for all topological properties which are hereditary with respect to closed sets.

3.7.26. THEOREM. *Let $\mathcal{P}$ be a topological property which is hereditary with respect to closed sets and invariant under Cartesian multiplication by a compact space. If there exists a perfect mapping $f: X \to Y$ of a Tychonoff space X to a space Y that has the property $\mathcal{P}$, then the space X also has the property $\mathcal{P}$.*

PROOF. By Theorem 3.2.6 there exists a homeomorphic embedding $g: X \to Z$ of the space X in a compact space Z. The diagonal $f \triangle g: X \to Y \times Z$ is both a homeomorphic embedding (see Theorem 2.3.20) and a perfect mapping (see Theorem 3.7.11). Hence, X is homeomorphic to a closed subspace of $Y \times Z$ and thus has the property $\mathcal{P}$.■

Let us observe that the property of being a Tychonoff space is hereditary with respect to closed sets and invariant under Cartesian multiplication by a compact space and yet is not an inverse invariant of perfect mappings. Thus in the last theorem the assumption that X is a Tychonoff space is essential.

3.7.27. THEOREM. *Let $\mathcal{P}$ be a topological property which is hereditary with respect to closed sets and finitely multiplicative. If for a space X there exists a one-to-one mapping $f: X \to Y$ to a Hausdorff space Y that has the property $\mathcal{P}$ and a perfect mapping $g: X \to Z$ to a space Z that has the property $\mathcal{P}$, then the space X also has the property $\mathcal{P}$.*

PROOF. The diagonal $f \triangle g: X \to Y \times Z$ is perfect and one-to-one, so that it is a homeomorphic embedding. Hence, X is a homeomorphic to a closed subspace of $Y \times Z$ and thus has the property $\mathcal{P}$.■

Straightforward examples show that neither first-countability nor second-countability are inverse invariants of perfect mappings (cf. Exercise 3.7.F). However, the last theorem yields

3.7.28. COROLLARY. *If $nw(X) \leq \mathfrak{m}$ and there exists a perfect mapping $f: X \to Y$ to a space Y such that $w(Y) \leq \mathfrak{m}$ (such that $\chi(Y) \leq \mathfrak{m}$), then $w(X) \leq \mathfrak{m}$ (then $\chi(X) \leq \mathfrak{m}$).*

PROOF. The validity of the corollary being obvious for $\mathfrak{m} < \aleph_0$, one can assume that $\mathfrak{m} \geq \aleph_0$. The properties "weight is $\leq \mathfrak{m}$" and "character is $\leq \mathfrak{m}$" are both hereditary and finitely multiplicative; furthermore X is a Hausdorff space, because a perfect mapping is defined on it. Hence the corollary follows from Lemma 3.1.18 and Theorem 3.7.27.■

3.7.29. THEOREM. *If a topological property $\mathcal{P}$ is an inverse invariant of perfect mappings and is hereditary with respect to open-and-closed sets, then $\mathcal{P}$ is hereditary with respect to closed subsets of Hausdorff spaces.*

PROOF. Let F be a closed subspace of a Hausdorff space X which has the property $\mathcal{P}$. The combination $i_F \triangledown \mathrm{id}_X$ is a perfect mapping of the sum $F \oplus X$ onto X. Since F is an open-and-closed subspace of $F \oplus X$, the space F has the property $\mathcal{P}$.■

Topological properties of Hausdorff spaces which are both invariants and inverse invariants of perfect mappings are called *perfect properties*; a class of all Hausdorff spaces that have a fixed perfect property is called a *perfect class of spaces*. From the theorems in this section it follows that classes of regular spaces, compact spaces, locally compact spaces and k-spaces are perfect.

Besides perfect mappings, a larger class of almost perfect mappings is sometimes considered. A continuous mapping $f: X \to Y$ is *almost perfect* if f is a closed mapping and all fibers $f^{-1}(y)$ are quasi-compact subsets of X. Hence, perfect mappings are almost perfect mappings that are defined on Hausdorff spaces. The reader can easily verify that 3.7.1–3.7.3, 3.7.6–3.7.9 and 3.7.14 remain valid, along with their proofs, when one replaces "perfect" by "almost perfect", "compact" by "quasi-compact" and "Hausdorff space" by "topological space". One can also readily verify that Proposition 3.7.5 and Theorem 3.7.11 remain valid for almost perfect mappings; however, the assumption that the mappings take values in Hausdorff spaces is essential in both of them (see Exercise 3.7.A(b)).

Considering mappings to one-point spaces, one easily observes that in theorems on inverse invariance of topological properties some assumptions about fibers are necessary. In this section it was shown that the assumption of compactness of fibers, along with the assumption that the mapping is closed, gives a series of inverse invariance theorems. It turns out that topological properties generally are not inverse invariants of open mappings with compact fibers (see Exercise 3.7.I), so that in the sequel, when discussing inverse invariance, we shall restrict ourselves to closed mappings with fibers satisfying various assumptions related to compactness.

Historical and bibliographic notes

The class of perfect mappings (in the realm of metric spaces) was first introduced by Vaĭnšteĭn in [1947]. Independently, perfect mappings were introduced and studied (in the realm of locally compact spaces) by Leray in [1950] and Bourbaki in [1951]; the last two authors defined this class of mappings by condition (iii) in Theorem 3.7.18. Theorem 3.7.2 was proved by Lubben in [1941] and Proposition 3.7.5 by Bourbaki in [1961]. The important Theorem 3.7.9 which shows that perfect mappings play among all continuous mappings, a role similar to that of compact spaces among all topological spaces, was proved by Frolík in [1960] and by Bourbaki in [1961]. Theorem 3.7.11 was proved by Arhangel'skiĭ in [1967a] (it was shown by Ponomarev in [1966] that the theorem holds in the realm of Tychonoff spaces); our proof is taken from Michael's paper [1971b]. Bourbaki in [1961] defined perfect mappings as mappings satisfying condition (iii) in Theorem 3.7.14 and proved the equivalence of all conditions in this theorem. Henriksen and Isbell's paper [1958] contains Theorem 3.7.16 (the implication (i)$\Rightarrow$(ii) was noted by Taĭmanov in (1955]) which is applied by these authors to prove invariance and inverse invariance under perfect mappings of many topological properties. Theorem 3.7.17 was stated by Isbell in [1962]. Kelley's book [1955] contains Theorems 3.7.19, 3.7.20 and 3.7.21 (Theorem 3.7.19 for $\mathfrak{m} = \aleph_0$ was proved by Whyburn in [1942]), Theorem 3.7.23 was proved by Henriksen and Isbell in [1958], Theorem 3.7.24 was, in principle, proved by Vaĭnšteĭn in [1952] (announcement in [1947]) and Theorem 3.7.25 was proved by Arhangel'skiĭ in [1965] (announcement in [1963]). Van der Slot's paper [1966] contains Theorem 3.7.26, Arhangel'skiĭ's paper [1967a] contains Theorem 3.7.27 and Henriksen and Isbell's paper [1958] contains Theorem 3.7.29. An example showing that complete regularity is not an inverse invariant of perfect mappings was given by Henriksen and Isbell in [1958]; Chaber in [1972] gave a simpler example, where the mapping is perfect and open.

Exercises

3.7.A. (a) Note that the diagonal of two almost perfect mappings is not necessarily closed.

(b) Check that the assumption that mappings take values in Hausdorff spaces is essential in Proposition 3.7.5 and Theorem 3.7.11.

(c) Show that a perfect mapping $f: X \to I$ may be continuously extendable over a T_1-space Y that contains X as a proper dense subspace.

3.7.B. (a) Verify that the sum $\bigoplus_{s\in S} f_s$ is perfect if and only if all mappings f_s are perfect.

(b) Prove that the mapping $f^*: X/E \to Y/E'$ defined in Exercise 2.4.B is perfect provided that both $f: X \to Y$ and $q': Y \to Y/E'$ are perfect and X/E is a Hausdorff space.

3.7.C. (a) Note that Theorem 3.7.21 and the second part of Theorem 3.7.24 follow from Theorems 3.6.1 and 3.7.16.

(b) Observe that if X and Y are locally compact, then a continuous mapping $f: X \to Y$ is perfect if and only if by assigning to the point in infinity of ωX the point in infinity of ωY one extends f to a continuous mapping.

3.7.D. Show that complete regularity is not an invariant of perfect mappings.

Hint. Consider a countable locally finite closed cover $\{M_i\}_{i=1}^{\infty}$ of the subspace M_0 of the space M in Example 1.5.9 and map onto M the space obtained by adjoining the point z_0 to the sum $\bigoplus_{i=1}^{\infty} M_i$.

3.7.E (Din' N'e T'ong [1963]). Let $f: X \to Y$ be a hereditarily quotient mapping with compact fibers defined on a Hausdorff space X. Prove that $w(Y) \le w(X)$ and that if X is locally compact and Y is a Hausdorff space, then Y also is locally compact.

3.7.F. (a) Prove that if $f: X \to Y$ is a closed mapping of a regular space X to a topological space Y and if for an $x \in X$ the inequalities $\chi(f(x), Y) \le \mathfrak{m}$ and $\chi(x, f^{-1}f(x)) \le \mathfrak{m}$ hold, then $\chi(x, X) \le \mathfrak{m}$.

(b) Note that if f is a perfect mapping, then the assumption of regularity of X can be omitted in part (a).

3.7.G (Čoban [1967]). (a) Observe that if there exists a perfect mapping of X onto Y, then $h(X) \le h(Y)$.

(b) Give an example of a perfect mapping $f: X \to Y$ of a first countable space X onto a space Y such that $h(Y) = 2^{\aleph_0}$.

Hint. Apply Exercise 3.1.I.

3.7.H (Hodel [1969a]). Let $\mathcal{P}$ be a topological property which is hereditary with respect to closed subsets and such that if a space X can be represented as the union of a locally finite family $\{X_s\}_{s\in S}$ of closed subspaces each of which has the property $\mathcal{P}$, then X also has the property $\mathcal{P}$ (cf. Theorem 3.7.22).

(a) Show that if a space X has an open cover $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$, where $\mathcal{U}_i$ is a locally finite family, such that $\overline{U}$ has the property $\mathcal{P}$ for every $U \in \mathcal{U}$, then the space X also has the property $\mathcal{P}$.

(b) Show that if a space X has an open cover $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$, where $\mathcal{U}_i$ is a locally finite family, such that every $U \in \mathcal{U}$ has the property $\mathcal{P}$ and can be represented as a countable union of open sets contained in U together with their closures (in particular, if every $U \in \mathcal{U}$ has the property $\mathcal{P}$ and is functionally open), then the space X also has the property $\mathcal{P}$.

(c) Show that if a Hausdorff space X has an open cover $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$, where $\mathcal{U}_i$ is a locally finite family, such that every $U \in \mathcal{U}$ has the property $\mathcal{P}$ and $\mathrm{Fr}\, U$ is compact for $U \in \mathcal{U}$, then the space X also has the property $\mathcal{P}$.

(d) Give an example of a non-regular Hausdorff space that can be represented as the union of two subspaces each of which is a normal second-countable space and one of which has a compact boundary.

3.7.I. Give an example of an open mapping with fibers homeomorphic to the interval I which transforms a non-normal completely regular space onto the interval I.

Hint. Restrict the mapping of the Niemytzki plane onto the real line defined in 1.4.15.

3.8. Lindelöf spaces

We say that a topological space X is a *Lindelöf space*, or has the *Lindelöf property*, if X is regular and every open cover of X has a countable subcover. Clearly, a regular space X is a Lindelöf space if and only if every open cover of X has a countable refinement. It follows from the definitions that every compact space is a Lindelöf space.

Theorem 1.1.14 gives

3.8.1. THEOREM. *Every regular second-countable space is a Lindelöf space.* ■

3.8.2. THEOREM. *Every Lindelöf space is normal.*

PROOF. By Lemma 1.5.15 it suffices to show that if X is a Lindelöf space, then for every closed set $F \subset X$ and every open $W \subset X$ that contains F there exists a sequence $W_1, W_2, \ldots$ of open subsets of X such that $F \subset \bigcup_{i=1}^{\infty} W_i$ and $\overline{W}_i \subset W$ for $i = 1, 2, \ldots$ By the regularity of X for any $x \in F$ there is an open U_x such that $x \in U_x \subset \overline{U}_x \subset W$. The open cover $(X \setminus F) \cup \{U_x\}_{x \in F}$ of the space X has a countable subcover $(X \setminus F) \cup \{U_{x_i}\}_{i=1}^{\infty}$. One easily sees that the sets $W_i = U_{x_i}$ satisfy the required conditions. ■

We say that a family $\mathcal{F} = \{F_s\}_{s \in S}$ of subsets of a set X has the *countable intersection property* if $\mathcal{F} \neq \emptyset$ and $\bigcap_{s \in S_0} F_s \neq \emptyset$ for every countable set $S_0 \subset S$.

The proofs of the following two theorems parallel those of Theorems 3.1.1 and 3.1.2; they are left to the reader.

3.8.3. THEOREM. *A regular space X is a Lindelöf space if and only if every family of closed subsets of X which has the countable intersection property has non-empty intersection.* ■

3.8.4. THEOREM. *Every closed subspace of a Lindelöf space is a Lindelöf space.* ■

One easily checks that if a subspace A of a topological space X has the Lindelöf property, then for every family $\{U_s\}_{s \in S}$ of open subsets of X such that $A \subset \bigcup_{s \in S} U_s$ there exists a countable set $\{s_1, s_2, \ldots\} \subset S$ such that $A \subset \bigcup_{i=1}^{\infty} U_{s_i}$; thus we have

3.8.5. THEOREM. *Every regular space which can be represented as a countable union of subspaces each of which has the Lindelöf property has itself the Lindelöf property.* ■

In particular, every regular space which can be represented as a countable union of compact subspaces (Hausdorff spaces with this property are called *σ-compact spaces*) has the Lindelöf property and is therefore normal.

The proof of the next theorem is left to the reader.

3.8.6. THEOREM. *The sum $\bigoplus_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, has the Lindelöf property if and only if all spaces X_s have the Lindelöf property and the set S is countable.* ■

The following two theorems are obtained by straightforward modifications in proofs of Theorems 3.1.10 and 3.7.2.

3.8.7. THEOREM. *If there exists a continuous mapping $f: X \to Y$ of a Lindelöf space X onto a regular space Y, then Y is a Lindelöf space.* ■

3.8.8. THEOREM. *If $f: X \to Y$ is a closed mapping defined on a regular space X and all fibers $f^{-1}(y)$ have the Lindelöf property, then for every subspace $Z \subset Y$ that has the Lindelöf property the inverse image $f^{-1}(Z)$ also has the Lindelöf property.* ■

3.8.9. THEOREM. *The class of Lindelöf spaces is perfect.*

PROOF. Invariance of the Lindelöf property under perfect mappings follows from Theorems 3.7.20 and 3.8.7; inverse invariance follows from Theorems 3.7.23 and 3.8.8. ■

3.8.10. COROLLARY. *The Cartesian product $X \times Y$ of a Lindelöf space X and a compact space Y is a Lindelöf space.* ■

In the next theorem an important property of Lindelöf spaces is established.

3.8.11. THEOREM. *Every open cover of a Lindelöf space has a locally finite open refinement.*

PROOF. Let $\mathcal{U}$ be an open cover of a Lindelöf space X. The space X being regular, for every $x \in X$ there exist open sets $U_x, V_x \subset X$ such that $x \in U_x \subset \overline{U}_x \subset V_x$ and V_x is contained in a member of $\mathcal{U}$. Let $\{U_{x_i}\}_{i=1}^{\infty}$ be a countable subcover of the cover $\{U_x\}_{x\in X}$ of the space X. The sets

$$W_i = V_{x_i} \setminus (\overline{U}_{x_1} \cup \overline{U}_{x_2} \cup \ldots \cup \overline{U}_{x_{i-1}}), \quad \text{where} \quad i = 1, 2, \ldots,$$

are open and constitute a cover of X. Indeed, for any $x \in X$ we have $x \in W_{i(x)}$, where $i(x)$ is the smallest integer i satisfying $x \in V_{x_i}$. The cover $\{W_i\}_{i=1}^{\infty}$ is a refinement of $\mathcal{U}$ and is locally finite because $U_{x_j} \cap W_i = \emptyset$ for $i > j$. ■

The notion of a Lindelöf space leads to the concept of the Lindelöf number: the smallest cardinal number $\mathfrak{m}$ such that every open cover of a space X has an open refinement of cardinality $\leq \mathfrak{m}$ is called the *Lindelöf number* of the space X and is denoted by $l(X)$. Thus a regular space X has the Lindelöf property if and only if $l(X) \leq \aleph_0$.

Remark 1.1.16 gives at once

3.8.12. THEOREM. *For every topological space X we have $l(X) \leq nw(X)$.* ■

3.8.13. EXAMPLES. The space $A(\mathfrak{m})$ for $\mathfrak{m} > \aleph_0$ is a Lindelöf space which is not separable.

The Niemytzki plane is a separable space which is not a Lindelöf space (an example of a normal space with similar properties is given in Exercise 3.8.E).

Since every countable regular space has the Lindelöf property, it follows from 3.3.24 that there exist Lindelöf spaces which are not k-spaces. The reader can easily check that the subspace $N \cup \{x\}$ of βN, where $x \in \beta N \setminus N$, is another example of such a space.■

3.8.14. EXAMPLE. We shall now show that the Sorgenfrey line K is a Lindelöf space. Let $\mathcal{U} = \{U_s\}_{s\in S}$ be an open cover of K and let V_s denote the interior of U_s with respect to the topology of the real line. One easily sees that the set $A = K \setminus \bigcup_{s\in S} V_s$ is countable. By virtue of Theorem 1.1.14 we have $\bigcup_{s\in S} V_s = \bigcup_{s\in S_0} V_s \subset \bigcup_{s\in S_0} U_s$, where $|S_0| \le \aleph_0$; adjoining to the family $\{U_s\}_{s\in S_0}$ countably many members of $\mathcal{U}$ whose union contains A we obtain a countable subcover of $\mathcal{U}$.

The space K is an example of a separable first-countable space with the Lindelöf property which is not second-countable.■

3.8.15. EXAMPLE. In Example 2.3.12 it was shown that the Cartesian product $K \times K$ is not normal. Applying Theorem 3.8.2, we infer that the Cartesian product of two Lindelöf spaces is not necessarily a Lindelöf space (cf. Exercises 3.8.G and 3.9.F(a)).■

Let us mention that one can define a space X all of whose finite powers are Lindelöf spaces and yet $X^{\aleph_0}$ is not normal. This implies, in particular, that the limit of an inverse sequence of Lindelöf spaces need not be a Lindelöf space (the last fact can be established in an easier way; cf. Problem 5.5.4(c)).

Historical and bibliographic notes

The notion of a Lindelöf space was introduced by Alexandroff and Urysohn in [1929] (Lindelöf proved in [1903] that any family of open subsets of R^n contains a countable subfamily with the same union). Theorem 3.8.9 was proved by Henriksen and Isbell in [1958]. Theorem 3.8.11 was proved by Morita in [1948] (under the additional assumptions of local compactness or metrizability by Dieudonné in [1944]). Examples 3.8.14 and 3.8.15 were given by Sorgenfrey in [1947]. The space X mentioned at the end of the section was defined by Przymusiński in [1980]. Earlier such a space was defined by Michael in [1971] under the assumption of the continuum hypothesis.

Exercises

3.8.A. (a) Note that the Lindelöf property is hereditary with respect to F_σ-sets.

(b) Observe that X is a hereditarily Lindelöf space if and only if all open subspaces of X have the Lindelöf property.

(c) Show that a Lindelöf space X is hereditarily Lindelöf if and only if X is perfectly normal; note that the Sorgenfrey line is hereditarily Lindelöf.

3.8.B (Smirnov [1950]). Show that if A and B are disjoint closed subsets of a regular space X which both have the Lindelöf property, then there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

3.8.C. (a) Observe that every hemicompact space is σ-compact, but not necessarily vice versa (cf. Exercise 3.4.E).

(b) Prove that for every locally compact space X the following conditions are equivalent:

(1) *The space X has the Lindelöf property.*

(2) *The space X is hemicompact.*

(3) *The space X is σ-compact.*

(4) *There exists a sequence $A_1, A_2, \ldots$ of compact subspaces of the space X such that $A_i \subset \operatorname{Int} A_{i+1}$ and $X = \bigcup_{i=1}^{\infty} A_i$.*

(5) *The space X is compact or $\chi(\Omega, \omega X) = \aleph_0$.*

3.8.D (M. E. Rudin and Klee [1956], Michael [1961]). Prove that if X and Y are second-countable spaces and Y is regular, then the space Y^X is hereditarily Lindelöf with respect to both compact-open topology and the topology of pointwise convergence (cf. Problem 5.5.13).

Hint. Apply Exercise 3.4.H(a).

3.8.E (Engelking [1968]). Let Y be a subspace of the Tychonoff cube $T = I^{\mathfrak{c}}$ which is homeomorphic to the space W_0 of all countable ordinals and let C be a countable dense subset of T. Consider the space $A(T)$ defined in Exercise 3.1.G and its subspace $X = Y_1 \cup C_2$. Prove that X is a normal separable space which is not a Lindelöf space (cf. Problem 3.12.17(c)).

Hint. If A and B are disjoint closed subsets of W_0, then one of them is compact.

Remark. First examples of spaces with similar properties were given by M. E. Rudin in [1956] and McAuley in [1956a].

3.8.F (Henriksen, Isbell and Johnson [1961]). (a) Prove that if for a subspace X of a compact space Z there exists a countable family $\{F_i\}_{i=1}^{\infty}$ of closed subsets of Z with the property that for every pair of points x, y such that $x \in X$ and $y \in Z \setminus X$ we have $x \in F_i$ and $y \notin F_i$ for some i, then X is a Lindelöf space.

Hint. Let $\{U_s\}_{s \in S}$ be an open cover of the space X. The sets $V_s = Z \setminus \overline{X \setminus U_s}$ are open in Z and $X \subset V = \bigcup_{s \in S} V_s$. Show that X is contained in the union of all finite intersections $F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_k}$ disjoint from $Z \setminus V$.

(b) Let X be a compact space and $\mathcal{L}$ the smallest family of subspaces of X that contains all closed sets and is closed with respect to countable unions and intersections. Prove that all members of $\mathcal{L}$ have the Lindelöf property.

3.8.G (Hager [1969]). Prove that the Cartesian product of countably many regular σ-compact spaces is a Lindelöf space.

Hint. Apply Exercise 3.8.F(a).

One can also observe that every regular σ-compact space is a continuous image of a closed subspace of the Cartesian product of $D(\aleph_0)$ and a compact space, and apply 3.2.4, 2.3.14, 3.8.1, 3.8.10 and 3.8.7.

3.8.H. Let $\mathcal{P}$ be a topological property which is hereditary with respect to closed subsets and such that if a space X can be represented as the union of a locally finite family $\{X_s\}_{s \in S}$ of closed subspaces each of which has the property $\mathcal{P}$, then X also has the property $\mathcal{P}$ (cf. Theorem 3.7.22 and Exercise 3.7.H).

(a) Show that if a regular space X has a locally finite open cover $\mathcal{U}$ such that every $U \in \mathcal{U}$ has the property $\mathcal{P}$ and $\operatorname{Fr} U$ has the Lindelöf property for $U \in \mathcal{U}$, then the space X also has the property $\mathcal{P}$.

Hint. Apply Exercise 3.8.B.

(b) Give an example of a non-normal Tychonoff space that can be represented as a countable union of open subspaces each of which is normal and has Lindelöf boundary.

(c) Show that if a normal space X has an open cover $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$, where $\mathcal{U}_i$ is a locally finite family, such that every $U \in \mathcal{U}$ has the property $\mathcal{P}$ and $\operatorname{Fr} U$ has the Lindelöf property for $U \in \mathcal{U}$, then the space X also has the property $\mathcal{P}$.

3.9. Čech-complete spaces

The following theorem, analogous to Theorem 3.5.8, will be the basis for our next definition.

3.9.1. THEOREM. *For every Tychonoff space X the following conditions are equivalent:*

(i) *For every compactification cX of the space X the remainder $cX \setminus c(X)$ is an F_σ-set in cX.*

(ii) *The remainder $\beta X \setminus \beta(X)$ is an F_σ-set in βX.*

(iii) *There exists a compactification cX of the space X such that the remainder $cX \setminus c(X)$ is an F_σ-set in cX.*

PROOF. Implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are obvious, so that it suffices to prove that (iii)$\Rightarrow$(i).

First we shall show that (iii)$\Rightarrow$(ii). It follows from the maximality of βX that there exists a continuous mapping $f: \beta X \to cX$ such that $f\beta = c$. By Theorem 3.5.7 we have $f^{-1}(cX \setminus c(X)) = \beta X \setminus \beta(X)$; hence, as $cX \setminus c(X)$ is an F_σ-set in cX, the remainder $\beta X \setminus \beta(X)$ is an F_σ-set in βX.

Now we shall show that (ii)$\Rightarrow$(i). Let $\beta X \setminus \beta(X) = \bigcup_{i=1}^{\infty} F_i$, where F_i are closed subsets of βX. Consider a compactification cX of the space X and a mapping $f: \beta X \to cX$ satisfying $f\beta = c$. By Theorem 3.5.7 we have $cX \setminus c(X) = \bigcup_{i=1}^{\infty} f(F_i)$ and since the sets $f(F_i)$ are closed in cX, the remainder is an F_σ-set in cX.■

A topological space X is *Čech-complete* if X is a Tychonoff space and satisfies condition (i), and hence all the conditions, in Theorem 3.9.1.

Note that every compact space is Čech-complete. Locally compact spaces are also Čech-complete, because a non-compact locally compact space has a compactification with one-point remainder. The space of all irrational numbers with the topology of a subspace of the real line is an example of a Čech-complete space that is not locally compact.

Our definition of Čech-complete spaces is an external definition; it characterizes Čech-complete spaces by their relations to other topological spaces, viz., their compactifications. We shall now establish an internal characterization of Čech-complete spaces.

To begin, we introduce an auxiliary concept. We shall say that the *diameter of a subset* A of a topological space X *is less than a cover* $\mathcal{A} = \{A_s\}_{s \in S}$ of the space X, and we shall write $\delta(A) < \mathcal{A}$, provided that there exists an $s \in S$ such that $A \subset A_s$.

3.9.2. THEOREM. *A Tychonoff space X is Čech-complete if and only if there exists a countable family $\{\mathcal{A}_i\}_{i=1}^{\infty}$ of open covers of the space X with the property that any family $\mathcal{F}$ of closed subsets of X, which has the finite intersection property and contains sets of diameter less than $\mathcal{A}_i$ for $i = 1, 2, \ldots$, has non-empty intersection.*

PROOF. Let us assume that a Tychnoff space $X \subset \beta X$ has a family $\{\mathcal{A}_i\}_{i=1}^{\infty}$ of open covers with the required property. Let $\mathcal{A}_i = \{U_{s,i}\}_{s\in S_i}$ for $i = 1,2,\ldots$ and let $V_{s,i}$ be an open subset of βX such that $U_{s,i} = X \cap V_{s,i}$ for $s \in S_i$ and $i = 1,2,\ldots$ Clearly

$$X \subset \bigcap_{i=1}^{\infty} \bigcup_{s\in S_i} V_{s,i};$$

to prove that X is Čech-complete it suffices to show that the reverse inclusion also holds.

Take a point $x \in \bigcap_{i=1}^{\infty} \bigcup_{s\in S_i} V_{s,i}$ and let $\mathcal{B}(x)$ be the family of all neighbourhoods of x in βX. The family $\mathcal{F} = \{X \cap \overline{V} : V \in \mathcal{B}(x)\}$, where $\overline{V}$ is the closure of V in βX, consists of closed subsets of the space X and has the finite intersection property. Since for every i there exists an $s \in S_i$ such that $x \in V_{s,i}$, it follows from regularity of βX that the family $\mathcal{F}$ contains sets of diameter less than $\mathcal{A}_i$ for $i = 1,2,\ldots$ By our assumption we have $X \cap \bigcap\{\overline{V} : V \in \mathcal{B}(x)\} \neq \emptyset$; as $\bigcap\{\overline{V} : V \in \mathcal{B}(x)\} = \{x\}$, it follows that $x \in X$.

Let us now consider a Čech-complete space $X \subset \beta X$; thus X is a G_δ-set in βX and there exists a family $\{G_i\}_{i=1}^{\infty}$ of open subsets of βX such that $X = \bigcap_{i=1}^{\infty} G_i$. For every $x \in X$ and $i = 1,2,\ldots$ choose an open set $V_{x,i} \subset \beta X$ such that $x \in V_{x,i} \subset \overline{V}_{x,i} \subset G_i$ and let $\mathcal{A}_i = \{X \cap V_{x,i}\}_{x\in X}$. We shall show that the family $\{\mathcal{A}_i\}_{i=1}^{\infty}$ of open covers of the space X has the required property.

Consider a family $\{F_s\}_{s\in S}$ of closed subsets of X which has the finite intersection property and contains sets of diameter less then $\mathcal{A}_i$ for $i = 1,2,\ldots$ As the family $\{\overline{F}_s\}_{s\in S}$ consists of closed subsets of βX and has the finite intersection property, there exists a point $x \in \bigcap_{s\in S} \overline{F}_s$; to prove that $x \in \bigcap_{s\in S} F_s$ it suffices to show that $x \in X$.

For every positive integer i choose an $s_i \in S$ such that $\delta(F_{s_i}) < \mathcal{A}_i$ and an $x_i \in X$ such that $F_{s_i} \subset X \cap V_{x_i,i}$. Since

$$x \in \overline{F}_{s_i} \subset \overline{X \cap V}_{x_i,i} \subset \overline{V}_{x_i,i} \subset G_i$$

for $i = 1,2,\ldots$, we have $x \in \bigcap_{i=1}^{\infty} G_i = X$. ■

The next theorem is important for its various applications; its name is due to the fact that countable unions of nowhere dense sets are sometimes called *first category sets.*

3.9.3. THE BAIRE CATEGORY THEOREM. *In a Čech-complete space X the union $A = \bigcup_{i=1}^{\infty} A_i$ of a sequence $A_1, A_2, \ldots$ of nowhere dense sets is a co-dense set, i.e., the complement $X \setminus A$ is dense in X.*

PROOF. We shall show that the set $G \setminus A$ is non-empty for every non-empty open set $G \subset X$.

Let $\{\mathcal{A}_i\}_{i=1}^{\infty}$ be a family of open covers of the space X that has the property stated in Theorem 3.9.2. Since the set A_1 is nowhere dense, there exist a point $x \in G \setminus \overline{A}_1$ and a neighbourhood G_1 of x such that

$$\overline{G}_1 \subset G \setminus \overline{A}_1 \quad \text{and} \quad \delta(\overline{G}_1) < \mathcal{A}_1.$$

Similarly, applying the fact that $G_1 \setminus \overline{A}_2$ is a non-empty open set, one obtains a non-empty open set G_2 such that

$$\overline{G}_2 \subset G_1 \setminus \overline{A}_2 \quad \text{and} \quad \delta(\overline{G}_2) < \mathcal{A}_2.$$

By induction one defines a sequence $G_1, G_2, \ldots$ of non-empty open subsets of X satisfying $G \supset \overline{G}_1 \supset \overline{G}_2 \supset \ldots$, $\overline{G}_i \cap A_i = \emptyset$ and $\delta(\overline{G}_i) < \mathcal{A}_i$ for $i = 1, 2, \ldots$

One easily checks that $\emptyset \neq \bigcap_{i=1}^{\infty} \overline{G}_i \subset G \setminus A$. ■

The following corollary is a dual version of the Baire category theorem.

3.9.4. COROLLARY. *In a Čech-complete space X the intersection $G = \bigcap_{i=1}^{\infty} G_i$ of a sequence $G_1, G_2, \ldots$ of dense open subsets is a dense set.* ■

If the Baire category theorem holds in a topological space X, i.e., if for each sequence $A_1, A_2, \ldots$ of nowhere dense subsets of X the union $\bigcup_{i=1}^{\infty} A_i$ is a co-dense set (or – equivalently – if for each sequence $G_1, G_2, \ldots$ of open dense subsets of X the intersection $\bigcap_{i=1}^{\infty} G_i$ is a dense set), we say that X is a *Baire space* (see Exercises 3.9.I, 3.9.J, 3.10.F(e) and 4.3.C(b)).

3.9.5. THEOREM. *Every Čech-complete space is a k-space.*

PROOF. Let X be a Čech-complete space and let $\{G_i\}_{i=1}^{\infty}$ be a family of open subsets of βX such that $X = \bigcap_{i=1}^{\infty} G_i$. Consider a set $A \subset X$ such that intersections of A with all compact subspaces of X are closed in X and assume that A is not closed in X, i.e., that there exists a point $x \in X \cap (\overline{A} \setminus A)$, where the bar denotes the closure in βX. Let $U_0 = \beta X, U_1, U_2, \ldots$ be a sequence of neighbourhoods of x in βX satisfying $\overline{U}_i \subset U_{i-1} \cap G_i$ for $i = 1, 2, \ldots$ The intersection $Z = \bigcap_{i=1}^{\infty} \overline{U}_i \subset X$ is compact, and so is the set $A \cap Z$; since $x \notin A \cap Z$, there exists a neighbourhood $V \subset \beta X$ of the point x such that $\overline{V} \cap A \cap Z = \emptyset$.

The neighbourhood $V \cap U_i$ of the point x intersects the set A; let us choose a point $x_i \in A \cap V \cap U_i$ for $i = 1, 2, \ldots$ and consider the set $B = \{x_1, x_2, \ldots\}$. From Corollary 3.1.5 it follows that

(1) if U is open in βX and $Z \subset U$, then for some i we have $U_i \subset U$,

and this implies that the set $Z \cup B$ is compact; thus the intersection $A \cap [\overline{V} \cap (Z \cup B)]$ also is compact. Since $B \subset A \cap V$, we have

$$A \cap [\overline{V} \cap (Z \cup B)] = (\overline{V} \cap A \cap Z) \cup (A \cap \overline{V} \cap B) = B,$$

which implies that $U = \beta X \setminus B$ is open in βX and $Z \subset U$. By virtue of (1) there exists an i such that $U_i \cap B = \emptyset$, which is impossible because $x_i \in U_i \cap B$. The contradiction shows that A is closed, i.e., that X is a k-space. ■

3.9.6. THEOREM. *Čech-completeness is hereditary with respect to closed subsets and with respect to G_δ-subsets.*

PROOF. The first part of the theorem follows from Theorem 3.9.2. The second part follows from the observation that if X is a G_δ-set in βX and A is a G_δ-set in X, then A is a G_δ-set in βX and, *a fortiori*, in the closure $\overline{A} \subset \beta X$ which is a compactification of A. ■

3.9.7. THEOREM. *The sum $\bigoplus_{s \in S} X_s$ is Čech-complete if and only if all spaces X_s are Čech-complete.*

PROOF. If the sum $\bigoplus_{s \in S} X_s$ is Čech-complete, then all spaces X_s are Čech-complete by 2.2.2 and 3.9.6.

Assume now that all spaces X_s are Čech-complete and take for every $s \in S$ a compactification c_sX_s of the space X_s; thus $c_sX_s \setminus X_s = \bigcup_{i=1}^{\infty} F_{s,i}$, where $F_{s,i}$ are closed subsets of c_sX_s for $s \in S$ and $i = 1, 2, \ldots$ By 3.3.12 the sum $X = \bigoplus_{s\in S} c_sX_s$ is locally compact, and thus Čech-complete; since $X \setminus \bigoplus_{s\in S} X_s = \bigcup_{i=1}^{\infty}(\bigcup_{s\in S} F_{s,i})$, and the sets $\bigcup_{s\in S} F_{s,i}$ are closed in X for $i = 1, 2, \ldots$, the sum $\bigoplus_{s\in S} X_s$ is Čech-complete by virtue of 3.9.6.■

3.9.8. THEOREM. *The Cartesian product of countably many Čech-complete spaces is Čech-complete.*

PROOF. Let $\{X_i\}_{i=1}^{\infty}$ be a family of Čech-complete spaces. Take for $i = 1, 2, \ldots$ a compactification c_iX_i of the space X_i; thus $c_iX_i \setminus X_i$ is an F_σ-set in c_iX_i for $i = 1, 2, \ldots$ By 2.3.5 and 3.2.4 the space $\prod_{i=1}^{\infty} c_iX_i$ is a compactification of the Cartesian product $\prod_{i=1}^{\infty} X_i$. The set $F_j = \prod_{i=1}^{\infty} A_{i,j}$, where $A_{j,j} = c_jX_j \setminus X_j$ and $A_{i,j} = c_iX_i$ whenever $i \neq j$, is an F_σ-set in $\prod_{i=1}^{\infty} c_iX_i$ for $j = 1, 2, \ldots$ Since $\prod_{i=1}^{\infty} c_iX_i \setminus \prod_{i=1}^{\infty} X_i = \bigcup_{j=1}^{\infty} F_j$, the Cartesian product $\prod_{i=1}^{\infty} X_i$ is Čech-complete.■

3.9.9. COROLLARY. *The limit of an inverse sequence of Čech-complete spaces is a Čech-complete space.*■

From Exercise 3.9.D(a) it follows that the assumption of countability is essential in the last theorem, and that the corollary does not hold for arbitrary inverse systems.

From the definition of Čech-complete spaces and Theorem 3.7.16 we obtain

3.9.10. THEOREM. *If X and Y are Tychonoff spaces and there exists a perfect mapping of X onto Y, then X is Čech-complete if and only if Y is Čech-complete.*■

On the other hand, Čech-completeness is not an invariant either of closed mappings or of open mappings onto Tychonoff spaces (see Exercise 3.9.D(b) and Problems 3.12.19(d) and 5.5.8(b)).

Let us note that the assumption that X and Y are Tychonoff spaces is essential in the last theorem. In fact, a perfect mapping can transform a Čech-complete space onto a space which is not a Tychonoff space (see Exercise 3.9.D(c)); there are also examples of open perfect mappings transforming a space which is not a Tychonoff space onto a Čech-complete space. Thus, the class of Čech-complete spaces is only "perfect in the realm of Tychonoff spaces".

Historical and bibliographic notes

Čech-complete spaces were defined in Čech's paper [1937]; the paper also contains a proof of the Baire category theorem, known earlier for the narrower class of all completely metrizable spaces (see notes to Section 4.3). The internal characterization of Čech-complete spaces in Theorem 3.9.2 was established independently by Frolík in [1960b] and by Arhangel'skiĭ in [1961]; a similar characterization was given by Šanin in [1943a]. Baire spaces were introduced by Bourbaki in [1948]. Theorem 3.9.5 was proved by Arhangel'skiĭ in [1965] and Theorem 3.9.10 by Henriksen and Isbell in [1958]. An example of an open perfect mapping transforming a space which is not a Tychonoff space onto a normal Čech-complete space was given by Chaber in [1972].

Exercises

3.9.A. Show that if a Čech-complete space X is a subspace of a Hausdorff space Y, then there exists a G_δ-set $Z \subset Y$ such that $X = \overline{X} \cap Z$.

Deduce that a subspace M of a Čech-complete space X is Čech-complete if and only if it can be represented in the form $F \cap Z$, where F is closed in X and Z is a G_δ-set in X.

3.9.B. (a) Show that the space of all rational numbers with the topology of a subspace of the real line is not Čech-complete. Deduce that the set of all irrational numbers is not an F_σ-set in the real line.

(b) Applying Theorem 3.9.2, show that the Niemytzki plane is Čech-complete.

Hint. Observe first that the subspace $L_2 \subset L$ is Čech-complete.

3.9.C. Prove the Baire category theorem applying the definition of Čech-complete spaces.

3.9.D. (a) Observe that the Cartesian product $N^{\aleph_1}$ is not Čech-complete.

Hint. Cf. Exercise 3.3.E(a).

(b) Note that every Čech-complete space is of pointwise countable type and deduce that Čech-completeness is not an invariant of closed mappings onto Tychonoff spaces.

(c) Show that Čech-completeness is not an invariant of perfect mappings.

Hint. See Exercise 3.7.D.

3.9.E (Arhangel'skiĭ [1960a], Arhangel'skiĭ and Holsztyński [1963]). For a Tychonoff space X we denote by $g(X)$ the smallest cardinal number $\mathfrak{m} \geq \aleph_0$ with the property that there exists a compact space Y containing X and a family $\mathcal{U}$ of open subsets of Y such that $|\mathcal{U}| = \mathfrak{m}$ and for any $x \in X$ and $y \in Y \setminus X$ there exists a $U \in \mathcal{U}$ satisfying $x \in U$ and $y \notin U$.

(a) Observe that $g(X) = \aleph_0$ for every Čech-complete space X. Show that $h(X) \leq g(X)$ for every Tychonoff space X (see Exercise 3.1.F) and give an example of a perfectly normal space X such that $h(X) < g(X)$.

(b) An *external base for a subspace* X of a space Y is a family $\mathcal{B}$ of open subsets of Y with the property that for every $x \in X$ and any neighbourhood U of x in the space Y there exists a $V \in \mathcal{B}$ such that $x \in V \subset U$. Prove that for every subspace X of a compact space Y and any cardinal number $\mathfrak{m} \geq \aleph_0$ the following conditions are equivalent:

(1) *The network weight of X is $\leq \mathfrak{m}$ and there exists a family $\mathcal{U}$ of open subsets of Y such that $|\mathcal{U}| \leq \mathfrak{m}$ and for any $x \in X$ and $y \in Y \setminus X$ one can find a $U \in \mathcal{B}$ satisfying $x \in U$ and $y \notin U$.*

(2) *There exists a family $\mathcal{F}$ of closed subsets of Y such that $|\mathcal{F}| \leq \mathfrak{m}$ and for any pair of distinct points $x_1 \in X, x_2 \in Y$ one can find disjoint sets $F_1, F_2 \in \mathcal{F}$ satisfying $x_1 \in F_1$ and $x_2 \in F_2$.*

(3) *There exists a family $\mathcal{V}$ of open subsets of Y such that $|\mathcal{V}| \leq \mathfrak{m}$ and for any pair of distinct points $x_1 \in X, x_2 \in Y$ one can find disjoint sets $V_1, V_2 \in \mathcal{V}$ satisfying $x_1 \in V_1$ and $x_2 \in V_2$.*

(4) *There exists an external base $\mathcal{B}$ for the subspace X of the space Y such that $|\mathcal{B}| \leq \mathfrak{m}$.*

(c) Deduce from (b) that $w(X) = nw(X)g(X)$ for every infinite Tychonoff space X and that $w(X) = nw(X)$ for every Čech-complete space X.

(d) Show that if a space X is embeddable in a perfectly normal compact space, then $w(X) = nw(X)$.

3.9.F. (a) (Frolík [1960], Engelking [1966]) Prove that the Cartesian product of countably many Čech-complete Lindelöf spaces is a Čech-complete Lindelöf space (cf. Problem 5.5.9(b)).

Hint (Hager [1969]). Apply Exercise 3.8.F(a).

One can also observe that every Čech-complete Lindelöf space is homeomorphic to a closed subspace of the Cartesian product of countably many regular σ-compact spaces and apply Exercise 3.8.G.

(b) Deduce from part (a) that the Sorgenfrey line is not a Čech-complete space (cf. Exercise 3.10.C(b)).

3.9.G (Zenor [1970]). Observe that a space is Čech-complete if and only if it is the limit of an inverse sequence of locally compact spaces.

3.9.H (Michael [1963a]; for metric spaces, Arens [1958]). Let $\mathrm{S} = \{X_i, \pi_j^i\}$ be an inverse sequence of non-empty Čech-complete spaces such that $\pi_j^i(X_i)$ is dense in X_j for any i, j satisfying $j \leq i$. Show that for every i the projection $\pi_i(X)$ of the limit $X = \varprojlim \mathrm{S}$ is dense in X_i, so that, in particular, the limit X is non-empty.

Hint. Consider the inverse sequence $\tilde{\mathrm{S}} = \{\tilde{X}_i, \tilde{\pi}_j^i\}$, where $\tilde{X}_i = \beta X_i$ and $\tilde{\pi}_j^i$ is the extension of π_j^i over βX_i and βX_j. Let $G_i = \tilde{\pi}_i^{-1}(X_i)$, where $\tilde{\pi}_i$ is the projection of the limit $\tilde{X} = \varprojlim \tilde{\mathrm{S}}$ to $\tilde{X}_i$; show that all the G_i's are dense G_δ-sets in $\tilde{X}$ and apply Corollary 3.9.4.

3.9.I (Fogelgren and McCoy [1971], Fletcher and Lindgren [1973]). Prove that a topological space X is a Baire space if and only if for every point-finite open cover of X (or – equivalently – for every countable point-finite open cover of X) the set of all points $x \in X$ which have a neighbourhood that meets only finitely many members of the cover is dense in X.

Hint. See Exercise 1.5.N.

3.9.J. (a) Note that every open subspace and every dense G_δ-subspace of a Baire space is a Baire space.

(b) Show that if a topological space X contains a dense subspace that is a Baire space, then X is a Baire space. Observe that a closed subspace of a Baire space is not necessarily a Baire space.

(c) (Kuratowski and Ulam [1932]) Prove that the Cartesian product $X \times Y$ of a Baire space X and a second-countable Baire space Y is a Baire space.

Hint. It suffices to show that for every sequence $G_1, G_2, \ldots$ of open dense subsets of $X \times Y$ there exists a point $(x_0, y_0) \in \bigcap_{i=1}^\infty G_i$. Let $\{U_j\}_{j=1}^\infty$ be a countable base for the space Y consisting of non-empty sets. Show that for $i, j = 1, 2, \ldots$ the projection of the set $G_i \cap (X \times U_j)$ onto the space X is open and dense in X and take a point x_0 in the intersection of all these projections. Observe that the sets $H_i = \{y \in Y : (x_0, y) \in G_i\}$, where $i = 1, 2, \ldots$, are open and dense in Y and take a point y_0 in their intersection.

(d) (Aarts and Lutzer [1973]) Prove that the Cartesian product $X \times Y$ of a Baire space X and a Čech-complete space Y is a Baire space.

Hint (White [1975]). By part (a) one can assume that Y is compact. It suffices to show that for every sequence $G_1, G_2, \ldots$ of open dense subsets of $X \times Y$ the intersection $\bigcap_{i=1}^\infty G_i$ is non-empty. To this end, applying the Teichmüller-Tukey lemma, define for $i = 1, 2, \ldots$ a family of non-empty open subsets $\{U_s \times V_s\}_{s \in S_i}$ of $X \times Y$ such that

(1) The set $U_s \times V_s$ is contained in G_i for every $s \in S_i$.

(2) The family $\{U_s\}_{s \in S_i}$ consists of pairwise disjoint sets.

(3) The union $\bigcup_{s\in S_i} U_s$ is dense in X.
(4) For every $s \in S_{i+1}$ there exists a $t \in S_i$ such that $U_s \times \overline{V}_s \subset U_t \times V_t$.

Remark. In the same way one can show that if X is a Baire space and Y is a regular countably compact space or a pseudocompact space (see Section 3.10), then the Cartesian product $X \times Y$ is a Baire space. Let us note that there exist Baire spaces X, Y such that the Cartesian product $X \times Y$ is not a Baire space. The first example of this kind was given by Oxtoby in [1961] under the assumption of the continuum hypothesis; Krom in [1974] has shown how any such example yields a pair of metrizable spaces with the same property. In [1976] P. E. Cohen has given an example without using the continuum hypothesis; simpler (metrizable) examples were given by Fleissner and Kunen in [1978] and by R. Pol in [1979].

(e) (Oxtoby [1961]; for completely metrizable spaces, Bourbaki [1948]) Prove that any Cartesian product of Čech-complete spaces is a Baire space.

3.10. Countably compact spaces, pseudocompact spaces and sequentially compact spaces

A topological space X is called a *countably compact space* if X is a Hausdorff space and every countable open cover of X has a finite subcover. Thus, every compact space is countably compact; more precisely:

3.10.1. THEOREM. *A topological space is compact if and only if it is a countably compact space with the Lindelöf property.* ■

Examples of non-compact countably compact spaces are given below (see Examples 3.10.16–3.10.19).

The next two theorems contain characterizations of countably compact spaces in terms of families with the finite intersection property and in terms of locally finite families. The first theorem can be obtained by a slight modification in the proof of Theorem 3.1.1.

3.10.2. THEOREM. *For every Hausdorff space X the following conditions are equivalent:*
(i) *The space X is countably compact.*
(ii) *Every countable family of closed subsets of X which has the finite intersection property has non-empty intersection.*
(iii) *For every decreasing sequence $F_1 \supset F_2 \supset \ldots$ of non-empty closed subsets of X the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty.* ■

3.10.3. THEOREM. *For every Hausdorff space X the following conditions are equivalent:*
(i) *The space X is countably compact.*
(ii) *Every locally finite family of non-empty subsets of X is finite.*
(iii) *Every locally finite family of one-point subsets of X is finite.*
(iv) *Every infinite subset of X has an accumulation point.*
(v) *Every countably infinite subset of X has an accumulation point.*

PROOF. First we shall show that (i)⇒(ii). Suppose that (ii) does not hold; thus there exists a locally finite family $\{A_i\}_{i=1}^{\infty}$ of non-empty subsets of X. One can readily verify that non-empty closed sets $F_1, F_2, \ldots$, where $F_i = \bigcup_{j=i}^{\infty} \overline{A}_j$, form a decreasing sequence and that $\bigcap_{i=1}^{\infty} F_i = \emptyset$. Hence, by the last theorem, the space X is not countably compact.

The implications (ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(v) are obvious; to conclude the proof it suffices to show that (v)$\Rightarrow$(i). Suppose that (i) does not hold; thus, by the last theorem, there exists a decreasing sequence $F_1 \supset F_2 \supset \ldots$ of non-empty closed subsets of X such that $\bigcap_{i=1}^{\infty} F_i = \emptyset$. Let us choose a point $x_i \in F_i$ for $i = 1, 2, \ldots$; the set $A = \{x_1, x_2, \ldots\}$ is infinite, because otherwise a point of A would belong to infinitely many F_i's and we would have $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$. We shall show that $A^{\mathrm{d}} = \emptyset$, i.e., that (v) does not hold. Indeed, for every $x \in X$ there exists an i such that $x \notin F_i$ and thus the set $U = (X \setminus F_i) \setminus [\{x_1, x_2, \ldots, x_{i-1}\} \setminus \{x\}]$ is a neighbourhood of x satisfying $U \cap A \subset \{x\}$. ■

The following theorem is a consequence of Theorem 3.10.2.

3.10.4. THEOREM. *Every closed subspace of a countably compact space is countably compact.* ■

Let us note that countable compactness does not imply normality; there even exist non-regular countably compact spaces (see Exercise 3.10.B or Example 5.1.40).

The proof of the next theorem, similar to the proof of Theorem 3.1.10, is left to the reader.

3.10.5. THEOREM. *If there exists a continuous mapping $f: X \to Y$ of a countably compact space X onto a Hausdorff space Y, then Y is a countably compact space.* ■

Every subspace of the real line is second-countable, so that, by Theorems 3.8.1 and 3.10.1, every countably compact subspace of the real line is compact. Applying Theorems 3.10.5 and 3.2.8 we obtain

3.10.6. THEOREM. *Every continuous real-valued function defined on a countably compact space is bounded and attains its bounds.* ■

3.10.7. THEOREM. *If X is a countably compact space and Y is a sequential space, in particular, a first-countable space, then the projection $p: X \times Y \to Y$ is closed.*

PROOF. Let F be a closed subset of $X \times Y$. Consider a sequence $y_1, y_2, \ldots$ of points of $p(F)$ and a point $y \in \lim y_i$. Choose a point $x_i \in X$ such that $(x_i, y_i) \in F$ for $i = 1, 2, \ldots$ If the set $A = \{x_1, x_2, \ldots\}$ is finite, then there exists an $x \in X$ such that $x_{k_i} = x$ for an infinite sequence $k_1 < k_2 < \ldots$ of integers, so that we have $(x, y) \in \lim(x_{k_i}, y_{k_i})$; this implies that $(x, y) \in \overline{F} = F$ which gives $y \in p(F)$. On the other hand, if A is infinite, then there exists an $x \in A^{\mathrm{d}}$; one readily checks that $(x, y) \in \overline{F} = F$ which gives $y \in p(F)$. Since Y is a sequential space, the set $p(F)$ is closed in Y. ■

A characterization of countably compact spaces in terms of projections, analogous to the characterization of compact spaces given in Theorem 3.1.16, is stated in Exercise 3.10.A(b).

Let us observe that in the last theorem the assumption that Y is a sequential space cannot be replaced by the assumption that Y is a k-space; indeed, one can even define a countably compact space X and a compact space Y such that the projection $p: X \times Y \to Y$ is not closed (see Example 3.10.16).

The reader can easily prove the following theorem.

3.10.8. THEOREM. *The sum $\bigoplus_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is countably compact if and only if all spaces X_s are countably compact and the set S is finite.*■

By a straightforward modification in the proof of Theorem 3.7.2 we obtain

3.10.9. THEOREM. *If $f: X \to Y$ is a closed mapping defined on a Hausdorff space X and all fibers $f^{-1}(y)$ are countably compact, then for every countable compact subspace $Z \subset Y$ the inverse image $f^{-1}(Z)$ is countably compact.*■

The last theorem, along with Theorems 3.10.5 and 3.7.20, yields

3.10.10. THEOREM. *The class of countably compact spaces is perfect.*■

Countable compactness is not finitely multiplicative: in Example 3.10.19 below we shall define two countably compact spaces whose Cartesian product is not countably compact. Now we shall show that if one of the factors is a k-space, then the Cartesian product of two countably compact spaces is countably compact.

3.10.11. LEMMA. *If $f: X \to Y$ is a perfect mapping, then for every locally finite family $\mathcal{A}$ of subsets of X, the family of images $\{f(A) : A \in \mathcal{A}\}$ is locally finite in Y.*

PROOF. From compactness of fibers of f it follows that for every $y \in Y$ there exists an open set $U(y) \subset X$ that contains $f^{-1}(y)$ and meets only finitely many members of $\mathcal{A}$. From closedness of f and Theorem 1.4.13 it follows that y has a neighbourhood $V(y)$ such that $f^{-1}(V(y)) \subset U(y)$; obviously, the neighbourhood $V(y)$ meets only finitely many members of the family $\{f(A) : A \in \mathcal{A}\}$.■

3.10.12. LEMMA. *Let $\{A_s \times B_s\}_{s\in S}$, where $|S| \geq \aleph_0$, be a locally finite family of non-empty subsets of the Cartesian product $X \times Y$, where X is a Hausdorff space and Y is a k-space. Then there exists an infinite subset $S_0 \subset S$ such that either the family $\{A_s\}_{s\in S_0}$ or the family $\{B_s\}_{s\in S_0}$ is locally finite.*

PROOF. If the family $\{\overline{B}_s\}_{s\in S}$ is locally finite, one can take $S_0 = S$; thus, we can suppose that the family $\{\overline{B}_s\}_{s\in S}$ is not locally finite.

We shall show that there exists a compact set $Z \subset Y$ and an infinite set $S_0 \subset S$ such that $Z \cap \overline{B}_s \neq \emptyset$ for $s \in S_0$. Let y be a point of Y every neighbourhood of which meets $\overline{B}_s$ for infinitely many $s \in S$. If the set $S(y) = \{s \in S : y \in \overline{B}_s\}$ is infinite we can define $Z = \{y\}$ and $S_0 = S(y)$. On the other hand, if the set $S(y)$ is finite, then the union $B = \bigcup\{\overline{B}_s : s \in S \setminus S(y)\}$ is not closed, because $y \in \overline{B} \setminus B$; in this case we take as Z an arbitrary compact subset of Y such that the intersection $Z \cap B = \bigcup\{Z \cap \overline{B}_s : s \in S \setminus S(y)\}$ is not closed and we let $S_0 = \{s \in S \setminus S(y) : Z \cap \overline{B}_s \neq \emptyset\}$, the last set being obviously infinite.

The family $\{A_s \times (Z \cap \overline{B}_s)\}_{s\in S_0}$ of non-empty subsets of the space $X \times Z$ is locally finite. Applying Theorem 3.7.1 and Lemma 3.10.11 we deduce that the family $\{A_s\}_{s\in S_0}$ also is locally finite.■

The equivalence of (i) and (iii) in 3.10.3 along with the last lemma yield

3.10.13. THEOREM. *The Cartesian product $X \times Y$ of a countably compact space X and a countably compact k-space Y is countably compact.*■

The next two corollaries follow easily from Theorem 3.10.13 (let us note that the first corollary also follows from Theorem 3.10.9 and the second one from Theorems 3.10.7 and 3.10.9).

3.10.14. COROLLARY. *The Cartesian product $X \times Y$ of a countably compact space X and a compact space Y is countably compact.* ■

3.10.15. COROLLARY. *The Cartesian product $X \times Y$ of a countably compact space X and a countably compact sequential space Y is countably compact.* ■

Countably compact spaces were defined and studied earlier than compact spaces. At first, they even seemed to be the more relevant class; to some extent, this was due to the fact that for the broad and important class of metrizable spaces both definitions are equivalent (see Theorem 4.1.17). Countably compact spaces were then called compact spaces, and our compact spaces were called *bicompact spaces*; at present, the terminology adopted here is used. The principal reason for the predominance of compact spaces over countably compact spaces is that the former class is multiplicative while the latter is not even finitely multiplicative (see Example 3.10.19).

3.10.16. EXAMPLE. The space W_0 of all countable ordinal numbers (cf. Example 3.1.27) is an example of a countably compact space which is not compact.

Clearly, the space W_0 is not compact, because when embedded in the space $W = W_0 \cup \{\omega_1\}$ it is not a closed subspace. On the other hand, for every countably infinite subset A of W_0 there exists an $x_0 < \omega_1$ such that $A \subset W_1 = [0, x_0] \subset W_0$ and – W_1 being compact as a closed subset of W – we have $A^d \neq \emptyset$, which shows that the space W_0 is countably compact.

One can easily verify that the projection $p: W_0 \times W \to W$ is not closed (cf. Theorem 3.10.7). ■

3.10.17. EXAMPLE. We shall now define a dense subspace of I^c which is countably compact but not compact. Let $I^c = \prod_{t \in R} I_t$, where $I_t = I$ for $t \in R$, and let X denote the subspace of I^c consisting of those points $\{x_t\}$ which have at most countably many coordinates distinct from zero. Since X is a proper dense subset of I^c, the space X is not compact. Let A be a countably infinite subset of X; by the definition of X, there exists a countable set $R_0 \subset R$ such that $p_t(x) = 0$ for every $x \in A$ and $t \in R \setminus R_0$, where p_t is the projection of I^c onto I_t. Hence, the set A is a subset of the Cartesian product $\prod_{t \in R} X_t$, where $X_t = I_t$ for $t \in R_0$ and $X_t = \{0\}$ for $t \in R \setminus R_0$. The Cartesian product $\prod_{t \in R} X_t$ being compact, the set A has an accumulation point in it. As $\prod_{t \in R} X_t \subset X$, the set A has an accumulation point in X, which shows that the space X is countably compact. ■

3.10.18. EXAMPLE. Still another example of a countably compact space which is not compact is the space $X = \beta N \setminus \{x\}$, where $x \in \beta N \setminus N$. Since X is a proper dense subset of βN, the space X is not compact. On the other hand, for every countably infinite subset A of $X \subset \beta N$, by virtue of Theorem 3.6.14 we have $|\overline{A}| = 2^c$ which implies that $A^d \cap X \neq \emptyset$; thus the set A has an accumulation point in X, which shows that the space X is countably compact. ■

3.10.19. EXAMPLE. We shall now define two countably compact Tychonoff spaces X and Y such that the Cartesian product $X \times Y$ is not countably compact. They are subspaces of βN satisfying the conditions $X \cup Y = \beta N$ and $X \cap Y = N$.

For every $M \subset \beta N$ let $\mathcal{P}(M)$ denote the family of all countably infinite subsets of the set M, and let f be a function assigning to every member A of $\mathcal{P}(\beta N)$ an accumulation point of the set A in the space βN.

Letting $X_0 = N$ and

$$X_\alpha = (\bigcup_{\gamma<\alpha} X_\gamma) \cup f[\mathcal{P}(\bigcup_{\gamma<\alpha} X_\gamma)] \quad \text{for} \quad 0 < \alpha < \omega_1,$$

we define by transfinite induction a transfinite sequence $X_0, X_1, \ldots, X_\alpha, \ldots,\ \alpha < \omega_1$ of subsets of βN. The space $X = \bigcup_{\alpha<\omega_1} X_\alpha$ is countably compact, because every $A \in \mathcal{P}(X)$ is contained in some X_α and thus has an accumulation point in $X_{\alpha+1}$ and, *a fortiori*, in X. Applying transfinite induction, one easily shows that $|X_\alpha| \leq \mathfrak{c} \cdot \mathfrak{c} + (\mathfrak{c} \cdot \mathfrak{c})^{\aleph_0} = \mathfrak{c}$, so that $|X| \leq \mathfrak{c}$. Let $Y = N \cup (\beta N \setminus X)$; since by Theorem 3.6.14 we have $|\overline{A}| = 2^{\mathfrak{c}}$ for every $A \in \mathcal{P}(Y)$, every countably infinite subset of Y has an accumulation point in Y, so that the space Y is countably compact.

Now, consider the Cartesian product $X \times Y$ and denote by Δ_0 the intersection of $X \times Y$ and the diagonal Δ of the Cartesian product $\beta N \times \beta N$; as $X \cap Y = N$ we have $\Delta_0 = \{(1,1),(2,2),\ldots\}$. Since $\{i\}$ is an open subset of βN for $i = 1,2,\ldots, \Delta_0$ is an open discrete subspace of $X \times Y$. On the other hand, since Δ is a closed subset of $\beta N \times \beta N$, the set Δ_0 is closed in $X \times Y$ and this shows that $X \times Y$ is not countably compact.

Let us observe that by Theorem 3.10.13 neither X nor Y is a k-space; in particular, X and Y are not Čech-complete (see Theorem 3.9.5). ■

Let us note in connection with the last example, that one can define a space X all of whose finite powers are countably compact and yet $X^{\aleph_0}$ is not countably compact. This implies, in particular, that the limit of an inverse sequence of countably compact spaces need not be countably compact (the last fact can be established directly by a modification of the construction in Example 3.10.19 leading to a decreasing sequence $X_1 \supset X_2 \supset \ldots$ of countably compact subspaces of βN whose intersection is equal to N).

We shall now study another class of spaces closely related to compact spaces.

A topological space X is called *pseudocompact* if X is a Tychonoff space and every continuous real-valued function defined on X is bounded. One can readily check that the last condition is equivalent to the condition that every continuous real-valued function defined on X attains its bounds. Theorem 3.10.6 yields

3.10.20. THEOREM. *Every countably compact Tychonoff space is pseudocompact.* ■

For normal spaces the reverse implication also is valid

3.10.21. THEOREM. *Every pseudocompact normal space is countably compact.*

PROOF. Assume that X is a normal space which is not countably compact. Thus there exists a set $A = \{x_1, x_2, \ldots\} \subset X$ such that $x_i \neq x_j$ whenever $i \neq j$ and $A^{\mathrm{d}} = \emptyset$. Clearly A is a discrete closed subspace of X and by the Tietze-Urysohn theorem there exists a continuous function $f: X \to R$ such that $f(x_i) = i$ for $i = 1, 2, \ldots$ Since f is not bounded, the space X is not pseudocompact. ■

The next two theorems contain characterizations of pseudocompact spaces in terms of locally finite families and in terms of families with the finite intersection property.

3.10.22. THEOREM. *For every Tychonoff space X the following conditions are equivalent:*
(i) *The space X is pseudocompact.*
(ii) *Every locally finite family of non-empty open subsets of X is finite.*
(iii) *Every locally finite open cover of X consisting of non-empty sets is finite.*
(iv) *Every locally finite open cover of X has a finite subcover.*

PROOF. First we shall show that (i)⇒(ii). Suppose that (ii) does not hold; thus there exists a locally finite family $\{U_i\}_{i=1}^{\infty}$ of non-empty open subsets of X. Let us choose a point $x_i \in U_i$ for $i = 1, 2, \ldots$ Since X is a Tychonoff space, for $i = 1, 2, \ldots$ there exists a continuous function $f_i: X \to R$ such that $f_i(x_i) = i$ and $f_i(X \setminus U_i) \subset \{0\}$. From the local finiteness of the family $\{U_i\}_{i=1}^{\infty}$ it follows that the formula $f(x) = \sum_{i=1}^{\infty} |f_i(x)|$ defines a continuous function $f: X \to R$; as f is not bounded, the space X is not pseudocompact.

The implications (ii)⇒(iii) and (iii)⇒(iv) are obvious; to conclude the proof it suffices to show that (iv)⇒(i). Let f be a continuous real-valued function defined on a space X satisfying (iv). Clearly, the family $\{f^{-1}((i-1, i+1)) : i = 0, \pm 1, \pm 2, \ldots\}$ is a locally finite open cover of X; the existence of a finite subcover implies that f is bounded. ∎

3.10.23. THEOREM. *For every Tychonoff space X the following conditions are equivalent:*
(i) *The space X is pseudocompact.*
(ii) *For every decreasing sequence $W_1 \supset W_2 \supset \ldots$ of non-empty open subsets of X the intersection $\bigcap_{i=1}^{\infty} \overline{W}_i$ is non-empty.*
(iii) *For every countable family $\{V_i\}_{i=1}^{\infty}$ of open subsets of X which has the finite intersection property the intersection $\bigcap_{i=1}^{\infty} \overline{V}_i$ is non-empty.*

PROOF. First we shall show that (i)⇒(ii). Let X be a peudocompact space and let $W_1 \supset W_2 \supset \ldots$ be a decreasing sequence of non-empty open subsets of X. It follows from Theorem 3.10.22 that the family $\{W_i\}_{i=1}^{\infty}$ is not locally finite; thus there exists a point $x \in X$ whose every neighbourhood meets infinitely many W_i's. Clearly $x \in \bigcap_{i=1}^{\infty} \overline{W}_i$.

To prove that (ii)⇒(iii) it suffices to consider the decreasing sequence $V_1, V_1 \cap V_2, \ldots$ Finally, (iii)⇒(i), because if there exists a non-bounded continuous function $f: X \to R$, then the family $\{V_i\}_{i=1}^{\infty}$, where $V_i = \{x : |f(x)| > i\}$, has the finite intersection property and yet $\bigcap_{i=1}^{\infty} \overline{V}_i = \emptyset$. ∎

The definition of pseudocompact spaces yields

3.10.24. THEOREM. *If there exists a continuous mapping $f: X \to Y$ of a pseudocompact space X onto a Tychonoff space Y, then Y is a pseudocompact space.* ∎

The reader can easily prove the following theorem.

3.10.25. THEOREM. *The sum $\bigoplus_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is pseudocompact if and only if all spaces X_s are pseudocompact and the set S is finite.* ∎

As shown in Example 3.10.29 below, pseudocompactness is not hereditary with respect to closed sets. Since, obviously, pseudocompactness is hereditary with respect to open-and-closed sets, it follows from Theorem 3.7.29 that pseudocompactness is not an inverse invariant of perfect mappings, even in the class of Tychonoff spaces (cf. Exercise 3.10.H).

Pseudocompactness is not finitely multiplicative. Indeed, the spaces X and Y defined in Example 3.10.19 are both pseudocompact, and the Cartesian product $X \times Y$ is not pseudocompact, because it contains $D(\aleph_0) = \Delta_0$ as an open-and-closed subset. It turns out, however, that if one of the factors is a k-space, then the Cartesian product of two pseudocompact spaces is pseudocompact.

3.10.26. THEOREM. *The Cartesian product $X \times Y$ of a pseudocompact space X and a pseudocompact k-space Y is pseudocompact.*

PROOF. It suffices to observe that every non-empty open subset of the Cartesian product $X \times Y$ contains a non-empty open subset of the form $U \times V$ and then apply Lemma 3.10.12 and the equivalence of conditions (i) and (ii) in Theorem 3.10.22. ■

3.10.27. COROLLARY. *The Cartesian product $X \times Y$ of a pseudocompact space X and a compact space Y is pseudocompact.* ■

3.10.28. COROLLARY. *The Cartesian product $X \times Y$ of a pseudocompact space X and a pseudocompact sequential space Y is pseudocompact.* ■

Let us note that one can define a space X all of whose finite powers are pseudocompact and yet $X^{\aleph_0}$ is not pseudocompact. This implies, in particular, that the limit of an inverse sequence of pseudocompact spaces need not be pseudocompact (the last fact can be established directly as indicated above in the discussion of countable compactness, or by an application of Problem 3.12.24(c); cf. the hint to Problem 6.3.25).

We shall now give an example of a pseudocompact space X which contains the discrete space $D(\aleph_0)$ as a closed subset. This example shows that a pseudocompact space is not necessarily countably compact and that pseudocompactness is not hereditary with respect to closed sets (cf. Exercise 3.10.F(c)).

3.10.29. EXAMPLE. By virtue of Corollary 3.6.8, the Čech-Stone compactification βN is a subspace of the Čech-Stone compactification βR. Let us consider the space $X = \beta R \setminus (\beta N \setminus N)$; clearly, X contains the discrete space $N = D(\aleph_0)$ as a closed subset. We shall show that X is pseudocompact. Assume that there exists a continuous function $f: X \to R$ which is not bounded. The set $R \setminus N$ being dense in X there exists a sequence $x_1, x_2, \ldots$ of distinct points of $R \setminus N$ such that $|f(x_i)| > i$ for $i = 1, 2, \ldots$ Since f is continuous, the set $A = \{x_1, x_2, \ldots\}$ has no accumulation point in X, which implies, in particular, that A is a closed subset of the real line R. It follows from Corollary 3.6.4 that $\overline{A} \cap \overline{N} = \overline{A} \cap \beta N = \emptyset$, i.e., that $\overline{A} \subset X$; thus A is an infinite subset of βR with no accumulation point. The contradiction shows that X is pseudocompact. ■

Sequential compactness is still another property related to compactness. A topological space X is called *sequentially compact* if X is a Hausdorff space and every sequence of points of X has a convergent subsequence. The equivalence of conditions (i) and (v) in Theorem 3.10.3 yields

3.10.30. THEOREM. *Every sequentially compact space is countably compact.* ■

The reverse implication does not hold; there exist even compact spaces which are not sequentially compact – by virtue of Corollary 3.6.15, the Čech-Stone compactification βN is such a space (cf. Example 3.10.38). We have however

3.10.31. THEOREM. *Sequential compactness and countable compactness are equivalent in the class of sequential spaces and, in particular, in the class of first-countable spaces.*

PROOF. It suffices to show that any sequene $x_1, x_2, \ldots$ of points of a countably compact sequential space X has a convergent subsequence; clearly, one can suppose that $x_i \neq x_j$ whenever $i \neq j$. Let x be an accumulation point of the infinite set $A = \{x_1, x_2, \ldots\}$. Since $x \in \overline{A \setminus \{x\}}$, the set $A \setminus \{x\}$ is not closed, so that, X being a sequential space, the set $A \setminus \{x\}$ contains a sequence converging to a point in the complement of $A \setminus \{x\}$. Rearranging, if necessary, the terms of this sequence we obtain a convergent subsequence of $x_1, x_2, \ldots$ ■

The last theorem implies that the space W_0 of all countable ordinal numbers is sequentially compact, because this is a countably compact first-countable space. This space is an example of a sequentially compact space which is not compact. Let us also observe, in connection with the last theorem, that there exist sequentially compact spaces which are not sequential spaces – the space of all ordinal numbers $\leq \omega_1$ is clearly such a space. There also exist non-regular sequentially compact spaces (see Exercise 3.10.B or Example 5.1.40).

The next three theorems follow directly from the definition of sequential compactness.

3.10.32. THEOREM. *If there exists a continuous mapping $f: X \to Y$ of a sequentially compact space X onto a Hausdorff space Y, then Y is a sequentially compact space.* ■

Considering a mapping of a compact but not sequentially compact space to a one-point space, we see that sequential compactness is not an inverse invariant of perfect mappings.

3.10.33. THEOREM. *Every closed subspace of a sequentially compact space is sequentially compact.* ■

3.10.34. THEOREM. *The sum $\bigoplus_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is sequentially compact if and only if all spaces X_s are sequentially compact and the set S is finite.* ■

We close this section with three theorems on Cartesian products of sequentially compact spaces.

3.10.35. THEOREM. *The Cartesian product of countably many sequentially compact spaces is sequentially compact.*

PROOF. Let $\{X_i\}_{i=1}^{\infty}$ be a countable family of sequentially compact spaces and let $z_1, z_2, z_3, \ldots$, where $z_j = \{x_i^j\}$, be a sequence of points of the Cartesian product $\prod_{i=1}^{\infty} X_i$. The sequence $x_1^1, x_1^2, x_1^3, \ldots$ of points of X_1 has a convergent subsequence $x_1^{k_1^1}, x_1^{k_2^1}, x_1^{k_3^1}, \ldots$; let x_1 be the limit of this subsequence. Similarly, the sequence $x_2^{k_1^1}, x_2^{k_2^1}, x_2^{k_3^1}, \ldots$ of points of X_2 has a convergent subsequence $x_2^{k_1^2}, x_2^{k_2^2}, x_2^{k_3^2}, \ldots$; let x_2 be the limit of this subsequence. Inductively we define for $i = 3, 4, \ldots$ a subsequence $x_i^{k_1^i}, x_i^{k_2^i}, x_i^{k_3^i}, \ldots$ of the sequence $x_i^{k_1^{i-1}}, x_i^{k_2^{i-1}}, x_i^{k_3^{i-1}}, \ldots$ converging to a point $x_i \in X_i$. Applying Proposition 2.3.34 and observing that after dropping the first $i-1$ terms the sequence $k_1^1, k_2^2, k_3^3, \ldots$ becomes a subsequence of $k_1^i, k_2^i, k_3^i, \ldots$, one can readily show that the point $\{x_i\} \in \prod_{i=1}^{\infty} X_i$ is the limit of the subsequence $z_{k_1^1}, z_{k_2^2}, z_{k_3^3}, \ldots$ of the sequence $z_1, z_2, z_3, \ldots$ ■

3.10.36. THEOREM. *The Cartesian product $X \times Y$ of a countably compact space X and a sequentially compact space Y is countably compact.*

PROOF. Consider a countably infinite set $A = \{z_1, z_2, \ldots\} \subset X \times Y$, where $z_i = (x_i, y_i)$ for $i = 1, 2, \ldots$ and $z_i \neq z_j$ whenever $i \neq j$, and let $y_{k_1}, y_{k_2}, \ldots$ be a subsequence of $y_1, y_2, \ldots$ that converges to a point $y \in Y$. If the set $\{x_{k_1}, x_{k_2}, \ldots\}$ is finite, then there exist a point $x \in X$ and a subsequence $k_{m_1}, k_{m_2}, \ldots$ of the sequence $k_1, k_2, \ldots$ such that $x_{k_{m_i}} = x$ for $i = 1, 2, \ldots$; if the set $\{x_{k_1}, x_{k_2}, \ldots\}$ is infinite, then it has an accumulation point $x \in X$. One readily sees that in both cases $(x, y) \in X \times Y$ is an accumulation point of the set A. ■

3.10.37. THEOREM. *The Cartesian product $X \times Y$ of a pseudocompact space X and a sequentially compact Tychonoff space Y is pseudocompact.*

PROOF. Suppose that there exists a non-bounded continuous function $f: X \times Y \to R$ and choose points $z_i = (x_i, y_i) \in X \times Y$ such that $|f(z_i)| \geq i$ for $i = 1, 2, \ldots$ Let $y_{k_1}, y_{k_2}, \ldots$ be a subsequence of $y_1, y_2, \ldots$ that converges to a point $y \in Y$. The subspace $Y_0 = \{y, y_{k_1}, y_{k_2}, \ldots\} \subset Y$ is compact, so that the space $X \times Y_0$ is pseudocompact by Corollary 3.10.27, and yet the function $f|X \times Y_0: X \times Y_0 \to R$ is not bounded. The contradiction shows that every continuous real-valued function on $X \times Y$ is bounded. ■

Since there exist sequentially compact spaces which are not k-spaces (see Exercise 3.10.I), Theorems 3.10.36 and 3.10.37 do not follow from Theorems 3.10.13 and 3.10.26.

3.10.38. EXAMPLE. As we observed above, applying Corollary 3.6.15, the space βN is not sequentially compact. This can be also proved directly: the sequence $1, 2, \ldots$ of points of βN does not contain any convergent subsequence, because for every increasing sequence $k_1 < k_2 < \ldots$ of positive integers the sets $A = \{k_1, k_3, k_5, \ldots\}$ and $B = \{k_2, k_4, k_6, \ldots\}$ have disjoint closures in βN and this implies that the sequence $k_1, k_2, k_3, \ldots$ does not converge. ■

As shown in Example 3.6.20, the space βN is embeddable in the Cantor cube $D^{\mathfrak{c}}$; hence, by Theorem 3.10.33, the Cantor cube $D^{\mathfrak{c}}$ is not sequentially compact and we see that the assumption of countability in Theorem 3.10.35 is essential.

From the fact that $D^{\mathfrak{c}}$ is not sequentially compact and from Example 2.5.3 it follows that the limit of an inverse system of sequentially compact spaces need not be sequentially compact. On the other hand, from Theorems 3.10.33 and 3.10.35 it follows immediately that the limit of an inverse sequence of sequentially compact spaces is sequentially compact.

Historical and bibliographic notes

Countably compact spaces were introduced by Fréchet in [1906]; the characterization of countable compactness by condition (iv) in Theorem 3.10.3 was established by Hausdorff in [1914]. Theorem 3.10.7 was proved by Fleischer and Franklin in [1967] (Hanai in [1961] and Isiwata in [1963] proved this theorem under the stronger assumptions that Y is first-countable and that Y is a Fréchet space, respectively). Theorem 3.10.10 was established by Henriksen and Isbell in [1958] and Theorem 3.10.13 by Noble in [1969] (Corollary 3.10.14 was proved by Smirnov in [1951d] and Corollary 3.10.15 by Franklin in [1965]). In Example 3.10.17, given by Pontrjagin in [1954], the important concept of a Σ-product is applied; it is discussed also in Exercise 3.10.D and Problems 2.7.14, 2.7.15, 3.12.24 and 4.5.12. Example 3.10.19 was constructed by Novák in [1953a] (announcement in Novák [1949]; our description follows Frolík's paper [1959]); a similar example was given by Terasaka in [1952]. The existence of a space X all of whose finite powers are countably compact and yet $X^{\aleph_0}$ is not countably compact was established by Frolík in [1967]. In Frolík's example the space $X^{\aleph_0}$ is even not pseudocompact; however, in the case of peudocompactness analogous examples can be obtained in a simpler way (see Comfort [1967] and Frolík [1967a]).

Pseudocompact spaces were defined by Hewitt in [1948]; in this paper Theorems 3.10.20 and 3.10.21 were proved. The equivalence to pseudocompactness of conditions (ii), (iii) and (iv) in Theorem 3.10.22 was observed by Glicksberg in [1952], Kerstan in [1957] and Smirnov in [1954], respectively. Theorem 3.10.23 was given by Colmez in [1951]. Theorem 3.10.26 was established by Tamano in [1960] (Corollary 3.10.27 was proved independently by Glicksberg in [1959] and by Mrówka in [1959]; Corollary 3.10.28 was proved by Bagley, Connell and McKnight in [1958] under the stronger assumption that Y is first-countable). Example 3.10.29 was given by Katětov in [1951a].

The concept of a sequentially compact space, as well as Theorems 3.10.30 and 3.10.32–3.10.35, belonged for a long time to topological folklore; the origin of the concept is related to a characterization of compactness in the class of metric spaces. Theorem 3.10.31 was proved by Franklin in [1965] (equivalence of sequential compactness and countable compactness in metric spaces was established by Hausdorff in [1914]). Theorem 3.10.36 was proved by Mrówka in [1959] (Ryll-Nardzewski in [1954] proved this theorem under the stronger assumption that Y is countably compact and first-countable); Theorem 3.10.37 appeared in Scarborough and A. H. Stone's paper [1966].

In connection with the facts that $D^{\mathfrak{c}}$ is not sequentially compact (see Example 3.10.38) and that every compact space of cardinality $< 2^{\aleph_1}$ is sequentially compact (see Problem 3.12.11(e)), the question arises whether $D^{\aleph_1}$ is a sequentially compact space. As shown by Booth in [1974], this is independent of the axioms of set theory.

Further results on Cartesian products of countably compact, pseudocompact and sequentially compact spaces can be found in Glicksberg [1959], Scarborough and A. H. Stone [1966], Kenderov [1968], Stephenson [1968] and Noble [1969]. J. E. Vaughan's paper [1984] is a survey of recent results on countably compact and sequentially compact spaces.

Exercises

3.10.A. (a) Note that a Hausdorff space X is countably compact if and only if every closed countable subspace of X is compact.

(b) (Hanai [1961]) Verify that a Hausdorff space X is countably compact if and only if the projection $p: X \times A(\aleph_0) \to A(\aleph_0)$ is closed (cf. Theorems 3.1.16 and 3.10.7 and Problem 3.12.14(a)).

(c) Show that if $f: X \to Y$ is a continuous mapping of a countably compact space X to a T_1-space Y, then $f(\bigcap_{i=1}^{\infty} A_i) = \bigcap_{i=1}^{\infty} f(A_i)$ for every decreasing sequence $A_1 \supset A_2 \supset \ldots$ of closed subsets of X.

3.10.B (Bourbaki [1948]). Show that the space obtained from the space W of all ordinal numers $\leq \omega_1$ by making the set Z of all countable limit numbers closed (cf. Example 1.5.6) is a non-regular countably compact (even sequentially compact) space (cf. Example 5.1.40).

3.10.C (Alexandroff and Urysohn [1929]). Let $X = C_0 \cup C_1 \subset R^2$, where $C_0 = \{(x,0) : 0 < x \leq 1\}$ and $C_1 = \{(x,1) : 0 \leq x < 1\}$, and let the topology on X be generated by the base consisting of sets of the form

$$\{(x,i) \in X : \ x_0 - 1/k < x < x_0 \quad \text{and} \quad i = 0,1\} \cup \{(x_0,0)\},$$

where $0 < x_0 \leq 1$ and $k = 1,2,\ldots$, and of sets of the form

$$\{(x,i) \in X : \ x_0 < x < x_0 + 1/k \quad \text{and} \quad i = 0,1\} \cup \{(x_0,1)\},$$

where $0 \leq x_0 < 1$ and $k = 1,2,\ldots$; the space X is called the *two arrows space*.

(a) Note the the subspaces C_0 and C_1 of the space X are homeomorphic to the Sorgenfrey line. Deduce that X is hereditarily separable and hereditarily Lindelöf and thus perfectly normal.

Hint. See Exercises 2.1.I and 3.8.A(c) and Example 3.8.14.

(b) Prove that X is a compact space and deduce that the Sorgenfrey line is not Čech-complete (cf. Exercise 3.9.F(b)).

Hint. Apply Theorem 3.10.1.

(c) Observe that X^2 is not hereditarily normal and deduce that hereditary and perfect normality are not inverse invariants of open perfect mappings with perfectly normal fibers.

3.10.D (Noble [1970]). Let $\{X_s\}_{s \in S}$ be a family of topological spaces and let $a = \{a_s\}$ be a point of the Cartesian product $\prod_{s \in S} X_s$; by $\Sigma(a)$ we denote the subspace of $\prod_{s \in S} X_s$ consisting of all points $x = \{x_s\}$ such that the set $S(x) = \{s \in S : x_s \neq a_s\}$ is countable. Prove that if all spaces X_s are first-countable, then $\Sigma(a)$ is a Fréchet space. This implies, in particular, that the space in Example 3.10.17 is a Fréchet space (and thus a k-space); verify that it is not a Čech-complete space.

Hint. Let $x = \{x_s\} \in \overline{A} \subset \Sigma(a)$ and let $\{U_{s,i}\}_{i=1}^{\infty}$ be a base for X_s at x_s. Define inductively a sequence $x_1, x_2, \ldots$ of points of A such that $x_i \in (\prod_{s \in S_i} U_{s,i} \times \prod_{s \in S \setminus S_i} X_s)$, where $S_i = \{s_{j,k} : 0 \leq j,k < i\}$ and the indices $s_{j,k}$ are determined by letting $S(x_i) = \{s_{i,0}, s_{i,1} \ldots\}$ and $x_0 = x$. Show that $x = \lim x_i$.

3.10.E (Kannan [1980]). Show that every sequential space which is embeddable in a hereditarily normal countably compact space is a Fréchet space.

Hint. To begin, observe that if X is a Hausdorff space and for an $x \in X$ the subspace $X \setminus \{x\}$ is normal, then for every sequence $x_1, x_2, \ldots$ of distinct points of $X \setminus \{x\}$ that converges to x there exist pairwise disjoint open sets $U_1, U_2, \ldots$ such that $x_i \in U_i$ for $i = 1,2,\ldots$ and for every sequence $x'_1, x'_2, \ldots$, where $x'_i \in U_i$, the set $\{x, x'_1, x'_2, \ldots\}$ is closed (see Theorems 2.1.14 and 5.1.17).

3.10.F. (a) (Hewitt [1948]) Show that a Tychonoff space X is pseudocompact if and only if the remainder $\beta X \setminus X$ does not contain non-empty G_δ-subsets of βX.

(b) (Colmez [1951], Glicksberg [1952]) Prove that a Tychonoff space X is pseudocompact if and only if X satisfies the Dini theorem, i.e., if for every sequence $\{f_i\}$ of continuous real-valued functions defined on X and satisfying $f_i(x) \leq f_{i+1}(x)$ for all $x \in X$ and $i = 1,2,\ldots,$

and for a function $f \in R^X$ such that $f(x) = \lim f_i(x)$ for every $x \in X$, we have $f = \lim f_i$, that is, the sequence $\{f_i\}$ is uniformly convergent to f (cf. Lemma 3.2.18).

(c) Note that if every closed subspace of a Tychonoff space X is pseudocompact, then the space X is countably compact.

(d) (Colmez [1951]) Show that pseudocompactness is hereditary with respect to closed domains.

(e) (Colmez [1952]) Observe that regular countably compact spaces and pseudocompact spaces are Baire spaces (cf. the remark to Exercise 3.9.J(d)).

3.10.G (Mrówka [1954]). Show that the space X defined in Exercise 3.6.I(a) is pseudocompact but not countably compact.

3.10.H (Hanai and Okuyama [1962]; for open perfect mappings, Ponomarev [1960a]). Prove that if there exists an open mapping $f: X \to Y$ of a Tychonoff space X onto a pseudocompact space Y which transforms functionally closed subsets of X to closed subsets of Y and has pseudocompact fibers, then the space X is pseudocompact.

Hint. Apply Exercise 1.5.L(a).

3.10.I. Following the pattern of Example 3.1.27, define a topology on the set of all ordinal numbers $\leq \omega_2$ (cf. Problem 1.7.4), and verify that the subspace obtained by removing all ordinal numbers cofinal with ω_1 is sequentially compact but is not a k-space.

3.10.J. A topological space X is *locally sequentially compact* if for every $x \in X$ and any neighbourhood U of x there exists a neighbourhood V of x such that $\overline{V}$ is sequentially compact and $\overline{V} \subset U$.

(a) Observe that a sequential space X is locally sequentially compact if and only if X is regular and for every $x \in X$ there exists a neighbourhood U of the point x such that $\overline{U}$ is a countably compact subspace of X. Give an example of a locally sequentially compact space which is not a sequential space.

(b) Verify that Theorem 3.3.17 holds under the assumptions that Y is a sequential space and X is locally sequentially compact.

(c) (Boehme [1965]) Prove that the Cartesian product $X \times Y$ of sequential spaces X and Y is sequential provided that the space X is locally sequentially compact (cf. Exercise 2.4.G(c)).

Hint. Apply (b) and Exercise 2.4.G(b).

(d) (Arhangel'skiĭ [1979] (announcement [1972]), Olson [1974]) Prove that the Cartesian product $X \times Y$ of a locally sequentially compact Fréchet space X and a first-countable space Y is a Fréchet space (cf. Exercises 2.3.K and 2.4.G(c)).

Hint. To begin, show that if a point (x_0, y_0) is in the closure of a subset A of the Cartesian product $X \times Y$, then x_0 belongs to the closure of the set consisting of all x in X which have the property that there exists a sequence of points of A converging to (x, y_0). Then, consider a decreasing sequence $U_1 \supset U_2 \supset \ldots$ of open subsets of Y that form a base at y_0 and a sequence $x_1, x_2, \ldots$ converging to x_0 such that for each i there exists a sequence $(x_1^i, y_1^i), (x_2^i, y_2^i), \ldots$ of points of A converging to (x_i, y_0); one can assume that $x_i \neq x_0$ for $i = 1, 2, \ldots$ and $y_j^i \in U_i$ for $j = 1, 2, \ldots$ Observe that x_0 is in the closure of the set $\{x_j^i : i, j = 1, 2, \ldots\}$, consider a sequence $x_{j_1}^{i_1}, x_{j_2}^{i_2}, \ldots$ converging to x_0 and show that the sequence $(x_{j_1}^{i_1}, y_{j_1}^{i_1}), (x_{j_2}^{i_2}, y_{j_2}^{i_2}), \ldots$ converges to (x_0, y_0).

3.11. Realcompact spaces

A topological space X is called a *realcompact space* if X is a Tychonoff space and there is no Tychonoff space $\tilde{X}$ which satisfies the following two conditions:

(RC1) *There exists a homeomorphic embedding $r: X \to \tilde{X}$ such that $r(X) \neq \overline{r(X)} = \tilde{X}$.*

(RC2) *For every continuous real-valued function $f: X \to R$ there exists a continuous function $\tilde{f}: \tilde{X} \to R$ such that $\tilde{f}r = f$.*

It follows directly from the definition that every compact space is realcompact; more precisely

3.11.1. THEOREM. *A topological space is compact if and only if it is a pseudocompact realcompact space.*

PROOF. Every compact space is pseudocompact and realcompact. If a non-compact space X is realcompact, then – since $X \neq \beta X$ – there exists a continuous function $f: X \to R$ which cannot be continuously extended over βX. Clearly the function f is not bounded, so that X is not pseudocompact. ■

3.11.2. EXAMPLES. The last theorem implies that spaces in Examples 3.10.16–3.10.19 and 3.10.29 are not realcompact. In particular, it follows that there exist locally compact spaces which are not realcompact. ■

We shall now establish an important characterization of realcompact spaces and deduce a few corollaries; two further characterizations of realcompact spaces will be given in Theorems 3.11.10 and 3.11.11 below.

3.11.3. THEOREM. *A topological space is realcompact if and only if it is homeomorphic to a closed subspace of a power $R^{\mathfrak{m}}$ of the real line.*

PROOF. Let X be a realcompact space. Denote by $\mathcal{F}$ the family of all continuous real-valued functions defined on X and consider the mapping $F = \triangle_{f \in \mathcal{F}} f: X \to \prod_{f \in \mathcal{F}} R_f$, where $R_f = R$ for $f \in \mathcal{F}$. Since X is a Tychonoff space, it follows from the diagonal theorem that F is a homeomorphic embedding; let $\tilde{X} = \overline{F(X)} \subset \prod_{f \in \mathcal{F}} R_f$. For every continuous real-valued function $f: X \to R$ there exists a continuous function $\tilde{f}: \tilde{X} \to R$ such that $\tilde{f}F = f$, namely the restriction $p_f|\tilde{X}$ of the projection $p_f: \prod_{f \in \mathcal{F}} R_f \to R_f$. Thus – by the definition of realcompact spaces – we have $\tilde{X} = F(X)$ and the space X is homeomorphic to the closed subspace $F(X)$ of the Cartesian product $\prod_{f \in \mathcal{F}} R_f = R^{|\mathcal{F}|}$.

Let us now consider a closed subspace X of the Cartesian product $\prod_{s \in S} R_s$, where $R_s = R$ for $s \in S$, a Tychonoff space $\tilde{X}$ and a homeomorphic embedding $r: X \to \tilde{X}$ for which condition (RC2) is satisfied; obviously, we can assume that $\overline{r(X)} = \tilde{X}$. For every $s \in S$ there exists a continuous function $\tilde{p}_s: \tilde{X} \to R_s = R$ such that $\tilde{p}_s r = p_s$; let $F = \triangle_{s \in S} \tilde{p}_s: \tilde{X} \to \prod_{s \in S} R_s$. Since $Fr(x) = x$ for every $x \in X$, we have $F(\tilde{X}) = F(\overline{r(X)}) \subset \overline{Fr(X)} = \overline{X} = X$, so that the restriction F_X maps $\tilde{X}$ onto X. For every $x \in X$ we have $rF_X(r(x)) = r(x)$; thus $rF_X: \tilde{X} \to \tilde{X}$, when restricted to $r(X)$, coincides with $\mathrm{id}_{r(X)}$ and Theorem 1.5.4 implies that $rF_X = \mathrm{id}_{\tilde{X}}$. As $rF_X(\tilde{X}) \subset r(X)$, it follows that $r(X) = \tilde{X}$. Hence there is no Tychonoff space $\tilde{X}$ satisfying (RC1) and (RC2), i.e., X is a realcompact space. ■

The next theorem follows immediately from Theorem 3.11.3.

3.11.4. THEOREM. *Every closed subspace of a realcompact space is realcompact.*■

Theorem 3.11.3, along with 3.11.4, 2.3.7 and 2.3.4, gives

3.11.5. THEOREM. *The Cartesian product $\prod_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is realcompact if and only if all spaces X_s are realcompact.*■

Theorems 3.11.4 and 3.11.5 yield

3.11.6. COROLLARY. *The limit of an inverse system of realcompact spaces is realcompact.*■

3.11.7. COROLLARY. *Let X be a topological space and $\{A_s\}_{s\in S}$ a family of subspaces of X; if all the A_s's are realcompact, then the intersection $\bigcap_{s\in S} A_s$ also is realcompact.*

PROOF. The space $\bigcap_{s\in S} A_s$ is homeomorphic to a closed subspace of the Cartesian product $\prod_{s\in S} A_s$ which is a realcompact space, namely to the intersection of $\prod_{s\in S} A_s$ with the diagonal of the Cartesian product $\prod_{s\in S} X_s$, where $X_s = X$ for $s \in S$ (see also Example 2.5.4).■

3.11.8. COROLLARY. *If $f: X \to Y$ is a continuous mapping of a realcompact space X to a Hausdorff space Y, then for every realcompact subspace B of Y the inverse image $f^{-1}(B) \subset X$ is realcompact.*

In particular, *realcompactness is hereditary with respect to functionally open subsets.*

PROOF. The graph $G(f_B)$ of the restriction $f_B: f^{-1}(B) \to B$ coincides with the intersection $(X \times B) \cap G(f)$ and thus is closed in the realcompact space $X \times B$. Since the inverse image $f^{-1}(B)$ is homeomorphic to $G(f_B)$, it is realcompact.■

3.11.9. LEMMA. *Let X be a topological space and A a subspace of X. If every continuous function $g: A \to R$ such that $g(x) \geq 1$ for all $x \in A$ is extendable over X, then any continuous function $f: A \to R$ is extendable over X.*

PROOF. Consider a continuous function $f: A \to R$ and let

$$g_1(x) = 1 + \max(f(x), 0) \quad \text{and} \quad g_2(x) = 1 - \min(f(x), 0);$$

clearly $g_i: A \to R$ is continuous and $g_i(x) \geq 1$ for all $x \in A$ and $i = 1, 2$. Since $f(x) = g_1(x) - g_2(x)$ for $x \in A$, the function $F = G_1 - G_2: X \to R$, where G_i is a continuous extension of g_i over X for $i = 1, 2$, is a continuous extension of f over X.■

3.11.10. THEOREM. *A Tychonoff space X is realcompact if and only if for every point $x_0 \in \beta X \setminus X$ there exists a continuous function $h: \beta X \to I$ such that $h(x_0) = 0$ and $h(x) > 0$ for any $x \in X$.*

PROOF. Let X be a realcompact space and $x_0 \in \beta X \setminus X$. Since $\tilde{X} = X \cup \{x_0\} \subset \beta X$ satisfies (RC1), there exists a continuous function $f: X \to R$ which cannot be continuously extended over $\tilde{X}$. By the last lemma one can suppose that $f(x) \geq 1$ for all $x \in X$. The function $1/f: X \to I$ is extendable to a continuous function $h: \beta X \to I$ which obviously satisfies $h(x) > 0$ for any $x \in X$. If we had $h(x_0) \neq 0$, we could define a continuous extension $\tilde{f}: \tilde{X} \to R$ of the function f by letting $\tilde{f}(x) = 1/h(x)$; hence we have $h(x_0) = 0$.

Let us now suppose that for every point $z \in \beta X \setminus X$ there exists a continuous function $h_z: \beta X \to I$ such that $h_z(z) = 0$ and $h_z(x) > 0$ for any $x \in X$. We have then $X = \bigcap_{z \in \beta X \setminus X} h_z^{-1}((0,1])$ and X is realcompact by 3.11.7 and the second part of 3.11.8.■

The characterization of realcompact spaces given in Theorems 3.11.3 and 3.11.10, as well as the definition of this class of spaces, are external; an internal characterization of realcompactness is contained in the next theorem. The symbol $\mathcal{D}_0(X)$ that appears in the formulation of the theorem denotes the family of all functionally closed subsets of a Tychonoff space X.

3.11.11. THEOREM. *A Tychonoff space X is realcompact if and only if every ultrafilter in $\mathcal{D}_0(X)$ which has the countable intersection property has non-empty intersection.*

PROOF. Consider a Tychonoff space X which is not realcompact, and take a point $x_0 \in \beta X \setminus X$ such that for every function f in the family $\mathcal{F} = \{f \in I^{\beta X} : f(x_0) = 0\}$ we have $Z_f = X \cap f^{-1}(0) \neq \emptyset$. The family $\mathcal{Z} = \{Z_f\}_{f \in \mathcal{F}} \subset \mathcal{D}_0(X)$ is closed with respect to countable intersections and thus it has the countable intersection property. Clearly, $\mathcal{Z}$ is a filter in $\mathcal{D}_0(X)$.

We shall show that $\mathcal{Z}$ is an ultrafilter in $\mathcal{D}_0(X)$. Consider any family $\mathcal{Z}' \subset \mathcal{D}_0(X)$ that has the finite intersection property and contains $\mathcal{Z}$. Take a $Z \in \mathcal{Z}'$, a continuous function $g: X \to I$ satisfying $g^{-1}(0) = Z$, and the continuous extension $G: \beta X \to I$ of g. If $x_0 \notin G^{-1}(0)$, then there exists an $f \in \mathcal{F}$ such that $f^{-1}(0) \cap G^{-1}(0) = \emptyset$ and

$$Z_f \cap Z = X \cap f^{-1}(0) \cap G^{-1}(0) = \emptyset,$$

which is impossible. Thus we have $x_0 \in G^{-1}(0)$, i.e., $G \in \mathcal{F}$, which implies that $Z = Z_G \in \mathcal{Z}$. Hence, $\mathcal{Z}$ is an ultrafilter in $\mathcal{D}_0(X)$ which has the countable intersection property; since $\bigcap \mathcal{Z} = \emptyset$, the space X does not satisfy the condition in the theorem.

Now, consider a realcompact space X and an ultrafilter $\mathcal{Z} = \{Z_s\}_{s \in S}$ in $\mathcal{D}_0(X)$ which has the countable intersection property. From the maximality of $\mathcal{Z}$ it follows that $\mathcal{Z}$ is closed with respect to countable intersections. The family $\{\overline{Z}_s\}_{s \in S}$ of closures of members of $\mathcal{Z}$ in βX has non-empty intersection. To conclude the proof it suffices to show that any $x_0 \in \bigcap_{s \in S} \overline{Z}_s$ belongs to X, because then $x_0 \in \bigcap_{s \in S} Z_s$.

Assume that $x_0 \in \beta X \setminus X$ and consider a continous function $f: \beta X \to I$ such that $f(x_0) = 0$ and $f(x) > 0$ for any $x \in X$. Let $Z_i = X \cap f^{-1}([0, 1/i])$, clearly $Z_i \in \mathcal{D}_0(X)$ for $i = 1, 2, \ldots$ Since $f^{-1}([0, 1/i))$ is a neighbourhood of x_0, we have $Z_i \cap Z_s \neq \emptyset$ for every $s \in S$ and $i = 1, 2, \ldots$ From the maximality of $\mathcal{Z}$ it follows that $Z_i \in \mathcal{Z}$ for $i = 1, 2, \ldots$, and thus $\bigcap_{i=1}^{\infty} Z_i \in \mathcal{Z}$. Since $\bigcap_{i=1}^{\infty} Z_i = \emptyset$, we have a contradiction which shows that $x_0 \in X$.■

Theorem 3.11.11 yields

3.11.12. THEOREM. *Every Lindelöf space is realcompact.*■

The last theorem, along with 3.8.13, shows that there exist realcompact spaces which are not k-spaces (cf. Exercise 3.3.E(a)).

3.11.13. EXAMPLE. Since the Sorgenfrey line K is a Lindelöf space (see Example 3.8.14) it is also a realcompact space. Thus the Cartesian product $K \times K$ is realcompact; this is an example of a non-normal realcompact space (see Example 2.3.12). The space $K \times K$ contains the discrete space $D(\mathfrak{c})$ as a closed subspace, so that $D(\mathfrak{c})$ is a realcompact space. ■

In connection with the last example one can ask whether every discrete space is realcompact. This question is of set-theoretic character; it is equivalent to the question if every cardinal number is non-measurable (see Exercise 3.11.D(a)). To define a non-measurable cardinal number we need an auxiliary notion.

Let $\mathcal{A}$ be a family of sets closed with respect to countable unions; by a *countably additive two-valued measure* defined on $\mathcal{A}$ we understand any function μ from $\mathcal{A}$ to $\{0,1\}$ satisfying the condition

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

whenever $A_i \in \mathcal{A}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Now, a cardinal number $\mathfrak{m}$ is called *non-measurable* provided that the only countably additive two-valued measure defined on the family of all subsets of a set X of cardinality $\mathfrak{m}$ which vanishes on all one-point sets is the trivial measure, identically equal to zero.

Obviously, the class $\mathcal{N}$ of all non-measurable cardinal numbers contains the number $\aleph_0$. One can prove that if $\mathfrak{m} \in \mathcal{N}$, then every cardinal number less than $\mathfrak{m}$, the sum of any family $\{\mathfrak{m}_s\}_{s\in S}$ of cardinal numbers from $\mathcal{N}$ with $|S| \leq \mathfrak{m}$ (i.e., the cardinality of the set $\bigcup_{s\in S} A_s$, where $|A_s| = \mathfrak{m}_s$ and $A_s \cap A_{s'} = \emptyset$ whenever $s \neq s'$), and the cardinal number $2^{\mathfrak{m}}$ also belong to $\mathcal{N}$. One can also prove that the first strongly inaccessible cardinal number belongs to $\mathcal{N}$ (a cardinal number $\aleph_\alpha$ is *strongly inaccessible* if it is weakly inaccessible and $2^{\mathfrak{m}} < \aleph_\alpha$ for every $\mathfrak{m} < \aleph_\alpha$). From the Remark to Exercise 3.1.H(a) it follows that the assumption that all cardinal numbers are non-measurable (and in fact that no cardinal number is strongly inaccessible) is consistent with the axioms of set theory.

Let us make a few observations on invariance and inverse invariance of realcompactness. Since the space of all countable ordinal numbers is a continuous image of the space $D(\mathfrak{c})$, it follows that realcompactness is not an invariant of continuous mappings. It turns out that realcompactness is not an invariant of either open mappings (see Exercise 3.11.E) or perfect mappings (cf. Exercise 3.11.G). The last result follows from the fact that there exists a Tychonoff space which is not realcompact but can be represented as the union of two closed realcompact subspaces (see Theorem 3.7.22). However, the construction of such a space is fairly complicated and will not be described in this book.

From Theorems 3.11.4, 3.11.5 and 3.7.26 we obtain

3.11.14. THEOREM. *If there exists a perfect mapping $f: X \to Y$ of a Tychonoff space X to a realcompact space Y, then the space X is realcompact.* ■

We close this section by constructing for every Tychonoff space X a realcompact space υX whose properties parallel those of βX.

3.11.15. LEMMA. *Let X be a topological space and A a subspace of X. If every continuous real-valued function on A is continuously extendable over X, then every continuous mapping of A to a power R^m of the real line also is continuously extendable over X. Moreover, if A is dense in X, then every continuous mapping of A to a closed subspace B of such a power is continuously extendable over X.*

PROOF. Consider a continuous mapping $f: A \to \prod_{s\in S} R_s$, where $R_s = R$ for $s \in S$. For every $s \in S$ there exists a continuous extension $\tilde{f}_s: X \to R_s$ of the composition $p_s f: A \to R_s$. One readily sees that the diagonal $F = \triangle_{s\in S} \tilde{f}_s: X \to \prod_{s\in S} R_s$ is a continuous extension of f. If $\overline{A} = X$ and $f: A \to B = \overline{B} \subset \prod_{s\in S} R_s$, then for a continuous extension $F: X \to \prod_{s\in S} R_s$ of the composition $i_B f: A \to \prod_{s\in S} R_s$ we have $F(X) = F(\overline{A}) \subset \overline{F(A)} \subset \overline{B} = B$, so that the restriction $F_B: X \to B$ is a continuous extension of f. ■

3.11.16. THEOREM. *For every Tychonoff space X there exists exactly one (up to a homeomorphism) realcompact space υX which satisfies the following two conditions:*

(i) *There exists a homeomorphic embedding $\upsilon: X \to \upsilon X$ such that $\overline{\upsilon(X)} = \upsilon X$.*

(ii) *For every continuous real-valued function $f: X \to R$ there exists a continuous function $\tilde{f}: \upsilon X \to R$ such that $\tilde{f}\upsilon = f$.*

The space υX satisfies also the condition

(iii) *For every continuous mapping $f: X \to Y$ of X to a realcompact space Y there exists a continuous mapping $\tilde{f}: \upsilon X \to Y$ such that $\tilde{f}\upsilon = f$.*

PROOF. For every $f: X \to R$ take the continuous extension $F: \beta X \to \omega R$, where ωR is the one-point compactification of the real line, and let $W_f = F^{-1}(R)$; one clearly has $X \subset W_f$. Let $\mathcal{F}$ be the family of all continuous real-valued functions defined on X and let $\upsilon X = \bigcap_{f\in\mathcal{F}} W_f$. The mapping $\upsilon: X \to \upsilon X$ defined by letting $\upsilon(x) = \beta(x)$ for $x \in X$ is a homeomorphic embedding, and $\overline{\upsilon(X)} = \upsilon(X)$, i.e., condition (i) is satisfied; it follows from the construction that υX also satisfies condition (ii). The fact that υX is realcompact follows from Corollaries 3.11.7 and 3.11.8.

From Theorem 3.11.3, properties (i) and (ii), and Lemma 3.11.15 it follows that υX satisfies also condition (iii).

If $\upsilon_1 X$ is a realcompact space satisfying conditions (i) and (ii), then $\upsilon_1 X$ satisfies also (iii), which implies – as in the proof of Theorem 3.5.4 – that $\upsilon_1 X$ is homeomorphic to υX. ■

The space υX is called the *Hewitt realcompactification of* X.

Let us observe that another construction of the Hewitt realcompactification of X is indicated in the proof of Theorem 3.11.3: the space υX can be obtained by taking in $\prod_{f\in\mathcal{F}} R_f$, where $\mathcal{F}$ is the family of all continuous real-valued functions defined on X and $R_f = R$ for $f \in \mathcal{F}$, the closure of the image of the space X under the mapping $F = \triangle_{f\in\mathcal{F}} f$ (cf. Exercise 3.11.F(b)).

Historical and bibliographic notes

Realcompact spaces were introduced by Hewitt in [1948]; they were originally defined by a condition close to the characterization given here in Problem 3.12.22(g); the equivalence of our definition to the original one was established by Katětov in [1951]. Nachbin in [1950] defined independently the same class of spaces in terms of uniformities, namely, by the condition that the coarsest uniformity making all continuous real-valued functions uniformly continuous is complete (see Examples 8.1.19 and 8.3.19 and Exercise 8.1.D). Hewitt's paper [1948] contains Theorems 3.11.1, 3.11.11, 3.11.12 and 3.11.16; Theorems 3.11.3, 3.11.4 and 3.11.5 were proved by Shirota in [1952] (Theorem 3.11.4 also by Katětov in [1951]). Theorem 3.11.10 was established by Mrówka in [1957]. The results on the invariance of the class $\mathcal{N}$ of non-measurable cardinal numbers under arithmetical operations mentioned in the text were obtained by Ulam in [1930]; non-measurability of the first strongly inaccessible aleph (and even some larger cardinal numbers) was proved by Keisler and Tarski in [1963]. The proofs of these results and further information can be found in Kuratowski and Mostowski's book [1976]. Some theorems on invariance of realcompactness under mappings were established by Isiwata in [1967] and by Kenderov in [1967]. A Tychonoff space which is not realcompact, but which can be represented as the union of its two closed realcompact subspaces, was described in Mrówka's paper [1958] (a correction in [1970]), a simpler example was given by Mysior in [1981]; it was observed by R. L. Blair (cited in Isbell [1962a]) that this implies the fact that realcompactness is not an invariant of perfect mappings. In our exposition of realcompactness we follow Engelking and Mrówka [1958] and van der Slot [1972]. Gillman and Jerison's book [1960] contains an important development of the theory of realcompact spaces. A survey of subsequent results can be found in Weir's book [1975].

Exercises

3.11.A (Gillman and Jerison [1960]). Let A be a realcompact subspace and B a compact subspace of a Tychonoff space X. Prove that $A \cup B \subset X$ is a realcompact space.

3.11.B. (a) (Shirota [1952], Gillman and Jerison [1960]) Prove that for every Tychonoff space X the following conditions are equivalent:

(1) *The space X is hereditarily realcompact.*

(2) *For every $x \in X$ the space $X \setminus \{x\}$ is realcompact.*

(3) *Every Tychonoff space that can be mapped onto X by a continuous mapping with compact fibers is realcompact.*

(4) *Every Tychonoff space that can be mapped onto X by a continuous one-to-one mapping is realcompact.*

Hint. Apply Exercise 3.11.A and Corollary 3.11.8.

(b) Deduce from (a) that the Niemytzki plane is realcompact.

(c) Deduce from (b) and the proof of Theorem 3.11.3 that R^c is not normal (cf. Exercises 2.3.E(a) and 3.1.H(a)).

3.11.C. Observe that a Tychonoff space X is pseudocompact if and only if $\upsilon X = \beta X$.

3.11.D. (a) (Mackey [1944], Hewitt [1950]) Prove that the space $D(\mathfrak{m})$ is realcompact if and only if $\mathfrak{m}$ is a non-measurable cardinal number.

Hint. Apply Theorem 3.11.11.

(b) (Gillman and Jerison [1960]) Show that the sum $\bigoplus_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is realcompact if and only if all spaces X_s and the space $D(|S|)$ are realcompact.

Hint. The sum $\bigoplus_{s\in S} X_s$ is homeomorphic to a closed subspace of the Cartesian product $D(|S|) \times \prod_{s\in S} X_s$.

3.11.E (Frolík [1961]). Observe that a normal space which is a continuous image of a realcompact space under an open mapping is not necessarily realcompact.

Hint. Take as the range the space W_0 of countable ordinal numbers and as the domain the sum $\bigoplus_{\alpha<\omega_1} X_\alpha$, where $X_\alpha = \{x \in W_0 : x \leq \alpha\}$.

3.11.F. (a) (Vulih [1952], Engelking [1964]) Prove that a continuous mapping $f: A \to Y$ of a dense subspace of a topological space X to a realcompact space Y has a continuous extension over X if and only if for every sequence $F_1, F_2, \ldots$ of functionally closed subsets of Y such that $\bigcap_{i=1}^{\infty} F_i = \emptyset$, we have $\bigcap_{i=1}^{\infty} \overline{f^{-1}(F_i)} = \emptyset$, where the bar denotes the closure in X.

Hint. Consider first the case of $Y = R$; apply Theorem 3.2.1.

(b) (Gillman and Jerison [1960]) Show that by taking in the construction of βX described in Exercise 3.6.K(a) only the ultrafilters in $\mathcal{D}_0(X)$ which have the countable intersection property, one obtains the Hewitt realcompactification υX.

3.11.G (Ponomarev [1959], Frolík [1961]). Prove that if there exists a perfect open mapping $f: X \to Y$ of a realcompact space X onto a Tychonoff space Y, then Y is realcompact.

Hint. Apply Exercise 1.5.L(a) and Theorems 3.7.16 and 3.11.10.

Remark. In [1984] Ohta defined, under an additional set-theoretic assumption, a closed-and-open mapping of a realcompact space onto a Tychonoff space that is not realcompact.

3.11.H (Pasynkov [1965]). Observe that a topological space X is realcompact if and only if it is homeomorphic to the limit of an inverse system of Lindelöf spaces (or – equivalently – of an inverse system of subspaces of the Hilbert cube $I^{\aleph_0}$).

3.11.I (Fine and Gillman [1960]). Prove that if X is a locally compact realcompact space, then every non-empty G_δ-set in $\beta X \setminus X$ has a non-empty interior.

Hint. Observe first that every non-empty G_δ-set in $\beta X \setminus X$ contains a non-empy set of the form $f^{-1}(0)$, where $f: \beta X \to I$. Then define a sequence $U_1, U_2, \ldots$ of non-empty open subsets of X such that $f(x) \leq 1/i$ for $x \in U_i$ and the closure of U_i is compact for $i = 1, 2, \ldots$, and show that the set $(\beta X \setminus X) \cap \overline{\bigcup_{i=1}^{\infty} U_i}$ is contained in $f^{-1}(0)$ and has a non-empty interior in $\beta X \setminus X$.

3.12. Problems

Further characterizations of compactness: complete accumulation points and the Alexander subbase theorem

3.12.1 (Kuratowski and Sierpiński [1921], Vietoris [1921], Alexandroff and Urysohn [1929] (announcement [1923])). A point x in a topological space X is called a *complete accumulation point of a set* $A \subset X$ if $|A \cap U| = |A|$ for every neighbourhood U of the point x.

Prove that for a Hausdorff space X the following conditions are equivalent:

(1) *The space X is compact.*

(2) *Every infinite subset of X has a complete accumulation point.*

(3) *For every decreasing transfinite sequence $F_0 \supset F_1 \supset \ldots \supset F_\xi \supset \ldots, \xi < \alpha$ of non-empty closed subsets of X the intersection $\bigcap_{\xi<\alpha} F_\xi$ is non-empty.*

Hint. For every limit ordinal number λ there exists an initial number cofinal with λ (see Kuratowski and Mostowski [1976]).

Remark. One readily shows that if every uncountable set $A \subset X$ has a complete accumulation point, then $l(X) \leq \aleph_0$; on the other hand the space of all ordinal numbers $< \omega_{\omega_0}$ with the topology induced by the natural linear order $<$ (see Problem 1.7.4) is a Lindelöf space that has no complete accumulation point. Alexandroff and Urysohn proved in [1929] that if $l(X) \leq \aleph_0$, then every uncountable set $A \subset X$ of a regular cardinality has a complete accumulation point. Miščenko in [1962a] defined a Tychonoff non-Lindelöf space X in which every uncountable set $A \subset X$ of a regular cardinality has a complete accumulation point and observed that every countably paracompact space with the last property is a Lindelöf space (cf. Problem 3.12.7(d)).

3.12.2 (Alexander [1939]). (a) Let X be a Hausdorff space and $\mathcal{P}$ a subbase for X; show that the space X is compact if and only if every cover of X by members of $\mathcal{P}$ has a finite subcover (this is the *Alexander subbase theorem*).

Hint. In the class of all families of open subsets of X the property of not containing a finite subfamily that covers X is a property of finite character. If X is not compact, then there exists a maximal family $\mathcal{R}$ which has the above property and is a cover of X. Show that if $G_1, G_2, \ldots, G_k$ are open sets and $G_i \notin \mathcal{R}$ for $i = 1, 2, \ldots, k$, then $G_1 \cap G_2 \cap \ldots \cap G_k \notin \mathcal{R}$ and that if $G_1 \notin \mathcal{R}$ and $G_1 \subset G_2$, then $G_2 \notin \mathcal{R}$. Deduce that $\mathcal{R} \cap \mathcal{P}$ is a cover of X.

(b) Prove the Tychonoff theorem applying the characterization of compactness established in part (a).

Linearly ordered spaces III (see Problems 1.7.4, 2.7.5, 3.12.12(f), 5.5.22, 6.3.2 and 8.5.13(j))

3.12.3. (a) (Haar and König [1910]) Show that a space X with the topology induced by a linear order $<$ is compact if and only if every subset $A \subset X$ has a least upper bound.

Hint. The least upper bound of the empty set is the smallest element of X.

(b) Prove that every space X with the topology induced by a linear order $<$ has a compactification cX whose topology is induced by a linear order $<'$ such that $x <' y$ is equivalent to $x < y$ whenever $x, y \in X$.

Hint. Consider the set of cuts of X; apply Problem 2.7.5(a).

Remark. Linearly ordered compactifications of linearly ordered spaces were investigated by Fedorčuk in [1969] (announcement [1966a]) and by Kaufman in [1967].

(c) Verify that the topology on the set of all ordinal numbers $\leq \omega_1$ defined in Example 3.1.27 coincides with the topology induced by the natural linear order $<$.

(d) Consider the linear order $<$ in the square I^2 defined by letting $(x_1, y_1) < (x_2, y_2)$ whenever $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$ (this order is called the *lexicographic order*). Prove that the square with the topology induced by this linear order is compact and first-countable but is neither separable nor perfectly normal; verify that the square thus topologized contains the two arrows space defined in Exercise 3.10.C.

3.12.4. (a) (Mardešić and Papić [1962]) Show that for every linearly ordered space X we have $\chi(X) \leq c(X)$.

(b) (Lutzer and Bennett [1969]; for compact X, Mardešić and Papić [1962]) Show that for every linearly ordered space X we have $hl(X) \leq c(X)$, where hl is the hereditary Lindelöf number (cf. Problem 2.7.9 and Section 3.8).

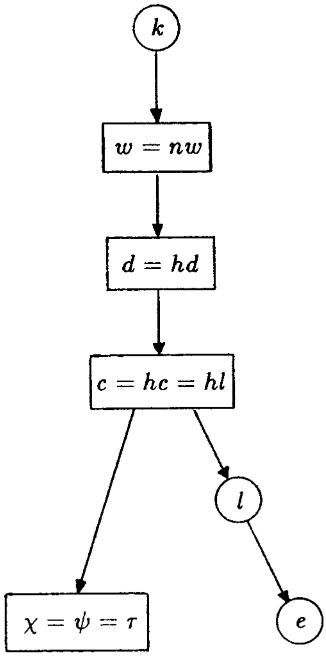
Hint. Applying 3.12.3(b) reduce the problem to the case of a compact space X. Observe that every open set $U \subset X$ can be represented as the union of a family of disjoint intervals, apply (a) and a suitable generalization of Exercise 3.8.A(b).

(c) (Skula [1965]) Show that for every linearly ordered space X we have $d(X) = hd(X)$.

Hint (Lutzer and Bennett [1969]). Let D be a dense subset of X such that $|D| = d(X)$. For an $A \subset X$ choose one point from each non-empty intersection $A \cap (a, b)$, where $a, b \in D$ and $a < b$; verify that the union E of the set thus obtained and the set of all isolated points of A is dense in A; apply (b) to show that $|E| \leq d(X)$.

(d) Show that for every linearly ordered space X we have $\chi(X) = \psi(X) = \tau(X)$ and $w(X) = nw(X)$.

(e) Verify that the following diagram (cf. Problem 1.7.12(a)) contains all equalities and inequalities between the cardinal functions appearing in it which hold in the class of all linearly ordered spaces (the symbol $k(X)$ denotes here the cardinality of X; for relations between $d(X)$ and $c(X)$ see the remark to Problem 2.7.9(f)).



Remark. Hajnal and Juhász proved in [1969] that for a linearly ordered space X we either have $d(X) = c(X)$ or $d(X) = c(X)^+$.

H-closed and H-minimal spaces

3.12.5. (a) (Alexandroff and Urysohn [1929] (announcement [1923]), Alexandroff [1939]) A Hausdorff space X is called *H-closed* if X is a closed subspace of every Hausdorff space in which it is contained.

Prove that for a Hausdorff space X the following conditions are equivalent:

(1) *The space X is H-closed.*

(2) *For every family $\{V_s\}_{s\in S}$ of open subsets of X which has the finite intersection property the intersection $\bigcap_{s\in S}\overline{V}_s$ is non-empty.*

(3) *Every ultrafilter in the family of all open subsets of X converges.*

(4) *Every open cover $\{U_s\}_{s\in S}$ of the space X contains a finite subfamily $\{U_{s_1}, U_{s_2}, \ldots, U_{s_k}\}$ such that $\overline{U}_{s_1}\cup\overline{U}_{s_2}\cup\ldots\cup\overline{U}_{s_k} = X$.*

Deduce that a regular space is H-closed if and only if it is compact and give an example of a non-regular H-closed space.

Note that H-closedness is not hereditary with respect to closed subsets.

(b) (Katětov [1940]) Note that H-closedness is hereditary with respect to closed domains and is invariant under continuous mappings onto Hausdorff spaces.

(c) (M. H. Stone [1937], Katětov [1940]) Prove that a Hausdorff space X is compact if and only if all closed subspaces of X are H-closed.

Hint (Katětov [1940]). Show that every family $\mathcal{F}$ of non-empty H-closed subspaces of a Hausdorff space X which is linearly ordered by inclusion has non-empty intersection. To this end, observe that the family of all open sets $U \subset X$ for which there exists an $F \in \mathcal{F}$ such that $F \subset \overline{F\cap U}$ has the finite intersection property.

(d) (Chevalley and O. Frink [1941]) Prove that the Cartesian product $\prod_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is H-closed if and only if all spaces X_s are H-closed.

(e) (Parhomenko [1939], Katětov [1940]) A Hausdorff space X is called *H-minimal* if every one-to-one continuous mapping of X onto a Hausdorff space is a homeomorphism.

Give an example of an H-minimal space which is not compact and prove that a Hausdorff space X is H-minimal if and only if X is semiregular and H-closed.

Hint. Apply Problem 1.7.8(b).

Remark. Besides H-closed and H-minimal spaces one considers, analogously defined, $\mathcal{P}$-closed and $\mathcal{P}$-minimal spaces for various topological properties $\mathcal{P}$. This topic has an extensive literature discussed in Berri, Porter and Stephenson's paper [1968].

3.12.6 (Katětov [1940]). Let $\mathcal{O}(X)$ denote the family of all open subsets of a Hausdorff space X. Consider the family $T(X)$ of all ultrafilters in $\mathcal{O}(X)$ which do not converge to a point of X and generate a topology on the set $\tau X = X \cup T(X)$ taking for neighbourhoods of a point $x \in X$ the family of all neighbourhoods of x in X and for neighbourhoods of an $\mathcal{F} \in T(X)$ the family of all sets $\{\mathcal{F}\} \cup U$, where $U \in \mathcal{F}$. The space τX thus obtained is called the *Katětov extension of X*.

(a) Observe that X is an open dense subspace of τX and that the subspace $\tau X \setminus X$ of τX is discrete.

(b) Prove that τX is an H-closed space and that for every continuous mapping $f: X \to Y$ of X to a Hausdorff space Y such that $\overline{f(X)} = Y$ there exist a subspace Z of τX that contains X and a continuous extension $F: Z \to Y$ of the mapping f satisfying $F(Z) = Y$.

Note that the above property characterizes τX (up to a homeomorphism) in the family of all Hausdorff spaces containing X as a dense subset.

(c) Show that if, in the situation described in (b), the space Y is compact or is an H-closed space containing X as a dense subset and $f = \mathrm{id}_X$, then $Z = \tau X$.

Cardinal functions III (see Problems 1.7.12, 1.7.13, 2.7.9–2.7.11, 3.12.4, 3.12.12(h), 3.12.12(j) and 8.5.17)

3.12.7. (a) Note that $hnw(X) = nw(X)$ and $h\psi(X) = \psi(X)$ for every topological space X, and that $\psi(X) \leq hl(X)$ for every Hausdorff space X.

(b) (Smirnov [1950]; for $\aleph_0$, Sierpiński [1921a]) Prove that a topological space X satisfies $hl(X) \leq \aleph_\alpha$ if and only if for every increasing transfinite sequence $U_0 \subset U_1 \subset \ldots \subset U_\xi \subset \ldots$, $\xi < \omega_{\alpha+1}$ of open subsets of X there exists a $\xi_0 < \omega_{\alpha+1}$ such that $U_\xi = U_{\xi_0}$ for every $\xi \geq \xi_0$ (cf. Problem 2.7.9(e)).

(c) (Sierpiński [1921a]) Give an example of a Hausdorff space X such that $hl(X) > hd(X) = \aleph_0$.

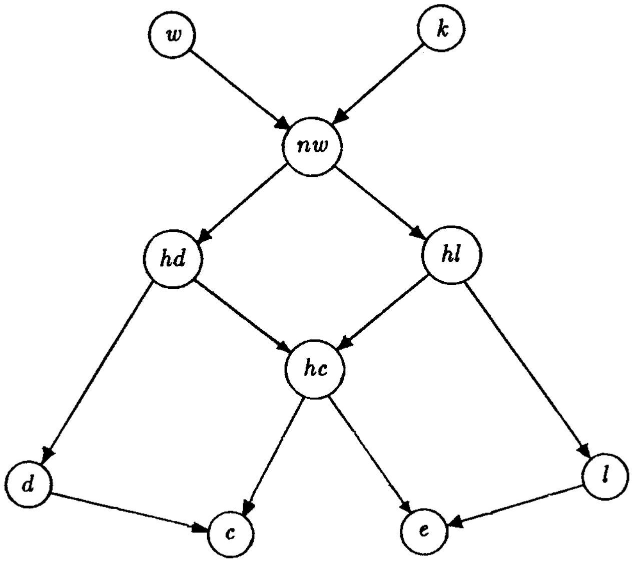
Hint. Consider a relation $\prec$ which well-orders the real line R and generate a topology on R taking as neighbourhoods of $x \in R$ the sets $U_i(x) = \{x\} \cup \{y \in R : |x - y| < 1/i$ and $y \prec x\}$ for $i = 1, 2, \ldots$

Remark. Todorčević described in [1986] a completely regular space X satisfying $hl(X) > hd(X) > \aleph_0$. A regular space X such that $hl(X) > hd(X) = \aleph_0$ is called an *S-space.* The first example of an S-space was given by M. E. Rudin in [1972]: she defined under the assumption that there exists a Souslin space (see the remark to Problem 2.7.9(f)) a normal hereditarily separable space X which is not Lindelöf (and satisfies $\psi(X) > \aleph_0$). An example of a hereditarily normal and hereditarily separable countably compact space X which is not a Lindelöf space was constructed in [1974] by Hajnal and Juhász under the assumption of the continuum hypothesis. Ostaszewski in [1976] deduced from a set-theoretic statement, consistent with the axioms, that there exists a hereditarily separable countably compact first-countable locally compact space which is not a Lindelöf space. Blending the last two constructions, Juhász, Kunen and M. E. Rudin gave in [1976] – under the assumption of the continuum hypothesis – a much simpler example of a hereditarily separable first-countable locally compact space which is not a Lindelöf space. From Ostaszewski's example and a result in Szentmiklóssy [1978] it follows that the existence of compact S-spaces is independent of the axioms of set theory. As shown by Todorčević in [1983], the existence of an S-space also is independent of the axioms of set theory (for a proof and further information see Roitman's survey [1984]).

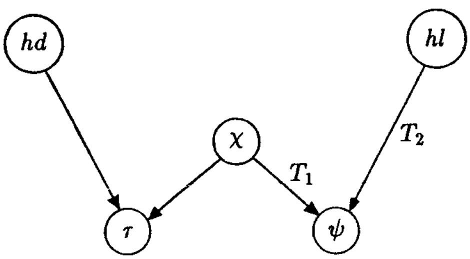
Todorčević also described in [1986] a completely regular space X satisfying $hd(X) > hl(X) > \aleph_0$. A regular space X such that $hd(X) > hl(X) = \aleph_0$ is called an *L-space.* No example of an L-space is known within the realm of classical set theory (the Hausdorff space X defined in the hint to Problem 2.7.9(f) satisfies $hd(X) > hl(X) = \aleph_0$). A Souslin space obviously is an L-space (cf. Problem 3.12.4(e)) and so are the spaces, constructed under the assumption of the continuum hypothesis, cited in the final paragraph of the remark to Problem 2.7.9(f) (the space constructed by Hajnal and Juhász satisfies $hl(X) < \tau(X)$). As shown by Juhász in [1970], the existence of a compact L-space is independent of the axioms of set theory (under the assumption of the continuum hypothesis a compact L-space was constructed by Kunen in [1981]).

(d) (Kuratowski and Sierpiński [1921]) Show that a regular space X is a hereditary Lindelöf space if and only if every uncountable set $A \subset X$ has a condensation point which is contained in A, i.e., if $A \cap A^0 \neq \emptyset$ (cf. Problem 1.7.11).

(e) Applying the remark to part (c) above, verify that the following diagram (cf. Problem 1.7.12(a)) contains all inequalities that hold between the cardinal functions that appear in it (the symbol $k(X)$ denotes here the cardinality of X).



(f) Applying the remark to part (c) above, verify that the following diagram contains all inequalities that hold between the cardinal functions that appear in it (the symbol T_i facing an arrow means that the corresponding inequality holds for T_i-spaces).



(g) For a cardinal function f we denote by $hclf$ (by $hopf$) the cardinal function whose value on a space X is equal to $\sup f(M)$, where the sup is taken over all closed (open) subspaces M of the space X.

Show that for every topological space X we have

$$\begin{aligned} hcld(X) &\le hd(X), & hopd(X) &= d(X),\\ hclc(X) &= hc(X), & hopc(X) &= c(X),\\ hcll(X) &= l(X), & hopl(X) &= hl(X),\\ hcle(X) &= e(X), & hope(X) &\le hc(X). \end{aligned}$$

Give an example of a Hausdorff space X such that $hcld(X) = \aleph_0 < hd(X)$ (cf. Problems 3.12.9(d) and (e)). Show that $hope(X) = hc(X)$ for every T_1-space X and give an example of a T_0-space X such that $hope(X) < hc(X)$.

Hint. To obtain the former example, modify the topology of the two arrows space (see Exercise 3.10.C) by declaring as open all sets of the form $U \setminus A$, where U is open in the original topology and A is a countable subset of C_0. Consider first a closed subspace F contained in C_0 and show that $|F| \leq \aleph_0$; apply Exercise 3.10.C(a).

Remark. The space defined in the hint is a modification of the first example of this kind given by Aull in [1975]. Malyhin has observed in [1986] that the union $X = X_0 \cup X_1 \subset D^{\aleph_1}$ of a countable dense subset X_0 and a non-separable subspace X_1 of $D^{\aleph_1}$ that intersects every nowhere dense subset of $D^{\aleph_1}$ in a countable set satisfies the inequality $hcld(X) = \aleph_0 < hd(X)$; X_1 can be defined under the assumption of the continuum hypothesis by a modification of the standard construction of a set with property L (see Kuratowski [1966]; in the modified construction Exercise 2.3.G is applied). We know no regular space X such that $hcld(X) < hd(X)$ defined without additional set-theoretic assumptions.

(h) (Zenor [1980]) Show that if a family $\{X_s\}_{s \in S}$ of topological spaces has the property that $hl(\prod_{s \in S_0} X_s) \leq \mathfrak{m}$ for every finite $S_0 \subset S$ and if $|S| \leq \mathfrak{m}$, then $hl(\prod_{s \in S} X_s) \leq \mathfrak{m}$.

(i) (Zenor [1980]) Prove that if $hc(X \times Y) \leq \mathfrak{m} \geq \aleph_0$, then $hl(X) \leq \mathfrak{m}$ or $hd(Y) \leq \mathfrak{m}$.

Hint. Apply part (b) and Problem 2.7.9(e).

3.12.8. (a) (Arhangel'skiĭ and Ponomarev [1968]) Show that if there exists a quotient mapping of X onto Y, then $\tau(Y) \leq \tau(X)$.

Hint. Apply Problem 1.7.13(b).

(b) Observe that if X is a k-space, then $\tau(X)$ is the least upper bound of all cardinal numbers of the form $\tau(F)$, where F is a compact subspace of X.

(c) The *tightness of a set* A in a topological space X is the smallest cardinal number $\mathfrak{m} \geq \aleph_0$ with the property that if $A \cap \overline{C} \neq \emptyset$, then there exists a $C_0 \subset C$ such that $|C_0| \leq \mathfrak{m}$ and $A \cap \overline{C_0} \neq \emptyset$; this cardinal number is denoted by $\tau(A, X)$.

Observe that for every topological space X and any set $A \subset X$ we have $\tau(A, X) \leq \chi(A, X)$.

Prove that if A_1 and A_2 are subsets of a space X such that $A_1 \subset A_2$ and for every set $F \subset A_2 \setminus A_1$ that is closed in A_2 there exist disjoint open subsets of X containing A_1 and F respectively, then $\tau(A_1, X) \leq \tau(A_1, A_2)\tau(A_2, X)$.

Deduce that for compact subsets F_1, F_2 of a Hausdorff space X such that $F_1 \subset F_2$ we have $\tau(F_1, X) \leq \tau(F_1, F_2)\tau(F_2, X)$ and that for every closed subset F of a regular space X and any point $x \in F$ we have $\tau(x, X) \leq \tau(x, F)\tau(F, X)$.

Hint. Let $\mathfrak{m} = \tau(A_1, A_2)\tau(A_2, X)$; assume that there exists a $C \subset X$ such that $A_1 \cap \overline{C} \neq \emptyset$ and yet $A_1 \cap [C]_{\mathfrak{m}} = \emptyset$, where $[C]_{\mathfrak{m}} = \bigcup\{\overline{M} : M \subset C \text{ and } |M| \leq \mathfrak{m}\}$. Verify that $A_1 \cap \overline{A_2 \cap [C]_{\mathfrak{m}}} = \emptyset$, take an open set $U \subset X$ such that $A_1 \subset U$ and $\overline{U} \cap A_2 \cap [C]_{\mathfrak{m}} = \emptyset$ and observe that $A_2 \cap \overline{U \cap C} \neq \emptyset$. Apply the inequality $\tau(A_2, X) \leq \mathfrak{m}$.

(d) Note that if $f: X \to Y$ is a closed mapping, then $\tau(f^{-1}(B), X) \leq \tau(B, Y)$ for every set $B \subset Y$.

Deduce that if $f: X \to Y$ is a closed mapping of a regular space X to a topological space Y and if for an $x \in X$ the inequalities $\tau(f(x), Y) \leq \mathfrak{m}$ and $\tau(x, f^{-1}f(x)) \leq \mathfrak{m}$ hold, then $\tau(x, X) \leq \mathfrak{m}$.

Note that if f is a perfect mapping, then the assumption of regularity of X can be omitted.

(e) (Arhangel'skiĭ [1972]) Define two normal spaces X and Y such that $\tau(X) = \tau(Y) = \aleph_0$ and $\tau(X \times Y) > \aleph_0$.

Hint. Take as X the sum of countably many copies of $A(\aleph_0)$ with all the accumulation points identified and as Y a similar space obtained from the sum of $\mathfrak{c}$ copies of $A(\aleph_0)$.

(f) (Malyhin [1972]) Show that if X is a locally compact space, then for every Hausdorff space Y we have $\tau(X \times Y) \leq \max(\tau(X), \tau(Y))$.

Prove that if a family $\{X_s\}_{s\in S}$ of topological spaces has the property that $\tau(\prod_{s\in S_0} X_s) \leq \mathfrak{m}$ for every finite $S_0 \subset S$ and if $|S| \leq \mathfrak{m}$, then $\tau(\prod_{s\in S} X_s) \leq \mathfrak{m}$.

Hint. Apply (d) in the proof of the first part.

(g) Show that for every family $\{X_s\}_{s\in S}$ of Hausdorff spaces such that $\tau(X_s) \leq \mathfrak{m}$ and $h(X_s) \leq \mathfrak{m}$ for $s \in S$ and $|S| \leq \mathfrak{m}$ we have $\tau(\prod_{s\in S} X_s) \leq \mathfrak{m}$.

3.12.9. (a) (Šapirovskiĭ [1972]) Prove that for every open cover $\{U_s\}_{s\in S}$ of a topological space X there exists an $S_0 \subset S$ with the property that for every $s \in S_0$ one can choose a point $x_s \in U_s$ in such a way that $x_s \neq x_{s'}$ whenever $s \neq s'$, the subspace $A = \{x_s : s \in S_0\}$ is discrete and $X = \bigcup_{s\in S_0} U_s \cup \overline{A}$.

Deduce that $X = \bigcup_{s\in S_0} U_s \cup \overline{A}$, where $|S_0| \leq hc(X)$ and $|A| \leq hc(X)$.

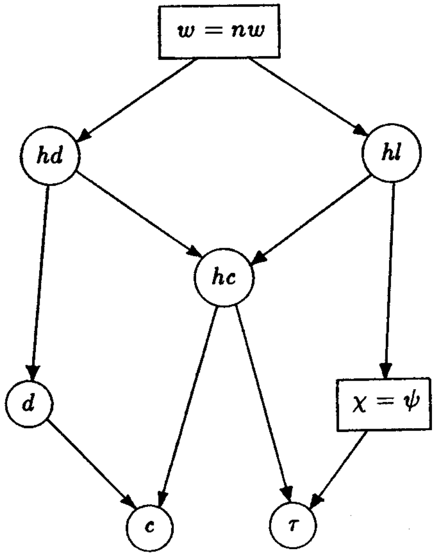
Hint. Applying transfinite induction definite points $x_0, x_1, \ldots, x_\xi, \ldots,\ \xi < \alpha$ of the space X and members $U_0, U_1, \ldots, U_\xi, \ldots,\ \xi < \alpha$ of the given cover such that $x_\xi \in U_\xi \setminus [\bigcup_{\gamma<\xi} U_\gamma \cup \overline{A}_\xi]$, where $A_\xi = \{x_\gamma : \gamma < \xi\}$, and $X = \bigcup_{\xi<\alpha} U_\xi \cup \overline{A}_\alpha$.

(b) (Šapirovskiĭ [1972]; for a compact X, Arhangel'skiĭ [1971]) Show that for every Hausdorff space X we have $\tau(X) \leq hc(X)h(X)$.

Deduce that $\tau(X) \leq hc(X)$ for every k-space X.

Hint. Applying 3.12.8(c) reduce the problem to the case of a compact X. Let $\mathfrak{m} = hc(X)$ and prove that if $\overline{C} \neq [C]_\mathfrak{m}$ for a $C \subset X$, then $[C]_\mathfrak{m} \neq C$. To this end, for a point $x_0 \in \overline{C} \setminus [C]_\mathfrak{m}$ take a family $\{U_s\}_{s\in S}$ of open subsets of X such that $x_0 \notin \overline{U}_s$ and $C \subset \bigcup_{s\in S} U_s$, and – applying (a) – choose $S_0 \subset S$ and $A \subset C$ such that $|S_0| \leq \mathfrak{m}$, $|A| \leq \mathfrak{m}$ and $C \subset \bigcup_{s\in S_0} U_s \cup \overline{A}$. Define a family $\{V_t\}_{t\in T}$ of open subsets of X such that $|T| \leq \mathfrak{m}$, $\{x_0\} = (C \cup \{x_0\}) \cap \bigcap_{t\in T} V_t$ and $\bigcap_{t\in T} \overline{V}_t = \bigcap_{t\in T} V_t$. Note that $\chi(Z, X) \leq \mathfrak{m}$ for $Z = \bigcap_{t\in T} \overline{V}_t$ and consider a base $\mathcal{B}$ for X at Z such that $|\mathcal{B}| \leq \mathfrak{m}$; form a set B by choosing one point from every intersection $C \cap U$, where $U \in \mathcal{B}$, and show that any $z_0 \in \overline{B} \cap Z$ satisfies $z_0 \in [C]_\mathfrak{m} \setminus C$.

(c) Applying the remark to Problem 3.12.7(c), verify that the diagram on the next page (cf. Problem 1.7.12(a)) contains all equalities and inequalities between the cardinal functions appearing in it which hold in the class of all compact spaces (clearly, for every compact space X we have $l(X) = e(X) = \aleph_0$).



Remark. Šapirovskiĭ proved in [1974] that for a compact space X we either have $hd(X) = hc(X)$ or $hd(X) = hc(X)^+$.

(d) Observe that for every space X we have $hd(X) \leq hcld(X)\tau(X)$. Note that $hcld(X) = hd(X)$ for every space X of countable tightness.

(e) Show that for every Hausdorff space X we have $hd(X) \leq hcld(X)h(X)$. Note that $hcld(X) = hd(X)$ for every space X of pointwise countable type.

Hint. Apply (b).

(f) Give examples of normal spaces X and Y such that $h(X) < \tau(X)$ and $\tau(Y) < h(Y)$.

3.12.10. (a) (Arhangel'skiĭ [1969]) Prove that for every Hausdorff space X we have $|X| \leq \exp[l(X)\chi(X)]$.

Hint (R. Pol [1974]). Modify the proof of Theorem 3.1.29.

(b) (Hajnal and Juhász [1967]) Prove that for every Hausdorff space X we have $|X| \leq \exp[c(X)\chi(X)]$.

Hint (R. Pol [1974]). Let $\mathfrak{m} = c(X)\chi(X)$. For every $x \in X$ choose a base $\mathcal{B}(x)$ at x satisfying $|\mathcal{B}(x)| \leq \mathfrak{m}$. Define an increasing transfinite sequence $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$, $\alpha < \tau$ of subsets of X, where τ is the initial number of cardinality $\mathfrak{m}^+$, such that $|A_\alpha| \leq 2^{\mathfrak{m}}$, and for every collection $\{\mathcal{U}_s\}_{s\in S}$ of subfamilies of $\bigcup\{\mathcal{B}(x) : x \in \bigcup_{\beta<\alpha} A_\beta\}$, where $|S| \leq \mathfrak{m}$ and $|\mathcal{U}_s| \leq \mathfrak{m}$ for $s \in S$, if $X \setminus \bigcup\{\overline{\bigcup \mathcal{U}_s} : s \in S\} \neq \emptyset$, then $A_\alpha \setminus \bigcup\{\overline{\bigcup \mathcal{U}_s} : s \in S\} \neq \emptyset$. Show that the union $A = \bigcup_{\alpha<\tau} A_\alpha$ equals X. To that end, assume that there exists a point $y \in X \setminus A$, let $\mathcal{B}(y) = \{V_s\}_{s\in S}$, where $|S| \leq \mathfrak{m}$, and for every $s \in S$ consider a maximal family $\mathcal{U}_s$ of pairwise disjoint open sets contained in the family $\{U \in \mathcal{B}(x) : x \in A \text{ and } U \cap V_s = \emptyset\}$.

(c) (Hajnal and Juhász [1967]) Prove that $|X| \leq \exp[hc(X)\psi(X)]$ holds for every T_1-space X (cf. part (i) below).

Hint (Hodel [1976]). Let $\mathfrak{m} = hc(X)\psi(X)$. For every $x \in X$ choose a family $\mathcal{U}(x)$ of open sets satisfying $\bigcap \mathcal{U}(x) = \{x\}$ and $|\mathcal{U}(x)| \leq \mathfrak{m}$. Define an increasing transfinite sequence $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$, $\alpha < \tau$ of subsets of X, where τ is the initial number of cardinality

$\mathfrak{m}^+$, such that $|A_\alpha| \le 2^{\mathfrak{m}}$ and for every subfamily $\{U_s\}_{s\in S}$ of $\bigcup\{\mathcal{U}(x) : x \in \bigcup_{\beta<\alpha} A_\beta\}$, where $|S| \le \mathfrak{m}$, and every family $\{B_t\}_{t\in T}$ of subsets of $\bigcup_{\beta<\alpha} A_\beta$, where $|T| \le \mathfrak{m}$ and $|B_t| \le \mathfrak{m}$ for $t \in T$, if $X \setminus [\bigcup_{s\in S} U_s \cup \bigcup_{t\in T} \overline{B_t}] \neq \emptyset$, then $A_\alpha \setminus [\bigcup_{s\in S} U_s \cup \bigcup_{t\in T} \overline{B_t}] \neq \emptyset$. Show that the union $A = \bigcup_{\alpha<\tau} A_\alpha$ equals X. To that end, assume that there exists a point $y \in X \setminus A$, let $X \setminus \{y\} = \bigcup_{t\in T} F_t$, where $|T| \le \mathfrak{m}$ and $\overline{F_t} = F_t$ for $t \in T$, and apply Problem 3.12.9(a) to obtain for every $t \in T$ a subfamily $\mathcal{V}_t$ of $\bigcup\{\mathcal{U}(x) : x \in A \cap F_t\}$ and a set $B_t \subset A \cap F_t$, both of cardinality $\le \mathfrak{m}$, such that $A \cap F_t \subset (\bigcup \mathcal{V}_t) \cup \overline{B_t} \subset X \setminus \{y\}$. Take the union $\bigcup_{t\in T} \mathcal{V}_t$ as the family $\{U_s\}_{s\in S}$.

(d) (Šapirovskiĭ [1972]) Show that for every Hausdorff space X we have $\psi(X) \le \exp hc(X)$.

Hint. Take a point $x_0 \in X$ and, applying Problem 3.12.9(a), find a family $\{U_s\}_{s\in S_0}$ of open subsets of X and a set $A \subset X_0 = X \setminus \{x_0\}$ such that $x_0 \notin \overline{U}_s$, $|S_0| \le hc(X)$, $|A| \le hc(X)$ and $X_0 = \bigcup_{s\in S_0} U_s \cup (\overline{A} \cap X_0)$. Applying Exercise 3.1.F(a) observe that for $X_1 = (\overline{A} \cap X_0) \cup \{x_0\}$ we have $\psi(X_1) \le \exp hc(X)$.

(e) (Hajnal and Juhász [1967]; for a regular X, de Groot [1965]) Prove that for every Hausdorff space X we have $|X| \le \exp\exp hc(X)$.

Hint. Apply parts (c) and (d).

Remark. The weaker inequality $|X| \le \exp\exp\exp hc(X)$ was proved for completely regular spaces by Isbell in [1964a] and for Hausdorff spaces by de Groot in [1965] and Efimov in [1965]. As proved by Todorčević in [1983], the assumption that every Hausdorff space X with $hc(X) \le \aleph_0$ satisfies $\psi(X) \le \aleph_0$, and thus $|X| \le \mathfrak{c}$, is consistent with the axioms of set theory; as shown by Hajnal and Juhász in [1973] (see also Juhász [1984]), the existence of a Hausdorff space X such that $hc(X) = \aleph_0$ and $|X| = 2^{\mathfrak{c}}$ also is consistent with the axioms of set theory.

(f) (Smirnov [1950]; for $hl(X) = \aleph_0$, Alexandroff and Urysohn [1929]) Prove that for every Hausdorff space X we have $|X| \le \exp hl(X)$.

Hint. Apply part (c).

(g) (Šapirovskiĭ [1972]) Prove that for every T_1-space X there exists a set $S \subset X$ of cardinality $\le [\psi(X)]^{hc(X)}$ such that $X = \bigcup\{\overline{M} : M \subset S \text{ and } |M| \le hc(X)\}$.

Hint. Let τ be the initial number of cardinality $hc(X)^+$; define a transfinite sequence $S_0, S_1, \ldots, S_\alpha, \ldots$ $\alpha < \tau$ of subsets of X and a transfinite sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_\alpha, \ldots$, $\alpha < \tau$ of families of open subsets of X such that for every $\alpha < \tau$

(1) $|S_\alpha| \le [\psi(X)]^{hc(X)}$ and $|\mathcal{U}_\alpha| \le [\psi(X)]^{hc(X)}$.

(2) If $\alpha > 0$ and $S_\alpha \subset (\bigcup \mathcal{U}) \cup \overline{A}$, where $\mathcal{U} \subset \bigcup_{\beta<\alpha} \mathcal{U}_\beta$, $A \subset \bigcup_{\beta<\alpha} S_\beta$, $|\mathcal{U}| \le hc(X)$ and $|A| \le hc(X)$, then $X = (\bigcup \mathcal{U}) \cup \overline{A}$.

(3) Every point in S_α is the intersection of a subfamily of $\mathcal{U}_\alpha$.

Then apply Problem 3.12.9(a).

(h) (Šapirovskiĭ [1972]) Show that for every Hausdorff space X there exists a set $S \subset X$ of cardinality $\le \exp hc(X)$ such that $X = \bigcup\{\overline{M} : M \subset S \text{ and } |M| \le hc(X)\}$.

(i) (Šapirovskiĭ [1972]) Observe that for a regular space X the inequality in part (c) follows from part (h) and Exercise 3.1.F(d).

(j) (Hajnal and Juhász [1967]) Show that for every Hausdorff space X we have $hd(X) \le \exp hc(X)$.

Hint. Apply part (h).

(k) (Šapirovskiĭ [1972]) Show that for every regular space X we have $nw(X) \leq \exp hc(X)$.

Deduce that $hl(X) \leq \exp hc(X)$ for every regular space X; note that the latter inequality follows from Problem 3.12.9(a) and Theorem 1.5.7.

Hint. Note that the sets $\overline{M}$ in (h) form a network for X.

(l) (Šapirovskiĭ [1972]) Prove that for every regular space X we have $w(X) \leq \exp[hc(X)h(X)]$.

Hint. Let $\mathfrak{m} = \exp[hc(X)h(X)]$ and note that $d(X) \leq \mathfrak{m}$ and $\chi(X) \leq \mathfrak{m}$ (here apply Exercise 3.1.E(a)). Take a base of cardinality $\leq \mathfrak{m}$ at every point of a dense set of cardinality $\leq \mathfrak{m}$, consider the union $\mathcal{U}$ of those bases and prove that the family $\{\operatorname{Int}(\overline{\bigcup \mathcal{U}_0}) : \mathcal{U}_0 \subset \mathcal{U}$ and $|\mathcal{U}_0| \leq c(X)\}$ is a base for X.

3.12.11. (a) (Čech and Pospíšil [1938]) Prove that if for every point x of a compact space X we have $\chi(x, X) \geq \mathfrak{m} \geq \aleph_0$, then $|X| \geq \exp \mathfrak{m}$ (here we consider the character as defined in the main body of the book, i.e., we disregard the convention, introduced in Problem 1.7.12, which makes the values of cardinal functions infinite).

Hint. Let τ be the initial number of cardinality $\mathfrak{m}$. For every $\alpha \leq \tau$ denote by $D(\alpha)$ the set of all transfinite sequences of type α taking only values 0 and 1. For every $f \in D(\alpha)$ and any $\beta < \alpha$ denote by f_β the element of $D(\beta)$ defined by letting $f_\beta(\gamma) = f(\gamma)$ for $\gamma < \beta$. For every $f \in D(\alpha)$, where $\alpha < \tau$, and $i = 0, 1$ denote by f^i the element of $D(\alpha + 1)$ defined by letting $f^i(\beta) = f(\beta)$ for $\beta < \alpha$ and $f^i(\alpha) = i$.

Applying transfinite induction define for every $\alpha < \tau$ and every $f \in D(\alpha)$ an open set $V_\alpha(f) \subset X$ such that

(1) $\overline{V_{\alpha+1}(f^i)} \subset V_\alpha(f)$ for $i = 0, 1$ and $f \in D(\alpha)$.
(2) $\overline{V_{\alpha+1}(f^0)} \cap \overline{V_{\alpha+1}(f^1)} = \emptyset$ for $f \in D(\alpha)$.
(3) $\bigcap_{\beta \leq \alpha} V_\beta(f_\beta) \neq \emptyset$ for $f \in D(\alpha)$.
(4) $V_\alpha(f) = X$ for each limit number α and $f \in D(\alpha)$.

To every $f \in D(\tau)$ assign a point in the intersection $\bigcap_{\alpha<\tau} V_\alpha(f_\alpha)$ and show that if f and f' differ at a non-limit ordinal number, then the points assigned to f and f' are different.

(b) (Čech and Pospíšil [1938]) Generalize (a) by showing that instead of assuming that X is a compact space one can assume that X is the intersection of $\aleph_0$ open subsets of a compact space (i.e., that X is a Čech-complete space).

(c) Note that the number $\aleph_0$ in (b) cannot be replaced by $\mathfrak{m}$.

Hint. Apply the construction described in Exercise 2.3.M to $X = A(\mathfrak{c})$.

(d) (Arhangel'skiĭ [1969]) Show that every first-countable compact space either is countable or has cardinality $\mathfrak{c}$.

Hint. Show that the set of condensation points of X is dense in itself and apply (a).

(e) (Franklin [1969]) Deduce from (b) that every compact space of cardinality $< 2^{\aleph_1}$ is sequentially compact.

Hint. It follows from (b) that every uncountable compact space of cardinality $< 2^{\aleph_1}$ contains uncountably many points whose character is $\leq \aleph_0$.

Dyadic spaces I (see Problems 4.5.9–4.5.11)

3.12.12. A compact space X is called a *dyadic space* (Alexandroff [1936]) if X is a continuous image of the Cantor cube $D^{\mathfrak{m}}$ for some $\mathfrak{m} \geq \aleph_0$.

(a) (Marczewski [1941], Tukey [1941]) Note that for every dyadic space X we have $c(X) = \aleph_0$ and deduce that $A(\mathfrak{m})$ is not a dyadic space if $\mathfrak{m} > \aleph_0$.

Hint. See Theorem 2.3.17.

(b) (Šanin [1948] (announcement [1946b])) Show that a dyadic space of weight $\mathfrak{m} \geq \aleph_0$ is a continuous image of $D^{\mathfrak{m}}$.

Hint (Engelking and Pełczyński [1963]). Apply Exercise 3.2.H(d).

(c) (Engelking and Pełczyński [1963]) Prove that for every continuous real-valued function $f: X \to R$ defined on a dyadic space X there exists a compact space $X_0 \subset X$ of weight $\aleph_0$ such that $f(X_0) = f(X)$.

Note that R can be replaced by any Tychonoff space of weight $\aleph_0$.

(d) (Engelking and Pełczyński [1963]) Deduce from (c) that the two arrows space is not dyadic and that if the Čech-Stone compactification of a space X is dyadic, then X is pseudocompact.

Hint. Show that βR is not a dyadic space and observe that any non-pseudocompact Tychonoff space can be continuously mapped onto a dense subset of R.

(e) (Esenin-Volpin [1949]) Show that for every dyadic space X we have $w(X) = \chi(X)$ (cf. parts (g) and (h) below).

Hint. Prove a counterpart of Problem 2.7.14(a) for a space Y whose one-point subsets are intersections of $\mathfrak{m} \geq \aleph_0$ open sets.

(f) (Šanin [1948] (announcement [1946b])) Prove that every linearly ordered dyadic space is second-countable.

Hint. Apply (e) and Problem 3.12.4(a).

(g) (Efimov [1963a]) Prove that if $\chi(x, X) \leq \mathfrak{m} \geq \aleph_0$ for every x in a dense subset of a dyadic space X, then $w(X) \leq \mathfrak{m}$.

Hint (E. Pol and R. Pol [1976]). Let $\chi(x, X) \leq \mathfrak{m}$ for every x in a set B dense in X. Consider a mapping $f: D^{\mathfrak{n}} \to X$ of a Cantor cube $D^{\mathfrak{n}} = \prod_{s \in S} D_s$ onto X and for every $a \in A = f^{-1}(B)$ choose an $S(a) \subset S$ such that $p_{S(a)}^{-1} p_{S(a)}(a) = f^{-1}f(a)$ and $|S(a)| \leq \mathfrak{m}$. Define inductively increasing sequences $S_1 \subset S_2 \subset \ldots$ and $A_1 \subset A_2 \subset \ldots$ of subsets of S and A respectively, such that $|S_i| \leq \mathfrak{m}$, $|A_i| \leq \mathfrak{m}$,

$$p_{S_i}(A) \subset \overline{p_{S_i}(A_i)} \quad \text{and} \quad S_{i+1} = S_i \cup \bigcup \{S(a) : a \in A_i\}.$$

Observe that for $S_0 = \bigcup_{i=1}^{\infty} S_i$ and $A_0 = \bigcup_{i=1}^{\infty} A_i$ we have $p_{S_0}(A) \subset \overline{p_{S_0}(A_0)}$ and $f(p_{S_0}^{-1} p_{S_0}(a)) = f(a)$ for every $a \in A$. Consider the set $A' = p_{S_0}(A) \times \prod_{s \in S \setminus S_0} \{a_s\}$, where $a_s = 0$ for $s \in S \setminus S_0$, and show that $f(\overline{A'}) = X$.

(h) (Arhangel'skiĭ and Ponomarev [1968]) Prove that for every dyadic space X we have $w(X) = \tau(X)$.

Hint (Arhangel'skiĭ [1969]). Consider a mapping $f: D^{\mathfrak{m}} \to X$ onto X and the set $\Sigma_{\mathfrak{n}} \subset D^{\mathfrak{m}}$, where $\mathfrak{n} = \tau(X)$, consisting of those points of $D^{\mathfrak{m}}$ which have at most $\mathfrak{n}$ coordinates distinct from zero; show that $f(\Sigma_{\mathfrak{n}}) = X$ and note that it suffices to prove that $d(X) \leq \mathfrak{n}$.

Making use of Theorem 2.3.15 observe that for every $\mathbf{k} \geq \mathbf{n}$ and a set $A \subset X$ of cardinality $\leq \exp \mathbf{k}$ there exists a set $B \subset X$ such that $A \subset \overline{B}$ and $|B| \leq \mathbf{k}$. Applying three times this observation and using Theorem 1.5.3 show that $|X| \leq \exp\exp \mathbf{n}$ (to this end, consider a set $A \subset X$ such that $|A| \leq \exp\exp\exp \mathbf{n}$), applying it twice more prove that $d(X) \leq \mathbf{n}$.

(i) (Engelking [1965]; for $\mathbf{m}$ of the form $\aleph_{\alpha+1}$, Efimov [1965a]) Prove that if X is a dyadic space and $\chi(x_0, X) = \mathbf{m} \geq \aleph_0$, then X contains a subspace M homeomorphic to $D(\mathbf{m})$ such that $M \cup \{x_0\}$ is homeomorphic to $A(\mathbf{m})$.

Hint. Consider a mapping $f\colon \prod_{s\in S} D_s \to X$ onto X, the fiber $A = f^{-1}(x_0)$ and the set $S_0 \subset S$ consisting of those $s \in S$ for which one can find points $a(s) \in A$ and $b(s) \notin A$ such that $p_{s'}(a(s)) = p_{s'}(b(s))$ for $s' \neq s$; for every $s \in S_0$ choose $a(s)$ and $b(s)$ with the above properties. Prove that $A = p_{S_0}(A) \times \prod_{s\in S\setminus S_0} D_s$ and deduce that $|S_0| \geq \mathbf{m}$. Observe that fibers of the function b from S_0 to $B = b(S_0)$ are finite and deduce that $|B| \geq \mathbf{m}$; verify that all accumulation points of B belong to A and let $M = f(B)$.

Remark. As proved by Hagler in [1975], Gerlits in [1976] and Efimov in [1977], every dyadic space of a regular weight $\mathbf{m}$ contains a subspace homeomorphic to $D^{\mathbf{m}}$ (the last two papers discuss also the case when the weight is non-regular). A similar result was obtained for Tychonoff cubes by Ščepin in [1979]: every Hausdorff space that is a continuous image of a Tychonoff cube and whose weight $\mathbf{m}$ is regular contains a subspace homeomorphic to $I^{\mathbf{m}}$.

(j) Verify that the following diagram (cf. Problem 1.7.12(a)) contains all equalities and inequalities between the cardinal functions appearing in it which hold in the class of all dyadic spaces (the symbol $k(X)$ denotes here the cadinality of X; from part (a) it follows that for every dyadic space X we have $c(X) = \aleph_0$, and clearly $l(X) = e(X) = \aleph_0$).

$$\begin{array}{c} k \\ \downarrow \\ \boxed{\begin{array}{c} nw=w=hd=hl=hc \\ \| \\ \chi=\psi=\tau \end{array}} \\ \downarrow \\ d \end{array}$$

(k) (Efimov [1963a]) Prove that every hereditarily normal dyadic space is second-countable.

Hint. Consider a mapping $f\colon D^{\mathbf{m}} \to X$ onto X and a Σ-product $\Sigma(a) \subset D^{\mathbf{m}}$. Show that if $f(\Sigma(a)) = X$, then $w(X) \leq \aleph_0$ (cf. Problem 3.12.24(f)). To this end, observe that every separable subspace of X is second-countable and so is every subspace of cardinality $\leq \mathbf{c}$; then apply (i) and (e). In the case when $f(\Sigma(a)) \neq X$, take a point $x \in X \setminus f(\Sigma(a))$ and show – applying (i) and Theorem 3.10.21 – that the space $X \setminus \{x\}$ is not normal.

Inverse systems II (see Problems 2.7.19 and 6.3.16)

3.12.13 (Mardešić [1971]). Let $\mathbf{S} = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$ be an inverse system of Hausdorff spaces and let A_σ be, for every $\sigma \in \Sigma$, a compact subspace of X_σ such that $\pi_\rho^\sigma(A_\sigma) \subset A_\rho$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$, i.e., such that $\mathbf{S}' = \{A_\sigma, \tilde{\pi}_\rho^\sigma, \Sigma\}$, where $\tilde{\pi}_\rho^\sigma(x) = \pi_\rho^\sigma(x)$ for $x \in A_\sigma$, is an inverse system. Prove that $\mathbf{S}'' = \{X_\sigma/A_\sigma, (\pi_\rho^\sigma)^*, \Sigma\}$, where $(\pi_\rho^\sigma)^*: X_\sigma/A_\sigma \to X_\rho/A_\rho$ is defined in Exercise 2.4.B, is an inverse system and that the limit $\varprojlim \mathbf{S}''$ is homeomorphic to the quotient space $(\varprojlim \mathbf{S})/(\varprojlim \mathbf{S}')$.

Verify that the assumption of compactness of the A_σ's cannot be omitted.

Around the Kuratowski and the Whitehead theorems

3.12.14. (a) (Mrówka [1959]) Prove that if X is a Hausdorff space and the projection $p: X \times Y \to Y$ is closed for every compact space Y such that $w(Y) \leq w(X)$, or – equivalently – the projection $p: X \times I^{w(X)} \to I^{w(X)}$ is closed, then the space X is compact.

Hint. Apply condition (3) in Problem 3.12.1, note that one can assume that $|\alpha| \leq w(X)$ and take as Y the space of all ordinal numbers $\leq \alpha$ with the topology induced by the natural linear order $<$.

(b) (Michael [1968]) Prove that if the Cartesian product $\mathrm{id}_X \times g$, where X is a regular space, is a quotient mapping for every quotient mapping g, then the space X is locally compact (cf. Theorem 3.3.17).

Hint. Apply the construction indicated in the hint to part (a) and use Example 2.4.20.

(c) (Michael [1968]) Prove that if the Cartesian product $X \times Y$, where X is a regular space, is a k-space for every k-space Y, then the space X is locally compact (cf. Theorem 3.3.27).

Hint. Apply (b) and Example 3.3.29.

(d) (Michael [1968]) Prove that if the Cartesian product $X \times Y$, where X is a regular space, is a sequential space for every sequential space Y, then the space X is locally sequentially compact (cf. Exercise 3.10.J(c)).

Hint. Use Theorem 3.10.33 and modify the construction indicated in the hint to (a).

Normality and related properties in Cartesian products II (see Problems 2.7.16, 3.12.20, 4.5.15, 4.5.16, 5.5.5, 5.5.6, 5.5.18, 5.5.19 and Exercise 2.3.E)

3.12.15 (Noble [1971]). (a) Prove that if for a topological space X the power $X^{\mathfrak{m}}$ is normal for $\mathfrak{m} = \aleph_1 \cdot w(X)$, then X is compact (cf. part (b) below).

Hint (Engelking [1988]). Apply Exercise 2.3.E(a) to show that X is countably compact. Suppose that X is not compact and consider a family $\{F_s\}_{s\in S}$ of closed subsets of X which has the finite intersection property and an empty intersection; one can assume that $|S| = \mathfrak{m}$. Note that the set $F = \prod_{s\in S} F_s \subset X^{\mathfrak{m}} = \prod_{s\in S} X_s$, where $X_s = X$, is closed and disjoint from the diagonal $\Delta \subset X^{\mathfrak{m}}$, consider an open set U containing F, a point $x_1 \in F$, and find a finite set $S_1 \subset S$ such that $p_{S_1}^{-1} p_{S_1}(x_1) \subset U$; define a point $x_2 \in F$ by letting $p_s(x_2) = a_1$ for $s \in S_1$, where a_1 is an arbitrary point in $\bigcap_{s\in S_1} F_s$, and $p_s(x_2) = p_s(x_1)$ for $s \notin S_1$, and enlarge S_1 to a finite set $S_2 \subset S$ such that $p_{S_2}^{-1} p_{S_2}(x_2) \subset U$; apply induction to obtain points $x_1, x_2, , \ldots$, finite sets $S_1 \subset S_2 \subset \ldots$ and points $a_1, a_2, \ldots$ Consider a point $a_0 \in X$ such that every neighbourhood of a_0 contains infinitely many a_i's and show that the set $\{y_1, y_2, \dots\}$, where $p_s(y_i) = p_s(x_i)$ for $s \in S_i$ and $p_s(y_i) = a_0$ for $s \notin S_i$, is contained in U and has an accumulation point in Δ, so that $\overline{U} \cap \Delta \neq \emptyset$.

(b) Apply Problem 2.7.13(a) to solve part (a).

Hint (Keesling [1972]). Observe that from Exercise 2.3.E(a) it follows that $X^{\mathfrak{m}}$ is countably compact and deduce from Problem 2.7.13(a) that every continuous real-valued function on $X^{\mathfrak{m}}$ depends on countably many coordinates. Assume that X is not compact and show that the sets F and Δ considered in the hint to part (a) are not completely separated.

(c) Prove that if for a non-constant mapping $f: X \to Y$ defined on a Hausdorff space X the power $f^{\mathfrak{m}}$ is closed for $\mathfrak{m} = w(X)$, then f is perfect.

Hint (Polkowski [1979]). From Remark 3.7.15 it follows that it suffices to show that $f \times \mathrm{id}_{I^{\mathfrak{m}}}$ is closed. Show first that the mapping $f \times \mathrm{id}_{D^{\mathfrak{m}}}$ is closed, then consider the mapping $\mathrm{id}_Y \times g^{\mathfrak{m}}$, where $g: D^{\aleph_0} \to I$ maps the Cantor set onto I (see Exercise 3.2.B), and observe that the composition $(f \times \mathrm{id}_{I^{\mathfrak{m}}})(\mathrm{id}_X \times g^{\mathfrak{m}}) = (\mathrm{id}_Y \times g^{\mathfrak{m}})(f \times \mathrm{id}_{D^{\mathfrak{m}}})$ is closed.

Compactifications

3.12.16. (a) (Hewitt [1947]) Prove that for every Tychonoff space X the following conditions are equivalent (cf. Problem 8.5.11):

(1) *The space X has a unique (up to equivalence) compactification.*

(2) *The space X is compact or $|\beta X \setminus X| = 1$.*

(3) *If two closed subsets of X are completely separated, then at least one of them is compact.*

(b) Show that every Tychonoff space that satisfies the conditions in part (a) is pseudocompact and locally compact (cf. Problem 3.12.20(a)).

(c) Give an example of a countably compact locally compact space which has infinitely many non-equivalent compactifications.

(d) Observe that if $f: X \to Y$ is a continuous mapping of a Tychonoff space X which has a unique compactification onto a non-compact Tychonoff space Y, then Y has a unique compactification and f is a perfect mapping.

(e) (Wisliceny and Flachsmeyer [1965]) Prove that if for each pair of compactifications of a first-countable Tychonoff space X there exists a greatest lower bound (i.e., if the family $\mathcal{C}(X)$ is a lattice), then the space X is locally compact. Show that the assumption of first-countability is essential.

Hint. Show that if there exists a sequence of points in $\beta X \setminus X$ converging to a point in X, then X has two compactifications $c_1 X$ and $c_2 X$ such that there is no compactification cX satisfying $cX \leq c_i X$ for $i = 1, 2$.

3.12.17 (Franklin and Rajagopalan [1971]). (a) Applying transfinite induction define a transfinite sequence $U_1 \subset U_2 \subset \dots \subset U_\alpha \subset \dots$, $\alpha < \omega_1$ of open-and-closed subsets of $\beta N \setminus N$ such that $U_\beta \neq U_\alpha$ if $\beta < \alpha$ and verify that the formula

$$f(x) = \begin{cases} \omega_1 & \text{for } x \in (\beta N \setminus N) \setminus \bigcup_{\alpha<\omega_1} U_\alpha, \\ \inf\{\alpha : x \in U_\alpha\} & \text{for } x \in \bigcup_{\alpha<\omega_1} U_\alpha, \end{cases}$$

defines a continuous mapping $f: \beta N \setminus N \to W$ onto the space W of all ordinal numbers $\leq \omega_1$.

Hint. Apply Exercise 3.6.A(b).

(b) Define a transfinite sequence $U_1 \subset U_2 \subset \ldots \subset U_\alpha \subset \ldots$, $\alpha < \delta$ of open-and-closed proper subsets of $\beta N \setminus N$ such that $U_\beta \neq U_\alpha$ if $\beta < \alpha$ and that the union $\bigcup_{\alpha<\delta} U_\alpha$ is dense in $\beta N \setminus N$. Observe that $\aleph_1 \leq |\delta| \leq \mathfrak{c}$ and that under the assumption of the continuum hypothesis one can have $\delta = \omega_1$. Define a continuous mapping $f: \beta N \setminus N \to W(\delta + 1)$ onto the space $W(\delta + 1)$ of all ordinal numbers $< \delta + 1$ with the topology induced by the natural linear order $<$.

(c) Show that there exists a compactification γN of the space N whose remainder coincides with W; note that γN is a separable compact space of cardinality $\aleph_1$ which is not a sequential space. Verify that the space $\gamma N \setminus \{\omega_1\}$ is a separable locally compact first-countable normal space which is not a Lindelöf space (cf. Problem 3.12.18(c)).

Hint. Apply (a) and Theorem 3.5.13.

(d) Show that there exists a compactification $\gamma' N$ of the space N whose remainder coincides with $W(\delta + 1)$ such that no sequence of points of N converges to δ; note that the space $\gamma' N \setminus \{\delta\}$ is a separable locally compact and sequentially compact normal space which is not a compact space.

Remark. The first example of a sequentially compact non-compact normal space that contains as a dense subspace a locally compact separable metric space was defined, under the assumption of the continuum hypothesis, by M. E. Rudin in [1965].

(e) Observe that under the assumption of the continuum hypothesis the space $\gamma' N \setminus \{\delta\}$ in part (d) is first-countable.

Parovičenko spaces

3.12.18. A compact space X is called a *Parovičenko space* if X contains no isolated points, has a base consisting of open-and-closed sets, every pair of disjoint open F_σ-sets in X has disjoint closures (i.e., X is an F-space (see Exercise 3.6.G(d)), and every non-empty G_δ-set in X has a non-empty interior.

(a) Show that if X is a locally compact realcompact space that can be represented as a countable union of compact subspaces and has a base consisting of open-and-closed sets, then the remainder $\beta X \setminus X$ is a Parovičenko space. Note that the remainder $\beta N \setminus N$ is a Parovičenko space.

Hint. Apply Exercises 3.6.G and 3.11.I; see Theorems 6.2.7 and 6.2.12.

(b) (Negrepontis [1969], Błaszczyk and Szymański [1980], Engelking [1985]) Prove that for every compact space X the following conditions are equivalent:

(1) *The space X is a Parovičenko space.*

(2) *For every continuous mapping f of X onto a second-countable compact space Y and every pair F_1, F_2 of closed subsets of Y such that $F_1 \cup F_2 = Y$ there exists an open-and-closed set $U \subset X$ such that $f(U) = F_1$ and $f(X \setminus U) = F_2$.*

(3) *For every continuous mapping f of X onto a second-countable compact space Y and every continuous mapping g of a second countable compact space Z onto Y there exists a continuous mapping h of X onto Z such that $gh = f$.*

Deduce that if X is a Parovičenko space, then every second-countable compact space is a continuous image of X.

Hint. When proving the implication (1) $\Rightarrow$ (2) assume that $F_1 \cap F_2 \neq \emptyset$, consider a countable dense subset A of $F_1 \cap F_2$, for each x in A take non-empty disjoint open-and-closed sets $U_x, V_x \subset f^{-1}(x)$ and find an open-and-closed set U such that $f^{-1}(X \setminus F_2) \cup \bigcup_{x\in A} U_x \subset U$ and $f^{-1}(X \setminus F_1) \cup \bigcup_{x\in A} V_x \subset X \setminus U$. When proving the implication (2) $\Rightarrow$ (3) apply Theorem 3.2.2 to reduce the problem to the case when $Z \subset D^{\aleph_0} = \prod_{i=1}^{\infty} D_i$, where $D_i = D$ for $i = 1, 2, \ldots$, (in fact, one can assume that $Z = D^{\aleph_0}$, cf. Problem 4.5.9(b)), then for each finite sequence $i_1, i_2, \ldots, i_k$ of zeros and ones consider the subset $F_{i_1 i_2 \ldots i_k} = Z \cap p_1^{-1}(i_1) \cap p_2^{-1}(i_2) \cap \ldots \cap p_k^{-1}(i_k)$ of Z, where $p_i: D^{\aleph_0} \to D_i$ is the projection, and define inductively open-and-closed sets $U_{i_1 i_2 \ldots i_k} \subset X$ such that $f(U_{i_1 i_2 \ldots i_k}) = g(F_{i_1 i_2 \ldots i_k})$, $U_0 \cup U_1 = X$, $U_0 \cap U_1 = \emptyset$, $U_{i_1 i_2 \ldots i_{k-1} 0} \cup U_{i_1 i_2 \ldots i_{k-1} 1} = U_{i_1 i_2 \ldots i_{k-1}}$ and $U_{i_1 i_2 \ldots i_{k-1} 0} \cap U_{i_1 i_2 \ldots i_{k-1} 1} = \emptyset$. Finally, apply Exercise 3.10.A(c) to show that if $x \in \bigcap_{k=1}^{\infty} U_{i_1 i_2 \ldots i_k}$, then the intersection $\bigcap_{k=1}^{\infty} F_{i_1 i_2 \ldots i_k}$ consists of a single point and let this point be $h(x)$.

(c) (Parovičenko [1963]) Prove that if X is a Parovičenko space, then every compact space Y of weight $\le \aleph_1$ is a continuous image of X. Deduce that for every compact space Y of weight $\le \aleph_1$ there exists a compactification of the space N with the remainder homeomorphic to the space Y.

Hint (Błaszczyk and Szymański [1980]). Represent Y as the limit of an inverse system $\{Y_\alpha, \pi_\beta^\alpha, W_0\}$ of second-countable compact spaces, where $\pi_\beta^\alpha(Y_\alpha) = Y_\beta$ for $\beta < \alpha$ and W_0 is the set of all countable ordinal numbers, and apply transfinite induction to define for each $\alpha < \omega_1$ a continuous mapping f_α of X onto Y_α such that $\pi_\beta^\alpha f_\alpha = f_\beta$ for $\beta < \alpha$.

(d) (Parovičenko [1963]) Prove that from the continuum hypothesis it follows that every Parovičenko space of weight $\mathfrak{c}$ is homeomorphic to the remainder $\beta N \setminus N$.

Hint (Engelking [1985]). It suffices to prove that any Parovičenko spaces X, Y of weight $\aleph_1$ are homeomorphic. Represent Y as indicated in the hint to part (c), consider a base $\{U_\alpha\}_{\alpha\in A}$ for the space X, where each U_α is open-and-closed, $U_1 = X$ and A is the set of all non-limit countable ordinal numbers, and apply transfinite induction to define for each $\alpha < \omega_1$ a countable ordinal number $\phi(\alpha) \ge \alpha$ and a continuous mapping f_α of X onto $Y_{\phi(\alpha)}$ such that

$$\phi(\beta) < \phi(\alpha) \quad \text{and} \quad \pi_{\phi(\beta)}^{\phi(\alpha)} f_\alpha = f_\beta \quad \text{for} \quad \beta < \alpha,$$

and

$$f_\alpha(U_\alpha) \cap f_\alpha(X \setminus U_\alpha) = \emptyset \quad \text{if} \quad \alpha \in A.$$

To define f_α for $\alpha = \beta + 1$, find an open-and-closed set $U \subset Y$ such that $\pi_{\phi(\beta)}(U) = f_\beta(U_\alpha)$ and $\pi_{\phi(\beta)}(Y \setminus U) = f_\beta(X \setminus U_\alpha)$ and an ordinal number $\phi(\alpha) > \phi(\beta)$ such that $\pi_{\phi(\alpha)}(U) \cap \pi_{\phi(\alpha)}(Y \setminus U) = \emptyset$, observe that both U_α and $X \setminus U_\alpha$ are Parovičenko spaces and consider continuous mappings f'_α of U_α onto $\pi_{\phi(\alpha)}(U)$ and f''_α of $X \setminus U_\alpha$ onto $\pi_{\phi(\alpha)}(Y \setminus U)$ such that $\pi_{\phi(\beta)}^{\phi(\alpha)} f'_\alpha(x) = f_\beta(x)$ for $x \in U_\alpha$ and $\pi_{\phi(\beta)}^{\phi(\alpha)} f''_\alpha(x) = f_\beta(x)$ for $x \in X \setminus U_\alpha$.

(e) (van Douwen and van Mill [1978]) Prove that the assumption that every Parovičenko space of weight $\mathfrak{c}$ is homeomorphic to the remainder $\beta N \setminus N$ is equivalent to the continuum hypothesis.

Hint. Show that the quotient space $X = (\beta N \setminus N)/F$, where $F = \bigcap_{\alpha<\omega_1} U_\alpha$ for a decreasing transfinite sequence $U_1 \supset U_2 \supset \ldots \supset U_\alpha \supset \ldots$, $\alpha < \omega_1$ of open-and-closed subsets of $\beta N \setminus N$ such that $U_\beta \ne U_\alpha$ if $\beta < \alpha$, and the remainder $Y = \beta Z \setminus Z$, where $Z = N \times D^{\mathfrak{c}}$, are Parovičenko spaces, and that $\chi(\{F\}, X) = \aleph_1$, while $\chi(y, Y) = \mathfrak{c}$ for every $y \in Y$. To establish the latter equality, let $D^{\mathfrak{c}} = \prod_{s\in S} D_s$, where $|S| = \mathfrak{c}$ and $D_s = D$ for $s \in S$, consider the restriction $f: Y \to D^{\mathfrak{c}}$ of the extension over βZ of the projection of Z

onto $D^{\mathfrak{c}}$, and show that for every open-and-closed set $U \subset Y$ there exists a countable set $S_0 \subset S$ such that $p_s f(U) = D_s$ for every $s \in S \setminus S_0$, where $p_s: D^{\mathfrak{c}} \to D_s$ is the projection.

The long line and the long segment

3.12.19. Let W_0 be the set of all countable ordinal numbers. In the set $V_0 = W_0 \times [0, 1)$ consider the linear order $<$ defined by letting $(\alpha_1, t_1) < (\alpha_2, t_2)$ whenever $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$ and $t_1 < t_2$; the set V_0 with the topology induced by the linear order $<$ is called the *long line*. Adjoining the point ω_1 to V_0 and assuming that $x < \omega_1$ for all $x \in V_0$, we obtain a linearly ordered set V; the set V with the topology induced by the linear order $<$ is called the *long segment*.

(a) Show that the long segment is the Čech-Stone compactification of the long line.

(b) Prove that for every $x_0 \in V_0$ the subspace $M = \{x \in V_0 : x \leq x_0\}$ of V_0 is homeomorphic to the interval $[0, 1]$.

Hint. Let Q be the set of all rational numbers in the interval $(0, 1)$. Show that the elements of Q and $M \cap (W_0 \times Q)$ can be arranged into sequences $r_1, r_2, \ldots$ and $s_1, s_2, \ldots$ in such a way that $r_i < r_j$ if and only if $s_i < s_j$; verify that the formula $f(r_i) = s_i$ defines a continuous mapping $f: Q \to M$ and apply Theorem 3.2.1.

(c) (Arhangel'skiĭ [1961], Frolík [1961]) A Tychonoff space X is called *locally Čech-complete* if every point $x \in X$ has a Čech-complete neighbourhood.

Show that the space obtained from the long line by removing all points $(\alpha, 0)$, where α is a non-limit ordinal number, is locally Čech-complete but is not Čech-complete (cf. Problem 5.5.8(c)).

(d) Observe that every locally Čech-complete space is a continuous image of a Čech-complete space under an open mapping and deduce that Čech-completeness is not an invariant of open mappings onto Tychonoff spaces (cf. Problem 5.5.8(b)).

The Tychonoff plank and related spaces

3.12.20. (a) (Tychonoff [1930], Hewitt [1948], Tong [1949]) Let W be the space of all ordinal numbers $\leq \omega_1$ and W' the subspace consisting of all ordinal numbers $\leq \omega_0$; the space $T = W \times W' \setminus \{(\omega_1, \omega_0)\}$ is called the *Tychonoff plank*.

Observe that $\beta T = W \times W'$ and deduce that T is not normal. Note that the Tychonoff plank has a unique compactification and is not a countably compact space (cf. Problem 3.12.16(b)).

(b) (Alexandroff and Urysohn [1929]) Prove that if to every countable ordinal number $\alpha > 0$ an ordinal number $\phi(\alpha)$ is assigned in such a way that $\phi(\alpha) < \alpha$ for every $\alpha < \omega_1$, then there exists an $\alpha_0 < \omega_1$ such that $|\phi^{-1}(\alpha_0)| = \aleph_1$.

(c) Prove that for every compact space X the Cartesian product $W \times X$ is the Čech-Stone compactification of $W_0 \times X$, where W_0 is the space of all countable ordinal numbers (cf. Problem 3.12.21(c)).

Hint (van Douwen [1978]). Let $f: W_0 \times X \to I$ be a continuous function. Consider a positive integer i and show that for every countable ordinal number $\alpha > 0$ there exists an ordinal number $\phi(\alpha) < \alpha$ such that $|f(\alpha', x) - f(\alpha, x)| < 1/i$ for each $x \in X$ whenever $\phi(\alpha) < \alpha' \leq \alpha$; then apply (b).

One can also modify the argument in Example 3.1.27 to show directly that for every positive integer i there exists an ordinal number α_i such that $|f(\alpha,x)-f(\alpha_i,x)|<1/i$ for every $\alpha \geq \alpha_i$ and every $x \in X$.

(d) (Dieudonné [1939b], Hewitt [1948]) Show that the Cartesian product $W \times W$ is the Čech-Stone compactification of $W_0 \times W$ and deduce that $W_0 \times W$ is not normal. Note that all powers of $W_0 \times W$ are countably compact (cf. Problem 3.12.15(a)).

(e) Deduce from (d) that normality is not an inverse invariant of perfect mappings. Give an example of a perfect mapping with fibers of cardinality ≤ 2 transforming a non-normal space onto a normal space.

Hint. Consider the natural mapping of the space $\{(\alpha,\beta) \in W_0 \times W : \alpha \leq \beta\} \subset W_0 \times W$ onto the quotient space obtained by identifying for each $\alpha \in W_0$ the points (α,α) and (α,ω_1).

The Čech-Stone compactification of Cartesian products

3.12.21. (a) (Isiwata [1969], Noble [1969a] and [1969b], Comfort and Hager cited in Noble [1969a]) Prove that for every Tychonoff spaces X and Y the following conditions are equivalent:

(1) *The projection $p: X \times Y \to X$ transforms functionally closed subsets of $X \times Y$ to closed subsets of X.*

(2) *Every bounded continuous function $f: X \times Y \to R$ can be continuously extended over $X \times \beta Y$.*

(3) *For every bounded continuous function $f: X \times Y \to R$ the formula $F(x) = \sup\{f(x,y) : y \in Y\}$ defines a continuous function $F: X \to R$.*

Hint (Comfort and Hager cited in Noble [1969a]). In the proof that (3) $\Rightarrow$ (1) for a functionally closed set $Z = g^{-1}(0) \subset X \times Y$ and a point $x_0 \in X \setminus p(Z)$ consider the function $f: X \times Y \to R$ defined by letting $f(x,y) = -\min(|g(x,y)/g(x_0,y)|, 1)$.

(b) (Tamano [1960]) Prove that the Cartesian product $X \times Y$ of Tychonoff spaces X and Y is pseudocompact if and only if X and Y are pseudocompact and the projection $p: X \times Y \to X$ transforms functionally closed subsets of $X \times Y$ to closed subsets of X.

Hint (Comfort and Hager [1971]). In the proof that if $X \times Y$ is pseudocompact, then $p(Z)$ is closed in X for every functionally closed set $Z \subset X \times Y$, consider a point $x_0 \in \overline{p(Z)} \setminus p(Z)$ and a continuous function $f: X \times Y \to I$ such that $Z = f^{-1}(0)$ and $f(x_0,y) = 1$ for every $y \in Y$ (cf. the hint to part (a)). Then define by induction a sequence $(x_1,y_1),(x_2,y_2),\ldots$ of points of Z, as well as two sequences $W_1, W_2, \ldots$ and $W_1', W_2', \ldots$ of open subsets of $X \times Y$, where $W_i = U_i \times V_i$ is a neighbourhood of (x_i,y_i) satisfying $f(W_i) \subset [0,1/3]$, and $W_i' = U_i' \times V_i$ is a neighbourhood of (x_0,y_i) satisfying $f(W_i') \subset [2/3,1]$, and where $U_{i+1} \cup U_{i+1}' \subset U_i'$ for $i = 1,2,\ldots$ Apply Theorem 3.10.22 to obtain a contradiction. In the proof of the reverse implication apply the equivalence of conditions (1) and (3) in part (a).

(c) (Glicksberg [1959]) Prove that if the Cartesian product $X \times Y$ of Tychonoff spaces X and Y is pseudocompact, then $\beta X \times \beta Y$ is the Čech-Stone compactification of $X \times Y$, i.e., every continuous function $f: X \times Y \to I$ is continuously extendable over $\beta X \times \beta X$; show that if $\beta X \times \beta Y$ is the Čech-Stone compactification of $X \times Y$ and both X and Y are infinite, then the Cartesian product $X \times Y$ is pseudocompact.

Hint (Comfort and Hager [1971]). Observe that if the projection $p: Z \times T \to Z$, where

Z is an infinite compact space, transforms functionally closed subsets of $Z \times T$ to closed subsets of Z, then T is pseudocompact. To this end, define a continuous function $f: Z \to I$ such that for a $z_0 \in Z$ we have $f(z_0) = 0$ and $z_0 \notin \operatorname{Int} f^{-1}(0)$ and – assuming that T is not pseudocompact – a continuous function $g: T \to (0,1]$ such that $\inf\{g(t) : t \in T\} = 0$. Consider the function $h: Z \times T \to R$ defined by letting $h(z,t) = -g(t)^{f(z)}$ and apply the equivalence of conditions (1) and (3) in part (a). In the proof of pseudocompactness of $X \times Y$ apply the above observation to the Cartesian products $\beta X \times Y$ and $X \times \beta Y$.

(d) (Glicksberg [1959]) Prove that if the Cartesian product $\prod_{s\in S} X_s$ of Tychonoff spaces X_s is pseudocompact, then $\beta \prod_{s\in S} X_s = \prod_{s\in S} \beta X_s$ and that if $\beta \prod_{s\in S} X_s = \prod_{s\in S} \beta X_s$, and the Cartesian product $\prod_{s\in S\setminus\{s_0\}} X_s$ is infinite for every $s_0 \in S$, then the Cartesian product $\prod_{s\in S} X_s$ is pseudocompact.

Hint. To begin, observe that if the Cartesian product $\prod_{s\in S} X_s$ is pseudocompact, then for every continuous function $f: \prod_{s\in S} X_s \to I$ and every $\epsilon > 0$ there exists a finite set $S_0 \subset S$ with the property that if for $x, y \in \prod_{s\in S} X_s$ we have $p_s(x) = p_s(y)$ for all $s \in S_0$, then $|f(x) - f(y)| < \epsilon$. To this end, assume that there is no such S_0 and define a sequence $S_1, S_2, \ldots$ of disjoint finite subsets of S and two sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ of points of $\prod_{s\in S} X_s$ such that $|f(x_i) - f(y_i)| \geq \epsilon/2$ and $p_s(x_i) = p_s(y_i)$ for $s \notin S_i$; find neighbourhoods $U_i = \prod_{s\in S} U_s^i$ and $V_i = \prod_{s\in S} V_s^i$ of x_i and y_i such that $U_s^i = V_s^i$ for $s \notin S_i$ and $|f(x) - f(y)| \geq \epsilon/4$ whenever $x \in U_i$ and $y \in V_i$ and deduce a contradiction.

Then consider a pair A, B of completely separated subsets of $\prod_{s\in S} X_s$ and for a continuous function f that separates A and B and $\epsilon = 1/3$ take a finite set $S_0 \subset S$ with the above property. Applying (c) extend f to a continuous function $f^*: \prod_{s\in S_0} \beta X_s \times \prod_{s\in S\setminus S_0} X_s \to I$ and note that the projections of A and B onto $\prod_{s\in S_0} X_s$ have disjoint closures in $\prod_{s\in S_0} \beta X_s$.

(e) (Glicksberg [1959]) Prove that if X is a pseudocompact space and $\{X_s\}_{s\in S}$ a family of locally compact pseudocompact spaces then the Cartesian product $X \times \prod_{s\in S} X_s$ is pseudocompact.

Hint. Observe first that it suffices to consider a countable family $\{X_i\}_{i=1}^\infty$ of non-compact spaces. Then show that no infinite sequence $U_1, U_2, \ldots$ of members of the canonical base for the Cartesian product $X \times \prod_{i=1}^\infty X_i$ is locally finite. To this end consider the Cartesian product $X \times \prod_{i=1}^\infty \omega X_i$ and the sequence $V_1, V_2, \ldots$ of its open subsets, where V_k is obtained from U_k by replacing each factor X_i by ωX_i.

Rings of continuous functions and compactifications

3.12.22 (M. H. Stone [1937], Gelfand and Kolmogoroff [1939], Hewitt [1948] and [1950]). For a Tychonoff space X the symbol $C(X)$ (the symbol $C^*(X)$) denotes the ring of all continuous real-valued (all bounded continuous real-valued) functions defined on X. An *ideal* in $C(X)$ (in $C^*(X)$) is a proper subset Δ of $C(X)$ (of $C^*(X)$) such that if $f, g \in \Delta$, then $f + g \in \Delta$ and if $f \in \Delta$ and $g \in C(X)$ (and $g \in C^*(X)$), then $f \cdot g \in \Delta$. An ideal Δ is a *maximal ideal* if for every ideal Δ' that contains Δ we have $\Delta' = \Delta$.

(a) Show that every ideal is contained in a maximal ideal.

(b) Prove that a Tychonoff space X is compact if and only if for every maximal ideal Δ in the ring $C(X)$, or – equivalently – for every maximal ideal Δ in the ring $C^*(X)$, there exists a point $x \in X$ such that the conditions $f(x) = 0$ and $f \in \Delta$ are equivalent.

Hint. Assume that for a compact space X there exists an ideal Δ which for every $x \in X$ contains a function f_x taking non-negative values and such that $f_x(x) = 1$; consider the cover $\{f_x^{-1}((1/2, 3/2))\}_{x\in X}$ of the space X.

For a non-compact Tychonoff space X take an open cover $\{U_s\}_{s\in S}$ that has no finite subcover and for every $x \in X$ choose an $s(x) \in S$ such that $x \in U_{s(x)}$ and a function $f_x \in C^*(X)$ such that $f_x(x) = 1$ and $f_x(X \setminus U_{s(x)}) = \{0\}$; consider an ideal containing all chosen functions.

(c) In the set $\mathcal{M}$ of all maximal ideals in the ring $C(X)$ (in the ring $C^*(X)$) generate a topology by the base consisting of all sets of the form $U_f = \{\Delta : f \notin \Delta\}$ and show that $\mathcal{M}$ is a compact space. Check that by assigning to every $x \in X$ the maximal ideal $\Delta(x)$ of all functions vanishing at x we define a homeomorphic embedding of X in $\mathcal{M}$. Prove that $\mathcal{M}$ is the Čech-Stone compactification of X.

Deduce that compact spaces X and Y are homeomorphic if and only if the rings $C(X)$ and $C(Y)$ are isomorphic, i.e., if there exists a one-to-one mapping Φ of $C(X)$ onto $C(Y)$ such that $\Phi(f+g) = \Phi(f) + \Phi(g)$ and $\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g)$ for all $f, g \in C(X)$. Show that first-countable Tychonoff spaces X and Y are homeomorphic if and only if the rings $C(X)$ and $C(Y)$, or – equivalently – the rings $C^*(X)$ and $C^*(Y)$, are isomorphic.

Hint. In the proof that $\mathcal{M}$ is the Čech-Stone compactification of X apply Theorem 3.2.1.

(d) Verify that if Δ is a maximal ideal in $C(X)$, then $\mathcal{F}(\Delta) = \{f^{-1}(0) : f \in \Delta\}$ is an ultrafilter in the family $\mathcal{D}_0(X)$ of all functionally closed subsets of X and that if $\mathcal{F}$ is an ultrafiler in $\mathcal{D}_0(X)$, then $\Delta(\mathcal{F}) = \{f \in C(X) : f^{-1}(0) \in \mathcal{F}\}$ is a maximal ideal in $C(X)$. Show that $\Delta(\mathcal{F}(\Delta)) = \Delta$ and $\mathcal{F}(\Delta(\mathcal{F})) = \mathcal{F}$, i.e. that there is a one-to-one correspondence between ultrafilters in $\mathcal{D}_0(X)$ and maximal ideals in $C(X)$. Observe that the construction of βX indicated in part (c) is in principle identical with the construction in Exercise 3.6.K(a).

Give an example of a maximal ideal Δ in $C^*(N)$ such that the family $\{f^{-1}(0) : f \in \Delta\}$ does not have the finite intersection property and an example of an ultrafiler $\mathcal{F}$ in $\mathcal{D}_0(N)$ such that the set $\{f \in C^*(N) : f^{-1}(0) \in \mathcal{F}\}$ is not a maximal ideal.

(e) A ring $P \subset C^*(X)$ will be called a *complete ring of functions* on the Tychonoff space X if P contains all constant functions, separates points and closed sets, and is closed with respect to uniform convergence. Prove that the construction in part (c) applied to the set $\mathcal{M}$ of all maximal ideals in a complete ring of functions P on a Tychonoff space X gives a compactification of X and that the family of all functions on X that are continuously extendable over this compactification coincides with P. Verify that in this way a one-to-one correspondence between all complete rings of functions on a Tychonoff space X and all compactifications of X is established.

Hint. In the proof that the space of all maximal ideals in P is a Hausdorff space, note that a function $f \in P$ satisfying $1/2 \le f(x) \le 3/2$ for every $x \in X$ does not belong to any ideal (apply the equality $1/f(x) = 1 + \sum_{i=1}^{\infty}(1 - f(x))^i$). Consider a compactification cX of X such that the family of all functions on X that are continuously extendable over cX coincides with P.

(f) A *linear-multiplicative functional* on $C(X)$ (on $C^*(X)$) is a functional ϕ assigning to every $f \in C(X)$ (to every $f \in C^*(X)$) a real number $\phi(f)$ in such a way that for all f_1, f_2 and all real numbers t_1, t_2

$$\phi(t_1 f_1 + t_2 f_2) = t_1\phi(f_1) + t_2\phi(f_2) \quad \text{and} \quad \phi(f_1 \cdot f_2) = \phi(f_1) \cdot \phi(f_2).$$

A functional ϕ is *non-trivial* if there exists an f such that $\phi(f) \neq 0$. A functional ϕ is *determined by a point* $x \in X$ if $\phi(f) = f(x)$ for all f.

Show that a Tychonoff space X is compact if and only if every non-trivial linear-multiplicative functional on $C^*(X)$ is determined by a point.

Observe that there is a one-to-one correspondence between non-trivial linear-multiplicative functionals on $C^*(X)$ and maximal ideals in $C^*(X)$, i.e., points of the Čech-Stone compactification of X.

Hint. For a non-trivial linear-multiplicative functional ϕ, the set $\Delta = \{f : \phi(f) = 0\}$ is a maximal ideal.

(g) Show that a Tychonoff space X is realcompact if and only if every non-trivial linear-multiplicative functional on $C(X)$ is determined by a point.

Observe that there is a one-to-one correspondence between non-trivial linear-multiplicative functionals on $C(X)$ and points of the Hewitt realcompactification of X.

Deduce that realcompact spaces X and Y are homeomorphic if and only if the rings $C(X)$ and $C(Y)$ are isomorphic.

Hint. Apply (f) and Theorem 3.11.10.

Remark. As $\upsilon X \subset \beta X$, the question arises to characterize maximal ideals in $C(X)$ (in $C^*(X)$) corresponding to points of υX; as we know, in the construction of βX using ultrafilters in $\mathcal{D}_0(X)$, ultrafilters which have the countable intersection property correspond to points of υX (cf. Exercise 3.11.F(b)). It turns out that these are ideals Δ (ideals $\Delta \cap C^*(X)$) for which the quotient ring $C(X)/\Delta$ is isomorphic to the field of real numbers; see Gillman and Jerison's book [1960].

Countable compactness

3.12.23. (a) (Aquaro [1965]) Prove that if every discrete family of non-empty subsets of a space X is finite (has cardinality $\leq \mathfrak{m} \geq \aleph_0$, i.e., $e(X) \leq \mathfrak{m}$), then every open cover $\mathcal{U}$ of the space X which is point-finite (has the property that no point of X belongs to more than $\mathfrak{m}$ members of $\mathcal{U}$) has a finite subcover (a subcover of cardinality $\leq \mathfrak{m}$).

Note that if X is a countably compact space, then every open cover $\mathcal{U}$ of the space X which is point-countable (has the property that no point of X belongs to more than $\mathfrak{m} \geq \aleph_0$ members of $\mathcal{U}$) has a finite subcover (a subcover of cardinality $\leq \mathfrak{m}$).

Hint. Consider a maximal set $A \subset X$ with the property that $|A \cap U| \leq 1$ for every $U \in \mathcal{U}$ and show that $X = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$.

(b) (Iséki and Kasahara [1957], Levšenko [1957]) Prove that a regular space X is countably compact if and only if every point-finite open cover of X has a finite subcover.

Hint. Apply Exercise 2.1.G.

(c) (Frolík [1960a]) Show that the assumption of regularity is essential in (b).

Hint. For $i = 1, 2, \ldots$ let A_i denote the set of all numbers of the form $k/2^i$, where $k = 1, 3, \ldots, 2^i - 1$. Generate a topology on the set $X = X_0 \cup \{x_1, x_2, \ldots\}$, where $X_0 = I \setminus \bigcup_{i=1}^{\infty} A_i$ and $x_i \notin I$ for $i = 1, 2, \ldots$, by inducing on X_0 the topology of a subspace of I, declaring X_0 to be an open subspace of X and taking as a base at x_i the family of all sets of the form $\{x_i\} \setminus (U \cap X_0)$, where U is an open subset of I such that $A_i \subset U$. Apply the Baire category theorem to show that for every increasing sequence $k_1 < k_2 < \ldots$ of positive integers and any family $\{U_i\}_{i=1}^\infty$ of open subsets of I such that $A_{k_i} \subset U_i$ for $i = 1, 2, \ldots$, there exists a point $x_0 \in X_0$ which belongs to infinitely many U_i's.

(d) (R. S. Houston cited in Fleischman [1970]; for regular spaces, Fleischman [1970]) Prove that a Hausdorff space X is countably compact if and only if for every open cover $\mathcal{U}$ of the space X there exists a finite set $F \subset X$ such that $X = \bigcup\{U \in \mathcal{U} : F \cap U \neq \emptyset\}$.

Hint. Let X be a Hausdorff space that contains a countably infinite set A with no accumulation point. Represent A as the union $\bigcup_{i=1}^\infty A_i$, where $|A_i| = i$ and $A_i \cap A_j = \emptyset$ if $i \neq j$, and for every $x \in A$ take a neighbourhood U_x such that $A \cap U_x = \{x\}$ and $U_x \cap U_y = \emptyset$ whenever x and y are distinct and belong to the same set A_i; consider the cover $\{U_x\}_{x \in A} \cup \{X \setminus A\}$.

(e) (Chaber [1976]) Prove that if X is a countably compact space and the diagonal Δ is a G_δ-set in the Cartesian product $X \times X$, then X is compact.

Hint. Define first a countable family $\{\mathcal{V}_i\}_{i=1}^\infty$ of open covers of the space X such that for every pair x, y of distinct points of X there exists a natural number i with the property that no member of $\mathcal{V}_i$ contains both x and y. Assume that X has an open cover $\mathcal{U}$ with no countable subcover. Observe that if a set $F \subset X$ cannot be covered by a countable subfamily of $\mathcal{U}$, then for each $x \in X$ there exists an i such that the set $F \setminus \bigcup\{V \in \mathcal{V}_i : x \in V\}$ cannot be covered by a countable subfamily of $\mathcal{U}$, either. Define a sequence $x_1, x_2, \ldots$ of points of X and a sequence $i_1, i_2, \ldots$ of positive integers such that for $k = 1, 2, \ldots$

(1) The point x_k belongs to the set $F_k = X \setminus \bigcup_{j=1}^{k-1} \bigcup\{V \in \mathcal{V}_{i_j} : x_j \in V\}$.

(2) The set $F_k \setminus \bigcup\{V \in \mathcal{V}_{i_k} : x_k \in V\}$ cannot be covered by a countable subfamily of $\mathcal{U}$.

(3) For every $x \in F_k$ and every $i < i_k$ the set $F_k \setminus \bigcup\{V \in \mathcal{V}_i : x \in V\}$ can be covered by a countable subfamily of $\mathcal{U}$.

Show that a number i occurs infinitely many times in the sequence $i_1, i_2, \ldots$ and obtain a contradiction.

(f) (Miščenko [1962], Corson and Michael [1964]) Prove that if a base $\mathcal{B}$ for a countably compact (compact) space X is point-countable (has the property that no point of X belongs to more than $\mathfrak{m} \geq \aleph_0$ members of $\mathcal{B}$), then $\mathcal{B}$ is countable (has cardinality $\leq \mathfrak{m}$).

Hint. Prove that $d(X) \leq \mathfrak{m}$. To this end define a sequence $C_1 \subset C_2 \subset \ldots$ of subsets of X such that $|C_i| \leq \mathfrak{m}$ for $i = 1, 2, \ldots$ and $\bigcup_{i=1}^\infty C_i$ is dense in X: take an arbitrary one-point subset of X as C_1 and define C_{i+1} as the set obtained from C_i by adjoining one point from every non-empty set of the form $X \setminus \bigcup \mathcal{V}$, where $\mathcal{V}$ is a finite subfamily of $\{B \in \mathcal{B} : B \cap C_i \neq \emptyset\}$.

In the case of a compact space one can also apply the fact, known as *Miščenko's lemma*, that if a family $\mathcal{A}$ of subsets of a set X has the property that no point of X belongs to more than $\mathfrak{m} \geq \aleph_0$ members of $\mathcal{A}$, then there are at most $\mathfrak{m}$ irreducible (i.e., not containing proper subcovers) finite covers of X by members of $\mathcal{A}$. To establish Miščenko's lemma, suppose that for an integer k the family $\mathbf{A}$ of all irreducible covers of X consisting of exactly k members of $\mathcal{A}$ has cardinality $> \mathfrak{m}$ and define by induction distinct members $A_1, A_2, \ldots, A_k$ of $\mathcal{A}$ such that for every $i \leq k$ the family $\mathbf{A}(A_1, A_2, \ldots, A_i)$ consisting of all covers in $\mathbf{A}$ that contain the sets $A_1, A_2, \ldots, A_i$ has cardinality $> \mathfrak{m}$.

(g) (Frolík [1959]) Show that a Hausdorff space X is countably compact if and only if every lower (upper) semicontinuous real-valued function defined on X is bounded below (above).

Σ-products II (see Problems 2.7.14, 2.7.15, 4.5.12 and Exercise 3.10.D)

3.12.24. (a) (Engelking [1966]) Let $\{X_s\}_{s\in S}$ be a family of topological spaces and let $a = \{a_s\}$ be a point of the Cartesian product $\prod_{s\in S} X_s$; by $\Sigma'(a)$ we denote the subspace of $\prod_{s\in S} X_s$ consisting of all points $\{x_s\}$ such that $x_s \neq a_s$ only for finitely many $s \in S$. Prove that if all finite Cartesian products $X_{s_1} \times X_{s_2} \times \ldots \times X_{s_k}$, where $s_1, s_2, \ldots, s_k \in S$, have the Lindelöf property, then for every family of points $\{x_t\}_{t\in T}$ in $\Sigma'(a)$, where $|T| > \aleph_0$, there exists an $x_0 \in \Sigma'(a)$ every neighbourhood of which contains points x_t for infinitely many $t \in T$.

Hint. Apply Problem 2.7.10(c).

(b) (Engelking [1966]) Prove that if $\{X_s\}_{s\in S}$ is a family of topological spaces such that all finite Cartesian products $X_{s_1} \times X_{s_2} \times \ldots \times X_{s_k}$, where $s_1, s_2, \ldots, s_k \in S$, have the Lindelöf property (see Exercise 3.8.G and 3.9.F(a)) and Y is a Hausdorff space such that the diagonal $\Delta \subset Y \times Y$ is a G_δ-set, then for every point $a \in \prod_{s\in S} X_s$ and every continuous mapping $f: \Sigma(a) \to Y$ there exists a countable set $S_0 \subset S$ and a continuous mapping $f_0: \prod_{s\in S_0} X_s \to Y$ such that f coincides with the composition $f_0(p_{S_0}|\Sigma(a))$ of the restriction of the projection $p_{S_0}: \prod_{s\in S} X_s \to \prod_{s\in S_0} X_s$ to $\Sigma(a)$ and the mapping f_0.

Hint. To begin, applying (a) show that the set S_0 consisting of those $s \in S$ for which there exist points $x, x' \in \Sigma'(a)$ which differ only at the sth coordinate and satisfy $f(x) \neq f(x')$ is countable. Then verify that if for two points $x = \{x_s\}$, $x' = \{x'_s\}$ in $\Sigma'(a)$ the set $\{s \in S : x_s \neq x'_s\}$ is finite and disjoint from S_0, then $f(x) = f(x')$. To conclude the proof apply the continuity of f and the fact that $\Sigma'(a)$ is dense in $\Sigma(a)$.

(c) (Glicksberg [1959]) Prove that for every family $\{X_s\}_{s\in S}$ of compact spaces and any point $a \in \prod_{s\in S} X_s$, the Cartesian product $\prod_{s\in S} X_s$ is the Čech-Stone compactification of the Σ-product $\Sigma(a)$.

(d) (Corson [1959], Engelking [1966]) Prove that for every family $\{X_s\}_{s\in S}$ of Tychonoff spaces such that all finite Cartesian products $X_{s_1} \times X_{s_2} \times \ldots \times X_{s_k}$, where $s_1, s_2, \ldots, s_k \in S$, have the Lindelöf property (see Exercises 3.8.G and 3.9.F(a)) and for any point $a \in \prod_{s\in S} X_s$, the Cartesian product $\prod_{s\in S} X_s$ is the Hewitt realcompactification of the Σ-product $\Sigma(a)$.

(e) (Corson [1959]) Give an example of a normal space whose Hewitt realcompactification is not normal.

Hint. Apply (d), Problem 2.7.15 and Exercise 2.3.E(a) or 3.1.H(a).

(f) (Efimov [1963a]) Let $\{X_s\}_{s\in S}$ be a family of compact spaces such that $w(X_s) \leq \mathfrak{m} \geq \aleph_0$ for $s \in S$ and let $a = \{a_s\}$ be a point of the Cartesian product $\prod_{s\in S} X_s$; by $\Sigma_{\mathfrak{m}}(a)$ we denote the subspace of $\prod_{s\in S} X_s$ consisting of all points $\{x_s\}$ such that the set $\{s \in S : x_s \neq a_s\}$ has cardinality $\leq \mathfrak{m}$. Prove that if a compact space is a continuous image of $\Sigma_{\mathfrak{m}}(a)$, then $w(X) \leq \mathfrak{m}$.

Hint. Apply (c) and the hint to Problem 3.12.12(h).

Normally placed sets III (see Problems 1.7.6 and 2.7.3)

3.12.25. (a) (Smirnov [1951c]) Show that if X is a Lindelöf space, then every normally placed subset of X with the subspace topology is a Lindelöf space. Prove that for every Tychonoff space X the following conditions are equivalent:

(1) *The space X has the Lindelöf property.*

(2) *The space X is normally placed in every compactification of X.*

(3) *The space X is normally placed in βX.*

(4) *The space X is normally placed in some compactification of X.*

Deduce that no point $x \in \beta N \setminus N$ is a G_δ-set in βN (cf. Exercise 3.6.A(b)).

(b) Verify that a Tychonoff space X has the Lindelöf property if and only if for every compact subset $Z \subset \beta X \setminus X$ there exists a continuous function $h: \beta X \to I$ such that $h(Z) \subset \{0\}$ and $h(x) > 0$ for any $x \in X$. Note that in the above characterization βX can be replaced by any compactification of X.

Regularly placed sets

3.12.26 (Mrówka [1957]). We say that a set A is *regularly placed* in a space X if for every point $x \in X \setminus A$ there exists in X an F_σ-set H such that $A \subset H \subset X \setminus \{x\}$.

(a) Note that every set normally placed in a T_1-space X is regularly placed in X.

(b) Show that a Tychonoff space X is realcompact if and only if X is regularly placed in βX. Give an example of a realcompact space X and of a compactification of X in which X is not regularly placed.

(c) Prove that if X is a realcompact space, then every regularly placed subset of X is realcompact.

(d) Deduce from (b) and (c) that the Cartesian product of every family of realcompact spaces is realcompact.

Spaces of closed subsets II (see Problems 2.7.20, 4.5.23, 6.3.22, 8.5.13(i) and 8.5.16)

3.12.27. (a) (Vietoris [1922]) Prove that if X is a compact space, then the exponential space 2^X also is compact and $w(2^X) = w(X)$. Note that if X is a T_1-space and 2^X is a compact space, then the space X also is compact.

Hint (Michael [1951]). Verify that the family of all sets of the form $\{B \in 2^X : B \subset U\}$ and $\{B \in 2^X : B \cap U \neq \emptyset\}$, where U is an open subset of X, is a subbase for 2^X and apply Problem 3.12.2(a).

Remark. Keesling proved in [1970] that normality of 2^{2^X} is equivalent to compactness of X and in [1970a] he proved that under the assumption of the continuum hypothesis normality of 2^X is equivalent to compactness of X. As shown by Veličko in [1975], the latter equivalence can be proved without the assumption of the continuum hypothesis. Let us also note that, as proved by Šapiro in [1976], the space 2^X is not necessarily dyadic for a dyadic space X.

(b) (Michael [1951]) Let $Z(X)$ denote for a T_1-space X the subspace of 2^X consisting of all non-empty compact closed subsets of X. Show that $Z(X)$ is a T_i-space if and only if X is a T_i-space for $i = 2, 3$ and $3\frac{1}{2}$. Give an example of a perfectly normal Lindelöf space X such that $Z(X)$ is not normal. Verify that $w(Z(X)) = w(X)$ for every T_1-space X.

(c) (Čoban [1971]) Verify that if X is a subspace of a compact space Z, then assigning to any $A \in Z(X)$ the same $A \in 2^Z$ defines a homeomorphic embedding of $Z(X)$ in 2^Z. Deduce that the space $Z(X)$ is locally compact if and only if X is locally compact. Note that 2^R is not locally compact.

Show that for every T_1-space X, assigning to any $A \in 2^X$ the closure $\overline{A}$ in the Wallman extension wX defines a homeomorphic embedding of 2^X in 2^{wX}. Prove that $w(2^X) = w(wX)$ for every T_1-space X.

(d) (Čoban [1971] (announcement [1969])); for $\mathfrak{m} = \aleph_0$, Wulbert [1968]) Prove that for a Hausdorff space X the inequality $\chi(Z(X)) \leq \mathfrak{m} \geq \aleph_0$ holds if and only if $d(A) \leq \mathfrak{m}$ and $\chi(A, X) \leq \mathfrak{m}$ for every $A \in Z(X)$.

Deduce that for a compact space X the inequality $\chi(2^X) \leq \mathfrak{m} \geq \aleph_0$ holds if and only if $hd(X) \leq \mathfrak{m}$ and $\chi(A, X) \leq \mathfrak{m}$ for every closed $A \subset X$.

Hint. Observe that $\mathcal{V}(U_1, U_2, \ldots, U_k) \subset \mathcal{V}(V_1, V_2, \ldots, V_m)$ if and only if $\bigcup_{i=1}^k U_i \subset \bigcup_{j=1}^m V_j$ and every V_j contains a U_i. Apply the fact that if $\chi(X) \leq \mathfrak{m}$ and $d(A) \leq \mathfrak{m}$ for every closed $A \subset X$, then $hd(X) \leq \mathfrak{m}$ (cf. Problem 3.12.9(d)).

(e) (Čoban [1971]) Verify that if X and Y are Hausdorff spaces and $f: X \to Y$ is a continuous mapping, then by letting $\tilde{f}(A) = f(A)$ for every $A \in Z(X)$ one defines a continuous mapping $\tilde{f}: Z(X) \to Z(Y)$. Show that if X and Y are Tychonoff spaces, then $\tilde{f}$ is perfect if and only if f is perfect.

Hint. Apply the Čech-Stone compactification.

(f) (Zenor [1970]; for compact spaces, Sirota [1968]) Prove that if $\mathbf{S} = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$ is an inverse system of Hausdorff spaces, then $\tilde{\mathbf{S}} = \{Z(X_\sigma), \tilde{\pi}_\rho^\sigma, \Sigma\}$, where $\tilde{\pi}_\rho^\sigma: Z(X_\sigma) \to Z(X_\rho)$ is defined in (e), also is an inverse system and $\varprojlim \tilde{\mathbf{S}}$ is homeomorphic to the space $Z(\varprojlim \mathbf{S})$.

Deduce that if $\mathbf{S} = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$ is an inverse system of compact spaces, then $\tilde{\mathbf{S}} = \{2^{X_\sigma}, \tilde{\pi}_\rho^\sigma, \Sigma\}$ also is an inverse system and $\varprojlim \tilde{\mathbf{S}}$ is homeomorphic to the space $2^{\varprojlim \mathbf{S}}$.

(g) (Čoban [1971] (announcement [1969]), Zenor [1970]) Prove that the space $Z(X)$ is Čech-complete if and only if X is Čech-complete.

Hint. Either apply (c) or use (f) and Exercise 3.9.G.

(h) (Zenor [1970]) Prove that the space $Z(X)$ is realcompact if and only if X is realcompact.

Hint. Either apply (c) and Problem 3.12.26(c) or use (f) and Exercise 3.11.H.

(i) (Marjanović [1966]) Prove that the Cartesian product $\prod_{s\in S} X_s$, where all spaces X_s are compact, is embeddable in the exponential space of the one-point compactification of the sum $\bigoplus_{s\in S} X_s$ and note that this implies the Tychonoff theorem.

(j) Show that if X is a compact space and Y a Hausdorff space, then assigning to each $f \in Y^X$ the graph $G(f) \subset X \times Y$ defines a homeomorphic embedding of the function space Y^X with the compact-open topology in the exponential space $2^{X\times Y}$.

(k) Verify that if X is a compact space and Y a Hausdorff space, then assigning to each pair $(f, A) \in Y^X \times 2^X$ the image $f(A) \in 2^Y$ defines a mapping of $Y^X \times 2^X$ to 2^Y which is continuous with respect to the compact-open topology in Y^X.

Set-valued mappings III (see Problems 1.7.17 and 2.7.21)

3.12.28. (a) (Engelking [1963]; for metric spaces, Kuratowski [1932]) Prove that for every family $\{F_s\}_{s\in S}$ of upper semicontinuous set-valued mappings, assigning to points of a topological space Y compact subsets of a Hausdorff space X, the intersection $F = \bigcap_{t\in T} F_t$, defined by the formula $F(y) = \bigcap_{t\in T} F_t(y)$, is upper semicontinuous.

Hint. Observe that $\{y : F(y) \subset U\} = \bigcup \bigcap_{t\in T}\{y : F_t(y) \subset U_t\}$, where the union is taken over all families $\{U_t\}_{t\in T}$ of open subsets of X such that $\bigcap_{t\in T} U_t = U$ and the set $\{t \in T : U_t \neq X\}$ is finite.

(b) (Engelking [1963]) For every $t \in T$ let F_t be a set-valued mapping assigning to points of a topological space Y closed subsets of a space X_t. Verify that if all mappings F_t are lower semicontinuous, then the Cartesian product $F = \prod_{t\in T} F_t$, defined by the formula $F(y) = \prod_{t\in T} F_t(y)$, is a lower semicontinuous set-valued mapping assigning to poins of Y closed subsets of the Cartesian product $\prod_{t\in T} X_t$.

Show that if all mappings F_t assume compact values and are upper semicontinuous, then the Cartesian product $F = \prod_{t\in T} F_t$ also is upper semicontinuous.

Deduce that for every family $\{X_t\}_{t\in T}$ of Hausdorff spaces, assigning to each $\{A_t\}$ in $\prod_{t\in T} \mathcal{Z}(X_t)$ the Cartesian product $\prod_{t\in T} A_t \in \mathcal{Z}(\prod_{t\in T} X_t)$ defines a continuous mapping.

(c) (Engelking [1963]) Prove that a Hausdorff space X is compact if and only if for every index-set T assigning to each $\{A_t\}$ in $\prod_{t\in T} 2^{X_t}$, where $X_t = X$ for $t \in T$ and 2^X is taken with the Vietoris topology, the intersection $\bigcap_{t\in T} A_t \subset X$ (or – equivalently – the Cartesian product $\prod_{t\in T} A_t \subset \prod_{t\in T} X_t$) defines an upper semicontinuous set-valued mapping.

Chapter 4
Metric and metrizable spaces

The concept of a topological space can be considered as an axiomatization of the notion of the closeness of a point to a set: a point is close to a set if it belongs to the closure of the set. In this chapter we shall study the theory of metric spaces which is an axiomatization of the notion of closeness of points: in a metric space, to every pair of points corresponds a real number, the distance between them, whose fundamental properties are described by a set of axioms. The distance between points can be used to define the distance from a point to a set; by agreeing that all points whose distance to a set A is equal to zero are close to A, and defining the closure of A as the set of all such points, we obtain a topological space. The topological spaces that can be obtained in this way are called metrizable spaces.

In Chapter 8, we shall discuss two further sets of axioms which describe related concepts: uniform spaces and proximity spaces. In uniform spaces one also considers the distance between pair of points, but it is measured in different way than in metric spaces. The theory of proximity spaces is an axiomatization of the notion of proximity of pairs of sets.

Those three notions – metric space, uniform space and proximity space – are therefore basically distinct from the notion of a topological space. The reason that they are studied in this book is that their numerous and interesting connections with topologial spaces have made them a part of general topology.

From a pureley logical point of view, we should postpone our study of metric spaces until Chapter 8, as we do with uniform and proximity spaces, and we should first complete the part of this book concerned exclusively with topological spaces. However, the class of metric spaces is intimately connected with the interesting class of metrizable spaces (which is a class of topological spaces), plays an important role in applications of general topology and – last but not least – helps to develop a proper topological intuition. Furthermore, our repertoire of topological notions is now large enough that we will be able to state the most important facts on metric and metrizable spaces.

Section 4.1 opens with definitions of metric and metrizable spaces; we show how a metric induces a topology and define two metrics to be equivalent if they induce the same topology. Then we show that every metrizable space is perfectly normal and first-countable, and prove that in metrizable spaces second-countability is equivalent to separability and to the Lindelöf property. The section closes with two important theorems asserting that in metrizable spaces compactness, countable compactness and sequential compactness are equivalent and that these conditions imply separability.

In Section 4.2 operations on metric spaces are defined and studied. We prove that subspaces, sums and countable Cartesian products of metrizable spaces are metrizable; furthermore it is shown that for a topological space X the function space R^X with the topology of uniform convergence is metrizable. We also give some conditions sufficient for metrizability of quotient spaces, of limits of inverse systems and of function spaces with the compact-open topology. The section ends with the definition of a metric on the set of all bounded mappings from a topological space to a metric space, and that leads to the definition of the topology of uniform convergence in this more general situation.

Section 4.3 is devoted to a study of two important classes of metric spaces: totally bounded spaces and complete spaces. Among other things we give topological characterizations of the classes of corresponding metrizable spaces, and for every metric space X we define the completion $\tilde{X}$, which is the smallest complete space that contains X. Metric properties of compact metrizable spaces are discussed at the end of this section.

Compared to the classical results of the first three sections, the results of Section 4.4. are relatively recent. They arose from A. H. Stone's theorem on locally finite refinement of open covers of metrizable spaces, and consist in topological conditions characterizing metrizability and conditions for invariance of metrizability under mappings.

4.1. Metric and metrizable spaces

A *metric space* is a pair (X, ρ) consisting of a set X and a function ρ defined on the set $X \times X$, assuming non-negative real values, and satisfying the following conditions:

(M1) $\rho(x, y) = 0$ *if and only if* $x = y$.
(M2) $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in X$.
(M3) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ *for all* $x, y, z \in X$.

The set X is called a *space*, the elements of X are called *points*, the function ρ is called a *metric* on the set X and the number $\rho(x, y)$ is called the *distance between* x *and* y. Condition (M1) asserts that the distance between two distinct points is positive and every point has distance zero from itself. Condition (M2) asserts that the distance is a symmetric function, i.e., does not depend on the order of points x and y. Condition (M3), called the *triangle inequality*, states, figuratively speaking, that the sum of two sides of a triangle is not smaller than the third side.

A function ρ defined on the set $X \times X$, assuming non-negative real values, satisfying conditions (M2), (M3) and condition

(M1′) $\rho(x, x) = 0$ *for every* $x \in X$

is called a *pseudometric* on the set X.

Let (X, ρ) be a metric space, x_0 a point of X and r a positive number; the set $B(x_0, r) = \{x \in X : \rho(x_0, x) < r\}$ is called the *open ball with centre* x_0 *and radius* r or, briefly, the *r-ball about* x_0. For a set $A \subset X$ and a positive number r, by the *r-ball about* A we mean the set $B(A, r) = \bigcup_{x \in A} B(x, r)$; let us note that $x \in B(x, r)$, so that $A \subset B(A, r)$. Let us also observe that if $x_1 \in B(x_0, r)$, then $B(x_1, r_1) \subset B(x_0, r)$ for $r_1 = r - \rho(x_0, x_1) > 0$. Indeed, if $x \in B(x_1, r_1)$, then by virtue of (M3) we have

$$\rho(x_0, x) \leq \rho(x_0, x_1) + \rho(x_1, x) < \rho(x_0, x_1) + r - \rho(x_0, x_1) = r.$$

From the last observation it follows that by letting $\mathcal{B}(x) = \{B(x, r) : r > 0\}$ for every $x \in X$ we define a collection of families of subsets of X which has properties (BP1)–(BP3) and thus – by Proposition 1.2.3 – we generate a topology $\mathcal{O}$ on the set X. Hence, every metric space (X,ρ) defines a topological space $(X,\mathcal{O})$; members of the topology $\mathcal{O}$, i.e., open subsets of the space $(X,\mathcal{O})$, are unions of open balls. Clearly, the family of all open balls is a base for $(X,\mathcal{O})$. The family of all $1/i$-balls about x_0, where $i=1,2,\ldots$, is a base for $(X,\mathcal{O})$ at the point x_0; this implies that the space $(X,\mathcal{O})$ is first-countable. The topology $\mathcal{O}$ on the set X is called the *topology induced by the metric ρ*.

Since for every pair x_1,x_2 of distinct poins of X we have $\rho(x_1,x_2)=r>0$, it follows from the triangle inequality that $B(x_1,r/2)$ and $B(x_2,r/2)$ are disjoint neighbourhoods of x_1 and x_2. Thus, any space with the topology induced by a metric is a Hausdorff space.

As in the case of topological spaces, we shall frequently denote the metric space (X,ρ) by the single letter X; it will always be clear from the context which metric on X is being considered.

An analogous construction can be carried on under the assumption that ρ is a pseudometric. The topological spaces obtained in this way form a very large class which will not be discussed here. Let us note, however, that a space X with a topology induced by a pseudometric ρ is a T_0-space if and only if ρ is a metric; indeed, if $\rho(x_1,x_2)=0$ for $x_1\neq x_2$, then every neighbourhood of x_1 contains x_2 and vice versa, so that X is not a T_0-space. One can readily verify that if X is a set containing more than one point, then by letting $\rho(x,y)=0$ for all $x,y\in X$ we define a pseudometric on X which induces the anti-discrete topology on X. Although not studied here for their own sake, in the sequel pseudometrics will be sometimes used as a convenient tool.

The notion of a metric space leads to an important topological notion, viz., the notion of a metrizable space. A topological space X is *metrizable* if there exists a metric ρ on the set X such that the topology induced by the metric ρ coincides with the original topology of X; metrics on the set X which induce the original topology of X will be called *metrics on the space X*.

We lay great stress on metric and metrizable spaces because many important topological spaces used in various branches of mathematics are metrizable and, moreover, their topology is often induced by a natural metric.

Let us observe that metrizability is a topological property but our definition of the class of metrizable spaces is not an internal definition. The question arises whether there exist internal characterizations of metrizable spaces. As we shall see later, the answer is positive. Theorems stating necessary and sufficient internal conditions for metrizabilty of topological spaces formulated in terms of topological invariants are called *metrization theorems*; two of these will be proved in Section 4.4 and others in Section 5.4.

Two metrics ρ_1 and ρ_2 on a set X are called *equivalent* if they induce the same topology on X; obviously, the relation thus defined is an equivalence relation. The reason for regarding two metrics inducing the same topology as equivalent objects is that we are interested in topologies on X and metrics are for us only auxiliary tools; their role could be compared with that of systems of coordinates in studying Euclidean spaces.

We shall give a convenient criterion for the equivalence of metrics in Theorem 4.1.2 below; first, however, we shall show how the closure of a set can be described in terms of a metric.

A sequence $x_1,x_2,\ldots$ of points of a metric space (X,ρ) *converges* to a point $x\in X$ if the sequence of real numbers $\rho(x,x_1), \rho(x,x_2), \ldots$ converges to zero; the point x is called the *limit* of the sequence $x_1, x_2, \ldots$ and is denoted by $\lim x_i$. It follows from (M1) and (M3) that any sequence of points of a metric space has at most one limit.

4.1.1. PROPOSITION. *A point x belongs to the closure $\overline{A}$ of a set $A \subset X$ with respect to the topology induced on X by a metric ρ if and only if there exists a sequence of points of A that converges to x.*

PROOF. Assume that $x \in \overline{A}$. For every natural number i take a point $x_i \in A \cap B(x, 1/i)$; clearly, we have $\rho(x, x_i) < 1/i$ and $x = \lim x_i$. On the other hand, if $x \notin \overline{A}$, then there exists an $r > 0$ such that $A \cap B(x,r) = \emptyset$; hence, we have $\rho(x,x') \geq r$ for every $x' \in A$ and there is no sequence of points of A that converges to X.■

Proposition 4.1.1 yields

4.1.2. THEOREM. *Two metrics ρ_1 and ρ_2 on a set X are equivalent if and only if they induce the same convergence, i.e., if for every $x \in X$ and any sequence $x_1, x_2, \ldots$ of points of X the conditions $\lim \rho_1(x, x_i) = 0$ and $\lim \rho_2(x, x_i) = 0$ are equivalent.*■

Let us observe, in connection with the notions studied in Section 1.6, that a sequence $x_1, x_2, \ldots$ in a metric space (X, ρ) converges to a point $x \in X$ if and only if $x = \lim x_i$ in the space X with the topology induced by the metric ρ; thus our use of the same terms and symbols in both cases is well justified. Clearly, Proposition 4.1.1 is an immediate consequence of the above observation and of Theorem 1.6.14. Let us also note that Proposition 1.6.15 – or Propositions 4.1.1 and 1.4.1 – imply that a mapping f of a metrizable space X to a metrizable space Y is continuous if and only if for any sequence $x, x_1, x_2, \ldots$ in the space X the equality $x = \lim x_i$ implies the equality $f(x) = \lim f(x_i)$.

The *diameter* of a non-empty set A in a metric space (X, ρ) is defined as the least upper bound of distances between points of A and is denoted by $\delta(A)$; it can be finite or equal to ∞. Thus

$$\delta(A) = \sup\{\rho(x_1, x_2) : \ x_1, x_2 \in A\}; \tag{1}$$

we also define $\delta(\emptyset) = 0$. One readily verifies that $\delta(A) = \delta(\overline{A})$.

A set A is said to be *bounded* if $\delta(A) < \infty$. A metric ρ on a set X is *bounded by a real number r (bounded)* if $\delta(X) < r$ (if $\delta(X) < \infty$). The notions of the diameter of a set with respect to a pseudometric, of a pseudometric bounded by a real number and of a bounded pseudometric are defined in a similar way.

4.1.3. THEOREM. *For every metric space (X, ρ) there exists a metric ρ_1 on the set X which is equivalent to ρ and bounded by* 1.

PROOF. Let us define

$$\rho_1(x,y) = \min(1, \rho(x,y)) \quad \text{for} \quad x, y \in X.$$

We shall verify that ρ_1 is a metric. The fact that ρ_1 satisfies conditions (M1) and (M2) follows directly from the fact that ρ satisfies these conditions. Let x, y and z be arbitrary points of

X and let $a = \rho(x,y)$, $b = \rho(y,z)$ and $c = \rho(x,z)$. Since each of the numbers 2, $1+a$, $1+b$ and $a+b$ is not less than either 1 or c, we have

$$\min(2, 1+a, 1+b, a+b) \geq \min(1,c),$$

and thus

$$\begin{aligned}\rho_1(x,y) + \rho_1(y,z) &= \min(1,a) + \min(1,b) \\ &= \min(2, 1+a, 1+b, a+b) \geq \min(1,c) = \rho_1(x,z).\end{aligned}$$

Hence ρ_1 also satisfies condition (M3). It is obvious that ρ_1 is bounded by 1 and its equivalence to ρ follows from Theorem 4.1.2.■

4.1.4. EXAMPLES. Let X be an arbitrary set and let us define, for $x, y \in X$,

$$\rho(x,y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

One can readily see that ρ is a metric on X. Since $B(x,1) = \{x\}$ for every $x \in X$, the metric ρ induces the discrete topology on X. Hence, every discrete space is metrizable. The term *discrete space* will also be applied to the metric space (X, ρ) described above.

The real line R and the closed unit interval I also are metrizable; we can define the distance between two points as the absolute value of their difference. It can easily be seen that in Section 1.1 we defined open sets in R and I as unions of the corresponding open balls.

As every metrizable space is first-countable, the space $A(\mathfrak{m})$ for $\mathfrak{m} > \aleph_0$ is an example of a non-metrizable space.■

4.1.5. EXAMPLE. Let $\mathfrak{m}$ be an infinite cardinal number, S a set of cardinality $\mathfrak{m}$, and let $I_s = I \times \{s\}$ for every $s \in S$. By letting

$$(x, s_1)E(y, s_2) \quad \text{whenever} \quad x = 0 = y \quad \text{or} \quad x = y \text{ and } s_1 = s_2$$

we define an equivalence relation E on the set $\bigcup_{s \in S} I_s$.

The reader can easily verify that the formula

$$\rho([(x,s_1)], [(y,s_2)]) = \begin{cases} |x-y|, & \text{if } s_1 = s_2, \\ x+y, & \text{if } s_1 \neq s_2, \end{cases}$$

defines a metric on the set of equivalence classes of E. For a fixed cardinal number $\mathfrak{m}$, the metric space thus obtained does not depend essentially on the choice of the set S; this space – as well as the corresponding metrizable space – will be called the *hedgehog of spininess* $\mathfrak{m}$ and will be denoted by $J(\mathfrak{m})$. One readily sees that for every $s \in S$ the mapping j_s of the interval I to $J(\mathfrak{m})$ defined by letting $j_s(x) = [(x,s)]$ is a homeomorphic embedding. The family of all balls with rational radii around points of the form $[(r,s)]$, where r is a rational number, is a base for $J(\mathfrak{m})$; so that $w(J(\mathfrak{m})) \leq \mathfrak{m}$. Since the subspace of $J(\mathfrak{m})$ consisting of all points of the form $[(1,s)]$ is a discrete space of cardinality $\mathfrak{m}$, it follows that $w(J(\mathfrak{m})) = \mathfrak{m}$.■

4.1.6. EXAMPLE. Let $X = R \times R$ be the plane; for every pair of points $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ in X define

$$\rho(z_1, z_2) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ |y_1| + |y_2| + |x_1 - x_2|, & \text{if } x_1 \neq x_2. \end{cases}$$

The reader can easily verify that ρ is a metric on X. Figuratively speaking, one can say that ρ measures distances in a wooded country with a river $y = 0$. The inhabitants of that country, in order to get access to the water, have cut out paths perpendicular to the river. If someone wishes there to go from (x_1, y_1) to (x_2, y_2), he would find out that the best solution is to go straight to the river, swim to get nearest to (x_2, y_2) and then walk again through the wood. The metric *"river"* on the plane is not equivalent to the metric defined in the first part of 4.1.4.■

4.1.7. EXAMPLE. Let H be the set of all infinite sequences $\{x_i\}$ of real numbers satisfying the condition $\sum_{i=1}^{\infty} x_i^2 < \infty$. We shall show that, letting

$$\rho(x, y) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2} \quad \text{for} \quad x = \{x_i\},\ y = \{y_i\},$$

we define a metric on H.

First of all, let us prove that ρ is well-defined, i.e., that the series in the definition of ρ is convergent. In the proof we shall apply the Cauchy inequality

$$\Big|\sum_{i=1}^{k} a_i b_i\Big| \le \sqrt{\sum_{i=1}^{k} a_i^2} \cdot \sqrt{\sum_{i=1}^{k} b_i^2},$$

which holds for all finite sequences $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ of real numbers.*

Let us note that for every pair of points $x = \{x_i\}$, $y = \{y_i\}$ in H and any positive integer k we have

$$\begin{aligned}
\sum_{i=1}^{k}(x_i - y_i)^2 &= \sum_{i=1}^{k} x_i^2 - 2\sum_{i=1}^{k} x_i y_i + \sum_{i=1}^{k} y_i^2 \\
&\le \sum_{i=1}^{k} x_i^2 + 2\sqrt{\sum_{i=1}^{k} x_i^2} \cdot \sqrt{\sum_{i=1}^{k} y_i^2} + \sum_{i=1}^{k} y_i^2 \\
&= \Big(\sqrt{\sum_{i=1}^{k} x_i^2} + \sqrt{\sum_{i=1}^{k} y_i^2}\Big)^2 \le \Big(\sqrt{\sum_{i=1}^{\infty} x_i^2} + \sqrt{\sum_{i=1}^{\infty} y_i^2}\Big)^2.
\end{aligned}$$

Since the last inequality holds for any positive integer k, the series in the definition of ρ is convergent and $\rho(x, y)$ is well-defined.

Obviously, ρ satisfies conditions (M1) and (M2); we shall show that condition (M3) is also satisfied.

Let $x = \{x_i\}$, $y = \{y_i\}$ and $z = \{z_i\}$ be any points of H; let

$$\begin{aligned}
x^k &= \{x_1, x_2, \ldots, x_k, 0, 0, \ldots\}, \\
y^k &= \{y_1, y_2, \ldots, y_k, 0, 0, \ldots\}, \\
z^k &= \{z_1, z_2, \ldots, z_k, 0, 0, \ldots\}
\end{aligned}$$

* Let $a = \sum_{i=1}^{k} a_i^2$, $b = \sum_{i=1}^{k} b_i^2$ and $c = \sum_{i=1}^{k} a_i b_i$; to prove the Cauchy inequality, i.e., the inequality $c^2 \le ab$, it suffices to note that the polynomial $ax^2 + 2cx + b = \sum_{i=1}^{k}(a_i x + b_i)^2$ has no distinct real roots, so that by the quadratic formula $4c^2 \le 4ab$.

and

$$a_i = x_i - y_i, \quad b_i = y_i - z_i, \quad c_i = x_i - z_i.$$

By the Cauchy inequality we have

$$\begin{aligned}
[\rho(x^k, z^k)]^2 &= \sum_{i=1}^k c_i^2 = \sum_{i=1}^k (a_i + b_i)^2 = \sum_{i=1}^k a_i^2 + 2\sum_{i=1}^k a_i b_i + \sum_{i=1}^k b_i^2 \\
&\le \sum_{i=1}^k a_i^2 + 2\sqrt{\sum_{i=1}^k a_i^2} \cdot \sqrt{\sum_{i=1}^k b_i^2} + \sum_{i=1}^k b_i^2 \\
&= \left(\sqrt{\sum_{i=1}^k a_i^2} + \sqrt{\sum_{i=1}^k b_i^2}\right)^2 = [\rho(x^k, y^k) + \rho(y^k, z^k)]^2.
\end{aligned}$$

It follows from the last inequality that for $k = 1, 2, \ldots$ we have

$$\rho(x, y) + \rho(y, z) \ge \rho(x^k, y^k) + \rho(y^k, z^k) \ge \rho(x^k, z^k),$$

and this implies that $\rho(x, y) + \rho(y, z) \ge \rho(x, z)$.

The space H is called *Hilbert space*. The set of all sequences $\{x_i\}$, where all the x_i's are rational numbers only finitely many of which are distinct from zero, is dense in H and countable, so that Hilbert space is separable. ■

The equivalence of conditions (i) and (iii) in Proposition 1.4.1 immediately implies the following characterization of continuity for mappings of metrizable spaces.

4.1.8. PROPOSITION. *A mapping f of a space X with the topology induced by a metric ρ to a space Y with the topology induced by a metric σ is continuous if and only if for every $x \in X$ and any $\epsilon > 0$ there exists a $\delta > 0$ such that $\sigma(f(x), f(x')) < \epsilon$ whenever $\rho(x, x') < \delta$.* ■

In the realm of metric spaces, one also studies uniformly continuous mappings. A mapping f of a space X with a metric ρ to a space Y with a metric σ is *uniformly continuous with respect to ρ and σ* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, x' \in X$ we have $\sigma(f(x), f(x')) < \epsilon$ whenever $\rho(x, x') < \delta$. Clearly, every uniformly continuous mapping is continuous but not vice versa. The concept of uniform continuity is not a topological concept; it refers to particular metrics on the spaces X and Y, and a mapping $f: X \to Y$ can be uniformly continuous with respect to some metrics and not be uniformly continuous with respect to other metrics (cf. Theorem 4.3.32). The concept of uniform continuity belongs to the theory of uniform spaces developed in Chapter 8 (see Exercise 8.1.A(a) and Problem 8.5.19(a)).

A mapping f of a space X with a metric ρ to a space Y with a metric σ is an *isometry* if $\sigma(f(x), f(y)) = \rho(x, y)$ for every pair of points $x, y \in X$; if there exists an isometry of X onto Y, then we say that the spaces X and Y are *isometric*. It is easily seen that an isometry is a uniformly continuous one-to-one mapping. Since the inverse mapping of an isometry onto is an isometry as well, it follows that isometries onto are homeomorphisms and that isometric spaces are homeomorphic.

The *distance $\rho(x, A)$ from a point x to a set A* in a metric space (X, ρ) is defined by letting

$$\rho(x, A) = \inf\{\rho(x, a) : a \in A\}, \quad \text{if} \quad A \ne \emptyset, \quad \text{and} \quad \rho(x, \emptyset) = 1.$$

Similarly, for a pair A, B of sets in a metric space (X, ρ) we define

$$\rho(A, B) = \inf\{\rho(a, b) : a \in A, b \in B\}, \quad \text{if} \quad A \neq \emptyset \neq B,$$

and

$$\rho(A, \emptyset) = 1 = \rho(\emptyset, B).$$

4.1.9. PROPOSITION. *For a pair of points x, y and a set A in a metric space (X, ρ) we have*

$$|\rho(x, A) - \rho(y, A)| \leq \rho(x, y).$$

PROOF. Clearly, we can suppose that $A \neq \emptyset$. For every $a \in A$ we have

$$\rho(x, A) \leq \rho(x, a) \leq \rho(x, y) + \rho(y, a),$$

and this implies, a being an arbitrary point of A, that $\rho(x, A) \leq \rho(x, y) + \rho(y, A)$, i.e., that

$$\rho(x, A) - \rho(y, A) \leq \rho(x, y).$$

By the symmetry of assumptions we also have

$$\rho(y, A) - \rho(x, A) \leq \rho(x, y). \blacksquare$$

Propositions 4.1.8 and 4.1.9 yield

4.1.10. THEOREM. *For every set A in a metric space (X, ρ) assigning to each point $x \in X$ the distance $\rho(x, A)$ defines a continuous function on X.* $\blacksquare$

From 4.1.1 and 4.1.10 we obtain

4.1.11. COROLLARY. *For every set A in a metric space (X, ρ) we have*

$$\overline{A} = \{x : \rho(x, A) = 0\}. \blacksquare$$

Let us note three further corollaries to Theorem 4.1.10.

4.1.12. COROLLARY. *Every closed subset of a metrizable space is functionally closed and, in particular, is a G_δ-set.*

PROOF. If $A = \overline{A}$, then letting $f(x) = \rho(x, A)$ we have $A = f^{-1}(0)$. $\blacksquare$

The above corollary and Theorem 1.5.19 yield

4.1.13. COROLLARY. *Every metrizable space is perfectly normal.* $\blacksquare$

Let us observe that normality of a metrizable space X follows directly from the fact that for every pair A, B of disjoint closed subsets of X and any metric ρ on the space X the formula

$$f(x) = \frac{\rho(x, A)}{\rho(x, A) + \rho(x, B)}$$

defines a continuous real-valued function on X such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.

4.1.14. COROLLARY. *In a metric space (X,ρ) for every compact set $A \subset X$ and any open set U containing A there exists an $r > 0$ such that $B(A,r) \subset U$.*

PROOF. The function $f: X \to R$, defined by letting $f(x) = \rho(x, X \setminus U)$ is positive on the set A. Hence, by Corollary 3.2.9, there exists an $r > 0$ such that $f(x) \geq r$ for every $x \in A$; clearly $B(A,r) \subset U$.■

As we already know, all metrizable spaces are first-countable, but not necessarily second-countable. Necessary and sufficient conditions for second-countability of a metrizable space will follow from our next theorem.

4.1.15. THEOREM. *For every metrizable space X and any cardinal number $\mathfrak{m}$ the following conditions are equivalent:*

(i) *The space X has a base of cardinality $\leq \mathfrak{m}$.*
(ii) *The space X has a network of cardinality $\leq \mathfrak{m}$.*
(iii) *Every open cover of the space X has a subcover of cardinality $\leq \mathfrak{m}$.*
(iv) *Every closed discrete subspace of the space X has cardinality $\leq \mathfrak{m}$.*
(v) *Every discrete subspace of the space X has cardinality $\leq \mathfrak{m}$.*
(vi) *Every family of pairwise disjoint non-empty open subsets of the space X has cardinality $\leq \mathfrak{m}$.*
(vii) *The space X has a dense subset of cardinality $\leq \mathfrak{m}$.*

PROOF. It is easily verified that if $\mathfrak{m}$ is a finite number, than all of the above conditions are equivalent to the inequality $|X| \leq \mathfrak{m}$; we shall assume that $\mathfrak{m} \geq \aleph_0$.

The implication (i)$\Rightarrow$(ii) is obvious and the fact that (ii)$\Rightarrow$(iii) follows from Theorem 3.8.12 or Remark 1.1.16.

Let A be a closed discrete subspace of X. For every $x \in A$ there exists an open set $U_x \subset X$ such that $A \cap U_x = \{x\}$. Since the open cover $\{U_x\}_{x \in A} \cup \{X \setminus A\}$ of the space X has no subcover of cardinality $< |A|$, it follows that (iii)$\Rightarrow$(iv).

We pass to the proof that (iv)$\Rightarrow$(v). Let A be a discrete subspace of X; one can readily verify that A is an open subset of the closure $\overline{A} \subset X$. From the perfect normality of X it follows that there exists a sequence $A_1, A_2, \ldots$ of closed subsets of $\overline{A}$ such that $A = \bigcup_{i=1}^{\infty} A_i$. Since the sets A_i are also closed in X, it follows from (iv) that $|A_i| \leq \mathfrak{m}$ for $i = 1, 2, \ldots$ and this implies that $|A| \leq \mathfrak{m}$.

To show that (v)$\Rightarrow$(vi) it suffices to observe that choosing a point from every member of a family of pairwise disjoint non-empty open subsets of the space X, we obtain a discrete subspace of X which has the same cardinality as the family of subsets under consideration.

To prove that (vi)$\Rightarrow$(vii) let us take a metric ρ on the space X and – applying the Teichmüller-Tukey lemma – let us choose for $i = 1, 2, \ldots$ a maximal subset $A_i \subset X$ with the property that $\rho(x,y) \geq 1/i$ whenever $x, y \in A_i$. Since $1/2i$-balls around points of A_i are pairwise disjoint, we have $|A_i| \leq \mathfrak{m}$ and it suffices to prove that the union $A = \bigcup_{i=1}^{\infty} A_i$ is dense in X. However, if there existed a point $x \in X \setminus \overline{A}$, we would have $\rho(x, A_{i_0}) \geq \rho(x, A) > 1/i_0$ for some positive integer i_0, and this is impossible by the maximality of A_{i_0}.

It remains to show that (vii)$\Rightarrow$(i). Let ρ be a metric on the space X and let A be a dense subset of X which has cardinality $\leq \mathfrak{m}$. Denote by $\mathcal{B}$ the family of all balls $B(x,r)$, where $x \in A$ and r is a rational number; clearly $|\mathcal{B}| \leq \mathfrak{m}$. We shall prove that $\mathcal{B}$ is a base for the space X. Take an arbitrary point $x \in X$ and a neighbourhood U of x; one can assume that $U = B(x,r)$. The set A being dense in X, there exists a point $x_0 \in A \cap B(x, r/3)$. For any rational number r_0 satisfying $r/3 < r_0 < r/2$ we have

$$x \in B(x_0, r_0) \subset B(x,r) = U \quad \text{and} \quad B(x_0, r_0) \in \mathcal{B}. \blacksquare$$

4.1.16. COROLLARY. *For every metrizable space X the following conditions are equivalent:*
(i) *The space X is second-countable.*
(ii) *The space X has the Lindelöf property.*
(iii) *The space X is separable.*
(iv) *Every family of pairwise disjoint non-empty open subsets of the space X is countable.* ■

The equivalence of conditions (iii) and (iv) in Theorem 4.1.15, along with Theorem 3.10.3, implies that every countably compact metrizable space has the Lindelöf property. Hence, by Theorems 3.10.1 and 3.10.31, we have

4.1.17. THEOREM. *For every metrizable space X the following conditions are equivalent:*
(i) *The space X is compact.*
(ii) *The space X is countably compact.*
(iii) *The space X is sequentially compact.* ■

Let us observe that, by virtue of Theorems 3.10.20 and 3.10.21, to the above set of equivalent conditions one can adjoin pseudocompactness of the space X.

The equivalence of conditions (iv) and (vii) in Theorem 4.1.15 yields

4.1.18. THEOREM. *Every compact metrizable space is separable.* ■

Historical and bibliographic notes

The class of metric spaces was the first class of abstract spaces to which several notions and results, discovered in the infancy of general topology in the study of subsets of the real line and of Euclidean spaces, were successfully generalized. The class of metric spaces is sufficiently large to include many objects studied in various branches of mathematics and thus describe them in geometric language, and, at the same time, the spaces in this class seem to be sufficiently simple to permit the use of geometric intuition. The notion of a metric space was introduced by Fréchet in his thesis [1906]. For many years topologists' attention was focused on metric spaces and, in particular, on separable metric spaces. No doubt, the latter class is the best explored class of topological spaces; Kuratowski's two-volume monograph [1966] and [1968] is a veritable encyclopaedia on this subject.

The equivalence of conditions (ii) and (iii) in Corollary 4.1.16 and of conditions (i) and (ii) in Theorem 4.1.17 was established by Gross in [1914]. Hausdorff in his [1914] book proved the equivalence of conditions (i) and (iii) in Corollary 4.1.16 and of conditions (ii) and (iii) in Theorem 4.1.17; he also proved Theorem 4.1.18 there. The hedgehog $J(\mathfrak{m})$ was discovered by Urysohn in [1927]. Hilbert space, described in Example 4.1.7 is – as shown by Anderson in [1966] – homeomorphic to the Cartesian product of $\aleph_0$ copies of the real line; this is a difficult and deep result (for a proof, see Bessaga and Pełczyński [1975] or Toruńczyk [1981]).

Exercises

4.1.A. Note that in a metric space (X,ρ) the closure of a ball $B(x_0,r)$ generally does not coincide with the set $\{x: \rho(x_0,x) \le r\}$.

4.1.B. (a) Show that for every metric space (X,ρ) the formula

$$\rho_2(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)} \quad \text{for } x,y \in X$$

defines a metric on the set X which is equivalent to ρ and bounded by 1 (cf. Theorem 4.1.3).

(b) We say that two metrics ρ_1 and ρ_2 on a set X are *uniformly equivalent* if for every $\epsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x, x' \in X$ we have $\rho_2(x,x') < \epsilon$ whenever $\rho_1(x,x') < \delta_1$ and $\rho_1(x,x') < \epsilon$ whenever $\rho_2(x,x') < \delta_2$.

Note that any uniformly equivalent metrics are equivalent and give an example of two metrics on the real line which are equivalent but not uniformly equivalent. Verify that metrics ρ_1 and ρ_2 on a space X are uniformly equivalent if and only if the mapping $\mathrm{id}_X: X \to X$ is uniformly continuous with respect to ρ_1 and ρ_2 and also with respect to ρ_2 and ρ_1, or – equivalently – if for every metric space (Y,σ) the class of all mappings of X to Y uniformly continuous with respect to ρ_1 and σ coincides with the class of all mappings of X to Y uniformly continuous with respect to ρ_2 and σ (cf. Exercise 8.1.A(b) and Problem 8.5.19(b)).

(c) Check that the metric ρ_1 in Theorem 4.1.3 and the metric ρ_2 in part (a) are uniformly equivalent to the metric ρ.

(d) Note that a metric uniformly equivalent to a bounded metric need not be bounded.

4.1.C. Show that the Hilbert cube $I^{\aleph_0}$ is homeomorphic to the subspace of Hilbert space consisting of all points $\{x_i\}$ satisfying $0 \le x_i \le 1/i$ for $i = 1,2,\ldots$

4.1.D. Give an example of a metrizable space which cannot be embedded in a locally compact metrizable space.

4.1.E. (a) Show that for a closed subset A of a metrizable space X the inequality $\chi(A,X) \le \aleph_0$ holds if and only if the set $\mathrm{Fr}\, A$ is compact. Deduce that a dense in itself metrizable space X is compact if and only if $\chi(A,X) \le \aleph_0$ for every closed $A \subset X$.

(b) (Henriksen and Isbell [1958]) Prove that if cX is a compactification of a metrizable space X, then the remainder $cX \setminus X$ has the Lindelöf property.

Hint. Apply (a) and Exercise 2.1.C(b).

4.1.F (Hausdorff [1919]). Verify that if A is a closed subspace of a metrizable space X, then for every continuous function $f: A \to I$ and every metric ρ on the space X the formula

$$F(x) = \begin{cases} \inf\left\{f(a) + \dfrac{\rho(x,a)}{\rho(x,A)} - 1 : a \in A\right\}, & \text{if } x \in X \setminus A, \\ f(x), & \text{if } x \in A, \end{cases}$$

defines a continuous extension F of f over X (cf. Theorem 2.1.8).

4.1.G. Observe that in Exercises 3.2.H(a) and (c) one can assume that the mapping f takes values in an arbitrary metrizable space.

Hint. Apply Theorem 4.1.18

4.1.H (Erdős and Tarski [1943]; implicity, Haratomi [1931]). (a) Prove that in every metrizable space of weight $\mathfrak{m}$ there exists a family of pairwise disjoint non-empty open sets which has cardinality $\mathfrak{m}$.

Hint. Difficulties arise when $w(X) = \mathfrak{m}$ is the least upper bound of cardinal numbers $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$, where $\aleph_0 \leq \mathfrak{m}_i < \mathfrak{m}$ for $i = 1, 2, \ldots$ First, consider the case when there exists a non-empty open set $U \subset X$ such that $w(V) = \mathfrak{m}$ for every non-empty open set $V \subset U$. Then, assuming that no such set exists, consider a maximal family $\{U_s\}_{s \in S}$ of pairwise disjoint non-empty open subsets of X such that $w(U_s) < \mathfrak{m}$ for every $s \in S$ and show that if $|S| < \mathfrak{m}$, then $\sup_{s \in S} w(U_s) = \mathfrak{m}$.

Remark. The problem of existence of a regular space in which the supremum of cardinalities of all families consisting of pairwise disjoint non-empty open sets is assumed by no such family is of set-theoretic character; for a discussion, see Comfort [1971].

(b) Deduce from part (a) that every metrizable space of weight $\mathfrak{m}$ contains a discrete subspace of cardinality $\mathfrak{m}$. Give an example of a metrizable space of weight $\mathfrak{m}$ which contains no closed discrete subspace of cardinality $\mathfrak{m}$.

Hint. Consider an appropriate subspace of $J(\aleph_\omega)$.

4.2. Operations on metrizable spaces

Let us begin with the simple observation that every subspace of a metrizable space is metrizable. Indeed, if X is a metrizable space and ρ a metric on the space X, then defining the distance between two points of a subset M of X to be the distance between these points with respect to the metric ρ, we introduce a metric on M; as one can readily verify, the topology induced by this metric coincides with the topology of M as a subspace of X. The restriction $\rho|M \times M$ of a metric ρ on X to a subset $M \subset X$ will usually be denoted by ρ_M and sometimes just by ρ; the metric space (M, ρ_M) will also be denoted by (M, ρ).

4.2.1. THEOREM. *The sum $\bigoplus_{s \in S} X_s$ is metrizable if and only if all spaces X_s are metrizable.*

PROOF. By the above observation, it suffices to prove that the sum of pairwise disjoint metrizable spaces $\{X_s\}_{s \in S}$ is metrizable. By virtue of Theorem 4.1.3, one can suppose that for every $s \in S$ the topology on X_s is induced by a metric ρ_s bounded by 1, i.e., that $\rho_s(x, y) \leq 1$ for $x, y \in X_s$ and $s \in S$. For every pair of points $x, y \in X = \bigoplus_{s \in S} X_s$ let

$$\rho(x, y) = \begin{cases} \rho_s(x, y), & \text{if } x, y \in X_s \text{ for some } s \in S, \\ 1, & \text{otherwise.} \end{cases}$$

We shall check that ρ is a metric on the set X. Obviously, conditions (M1) and (M2) are satisfied, it remains to show that condition (M3) is also satisfied. Let x, y and z be any points of X; we shall show that

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z). \tag{1}$$

If $x, z \in X_s$ for some $s \in S$, then the left-hand side of (1) is equal to $\rho_s(x, z)$ and the right-hand side is equal to $\rho_s(x, y) + \rho_s(y, z)$, if $y \in X_s$, or to 2, if $y \notin X_s$ – so that (1) holds. On the other hand, if $x \in X_{s_1}$ and $z \in X_{s_2}$ for $s_1 \neq s_2$, then the left-hand side of (1) is equal to 1 and the right-hand side is not less than 1 because either $y \notin X_{s_1}$ or $y \notin X_{s_2}$ – so that (1) holds again.

One easily sees that, for every $s \in S$, the set X_s is open in the space X with the topology induced by ρ. Since $\rho_{X_s} = \rho_s$ induces the original topology on X_s, it follows from Proposition 2.2.4 that ρ induces on X the topology of the sum of topological spaces $\{X_s\}_{s \in S}$. ■

Let $X_1, X_2, \ldots$ be a sequence of metrizable spaces and let ρ_i be a metric on the space X_i bounded by 1 for $i = 1, 2, \ldots$ Consider the set $X = \prod_{i=1}^{\infty} X_i$ and for every pair $x = \{x_i\}$, $y = \{y_i\}$ of points of X let

(2) $$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_i(x_i, y_i).$$

The reader can easily verify that ρ satisfies conditions (M1)–(M3), so that the question arises whether the topology on X induced by the metric ρ coincides with the Tychonoff topology of the Cartesian product $\prod_{i=1}^{\infty} X_i$. The positive answer is contained in the following theorem.

4.2.2. THEOREM. *Let $\{X_i\}_{i=1}^{\infty}$ be a family of metrizable spaces and let ρ_i be a metric on the space X_i bounded by 1. The topology induced on the set $X = \prod_{i=1}^{\infty} X_i$ by the metric ρ defined in (2) coincides with the topology of the Cartesian product of the spaces $\{X_i\}_{i=1}^{\infty}$.*

PROOF. For $x = \{x_i\}$, $y = \{y_i\}$ we clearly have $\rho_i(x_i, y_i) < \epsilon$ whenever $\rho(x, y) < \epsilon/2^i$, so that, by Proposition 4.1.8, the projection p_i of X onto X_i is continuous with respect to the topology induced on X by ρ. It follows from the definition of the Tychonoff topology that the topology induced by ρ is finer than the topology of the Cartesian product on X.

Now, we shall show that every set $U \subset X$ open with respect to the topology induced by ρ is also open with respect to the topology of the Cartesian product and this will conclude the proof. Let us take a point $x = \{x_i\} \in U$; there exists an $r > 0$ such that $B(x, r) \subset U$. It suffices to find a positive integer k and open sets $U_i \subset X_i$, where $i = 1, 2, \ldots, k$, such that

(3) $$x \in \bigcap_{i=1}^{k} p_i^{-1}(U_i) \subset B(x, r) \subset U.$$

Let k be a positive integer satisfying

(4) $$\sum_{i=k+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^k} < \frac{1}{2} r$$

and for $i = 1, 2, \ldots, k$ let

$$U_i = B(x_i, r/2) = \{z \in X_i : \ \rho_i(x_i, z) < r/2\}.$$

For every $y = \{y_i\} \in \bigcap_{i=1}^{k} p_i^{-1}(U_i)$ we have $\rho_i(x_i, y_i) < r/2$ whenever $i \le k$, so that – by (2) and (4) –

$$\rho(x, y) = \sum_{i=1}^{k} \frac{1}{2^i} \rho_i(x_i, y_i) + \sum_{i=k+1}^{\infty} \frac{1}{2^i} \rho_i(x_i, y_i) < \frac{1}{2} r + \frac{1}{2} r = r,$$

which proves (3).■

In the last theorem, the assumption that the metrics ρ_i are bounded by 1 and the factors $1/2^i$ are needed only to guarantee that the series in (2) is convergent. For a finite sequence $X_1, X_2, \ldots, X_k$ of metrizable spaces one can define the distance between two points $x = \{x_i\}$ and $y = \{y_i\}$ of the set $X = X_1 \times X_2 \times \ldots \times X_k$ by letting

$$\rho(x, y) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2) + \ldots + \rho_k(x_k, y_k),$$

where ρ_i is any metric on the space X_i; one can easily verify that ρ is a metric on the set X which induces the topology of the Cartesian product (cf. Example 4.2.11).

From Theorem 4.2.2 several corollaries follow:

4.2.3. COROLLARY. *The Hilbert cube $I^{\aleph_0}$ is metrizable.* ∎

4.2.4. COROLLARY. *The Cartesian product $\prod_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is metrizable if and only if all spaces X_s are metrizable and there exists a countable set $S_0 \subset S$ such that X_s is a one-point space for $s \in S \setminus S_0$.*

PROOF. Theorem 2.3.24 implies that the Cartesian product of uncountably many metrizable spaces, each of which contains at least two points, is not first-countable. ∎

4.2.5. COROLLARY. *The limit of an inverse sequence of metrizable spaces is metrizable.* ∎

4.2.6. COROLLARY. *For every metric space (X,ρ) assigning to each point $(x,y) \in X \times X$ the distance $\rho(x,y)$ defines a continuous function from the Cartesian product $X \times X$ to R.*

PROOF. One readily verifies that

$$|\rho(x,y) - \rho(x',y')| \leq \rho(x,x') + \rho(y,y'). \blacksquare$$

From Theorem 4.2.2 one can derive the following corollary by a simple calculation; the corollary can be also deduced directly from Proposition 2.3.34.

4.2.7. COROLLARY. *A sequence $\{x_i^1\}, \{x_i^2\}, \ldots$ in the Cartesian product $\prod_{i=1}^\infty X_i$ of metrizable spaces converges to $x = \{x_i\} \in \prod_{i=1}^\infty X_i$ if and only if the sequence $x_i^1, x_i^2, \ldots$ converges to x_i for $i = 1, 2, \ldots$* ∎

4.2.8. THEOREM. *A compact space is metrizable if and only if it is a second-countable space.*

PROOF. From 4.1.18 and 4.1.16 it follows that every compact metrizable space is second-countable. Since every compact space is a Tychonoff space, it follows from 2.3.23 and 4.2.3 that every compact second-countable space is metrizable. ∎

4.2.9. THEOREM. *A second-countable space is metrizable if and only if it is a regular space.*

PROOF. From 4.1.13 it follows that every metrizable space is regular. Since every second-countable regular space is – by 1.5.16 – a Tychonoff space, it follows from 2.3.23 and 4.2.3 that every second-countable regular space is metrizable. ∎

4.2.10. THEOREM. *The Hilbert cube $I^{\aleph_0}$ is universal for all compact metrizable spaces and for all separable metrizable spaces.* ∎

Theorems 4.2.8 and 4.2.9 are metrization theorems: they state in terms of topological invariants (second-countability and regularity respectively) conditions for metrizability necessary and sufficient within two special classes of topological spaces; viz., compact spaces and second-countable spaces.

4.2.11. EXAMPLE. Let $X_1, X_2, \ldots, X_k$ be a finite sequence of metrizable spaces and let ρ_i be a metric on the space X_i. Define the distance between two points $x = \{x_i\}$ and $y = \{y_i\}$ of the set $X = X_1 \times X_2 \times \ldots \times X_k$ by letting

$$\rho(x, y) = \sqrt{\sum_{i=1}^{k} (\rho_i(x_i, y_i))^2}. \tag{5}$$

As in Example 4.1.7, one can verify that ρ satisfies the triangle inequality. Since conditions (M1) and (M2) are clearly satisfied, ρ is a metric on the set X. The reader can easily check that ρ induces the topology of the Cartesian product on X. This implies, in particular, that the Cartesian product topology on Euclidean n-space R^n, as defined in 2.3.9, coincides with the topology induced by the natural metric on this space, i.e., the metric ρ defined by (5), where $\rho_i(x_i, y_i) = |x_i - y_i|$. The reader can also verify that the notion of a bounded subset of R^n defined in Section 3.2 coincides with the notion of a bounded set in the metric space (R^n, ρ). ∎

4.2.12. EXAMPLE. Let X_i be for $i = 1, 2, \ldots$ the discrete space $D(\mathfrak{m})$ of cardinality $\mathfrak{m} \geq \aleph_0$ with the metric ρ_i defined by letting

$$\rho_i(x, y) = 1, \text{ if } x \neq y, \quad \text{and} \quad \rho_i(x, x) = 0.$$

It follows from Theorem 4.2.2 that the space $B(\mathfrak{m}) = [D(\mathfrak{m})]^{\aleph_0} = \prod_{i=1}^{\infty} X_i$ is metrizable and that the formula

$$\sigma(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_i(x_i, y_i)$$

defines a metric on the space $\prod_{i=1}^{\infty} X_i$.

One can readily verify that the formula

$$\rho(\{x_i\}, \{y_i\}) = \begin{cases} 1/k, & \text{if } x_k \neq y_k \text{ and } x_i = y_i \text{ for } i < k, \\ 0, & \text{if } x_i = y_i \text{ for all } i, \end{cases} \tag{6}$$

defines a metric ρ on the set $\prod_{i=1}^{\infty} X_i$. A sequence $\{x_i^1\}, \{x_i^2\}, \ldots$ in the space $B(\mathfrak{m})$ converges to a point $x = \{x_i\}$ if and only if for every i there exists a $k(i)$ such that $x_i^j = x_i$ whenever $j \geq k(i)$. The same condition is necessary and sufficient for convergence of the sequence $\{x_i^1\}, \{x_i^2\}, \ldots$ to the point $x = \{x_i\}$ in the space $(B(\mathfrak{m}), \rho)$, so that, by Theorem 4.1.2, ρ is a metric on the space $B(\mathfrak{m})$.

The space $B(\mathfrak{m})$ is called the *Baire space of weight* $\mathfrak{m}$; the reader can easily check that the weight of $B(\mathfrak{m})$ is indeed equal to $\mathfrak{m}$. When speaking of a metric on a Baire space, we shall always have in mind the metric ρ defined by (6). ∎

Quotient spaces of metrizable spaces are not necessarily metrizable; Examples 1.4.17 and 2.4.12. show that for a separable metrizable space X and a closed equivalence relation E on X, the quotient space X/E need not be first-countable. In Section 4.4 we shall show that if E is a closed equivalence relation on a metrizable space X and if the quotient space X/E is first-countable, or if the equivalence classes of E are compact, then X/E is a metrizable space. For the time being, let us observe that Theorems 3.7.19, 3.7.20 and 4.2.9 yield the following weaker result.

4.2.13. THEOREM. *If E is a closed equivalence relation on a separable metrizable space X and the equivalence classes of E are compact, then the quotient space X/E is metrizable.* ∎

The final part of this section is devoted to function spaces.

Let X be a topological space and (Y, ρ) a metric space; we shall say that a mapping f of X to Y is *bounded* if the set $f(X) \subset Y$ is bounded. On the set $B(X, (Y, \rho))$ of all bounded continuous mappings of X to Y one can define a metric $\hat{\rho}$ by letting

$$\hat{\rho}(f, g) = \sup_{x \in X} \rho(f(x), g(x)). \tag{7}$$

The sets $f(X)$ and $g(X)$ being bounded, $\hat{\rho}(f, g)$ is a well-defined real number, and one easily checks that $\hat{\rho}$ satisfies conditions (M1)–(M3).

The notion of a bounded mapping is not a topological notion: it depends on the choice of a particular metric on Y. It follows from Theorem 4.1.3 that there exists a metric on Y with respect to which all mappings of X to Y are bounded; thus, there exists a metric on the set Y^X of all continuous mappings of X to Y, and this metric induces a topology on Y^X. Unfortunately, this topology on Y^X depends on the choice of a bounded metric on Y.

4.2.14. EXAMPLE. Let ρ_1 be a metric on the set R of real numbers defined by letting $\rho_1(x, y) = \min(1, |x - y|)$ and let ρ_2 be a metric on the set R defined by letting $\rho_2(x, y) = \rho(h(x), h(y))$, where $h: R \to S^1 \setminus \{(0, 1)\} \subset R^2$ is a homeomorphism and ρ is the natural metric on R^2. Clearly the two metrics ρ_1 and ρ_2 are equivalent; however, we have $\rho_1(-i, i) = 1$ for $i = 1, 2, \ldots$, while $\lim \rho_2(-i, i) = 0$.

For $X = N$ and $Y = R$ the set Y^X consists of all mappings of X to Y. The two metrics ρ_1 and ρ_2 on R yield two metrics $\hat{\rho}_1$ and $\hat{\rho}_2$ on Y^X; we shall show that the metrics $\hat{\rho}_1$ and $\hat{\rho}_2$ are not equivalent.

For $i = 1, 2, \ldots$ consider the mapping $f_i: X \to Y$ defined by the formulas

$$f_i(i) = -i \quad \text{and} \quad f_i(k) = k \quad \text{for } k \neq i.$$

For the mapping $f: X \to Y$ defined by letting $f(i) = i$ we clearly have $\hat{\rho}_1(f, f_i) = 1$ for $i = 1, 2, \ldots$, while $\lim \hat{\rho}_2(f, f_i) = 0$. Hence the metrics $\hat{\rho}_1$ and $\hat{\rho}_2$ are not equivalent by Theorem 4.1.2.■

From the topological point of view there is, therefore, no reason to consider the topology on Y^X induced by the metric $\hat{\rho}$ defined in (7). Nevertheless, the space of all bounded continuous mappings of X to Y with the topology induced by $\hat{\rho}$ has interesting properties and is very useful, mainly because – as we shall see in the next section – the Baire category theorem holds there whenever (Y, ρ) is a complete space.

Let us observe that if X is a compact space, then by 4.2.6 and 3.2.9 every continuous mapping of X to a metrizable space Y is bounded with respect to any metric ρ on the space Y, so that in this case $\hat{\rho}$ is a metric on the whole set Y^X. As shown in Theorem 4.2.17 below, for a compact space X the topology induced by $\hat{\rho}$ coincides with the compact-open topology and thus is independent of the choice of a particular metric ρ on the space Y.

We shall now prove

4.2.15. THEOREM. *For every topological space X and any metric space (Y, ρ) such that $B(X, (Y, \rho)) = Y^X$ the topology on Y^X induced by $\hat{\rho}$ is admissible.*

PROOF. By virtue of Proposition 2.6.11 it suffices to show that the evaluation mapping of Y is continuous, i.e., $\Omega: Y^X \times X \to Y$. Consider a point $(f_0, x_0) \in Y^X \times X$ and let

$f_0(x_0) = y_0$. Since f_0 is continuous, for every $\epsilon > 0$ there exists a neighbourhood $V \subset X$ of x_0 such that $f_0(V) \subset B(y_0, \epsilon/2)$. One readily verifies that $\Omega(B(f_0, \epsilon/2) \times V) \subset B(y_0, \epsilon)$, and this shows that the evaluation mapping Ω is continuous.■

The last theorem, along with 3.4.1 and 2.6.12, yields the following corollary (which is also an easy consequence of 4.1.14).

4.2.16. COROLLARY. *For every topological space X and any metric space (Y, ρ) such that $B(X, (Y, \rho)) = Y^X$ the topology on Y^X induced by $\hat{\rho}$ is finer than the compact-open topology.*■

4.2.17. THEOREM. *For every compact space X, a metrizable space Y and any metric ρ on the space Y, the topology on Y^X induced by $\hat{\rho}$ coincides with the compact-open topology and is independent of the choice of the metric ρ.*

PROOF. By virtue of 4.2.16 it suffices to show that, for every $f \in Y^X$ and any $r > 0$, there exists a set $V \subset Y^X$, open with respect to the compact-open topology, such that

$$f \in V \subset B(f, r). \tag{8}$$

The family $\{U_x\}_{x \in X}$, where $U_x = f^{-1}(B(f(x), r/4))$, is an open cover of the compact space X; hence there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset X$ such that

$$X = U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_k}. \tag{9}$$

For $i = 1, 2, \ldots, k$ let

$$C_i = \overline{U}_{x_i} = \overline{f^{-1}(B(f(x_i), r/4))} \quad \text{and} \quad V_i = B(f(x_i), r/3). \tag{10}$$

Since the subsets $C_1, C_2, \ldots, C_k$ of the space X are compact and the subsets $V_1, V_2, \ldots, V_k$ of the space Y are open, the set

$$V = \bigcap_{i=1}^{k} M(C_i, V_i)$$

is open with respect to the compact-open topology on Y^X; we shall show that V satisfies (8).

It follows from (10) that $f(C_i) \subset V_i$ for $i = 1, 2, \ldots, k$, so that $f \in V$. Let us now consider an arbitrary $g \in V$. By (9) and (10) for any point $x \in X$ there exists an $i \le k$ such that $x \in C_i$; clearly $g(x) \in V_i$ and $f(x) \in V_i$. Since $\delta(V_i) \le 2r/3$, we have $\rho(f(x), g(x)) \le 2r/3$ and – the point x being arbitrary – $\hat{\rho}(f, g) < r$, which concludes the proof of (8).■

Theorems 4.2.17 and 3.4.16 yield

4.2.18. COROLLARY. *For every compact metrizable space X and any separable metric space (Y, ρ), the space $(Y^X, \hat{\rho})$ is separable.*■

The reader can easily check that in the last corollary compactness of X cannot be weakened to local compactness and separability.

We shall close this section with a theorem clarifying the nature of the topology of uniform convergence on R^X introduced in Section 2.6. To begin, we shall generalize the notion of a uniformly convergent sequence of functions.

Let X be a topological space, (Y, ρ) a metric space and $\{f_i\}$ a sequence of mappings of X to Y. We say that the sequence $\{f_i\}$ is *uniformly convergent* to a mapping f of X to Y

if for every $\epsilon > 0$ there exists a k such that we have $\rho(f(x), f_i(x)) < \epsilon$ for every $x \in X$ and $i \geq k$. The proof of the following theorem, similar to that of Theorem 1.4.7, is left to the reader.

4.2.19. THEOREM. *Let X be a topological space, (Y, ρ) a metric space and $\{f_i\}$ a sequence of continuous mappings of X to Y. If the sequence $\{f_i\}$ is uniformly convergent to a mapping f, then f is a continuous mapping of X to Y. If all the f_i's are bounded, then f also is bounded.* ■

One easily sees that a sequence $\{f_i\}$ of continuous mappings of X to (Y, ρ) is uniformly convergent to a mapping $f: X \to Y$ if and only if $\lim \hat{\rho}_1(f, f_i) = 0$, where $\rho_1(x, y) = \min(1, \rho(x, y))$, i.e., if and only if $f = \lim f_i$ in the metric space $(Y^X, \hat{\rho}_1)$. Hence, Proposition 4.1.1 and formula (1) in Section 2.6 yield

4.2.20. THEOREM. *For every topological space X the function space R^X with the topology of uniform convergence is metrizable.*

More exactly, *the topology of uniform convergence on R^X is induced by the metric $\hat{\rho}$, where ρ is the metric on the real line defined by letting $\rho(x, y) = \min(1, |x - y|)$.* ■

Historical and bibliographic notes

Information on the origins of operations on metric and metrizable spaces is included to notes in Chapter 2. Theorems 4.2.8 and 4.2.9 were proved by Urysohn in [1924] (announcement in [1923]) and in [1925a] respectively. In the original formulation of the latter, regularity was replaced by normality; it was shown by Tychonoff in [1925] that the theorem holds in the stronger form given here (cf. Theorem 1.5.16). A topological characterization of compact metrizable spaces was given also by Chittenden and Pitcher in [1919]. The Baire space $B(\mathfrak{m})$ was defined by Baire in [1909] for $\mathfrak{m} = \aleph_0$. Theorem 4.2.13 was proved by Whyburn in [1942]. The space of bounded mappings with the metric $\hat{\rho}$ defined by formula (7) was studied by Fréchet in [1906] (the idea is ascribed there to Weierstrass). Corollary 4.2.16 was noted first in Jackson's paper [1952]. Theorem 4.2.17 was proved by Arens in [1946].

Exercises

4.2.A. (a) Note that if two metrics ρ and σ on a set X are uniformly equivalent, then for every $M \subset X$ the metrics ρ_M and σ_M on the subspace M are also uniformly equivalent.

(b) Verify that if for $i = 1, 2, \ldots$ metrics ρ_i and σ_i on a set X_i are uniformly equivalent and bounded by 1, then the metrics ρ and σ on $\prod_{i=1}^{\infty} X_i$, defined as in formula (2), also are uniformly equivalent.

(c) Verify that if two metrics ρ and σ on a set Y are uniformly equivalent and bounded, then for every space X the metrics $\hat{\rho}$ and $\hat{\sigma}$ on Y^X, defined as in formula (7), also are uniformly equivalent and thus induce the same topology on Y^X (cf. Exercise 4.3.I(b)).

4.2.B (Šneĭder [1945]). Show that a compact space X is metrizable if and only if the diagonal Δ is a G_δ-set in the Cartesian product $X \times X$ (it suffices to assume that X is countably compact, see Problem 3.12.23(e); cf. Problem 4.5.15 and Exercise 5.1.I).

Hint. Define a countable family $\{\mathcal{V}_i\}_{i=1}^{\infty}$ of finite open covers of the space X such that for every pair x, y of distinct points of X there exists a natural number i with the property that the closure of no member of $\mathcal{V}_i$ contains both x and y. Check that the family of all finite intersections $V_1 \cap V_2 \cap \ldots \cap V_k$, where $V_i \in \mathcal{V}_i$ for $i = 1, 2, \ldots, k$ and $k = 1, 2, \ldots$, is a base for the space X.

One can also take a function $f: X \times X \to I$ such that $\Delta = f^{-1}(0)$ and let $\rho(x, y) = \sup\{|f(x, z) - f(y, z)| : z \in X\}$.

4.2.C (H. Vaughan [1937]). Prove that for a metrizable space X, separability and local compactness is a necessary and sufficient condition for existence on the space X of a metric with the property that a subspace $A \subset X$ is compact if and only if the set A is closed and bounded.

4.2.D. (a) (Ponomarev [1960]) Prove that a T_0-space X is first countable if and only if X is a continuous image of a metrizable space under an open mapping.

Hint. Let X be a first-countable space and let $\{U_s\}_{s \in S}$ be a base for X. Consider the Baire space $B(\mathfrak{m}) = \prod_{i=1}^{\infty} X_i$, where $X_i = S$ with the discrete topology, and the subset $T \subset B(\mathfrak{m})$ consisting of all points $\{s_i\}$ such that $\{U_{s_i}\}_{i=1}^{\infty}$ is a base at a point $x \in X$; assign the point $x \in X$ to the point $\{s_i\} \in T$.

(b) (Michael [1971a]) Show that every first-countable space X is a continuous image of a first-countable Hausdorff space under an open mapping, and deduce that in (a) the assumption that X is a T_0-space can be omitted.

Hint (Shimrat [1956]). Define in $B(\mathfrak{m})$, where $\mathfrak{m} = |X|$, a family $\{A_x\}_{x \in X}$ of pairwise disjoint dense subsets and consider the subset $\bigcup_{x \in X}(\{x\} \times A_x)$ of the Cartesian product $X \times B(\mathfrak{m})$.

Remark. The construction in the above hint shows that every topological space is a continuous image of a Hausdorff space under an open mapping. Isbell proved more in [1969]: every topological space is a continuous image of a hereditarily paracompact and hereditarily strongly zero-dimensional space under an open mapping. Further information related to parts (a) and (b) can be found in Junnila [1978] and R. Pol [1981].

(c) (Arhangel'skiĭ[1963], Franklin [1965]) Observe that the construction outlined in Exercise 2.4.G(b) shows that sequential spaces can be characterized as the images of metrizable spaces under quotient mappings and Fréchet spaces can be characterized as the images of metrizable spaces under hereditarily quotient mappings.

4.2.E (Kuratowski [1947]; for real-valued functions, Hahn [1921]). Prove that if X is a compact metrizable space and (Y, ρ) a metric space, then a sequence $\{f_i\}$ of continuous mappings of X to Y is uniformly convergent to a mapping $f \in Y^X$ if and only if for every sequence $x_1, x_2, \ldots$ of points of X which converges to a point $x \in X$, the sequence $f_i(x_i)$ converges to $f(x)$ (cf. Exercise 2.6.C).

4.2.F (Kuratowski [1931]). Show that if X is a compact metrizable space and Y a Hausdorff space, then all topological embeddings of X in Y form a G_δ-set in the space Y^X with the compact-open topology.

4.2.G (Jackson [1952]). Let X be a Tychonoff space, Y a metrizable space that contains a subspace homemorphic to R, and ρ a bounded metric on the space Y. Show that if the metric $\hat{\rho}$ induces the compact-open topology on Y^X, then X is a compact space.

Hint. Modify the argument in the hint to Exercise 3.4.A.

4.2.H (Arens [1946]). Prove that if X is a hemicompact space, then for every metrizable space Y the space Y^X with the compact-open topology is metrizable (cf. Exercises 3.4.E, 3.5.G(b), 3.8.C(b) and 4.3.F).

4.2.I. Verify that if ρ is a pseudometric on a set X, then by letting $xE(\rho)y$ whenever $\rho(x,y) = 0$ we define an equivalence relation $E(\rho)$ on the set X; check that the formula $\overline{\rho}([x],[y]) = \rho(x,y)$ defines a metric $\overline{\rho}$ on the set of all equivalence classes of $E(\rho)$; the metric space thus obtained is denoted by X/ρ.

Show that if X is a topological space and ρ a pseudometric on the set X such that $\rho: X \times X \to R$ is a continuous function, then assigning the class $[x] \in X/\rho$ to the point $x \in X$ defines a continuous mapping $f: X \to X/\rho$. Observe that generally the topology induced on X/ρ by the metric $\overline{\rho}$ is coarser than the topology of the quotient space $X/E(\rho)$ and that the two topologies coincide if the topology of X is induced by ρ.

Deduce that if X is a topological space and ρ a pseudometric on the set X such that $\rho: X \times X \to R$ is a continuous function, then the topology on X induced by ρ is coarser than the original topology.

4.3. Totally bounded and complete metric spaces. Compactness in metric spaces

Let (X,ρ) be a metric space and A a subset of X; we say that A is *ϵ-dense* in (X,ρ) if for every $x \in X$ there exists an $x' \in A$ such that $\rho(x,x') < \epsilon$.

A metric space (X,ρ) is *totally bounded* if for every $\epsilon > 0$ there exists a finite set $A \subset X$ which is ϵ-dense in (X,ρ); a metric ρ on a set X is *totally bounded* if the space (X,ρ) is totally bounded. A topological space X is *metrizable by a totally bounded metric* if there exists a totally bounded metric on the space X.

The first definition in the last paragraph defines a class of metric spaces, the third one – a class of topological spaces which contains together with a space X all spaces homeomorphic to X. However, the definition of the latter class is not internal, because the notion of a metric is used in it.

We shall now briefly discuss the class of totally bounded metric spaces, and we shall show that the class of all spaces metrizable by a totally bounded metric coincides with the class of all separable metrizable spaces; as we shall see, the last fact yields an internal characterization of the class of all spaces metrizable by a totally bounded metric.

To begin, we discuss a few examples.

4.3.1. EXAMPLES. The discrete space (X,ρ) defined in 4.1.4 is not totally bounded unless the set X is finite; in fact, if X is infinite, no finite set is ϵ-dense in (X,ρ) for $\epsilon = 1$.

The reader can easily verify that the real line R, the hedgehog $J(\mathfrak{m})$, the plane with the "river" metric, Hilbert space H and the Baire space $B(\mathfrak{m})$ are not totally bounded.

On the other hand, every closed interval $J = [a,b] \subset R$ is totally bounded. Indeed, for every $\epsilon > 0$ the set $J \cap \{i/k : i = 0, \pm 1, \pm 2, \ldots\}$, where k is a natural number satisfying $1/k < \epsilon$, is finite and ϵ-dense in J. More generally, every compact metric space is totally bounded (see Theorem 4.3.27). Also every open interval is totally bounded; thus – since the real line is homeomorphic to the interval $(-1,1)$ – we see that a space homeomorphic to a totally bounded space need not be totally bounded, it is, however, metrizable by a totally bounded metric. The reader can readily verify that a space isometric to a totally bounded space is totally bounded. ■

4.3.2. THEOREM. *If (X, ρ) is a totally bounded space, then for every subset M of X the space (M, ρ) is totally bounded.*

If (X, ρ) is an arbitrary metric space and for a subset M of X the space (M, ρ) is totally bounded, then the space $(\overline{M}, \rho)$ also is totally bounded.

PROOF. Take an $\epsilon > 0$ and a finite set $A = \{x_1, x_2, \ldots, x_k\}$ which is $\epsilon/2$-dense in (X, ρ). Let $\{x_{m_1}, x_{m_2}, \ldots, x_{m_l}\}$ be the subset of A consisting of all points whose distance from M is less than $\epsilon/2$ and let $x'_1, x'_2, \ldots, x'_l$ be arbitrary points of M satisfying

(1) $$\rho(x'_j, x_{m_j}) < \epsilon/2 \quad \text{for} \quad j = 1, 2, \ldots, l.$$

We shall show that the set $A' = \{x'_1, x'_2, \ldots, x'_l\}$ is ϵ-dense in M. Let x be a point of M; by the definition of A there exists an $i \le k$ such that

(2) $$\rho(x, x_i) < \epsilon/2.$$

Hence $x_i = x_{m_j}$ for some $j \le l$ and, by (1) and (2), we have $\rho(x, x'_j) < \epsilon$.

The second part of the theorem follows from the easily observed fact that any set which is $\epsilon/2$-dense in (M, ρ) is ϵ-dense in $(\overline{M}, \rho)$. ■

The reader can easily check that if $\{(X_s, \rho_s)\}_{s \in S}$ is a family of non-empty metric spaces such that the metric ρ_s is bounded by 1 for every $s \in S$, then the sum $\bigoplus_{s \in S} X_s$ with the metric ρ defined as in the proof of Theorem 4.2.1 is totally bounded if and only if all spaces (X_s, ρ_s) are totally bounded and $|S| < \aleph_0$.

4.3.3. THEOREM. *Let $\{(X_i, \rho_i)\}_{i=1}^{\infty}$ be a family of non-empty metric spaces such that the metric ρ_i is bounded by 1 for $i = 1, 2, \ldots$ The Cartesian product $\prod_{i=1}^{\infty} X_i$ with the metric ρ defined by formula (2) in Section 4.2 is totally bounded if and only if all spaces (X_i, ρ_i) are totally bounded.*

PROOF. Assume that the space $(\prod_{i=1}^{\infty} X_i, \rho)$ is totally bounded. The subspace $X_j^* = \prod_{i=1}^{\infty} A_i$ of $\prod_{i=1}^{\infty} X_i$, where $A_j = X_j$ and A_i is an arbitrarily chosen one-point subset of X_i for $i \ne j$, is totally bounded by virtue of Theorem 4.3.2. One can easily verify that if a set A is $\epsilon/2^j$-dense in (X_j^*, ρ), then the set $p_j(A)$ is ϵ-dense in (X_j, ρ_j), so that the space (X_j, ρ_j) is totally bounded.

Assume now that all spaces (X_i, ρ_i) are totally bounded. Take an $\epsilon > 0$ and a natural number k such that $1/2^k < \epsilon/2$. For every $i \le k$ choose a finite set $\{x_1^i, x_2^i, \ldots, x_{m(i)}^i\}$ which is $\epsilon/2$-dense in X_i and for every $i > k$ choose an arbitrary point $x_0^i \in X_i$.

The set $A \subset \prod_{i=1}^{\infty} X_i$ consisting of all points of the form

(3) $$y = \{x_{j_1}^1, x_{j_2}^2, \ldots, x_{j_k}^k, x_0^{k+1}, x_0^{k+2}, \ldots\}, \quad \text{where} \quad 1 \le j_i \le m(i) \ \text{ for } \ i \le k,$$

is finite; to conclude the proof it suffices to show that A is ϵ-dense in the space $(\prod_{i=1}^{\infty} X_i, \rho)$.

Let $x = \{x_i\}$ be an arbitrary point of $\prod_{i=1}^{\infty} X_i$. For every $i \leq k$ there exists a $j_i \leq m(i)$ such that $\rho_i(x_i, x_{j_i}^i) < \epsilon/2$ and for the point y defined in (3) we have

$$\rho(x,y) = \sum_{i=1}^{k} \frac{1}{2^i}\rho_i(x_i, x_{j_i}^i) + \sum_{i=k+1}^{\infty} \frac{1}{2^i}\rho_i(x_i, x_0^i) < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon,$$

so that the set A is ϵ-dense in $(\prod_{i=1}^{\infty} X_i, \rho)$. ∎

4.3.4. COROLLARY. *The Hilbert cube $I^{\aleph_0}$ with the metric ρ defined by letting*

$$\rho(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i - y_i|, \quad \textit{where } x = \{x_i\} \textit{ and } y = \{y_i\},$$

is a totally bounded space. ∎

4.3.5. THEOREM. *A metrizable space is metrizable by a totally bounded metric if and only if it is a separable space.*

PROOF. Sufficiency of separability follows from 4.2.10, 4.3.4 and 4.3.2. To prove that separability also is a necessary condition, it is enough to observe that if ρ is a totally bounded metric on a space X and $A_i \subset X$ is a finite set which is $1/i$-dense in (X, ρ), then the union $A = \bigcup_{i=1}^{\infty} A_i$ is a countable dense subset of X. ∎

The last theorem, along with 4.1.16 and 4.2.9, yields

4.3.6. COROLLARY. *A topological space is metrizable by a totally bounded metric if and only if it is a regular second-countable space.* ∎

Now, we pass to the class of complete metric spaces.

Let (X, ρ) be a metric space and $\{x_i\}$ a sequence of points of X; we say that $\{x_i\}$ is a *Cauchy sequence* in (X, ρ) if for every $\epsilon > 0$ there exists a natural number k such that $\rho(x_i, x_k) \leq \epsilon$ whenever $i \geq k$. As can be easily seen, every convergent sequence in a metric space is a Cauchy sequence.

A metric space (X, ρ) is *complete* if every Cauchy sequence in (X, ρ) is convergent to a point of X; a metric ρ on a set X is *complete* if the space (X, ρ) is complete. A topological space X is *completely metrizable* if there exists a complete metric on the space X.

The observations made at the beginning of this section in connection with totally bounded spaces and spaces metrizable by a totally bounded metric apply also to the classes of complete and completely metrizable spaces.

We shall first discuss the class of complete metric spaces, then the class of completely metrizable spaces, and we shall conclude the subject with an internal characterization of completely metrizable spaces.

To begin, we discuss a few examples.

4.3.7. EXAMPLES. The discrete space (X, ρ) defined in 4.1.4 is complete; in fact, every Cauchy sequence in (X, ρ) is eventually constant.

We shall show that the real line R with its natural metric is a complete space. Let $\{x_i\}$ be a Cauchy sequence in R. There exists a natural number k such that $|x_i - x_k| \leq 1$ whenever

$i \geq k$, so that all terms of the sequence $\{x_i\}$, beginning with the kth, are contained in the closed interval $J = [x_k - 1, x_k + 1]$. It follows from compactness of J and from Theorem 4.1.17 that there exists a subsequence $\{x_{k_i}\}$ of the sequence $\{x_i\}$ convergent to a point $x \in J$; a straightforward evaluation shows that the sequence $\{x_i\}$ also is convergent to the point x.

All closed intervals are also complete. More generally, every compact metric space is complete (see Theorem 4.3.28). On the other hand, the open interval $(-1,1)$ – as well as any open interval – is not complete; indeed, the sequence $\{1 - 1/n\}$ is a Cauchy sequence in $(-1,1)$ and it does not converge to any point of $(-1,1)$. Since the interval $(-1,1)$ is homeomorphic to the real line, we see that a space homeomorphic to a complete space need not be complete, it is, however, a completely metrizable space. The reader can easily verify that a space isometric to a complete space is complete.

Let us also observe that – as shown by the subspace $(-1,1)$ of R – an open subset of a complete space is not necessarily complete. ■

4.3.8. PROPOSITION. *For every complete metric space (X,ρ) there exists a complete metric ρ_1 on the set X which is equivalent to ρ and bounded by* 1.

PROOF. The metric ρ_1 defined by letting

$$\rho_1(x,y) = \min(1, \rho(x,y)) \quad \text{for } x, y \in X$$

(cf. Theorem 4.1.3) is complete because a sequence $\{x_i\}$ of points of X is a Cauchy sequence in (X,ρ) if and only if it is a Cauchy sequence in (X,ρ_1). ■

The next two theorems contain two characterizations of completeness; the first is analogous to the characterization of countable compactness given in Theorem 3.10.2, condition (iii), and the second to the characterization of compactness given in Theorem 3.1.1.

4.3.9. THE CANTOR THEOREM. *A metric space (X,ρ) is complete if and only if for every decreasing sequence $F_1 \supset F_2 \supset \ldots$ of non-empty closed subsets of X, such that* $\lim \delta(F_i) = 0$, *the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty.*

PROOF. Let (X,ρ) be a complete space and $F_1, F_2, \ldots$ a sequence of non-empty closed subsets of X such that

$$\lim \delta(F_i) = 0 \quad \text{and} \quad F_{i+1} \subset F_i \quad \text{for } i = 1, 2, \ldots \tag{4}$$

Let us choose a point $x_i \in F_i$ for $i = 1, 2, \ldots$ One readily sees that all terms of the sequence $\{x_i\}$, beginning with the ith, are contained in the set F_i, so that – by the first part of (4) – $\{x_i\}$ is a Cauchy sequence and thus is convergent to a point $x \in X$. The sets F_i being closed, we have $x \in F_i$ for $i = 1, 2, \ldots$; hence $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$.

Now, let (X,ρ) be a metric space satisfying the condition in our theorem and let $\{x_i\}$ be a Cauchy sequence in (X,ρ). The sets $F_i = \overline{\{x_i, x_{i+1}, \ldots\}}$, where $i = 1, 2, \ldots$, are closed and satisfy (4), so that there exists a point $x \in \bigcap_{i=1}^{\infty} F_i$; one readily shows that $x = \lim x_i$. Hence the space (X,ρ) is complete. ■

4.3.10. THEOREM. *A metric space (X,ρ) is complete if and only if every family of closed subsets of X which has the finite intersection property and for every $\epsilon > 0$ contains a set of diameter less than ϵ has non-empty intersection.*

PROOF. Sufficiency of the condition in our theorem for completeness of (X, ρ) follows from the Cantor theorem.

We shall show that the condition holds in every complete space (X, ρ). Consider a family $\{F_s\}_{s\in S}$ of closed subsets of X which has the finite intersection property and which for every integer j contains a set F_{s_j} such that $\delta(F_{s_j}) < 1/j$. One easily sees that the sequence $F_1, F_2, \ldots$, where $F_i = \bigcap_{j\le i} F_{s_j}$, satisfies the condition of the Cantor theorem, so that there exists an $x \in \bigcap_{i=1}^{\infty} F_i$. Clearly, we have $\bigcap_{i=1}^{\infty} F_i = \{x\}$. Now, let us take an arbitrary $s_0 \in S$; letting $F_i' = F_{s_0} \cap F_i$ for $i = 1, 2, \ldots$ we obtain again a sequence $F_1', F_2', \ldots$ satisfying the conditions of the Cantor theorem. Since

$$\emptyset \neq \bigcap_{i=1}^{\infty} F_i' = F_{s_0} \cap \bigcap_{i=1}^{\infty} F_i = F_{s_0} \cap \{x\},$$

we have $x \in F_{s_0}$. Hence $x \in \bigcap_{s\in S} F_s$.■

We proceed to a discussion of operations on complete spaces.

4.3.11. THEOREM. *If (X, ρ) is a complete space, then for every closed subset M of X the space (M, ρ) is complete.*

If (X, ρ) is an arbitrary metric space and for a subset M of X the space (M, ρ) is complete, then M is closed in X.

PROOF. The fact that if (X, ρ) is complete and M is closed in X, then the space (M, ρ) is complete follows readily from the definition of completeness.

Assume that the space (M, ρ) is complete and consider a point $x \in \overline{M}$. Let $\{x_i\}$ be a sequence of points of M that converges to x. Since $\{x_i\}$ is a Cauchy sequence in (M, ρ), it has a limit in M, and this limit is the point x; thus $x \in M$.■

The reader can easily check that if $\{(X_s, \rho_s)\}_{s\in S}$ is a family of metric spaces such that the metric ρ_s is bounded by 1 for every $s \in S$, then the sum $\bigoplus_{s\in S} X_s$ with the metric ρ defined as in the proof of Theorem 4.2.1 is complete if and only if all spaces (X_s, ρ_s) are complete. Hence, the sum $\bigoplus_{s\in S} X_s$ of completely metrizable spaces X_s is completely metrizable.

4.3.12. THEOREM. *Let $\{(X_i, \rho_i)\}_{i=1}^{\infty}$ be a family of non-empty metric spaces such that the metric ρ_i is bounded by 1 for $i = 1, 2, \ldots$ The Cartesian product $\prod_{i=1}^{\infty} X_i$ with the metric ρ defined by formula (2) in Section 4.2 is complete if and only if all spaces (X_i, ρ_i) are complete.*

PROOF. Assume that the space $(\prod_{i=1}^{\infty} X_i, \rho)$ is complete. The subspace $X_j^* = \prod_{i=1}^{\infty} A_i$ of $\prod_{i=1}^{\infty} X_i$ where $A_j = X_j$ and A_i is an arbitrarily chosen one-point subset of X_i for $i \neq j$, is closed and, by virtue of Theorem 4.3.11, complete. One can easily verify that $p_j^* = p_j|X_j^*\colon X_j^* \to X_j$ is a homeomorphism and that for every Cauchy sequence $\{x_i\}$ in (X_j, ρ_j), the sequence $\{p_j^{*-1}(x_i)\}$ is a Cauchy sequence in X_j^*. The image of the limit of $\{p_j^{*-1}(x_i)\}$ under p_j^* is the limit of the sequence $\{x_i\}$, so that the space (X_j, ρ_j) is complete.

Assume now that all spaces (X_i, ρ_i) are complete. If $\{x_i^1\}, \{x_i^2\}, \ldots$ is a Cauchy sequence in $(\prod_{i=1}^{\infty} X_i, \rho)$, then for $i = 1, 2, \ldots$ the sequence $x_i^1, x_i^2, \ldots$ is a Cauchy sequence in (X_i, ρ_i) and thus converges to a point $x_i \in X_i$. From Corollary 4.2.7 it follows that the sequence $\{x_i^1\}, \{x_i^2\}, \ldots$ converges to the point $x = \{x_i\} \in \prod_{i=1}^{\infty} X_i$, so that $(\prod_{i=1}^{\infty} X_i, \rho)$ is a complete space.■

4.3.13. THEOREM. *For every topological space X and any complete metric space (Y, ρ), the space of all bounded continuous mappings of X to Y with the metric $\hat{\rho}$, defined by formula* (7) *in Section* 4.2, *is complete.*

PROOF. Let $\{f_i\}$ be a Cauchy sequence in the space under consideration. Clearly $\{f_i(x)\}$ is a Cauchy sequence in (Y, ρ) for every $x \in X$; assigning the limit of $\{f_i(x)\}$ to $x \in X$ defines a mapping f of X to Y. One easily sees that the sequence $\{f_i\}$ is uniformly convergent to f, so that – by Theorem 4.2.19 – the mapping f is bounded and continuous. Hence, we have $\lim \hat{\rho}(f, f_i) = 0$, i.e., the sequence $\{f_i\}$ converges to f. ■

4.3.14. THEOREM. *Every metric space is isometric to a subspace of a complete metric space.*

PROOF. Let (X, ρ) be an arbitrary metric space. By virtue of the last theorem, the space (Y, σ) of all bounded continuous real-valued functions on X, where $\sigma(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$, is complete. Let us fix a point $a \in X$ and assign to every point $x \in X$, the function $f_x \in R^X$ defined by letting

$$f_x(z) = \rho(z, x) - \rho(z, a) \quad \text{for } z \in X.$$

Since $|f_x(z)| \leq \rho(a, x)$ by the triangle inequality, $f_x \in Y$ for every $x \in X$. We shall show that

$$(5) \qquad \sigma(f_x, f_y) = \rho(x, y) \quad \text{for all } x, y \in X.$$

To begin, let us note that for any $z \in X$ we have

$$f_x(z) - f_y(z) = \rho(z, x) - \rho(z, a) - \rho(z, y) + \rho(z, a) \leq \rho(y, x);$$

the symmetry of assumptions implies that $|f_x(z) - f_y(z)| \leq \rho(x, y)$, i.e., we have

$$\sigma(f_x, f_y) \leq \rho(x, y).$$

Since $f_x(y) - f_y(y) = \rho(y, x) - \rho(y, a) + \rho(y, a) = \rho(y, x)$, we also have

$$\sigma(f_x, f_y) \geq \rho(x, y),$$

which concludes the proof of (5) and of the theorem. ■

4.3.15. COROLLARY. *Every metrizable space is embeddable in a completely metrizable space.* ■

Let us observe, in connection with the last proof, that if the space (X, ρ) is bounded, then an isometry of (X, ρ) into (Y, σ) can be obtained in a simpler way; it suffices to assign to every point $x \in X$ the function $f_x \in Y$ defined by letting $f_x(z) = \rho(z, x)$.

Clearly, for a given metric space (X, ρ) there exist many complete metric spaces (Y, σ) containing a subspace isometric to (X, ρ). It turns out, however, that if we add the condition that the subspace isometric to (X, ρ) should be dense in (Y, σ), then the space (Y, σ) is uniquely determined (up to an isometry). In the proof of this fact we shall need a theorem on extendability of mappings into complete metric spaces.

Let X be a topological space, (Y, σ) a metric space and $f: A \to Y$ a continuous mapping defined on a dense subset A of the space X; we say that the *oscillation of the mapping f at a point $x \in X$ is equal to zero* if for every $\epsilon > 0$ there exists a neighbourhood U of the point x

such that $\delta(f(A \cap U)) < \epsilon$. Since for every integer i the set of all points $x \in X$ which have a neighbourhood U such that $\delta(f(A \cap U)) < 1/i$ is open and contains A, the set of all points at which the oscillation of f is equal to zero is a G_δ-set containing A.

4.3.16. LEMMA. *If X is a topological space, (Y, σ) a complete metric space and $f: A \to Y$ a continuous mapping defined on a dense subset A of the space X, then the mapping f is extendable to a continuous mapping $F: B \to Y$ defined on the set B consisting of all points of X at which the oscillation of f is equal to zero.*

PROOF. For every $x \in B$ denote by $\mathcal{B}(x)$ the family of all neighbourhoods of x and consider the family $\{\overline{f(A \cap U)}\}_{U \in \mathcal{B}(x)}$ of closed subsets of Y. The latter family has the finite intersection property and for every $\epsilon > 0$ contains a set of diameter less than ϵ, so that – by Theorem 4.3.10 – the intersection $\bigcap_{U \in \mathcal{B}(x)} \overline{f(A \cap U)}$ is non-empty. Since the diameter of that intersection equals zero, the intersection consists of a single point $F(x)$. Obviously, $F(x) = f(x)$ whenever $x \in A$. Assigning to $x \in B$ the point $F(x)$ we define a mapping F of B to Y which is an extension of f; it remains to prove that F is continuous. Let us take an $x \in B$ and an $\epsilon > 0$. From the definition of B it follows that there exists a $U \in \mathcal{B}(x)$ satisfying $\delta(\overline{f(A \cap U)}) < \epsilon$. For every $x' \in B \cap U$ we have $U \in \mathcal{B}(x')$ and thus $F(x') \in \overline{f(A \cap U)}$; since $F(x) \in \overline{f(A \cap U)}$ as well, we have $\sigma(F(x), F(x')) \le \delta(\overline{f(A \cap U)}) < \epsilon$, which proves that F is continuous. ■

4.3.17. THEOREM. *If (X, ρ) is a metric space and (Y, σ) a complete metric space, then every mapping $f: A \to Y$ from a dense subset A of the space X to the space Y which is uniformly continuous with respect to ρ_A and σ is extendable to a mapping $F: X \to Y$ uniformly continuous with respect to ρ and σ.*

PROOF. It follows from uniform continuity of f that the oscillation of f at every point of X is equal to zero. Hence, by the lemma, there exists a continuous extension $F: X \to Y$ of the mapping f; it remains to prove that F is uniformly continuous.

Take an arbitrary $\epsilon > 0$ and a $\delta > 0$ such that for all $x, x' \in A$ we have $\sigma(f(x), f(x')) < \epsilon/2$ whenever $\sigma(x, x') < \delta$. For every pair x_0, x_0' of points of X satisfying $\rho(x_0, x_0') < \delta$, the set $U = B(x_0, r) \cup B(x_0', r)$, where $r = (\delta - \rho(x_0, x_0'))/3$, is open and has diameter less than δ. Hence, $\delta(f(A \cap U)) \le \epsilon/2$ and $\delta(\overline{f(A \cap U)}) \le \epsilon/2 < \epsilon$; since $F(x_0)$ and $F(x_0')$ belong to $f(\overline{A \cap U}) \subset \overline{f(A \cap U)}$, we have $\sigma(F(x_0), F(x_0')) < \epsilon$. ■

As isometries are uniformly continuous, the last theorem – along with 1.5.4 and 4.2.6 – yields

4.3.18. COROLLARY. *If (X, ρ) and (Y, σ) are complete metric spaces, then every isometry of (A, ρ_A) onto (B, σ_B), where A and B are dense subsets of X and Y respectively, is extendable to an isometry of (X, ρ) onto (Y, σ).* ■

4.3.19. THEOREM. *For every metric space (X, ρ) there exists exactly one (up to an isometry) complete metric space $(\tilde{X}, \tilde{\rho})$ such that $\tilde{X}$ contains a dense subspace isometric to (X, ρ). Moreover, we have $w(\tilde{X}) = w(X)$, and if (X, ρ) is a totally bounded space, then $(\tilde{X}, \tilde{\rho})$ also is totally bounded.*

PROOF. Let (X,ρ) be a metric space; by virtue of Theorem 4.3.14 there exists a complete metric space (Y,σ) and an isometry $f: X \to Y$. Letting $\tilde{X} = \overline{f(X)} \subset Y$ and $\tilde{\rho} = \sigma_{\tilde{X}}$ we obtain a metric space $(\tilde{X},\tilde{\rho})$ with the required properties. Uniqueness of $(\tilde{X},\tilde{\rho})$ follows from Corollary 4.3.18.

The equality $w(\tilde{X}) = w(X)$ is a consequence of Theorem 4.1.15 and, by the second part of Theorem 4.3.2, if (X,ρ) is a totally bounded space, then $(\tilde{X},\tilde{\rho})$ also is totally bounded.∎

The space $(\tilde{X},\tilde{\rho})$ satisfying the conditions in Theorem 4.3.19 is called the *completion of the metric space* (X,ρ).

Now, we are going to consider the class of completely metrizable spaces. Let us begin with a simple theorem on extending mappings.

4.3.20. THEOREM. *If Y is a completely metrizable space, then every continuous mapping $f: A \to Y$ from a dense subset A of a topological space X to the space Y is extendable to a continuous mapping $F: B \to Y$ defined on a G_δ-set $B \subset X$ containing the set A.*

PROOF. It suffices to consider any complete metric on the space Y, take as B the set of all points at which the oscillation of f is equal to zero and apply Lemma 4.3.16.∎

Our next theorem is an important result on extending homeomorphisms.

4.3.21. THE LAVRENTIEFF THEOREM. *Let X and Y be completely metrizable spaces and let $A \subset X$ and $C \subset Y$ be arbitrary subspaces. Every homeomorphism $f: A \to C$ is extendable to a homeomorphism $F: B \to D$, where $A \subset B \subset X$, $C \subset D \subset Y$ and B and D are G_δ-sets in X and Y respectively.*

PROOF. Without loss of generality one can assume that $\overline{A} = X$ and $\overline{C} = Y$. Let $g: C \to A$ be the inverse mapping of f. It follows from 4.3.20 that there exist extensions

$$F_0: B_0 \to Y \quad \text{and} \quad G_0: D_0 \to X$$

of f and g over G_δ-sets $B_0 \subset X$ and $D_0 \subset Y$. The intersections $B = B_0 \cap F_0^{-1}(D_0)$ and $D = D_0 \cap G_0^{-1}(B_0)$ are G_δ-sets in X and Y respectively; clearly $A \subset B$ and $C \subset D$. We shall show that

$$G_0F_0(x) = x \quad \text{for every } x \in B \quad \text{and} \quad F_0(B) \subset D. \tag{6}$$

By Theorem 1.5.4, the fact that the restriction of $G_0F_0|B$ to A coincides with the embedding $i_A: A \to X$ implies the first part of (6) which, in turn, implies that $F_0(B) \subset G_0^{-1}(B) \subset G_0^{-1}(B_0)$. Since $F_0(B) \subset D_0$, we have the second part of (6). By the symmetry of our assumptions,

$$F_0G_0(y) = y \quad \text{for every } y \in D \quad \text{and} \quad G_0(D) \subset B. \tag{7}$$

Now, (6) and the second part of (7) yield $G_0(D) = B$; similarly, $F_0(B) = D$. It follows that $F = F_0|B: B \to D$ and $G = G_0|D: D \to B$ are inverse to each other, and this implies that F is a homeomorphism.∎

In the following two theorems we discuss the subspace operation.

4.3.22. LEMMA. *Every G_δ-set in a metrizable space X is homeomorphic to a closed subspace of the Cartesian product $X \times R^{\aleph_0}$.*

PROOF. Let A be a G_δ-set in X and ρ an arbitrary metric on the space X. Represent the complement $X \setminus A$ as a union $\bigcup_{i=1}^\infty F_i$ of closed sets F_i and consider the mapping $f = \triangle_{i=1}^\infty f_i : X \to \prod_{i=1}^\infty X_i = R^{\aleph_0}$, where $f_i : X \to X_i = R$ is defined by letting $f_i(x) = \rho(x, F_i)$ for $i = 1, 2, \ldots$ By Corollary 2.3.22 the graph $G(f) = \{(x, f_1(x), f_2(x), \ldots) : x \in X\} \subset X \times R^{\aleph_0}$ of the mapping f, i.e., the image of X under the homeomorphic embedding $g = \mathrm{id}_X \triangle f : X \to X \times R^{\aleph_0}$, is a closed subspace of $X \times R^{\aleph_0}$. One readily checks that $g(A) = g(X) \cap (X \times R_0^{\aleph_0})$, where $R_0 = \{x \in R : x > 0\}$, and thus $g(A)$ is closed in $X \times R_0^{\aleph_0}$. To conclude the proof it suffices to note that $X \times R_0^{\aleph_0}$ and $X \times R^{\aleph_0}$ are homeomorphic.■

The lemma, together with Proposition 4.3.8 and Theorems 4.3.11 and 4.3.12, yields (cf. Exercise 2.3.L(a))

4.3.23. THEOREM. *Complete metrizability is hereditary with respect to G_δ-sets.*■

4.3.24. THEOREM. *If a subspace M of a metrizable space X is completely metrizable, then M is a G_δ-set in X.*

PROOF. By virtue of Theorem 4.3.20, the mapping $\mathrm{id}_M : M \to M$ from the dense subset M of the space $\overline{M} \subset X$ is extendable to a continuous mapping $F: B \to M$, defined on a G_δ-set $B \subset \overline{M}$ containing the set M. The assumption that there exists a point $x \in B \setminus M$ contradicts the continuity of F, so that $B = M$ and M is a G_δ-set in $\overline{M}$ and thus a G_δ-set in X.■

Since every separable metrizable space is embeddable in $R^{\aleph_0}$, from 4.3.22-4.3.24 we obtain

4.3.25. COROLLARY. *A separable metrizable space is completely metrizable if and only if it is embeddable in $R^{\aleph_0}$ as a closed subspace.*■

It follows from Theorems 4.3.14, 4.3.23 and 4.3.24 that, in the realm of metrizable spaces, completely metrizable spaces can be characterized as *absolute G_δ's*, i.e., spaces which are G_δ-sets in any metrizable space in which they are embedded.

4.3.26. THEOREM. *A topological space is completely metrizable if and only if it is a Čech-complete metrizable space.*

PROOF. Let X be a completely metrizable space and ρ a complete metric on the space X. From Theorem 4.3.10 it follows that any family of closed subsets of X which has the finite intersection property and which contains sets of diameter less than the covering $\mathcal{A}_i = \{B(x, 1/i)\}_{x \in X}$ for $i = 1, 2, \ldots$ has non-empty intersection. Hence, the space X is Čech-complete by Theorem 3.9.2.

Let us now consider a metrizable space X which is Čech-complete. Take the completion $\tilde{X}$ of the space X and a compactification $c\tilde{X}$ of the space $\tilde{X}$; clearly, $c\tilde{X}$ is a compactification of X. From Theorem 3.9.1 it follows that X is a G_δ-set in $c\tilde{X}$, and thus a G_δ-set in $\tilde{X}$; Theorem 4.3.23 implies that X is completely metrizable.■

Observe that, as the classes of metrizable spaces and Čech-complete spaces can be characterized internally (cf. Sections 4.4 and 5.4 and Theorem 3.9.2), the last theorem yields an internal characterizations of completely metrizable spaces.

Theorems 4.3.26 and 3.9.10 imply that complete metrizability is an inverse invariant of perfect mappings defined on metrizable spaces. It turns out that complete metrizability is also an invariant of closed mappings and open mappings onto metrizable spaces (see Problems 4.5.13(e) an 5.5.8(d)).

We shall discuss now properties of metrics on compact spaces.

4.3.27. THEOREM. *Every metric on a compact space is totally bounded.*

PROOF. Let ρ be a metric on a compact space X. Take an $\epsilon > 0$. The open cover $\{B(x,\epsilon)\}_{x\in X}$ of X has a finite subcover, i.e., there exists a finite set $A = \{x_1, x_2, \ldots, x_k\} \subset X$ such that

$$X = B(x_1,\epsilon) \cup B(x_2,\epsilon) \cup \ldots \cup B(x_k,\epsilon).$$

One readily sees that the set A is ϵ-dense in X. ∎

The Cantor theorem yields

4.3.28. THEOREM. *Every metric on a compact space is complete.* ∎

Our next theorem gives a characterization of compactness in terms of metrics, valid within the realm of metrizable spaces.

4.3.29. THEOREM. *A metrizable space X is compact if and only if on the space X there exists a metric ρ which is both totally bounded and complete.*

PROOF. Necessity of conditions in the theorem follows from Theorems 4.3.27 and 4.3.28.

Now, consider a metric space (X,ρ) which is both totally bounded and complete. For every natural number i take a finite set $\{x_1^i, x_2^i, \ldots, x_{k(i)}^i\}$ which is $1/2i$-dense in (X,ρ). Letting $B_j^i = B(x_j^i, 1/2i)$ for $j = 1,2,\ldots,k(i)$, we have

$$X = \bigcup_{j=1}^{k(i)} B_j^i \quad \text{and} \quad \delta(B_j^i) \leq 1/i \quad \text{for} \quad j \leq k(i). \tag{8}$$

We shall show that every infinite subset A of the space X has an accumulation point; by Theorems 3.10.3 and 4.1.17 this will conclude the proof.

Since the set A is infinite, by the first part of (8) there exists a $j \leq k(1)$ such that the intersection $A \cap B_j^1$ is infinite; let us denote this intersection by A_1. Thus we have

$$A \supset A_1, \quad \delta(A_1) \leq 1 \quad \text{and} \quad |A_1| \geq \aleph_0.$$

In a similar way, we obtain a set A_2 – equal to one of the intersections $A_1 \bigcap B_j^2$ – satisfying

$$A \supset A_1 \supset A_2, \quad \delta(A_2) \leq 1/2 \quad \text{and} \quad |A_2| \geq \aleph_0. \tag{9}$$

By induction we define a sequence $A_1, A_2, \ldots$ of sets satisfying

$$A \supset A_1 \supset A_2 \supset \ldots, \quad \delta(A_i) \leq 1/i \quad \text{and} \quad |A_i| \geq \aleph_0.$$

From the Cantor theorem it follows that there exists an $x \in \bigcap_{i=1}^{\infty} \overline{A}_i$, and from (9) it follows that every neighbourhood of x contains infinitely many points of the set A. Hence x is an accumulation point of A.■

4.3.30. COROLLARY. *The completion of a metric space (X, ρ) is compact if and only if (X, ρ) is a totally bounded space.*■

We conclude this section with a simple but useful theorem on open covers of compact metric spaces.

4.3.31. THE LEBESGUE COVERING THEOREM. *For every open cover $\mathcal{A}$ of a compact metric space X there exits an $\epsilon > 0$ such that the cover $\{B(x, \epsilon)\}_{x \in X}$ is a refinement of $\mathcal{A}$.*

PROOF. For every $x \in X$ take an $\epsilon_x > 0$ such that the ball $B(x, 2\epsilon_x)$ is contained in a member of $\mathcal{A}$. The open cover $\{B(x, \epsilon_x)\}_{x \in X}$ of X has a finite subcover, i.e., there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset X$ such that

$$X = B(x_1, \epsilon_{x_1}) \cup B(x_2, \epsilon_{x_2}) \cup \ldots \cup B(x_k, \epsilon_{x_k}).$$

One can readily see that the number $\epsilon = \min(\epsilon_{x_1}, \epsilon_{x_2}, \ldots, \epsilon_{x_k})$ has the required property.■

Any positive number ϵ satisfying the last theorem is called a *Lebesgue number* of the cover $\mathcal{A}$.

The Lebesgue covering theorem yields

4.3.32. THEOREM. *Every continuous mapping $f: X \to Y$ of a compact metrizable space X to a metrizable space Y is uniformly continuous with respect to any metrics ρ and σ on the spaces X and Y respectively.*

PROOF. For $\epsilon > 0$ let δ be a Lebesgue number of the open cover $\{f^{-1}(B(y, \epsilon/2))\}_{y \in Y}$ of the space X. If $\rho(x, x') < \delta$, then there exits $y \in Y$ such that $\{x, x'\} \subset f^{-1}(B(y, \epsilon/2))$ and we have $\sigma(f(x), f(x')) < \epsilon$.■

Historical and bibliographic notes

Totally bounded spaces were introduced by Hausdorff in [1914], and complete spaces and completely metrizable spaces were defined by Fréchet in [1906]. Kuratowski in [1933] proved Theorems 4.3.3, 4.3.10 and 4.3.13; in [1930] he observed that the condition in the Cantor theorem is sufficient for completeness. Necessity of this condition was established by Hausdorff in [1914] (Cantor proved the theorem for the real line in 1880). The observation that every totally bounded metric space is separable can be found in Hausdorff's [1927] book. Theorem 4.3.14 was proved by Hausdorff in [1914]; our proof is taken from Kuratowski's paper [1935a] (a similar construction was used by Fréchet in (1910]). The completion of a metric space was described by Hausdorff in [1914] (uniqueness was noted in (1927]); his construction (see problem 4.5.6) is related to the Cantor-Méray theory of real numbers. Lavrentieff proved Theorems 4.3.20 and 4.3.21 in [1924]. Theorem 4.3.23 was proved by Alexandroff in [1924] for separable spaces, and generalized to arbitrary metrizable spaces by Hausdorff in [1924]; Lemma 4.3.22 was established in Kuratowski's book [1933]. Theorem 4.3.24 follows easily from the fact, established by Lavrentieff in [1924] (for subsets of Euclidean spaces by Mazurkiewicz in [1916]), that the property of being a G_δ-set in a complete space is topologically invariant (see problem 4.5.7(a)). Theorem 4.3.26 was proved by Čech in [1937]; it implies that the Baire category theorem holds in completely metrizable spaces (cf. Exercise 4.3.C) – this was first proved by Hausdorff in [1914] (Baire proved the theorem for the real line in 1889). The Baire category theorem applied to function spaces (see Theorem 4.3.13) or to spaces of closed subsets (see Problems 4.5.23(c) and (d)) is an efficient and often used method of establishing the existence of some mathematical objects (it is called the *category method*). The reader can try to prove, using the category method, that there exists a continuous function $f: R \to R$ which is not differentiable at any point of R (this fact is a classical result of Weierstrass, a proof using the category method was given by Banach in [1931]). Theorems 4.3.27-4.3.29 were proved by Fréchet in [1910a], but he did not consider the class of totally bounded spaces. Lebesgue proved Theorem 4.3.31 in his paper [1921] (the theorem is stated there in a dual formulation); our proof is based on an idea appearing in Mazurkiewicz's paper [1920a]. Theorem 4.3.32 can be found in Hausdorff's book [1914].

Exercises

4.3.A. (a) Prove that for every metric space X the following conditions are equivalent:

(1) *The space (X, ρ) is totally bounded.*

(2) *For every $\epsilon > 0$ there exists a finite cover of X by sets of diameter less than ϵ.*

(3) *Every sequence of points of X contains a subsequence which is a Cauchy sequence.*

(b) Verify that if a mapping $f: X \to Y$ of a metrizable space X onto a metrizable space Y is uniformly continuous with respect to metrics ρ and σ, on the spaces X and Y respectively, and the space (X, ρ) is totally bounded, then the space (Y, σ) also is totally bounded.

(c) Check that if a space (X, ρ) is totally bounded and the metric ρ is uniformly equivalent to a metric σ on the space X, then the space (X, σ) also is totally bounded.

4.3.B. (a) Give an example of a complete space (X, ρ) which can be mapped onto a non-complete space (Y, σ) by a mapping uniformly continuous with respect to ρ and σ.

(b) Check that if a space (X, ρ) is complete and the metric ρ is uniformly equivalent to a metric σ on the space X, then the space (X, σ) also is complete.

(c) Verify that the hedgehog $J(\mathfrak{m})$, the plane with the "river" metric, Hilbert space H, and the Baire space $B(\mathfrak{m})$ are all complete.

4.3.C. (a) Give a direct proof of the Baire category theorem for completely metrizable spaces (cf. Exercise 4.4.F(e)).

Hint. As in the proof of Theorem 3.9.3, for a non-empty open set G define a sequence $G_1, G_2, \ldots$ of non-empty open sets satisfying

$$G \supset \overline{G}_1 \supset \overline{G}_2 \supset \ldots, \quad \overline{G}_i \cap A_i = \emptyset \quad \text{and} \quad \delta(\overline{G}_i) < 1/i \quad \text{for } i = 1, 2, \ldots$$

(b) Give an example of a subspace X of the plane which is a Baire space, and yet is not completely metrizable.

Hint. See Exercise 3.9.J(b).

4.3.D (Lindenbaum [1926]). (a) Show that if (X, ρ) is a totally bounded space, then for every isometry $f: X \to X$ the image $f(X)$ is dense in X (cf. Problem 4.5.4).

Hint. Check that for every $x \in X$ the sequence $f(x), ff(x), \ldots$ contains a subsequence converging to x.

(b) Deduce from (a) that if (X, ρ) is a compact space, then every isometry $f: X \to X$ is a mapping onto. Observe that the assumption of compactness cannot be weakened to the assumption that (X, ρ) is totally bounded.

4.3.E. (a) Let (X, ρ) and (Y, σ) be metric spaces and let $f: X \to Y$ be a continuous mapping. Verify that the formula

$$\overline{\rho}(x, y) = \rho(x, y) + \sigma(f(x), f(y)) \quad \text{for } x, y \in X$$

defines a metric $\overline{\rho}$ on the space X which is equivalent to the metric ρ, and that the mapping f is uniformly continuous with respect to $\overline{\rho}$ and σ. Check that if the metrics ρ and σ are totally bounded, then $\overline{\rho}$ also is totally bounded.

(b) Verify that if (X, ρ) is a metric space with a bounded metric ρ and $f: X \to X$ is a continuous mapping of X to itself, then the formula

$$\overline{\rho}(x, y) = \rho(x, y) + \sum_{i=1}^{\infty} \frac{1}{2^i} \rho(f^i(x), f^i(y)) \quad \text{for } x, y \in X,$$

where $f^i(x) = f(f^{i-1}(x))$ for $i \geq 1$ and $f^0(x) = x$, defines a metric $\overline{\rho}$ on the space X which is equivalent to ρ, and that the mapping f is uniformly continuous with respect to $\overline{\rho}$ and $\overline{\rho}$. Check that if the metric ρ is totally bounded, then $\overline{\rho}$ also is totally bounded.

(c) Show that a metrizable space X is compact if and only if every metric on the space X is totally bounded or – equivalently – every metric on the space X is bounded (cf. Problem 4.5.21(d)).

(d) (Niemytzki and Tychonoff [1928]) Prove that a metrizable space X is compact if and only if every metric on the space X is complete (cf. Exercise 4.4.E(b) and Problem 4.5.21(d)).

Hint. Let X be a non-compact metrizable space and $F_1 \supset F_2 \supset \ldots$ a decreasing sequence of non-empty closed subsets of X such that the intersection $\bigcap_{i=1}^{\infty} F_i$ is empty. Take a metric σ on the space X which is bounded by 1 and verify that for $i = 1, 2, \ldots$ the formula

$$\rho_i(x, y) = |\sigma(x, F_i) - \sigma(y, F_i)| + [\min(\sigma(x, F_i), \sigma(y, F_i))]\sigma(x, y) \quad \text{for } x, y \in X$$

defines a pseudometric on the set X; using thus constructed pseudometrics define a metric ρ on the space X such that $\delta(F_i) \leq 1/2^i$ for $i = 1, 2, \ldots$ (cf. Lemma 4.4.6).

4.3.F. (a) (Arens [1946]) Prove that if X is a locally compact Lindelöf space, then for every completely metrizable space Y the space Y^X with the compact-open topology is completely metrizable (cf. Exercises 4.2.H and 3.8.C(b)).

(b) Give an example of a hemicompact space X such that the space I^X with the compact-open topology is not completely metrizable (cf. Exercise 4.2.H).

Hint. Take as X the space Y in Example 1.6.20 and consider the set consisting of all functions $f \in I^X$ that assume only the values 0 and 1 and are constant on every subspace of the form $Y \cap (1/i, 1/i + 1/i^2]$.

4.3.G (Baire [1909]). Prove that the space P of all irrational numbers (with the topology of a subspace of the real line) is homeomorphic to the Baire space $B(\aleph_0) = N^{\aleph_0}$ (cf. Exercise 6.2.A).

Hint. Show first that for every metric ρ on the space P, every $\epsilon > 0$ and any non-empty open set $U \subset P$ there exists a sequence $F_1, F_2, \ldots$ of pairwise disjoint non-empty closed-and-open subsets of P such that $\delta(F_i) < \epsilon$ for $i = 1, 2, \ldots$ and $U = \bigcup_{i=1}^{\infty} F_i$. Then take a complete metric ρ on the space P and for every finite sequence $i_1, i_2, \ldots, i_k$ of natural numbers define inductively a non-empty closed-and-open set $F_{i_1 i_2 \ldots i_k} \subset P$ with diameter less than $1/k$ such that

$$P = \bigcup_{i=1}^{\infty} F_i, \quad F_{i_1 i_2 \ldots i_k} = \bigcup_{i=1}^{\infty} F_{i_1 i_2 \ldots i_k i} \quad \text{and} \quad F_{i_1 i_2 \ldots i_k} \cap F_{j_1 j_2 \ldots j_k} = \emptyset$$

whenever the sequences of indices are distinct.

4.3.H. (a) (Brouwer [1913a]; implicitly, Fréchet [1910]) Prove that for any two countable dense subsets A, B of the real line R there exists a homeomorphism $f: R \to R$ such that $f(A) = B$ (cf. Problem 4.5.2).

Hint. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$; define inductively a function f from A to R by letting $f(a_1) = b_1$ and taking as $f(a_i)$ an element of B, with the smallest possible index, such that the conditions $a_j < a_k$ and $f(a_j) < f(a_k)$ are equivalent for $j, k \leq i$. Extend f over R and prove that this extension is a homeomorphism.

(b) (Fréchet [1910]) Prove that the space Q of all rational numbers (with the topology of a subspace of the real line) is a universal space for all countable metrizable spaces.

Hint. Observe that countable metrizable spaces are embeddable in R; for a countable $X \subset R$ consider the sets $X \cup Q$ and Q and apply (a).

(c) Prove that if A_1, A_2, B_1 and B_2 are countable dense subsets of R satisfying the condition $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$, then there exists a homeomorphism $f: R \to R$ such that $f(A_i) = B_i$ for $i = 1, 2$.

(d) Show that for any two countable dense subsets A, B of the space P of irrational numbers there exists a homeomorphism $f: P \to P$ such that $f(A) = B$.

(e) Show that for any two countable dense subsets A, B of the Cantor set C there exists a homeomorphism $f: C \to C$ such that $f(A) = B$.

Hint. Observe that for every countable set $A \subset C$ there exists a homeomorphism $g: C \to C$ such that the set $g(A)$ is disjoint from the set of end-points of all intervals removed from I in the construction of the Cantor set and apply Exercise 3.2.B.

4.3.I. (a) Note that any two metrics on a compact metrizable space are uniformly equivalent.

(b) Check that if Y is a compact metrizable space, then for every topological space X the topology on Y^X induced by the metric $\hat{\rho}$, defined by formula (7) in Section 4.2, does not depend on the choice of the metric ρ on the space Y. Observe that the above defined topology on Y^X does not necessarily coincide with the compact-open topology.

Hint. See Exercises 4.2.A(c) and 4.2.G.

4.3.J (Banach [1922]). Let (X, ρ) be a metric space; a mapping f of X to itself is *contractive* if there exists a number $c \in [0, 1)$ such that $\rho(f(x), f(y)) \leq c\rho(x, y)$ for all $x, y \in X$.

Show that for every contractive mapping f of a complete metric space X to itself there exists exactly one point $x_0 \in X$ such that $f(x_0) = x_0$ (this is the *Banach fixed-point theorem*).

Hint. Choose a point $a \in X$ and show that $f(a), ff(a), \ldots$ is a Cauchy sequence.

4.4. Metrization theorems I

A family of subsets of a topological space is called *σ-locally finite* (*σ-discrete*) if it can be represented as a countable union of locally finite (discrete) families.

In the following theorem, one of the most important properties of metrizable spaces is established.

4.4.1. THE STONE THEOREM. *Every open cover of a metrizable space has an open refinement which is both locally finite and σ-discrete.*

PROOF. Let $\{U_s\}_{s\in S}$ be an open cover of a metrizable space X; take a metric ρ on the space X and a well-ordering relation $<$ on the set S. Define inductively families $\mathcal{V}_i = \{V_{s,i}\}_{s\in S}$ of subsets of X by letting

$$V_{s,i} = \bigcup B(c, 1/2^i),$$

where the union is taken over all points $c \in X$ satisfying the following conditions:

(1) $\qquad s$ is the smallest element of S such that $c \in U_s$.

(2) $\qquad c \notin V_{t,j}$ for $j < i$ and $t \in S$.

(3) $\qquad B(c, 3/2^i) \subset U_s$.

It follows from the definition that the sets $V_{s,i}$ are open, and (3) implies that $V_{s,i} \subset U_s$. Let x be a point of X; take the smallest $s \in S$ such that $x \in U_s$ and a natural number i such that $B(x, 3/2^i) \subset U_s$. Clearly, we either have $x \in V_{t,j}$ for a $j < i$ and a $t \in S$ or $x \in V_{s,i}$. Hence, the union $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ is an open refinement of the cover $\{U_s\}_{s\in S}$.

We shall prove that for every i

(4) $\qquad$ if $x_1 \in V_{s_1,i}$, $x_2 \in V_{s_2,i}$ and $s_1 \neq s_2$, then $\rho(x_1, x_2) > 1/2^i$,

and this will show that the families $\mathcal{V}_i$ are discrete, because every $1/2^{i+1}$-ball meets at most one member of $\mathcal{V}_i$.

Let us assume that $s_1 < s_2$. By the definition of $V_{s_1,i}$ and $V_{s_2,i}$ there exist points c_1, c_2 satisfying (1)-(3) such that $x_k \in B(c_k, 1/2^i) \subset V_{s_k,i}$ for $k = 1, 2$. From (3) it follows that $B(c_1, 3/2^i) \subset U_{s_1}$ and from (1) we see that $c_2 \notin U_{s_1}$, so that $\rho(c_1, c_2) \geq 3/2^i$. Hence,

$$\rho(x_1, x_2) \geq \rho(c_1, c_2) - \rho(c_1, x_1) - \rho(c_2, x_2) > 1/2^i,$$

which proves (4).

To conclude the proof of the theorem it suffices to show that for every $t \in S$ and any pair k, j of natural numbers

(5) $\qquad$ if $B(x, 1/2^k) \subset V_{t,j}$, then $B(x, 1/2^{i+k}) \cap V_{s,i} = \emptyset$ for $i \geq j + k$ and $s \in S$,

because for every $x \in X$ there exist k, j and t such that $B(x, 1/2^k) \subset V_{t,j}$ and thus the ball $B(x, 1/2^{j+k})$ meets at most $j + k - 1$ members of $\mathcal{V}$.

It follows from (2) that the points c in the definition of $V_{s,i}$ do not belong to $V_{t,j}$ whenever $i \geq j + k$; since $B(x, 1/2^k) \subset V_{t,j}$, we have $\rho(x, c) \geq 1/2^k$ for any such c. The inequalities $j + k \geq k + 1$ and $i \geq k + 1$ imply that $B(x, 1/2^{j+k}) \cap B(c, 1/2^i) = \emptyset$, and this yields (5).■

4.4.2. REMARK. Let us note that the proof of the Stone theorem gives more: every open cover of a space X whose topology is induced by a pseudometric has an open refinement which is both locally finite and σ-discrete.

4.4.3. THEOREM. *Every metrizable space has a σ-discrete base.*

PROOF. Let ρ be any metric on a metrizable space X and let $\mathcal{B}_i$ be an open σ-discrete refinement of the open cover of X consisting of all $1/i$-balls. One easily verifies that the σ-discrete family $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a base for X.■

4.4.4. COROLLARY. *Every metrizable space has a σ-locally finite base.*■

We shall prove that the existence of a σ-locally finite base is also sufficient for metrizability of a regular space. We start with a lemma generalizing Theorem 1.5.16.

4.4.5. LEMMA. *Every regular space which has a σ-locally finite base is perfectly normal.*

PROOF. Let $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$, where the families $\mathcal{B}_i$ are locally finite, be a base for a regular space X. Consider an arbitrary open set $W \subset X$. For every $x \in W$ there exists a natural number $i(x)$ and an open set $U(x) \in \mathcal{B}_{i(x)}$ such that $x \in U(x) \subset \overline{U(x)} \subset W$. Letting $W_i = \bigcup\{U(x) : i(x) = i\}$ we obtain a sequence $W_1, W_2, \ldots$ of open subsets of X such that $W = \bigcup_{i=1}^{\infty} W_i$ and – by Theorem 1.1.11 – $\overline{W}_i \subset W$ for $i = 1, 2, \ldots$ Normality of X now follows from Lemma 1.5.15; since $W = \bigcup_{i=1}^{\infty} \overline{W}_i$, the space X is perfectly normal.■

4.4.6. LEMMA. *Let X be a T_0-space and $\{\rho_i\}_{i=1}^{\infty}$ a countable family of pseudometrics on the set X which all are bounded by 1 and satisfy the following two conditions:*

(i) *$\rho_i \colon X \times X \to R$ is a continuous function for $i = 1, 2, \ldots$*

(ii) *For every $x \in X$ and every non-empty closed set $A \subset X$ such that $x \notin A$ there exists an i such that $\rho_i(x, A) = \inf\{\rho_i(x, a) : a \in A\} > 0$.*

Then the space X is metrizable and the function ρ defined by

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_i(x, y) \quad \textit{for } x, y \in X$$

is a metric on the space X.

PROOF. One readily sees that $\rho(x, x) = 0$ for every $x \in X$ and that ρ satisfies conditions (M2) and (M3). Since X is a T_0-space, for every pair of distinct points $x, y \in X$ we either have $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$; hence $\rho(x, y) > 0$ by (ii), i.e., ρ is a metric on the set X.

By Corollary 4.1.11, to prove that ρ is a metric on the space X it suffices to show that

$$\rho(x, A) = 0 \quad \text{if and only if} \quad x \in \overline{A}.$$

If $x \notin \overline{A}$, then by (ii) there exists an i such that $\rho_i(x, \overline{A}) = r > 0$ and we have $\rho(x, A) \geq \rho(x, \overline{A}) \geq r/2^i > 0$. On the other hand, it follows from (i) by virtue of Theorem 1.4.7 that

$\rho: X \times X \to R$ is a continuous function, and Proposition 4.1.9 implies that the function $f: X \to R$, defined by letting $f(x) = \rho(x, A)$, also is continuous. Hence, if $x \in \overline{A}$, then $f(x) \in f(\overline{A}) \subset \overline{f(A)} = \{0\}$, i.e., $\rho(x, A) = 0$. ■

4.4.7. THE NAGATA-SMIRNOV METRIZATION THEOREM. *A topological space is metrizable if and only if it is regular and has a σ-locally finite base.*

PROOF. Necessity of conditions in the theorem follows from 4.1.13 and 4.4.4. We shall show that the conditions are also sufficient.

Consider a regular space X which has a base $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$, where $\mathcal{B}_i = \{U_s\}_{s \in S_i}$ is a locally finite family. For each natural number i and any $s \in S_i$ by virtue of Lemma 4.4.5 and Corollary 1.5.13 there exists a continuous function $f_s: X \to I$ such that $U_s = f_s^{-1}((0, 1])$. Since the family $\{W_s\}_{s \in S_i}$, where $W_s = (U_s \times X) \cup (X \times U_s)$, is locally finite in $X \times X$ and $|f_s(x) - f_s(y)| = 0$ if $(x, y) \notin W_s$, it follows from Corollary 2.1.12 that by letting

$$g_i(x, y) = \sum_{s \in S_i} |f_s(x) - f_s(y)| \quad \text{for } (x, y) \in X \times X$$

we define a continuous function $g_i: X \times X \to R$. For $i = 1, 2, \ldots$ the formula $\rho_i(x, y) = \min(1, g_i(x, y))$ defines a pseudometric on the set X which is bounded by 1, and the family $\{\rho_i\}_{i=1}^{\infty}$ satisfies condition (i) in 4.4.6. Condition (ii) also is satisfied. Indeed, for every $x \in X$ and every non-empty closed set $A \subset X$ such that $x \notin A$ there exists a $U \in \mathcal{B}$ such that $x \in U$ and $A \subset X \setminus U$; now, $U = U_s \in \mathcal{B}_i$ for some i and some $s \in S_i$ and since $f_s(x) > 0$ and $f_s(A) = \{0\}$, we have $\inf\{\rho_i(x, a) : a \in A\} \geq f_s(x) > 0$. Metrizability of X now follows from Lemma 4.4.6 (cf. Exercise 4.4.A(d)). ■

Let us observe that from the last theorem it follows immediately that the conditions in Theorems 4.2.8 and 4.2.9 are sufficient for metrizability.

Theorems 4.4.3 and 4.4.7 yield

4.4.8. THE BING METRIZATION THEOREM. *A topological space is metrizable if and only if it is regular and has a σ-discrete base.* ■

We shall show now that there exists a universal space for all metrizable spaces of weight $\mathfrak{m} \geq \aleph_0$ (cf. Exercise 4.4.K).

4.4.9. THEOREM. *The Cartesian product $[J(\mathfrak{m})]^{\aleph_0}$ of $\aleph_0$ copies of the hedgehog $J(\mathfrak{m})$ is universal for all metrizable spaces of weight $\mathfrak{m} \geq \aleph_0$.*

PROOF. Clearly $[J(\mathfrak{m})]^{\aleph_0}$ is a metrizable space of weight $\mathfrak{m}$.

Let X be a metrizable space of weight $\mathfrak{m} \geq \aleph_0$. By Theorem 4.4.3 there exists a base $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$, where $\mathcal{B}_i = \{U_s\}_{s \in S_i}$ is a discrete family. By virtue of Theorem 1.1.15 one can assume that the set $S = \bigcup_{i=1}^{\infty} S_i$ has cardinality $\mathfrak{m}$; with no loss of generality one can also assume that the set S coincides with the set used in Example 4.1.5 in the construction of the hedgehog $J(\mathfrak{m})$.

For each natural number i and any $s \in S_i$ by virtue of Corollary 4.1.12 there exists a continuous function $f_s: X \to I$ such that $U_s = f_s^{-1}((0, 1])$. Since the family $\{\overline{U}_s\}_{s \in S_i}$ is discrete, it follows from Proposition 2.1.13 that by letting

$$f_i(x) = j_s f_s(x) \text{ for } x \in \overline{U}_s \quad \text{and} \quad f_i(x) = j_{s_0}(0) \text{ for } x \in X \setminus \bigcup_{s \in S_i} U_s,$$

where s_0 is a fixed element of S, we define a continuous function $f_i: X \to J(\mathfrak{m})$. One can easily check that the family $\{f_i\}_{i=1}^{\infty}$ separates points and closed sets. Thus by the diagonal theorem, the space X is embeddable in $[J(\mathfrak{m})]^{\aleph_0}$. ■

No doubt, the reader observed that in the above proof of embeddability of the space X in $[J(\mathfrak{m})]^{\aleph_0}$, only perfect normality of X and the fact that X has a σ-discrete base were used. Since, by Lemma 4.4.5, in regular spaces perfect normality follows from the existence of a σ-discrete base, we have incidentally given another proof of the Bing metrization theorem.

The final part of this section is devoted to a study of invariance of metrizability under mappings. Let us start by reminding the reader that in Example 1.4.17 we defined a closed mapping of the real line onto a space which is not first-countable, so that metrizability is not an invariant of closed mappings (cf. Theorem 4.4.17). Now we shall give two examples to show that metrizability is not an invariant of open mappings, either (cf. Exercise 4.2.D and Problem 4.5.17).

4.4.10. EXAMPLE. Consider the subspace of the plane $X = \{(0,0)\} \cup \{(k, 1/i + 1/i \cdot j) : k = 0,1,\ i = 1,2,\ldots,\ j = i, i+1,\ldots\} \cup \{(1,1/i) : i = 1,2,\ldots\}$ and define on X an equivalence relation E by letting $(x_1,y_1)E(x_2,y_2)$ whenever $y_1 = y_2$. The natural quotient mapping $q: X \to X/E = Y$ is open. Indeed, for each open set $A \subset X$ the set $q^{-1}q(A)$ is open, because it is obtained from A by adjoining some isolated points of X, and thus the set $q(A) \subset Y$ also is open. One readily sees that Y is a Hausdorff space; we shall show that Y is not regular. The set $q(F)$, where $F = \{(1,1/i) : i = 1,2,\ldots\}$ is closed in Y and does not contain the point $p = q((0,0))$, and yet for every pair U, V of open subsets of Y such that $p \in U$ and $q(F) \subset V$ we have $U \cap V \neq \emptyset$. Indeed, there exist an i_0 such that $(0, 1/i + 1/i \cdot j) \in q^{-1}(U)$ whenever $j \geq i \geq i_0$ and a $j_0 \geq i_0$ such that $(1, 1/i_0 + 1/i_0 \cdot j) \in q^{-1}(V)$ whenever $j \geq j_0$; thus for $y = 1/i_0 + 1/i_0 \cdot j_0$ we have $(0,y) \in q^{-1}(U)$ and $(1,y) \in q^{-1}(V)$, and this implies that $q(0,y) = q(1,y) \in U \cap V$. Let us note that the fibers of q are one-point and two-point sets. ■

4.4.11. EXAMPLE. Let $Y = W_0$ be the space of all ordinal numbers less than ω_1 (see Example 3.1.27) and let $X = \bigoplus_{\alpha<\omega_1} X_\alpha$, where $X_\alpha = \{x \in W_0 : x \leq \alpha\}$ with the subspace topology. Since all the X_α's are open, the combination $f = \nabla_{\alpha<\omega_1} i_\alpha: X_\alpha \to Y$, where $i_\alpha: X_\alpha \to Y$ is the embedding, is open by Propositions 2.1.11 and 2.1.15. The spaces X_α are second-countable, so that they are metrizable by Theorem 4.2.9; it follows then from Theorem 4.2.1 that the space X also is metrizable. On the other hand, the space $Y = f(X)$ is not metrizable, because it is a countably compact non-compact space (see Example 3.10.16 and Theorem 4.1.17). Let us note that the fibers of f are all homeomorphic to the discrete space $D(\aleph_1)$. ■

We shall now prove that metrizability is an invariant of perfect mappings. In the proof two lemmas will be applied (the second one is an important particular case of Theorem 5.1.33).

4.4.12. LEMMA. *If every open cover of a topological space X has a locally finite closed refinement, then every open cover of X has also a locally finite open refinement.*

PROOF. Let $\mathcal{U}$ be an open cover of the space X; take a locally finite refinement $\mathcal{A} = \{A_s\}_{s\in S}$ of $\mathcal{U}$ and for every $x \in X$ choose a neighbourhood V_x of the point x which meets only finitely many members of $\mathcal{A}$. Let $\mathcal{F}$ be a locally finite closed refinement of the open cover $\{V_x\}_{x\in X}$ and for every $s \in S$ let

$$W_s = X \setminus \bigcup\{F \in \mathcal{F} : F \cap A_s = \emptyset\}.$$

Clearly, the set W_s is open and contains A_s; furthermore for every $s \in S$ and any $F \in \mathcal{F}$ we have

$$(6) \qquad W_s \cap F \neq \emptyset \text{ if and only if } A_s \cap F \neq \emptyset.$$

For every $s \in S$ take a $U(s) \in \mathcal{U}$ such that $A_s \subset U(s)$ and let $V_s = W_s \cap U(s)$. The family $\{V_s\}_{s\in S}$ is an open refinement of the cover $\mathcal{U}$. Since every point $x \in X$ has a neighbourhood that meets only finitely many members of $\mathcal{F}$ and every member of $\mathcal{F}$ meets only finitely many members of $\mathcal{A}$, it follows from (6) that the cover $\{V_s\}_{s\in S}$ is locally finite. ■

4.4.13. LEMMA. *If there exists a perfect mapping $f: X \to Y$ of a metrizable space X onto a space Y, then every open cover of the space Y has an open locally finite refinement.*

PROOF. Let $\mathcal{V}$ be an open cover of the space Y. By virtue of the Stone theorem the open cover $\{f^{-1}(V)\}_{V\in\mathcal{V}}$ of the space X has a locally finite open refinement $\{U_s\}_{s\in S}$. Applying Theorem 1.5.18, take a closed cover $\mathcal{F} = \{F_s\}_{s\in S}$ of the space X such that $F_s \subset U_s$ for every $s \in S$. Since $\mathcal{F}$ is locally finite, it follows from Lemma 3.10.11 that the family $\{f(F_s)\}_{s\in S}$ is a locally finite refinement of the cover $\mathcal{V}$ of Y. The lemma follows now from 4.4.12. ■

4.4.14. REMARK. Let us observe that in the proof of 4.4.13 only normality of the space X and the fact that every open cover of X has a locally finite open refinement were used.

4.4.15. THEOREM. *Metrizability is an invariant of perfect mappings.*

PROOF. Let $f: X \to Y$ be a perfect mapping of a metrizable space X onto a space Y. Take a metric ρ on the space X and for $i = 1, 2, \ldots$ and every $y \in Y$ consider the open sets

$$U_i(y) = B(f^{-1}(y), 1/i), \quad W_i(y) = Y \setminus f(X \setminus U_i(y)), \quad V_i(y) = f^{-1}(W_i(y)) \subset U_i(y).$$

It follows from the definition that

$$(7) \qquad U_j(y) \subset U_i(y) \quad \text{whenever} \quad j \geq i.$$

The family $\mathcal{W}_i = \{W_i(y)\}_{y\in Y}$ is an open cover of Y; let us observe that for every $y \in Y$

$$(8) \qquad \{W_i(y)\}_{i=1}^{\infty} \text{ is a base for } Y \text{ at the point } y.$$

Indeed, for any neighbourhood V of the point y we have $f^{-1}(y) \subset f^{-1}(V)$ and by 4.1.14 there exists an i such that $U_i(y) \subset f^{-1}(V)$ which implies that $W_i(y) \subset V$.

We shall now show that

$$(9) \qquad \text{for each set } W_i(y) \text{ there exists a } j \text{ such that } \bigcup\{W_j(z) : y \in W_j(z)\} \subset W_i(y).$$

By 4.1.14 and (7) there exists a $j \geq 2i$ such that $U_j(y) \subset V_{2i}(y)$. Consider a point $z \in Y$ such that $y \in W_j(z)$; since $f^{-1}(y) \subset V_j(z) \subset U_j(z)$, there exist $x \in f^{-1}(z)$ and $x' \in f^{-1}(y)$

satisfying $\rho(x, x') < 1/j$. This implies that $U_j(y) \cap f^{-1}(z) \neq \emptyset$ and $f^{-1}(z) \subset V_{2i}(y)$, because the last set together with any point x contains the fiber $f^{-1}f(x)$.

Let us now take a $t \in W_j(z)$; since $f^{-1}(t) \subset U_j(z)$, for every $x \in f^{-1}(t)$ there exists an $x' \in f^{-1}(z)$ such that $\rho(x, x') < 1/j \leq 1/2i$. As shown in the last paragraph, $f^{-1}(z) \subset V_{2i}(y) \subset U_{2i}(y)$; therefore there exists an $x'' \in f^{-1}(y)$ such that $\rho(x', x'') < 1/2i$. We have $\rho(x, x'') < 1/i$ which implies that $f^{-1}(t) \subset U_i(y)$, i.e., that $t \in W_i(y)$, and the proof of (9) is concluded.

By virtue of Lemma 4.4.13, the cover $\mathcal{W}_i$ of the space Y has an open locally finite refinement $\mathcal{B}_i$. It follows from (8) and (9) that the union $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a base for Y, so that – being normal by Theorem 1.5.20 – the space Y is metrizable by virtue of the Nagata-Smirnov metrization theorem. ∎

Our next theorem gives necessary and sufficient conditions for metrizability of closed images of metrizable spaces.

4.4.16. VAĬNŠTEĬN'S LEMMA. *If $f: X \to Y$ is a closed mapping of a metrizable space X onto a space Y, then for every $y \in Y$ such that $\chi(y, Y) \leq \aleph_0$ the set $\operatorname{Fr} f^{-1}(y)$ is compact.*

PROOF. Let $A = \{x_1, x_2, \ldots\}$ be a subset of $F = \operatorname{Fr} f^{-1}(y)$ such that $|A| = \aleph_0$ and let $\{V_i\}_{i=1}^{\infty}$ be a base for Y at the point y. Consider a metric ρ on the space X and for $i = 1, 2, \ldots$ choose a point $x_i' \in f^{-1}(V_i) \setminus f^{-1}(y)$ such that $\rho(x_i, x_i') < 1/i$; such a point certainly exists, because – the mapping f being closed – the sets $\{y\}$ and $f^{-1}(y)$ are closed, which implies that $x_i \in F \subset f^{-1}(y) \subset f^{-1}(V_i)$, and thus the set $B(x_i, 1/i) \cap f^{-1}(V_i)$ is a neighbourhood of x_i. For the set $B = \{x_1', x_2', \ldots\}$ we have $y \in \overline{f(B)} \setminus f(B)$; as f is a closed mapping, $B \neq \overline{B}$ and $B^{\mathrm{d}} \neq \emptyset$. Since $\rho(x_i, x_i') < 1/i$ for $i = 1, 2, \ldots$, we have $A^{\mathrm{d}} = B^{\mathrm{d}} \neq \emptyset$. Hence, every countably infinite subset of F has an accumulation point, and this – by Theorems 3.10.3 and 4.1.17 – implies that the set F is compact. ∎

4.4.17. THE HANAI-MORITA-STONE THEOREM. *For every closed mapping $f: X \to Y$ of a metrizable space X onto a space Y the following conditions are equivalent:*

(i) *The space Y is metrizable.*

(ii) *The space Y is first-countable.*

(iii) *For every $y \in Y$ the set $\operatorname{Fr} f^{-1}(y)$ is compact.*

PROOF. The implication (i)⇒(ii) is obvious and the implication (ii)⇒(iii) follows from the above lemma. We shall show that (iii)⇒(i). For each $y \in Y$ let $A(y) \subset X$ be equal to $\operatorname{Fr} f^{-1}(y)$, if the latter set is non-empty, and be an arbitrary one-point subset of $f^{-1}(y)$, if $\operatorname{Fr} f^{-1}(y) = \emptyset$, i.e., if $f^{-1}(y)$ is a closed-and-open subset of X. The set $A = \bigcup_{y \in Y} A(y)$ is closed in X, because its complement is the union of open sets $\operatorname{Int} f^{-1}(y)$ with at most one point removed from each. The restriction $f|A: A \to Y$ is a closed mapping of A onto Y; since the fibers of $f|A$ are either of the from $\operatorname{Fr} f^{-1}(y)$ or are one-point sets, the mapping $f|A$ is perfect. Metrizability of Y follows now from Theorem 4.4.15. ∎

Theorems 1.4.16 and 4.4.17 yield

4.4.18. THEOREM. *Metrizability is an invariant of closed-and-open mappings.* ∎

Let us observe that metrizability is not an inverse invariant of perfect open mappings, as shown by identification of any non-metrizable compact space to a point.

The following theorem, last in this section, is a consequence of Theorems 3.7.22, 4.2.1 and 4.4.15.

4.4.19. THEOREM. *If a topological space X has a locally finite closed cover consisting of metrizable subspaces, then X is itself metrizable.* ■

Historical and bibliographic notes

Theorem 4.4.1 was proved in A. H. Stone's paper [1948]; it is one of the most important theorems in general topology (our proof was given by M. E. Rudin in [1969a]). Nagata and Smirnov proved Theorem 4.4.7 in [1950] and [1951a] respectively; Theorem 4.4.8 was proved by Bing in [1951]. Further metrization theorems will be given in Section 5.4 (see also the notes to that section). Theorem 4.4.9 is due to Kowalsky and was proved in [1957]; the proof given here is taken from Swardson [1979]. Example 4.4.10 is taken from Alexandroff and Hopf's book [1935] and Example 4.4.11 from A. H. Stone's paper [1956]. Theorems 4.4.15 and 4.4.17 were proved independently by Morita and Hanai in [1956] and by A. H. Stone in [1956] (for separable spaces, Whyburn proved the former in [1942] and the equivalence of (i) and (ii) in the latter in (1950]). Lemma 4.4.12 was proved by Michael in [1953], Lemma 4.4.13 by Morita and Hanai in [1956], and Lemma 4.4.16 by Vaĭnšteĭn in [1947]. Theorem 4.4.18 was proved by Balachandran in [1955] and Theorem 4.4.19 by Nagata in [1950].

Exercises

4.4.A. (a) Show that a locally finite open cover of a metrizable space is not necessarily σ-discrete.

Hint. Consider the discrete space $D(\mathfrak{c})$.

(b) Check that the space X in Example 1.5.6 is second-countable and thus has a σ-discrete base.

(c) Show that a T_1-space X has a locally finite base if and only if X is discrete.

(d) Consider the pseudometrics ρ_i defined in the proof of the Nagata-Smirnov metrization theorem and observe that the family of mappings $\{f_i\}_{i=1}^{\infty}$, where $f_i: X \to X/\rho_i$ is defined as in Exercise 4.2.I, separates points and closed sets. Applying the diagonal theorem, deduce that X is metrizable.

4.4.B. Prove that every completely metrizable space of weight $\leq \mathfrak{m} \geq \aleph_0$ is homeomorphic to a closed subspace of the Cartesian product $[J(\mathfrak{m})]^{\aleph_0}$.

Hint. Observe that the real line is homeomorphic to a closed subspace of $[J(\aleph_0)]^2$ and apply Lemma 4.3.22.

4.4.C. (a) Prove that for every metrizable space X of weight $\leq \mathfrak{c}$ there exists a one-to-one continuous mapping $f: X \to Y$ onto a separable metrizable space Y.

Hint. Consider first the space $X = J(\mathfrak{c})$, then apply Theorem 4.4.9.

(b) (Vidossich [1972]) Prove that if X is a Tychonoff space and there exists a one-to-one continuous mapping of X onto a metrizable space, then every family of pairwise disjoint non-empty open subsets of the space R^X with the compact-open topology is countable.

Deduce that in the space R^m every family of pairwise disjoint non-empty open sets is countable (cf. Corollary 2.3.18).

Hint. Consider first the case of a metrizable space X and modify the proof of Theorem 2.3.17 applying (a) and Exercise 3.4.H(b), then apply the hint to Exercise 3.4.H(b).

4.4.D. (a) A continuous mapping $f: X \to Y$ is a *local homeomorphism* if for every $x \in X$ there exists a neighbourhood U of the point x such that $f|U$ is a homeomorphism of U onto an open subspace of Y.

Verify that every open mapping $f: X \to Y$, defined on a Hausdorff space X, such that all fibers of f are finite and have the same cardinality is a local homeomorphism.

(b) (Arhangel'skiĭ [1966]) Show that every local homeomorphism $f: X \to Y$, defined on a Hausdorff space X, such that all fibers of f are finite and have the same cardinality is a closed mapping.

(c) Show that if there exists an open mapping $f: X \to Y$ of a metrizable space X onto a space Y, such that all fibers of f are finite and have the same cardinality, then the space Y is metrizable.

4.4.E. (a) Apply the Nagata-Smirnov metrization theorem to solve Exercise 4.4.D(c).

(b) Apply the Nagata-Smirnov metrization theorem to solve Exercise 4.3.E(d).

Hint. For a non-compact metrizable space X define a sequence $x_1, x_2, \ldots$ of points of X such that for a metric ρ on the space X we have $\rho(x_i, x_j) \geq 1$ whenever $i \neq j$. Adjoin a point x_0 to the space X in such a way that the space $X \cup \{x_0\}$ is metrizable and $x_0 = \lim x_i$.

4.4.F. A topological space X is *locally separable at a point* $x \in X$ if x has a separable neighbourhood. A topological space X is *locally separable* if X is locally separable at every point $x \in X$.

(a) Verify that the hedgehog $J(\mathfrak{c})$ is not locally separable.

(b) (Urysohn [1927a]) Give an example of a metrizable space of weight $\mathfrak{c}$ that is not locally separable at any point.

(c) (Alexandroff [1924b]) Show that if X is a locally separable metrizable space, then $X = \bigoplus_{s \in S} X_s$, where all spaces X_s are separable (cf. Exercise 5.3.A(b)).

Hint. Take a locally finite open refinement $\mathcal{U}$ of an open cover of X by separable sets. Define an equivalence relation E on $\mathcal{U}$ letting UEU' whenever there exists a finite sequence $U_0, U_1, \ldots, U_k$ of members of $\mathcal{U}$ such that $U_0 = U, U_k = U'$ and $U_i \bigcap U_{i+1} \neq \emptyset$ for $i = 0, 1, \ldots, k-1$. For every equivalence class $\mathcal{U}_s \subset \mathcal{U}$ of the relation E consider the union $X_s = \bigcup \mathcal{U}_s$.

(d) (Štěpánek and Vopěnka [1967]) Show that every metrizable space X which is not locally separable at any point can be represented as the union of an increasing sequence $A_1 \subset A_2 \subset \ldots \subset A_\alpha \subset \ldots$, $\alpha < \omega_1$ consisting of nowhere dense sets.

Hint (Kulpa and Szymański [1977], Engelking [1978a]). Let $\{U_s\}_{s \in S}$ be a σ-discrete base for X and for each $s \in S$ let $\{U_{s,\alpha} : \alpha < \omega_1\}$ be a family of pairwise disjoint non-empty open sets contained in U_s; consider the sets $A_\alpha = X \setminus \bigcup_{s \in S} \bigcup_{\beta > \alpha} U_{s,\beta}$.

(e) (Štěpánek and Vopěnka [1967]) Show that every metrizable space without isolated points can be represented as the union of a family of cardinality $\leq \mathfrak{c}$ consisting of nowhere dense sets.

4.4.G. Note that Vaĭnšteĭn's lemma follows easily from Exercises 2.1.C(d) and 4.1.E(a).

4.4.H. (a) (Arhangel'skiĭ [1960]; for locally compact spaces, Smirnov [1956a]) Observe that if a Čech-complete space X has a countable cover consisting of separable metrizable spaces, then X is metrizable.

Hint. See Exercise 3.9.E(c).

(b) Verify that the assumption of separability in (a) is essential even in the case of a two-element cover.

4.4.I. Show that if there exists a closed mapping $f: X \to Y$ of a metrizable space X onto a space Y of pointwise countable type (in particular, onto a Čech-complete space Y), then Y is metrizable.

Hint. Apply Exercise 3.1.F(b).

4.4.J (Morita [1955]). Prove that every metrizable space X of weight $\mathfrak{m} \geq \aleph_0$ is a continuous image of a subspace of the Baire space $B(\mathfrak{m})$ under a perfect mapping (cf. the hint to Exercise 4.2.D(a)).

Hint. Applying the Stone theorem, define a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally finite closed covers $\mathcal{F}_i = \{F_{s,i}\}_{s\in S}$ of the space X such that $|S| = \mathfrak{m}$ and $\delta(F_{s,i}) \leq 1/i$ for every $s \in S$. Consider the subset $T \subset B(\mathfrak{m}) = S^{\aleph_0}$ consisting of all points $\{s_i\}$ such that $\bigcap_{i=1}^{\infty} F_{s_i,i} \neq \emptyset$ and assign the unique point in $\bigcap_{i=1}^{\infty} F_{s_i,i} \subset X$ to the point $\{s_i\} \in T$. Observe that $f^{-1}(x) = \prod_{i=1}^{\infty}\{s \in S : x \in F_{s,i}\}$ for every $x \in X$.

4.4.K (Dowker [1947a]). Let $\mathfrak{m}$ be an infinite cardinal number and S a set of cardinality $\mathfrak{m}$. Consider the set of all real-valued funtions x defined on S such that $|\{s \in S : x_s \neq \emptyset\}| \leq \aleph_0$ and $\sum_{s\in S} x_s^2 < \infty$, where x_s stands for the value of the function x at s; the formula $\rho(x,y) = \sqrt{\sum_{s\in S}(x_s - y_s)^2}$ defines a metric on this set. The metric space thus obtained does not depend (up to an isometry) on the choice of the set S; it is called the *Hilbert space of weight* $\mathfrak{m}$ and is denoted by $H(\mathfrak{m})$. Clearly, the Hilbert space defined in Example 4.1.7 coincides with $H(\aleph_0)$.

Prove that the Hilbert space $H(\mathfrak{m})$ is universal for all metrizable spaces of weight $\mathfrak{m} \geq \aleph_0$.

Remark. In Dowker's paper this result was obtained under the additional assumption of paracompactness, the fact that all metrizable spaces are paracompact was not known then. It has been established by Toruńczyk in [1981] (correction [1985]) that for each $\mathfrak{m} \geq \aleph_0$ the Hilbert space $H(\mathfrak{m})$ is homeomorphic to $[J(\mathfrak{m})]^{\aleph_0}$; this is a difficult and deep result.

4.5. Problems

Extending open and closed sets

4.5.1 (Kuratowski [1948a]). (a) Show that for every family $\{V_s\}_{s\in S}$ of open subsets of a subspace M of a metrizable space X there exists a family $\{U_s\}_{s\in S}$ of open subsets of X such that $V_s = M \cap U_s$ for every $s \in S$ and for every finite set $S_0 \subset S$ satisfying $\bigcap_{s\in S_0} V_s = \emptyset$ we have $\bigcap_{s\in S_0} U_s = \emptyset$.

Hint. Let $U_s = \{x \in X : \rho(x, V_s) < \rho(x, M \setminus V_s)\}$, where ρ is a metric on the space X.

(b) Show that for every family $\{A_s\}_{s\in S}$ of closed subsets of a subspace M of a metrizable space X there exists a family $\{F_s\}_{s\in S}$ of closed subsets of X such that $A_s = M \cap F_s$ for every $s \in S$ and for every finite set $S_0 \subset S$ satisfying $\bigcup_{s\in S_0} A_s = M$ we have $\bigcup_{s\in S_0} F_s = X$.

R^n is homogeneous with respect to countable dense subsets

4.5.2 (Brouwer [1913a]; implicity, Fréchet [1910]). Prove that for any two countable dense subsets A, B of Euclidean n-space R^n there exists a homeomorphism $f: R^n \to R^n$ such that $f(A) = B$ (cf. Exercise 4.3.H(a)).

Hint. A set $A \subset R^n$ is in *general position with respect to the coordinate axes* if for every pair $x = \{x_i\}$, $y = \{y_i\}$ of distinct points in A the difference $x_i - y_i$ does not vanish for $i = 1, 2, \ldots, n$. Prove first that for every countable set $A \subset R^n$ there exists a homeomorphism of R^n onto itself that maps A onto a set in general position with respect to the coordinate axes. Then show that the elements of two countably infinite sets in general position with respect to the coordinate axes can be arranged into two sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, where $x_j = \{x_i^j\}$ and $y_j = \{y_i^j\}$ for $j = 1, 2, \ldots$, in such a way that the differences $x_i^j - x_i^k$ and $y_i^j - y_i^k$ have the same sign for distinct $j, k = 1, 2, \ldots$ and $i = 1, 2, \ldots, n$.

The topology of pointwise convergence and metrics

4.5.3. (a) Observe that the space I^I with the topology of pointwise convergence is not metrizable.

Hint. Apply Exercise 2.1.C(a).

(b) Define two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ on a countable set X in such a way that the space $(X, \mathcal{O}_1)$ is metrizable, the space $(X, \mathcal{O}_2)$ is not metrizable, and yet $x = \lim x_i$ with respect to $\mathcal{O}_1$ if and only if $x = \lim x_i$ with respect to $\mathcal{O}_2$.

(c) (Fort [1951]) Prove that there is no metric ρ on the set I^I with the property that $\lim \rho(f, f_i) = 0$ if and only if $f = \lim f_i$ with respect to the topology of pointwise convergence.

Hint (Dugundji [1966]). Suppose that there exists a metric ρ on I^I which has the above property. For every $x \in I$ and $i = 1, 2, \ldots$ let $d_i(x) = \sup\{f(x) : f \in I^I \text{ and } \rho(f, f_0) < 1/i\}$, where $f_0 \in I^I$ is identically equal to zero, and check that $\lim d_i(x) = 0$ for every $x \in X$. Observe that for an integer i_0 there exist a sequence $x_1, x_2, \ldots$ of points of I and a sequence $U_1, U_2, \ldots$ of open subsets of I such that $d_{i_0}(x_i) < 1$ and $x_i \in U_i$ for $i = 1, 2, \ldots$ and $U_i \cap U_j = \emptyset$ whenever $i \neq j$.

For $i = 1, 2, \ldots$ choose a continuous function f_i of I to itself such that $f_i(I \setminus U_i) = \{0\}$ and $f_i(x_i) = 1$, and consider the sequence $f_1, f_2, \ldots$

Expanding and contracting mappings of metric spaces

4.5.4 (Freudenthal and Hurewicz [1936]). (a) Show that if (X, ρ) is a totally bounded space, then every mapping f of X to itself, satisfying the condition $\rho(x, y) \le \rho(f(x), f(y))$ for all $x, y \in X$, is an isometry (cf. Exercise 4.3.D).

Hint. Check that f has the property stated in the hint to Exercise 4.3.D(a) and apply this fact to the mapping $f \times f$.

(b) Show that if (X, ρ) is a totally bounded space, then every mapping f of X to itself, satisfying the condition $\rho(x, y) \ge \rho(f(x), f(y))$ for all $x, y \in X$ and such that $f(X)$ is dense in X, is an isometry.

Hint. Extend f to $\tilde{f}: \tilde{X} \to \tilde{X}$, where $\tilde{X}$ is the completion of X, and apply (a).

Every dense in itself completely metrizable space contains the Cantor set

4.5.5 (Hausdorff [1914]). (a) Show that every non-empty dense in itself completely metrizable space contains a subspace homeomorphic to the Cantor set.

Hint. Let ρ be a complete metric on a non-empty dense in itself space X. For every finite sequence $i_1, i_2, \ldots, i_k$ of zeros and ones define inductively a non-empty open set $V_{i_1 i_2 \ldots i_k} \subset X$ with diameter less than $1/k$ such that

$$\overline{V}_{i_1 i_2 \ldots i_k 0} \cap \overline{V}_{i_1 i_2 \ldots i_k 1} = \emptyset \quad \text{and} \quad \overline{V}_{i_1 i_2 \ldots i_k i} \subset V_{i_1 i_2 \ldots i_k} \quad \text{for } i = 0, 1.$$

(b) Show that every separable completely metrizable space either is countable or has cardinality $\mathfrak{c}$.

Hint. Apply Problem 1.7.11 and (a) or observe that this result follows directly from Problem 3.12.11(b).

A direct construction of the completion

4.5.6 (Hausdorff [1914]). Let (X, ρ) be a metric space. Define an equivalence relation E on the set of all Cauchy sequences in the space (X, ρ) letting $\{x_i\}E\{y_i\}$ whenever $\lim \rho(x_i, y_i) = 0$. Check that the formula $\tilde{\rho}([\{x_i\}], [\{y_i\}]) = \lim \rho(x_i, y_i)$ defines a metric on the set $\tilde{X}$ of all equivalence classes of E. To every point $x \in X$ assign the point $[\{x_i\}] \in \tilde{X}$, where $x_i = x$ for $i = 1, 2, \ldots$, and show that $(\tilde{X}, \tilde{\rho})$ is the completion of the space (X, ρ).

Borel sets II (see Problems 1.7.5 and 7.4.22)

4.5.7. A metrizable space X is *absolutely of the multiplicative (the additive) class* α, where $\alpha < \omega_1$, if for every homeomorphic embedding $h: X \to Y$ of X in a metrizable space Y the image $h(X)$ is a set of the multiplicative (the additive) class α in Y.

(a) (Lavrentieff [1924]) Prove that for every $\alpha > 0$ (every $\alpha > 1$) a metrizable space X is absolutely of the multiplicative (the additive) class α if and only if there exists a homeomorphic embedding $h: X \to Y$ of X in a completely metrizable space Y such that the image $h(X)$ is a set of the multiplicative (the additive) class α in Y.

Hint. Apply the Lavrentieff theorem.

(b) Observe that (a) does not hold either for the multiplicative class 0 or for the additive classes 0 and 1. Note that a metrizable space X is absolutely of the multiplicative (the additive) class 0 if and only if X is compact (empty).

(c) Show that a metrizable separable space X is absolutely of the additive class 1 if and only if X is σ-compact.

Remark. A. H. Stone proved in [1962] that a metrizable space X is absolutely of the additive class 1 if and only if X can be represented as a countable union of locally compact, or – equivalently – closed locally compact, subspaces.

4.5.8. (a) (Michael [1954]) Prove that in a metrizable space X the union of a locally finite family consisting of sets of the multiplicative (the additive) class α is a set of the same class.

Hint. Apply the Stone theorem and transfinite induction.

Remark. The same result (and the proof) holds if X is a perfectly normal paracompact space.

(b) (Montgomery [1935]) Show that if every point in a subset A of a metrizable space X has a neighbourhood U in the space X such that the intersection $A \cap U$ is a set of the multiplicative class $\alpha > 0$ (the additive class α) in the subspace U of X, then A is a set of the same class (cf. Problem 2.7.1).

Hint (Michael [1954]). Apply (a) and the fact that X has a σ-locally finite base.

(c) (Montgomery [1935]) Prove that if X and Y are metrizable spaces and $f: X \to Y$ is a measurable mapping of class α, then the graph $G(f)$ is a set of the multiplicative class α in the Cartesian product $X \times Y$.

Hint (Engelking [1967]). Show that for any base $\{B_s\}_{s\in S}$ for the space Y there exists a family $\{A_s\}_{s\in S}$ of open subsets of Y such that $(X \times Y) \setminus G(f) = \bigcup_{s\in S}(f^{-1}(A_s) \times B_s)$. Apply (a) and the fact that Y has a σ-locally finite base.

Dyadic spaces II (see Problem 3.12.12)

4.5.9. (a) (Sierpiński [1928]) Show that every non-empty closed subset A of the Cantor set C is a retract of C.

Hint (Halmos [1963]). Check that the metric σ on the set $D^{\aleph_0}$, defined by letting

$$\sigma(x, y) = \sum_{i=1}^{\infty} \frac{1}{10^i}|x_i - y_i| \quad \text{for} \quad x = \{x_i\},\ y = \{y_i\},$$

induces the topology of the Cartesian product. Observe that if $\sigma(x, y) = \sigma(x, z)$, then $y = z$ and deduce that for every $x \in D^{\aleph_0}$ there exists exactly one point $a \in A$ such that $\sigma(x, a) = \sigma(x, A)$.

(b) (Alexandroff [1927] (announcement [1925]), Hausdorff [1927]) Observe that from (a) and Theorem 3.2.2 it follows that every non-empty compact metrizable space is a continuous image of the Cantor set, i.e., is a dyadic space (cf. Theorem 3.2.2 and Problem 3.12.12(a)).

4.5.10 (Efimov [1963]). Show that every non-empty closed G_δ-set $F \subset D^{\mathfrak{m}}$ is a retract of $D^{\mathfrak{m}}$. Deduce that dyadicity is hereditary with respect to non-empty closed G_δ-sets.

Hint (Engelking and Pełczyński [1963]). Take a function $f: D^{\mathfrak{m}} \to R$ such that $F = f^{-1}(0)$, apply Exercise 3.2.H(a) and Problem 4.5.9(a).

4.5.11 (Efimov [1963a]). Show that every dyadic compactification cX of a metrizable space X is second-countable, i.e., is metrizable.

Hint (Engelking and Pełczyński [1963]). Observe first that the space X is separable, then apply Exercise 3.5.F and Problem 3.12.12(c).

One can also apply Problem 3.12.12(g) and Exercise 2.1.C(a).

Σ-products III (see Problems 2.7.14, 2.7.15, 3.12.24 and Exercise 3.10.D)

4.5.12. (a) (Gul'ko [1977], M. E. Rudin [1977]) Let $\Sigma(a)$ be a Σ-product of metrizable spaces $\{X_s\}_{s\in S}$, where $a = \{a_s\} \in \prod_{s\in S} X_s$. Prove that for every discrete family $\mathcal{F}$ of closed subsets of $\Sigma(a)$ there exists an open σ-locally finite cover $\mathcal{U}$ of $\Sigma(a)$ such that the closure of each member of $\mathcal{U}$ intersects at most one member of $\mathcal{F}$.

Hint. For each intersection $U = \Sigma(a) \cap \prod_{s\in S} U_s$ of $\Sigma(a)$ and a member $\prod_{s\in S} U_s$ of the canonical base $\mathcal{B}$ for the Cartesian product $\prod_{s\in S} X_s$ let $S(U) = \{s \in S : U_s \neq X_s\}$, for each

$x = \{x_s\} \in \Sigma(a)$ let $\{s \in S : x_s \neq a_s\} = \{s_{x,1}, s_{x,2}, \ldots\}$, and for each open subset U of $\Sigma(a)$ that intersects more than one member of $\mathcal{F}$ let $A(U) \subset U$ consists of two points chosen in two distinct members of $\mathcal{F}$. For every $s \in S$ define a sequence $\mathcal{U}_{s,1}, \mathcal{U}_{s,2}, \ldots$ of locally finite open covers of the space X_s such that each member of $\mathcal{U}_{s,i}$ has diameter less than $1/i$ and can be represented as the union of members of $\mathcal{U}_{s,i+1}$, and let $\mathcal{B}_i = \{U = \Sigma(a) \cap \prod_{s\in S} U_s : \emptyset \neq \prod_{s\in S} U_s \in \mathcal{B}$ and $U_s \in \mathcal{U}_{s,i}$ for $s \in S(U)\}$. Assume that $|\mathcal{F}| \geq 2$ and consider the family $\mathcal{S}$ of all finite sequences $U_0, U_1, \ldots, U_n$ of open subsets of $\Sigma(a)$, where $\Sigma(a) = U_0 \supset U_1 \supset \ldots \supset U_n$, $U_i \in \mathcal{B}_i$ for $i = 1, 2, \ldots, n$, each U_i with $i \leq n-1$ intersects more than one member of $\mathcal{F}$, and $S(U_i) = \{s_{x,j} : x \in A(U_0) \cup A(U_1) \cup \ldots \cup A(U_{i-1})$ and $j = 1, 2, \ldots, i\}$ for $i = 1, 2, \ldots, n$. Consider the family $\mathcal{U}_0$ consisting of the last terms U_n of sequences in $\mathcal{S}$ that intersect at most one member of $\mathcal{F}$ and show that $\mathcal{U}_0$ is a σ-locally finite open cover of $\Sigma(a)$. To prove that $\mathcal{U}_0$ is a cover, assume that $x = \{x_s\} \in \Sigma(a) \setminus \bigcup \mathcal{U}_0$, consider an infinite sequence $U_0, U_1, U_2, \ldots$ such that $x \in U_i$ for $i = 0, 1, 2, \ldots$ and $U_0, U_1, \ldots, U_n$ belongs to $\mathcal{S}$ for $n = 0, 1, 2, \ldots$, define $x' = \{x'_s\} \in \Sigma(a)$ by letting $x'_s = x_s$ for $s \in \bigcup_{i=0}^{\infty} S(U_i)$ and $x'_s = a_s$ otherwise, and deduce a contradiction by showing that $x' \in \overline{\{a_0, a_1, a_2, \ldots\}}$, where $a_i \in A(U_i) \cap \bigcup\{F \in \mathcal{F} : x' \notin F\}$; then show by induction that the family of the last terms of sequences in $\mathcal{S}$ which have length $n+1$ is locally finite for $n = 0, 1, 2, \ldots$ To obtain $\mathcal{U}$, observe that the members of $\mathcal{U}_0$ are functionally open and replace each of them by countably many appropriate open sets.

(b) (Gul'ko [1977], M. E. Rudin [1977]; for completely metrizable spaces, Corson [1959]) Prove that any Σ-product of metrizable spaces is normal.

Hint. Let $\mathcal{F} = \{F_1, F_2\}$, where F_1, F_2 are disjoint closed subsets of the Σ-product, apply (a) and Lemma 1.5.15.

Remark. A similar argument shows that any Σ-product of metrizable spaces is collectionwise normal (see Lemma 5.1.10).

Extending closed and open mappings

4.5.13. (a) (Vaĭnšteĭn [1952]) A continuous mapping $f: X \to Y$ is *closed at a point* $x \in Y$ if for every open set $U \subset X$ which contains $f^{-1}(y)$ there exists in Y a neighbourhood V of y such that $f^{-1}(V) \subset U$; the set of all points of the space Y at which a mapping $f: X \to Y$ is closed is denoted by $C(f)$.

Show that $C(f) \cap (\overline{f(X)} \setminus f(X)) = \emptyset$ and that $B \cap C(f) \subset C(f_B)$ for every set $B \subset Y$. Observe that $C(f|A) \subset C(f)$ for every continuous mapping $f: X \to Y$ of a normal space X to a T_1-space Y and every dense set $A \subset X$.

(b) (Engelking [1971]) Prove that for every continuous mapping $f: X \to Y$ of a completely metrizable space X to a first-countable Hausdorff space Y, the set $C(f)$ of all points at which f is closed is a G_δ-set in Y.

Hint. Observe that for every $y \in C(f)$ the set $\mathrm{Fr}\, f^{-1}(y)$ is compact. Take a complete metric ρ on the space X and for $i = 1, 2, \ldots$ consider the set W_i consisting of all points $y \in Y$ which have a neighbourhood W with the property that every set $K \subset f^{-1}(W)$, such that $\rho(x, x') \geq 1/i$ and $f(x) \neq f(x')$ for any pair x, x' of distinct points of K, is finite; show that $C(f) = \bigcap_{i=1}^{\infty} W_i$.

(c) (Vaĭnšteĭn [1952] (announcement [1947])) Prove that if $f: X \to Y$ is a continuous mapping of a completely metrizable space X to a first-countable Hausdorff space Y, then for every set $A \subset X$ such that the restriction $f|A: A \to f(A)$ is closed there exists a G_δ-set $B \subset X$ such that $A \subset B$ and the restriction $f|B: B \to f(B)$ is closed. Observe that if the restriction $f|A$ is perfect, then there exists a G_δ-set $B \subset X$ such that $A \subset B$ and the restriction $f|B$ is perfect.

Hint. Apply (a) and (b).

(d) Let X and Y be completely metrizable spaces and let $A \subset X$ and $C \subset Y$ be arbitrary subspaces. Deduce from (c) that every closed mapping $f: A \to C$ of A onto C is extendable to a closed mapping $F: B \to D$ of B onto D, where $A \subset B \subset X$, $C \subset D \subset Y$ and B is a G_δ-set in X. Observe that if f is a perfect mapping, then there exist such B and D that $F: B \to D$ is a perfect mapping.

Remark. It follows from part (e) below and from Theorems 4.3.23 and 4.3.24 that D is a G_δ-set in Y.

(e) (Vaĭnšteĭn [1952] (announcement [1947])) Prove that if a metrizable space Y is a continuous image of a completely metrizable space X under a closed mapping, then Y is completely metrizable.

Hint. Apply (a) and (b).

One can also apply Theorems 4.3.26 and 3.9.10 and restrict the closed mapping to a perfect mapping as in the proof of Theorem 4.4.17.

4.5.14. (a) (Mazurkiewicz [1932]) Prove that if $f: X \to Y$ is a continuous mapping of a completely metrizable separable space X to a metrizable space Y, then for every set $A \subset X$ such that the restriction $f|A: A \to f(A)$ is open there exists a G_δ-set $B \subset X$ such that $A \subset B$ and the restriction $f|B: B \to f(B)$ is open.

Hint (Hausdorff [1934]). Without loss of generality one can assume that $\overline{A} = X$ and $\overline{C} = Y$, where $C = f(A)$. Take a complete metric ρ on the space X and any metric σ on the space Y; for a fixed countable base $\mathcal{B}$ for the space X let $\mathcal{B}_k = \{U \in \mathcal{B} : \delta(U) < 1/k \text{ and } \delta(f(U)) < 1/k\}$. For every finite sequence $i_1, i_2, \ldots, i_k$ of natural numbers define inductively a set $U_{i_1 i_2 \ldots i_k} \in \mathcal{B}_k$ such that

$$X = \bigcup_{i=1}^{\infty} U_i, \quad U_{i_1 i_2 \ldots i_k} = \bigcup_{i=1}^{\infty} U_{i_1 i_2 \ldots i_k i} \quad \text{and} \quad \overline{U}_{i_1 i_2 \ldots i_k i} \subset U_{i_1 i_2, \ldots i_k} \quad \text{for } i = 1, 2, \ldots$$

Choose open sets $V_{i_1 i_2 \ldots i_k} \subset Y$ with diameter less than $1/k$ satisfying

$$f(A \cap U_{i_1 i_2 \ldots i_k}) = C \cap V_{i_1 i_2 \ldots i_k} \text{ and } V_{i_1 i_2 \ldots i_k i} \subset V_{i_1 i_2 \ldots i_k} \text{ for } i = 1, 2, \ldots;$$

let

$$W = \bigcup_{i=1}^{\infty} V_i \quad \text{and} \quad W_{i_1 i_2 \ldots i_k} = \bigcup_{i=1}^{\infty} V_{i_1 i_2 \ldots i_k i}$$

and observe that

$$E = (Y \setminus W) \cup \bigcup_{k=1}^{\infty} \bigcup_{i_1, i_2, \ldots, i_k = 1}^{\infty} (V_{i_1 i_2 \ldots i_k} \setminus W_{i_1 i_2 \ldots i_k})$$

is an F_σ-set disjoint from C. Consider the set

$$B = f^{-1}(Y \setminus E) \cap [\bigcap_{k=1}^{\infty} \bigcup_{i_1, i_2, \ldots, i_k = 1}^{\infty} U_{i_1 i_2 \ldots i_k} \cap f^{-1}(V_{i_1 i_2 \ldots i_k})].$$

(b) Let X be a completely metrizable separable space, Y a completely metrizable space and let $A \subset X$ and $C \subset Y$ be arbitrary subspaces. Deduce from (a) that every open mapping $f: A \to C$ of A onto C is extendable to an open mapping $F: B \to D$ of B onto D, where $A \subset B \subset X$, $C \subset D \subset Y$ and B is a G_δ-set in X.

Remark. It follows from Problem 5.5.8(d) and from Theorems 4.3.23 and 4.3.24 that D is a G_δ-set in Y.

(c) (R. Pol [1981]) Prove that the assumption of separability in parts (a) and (b) is essential.

Hint. Let Q denote the set of all rational numbers, $P = R \setminus Q$, $S = D(\mathfrak{c})$, and $\{C_s\}_{s \in S}$ be the family of all countable dense subsets of P. Consider the subspace $X = \{(s,t) \in S \times R : t \in R \setminus C_s\}$ of $S \times R$ and the mapping $f: X \to R$ defined by letting $f(s,t) = t$. Note that the restriction $f|A$, where $A = S \times Q$, is open, consider any G_δ-set $B \subset X$ such that $A \subset B$, observe that there exists a G_δ-set $C \subset R$ such that $Q \subset C \subset f(B)$, and consider the set $f(B \cap [\{s\} \times (R \setminus C_s)])$, where $C_s \subset C$.

(d) Prove that if $X \to Y$ is a continuous mapping of a metrizable space X to a metrizable space Y, then for every set $A \subset X$ such that the restriction $f|A: A \to f(A)$ is open and $f(A)$ is a G_δ-set in Y there exists a G_δ-set $B \subset X$ such that $A \subset B$ and the restriction $f|B: B \to f(B)$ is open.

Hint. One can assume that $f(A) = Y$.

Normality and related properties in Cartesian products III (see Problems 2.7.16, 3.12.15, 3.12.20, 5.5.5, 5.5.6, 5.5.18, 5.5.19 and Exercise 2.3.E)

4.5.15. (a) (Katětov [1948]) Show that a compact space X is metrizable if and only if the Cartesian product $X \times X \times X$ is hereditarily normal (cf. Exercise 4.2.B).

Hint. Apply Problem 2.7.16(a) and Exercise 4.2.B.

Remark. Nyikos proved in [1977] that under additional set-theoretic assumptions there exist a compact non-metrizable space X such that the Cartesian product $X \times X$ is hereditarily normal.

(b) (Chaber [1976]) Show that the assumption of compactness in part (a) can be relaxed to countable compactness.

Hint. See Problem 3.12.23(e).

4.5.16. (a) (Michael [1953]) Prove that the Cartesian product $X \times Y$ of a perfect space X and a metrizable space Y is a perfect space.

Hint. Apply the fact that Y has a σ-locally finite base.

(b) (Morita [1963]) Prove that the Cartesian product $X \times Y$ of a perfectly normal space X and a metrizable space Y is a perfectly normal space.

Hint. Apply Exercise 1.5.K.

Remark. As shown in Example 5.1.32, normality and hereditary normality are not invariant under Cartesian multiplication by a metrizable space.

(c) (Bourbaki [1958], Dieudonné [1958]; announcement A. H. Stone [1948]) Prove that the Cartesian product $X \times Y$ of a countably compact normal space X and a metrizable space Y is a normal space.

Hint. Modify the proof of Lemma 5.2.7; apply Theorem 3.10.7.

Remark. The same result (and the proof) holds if Y is a first-countable paracompact space (cf. Example 5.1.40).

(d) (Willard [1971]) Prove that the Cartesian product $X \times Y$ of a hereditarily Lindelöf space X and a separable metrizable space Y is a hereditarily Lindelöf space.

Hint. Apply (a) and Corollary 3.8.10.

Remark. Morita gave in [1964] a characterization of topological spaces whose Cartesian product with every metric space is normal.

Invariance of metrizability under open and quotient mappings

4.5.17 (A. H. Stone [1956]). Prove that if $f: X \to Y$ is an open mapping of a locally separable metrizable space X onto a regular space Y and all fibers of f are separable, then the space Y is metrizable (cf. Problem 5.5.8(d)).

Hint. Let $X = \bigoplus_{s \in S} X_s$, where all spaces X_s are separable (cf. Exercise 4.4.F(c)). Define an equivalence relation E on the set S letting sEs' if there exists a finite sequence $s_0, s_1, \ldots, s_k$ such that $s_0 = s, s_k = s'$ and $f(X_{s_i}) \cap f(X_{s_{i+1}}) \neq \emptyset$ for $i = 0, 1, \ldots, k-1$. Check that unions of the X_s's corresponding to an equivalence class of E are separable subspaces of X whose images under f are open and pairwise disjoint.

4.5.18 (Čoban [1966]; for separable X, A. H. Stone [1956]). Prove that if $f: X \to Y$ is a quotient mapping of a metrizable space X onto a first-countable separable regular space Y and all fibers of f are separable, then the space Y is metrizable.

Hint. Show that for any countable dense set $B \subset Y$, the subspace $A = \overline{f^{-1}(B)} \subset X$ is separable. Take a countable base $\mathcal{B}$ for the space A which is closed with respect to finite unions and prove that the family $\{\operatorname{Int} \overline{f(U)} : U \in \mathcal{B}\}$ is a base for the space Y.

Closed images of metrizable spaces

4.5.19 (Lašnev [1965]). Prove that if $f: X \to Y$ is a closed mapping of a metrizable space X onto a space Y, then $Y = Y_0 \cup \bigcup_{i=1}^{\infty} Y_i$, where $f^{-1}(y)$ is compact for each $y \in Y_0$ and Y_i is a discrete closed subspace of Y for $i = 1, 2, \ldots$ (this fact is known as *Lašnev's theorem*).

Hint. For $i = 1, 2, \ldots$ let $Y_i = \{y \in Y :$ there is no $y' \neq y$ such that $f^{-1}(y) \subset B(f^{-1}(y'), 1/i)\}$. To show that $f^{-1}(y_0)$ is compact for each $y_0 \in Y_0 = Y \setminus \bigcup_{i=1}^{\infty} Y_i$, choose a sequence $y_1, y_2, \ldots$, where $f^{-1}(y_0) \subset B(f^{-1}(y_i), 1/i)$ for $i = 1, 2, \ldots$ and $y_i \neq y_j$ for $i \neq j$, observe that $f^{-1}(y_0) \subset \overline{\bigcup_{i=1}^{\infty} f^{-1}(y_i)}$ and apply Lemma 4.4.16 to the restriction $f_Z: f^{-1}(Z) \to Z$ of f to the subspace $Z = \{y_0, y_1, \ldots\}$ of Y.

Remark. For generalizations of Lašnev's theorem see Burke [1980] (this is a survey of results on closed mappings) and Chaber [1983].

Extending functions and metrics

4.5.20. (a) (Dugundji [1951]; for separable X, Borsuk [1933]) Let X be a metrizable space and M a closed subspace of X. Prove that one can assign to every continuous function

$f: M \to R$ a continuous extension $e(f) = F: X \to R$ in such a way that

$$\sup_{x \in X} |[e(f)](x)| = \sup_{x \in M} |f(x)| \quad \text{for every } f: M \to R$$

and

$$e(t_1 f_1 + t_2 f_2) = t_1 e(f_1) + t_2 e(f_2) \quad \text{for all } f_1, f_2: M \to R \text{ and real numbers } t_1, t_2.$$

Hint (Dugundji [1951], Arens [1952]). Take a metric ρ on the space X and a locally finite open refinement $\{V_s\}_{s\in S}$ of the cover $\{B(x, \rho(x, M)/4)\}_{x \in X \setminus M}$ of the subspace $X \setminus M$. For every $s \in S$ choose a point $x_s \in X \setminus M$ such that $V_s \subset B(x_s, \rho(x_s, M)/4)$ and a point $a_s \in M$ such that $\rho(a_s, x_s) < 5\rho(x_s, M)/4$. Applying Theorem 1.5.18 define functions $g_s: X \setminus M \to I$ such that $g_s(X \setminus V_s) \subset \{0\}$ for every $s \in S$ and $\sum_{s\in S} g_s(x) = 1$ for every $x \in X \setminus M$; let

$$[e(f)](x) = \begin{cases} f(x) & \text{for } x \in M, \\ \sum_{s \in S} f(a_s) \cdot g_s(x) & \text{for } x \in X \setminus M. \end{cases}$$

In the proof of continuity of $e(f)$ observe that for every $a \in M$ and any $x \in V_s$ the inequality $\rho(a, a_s) < 3\rho(a, x)$ holds.

(b) (Dugundji [1951]) Let X be a metrizable space and M a closed subspace of X. Prove that for every continuous mapping $f: M \to C^*(Y)$, where $C^*(Y)$ is the ring of all bounded continuous real-valued functions defined on a topological space Y with the topology induced by the metric $\hat{\sigma}$ defined by formula (7) in Section 4.2 for the natural metric σ on the real line, there exists a continuous extension $F: X \to C^*(Y)$ of the mapping f over the space X.

Hint. Follow the construction sketched in the hint to part (a) and define F by the formula defining $e(f)$ there.

Remark. The same construction yields a linear extension of functions with values in any locally convex linear topological space.

(c) (Gęba and Semadeni [1960]) Show that there is no extension operator e assigning to every continuous function $f: \beta N \setminus N \to R$ a continuous extension $e(f): \beta N \to R$ in a such a way that

$$\sup_{x \in \beta N} |[e(f)](x)| = \sup_{x \in \beta N \setminus N} |f(x)| \quad \text{for every } f: \beta N \setminus N \to R$$

and

$$e(f_1 + f_2) = e(f_1) + e(f_2) \quad \text{for all } f_1, f_2: \beta N \setminus N \to R.$$

Hint. Consider the family $\{U_t\}_{t\in I}$ of open subsets of $\beta N \setminus N$ defined in Example 3.6.18 and the family of functions $\{f_t\}_{t \in I}$, where $f_t: \beta N \setminus N \to R$ is defined by the formula

$$f_t(x) = \begin{cases} 1, & \text{if } x \in U_t, \\ 0, & \text{if } x \in (\beta N \setminus N) \setminus U_t. \end{cases}$$

4.5.21. (a) (Hausdorff [1938]) Let M be a closed subspace of a metrizable space X and let $f: M \to L$ be a continuous mapping of M onto a metric space L. Prove that the space L can be isometrically embedded as a closed subset in a metric space Y in such a way that the mapping f is extendable to a continuous mapping $F: X \to Y$ such that the restriction $F|X \setminus M$ is a homeomorphism of $X \setminus M$ onto $Y \setminus L$.

Hint (Kuratowski [1938], Arens [1952]). Extend the composition $f': M \to C^*(L)$ of the mapping f and the isometrical embedding of L in $C^*(L)$ defined in the proof of Theorem 4.3.14 to a continuous mapping $F': X \to C^*(L)$ (cf. Problem 4.5.20(b)). Consider the Cartesian product $Z = C^*(L) \times R \times C^*(X)$ and the mapping $F: X \to Z$ defined by letting $F(x) = (F'(x), \rho(x, M), \rho(x, M) \cdot f_x)$, where ρ is a bounded metric on the space X and $f_x(y) = \rho(x, y)$. Check that one can take as Y the subspace $F(X) \subset Z$.

(b) Verify that if the mapping f in (a) is a homeomorphism, then the extension F defined in the above hint also is a homeomorphism.

Hint. Show that if $\{V_s\}_{s \in S}$ is the open cover of $X \setminus M$ and $\{a_s : s \in S\}$ the subset of M described in the hint to Problem 4.5.20(a), then for every $a \in M$ and any $x \in V_s$ we have the inequality $\rho(a, x) < \rho(a, a_s) + 2\rho(x, M)$.

(c) (Hausdorff [1930]) Prove that if M is a closed subspace of a merizable space X, then every metric on the subspace M is extendable to a metric on the space X.

(d) Note that (c) solves immediately (c) and (d) in Exercise 4.3.E.

(e) Show that if M is a closed subspace of a space X metrizable by a totally bounded metric, then every totally bounded metric on the subspace M is extendable to a totally bounded metric on the space X.

(f) (Bacon [1968]) Show that if M is a closed subspace of a completely metrizable space X, then every complete metric on the subspace M is extendable to a complete metric on the space X.

Hint. Let ρ be a complete metric on the subspace M; extend ρ to a metric $\overline{\rho}$ on the space X and take the completion $(\tilde{X}, \tilde{\rho})$ of $(X, \overline{\rho})$. Represent the difference $\tilde{X} \setminus X$ as a countable union $\bigcup_{i=1}^{\infty} F_i$ of closed subsets of $\tilde{X}$ and consider the metric σ on X defined by the formula

$$\sigma(x, y) = \overline{\rho}(x, y) + \sum_{i=1}^{\infty} \frac{1}{2^i} \min\left(1, \left|\frac{\tilde{\rho}(x, M)}{\tilde{\rho}(x, F_i)} - \frac{\tilde{\rho}(y, M)}{\tilde{\rho}(y, F_i)}\right|\right).$$

The space R^I is metrically universal for all separable metric spaces

4.5.22. (a) Show that for every countable family $\{f_i\}_{i=1}^{\infty}$ of bounded continuous real-valued functions defined on a separable metrizable space X there exists a metrizable compactification cX of the space X such that all functions f_i are continuously extendable over cX.

Hint. See Exercise 4.3.E(a).

(b) Observe that for every compact metric space (X, ρ) the space $(R^X, \hat{\sigma})$, where σ is the natural metric on the real line, is isometric to a subspace of the space $(R^C, \hat{\sigma})$, where C is the Cantor set.

Hint. Apply Problem 4.5.9(b).

(c) (Banach and Mazur, cited in Banach [1932]) Prove that every separable metric space (X, ρ) is isometric to a subspace of the space $(R^I, \hat{\sigma})$, where σ is the natural metric on the real line.

Hint. Observe first that the space $(R^C, \hat{\sigma})$ is isometric to a subspace of $(R^I, \hat{\sigma})$. Then take a countable dense subset $A = \{a_1, a_2, \ldots\}$ of the space X and consider the family $\{f_i\}_{i=1}^{\infty}$ of real-valued functions on X defined by letting $f_i(x) = \rho(x, a_i) - \rho(x, a)$, where $a \in X$ is a fixed point. Finally, apply (a), (b) and Theorem 4.3.17.

Remark. A metric space which is metrically universal for all separable metric spaces was first described by Urysohn in [1927].

Spaces of closed subsets III (see Problems 2.7.20, 3.12.27, 6.3.22, 8.5.13(i) and 8.5.16)

4.5.23. The *Hausdorff metric* on the family of all bounded, non-empty closed subsets of a metric space (X, ρ) is defined by letting

$$\rho_H(A, B) = \max\{\sup_{a\in A} \rho(a, B)\,,\ \sup_{b\in B} \rho(b, A)\,\}.$$

(a) (Hausdorff [1914], Michael [1951]) Check that ρ_H is a metric on the family of all bounded, non-empty closed subsets of a metric space (X, ρ); verify that (X, ρ) is isometric to a closed subspace of the metric space thus obtained. Give an example of two equivalent totally bounded metrics ρ and σ on a space X such that the topologies on 2^X induced by ρ_H and σ_H are different. Give an example of a bounded separable metric space (X, ρ) such that the Vietoris topology on 2^X and the topology induced by the metric ρ_H are incomparable. Show that the topology induced by the Hausdorff metric ρ_H on the family $\mathcal{Z}(X)$ of all non-empty compact subspaces of a metric space (X, ρ) coincides with the Vietoris topology on $\mathcal{Z}(X)$.

Remark. Pompéiu in [1905] investigated convergence induced by the metric $\rho_P(A, B) = \sup_{a\in A} \rho(a, B) + \sup_{b\in B} \rho(b, A)$ on the family of all bounded, non-empty closed subsets of the plane.

(b) Verify that if the space (X, ρ) is totally bounded, then the space $(2^X, \rho_H)$ also is totally bounded. Give an example of a bounded separable metric space (X, ρ) such that the space $(2^X, \rho_H)$ is not separable.

(c) (Hahn [1932]) Show that if the space (X, ρ) is complete, then the space of all bouned, non-empty closed subsets of the space (X, ρ) with the Hausdorff metric ρ_H also is complete.

(d) (Kuratowski [1956]) Show that if the space (X, ρ) is complete, then the space $(\mathcal{Z}(X), \rho_H)$ also is complete.

(e) Observe that if X is a T_1-space, then the exponential space 2^X is metrizable if and only if the space X is metrizable and compact; note that the space $\mathcal{Z}(X)$ is metrizable if and only if the space X is metrizable.

Hint. Apply Problem 2.7.20(f).

(f) (Kuratowski [1948]) Let (X, ρ) be a metric space and a an arbitrary point of X; to every bounded, non-empty closed subset A of X assign the function $f_A \in R^X$ defined by letting $f_A(x) = \rho(x, A) - \rho(x, a)$. Show that such assignment defines an isometry of the space of all bounded, non-empty closed subsets of (X, ρ) with the Hausdorff metric ρ_H, to the space of all bounded continuous real-valued functions on X with the metric $\hat{\sigma}$ defined by formula (7) in Section 4.2 for the natural metric σ on the real line.

Deduce that for every compact metrizable space X, the exponential space 2^X with the Vietoris topology is embeddable in the function space R^X with the compact-open topology. Observe that assumptions of metrizability and compactness of X are both essential (cf. Problem 3.12.27(j)).

Chapter 5
Paracompact spaces

Paracompact spaces simultaneously generalize both compact spaces and metrizable spaces; although defined much later than the two latter classes, paracompact spaces quickly became popular among topologists and analysts, and now are considered to be one of the most important classes of topological spaces. Due to the introduction of paracompactness, many theorems in topology and analysis were generalized and many proofs were simplified. It also turned out that the notion of a locally finite family and notions related to it are very efficient and natural tools for studying topological spaces.

Section 5.1 is devoted to paracompact spaces. We start with three theorems containing various characterizations of paracompactness (the characterizations in terms of partitions of unity is particularly important for analysis). Then we prove that paracompact spaces have the property of collectionwise normality, which is much stronger than just normality, and we give a few examples. In the second part of the section we study operations on paracompact spaces and the behaviour of this class of spaces under mappings. The section concludes with the Tamano theorem establishing an interesting external characterization of paracompactness.

In Section 5.2 we study the class of countably paracompact spaces. The theorems in that section contain various characterizations of countable paracompactness.

Section 5.3 is devoted to weakly and strongly paracompact spaces. Like the class of countably paracompact spaces, those two classes are of much less importance than the class of paracompact spaces; however, they do play a role in dimension theory and in algebraic topology. Among the theorems in that section, the most important are the Nagami-Michael theorem stating that every collectionwise normal weakly paracompact space is paracompact, and the Worrell theorem establishing invariance of weak paracompactness under closed mappings.

The last section is a continuation of Section 4.4; five further metrization theorems are given there.

5.1 Paracompact spaces

The notion of a locally finite family of sets, introduced in Chapter 1, leads to the definition of an important class of topological spaces, the paracompact spaces. A topological space X is called a *paracompact space* if X is a Hausdorff space and every open cover of X has a locally finite open refinement.

Let us observe that, in contrast to the definition of compactness, in the definition of paracompactness the term "refinement" cannot be replaced by the term "subcover". In fact, one readily sees that every discrete space is paracompact – the cover consisting of all one-point sets is open and locally finite and refines any other cover of the space – and yet the open cover $\{N \cap [1, i]\}_{i=1}^{\infty}$ of the space of natural numbers N has no locally finite subcover (cf. Exercise 5.1.A(d)).

The definition of paracompactness yields

5.1.1. THEOREM. *Every compact space is paracompact.* ■

Using the notion of paracompactness, Theorems 3.8.11 and 4.4.1 can be stated as follows:

5.1.2. THEOREM. *Every Lindelöf space is paracompact.* ■

5.1.3. THEOREM. *Every metrizable space is paracompact.* ■

The reader can easily deduce from Theorem 5.1.12 and Remark 5.1.7 that the existence of open refinements which are both locally finite and σ-discrete, established for metrizable spaces in Theorem 4.4.1, is only formally stronger than paracompactness.

5.1.4. LEMMA. *Let X be a paracompact space and A, B a pair of closed subsets of X. If for every $x \in B$ there exist open sets U_x, V_x such that $A \subset U_x$, $x \in V_x$ and $U_x \cap V_x = \emptyset$, then there also exist open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

PROOF. The family $\{V_x\}_{x \in B} \cup \{X \setminus B\}$ is an open cover of the space X, so that it has a locally finite open refinement $\{W_s\}_{s \in S}$. Letting $S_0 = \{s \in S : W_s \cap B \neq \emptyset\}$ we have

$$A \cap \overline{W}_s = \emptyset \text{ for every } s \in S_0 \quad \text{and} \quad B \subset \bigcup_{s \in S_0} W_s.$$

By virtue of Theorem 1.1.11 the set $U = X \setminus \bigcup_{s \in S_0} \overline{W}_s$ is open; one readily sees that U and $V = \bigcup_{s \in S_0} W_s$ have all the required properties. ■

5.1.5. THEOREM. *Every paracompact space is normal.*

PROOF. Substituting one-point sets for A in the above lemma, we see that every paracompact space is regular; using this fact and applying the lemma again we obtain the theorem. ■

Let us observe that the last theorem is a common generalization of Theorems 1.5.16, 3.1.9 and 3.8.2.

A family $\{f_s\}_{s \in S}$ of continuous functions from a space X to the closed unit interval I is called a *partition of unity* on the space X if $\sum_{s \in S} f_s(x) = 1$ for every $x \in X$. The last equality means that for each $x_0 \in X$ only countably many functions f_s do not vanish at x_0 and that the series $\sum_{i=1}^{\infty} f_{s_i}(x_0)$, where $\{s_1, s_2, \ldots\} = \{s \in S : f_s(x_0) \neq 0\}$, converges to 1; since the sequence is absolutely convergent, the arrangement of terms does not matter and convergence to 1 means that 1 is the least upper bound of the set consisting of all numbers of the form $f_{s_1}(x_0) + f_{s_2}(x_0) + \ldots + f_{s_k}(x_0)$, where $s_1, s_2, \ldots, s_k \in S$ and $k = 1, 2, \ldots$

We say that a partition of unity $\{f_s\}_{s \in S}$ on a space X is *locally finite* if the cover $\{f_s^{-1}((0, 1])\}_{s \in S}$ of the space X is locally finite. This means that for each $x_0 \in X$ there exists a neighbourhood U_0 of the point x_0 and a finite set $S_0 = \{s_1, s_2, \ldots, s_k\} \subset S$ such that for every $x \in U_0$ we have $f_s(x) = 0$ whenever $s \in S \setminus S_0$, and $f_{s_1}(x) + f_{s_2}(x) + \ldots + f_{s_k}(x) = 1$.

A partition of unity $\{f_s\}_{s \in S}$ on a space X is *subordinated to a cover $\mathcal{A}$* of X if the cover $\{f_s^{-1}((0, 1])\}_{s \in S}$ of the space X is a refinement of $\mathcal{A}$.

Our next theorem contains two characterizations of paracompactness in terms of partitions of unity; these characterizations are very useful not only in topology but also in analysis and differential geometry. The theorem will be preceded by two lemmas. The first one will also be applied later and is stated in a form slightly more general than needed here; in our theorem instead of using this lemma one could apply – less elementary – Theorem 1.5.18.

5.1.6. LEMMA. *If every open cover of a regular space X has a locally finite refinement (consisting of arbitrary sets), then for every open cover $\{U_s\}_{s\in S}$ of the space X there exists a closed locally finite cover $\{F_s\}_{s\in S}$ of X such that $F_s \subset U_s$ for every $s \in S$.*

PROOF. By regularity of X there exists an open cover $\mathcal{W}$ of the space X such that $\{\overline{W} : W \in \mathcal{W}\}$ is a refinement of $\{U_s\}_{s\in S}$. Take a locally finite refinement $\{A_t\}_{t\in T}$ of the cover $\mathcal{W}$, for every $t \in T$ choose an $s(t) \in S$ such that $\overline{A}_t \subset U_{s(t)}$, and let $F_s = \bigcup_{s(t)=s} \overline{A}_t$. From Theorems 1.1.11 and 1.1.13 it follows readily that $\{F_s\}_{s\in S}$ is a closed locally finite cover of X and the definition of the F_s's implies that $F_s \subset U_s$ for every $s \in S$.■

5.1.7. REMARK. Let us note that if the cover $\{A_t\}_{t\in T}$ in the last proof is open, then the sets $V_s = \bigcup_{s(t)=s} A_t$ are open and $\overline{V}_s = F_s$. Hence, for every open cover $\{U_s\}_{s\in S}$ of a paracompact space there exists a locally finite open cover $\{V_s\}_{s\in S}$ such that $\overline{V}_s \subset U_s$ for every $s \in S$.

5.1.8. LEMMA. *If for an open cover $\mathcal{U}$ of a space X there exists a partition of unity $\{f_s\}_{s\in S}$ subordinated to it, then $\mathcal{U}$ has an open localy finite refinement.*

PROOF. To begin, let us observe that for every continuous function $g : X \to I$ and any point $x_0 \in X$ satisfying $g(x_0) > 0$ there exists a neighbourhood U_0 of the point x_0 and a finite set $S_0 \subset S$ such that

$$f_s(x) < g(x) \quad \text{for } x \in U_0 \text{ and } s \in S \setminus S_0. \tag{1}$$

Indeed, one easily verifies that any set $S_0 = \{s_1, s_2, \ldots, s_k\} \subset S$ such that

$$1 - \sum_{i=1}^{k} f_{s_i}(x_0) < g(x_0)$$

and the open set $U_0 = \{x \in X : 1 - \sum_{i=1}^{k} f_{s_i}(x) < g(x)\}$ satisfy (1).

For every $x \in X$ there exists an $s(x) \in S$ such that $f_{s(x)}(x) > 0$. Letting $g = f_{s(x)}$ in the above observation we infer from 2.1.12 that the formula $f(x) = \sup_{s\in S} f_s(x)$ defines a continuous function $f : X \to (0,1]$. For every $s \in S$ the set

$$V_s = \{x \in X : f_s(x) > \frac{1}{2} f(x)\}$$

is open, and the family $\mathcal{V} = \{V_s\}_{s\in S}$ is a refinement of $\mathcal{U}$. Letting $g = \frac{1}{2}f$ in our original observation we infer that $\mathcal{V}$ is a locally finite family.■

5.1.9. THEOREM. *For every T_1-space X the following conditions are equivalent:*

(i) *The space X is paracompact.*

(ii) *Every open cover of the space X has a locally finite partition of unity subordinated to it.*

(iii) *Every open cover of the space X has a partition of unity subordinated to it.*

PROOF. Assume that X is paracompact and consider an open cover $\mathcal{A}$ of X. Let $\mathcal{U} = \{U_s\}_{s\in S}$ be a locally finite open refinement of $\mathcal{A}$. By virtue of Lemma 5.1.6 there exists a closed cover $\{F_s\}_{s\in S}$ of the space X such that $F_s \subset U_s$ for every $s \in S$. From Urysohn's lemma it follows that for every $s \in S$ one can find a continuous function $g_s : X \to I$ such that $g_s(x) = 0$ for $x \in X \setminus U_s$ and $g_s(x) = 1$ for $x \in F_s$. The family $\mathcal{U}$ being locally finite, by letting $g(x) = \sum_{s\in S} g_s(x)$ we define a continuous function $g : X \to R$. One easily sees that $\{f_s\}_{s\in S}$, where $f_s = g_s/g$, is a locally finite partition of unity subordinated to $\mathcal{A}$. Hence the implication (i)$\Rightarrow$(ii) is established.

Since the implication (ii)$\Rightarrow$(iii) is obvious, to conclude the proof it suffices to show that (iii)$\Rightarrow$(i), which – by virtue of Lemma 5.1.8 – reduces to showing that every T_1-space X satisfying (iii) is a Hausdorff space. Consider a pair of distinct poins $x_1, x_2 \in X$. The open cover $\mathcal{U} = \{X \setminus \{x_1\}, X \setminus \{x_2\}\}$ of the space X has a partition of unity $\{f_s\}_{s\in S}$ subordinated to it. Take an $s_0 \in S$ such that $f_{s_0}(x_1) = a > 0$; since the set $f_{s_0}^{-1}((0,1])$ is contained in $X \setminus \{x_2\}$, we have $f_{s_0}(x_2) = 0$. The open sets $U_1 = f_{s_0}^{-1}((a/2,1])$ and $U_2 = f_{s_0}^{-1}([0,a/2))$ are disjoint and contain x_1 and x_2 respectively. ■

Three further characterizations of paracompactness are stated in the next theorem.

5.1.10. LEMMA. *Every open σ-locally finite cover $\mathcal{V}$ of a topological space X has a locally finite refinement.*

PROOF. Let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$, where $\mathcal{V}_i = \{V_s\}_{s\in S_i}$ is a locally finite family of open sets and $S_i \cap S_j = \emptyset$ whenever $i \neq j$. For every $s_0 \in S_i$ let

$$A_{s_0} = V_{s_0} \setminus \bigcup_{k<i} \bigcup_{s\in S_k} V_s;$$

the family $\mathcal{A} = \{A_s\}_{s\in S}$, where $S = \bigcup_{i=1}^{\infty} S_i$, covers X and is a refinement of $\mathcal{V}$. We shall show that $\mathcal{A}$ is locally finite. Consider a point $x \in X$, denote by k the smallest natural number such that $x \in \bigcup_{s\in S_k} V_s$, and take an $s_0 \in S_k$ satisfying $x \in V_{s_0}$; clearly V_{s_0} is a neighbourhood of x disjoint from all sets A_s with $s \in \bigcup_{i>k} S_i$. Since the families $\mathcal{V}_i$ are locally finite, for every $i \leq k$ there exists a neighbourhood U_i of x which meets only finitely many members of $\mathcal{V}_i$. The neighbourhood $U_1 \cap U_2 \cap \ldots \cap U_k \cap V_{s_0}$ of the point x meets only finitely many members of $\mathcal{A}$. ■

5.1.11. THEOREM. *For every regular space X the following conditions are equivalent:*
(i) *The space X is paracompact.*
(ii) *Every open cover of the space X has an open σ-locally finite refinement.*
(iii) *Every open cover of the space X has a locally finite refinement.*
(iv) *Every open cover of the space X has a closed locally finite refinement.*

PROOF. The theorem follows from 5.1.10, 5.1.6 and 4.4.12. ■

Let us note that the last theorem immediately implies that every Lindelöf space is paracompact. It should be also noted that in (ii) the second "open" cannot be replaced by "closed" (see the remark to Problem 5.5.3(a)).

We shall now introduce some notions related to the notion of a cover, which will be applied to establish further characterizations of paracompactness. Let $\mathcal{A} = \{A_s\}_{s\in S}$ be a cover of a set X; the *star of a set $M \subset X$ with respect to $\mathcal{A}$* is the set $\mathrm{St}(M, \mathcal{A}) = \bigcup\{A_s :$

$M \cap A_s \neq \emptyset\}$. The star of a one-point set $\{x\}$ with respect to a cover $\mathcal{A}$ is called the *star of the point x with respect to $\mathcal{A}$* and is denoted by $\mathrm{St}(x, \mathcal{A})$. We say that a cover $\mathcal{B} = \{B_t\}_{t\in T}$ of a set X is a *star refinement* of another cover $\mathcal{A} = \{A_s\}_{s\in S}$ of the same set X if for every $t \in T$ there exists an $s \in S$ such that $\mathrm{St}(B_t, \mathcal{B}) \subset A_s$; if for every $x \in X$ there exists an $s \in S$ such that $\mathrm{St}(x, \mathcal{B}) \subset A_s$, then we say that $\mathcal{B}$ is a *barycentric refinement* of $\mathcal{A}$. Clearly, every star refinement is a barycentric refinement and every barycentric refinement is a refinement.

The next theorem contains yet three characterizations of paracompactness; it will be deduced directly from Lemmas 5.1.13, 5.1.15, and 5.1.16 that are stated and proved below.

5.1.12. THEOREM. *For every T_1-space X the following conditions are equivalent:*

(i) *The space X is paracompact.*

(ii) *Every open cover of the space X has an open barycentric refinement.*

(iii) *Every open cover of the space X has an open star refinement.*

(iv) *The space X is regular and every open cover of X has an open σ-discrete refinement.*

5.1.13. LEMMA. *If an open cover $\mathcal{U}$ of a topological space X has a closed locally finite refinement, then $\mathcal{U}$ has also an open barycentric refinement.*

PROOF. Let $\mathcal{F} = \{F_t\}_{t\in T}$ be a closed locally finite refinement of $\mathcal{U} = \{U_s\}_{s\in S}$. For every $t \in T$ choose an $s(t) \in S$ such that $F_t \subset U_{s(t)}$. It follows from the local finiteness of $\mathcal{F}$ that the set $T(x) = \{t \in T : x \in F_t\}$ is finite for every $x \in X$, and this implies that the set

$$V_x = \bigcap_{t\in T(x)} U_{s(t)} \cap (X \setminus \bigcup_{t\notin T(x)} F_t) \tag{2}$$

is open for every $x \in X$. As $x \in V_x$, the family $\mathcal{V} = \{V_x\}_{x\in X}$ is an open cover of X. Let x_0 be a point of X and t_0 an element of $T(x_0)$; it follows from (2) that if $x_0 \in V_x$, then $t_0 \in T(x)$, and thus $V_x \subset U_{s(t_0)}$. Hence we have $\mathrm{St}(x_0, \mathcal{V}) \subset U_{s(t_0)}$ which shows that $\mathcal{V}$ is a barycentric refinement of $\mathcal{U}$. ■

5.1.14. REMARK. The same proof shows that if a locally finite open cover of a topological space has a closed locally finite refinement then it has also a locally finite open barycentric refinement; indeed, if the cover $\mathcal{U}$ is locally finite, then the family of all sets of the form (2) is a locally finite open barycentric refinement of $\mathcal{U}$.

5.1.15. LEMMA. *If a cover $\mathcal{A} = \{A_s\}_{s\in S}$ of a set X is a barycentric refinement of a cover $\mathcal{B} = \{B_t\}_{t\in T}$ of X, and $\mathcal{B}$ is a barycentric refinement of a cover $\mathcal{C} = \{C_z\}_{z\in Z}$ of the same set, then $\mathcal{A}$ is a star refinement of $\mathcal{C}$.*

PROOF. Let us take an $s_0 \in S$ and for every $x \in A_{s_0}$ let us choose a $t(x) \in T$ such that

$$\mathrm{St}(x, \mathcal{A}) \subset B_{t(x)}. \tag{3}$$

Thus we have

$$\mathrm{St}(A_{s_0}, \mathcal{A}) = \bigcup_{x\in A_{s_0}} \mathrm{St}(x, \mathcal{A}) \subset \bigcup_{x\in A_{s_0}} B_{t(x)}. \tag{4}$$

Let x_0 be a fixed element of A_{s_0}; from (3) it follows that $x_0 \in B_{t(x)}$ for every $x \in A_{s_0}$, so that

$$\bigcup_{x\in A_{s_0}} B_{t(x)} \subset \mathrm{St}(x_0, \mathcal{B}).$$

Since $\mathrm{St}(x_0, \mathcal{B}) \subset C_z$ for a $z \in Z$, the last inclusion, along with (4), implies that $\mathcal{A}$ is a star refinement of $\mathcal{C}$. ■

5.1.16. LEMMA. *If every open cover of a topological space X has an open star refinement, then every open cover of X has also an open σ-discrete refinement.*

PROOF. Consider an open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of the space X. Let $\mathcal{U}_0 = \mathcal{U}$ and denote by $\mathcal{U}_1, \mathcal{U}_2, \ldots$ a sequence of open covers of X such that

(5) $$\mathcal{U}_{i+1} \text{ is a star refinement of } \mathcal{U}_i \text{ for } i = 0, 1, \ldots$$

For every $s \in S$ and $i = 1, 2, \ldots$ take the open set

$$U_{s,i} = \{x \in X : x \text{ has a neighbourhood } V \text{ such that } \mathrm{St}(V, \mathcal{U}_i) \subset U_s\}.$$

The family $\{U_{s,i}\}_{s\in S}$ is an open refinement of $\mathcal{U}$ for $i = 1, 2, \ldots$ Let us observe that

(6) $$\text{if } x \in U_{s,i} \text{ and } y \notin U_{s,i+1}, \text{ then there is no } U \in \mathcal{U}_{i+1} \text{ such that } x, y \in U.$$

Indeed, it follows from (5) that for every $U \in \mathcal{U}_{i+1}$ there exists a $W \in \mathcal{U}_i$ such that $\mathrm{St}(U, \mathcal{U}_{i+1}) \subset W$; therefore if $x \in U \cap U_{s,i}$, then $W \subset \mathrm{St}(x, \mathcal{U}_i) \subset U_s$ which implies that $\mathrm{St}(U, \mathcal{U}_{i+1}) \subset U_s$ and $U \subset U_{s,i+1}$.

Take a well-ordering relation $<$ on the set S and let

(7) $$V_{s_0,i} = U_{s_0,i} \setminus \overline{\bigcup_{s<s_0} U_{s,i+1}}.$$

For every pair s_1, s_2 of distinct elements of S we have either $s_1 < s_2$ or $s_2 < s_1$; depending on which part of the alternative holds, by virtue of (7) we have

$$\text{either} \quad V_{s_2,i} \subset X \setminus U_{s_1,i+1} \quad \text{or} \quad V_{s_1,i} \subset X \setminus U_{s_2,i+1}.$$

Hence, it follows from (6) that if $x \in V_{s_1,i}$ and $y \in V_{s_2,i}$, where $s_1 \neq s_2$, then there is no $U \in \mathcal{U}_{i+1}$ such that $x, y \in U$. Thus the family of open sets $\{V_{s,i}\}_{s\in S}$ is discrete for $i = 1, 2, \ldots$

To conclude the proof it suffices to show that the family $\{V_{s,i}\}_{i=1,s\in S}^{\infty}$ is a cover of X. Let x be a point of X; denote by $s(x)$ the smallest element in S such that $x \in U_{s(x),i}$ for some positive integer i – the existence of $s(x)$ follows from the fact that for $i = 1, 2, \ldots$ the family $\{U_{s,i}\}_{s\in S}$ is a cover of X. Since $x \notin U_{s,i+2}$ for $s < s(x)$, it follows from (6) that

$$\mathrm{St}(x, \mathcal{U}_{i+2}) \cap \bigcup_{s<s(x)} U_{s,i+1} = \emptyset,$$

and this shows that $x \in V_{s(x),i}$. ■

PROOF OF THEOREM 5.1.12. By virtue of the last three lemmas and by Theorem 5.1.11, it suffices to show that every T_1-space X satisfying (iii) is regular. Consider a point $x \in X$ and a closed set $F \subset X$ such that $x \notin F$ and take an open star refinement $\mathcal{U}$ of the open cover $\{X\setminus F, X\setminus\{x\}\}$ of the space X. Let U be a member of $\mathcal{U}$ that contains x. As $\mathrm{St}(U, \mathcal{U}) \subset X\setminus F$, we have $\overline{U} \cap F = \emptyset$, so that the space X is regular. ■

The concept of a star refinement leads to the notion of a normal cover (cf. Exercises 5.4.H(c) and (d)). An open cover $\mathcal{W}$ of a space X is *normal* if there exists a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers of X such that $\mathcal{W}_1 = \mathcal{W}$ and $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$ Theorem 5.1.12 implies that in the realm of T_1-spaces, the condition that every open cover is normal characterizes paracompact spaces. Theorem 1.5.18, Lemmas 5.1.13, 5.1.15, and Remark 5.1.14 imply that every locally finite open cover of a normal space is normal; it turns out that this property characterizes normal spaces in the realm of T_1-spaces (see Exercise 5.1.A(a)). Hence, although the name and the definition stress an analogy between paracompactness and compactness, paracompactness can be also viewed as a considerable strengthening of normality.

Collectionwise normality is another strengthening of normality, weaker than paracompactness. A topological space X is called *collectionwise normal* if X is a T_1-space and for every discrete family $\{F_s\}_{s\in S}$ of closed subsets of X there exists a discrete family $\{V_s\}_{s\in S}$ of open subsets of X such that $F_s \subset V_s$ for every $s \in S$. Clearly, every collectionwise normal space is normal.

5.1.17. THEOREM. *A T_1-space X is collectionwise normal if and only if for every discrete family $\{F_s\}_{s\in S}$ of closed subsets of X there exists a family $\{U_s\}_{s\in S}$ of open subsets of X such that $F_s \subset U_s$ for every $s \in S$ and $U_s \cap U_{s'} = \emptyset$ whenever $s \neq s'$.*

PROOF. It suffices to prove that any T_1-space X satisfying the condition in the theorem is collectionwise normal. Clearly, the space X is normal, so that for a discrete family $\{F_s\}_{s\in S}$ of closed subsets of X and the family $\{U_s\}_{s\in S}$ of pairwise disjoint open sets, the closed sets $A = \bigcup_{s\in S} F_s$ and $B = X \setminus \bigcup_{s\in S} U_s$ are respectively contained in disjoint open sets U and V. One easily checks that the family $\{V_s\}_{s\in S}$, where $V_s = U_s \cap U$, is discrete.■

Since by Theorem 3.10.3 every locally finite family of non-empty subsets of a countably compact space is finite, it follows from Theorem 2.1.14 that every countably compact normal space is collectionwise normal.

5.1.18. THEOREM. *Every paracompact space is collectionwise normal.*

PROOF. Let $\{F_s\}_{s\in S}$ be a discrete family of closed subsets of a paracompact space X. For every $x \in X$ choose a neighbourhood H_x of the point x whose closure meets at most one set F_s, consider an open locally finite refinement $\mathcal{W}$ of the cover $\{H_x\}_{x\in X}$, and for every $s \in S$ let $V_s = X \setminus \bigcup\{\overline{W} : W \in \mathcal{W} \text{ and } \overline{W} \cap F_s = \emptyset\}$. Clearly $F_s \subset V_s$, so that to conclude the proof it suffices to show that every $W \in \mathcal{W}$ meets at most one element of the family $\{V_s\}_{s\in S}$. This, however, follows from the fact that $\overline{W}$ meets at most one set F_s.■

5.1.19. REMARK. One easily checks that if $\{F_s\}_{s\in S}$ is a locally finite family of closed subsets of a paracompact space, then the construction in the last proof yields a locally finite family $\{V_s\}_{s\in S}$ of open sets such that $F_s \subset V_s$ for every $s \in S$ (cf. Problems 5.5.16 and 5.5.17).

As shown in Theorem 4.1.17, compactness and countable compactness are equivalent within the realm of metrizable spaces. We shall now show that the same is true in the larger class of paracompact spaces (cf. Theorem 5.3.2).

5.1.20. THEOREM. *Every countable compact paracompact space is compact.*

PROOF. Let $\mathcal{A}$ be an open cover of a countably compact paracompact space X. It follows from Theorem 3.10.3 that any locally finite open refinement $\mathcal{B}$ of $\mathcal{A}$ is finite, so that the space X is compact. ■

We are now going to discuss a few examples.

5.1.21. EXAMPLE. From the last theorem and Example 3.10.16 it follows that the space W_0 of all countable ordinal numbers is not paracompact. Since W_0 is countably compact and normal, it is a collectionwise normal space. ■

The construction of a normal space which is not collectionwise normal is much more difficult. It will be preceded by an auxiliary example in which we discuss a simple operation on topological spaces which has proved useful in constructing counterexamples.

5.1.22. EXAMPLE. Let M be a subspace of a topological space X. One easily checks that the family of all sets of the form $U \cup K$, where U is an open subset of X and $K \subset X \setminus M$, is a topology on X; the set X with this new topology will be denoted by X_M. Thus, the spaces X and X_M have the same underlying set, but their topologies are in general distinct: the topology of X_M is finer than the topology of X. The set $X \setminus M$ and all its subsets are open in X_M, so that $X \setminus M$ is an open discrete subspace of X_M. The subspace $M \subset X_M$ is closed and its topology coincides with the topology induced on M by the topology of X.

Some properties of the space X are shared by the space X_M; e.g., one easily sees that if X is a T_i-space with $i = 0, 1$ or 2, then X_M also is a T_i-space. The reader can readily verify that the same holds for $i = 3$ and $3\frac{1}{2}$. On the other hand, the space X_M need not be normal if X is a normal space, or even a compact space (see Exercise 5.1.D).

We shall now show that X_M is normal, provided that X is a T_1-space and for every pair of disjoint closed subsets A_1, B_1 of the subspace $M \subset X$ there exist open subsets U, V of the space X such that

$$A_1 \subset U, \quad B_1 \subset V \quad \text{and} \quad U \cap V = \emptyset; \tag{8}$$

in particular, X_M is normal if X is normal and M is a closed subspace of X. As we already know X_M is a T_1-space. Let A, B be a pair of disjoint closed subsets of X_M; the sets $A_1 = A \cap M$ and $B_1 = B \cap M$ are closed in the subspace $M \subset X$ and disjoint, so that there exist open sets $U, V \subset X$ satisfying (8). One easily sees that the sets

$$(U \setminus B) \cup (A \setminus M) \quad \text{and} \quad (V \setminus A) \cup (B \setminus M)$$

are open in X_M, disjoint, and contain the sets A and B respectively. Hence the space X_M is normal.

We shall also show, for the sake of a later application, that if X is a hereditarily paracompact space (i.e., if every subspace of X is paracompact, which holds for example for a metrizable space X), then the space X_M also is hereditarily paracompact. Since every subspace of X_M is of the form $X'_{M'}$, where $X' \subset X$ and $M' = M \cap X'$, it suffices to verify that X_M is a paracompact space. Every open cover of the space X_M is of the form $\{U_s \cup K_s\}_{s \in S}$ where U_s is open in X and $K_s \subset X \setminus M$. The open cover $\{U_s\}_{s \in S}$ of the space $U = \bigcup_{s \in S} U_s \subset$

X has a locally finite open refinement $\{V_t\}_{t\in T}$; one can easily check that adjoining to this refinement all one-point subsets of $X \setminus U$ one obtains a locally finite open cover of X_M which refines the cover $\{U_s \cup K_s\}_{s\in S}$.

Further properties of the operation X_M defined in this example can be found in Problem 5.5.2. ■

We shall now describe a normal space which is not collectionwise normal.

5.1.23. BING'S EXAMPLE. Denote by $\mathcal{F}$ the family of all mappings of the discrete space $D(\mathfrak{c})$ of cardinality $\mathfrak{c}$ to the two-point discrete space $D = \{0,1\}$; clearly $|\mathcal{F}| = 2^{\mathfrak{c}}$. By Theorem 2.3.20 the diagonal $F = \triangle_{f\in\mathcal{F}} f : D(\mathfrak{c}) \to D^{2^{\mathfrak{c}}} = \prod_{f\in\mathcal{F}} D_f$, where $D_f = D$ for every $f \in \mathcal{F}$, is a homeomorphic embedding. For every pair of disjoint closed subsets A_1, B_1 of the subspace $M = F(D(\mathfrak{c})) \subset D^{2^{\mathfrak{c}}}$ the sets

$$U = p_{f_A}^{-1}(1) \quad \text{and} \quad V = p_{f_A}^{-1}(0),$$

where $A = F^{-1}(A_1)$ and $f_A \in \mathcal{F}$ is defined by letting $f_A(x) = 1$ if $x \in A$ and $f(x) = 0$ if $x \in D(\mathfrak{c}) \setminus A$, satisfy (8), so that – by virtue of Example 5.1.22 – the space $X = D_M^{2^{\mathfrak{c}}}$ is normal.

It remains to show that the space X is not collectionwise normal. Assume that X is collectionwise normal. As $M \subset X$ is discrete and $X \setminus M$ is open, the family $\{\{x\}\}_{x\in M}$ of one-point subsets of X is discrete; thus there exists a family $\{V_x\}_{x\in M}$ of pairwise disjoint open subsets of X such that $x \in V_x$ for every $x \in M$. Since $V_x = U_x \cup K_x$, where U_x is open in $D^{2^{\mathfrak{c}}}$ and $K_x \subset D^{2^{\mathfrak{c}}} \setminus M$, we have $x \in U_x$; hence $\{U_x\}_{x\in M}$ is a family of cardinality $\mathfrak{c}$ consisting of pairwise disjoint non-empty open subsets of $D^{2^{\mathfrak{c}}}$ and this contradicts Corollary 2.3.18.

As shown in Example 3.6.20, the closure of M in $D^{2^{\mathfrak{c}}}$ is the Čech-Stone compactification of $D(\mathfrak{c})$. One readily verifies that a normal space which is not collectionwise normal can be obtained in the same way by taking an arbitrary homeomorphic embedding h of $D(\mathfrak{c})$ in $D^{2^{\mathfrak{c}}}$ or $I^{2^{\mathfrak{c}}}$ satisfying $\overline{h(D(\mathfrak{c}))} = \beta D(\mathfrak{c})$, instead of the specific embedding F used above (see Theorems 3.6.11, 3.6.13 and 6.2.16 or 3.6.11 and 2.3.23). ■

The next two theorems describe relations between paracompactness and the Lindelöf property.

5.1.24. LEMMA. *Every locally finite family of non-empty subsets of a Lindelöf space is countable.*

PROOF. Let $\mathcal{A}$ be a locally finite family of non-empty subsets of a Lindelöf space X. For every $x \in X$ choose a neighbourhood U_x of the point x which meets only finitely many members of $\mathcal{A}$ and take a countable subcover $\mathcal{U}$ of the cover $\{U_x\}_{x\in X}$ of X. Since every member of $\mathcal{A}$ meets a $U \in \mathcal{U}$, it follows that $|\mathcal{A}| \le \aleph_0$. ■

5.1.25. THEOREM. *If a paracompact space X contains a dense subspace A which has the Lindelöf property, then X is a Lindelöf space.*

PROOF. Let $\mathcal{U} = \{U_s\}_{s\in S}$ be an open cover of the space X; by virtue of Remark 5.1.7 there exists a locally finite open cover $\{V_s\}_{s\in S}$ of the space X such that $\overline{V}_s \subset U_s$ for every $s \in S$. By the above lemma the set $S_0 = \{s \in S : A \cap V_s \neq \emptyset\}$ is countable. Since $A = \bigcup_{s\in S_0} A \cap V_s$, we have

$$X = \overline{A} = \overline{\bigcup_{s\in S_0} A \cap V_s} = \bigcup_{s\in S_0} \overline{A \cap V_s} \subset \bigcup_{s\in S_0} \overline{V}_s \subset \bigcup_{s\in S_0} U_s;$$

thus $\mathcal{U}$ has a countable subcover.■

5.1.26. COROLLARY. *Every separable paracompact space is a Lindelöf space.*■

5.1.27. THEOREM. *Every locally compact paracompact space X can be represented as the union of a family of disjoint closed-and-open subspaces of X each of which has the Lindelöf property.*

PROOF. For every $x \in X$ choose a neighbourhood U_x of the point x such that $\overline{U}_x$ is compact and take a locally finite open refinement $\mathcal{V}$ of the cover $\{U_x\}_{x\in X}$ of X. For every $V \in \mathcal{V}$ and any $x \in \overline{V}$ there exists a neighbourhood W_x of the point x which meets only finitely many members of $\mathcal{V}$. Since $V \subset \overline{V} \subset \bigcup_{x\in\overline{V}} W_x$ and $\overline{V}$ is compact, the set V is contained in a finite union of the W_x's, so that every $V \in \mathcal{V}$ meets only finitely many members of $\mathcal{V}$.

For a $V_0 \in \mathcal{V}$ let $\mathcal{S}_k(V_0) \subset \mathcal{V}$ consist of those $V \in \mathcal{V}$ for which there exists a sequence $V_1, V_2, \ldots, V_k$ of members of $\mathcal{V}$ such that $V_k = V$ and $V_i \cap V_{i+1} \neq \emptyset$ for $i = 0, 1, \ldots, k-1$; furthermore let $\mathcal{S}(V_0) = \bigcup_{k=1}^{\infty} \mathcal{S}_k(V_0)$ and $S(V_0) = \bigcup \mathcal{S}(V_0)$. One easily sees that the families $\mathcal{S}_k(V_0)$ are all finite which implies that $|\mathcal{S}(V_0)| \leq \aleph_0$. For $V_0, V_0' \in \mathcal{V}$ the sets $S(V_0)$ and $S(V_0')$ either coincide or are disjoint so that all sets $S(V_0)$ are closed-and-open. The equality $\overline{S(V_0)} = S(V_0)$ implies that $S(V_0) = \bigcup\{\overline{V} : V \in \mathcal{S}(V_0)\}$. Thus, by Theorem 3.8.5, all subspaces $S(V_0)$ have the Lindelöf property.■

Now we are going to discuss operations on paracompact spaces. We start with two theorems.

5.1.28. THEOREM. *Paracompactness is hereditary with respect to F_σ-sets.*

PROOF. Let M be a subspace of a paracompact space X and $M = \bigcup_{i=1}^{\infty} F_i$, where the F_i's are closed in X. Consider an open cover $\{U_s\}_{s\in S}$ of the space M and take a family $\{V_s\}_{s\in S}$ of open subsets of X such that $U_s = M \cap V_s$ for all $s \in S$. For $i = 1, 2, \ldots$ the family $\{V_s\}_{s\in S} \cup \{X \setminus F_i\}$ is an open cover of the space X, so that it has an open locally finite refinement $\mathcal{A}_i$. Let

$$\mathcal{B}_i = \{M \cap U : U \in \mathcal{A}_i \text{ and } U \cap F_i \neq \emptyset\};$$

clearly $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is an open σ-locally finite cover of M which refines $\{U_s\}_{s\in S}$. Hence the space M is paracompact by virtue of Theorem 5.1.11.■

The last theorem yields the following corollary which can be also deduced directly from the definition of paracompactness.

5.1.29. COROLLARY. *Every closed subspace of a paracompact space is paracompact.*■

5.1.30. THEOREM. *The sum $\bigoplus_{s\in S} X_s$ is paracompact if and only if all spaces X_s are paracompact.*

PROOF. If the sum $\bigoplus_{s\in S} X_s$ is paracompact, then all the X_s's are paracompact by virtue of the last corollary.

Conversely, if all the X_s's are paracompact, then for every open cover $\mathcal{V} = \{V_t\}_{t\in T}$ of the sum $\bigoplus_{s\in S} X_s$, the family $\bigcup_{s\in S} \mathcal{A}_s$ – where $\mathcal{A}_s$ is a locally finite open refinement of the open cover $\{X_s \cap V_t\}_{t\in T}$ of the subspace X_s – is an open locally finite cover of the sum $\bigoplus_{s\in S} X_s$ that refines $\mathcal{V}$.■

As shown in the example below, paracompactness is not a multiplicative property.

5.1.31. EXAMPLE. It follows from Example 3.8.14 and Theorem 5.1.2 that the Sorgenfrey line K is a paracompact space. Since the Cartesian product $K \times K$ is not normal (see Example 2.3.12), we infer from Theorem 5.1.5 that the Cartesian product of two paracompact spaces is not necessarily paracompact (cf. Problems 5.5.5 and 5.5.6).■

We shall show now that even the Cartesian product of a paracompact space and a separable metrizable space is not necessarily paracompact; on the other hand, any Cartesian product of a paracompact space and a compact space is paracompact (see Theorem 5.1.36).

5.1.32. MICHAEL'S EXAMPLE. Denote by Q and P the subspaces of R consisting of all rational and all irrational numbers respectively. By virtue of Example 5.1.22 the space $X = R_Q$ is hereditarily paracompact (the space X is called the *Michael line*). We shall prove that the Cartesian product $X \times Y$, where $Y = P$, is not normal.

Consider the pair A, B of disjoint closed subsets of $X \times Y$ defined by letting

$$A = \{(y, y) : y \in P\} \quad \text{and} \quad B = Q \times P;$$

if suffices to show that for every open set $U \subset X \times Y$ which contains A we have $B \cap \overline{U} \neq \emptyset$. Let $\{p_1, p_2, \ldots\}$ be a countable set dense in P. As $A \subset U$, for every $y \in P$ there exist a neighbourhood $U_y \subset P$ of the point y such that

$$\{y\} \times U_y \subset U \tag{9}$$

and a natural number $i(y)$ satisfying

$$p_{i(y)} \in U_y; \tag{10}$$

let $P_i = \{y \in P : i(y) = i\}$ for $i = 1, 2, \ldots$ The Baire category theorem readily implies that P is not an F_σ-set in R (cf. Exercise 3.9.B(a)), so that there exist a natural number i_0 and a $q_0 \in Q$ such that $q_0 \in \overline{P}_{i_0}$, where the bar denotes the closure in R; clearly $(q_0, p_{i_0}) \in B$. Let $V \times W$ be any neighbourhood of (q_0, p_{i_0}) in the Cartesian product $X \times Y = R_Q \times P$. Since $q_0 \in \overline{P}_{i_0}$, we have $V \cap P_{i_0} \neq \emptyset$; i.e., there exists a $y \in V \cap P_{i_0}$. As $i(y) = i_0$, by (10) we have $p_{i_0} \in U_y$ which, along with (9), implies that $(y, p_{i_0}) \in U$. Hence – as obviously $(y, p_{i_0}) \in V \times W$ – every neighbourhood of the point $(q_0, p_{i_0}) \in B$ meets the set U, i.e., we have $B \cap \overline{U} \neq \emptyset$.

Let us observe that in the above proof we applied only three properties of P and Q, viz., the facts that P and Q are complementary subsets of a topological space, P is not an F_σ-set and P is separable.■

We pass now to a discussion of invariance of paracompactness under mappings. To begin, let us note that paracompactness is not an invariant of continuous mappings: indeed, every topological space is a continuous image of a discrete, hence paracompact, space. Example 4.4.10 or Examples 4.4.11 and 5.1.21 show that paracompactness is not an invariant of open mappings, either. On the other hand, we have

5.1.33. THE MICHAEL THEOREM. *Paracompactness is an invariant of closed mappings.*

PROOF. Let $f : X \to Y$ be a closed mapping of a paracompact space X onto a topological space Y. It follows from Theorems 1.5.20 and 5.1.5 that the space Y is normal, so that – by Theorem 5.1.11 – it suffices to prove that any open cover $\{U_s\}_{s\in S}$ of the space Y has an open σ-discrete refinement.

Let $<$ be a well-ordering relation on the set S; we shall define inductively for $i = 1, 2, \dots$ a closed locally finite cover $\mathcal{F}_i = \{F_{s,i}\}_{s\in S}$ of the space X satisfying the conditions:

(11) $$F_{s,i} \subset f^{-1}(U_s) \quad \text{for } s \in S \text{ and } i = 1, 2, \dots$$

(12) $$f(F_{s,i}) \cap f(E_{s,i-1}) = \emptyset, \quad \text{where } E_{s,i-1} = \textstyle\bigcup_{t<s} F_{t,i-1}, \text{ for } i > 1.$$

The existence of $\mathcal{F}_1$ follows from Lemma 5.1.6. Assume that the covers $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{k-1}$ are already defined and satisfy (11) and (12) for $i < k$. Since the cover $\mathcal{F}_{k-1}$ is locally finite and the mapping f is closed, the sets

(13) $$W_{s,k} = f^{-1}(U_s) \setminus f^{-1}f(E_{s,k-1}) \subset f^{-1}(U_s)$$

are open. For every $x \in X$ denote by $s(x)$ the smallest element in S such that $x \in f^{-1}(U_{s(x)})$; since $E_{s(x),k-1} \subset \bigcup_{s<s(x)} f^{-1}(U_s)$ by (11), we have $f^{-1}f(E_{s(x),k-1}) \subset \bigcup_{s<s(x)} f^{-1}(U_s)$. Hence, $x \in W_{s(x),k}$; i.e., $\{W_{s,k}\}_{s\in S}$ is an open cover of X. Applying Lemma 5.1.6 we obtain a locally finite closed cover $\mathcal{F}_k = \{F_{s,k}\}_{s\in S}$ of the space X such that $F_{s,k} \subset W_{s,k}$ for every $s \in S$. It follows from (13) that the cover $\mathcal{F}_k$ satisfies conditions (11) and (12) for $i = k$, so that the construction of the covers $\mathcal{F}_i$ is concluded.

Let us consider the open sets $V_{s,i} = Y \setminus [f(\bigcup_{t\neq s} F_{t,i}) \cup f(\bigcup_{t<s} F_{t,i+1})]$. Since the families $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ are covers of X and f maps X onto Y, we have

(14) $$V_{s,i} \subset f(F_{s,i}) \quad \text{and} \quad V_{s,i} \subset f(\bigcup_{t\geq s} F_{t,i+1}).$$

We shall show that the family $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$, where $\mathcal{V}_i = \{V_{s,i}\}_{s\in S}$, is a σ-discrete refinement of $\{U_s\}_{s\in S}$. For every $y \in Y$ denote by $s(y)$ the smallest element in S such that $y \in f(F_{s(y),i})$ for some positive integer i and take an integer $i(y)$ such that $y \in f(F_{s(y),i(y)-1})$. Hence $y \in f(E_{s,i(y)-1})$ for every $s > s(y)$ and, by virtue of (12), $y \notin f(F_{s,i(y)})$ whenever $s > s(y)$. On the other hand, $y \notin f(F_{s,i(y)})$ and $y \notin f(F_{s,i(y)+1})$ whenever $s < s(y)$, so that $y \in V_{s(y),i(y)}$. Thus $\mathcal{V}$ is a cover of Y. The inclusion $V_{s,i} \subset f(F_{s,i})$, along with (11), implies that $\mathcal{V}$ is a refinement of $\{U_s\}_{s\in S}$. To conclude the proof it suffices to show that $\mathcal{V}_i$ is discrete for $i = 1, 2, \dots$ Consider a point $y \in Y$ and denote by $t(y)$ the smallest element in S such that $y \in f(F_{t(y),i})$; since – by virtue of (12) – $f(F_{s,i+1}) \cap f(F_{t(y),i}) = \emptyset$ for every $s > t(y)$, the set

$$U = Y \setminus (f(\bigcup_{s<t(y)} F_{s,i}) \cup f(\bigcup_{s>t(y)} F_{s,i+1}))$$

is a neighbourhood of the point y. From the first part of (14) it follows that $U \cap V_{s,i} = \emptyset$ for $s < t(y)$, and from the second part that $U \cap V_{s,i} = \emptyset$ for $s > t(y)$; hence $\mathcal{V}_i$ is discrete.∎

Let us note, in connection with the Michael theorem, that there are characterizations of paracompact spaces which immediately imply invariance of paracompactness under closed mappings; such a characterization is used implicitly in the above proof (see Exercise 5.1.G). Let us note that an important particular case of the Michael theorem – the invariance of paracompactness under perfect mappings – can be established in a much simpler way (see Lemma 4.4.13 and Remark 4.4.14).

Invariance of paracompactness under perfect mappings, along with Theorems 5.1.30 and 3.7.22, yields

5.1.34. THEOREM. *If a topological space X has a locally finite closed cover consisting of paracompact subspaces, then X is itself paracompact.*∎

The last theorem can be also deduced from the equivalence of conditions (i) and (iv) in Theorem 5.1.11.

5.1.35. THEOREM. *Paracompactess is an inverse invariant of perfect mappings.*

PROOF. Let $f : X \to Y$ be a perfect mapping onto a paracompact space Y. Consider an open cover $\{U_s\}_{s\in S}$ of the space X and for every $y \in Y$ choose a finite set $S(y) \subset S$ such that $f^{-1}(y) \subset \bigcup_{s\in S(y)} U_s$. Applying Theorem 1.4.13, take a neighbourhood V_y of the point y such that

$$f^{-1}(y) \subset f^{-1}(V_y) \subset \bigcup_{s\in S(y)} U_s.$$

The open cover $\{V_y\}_{y\in Y}$ of Y has an open locally finite refinement $\{W_t\}_{t\in T}$. The family $\{f^{-1}(W_t)\}_{t\in T}$ is an open locally finite cover of X and for every $t \in T$ there exists a $y_t \in Y$ satisfying

$$f^{-1}(W_t) \subset f^{-1}(V_{y_t}) \subset \bigcup_{s\in S(y_t)} U_s.$$

One easily checks that the family $\{f^{-1}(W_t) \cap U_s : t \in T \text{ and } s \in S(y_t)\}$ is an open locally finite refinement of $\{U_s\}_{s\in S}$.∎

Theorem 3.7.1 and 5.1.35 yield

5.1.36. THEOREM. *The Cartesian product $X \times Y$ of a paracompact space X and a compact space Y is paracompact.*∎

Theorem 5.1.35 and the Michael theorem yield

5.1.37. THEOREM. *The class of paracompact spaces is perfect.*∎

We conclude this section with an interesting external characterization of paracompactness:

5.1.38. THE TAMANO THEOREM. *For every Tychonoff space X the following conditions are equivalent:*

(i) *The space X is paracompact.*

(ii) *For every compactification cX of the space X the Cartesian product $X \times cX$ is normal.*
(iii) *The Cartesian product $X \times \beta X$ is normal.*
(iv) *There exists a compactification cX of the space X such that the Cartesian product $X \times cX$ is normal.*

PROOF. The implication (i)$\Rightarrow$(ii) follows from Theorem 5.1.36. Obviously (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(iv), so that it remains to show that (iv)$\Rightarrow$(i).

Let $\{U_s\}_{s\in S}$ be a open cover of the space X. For every $s \in S$ choose an open set $V_s \subset cX$ such that $U_s = X \cap V_s$. Since the complement $Z = cX \setminus \bigcup_{s\in S} V_s$ is a compact subset of the remainder $cX \setminus X$, the diagonal $\Delta \subset X \times X$ and the Cartesian product $X \times Z$ form a pair of disjoint closed subsets of $X \times cX$. Hence, by (iv), there exists a continuous function $f : X \times cX \to I$ such that

$$f(\Delta) \subset \{0\} \quad \text{and} \quad f(X \times Z) \subset \{1\}.$$

Letting

$$\rho(x, y) = \sup_{z\in cX} |f(x, z) - f(y, z)|$$

we define a pseudometric ρ on the set X. The topology $\mathcal{O}_1$ induced by the pseudometric ρ is coarser than the original topology $\mathcal{O}_2$. Indeed, for every $x_0 \in X$ and any $\epsilon > 0$ there exist open sets $G_1 \times H_1, G_2 \times H_2, \ldots, G_k \times H_k \subset X \times cX$ such that $x_0 \in G_i$ and $\delta(f(G_i \times H_i)) < \epsilon$ for $i = 1, 2, \ldots, k$, and $\{x_0\} \times cX \subset \bigcup_{i=1}^k (G_i \times H_i)$; hence we have $x_0 \in \bigcap_{i=1}^k G_i \subset B(x_0, \epsilon) = \{x \in X : \rho(x_0, x) < \epsilon\}$ which implies that all open balls with respect to ρ belong to $\mathcal{O}_2$, i.e., that $\mathcal{O}_1 \subset \mathcal{O}_2$.

From Remark 4.4.2 it follows that the cover $\{B(x, 1/2)\}_{x\in X}$ of the set X has a refinement $\{W_t\}_{t\in T}$ open and locally finite with respect to the topology $\mathcal{O}_1$ and thus also with respect to the topology $\mathcal{O}_2$. For every $x \in X$ and $y \in B(x, 1/2)$ we have $f(x, y) = |f(x, y) - f(y, y)| \leq \rho(x, y) < 1/2$, so that $f(x, y) \leq 1/2$ whenever $y \in \overline{B(x, 1/2)}$, where the bar denotes the closure in cX. As $f(x, z) = 1$ for $z \in Z$, we have $\overline{W}_t \cap Z = \emptyset$ for every $t \in T$. The set $\overline{W}_t$ being compact, there exists a finite set $S(t) \subset S$ such that $W_t \subset \bigcup_{s\in S(t)} U_s$. One easily checks that the family $\{W_t \cap U_s : t \in T, s \in S(t)\}$ is a locally finite open cover of X which refines $\{U_s\}_{s\in S}$. ■

Theorems 5.1.36, 3.5.2 and 5.1.38 yield

5.1.39. THEOREM. *For every topological space X the following conditions are equivalent:*
(i) *The space X is paracompact.*
(ii) *For every compact space Y the Cartesian product $X \times Y$ is normal.*
(iii) *For every compact space Y such that $w(Y) \leq w(X)$ the Cartesian product $X \times Y$ is normal.*
(iv) *The Cartesian product $X \times I^{w(X)}$ is normal.* ■

5.1.40. EXAMPLE. The Tamano theorem implies that the Cartesian product $W_0 \times W$, where W_0 is the space of all countable ordinal numbers and W the space of all ordinal numbers $\leq \omega_1$, is not normal (this can be also proved in a direct way, see Problem 3.12.20(d)); indeed, W is a compactification of W_0 and it was shown in Example 5.1.21 that W_0 is not paracompact.

Hence in the space $W_0 \times W$ there exists a pair A, B of disjoint closed subsets which cannot be separated by disjoint open sets. One can easily check – remembering that the space $W_0 \times W$ is countably compact (see Corollary 3.10.14) – that the quotient space $(W_0 \times W)/A$, obtained by identifying the set A to a point (see Example 2.4.12), is an example of a countably compact (even sequentially compact) non-regular space (cf. Exercise 3.10.B).

Let us observe that the projection of $W_0 \times W$ onto W_0 shows that normality is not an inverse invariant of open perfect mappings.■

Historical and bibliographic notes

The concept of paracompactness was introduced by Dieudonné in [1944]; the same paper contains Theorems 5.1.5 and 5.1.36 as well as Corollary 5.1.29. As mentioned above, Theorem 5.1.2 was established by Morita in [1948] and Theorem 5.1.3 by A. H. Stone in [1948] (paracompactness of second-countable metrizable spaces and locally compact metrizable spaces was established by Dieudonné in [1944]). Theorems 5.1.9 and 5.1.11, as well as the equivalence of conditions (i) and (iv) in Theoem 5.1.12, were proved by Michael in [1953]; the simple proof of Lemma 5.1.8 is due to Mather [1964]. The equivalence of conditions (i) and (iii) in Theorem 5.1.12 was established by A. H. Stone in [1948]. Tukey introduced in [1940] the class of *fully normal spaces*, defined as spaces satisfying condition (iii) in Theorem 5.1.12; he proved that every metrizable space is fully normal (cf. Exercise 5.1.A(c)), established the equivalence of conditions (ii) and (iii) in Theorem 5.1.12 and introduced the notion of a normal cover. The class of collectionwise normal spaces was defined by Bing in [1951] (using the condition appearing here in Theorem 5.1.17; the equivalence of this condition to our definition was established by Dowker in [1952a]). Bing's paper also contains Theorem 5.1.18 and Example 5.1.23; besides the latter example and related spaces (cf. Problem 5.5.3) no normal spaces which are not collectionwise normal are known. Tall's paper [1984] is a survey of recent results on relations between normality and collectionwise normality. The modification of topologies described in Example 5.1.22 was introduced by Bing in [1951] and by Hanner in [1951]; it was applied again by Michael in [1963], where Example 5.1.32 was given (for further information see Problems 5.5.2 and 5.5.4). Examples 5.1.31 (due to Sorgenfrey [1947]) and 5.1.32 suggest the question whether normality of a Cartesian product of two paracompact spaces implies paracompactness of this product; as shown by Przymusiński in [1980], the answer is negative. On the other hand, normality of a Cartesian product of a metric space and a paracompact space implies paracompactness of that product (see Problem 5.5.18(d)). Theorem 5.1.25 was noted by Willard in [1971]; one can easily show that every paracompact space which has the Souslin property (see Problem 1.7.12) also is a Lindelöf space. Morita's paper [1948], cited already above, contains also Theorem 5.1.27. Theorems 5.1.28 and 5.1.34 were proved by Michael in [1953], and Theorem 5.1.33 by the same author in [1957] (for perfect mappings by Morita and Hanai in [1956]). Theorem 5.1.35 was, in principle, proved by Hanai in [1956]. The equivalence of conditions (i) an (iii) in Theorem 5.1.38 was established by Tamano in [1960a]; the equivalence of conditions (i) and (iv) was obtained independently by Corson in [1962], Morita in [1962] and Tamano in [1962].

Exercises

5.1.A. (a) Note that if every two-element open cover of a T_1-space X has an open star refinement, then X is a normal space.

(b) Show directly that every finite open cover of a normal space has a finite open star refinement.

Hint. First consider a two-element cover.

(c) (Tukey [1940]) Show directly that every open cover of a metrizable space has an open star refinement.

Hint. Let $\mathcal{U}$ be an open cover of a metrizable space X. Consider a metric on the space X and for every $x \in X$ take an $\epsilon_x \in (0,1]$ such that the ball $B(x, 4\epsilon_x)$ is contained in a member of $\mathcal{U}$. Check that the cover $\{B(x,\epsilon_x)\}_{x\in X}$ is a barycentric refinement of $\mathcal{U}$.

(d) Note that a Hausdorff space X is compact if and only if every open cover of X has a locally finite (or – equivalently – a point-finite) subcover.

5.1.B (Arhangel'skiĭ [1965]). (a) Show that a regular k-space X is paracompact if and only if every open cover $\mathcal{U}$ of the space X has a closed refinement $\mathcal{F}$ such that every compact subset of X meets only finitely many members of $\mathcal{F}$.

(b) Show that a normal k-space X is paracompact if and only if every open cover $\mathcal{U}$ of the space X has an open refinement $\mathcal{V}$ such that every compact subset of X meets only finitely many members of $\mathcal{V}$.

5.1.C. (a) Check that collectionwise normality is hereditary with respect to closed subsets (cf. Problem 5.5.1(b)) and is an additive property.

(b) Show that collectionwise normality is an invariant of closed mappings, but is not an invariant of open mappings (cf. Exercise 4.2.D).

(c) Note that collectionwise normality is not an inverse invariant of perfect mappings.

(d) Give an example of an inverse sequence of paracompact spaces whose limit is not normal (cf. Problem 5.5.4(c)).

Hint. Make use of Example 5.1.32 and Exercise 4.3.G.

5.1.D. Give an example of a compact space X and a subspace $M \subset X$ such that the space X_M defined in Example 5.1.22 is not normal.

Hint. Consider the space $X \times Y$ in Example 2.3.36 or another compact space which is not hereditarily normal.

5.1.E. Observe that every point of the space X in Example 5.1.23 has a paracompact closed-and-open neighbourhood.

5.1.F. (a) (Dieudonné [1944]) Observe that X is a hereditarily paracompact space if and only if all open subspaces of X are paracompact.

(b) (Dowker [1947a]) Note that every perfectly normal paracompact space is hereditarily paracompact.

(c) Show that every separable hereditarily paracompact space is a hereditarily Lindelöf space, and thus is perfectly normal (see Exercise 3.8.A(c)). Give an example of a hereditarily paracompact space which is not perfectly normal.

5.1.G (Michael [1957]). A family $\{A_s\}_{s\in S}$ of subsets of a topological space is *closure-preserving* if $\overline{\bigcup_{s\in S_0} A_s} = \bigcup_{s\in S_0} \overline{A_s}$ for every $S_0 \subset S$.

Prove that for a regular space X the following conditions are equivalent:

(1) *The space X is paracompact.*
(2) *Every open cover of the space X has an open closure-preserving refinement.*
(3) *Every open cover of the space X has a closure-preserving refinement.*
(4) *Every open cover of the space X has a closed closure-preserving refinement.*

Hint. When proving the implication (4)⇒(1), observe first that for every open cover $\{U_s\}_{s\in S}$ of the space X there exists a closed closure-preserving cover $\{F_s\}_{s\in S}$ such that $F_s \subset U_s$ for every $s \in S$, then note that X is normal and mimic the proof of Theorem 5.1.33 for $Y = X$ and $f = \mathrm{id}_X$.

5.1.H. Prove that if X is a metrizable space and the space R^X with the compact-open topology is first-countable, then X is locally compact and second-countable; observe that the space R^X also is second-countable (see Theorem 3.4.16).

Hint. First apply Exercise 3.4.E to show that X is locally compact, then make use of 5.1.27, 3.4.4 and 2.3.F(b).

5.1.I (Nagata [1950]). Show that a paracompact Čech-complete space X is metrizable if and only if the diagonal Δ is a G_δ-set in the Cartesian product $X \times X$ (cf. Exercise 4.2.B and Problem 5.5.9(c)).

Hint. Define a countable family $\{\mathcal{V}_i\}_{i=1}^{\infty}$ of locally finite open covers of the space X such that

(1) For every family $\mathcal{A}$ of subsets of X which has the finite intersection property and contains sets of diameter less than $\mathcal{V}_i$ for $i = 1, 2, \ldots$, the intersection $\bigcap\{\overline{A} : A \in \mathcal{A}\}$ is non-empty.
(2) For every pair x, y of distinct points of X there exists a natural number i with the property that the closure of no member of $\mathcal{V}_i$ contains both x and y.

Check that the family of all finite intersections $V_1 \cap V_2 \cap \ldots \cap V_k$, where $V_i \in \mathcal{V}_i$ for $i = 1, 2, \ldots, k$ and $k = 1, 2, \ldots$, is a base for the space X.

One can also take a function $f : X \times \beta X \to I$ such that $\Delta = f^{-1}(0)$ and let $\rho(x, y) = \sup\{|f(x, z) - f(y, z)| : z \in \beta X\}$.

5.1.J (Tukey [1940], A. H. Stone [1948], Michael [1953], Morita [1964]). (a) Show that for every σ-locally finite cover $\{U_s\}_{s\in S}$ of a space X consisting of functionally open sets there exist a pseudometric ρ on the set X, such that $\rho : X \times X \to R$ is a continuous mapping, and a cover $\{V_s\}_{s\in S}$ of the set X, which is open with respect to the topology on X induced by the pseudometric ρ, such that $V_s \subset U_s$ for every $s \in S$.

Hint. Consider functions $f_s : X \to I$ satisfying $U_s = f_s^{-1}((0, 1])$.

(b) Deduce from (a) that for every σ-locally finite cover $\{U_s\}_{s\in S}$ of a space X consisting of functionally open sets there exist a continuous mapping $f : X \to Y$ onto a metrizable space Y and an open cover $\{W_s\}_{s\in S}$ of the space Y such that $f^{-1}(W_s) \subset U_s$ for every $s \in S$.

Hint. Apply Exercise 4.2.I.

(c) Observe that (a) and (b) hold for any open cover $\{U_s\}_{s\in S}$ of a space X which has a partition of unity subordinated to it (see Exercise 5.4.H(c)).

(d) Observe that (a) and (b) hold if X is a normal space and the open cover $\{U_s\}_{s\in S}$ is either locally finite or point-finite and countable (cf. Exercise 5.2.F(b)).

(e) Prove that a T_1-space X is paracompact if and only if for every open cover $\{U_s\}_{s\in S}$ of the space X there exist a continuous mapping $f : X \to Y$ onto a metrizable space Y and an open cover $\{W_s\}_{s\in S}$ of the space Y such that $f^{-1}(W_s) \subset U_s$ for every $s \in S$.

(f) Deduce from part (e) that every paracompact space is homeomorphic to a closed subspace of a Cartesian product of metrizable spaces (cf. Problem 8.5.13).

5.2. Countably paracompact spaces

A topological space X is called *countably paracompact* if X is a Hausdorff space and every countable open cover of X has a locally finite open refinement. Clearly, every paracompact space is countably paracompact and so is every countably compact space. It follows from the last observation that there exist collectionwise normal countably paracompact spaces which are not paracompact (see Example 5.1.21), as well as countably paracompact non-regular spaces (see Example 5.1.40). Theorem 3.10.22 implies that every pseudocompact countably paracompact space is countably compact. Hence, the space X in Example 3.10.29 is a Tychonoff space which is not countably paracompact (see also Exercise 5.2.C). On the other hand, finding a normal space which is not countably paracompact was an outstanding problem in general topology for about twenty years; such spaces are now known but they are too complicated to be described here.

This section contains a few theorems establishing various characterizations of countably paracompact spaces and of normal countably paracompact spaces; further information can be found in the Exercises (see also Problem 5.5.15).

5.2.1. THEOREM. *For every Hausdorff space X the following conditions are equivalent:*

(i) *The space X is countably paracompact.*

(ii) *For every countable open cover $\{U_i\}_{i=1}^\infty$ of the space X there exists a locally finite open cover $\{V_i\}_{i=1}^\infty$ of X such that $V_i \subset U_i$ for $i = 1, 2, \ldots$*

(iii) *For every increasing sequence $W_1 \subset W_2 \subset \ldots$ of open subsets of X satisfying $\bigcup_{i=1}^\infty W_i = X$ there exists a sequence $F_1, F_2, \ldots$ of closed subsets of X such that $F_i \subset W_i$ for $i = 1, 2, \ldots$ and $\bigcup_{i=1}^\infty \operatorname{Int} F_i = X$.*

(iv) *For every decreasing sequence $F_1 \supset F_2 \supset \ldots$ of closed subsets of X satisfying $\bigcap_{i=1}^\infty F_i = \emptyset$ there exists a sequence $W_1, W_2, \ldots$ of open subsets of X such that $F_i \subset W_i$ for $i = 1, 2, \ldots$ and $\bigcap_{i=1}^\infty \overline{W}_i = \emptyset$.*

PROOF. To show that (i)$\Rightarrow$(ii) it suffices to take a locally finite open refinement $\mathcal{V}$ of the cover $\{U_i\}_{i=1}^\infty$, for every $V \in \mathcal{V}$ choose a natural number $i(V)$ such that $V \subset U_{i(V)}$, and let $V_i = \bigcup\{V : i(V) = i\}$.

We shall show that (ii)$\Rightarrow$(iii). As $\{W_i\}_{i=1}^\infty$ is a countable open cover of X, there exists an open locally finite cover $\{V_i\}_{i=1}^\infty$ of X such that $V_i \subset W_i$ for $i = 1, 2, \ldots$ The sets $F_i = X \setminus \bigcup_{j>i} V_j \subset \bigcup_{j\le i} V_j$ are closed and – since $\bigcup_{j\le i} V_j \subset \bigcup_{j\le i} W_j = W_i$ – we have $F_i \subset W_i$ for $i = 1, 2, \ldots$ The family $\{V_i\}_{i=1}^\infty$ being locally finite, every point $x \in X$ has a neighbourhood which is contained in some F_i, i.e., $\bigcup_{i=1}^\infty \operatorname{Int} F_i = X$.

From De Morgan's laws it follows easily that conditions (iii) and (iv) are equivalent; hence, to conclude the proof it suffices to prove that (iii)$\Rightarrow$(i). Let $\{U_i\}_{i=1}^\infty$ be a countable open cover of the space X. Consider the increasing sequence $W_1 \subset W_2 \subset \ldots$ of open subsets of X, where $W_i = \bigcup_{j\le i} U_j$; as $\bigcup_{i=1}^\infty W_i = X$, there exists a sequence $F_1, F_2, \ldots$ of closed subsets of X such that $F_i \subset W_i$ and $\bigcup_{i=1}^\infty \operatorname{Int} F_i = X$. The set $V_i = U_i \setminus \bigcup_{j<i} F_j \subset U_i$ is open

for $i = 1, 2, \ldots$; since $\bigcup_{j<i} F_j \subset \bigcup_{j<i} W_j \subset \bigcup_{j<i} U_j$, we have $U_i \setminus \bigcup_{j<i} U_j \subset V_i$ which implies that the family $\{V_i\}_{i=1}^{\infty}$ is a cover of X. Every point $x \in X$ has a neighbourhood of the form $\operatorname{Int} F_j$; this neighbourhood is disjoint from all sets V_i for $i > j$, so that the cover $\{V_i\}_{i=1}^{\infty}$ is locally finite. ■

5.2.2. COROLLARY. *A normal space X is countably paracompact if and only if for every decreasing sequence $F_1 \supset F_2 \supset \ldots$ of closed subsets of X satisfying $\bigcap_{i=1}^{\infty} F_i = \emptyset$ there exists a sequence $W_1, W_2, \ldots$ of open subsets of X such that $F_i \subset W_i$ for $i = 1, 2, \ldots$ and $\bigcap_{i=1}^{\infty} W_i = \emptyset$.* ■

5.2.3. THEOREM. *For every T_1-space X the following conditions are equivalent:*

(i) *The space X is normal and countably paracompact.*

(ii) *For every countable open cover $\{U_i\}_{i=1}^{\infty}$ of the space X there exists a locally finite open cover $\{V_i\}_{i=1}^{\infty}$ of X such that $\overline{V}_i \subset U_i$ for $i = 1, 2, \ldots$*

(iii) *For every countable open cover $\{U_i\}_{i=1}^{\infty}$ of the space X there exists a closed cover $\{F_i\}_{i=1}^{\infty}$ of X such that $F_i \subset U_i$ for $i = 1, 2, \ldots$*

PROOF. To prove that (i)$\Rightarrow$(ii) it suffices to observe that, by virtue of Theorem 5.2.1, there exists a locally finite open cover $\{W_i\}_{i=1}^{\infty}$ of X such that $W_i \subset U_i$ for $i = 1, 2, \ldots$, and to apply Theorem 1.5.18.

The implication (ii)$\Rightarrow$(iii) is obvious; we shall show that (iii)$\Rightarrow$(i). Consider a T_1-space X which satisfies condition (iii). Let us observe first that X is normal; indeed, if for a pair U, V of open subsets of X such that $U \cup V = X$ we let $U_1 = U$, $U_2 = V$, $U_3 = U_4 = \ldots = \emptyset$ then (iii) yields closed subsets $F_1, F_2 \subset X$ such that $F_1 \subset U$, $F_2 \subset V$ and $F_1 \cup F_2 = X$. Now, it follows from De Morgan's laws that, for every sequence $F_1, F_2, \ldots$ of closed subsets of the space X satisfying $\bigcap_{i=1}^{\infty} F_i = \emptyset$, there exists a sequence $W_1, W_2, \ldots$ of open subsets of X such that $F_i \subset W_i$ for $i = 1, 2, \ldots$ and $\bigcap_{i=1}^{\infty} W_i = \emptyset$; hence, X is countably paracompact by Corollary 5.2.2. ■

Besides locally finite and point-finite families of sets, one also considers star-finite and star-countable families: a family $\{A_s\}_{s\in S}$ of subsets of a set X is *star-finite (star-countable)* if for every $s_0 \in S$ the set $\{s \in S : A_s \cap A_{s_0} \neq \emptyset\}$ is finite (countable). Clearly, any star-finite family is point-finite. Let us note that a star-finite family of subsets of a topological space is not necessarily locally finite; however, a star-finite open cover of a topological space is locally finite.

Our next theorem, containing two further conditions equivalent to countable paracompactness in the realm of normal spaces, is preceded by a lemma.

5.2.4. LEMMA. *Every countable cover $\{U_i\}_{i=1}^{\infty}$ of a topological space X, where each U_i is functionally open, has a countable star-finite refinement consisting of functionally open sets.*

PROOF. Let $U_i = f_i^{-1}((0,1])$, where $f_i : X \to I$; letting $f(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x)$, we define a continuous function $f : X \to I$. As $\bigcup_{i=1}^{\infty} U_i = X$, we have $f(x) > 0$ for every $x \in X$. The families $\{V_k\}_{k=1}^{\infty}$ and $\{F_k\}_{k=1}^{\infty}$, where

$$V_k = f^{-1}((1/k, 1]) \quad \text{and} \quad F_k = f^{-1}([1/k, 1]),$$

are covers of X consisting of functionally open and functionally closed sets respectively.

We shall prove that the functionally open sets

$$U_{k,j} = U_j \cap (V_{k+1} \setminus F_{k-1}), \quad \text{where } 1 \le j \le k,\ k = 1,2,\dots \text{ and } F_0 = \emptyset,$$

form a star-finite cover of X. Let x be a point of X; denote by k the smallest integer k such that $x \in F_k$. Since – as one readily checks – $F_k \subset \bigcup_{j\le k} U_j$, there exists a $j \le k$ such that $x \in U_j$ which implies that $x \in U_j \cap (F_k \setminus F_{k-1}) \subset U_{k,j}$. For every $j \le k$ we have $U_{k,j} \subset V_{k+1} \subset F_{k+1}$, so that

(1) $$U_{k,j} \cap U_{m,i} = \emptyset \quad \text{for } m \ge k+2 \text{ and } i \le m.$$

It follows from (1) that the cover $\{U_{k,j}\}_{k=1,j\le k}^{\infty}$ is star-finite. ∎

Let us note that the above lemma and Theorem 1.5.19 yield

5.2.5. COROLLARY. *Every perfectly normal space is countable paracompact.* ∎

5.2.6. THEOREM. *For every normal space X the following conditions are equivalent:*
(i) *The space X is countably paracompact.*
(ii) *Every countable open cover of the space X has a star-finite open refinement.*
(iii) *Every countable open cover of the space X has a point-finite open refinement.*

PROOF. The implication (i)⇒(ii) follows from Theorem 5.2.3, Urysohn's lemma and Lemma 5.2.4. The implication (ii)⇒(iii) is obvious, and the fact that (iii)⇒(i) follows from Theorems 1.5.18 and 5.2.3. ∎

We conclude this section with an interesting characterization of countably paracompact normal spaces in terms of Cartesian products.

5.2.7. LEMMA. *The Cartesian product $X \times Y$ of a countably paracompact normal space X and a compact second-countable space Y is normal.*

PROOF. Let $\{W_i\}_{i=1}^{\infty}$ be a base for the space Y; denote by $\mathcal{S}$ the family of all finite sets of natural numbers and for every $S \in \mathcal{S}$ let $W_S = \bigcup_{i\in S} W_i$. By virtue of Theorem 3.1.16, the projection $p: X \times Y \to X$ is a closed mapping; let $q: X \times Y \to Y$ be the other projection, and for each $M \subset X \times Y$ let $M_x = q[(\{x\} \times Y) \cap M]$.

Consider a pair A, B of disjoint closed subsets of $X \times Y$, and for every $S \in \mathcal{S}$ define

(2) $$U_S = \{x \in X : A_x \subset W_S \subset \overline{W}_S \subset Y \setminus B_x\}.$$

Since

$$\begin{aligned} X \setminus U_S &= \{x \in X : A_x \cap (Y \setminus W_S) \ne \emptyset\} \cup \{x \in X : B_x \cap \overline{W}_S \ne \emptyset\} \\ &= p[A \cap (X \times (Y \setminus W_S))] \cup p[B \cap (X \times \overline{W}_S)], \end{aligned}$$

the sets U_S are open. One readily checks, using compactness of Y, that the family $\{U_S\}_{S\in\mathcal{S}}$ is a cover of X; clearly, it is a countable open cover. Hence, by Theorem 5.2.3, there exists a locally finite open cover $\{V_S\}_{S\in\mathcal{S}}$ of X such that $\overline{V}_S \subset U_S$ for every $S \in \mathcal{S}$.

The set $U = \bigcup_{S\in\mathcal{S}} (V_S \times W_S)$ is open; to conclude the proof it suffices to show that

(3) $$A \subset U \quad \text{and} \quad B \subset V = (X \times Y) \setminus \overline{U}.$$

For every point $(x,y) \in A$ there exists an $S \in \mathcal{S}$ such that $x \in V_S \subset U_S$. From (2) it follows that $(x,y) \in V_S \times W_S \subset U$, so that the first part of (3) holds. The cover $\{V_S\}_{S\in\mathcal{S}}$ being locally finite, $\{V_S \times W_S\}_{S\in\mathcal{S}}$ is a locally finite family of subsets of $X \times Y$; hence,

$$(4) \qquad \overline{U} = \overline{\bigcup_{S\in\mathcal{S}} (V_S \times W_S)} = \bigcup_{S\in\mathcal{S}} \overline{V_S \times W_S} = \bigcup_{S\in\mathcal{S}} (\overline{V}_S \times \overline{W}_S) \subset \bigcup_{S\in\mathcal{S}} (U_S \times \overline{W}_S).$$

From (2) it follows that $B \cap (U_S \times \overline{W}_S) = \emptyset$ for each $S \in \mathcal{S}$, thus by virtue of (4) we have $B \cap \overline{U} = \emptyset$, i.e., the second part of (3) holds. ■

Let us note that – as shown by Example 5.1.40 – the assumption of second-countability (i.e. metrizability) of Y in Lemma 5.2.7 is essential (cf. Exercise 5.2.G(b)).

5.2.8. THEOREM. *A topological space X is normal and countably paracompact if and only if the Cartesian product $X \times I$ of X and the closed unit interval I is normal.*

PROOF. The above lemma implies that for every countably paracompact normal space X the Cartesian product $X \times I$ is normal.

Consider now a topological space X such that $X \times I$ is a normal space. Since X is homeomorphic to the closed subspace $X \times \{0\}$ of $X \times I$, the space X is normal. We shall show that X satisfies condition (iv) in Theorem 5.2.1. Let $F_1 \supset F_2 \supset \ldots$ be a decreasing sequence of closed subsets of X such that $\bigcap_{i=1}^{\infty} F_i = \emptyset$. The sets

$$A = \bigcup_{i=1}^{\infty} (F_i \times \{1/i\}) \quad \text{and} \quad B = X \times \{0\}$$

are disjoint and closed in $X \times Y$. Therefore, there exist open sets $U, V \subset X \times Y$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. The sets $W_i = \{x \in X : (x, 1/i) \in U\}$ are open for $i = 1, 2, \ldots$, and $\bigcap_{i=1}^{\infty} \overline{W}_i = \emptyset$ because $\overline{U} \cap B = \emptyset$. To conclude the proof, it suffices to note that $F_i \subset W_i$ for $i = 1, 2, \ldots$ ■

5.2.9. REMARK. In the last proof only the existence of a non-trivial convergent sequence in the space I was used, so that we proved a little more, viz., that if the Cartesian product $X \times A(\aleph_0)$ is normal, then the space X is normal and countably paracompact.

Historical and bibliographic notes

The class of countably paracompact spaces was introduced independently by Dowker in [1951] and by Katětov in [1951]. The attention of topologists was focused on countably paracompact spaces for twenty years after 1951 in connection with the search for a normal space which is not countably paracompact; such spaces are now called *Dowker spaces* (they are discussed in M. E. Rudin's survey [1984]). A Dowker space was finally described by M. E. Rudin in [1971], it is even collectionwise normal (in [1955] M. E Rudin gave an example of a Dowker space under the assumption that there exists a Souslin space (see the remark to Problem 2.7.9(f))). The class of countably paracompact spaces is interesting mainly because it delineates the realm where some important theorems hold (see Problems 5.5.17(a) and 5.5.20(a)). In the papers cited above, Dowker and Katětov independently proved Corollary 5.2.2, Theorem 5.2.3, Corollary 5.2.5 and the equivalence of conditions (i) an (iii) in Theorem

5.2.6. Theorem 5.2.1 was proved by Ishikawa in [1955]. Lemma 5.2.4 was, in principle, proved in Morita's paper [1948]. The equivalence of conditions (i) and (ii) in Theorem 5.2.6 is due to Iséki [1954]. Theorem 5.2.8 and Lemma 5.2.7 were proved by Dowker in [1951].

Exercises

5.2.A. Show that a T_1-space X is countably paracompact and normal if and only if every countable open cover of the space X has a partition of unity, or – equivalently – a locally finite partition of unity, subordinated to it.

Hint. Show, modifying the proof of Lemma 5.1.8, that if every two-element open cover of a T_1-space X has a partition of unity subordinated to it, then the space X is normal.

5.2.B. Check that countable paracompactness is hereditary with respect to closed subsets and is an additive property (cf. Problem 5.5.14).

5.2.C. (a) Verify that the space $A(\aleph_0) \times A(\mathfrak{c}) \setminus \{(x_0, y_0)\}$, where x_0 and y_0 are the unique accumulation points of $A(\aleph_0)$ and $A(\mathfrak{c})$ respectively, is not countably paracompact (cf. Example 2.3.36).

(b) (Fleissner [1978]) Prove that no separable countably paracompact space contains a closed discrete subspace of cardinality continuum, or – more generally – that for every countably paracompact space X and every closed discrete subspace A of X we have $|A| < \exp d(X)$. Deduce that the Niemytzki plane, the square of the Sorgenfrey line and the space $N^{\mathfrak{c}}$ are not countably paracompact (cf. part (c)).

Hint. Assume that a countably paracompact space X contains a countable dense set and a closed discrete subspace A of cardinality $\mathfrak{c}$. Observe that the family of all distinct sequences $\overline{W}_1, \overline{W}_2, \ldots$, where W_i is open in X, has cardinality $\leq \mathfrak{c}$ and index all such sequences satisfying $\bigcap_{i=1}^{\infty} \overline{W}_i = \emptyset$ as $W_{a,1}, W_{a,2}, \ldots$ with $a \in A$. For $k = 1, 2, \ldots$ consider the set $A_k = \{a \in A : k$ is the smallest integer satisfying $a \notin \overline{W}_{a,k}\}$, observe that $A = \bigcup_{k=1}^{\infty} A_k$, let $F_i = \bigcup_{k=i}^{\infty} A_k$ for $i = 1, 2, \ldots$ and apply Theorem 5.2.1 to obtain a contradiction.

To deduce that $N^{\mathfrak{c}}$ is not countably paracompact apply Exercise 3.1.H(a).

(c) (Nagami [1972]) Prove that the Cartesian product $N^{\aleph_1}$ is not countably paracompact.

Hint. Let $N^{\aleph_1} = \prod_{s \in S} N_s$, where $N_s = N$ and $|S| = \aleph_1$. For every positive integer k consider the set $A_k \subset \prod_{s \in S} N_s$ consisting of all points $\{j_s\}$ such that for every $j \neq k$ the equality $j_s = j$ holds for at most one $s \in S$, let $F_i = \bigcup_{k=i}^{\infty} A_k$ for $i = 1, 2, \ldots$, apply Theorem 5.2.1 and modify the construction in the hint to Exercise 2.3.E(a) to obtain a contradiction (one can also obtain a contradiction by applying Theorem 5.2.1 and Problem 2.7.12(a)).

(d) Give an example of a non-normal Tychonoff space which is countably paracompact.

(e) Verify that the space X in Example 5.1.23 is countably paracompact.

(f) Show that the Cartesian product $X \times Y$ in Example 5.1.32 is not countably paracompact (cf. Problem 5.5.18(a)) and deduce that the limit of an inverse sequence of countably paracompact normal spaces need not be countably paracompact or normal (see Exercise 5.1.C(d)).

5.2.D (Aull [1965]; for countably compact spaces, Alexandroff and Urysohn [1929]). Show that every countably paracompact first-countable space is regular.

5.2.E. Prove that for every countable cover $\{U_i\}_{i=1}^{\infty}$ of a topological space X, where each U_i is functionally open, there exists a cover $\{F_i\}_{i=1}^{\infty}$ consisting of functionally closed sets such that $F_i \subset U_i$ for $i = 1, 2, \ldots$

Hint. For $i = 1, 2, \ldots$ take a function $f_i: X \to I$ such that $U_i = f_i^{-1}((0,1])$, and consider the functions f_i/f, where $f(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x)$.

One can also apply Exercise 5.1.J(b).

5.2.F (Morita [1964]). (a) Show that every σ-locally finite open cover of a countably paracompact normal space has a locally finite open refinement.

(b) Observe that (a) and (b) in Exercise 5.1.J hold if X is a countably paracompact normal space and the open cover $\{U_s\}_{s\in S}$ is σ-locally finite.

5.2.G. (a) (Hanai [1956], Henriksen and Isbell [1958]) Show that, in the realm of Hausdorff spaces, countable paracompactness is both an invariant and an inverse invariant of closed mappings with countably compact fibers. Note that the class of countably paracompact spaces is perfect.

Hint. When proving the invariance, use condition (iii) in Theorem 5.2.1; when proving the inverse invariance, modify the proof of Theorem 5.1.35, replacing the sets V_y by the sets $Y \setminus f(X \setminus \bigcup_{s\in S(y)} U_s)$.

(b) (Dowker [1951]) Deduce from (a) that the Cartesian product $X \times Y$ of a countably paracompact space X and a compact space Y is countably paracompact. Observe that this is also a consequence of Theorem 3.1.16 and condition (iv) in Theorem 5.2.1.

(c) (Zenor [1969]) Prove that countable paracompactness is not an invariant of closed mappings in the realm of Hausdorff spaces.

Hint. Let Z be a non-normal Tychonoff space which is countably paracompact, and A, B a pair of disjoint closed subsets of Z which cannot be separated by disjoint open sets (see Exercise 5.2.C(d)). Consider the natural quotient mapping of the sum $X = \bigoplus_{i=1}^{\infty} X_i$, where $X_i = Z \times \{i\}$, onto the space Y obtained from X by identifying the set $\bigcup_{i=1}^{\infty}(B \times \{i\})$ to a point.

(d) Show that countable paracompactness is an invariant of closed-and-open mappings onto Hausdorff spaces.

(e) (Hanai [1956]) Show that the property of being a countably paracompact normal space is an invariant of closed mappings.

Hint. Apply Corollary 5.2.2.

(f) Note that the property of being a countably paracompact normal space is not an inverse invariant of perfect mappings.

5.2.H (implicitly, Morita [1961]). Prove that if the Cartesian product $X \times Y$ of a topological space X and a non-discrete compact space Y is normal, then X is countably paracompact.

Hint. Consider a countably infinite discrete subspace A of the space Y, note that the Cartesian product $X \times \overline{A}$ is normal, define a continuous mapping of $\overline{A}$ onto $A(\aleph_0)$, and apply 3.7.14, 1.5.20 and 5.2.9.

5.3. Weakly and strongly paracompact spaces

A topological space X is called *weakly paracompact* * if X is a Hausdorff space and every open cover of X has a point-finite open refinement. Every paracompact space is weakly paracompact, but not vice versa (see Example 5.3.4 and Problem 5.5.3(c)). Theorem 5.2.6 implies that every weakly paracompact normal space is countably paracompact; the assumption of normality is essential (see Exercises 5.2.C(a) and 5.3.B(b)).

Our discussion of weakly paracompact spaces will be preceded by a simple theorem on point-finite covers. A cover $\{A_s\}_{s\in S}$ of a space X will be called *irreducible* if $\bigcup_{s\in S_0} A_s \neq X$ for every proper subset S_0 of the set S.

5.3.1. THEOREM. *Every point-finite cover $\{A_s\}_{s\in S}$ of a space X has an irreducible subcover.*

PROOF. Let $\mathcal{G}$ be the family of all functions G from the set S to the family of all subsets of X subject to the conditions:

$$G(s) = A_s \quad \text{or} \quad G(s) = \emptyset \tag{1}$$

and

$$\bigcup_{s\in S} G(s) = X. \tag{2}$$

Let us order the family $\mathcal{G}$ by defining that $G_1 \leq G_2$ whenever $G_2(s) = \emptyset$ for every $s \in S$ such that $G_1(s) = \emptyset$. As in the proof of Theorem 1.5.18, one readily checks that, for each linearly ordered subfamily $\mathcal{G}_0 \subset \mathcal{G}$, the formula $G_0(s) = \bigcap_{G\in\mathcal{G}_0} G(s)$ for $s \in S$ defines a member of $\mathcal{G}$ and that $G \leq G_0$ for every $G \in \mathcal{G}_0$.

From the Kuratowski-Zorn lemma it follows that in $\mathcal{G}$ there exists a maximal element G. The cover $\{A_s\}_{s\in S_1}$, where $S_1 = \{s \in S : G(s) \neq \emptyset\}$, is an irreducible subcover of $\{A_s\}_{s\in S}$. ■

Generalizing Theorems 4.1.17 and 5.1.20, we shall show that compactness and countable compactness are equivalent within the realm of weakly paracompact spaces (cf. Problem 5.5.23).

5.3.2. THEOREM. *Every countably compact weakly paracompact space is compact.*

PROOF. Let $\mathcal{U}$ be an open cover of a countably compact weakly paracompact space X. By virtue of Theorem 5.3.1, there exists an irreducible point-finite open refinement $\mathcal{V} = \{V_t\}_{t\in T}$ of the cover $\mathcal{U}$. The cover $\mathcal{V}$ being irreducible, for every $t \in T$ there exists a point $x_t \in V_t \setminus \bigcup_{t'\neq t} V_{t'}$. Since the sets V_t cover the space X, every point $x \in X$ has a neighbourhood which contains exactly one point of the set $A = \{x_t : t \in T\}$. Hence $A^{\mathrm{d}} = \emptyset$ and it follows from the countable compactness of X that the set A is finite; thus the set T also is finite and $\mathcal{V}$ is a finite open refinement of $\mathcal{U}$. ■

5.3.3. THE MICHAEL-NAGAMI THEOREM. *Every weakly paracompact collectionwise normal space is paracompact.*

PROOF. By virtue of Theorem 5.1.11, it suffices to show that every point-finite open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of a collectionwise normal space X has a σ-locally finite open refinement.

* The terms *metacompact* and *pointwise paracompact* are also used.

We shall define inductively for $i = 0, 1, \ldots$ a discrete family $\mathcal{V}_i = \{V_T\}_{T\in\mathcal{T}_i}$ of open subsets of X such that every member of $\mathcal{V}_i$ is contained in a U_s and the sets $W_i = \bigcup \mathcal{V}_i$ satisfy the condition

$$\text{(3)} \qquad \text{if} \quad |\{s \in S : x \in U_s\}| \leq i, \quad \text{then} \quad x \in \bigcup_{j=0}^{i} W_j.$$

Let $\mathcal{V}_0 = \{\emptyset\}$ and assume that the families $\mathcal{V}_i$ satisfying (3) are already defined for $i \leq k$. Denote by $\mathcal{T}_{k+1}$ the family of all subsets of S that have exactly $k+1$ elements, and for every $T \in \mathcal{T}_{k+1}$ let

$$\text{(4)} \qquad A_T = (X \setminus \bigcup_{j=0}^{k} W_j) \cap (X \setminus \bigcup_{s \notin T} U_s).$$

Note that

$$\text{(5)} \qquad A_T \subset \bigcap_{s\in T} U_s \quad \text{for every} \quad T \in \mathcal{T}_{k+1}.$$

Indeed, if we had $x \notin U_{s_0}$ for an $x \in A_T$ and an $s_0 \in T$, then by virtue of (4) the point x would belong to at most k members of $\mathcal{U}$ and this would contradict (3).

We shall now show that every point $x \in X$ has a neighbourhood $V(x)$ which meets at most one member of the family $\{A_T\}_{T\in\mathcal{T}_{k+1}}$. If x belongs to $k+2$ members of $\mathcal{U}$, say $U_{s_1}, U_{s_2}, \ldots, U_{s_{k+2}}$, then letting $V(x) = \bigcap_{i=1}^{k+2} U_{s_i}$ we have, by virtue of (4), $V(x) \cap A_T = \emptyset$ for every $T \in \mathcal{T}_{k+1}$. If x belongs only to $i \leq k$ members of $\mathcal{U}$, then by virtue of (3) the set $V(x) = \bigcup_{j=0}^{k} W_j$ is a neighbourhood of x disjoint from all sets A_T. Finally, if x belongs to exactly $k+1$ members of $\mathcal{U}$, say $U_{s_1}, U_{s_2}, \ldots, U_{s_{k+1}}$, then the neighbourhood $V(x) = \bigcap_{i=1}^{k+1} U_{s_i}$ of the point x meets at most one member of the family $\{A_T\}_{T\in\mathcal{T}_{k+1}}$, viz., the set A_{T_0}, where $T_0 = \{s_1, s_2, \ldots, s_{k+1}\}$.

Hence $\{A_T\}_{T\in\mathcal{T}_{k+1}}$ is a discrete family of closed subsets of X; let $\{G_T\}_{T\in\mathcal{T}_{k+1}}$ be a discrete family of open subsets of X such that $A_T \subset G_T$ for every $T \in \mathcal{T}_{k+1}$. We shall show that the family $\mathcal{V}_{k+1} = \{V_T\}_{T\in\mathcal{T}_{k+1}}$, where

$$\text{(6)} \qquad V_T = G_T \cap \bigcap_{s\in T} U_s,$$

has all the required properties.

Clearly the family $\mathcal{V}_{k+1}$ is discrete, consists of open subsets of X and every member of $\mathcal{V}_{k+1}$ is contained in a U_s. Consider a point $x \in X$ which belongs to at most $k+1$ members of $\mathcal{U}$; there exists then a $T \in \mathcal{T}_{k+1}$ such that $x \in X \setminus \bigcup_{s\notin T} U_s$ and by (4) we have

$$x \in X \setminus \bigcup_{s\notin T} U_s = [(X \setminus \bigcup_{j=0}^{k} W_j) \cup \bigcup_{j=0}^{k} W_j] \cap (X \setminus \bigcup_{s\notin T} U_s) \subset A_T \cup \bigcup_{j=0}^{k} W_j.$$

The last formula, along with (5), (6) and the inclusion $A_T \subset G_T$, implies that $x \in \bigcup_{j=0}^{k+1} W_j$.

Since the cover $\mathcal{U}$ is point-finite, it follows from (3) that $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a σ-locally finite open refinement of $\mathcal{U}$.■

5.3.4. EXAMPLE. The non-regular Hausdorff space X described in Example 1.5.6 is weakly paracompact.

Let $\mathcal{U} = \{U_s\}_{s\in S}$ be an open cover of the space X. The open subspace $X \setminus Z$ of the space X, where Z is the set of reciprocals of all integers different from zero, has the topology of a subspace of the real line, so that $X \setminus Z$ is a paracompact space. Hence the open cover $\{U_s \setminus Z\}_{s\in S}$ of $X \setminus Z$ has a locally finite open refinement $\mathcal{V}$. Now, for every $z \in Z$ choose an $s(z) \in S$ such that $z \in U_{s(z)}$, and let

$$V_z = \begin{cases} (0, 2z) \cap U_{s(z)}, & \text{if } z > 0, \\ (2z, 0) \cap U_{s(z)}, & \text{if } z < 0. \end{cases}$$

One readily sees that the family $\mathcal{V} \cup (V_z\}_{z\in Z}$ is a point-finite open refinement of $\mathcal{U}$. ■

There exists collectionwise normal spaces which are not weakly paracompact: by virtue of Theorem 5.3.2 and Example 5.1.21, the space W_0 of all countable ordinal numbers is such a space. There also exist weakly paracompact normal spaces which are not collectionwise normal (see Problem 5.5.3(c)).

We conclude the first part of this section with a theorem on invariance of weak paracompactness under closed mappings onto Hausdorff spaces. Let us observe that this result, along with Theorem 5.3.3 and the invariance of collectionwise normality under closed mappings (see Exercise 5.1.C(b)), yields another proof of Theorem 5.1.33. Still another proof of Theorem 5.1.33 can be obtained by introducing small modifications, suggested in Exercise 5.3.E below, in the proof of the theorem on invariance of weak paracompactness. However, both these proofs are more complicated than the original proof in Section 5.1.

5.3.5. LEMMA. *For every open cover $\{U_s\}_{s\in S}$ of a weakly paracompact space X there exists a point-finite open cover $\{V_s\}_{s\in S}$ of X such that $V_s \subset U_s$ for every $s \in S$.* ■

5.3.6. LEMMA. *If there exists a closed mapping $f: X \to Y$ of a weakly paracompact space X onto a space Y, then every open cover of Y which can be represented as a countable union of point-finite families has a point-finite open refinement.*

PROOF. Let $\mathcal{U} = \bigcup_{i=1}^\infty \mathcal{U}_i$, where $\mathcal{U}_i$ is a point-finite family, be an open cover of Y. By virtue of Lemma 5.3.5 there exists a point-finite open cover $\{G_i\}_{i=1}^\infty$ of the space X such that $G_i \subset f^{-1}(U_i)$ for $i = 1, 2, \ldots$, where $U_i = \bigcup \mathcal{U}_i$. The set $E_k = X \setminus \bigcup_{i\geq k} G_i$ is closed for $k = 1, 2, \ldots$; one easily sees that $E_1 \subset E_2 \subset \ldots$ and that $\{E_k\}_{k=1}^\infty$ is a cover of X. Moreover

$$f(E_k) \subset f(\bigcup_{i<k} G_i) \subset \bigcup_{i<k} f(G_i) \subset \bigcup_{i<k} U_i \quad \text{for} \quad k = 1, 2, \ldots$$

To conclude the proof, it suffices to show that the family $\mathcal{W} = \bigcup_{i=1}^\infty \mathcal{W}_i$, where

$$\mathcal{W}_i = \{U \setminus f(E_i) : U \in \mathcal{U}_i\},$$

is a point-finite open cover of Y.

Since the mapping f is closed, members of $\mathcal{W}$ are open; we shall prove that $\mathcal{W}$ is a point-finite family. As $Y = \bigcup_{k=1}^\infty f(E_k)$, every point $y \in Y$ belongs to a set $f(E_k)$, so that y belongs to no member of $\mathcal{W}_i$ for $i \geq k$. The families $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{k-1}$ being point-finite, y belongs only to finitely many members of $\mathcal{W}$. It remains to show that $\mathcal{W}$ is a cover of Y.

Consider a point $y \in Y$ and denote by $i(y)$ the smallest natural number such that $y \in U_{i(y)}$. Since $f(E_{i(y)}) \subset \bigcup_{i<i(y)} U_i$, there exists a $U \in \mathcal{U}_{i(y)}$ such that $y \in U \setminus f(E_{i(y)})$ and thus $y \in \bigcup \mathcal{W}$. ■

5.3.7. THE WORRELL THEOREM. *If there exists a closed mapping $f: X \to Y$ of a weakly paracompact space X onto a Hausdorff space Y, then Y is a weakly paracompact space.*

PROOF. By Lemma 5.3.6 it suffices to prove that every open cover $\{U_s\}_{s\in S}$ of the space Y has an open refinement which can be represented as a countable union of point-finite families.

Let $<$ be a well-ordering relation on the set S; we shall define inductively for $i = 1,2,\ldots$ an open point-finite cover $\mathcal{G}_i = \{G_{s,i}\}_{s\in S}$ of the space X satisfying the conditions:

(7) $$G_{s,i} \subset f^{-1}(U_s) \quad \text{for} \quad s \in S \quad \text{and} \quad i = 1,2,\ldots$$

(8) $$f(G_{s,i}) \cap f(E_{s,i-1}) = \emptyset, \quad \text{where} \quad E_{s,i-1} = X \setminus \bigcup_{t\geq s} G_{t,i-1}, \quad \text{for} \quad i > 1.$$

The existence of $\mathcal{G}_1$ follows from Lemma 5.3.5. Assume that the covers $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{k-1}$ are already defined and satisfy (7) and (8) for $i < k$. Since the mapping f is closed, the sets

(9) $$W_{s,k} = f^{-1}(U_s) \setminus f^{-1}f(E_{s,k-1}) \subset f^{-1}(U_s)$$

are open. For every x denote by $s(x)$ the smallest element in S such that $x \in f^{-1}(U_{s(x)})$; since $E_{s(x),k-1} \subset \bigcup_{s<s(x)} G_{s,k-1}$ and, by (7), $\bigcup_{s<s(x)} G_{s,k-1} \subset \bigcup_{s<s(x)} f^{-1}(U_s)$, we have $f^{-1}f(E_{s(x),k-1}) \subset \bigcup_{s<s(x)} f^{-1}(U_s)$. Hence $x \in W_{s(x),k}$; i.e., $\{W_{s,k}\}_{s\in S}$ is an open cover of X. Applying Lemma 5.3.5, we obtain a point-finite open cover $\mathcal{G}_k = \{G_{s,k}\}_{s\in S}$ of the space X such that $G_{s,k} \subset W_{s,k}$ for every $s \in S$. It follows from (9) that the cover $\mathcal{G}_k$ satisfies conditions (7) and (8) for $i = k$, so that the construction of the covers $\mathcal{G}_i$ is concluded.

Observe that for every $s_0 \in S$ and $i = 1,2,\ldots$

(10) $$E_{s_0,i} = X \setminus \bigcup_{s\geq s_0} G_{s,i} \subset \bigcup_{s<s_0} (X \setminus \bigcup_{t>s} G_{t,i}).$$

Indeed, if $x \notin \bigcup_{s\geq s_0} G_{s,i}$, then there exists an $s < s_0$ such that $x \notin \bigcup_{t>s} G_{t,i}$, namely the largest $s \in S$ such that $x \in G_{s,i}$. Similarly,

(11) $$X = \bigcup_{s\in S} (X \setminus \bigcup_{t>s} G_{t,i}) \quad \text{for} \quad i = 1,2,\ldots$$

Let us consider the open sets $V_{s,i} = Y \setminus f(X \setminus G_{s,i})$. Since

(12) $$f^{-1}(V_{s,i}) \subset G_{s,i} \quad \text{for} \quad s \in S \quad \text{and} \quad i = 1,2,\ldots,$$

the family $\mathcal{V}_i = \{V_{s,i}\}_{s\in S}$ is point-finite for $i = 1,2,\ldots$ We shall show that the family $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a cover of the space Y. For every $y \in Y$ denote by $s(y)$ the smallest element in S, which exists by virtue of (11), such that $y \in f(X \setminus \bigcup_{t>s(y)} G_{t,i})$ for some positive integer i and take an integer $i(y)$ such that $y \in f(X \setminus \bigcup_{t>s(y)} G_{t,i(y)-1})$. Hence $y \in f(E_{s,i(y)-1})$

for every $s > s(y)$ and, by virtue of (8), $y \notin \bigcup_{s>s(y)} f(G_{s,i(y)}) = f(\bigcup_{s>s(y)} G_{s,i(y)})$, i.e., $f^{-1}(y) \cap \bigcup_{s>s(y)} G_{s,i(y)} = \emptyset$. On the other hand, by virtue of (10),

$$f(X \setminus \bigcup_{s \geq s(y)} G_{s,i(y)}) \subset f(\bigcup_{s<s(y)} (X \setminus \bigcup_{t>s} G_{t,i(y)})) = \bigcup_{s<s(y)} f(X \setminus \bigcup_{t>s} G_{t,i(y)}).$$

Since y does not belong to the last union, we have $f^{-1}(y) \subset \bigcup_{s \geq s(y)} G_{s,i(y)}$; so that $f^{-1}(y) \subset G_{s(y),i(y)}$, i.e., $y \in V_{s(y),i(y)}$. Hence $\mathcal{V}$ is a cover of Y. To conclude the proof it suffices to observe that from (7) and (12) it follows that $\mathcal{V}$ is a refinement of $\{U_s\}_{s\in S}$.■

Now, we pass to the class of strongly paracompact spaces. A topological space X is called *strongly paracompact* * if X is a Hausdorff space and every open cover of X has a star-finite open refinement. Every strongly paracompact space is paracompact (and, *a fortiori*, normal) but not vice versa (see Exercises 5.3.F(b) and 6.1.E).

Our discussion of strongly paracompact spaces will be preceded by two simple lemmas on families of sets. Let $\mathcal{A} = \{A_s\}_{s\in S}$ be a family of subsets of a set X; by a *chain* from A_s to $A_{s'}$ we mean a finite sequence $A_{s_1}, A_{s_2}, \ldots, A_{s_k}$ of members of $\mathcal{A}$ such that $s_1 = s, s_k = s'$ and $A_{s_i} \cap A_{s_{i+1}} \neq \emptyset$ for $i = 1, 2, \ldots, k-1$. We say that the family $\mathcal{A}$ is *connected* if for every pair $A_s, A_{s'}$ of members of $\mathcal{A}$ there exists a chain from A_s to $A_{s'}$. For any family $\mathcal{A}$ the *components* of $\mathcal{A}$ are defined as maximal connected subfamilies of $\mathcal{A}$, i.e., connected subfamilies of $\mathcal{A}$ which are not proper subsets of any connected subfamily of $\mathcal{A}$.

5.3.8. LEMMA. *Every family $\mathcal{A}$ of sets decomposes into the union of its components. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are distinct components of $\mathcal{A}$, then $(\bigcup \mathcal{A}_1) \cap (\bigcup \mathcal{A}_2) = \emptyset$.*■

5.3.9. LEMMA. *Every connected star-countable family of sets is countable.*■

5.3.10. THEOREM. *For every regular space X the following conditions are equivalent:*

(i) *The space X is strongly paracompact.*

(ii) *Every open cover of the space X has a closed refinement which is both locally finite and star-finite.*

(iii) *Every open cover of the space X has a closed refinement which is both locally finite and star-countable.*

(iv) *Every open cover of the space X has a star-countable open refinement.*

PROOF. We shall show that (i)$\Rightarrow$(ii). Let $\mathcal{U}$ be an open cover of the space X and $\mathcal{V} = \{V_s\}_{s\in S}$ a star-finite open refinement of $\mathcal{U}$. Since the space X is normal and the cover $\mathcal{V}$ is locally finite, by virtue of Theorem 1.5.18 there exists a closed cover $\mathcal{F} = \{F_s\}_{s\in S}$ of the space X such that $F_s \subset V_s$ for every $s \in S$. Clearly, $\mathcal{F}$ refines $\mathcal{U}$ and is both locally finite and star-finite.

The implication (ii)$\Rightarrow$(iii) is obvious; let us show that (iii)$\Rightarrow$(iv). Consider an open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of the space X and a closed refinement $\mathcal{F}$ which is both locally finite and star-countable. Let $\mathcal{F} = \bigcup_{t\in T} \mathcal{F}_t$, where the $\mathcal{F}_t$'s are the components of $\mathcal{F}$. It follows from Lemma 5.3.9 that all families $\mathcal{F}_t$ are countable; let $\mathcal{F}_t = \{F_{t,i}\}_{i=1}^{\infty}$. The sets $C_t = \bigcup \mathcal{F}_t$ are pairwise disjoint and – by the local finiteness of $\mathcal{F}$ – are closed-and-open. For every

* The term *hypocompact* is also used.

$t \in T$ and any natural number i take an $s(t,i) \in S$ such that $F_{t,i} \subset U_{s(t,i)}$. The family $\{C_t \cap U_{s(t,i)}\}_{i=1,t\in T}^{\infty}$ is a star-countable open refinement of $\mathcal{U}$.

The first step in the proof that (iv)$\Rightarrow$(i) is to show that every regular space X satisfying (iv) is paracompact. Let $\mathcal{U}$ be an open cover of the space X and $\mathcal{V}$ a star-countable open refinement of $\mathcal{U}$. Let $\{\mathcal{V}_t\}_{t\in T}$ be the family of all components of $\mathcal{V}$; by virtue of Lemma 5.3.9 we have $\mathcal{V}_t = \{V_{t,i}\}_{i=1}^{\infty}$ for every $t \in T$. The family $\{V_{t,i}\}_{t\in T}$ is discrete for $i = 1, 2, \ldots$, so that $\mathcal{V}$ is a σ-locally finite open refinement of $\mathcal{U}$. Hence the space X is paracompact by Theorem 5.1.11.

Now, consider an open cover $\mathcal{U}$ of the space X, and take a star-countable open refinement $\mathcal{V}$ of $\mathcal{U}$. Let $\{\mathcal{V}_t\}_{t\in T}$ be the family of all components of $\mathcal{V}$ and $C_t = \bigcup \mathcal{V}_t$ for every $t \in T$. The sets C_t are open and pairwise disjoint so that they are closed-and-open. It follows from the last paragraph that the space C_t is paracompact for every $t \in T$, and by virtue of Theorem 5.2.6 the countable open cover $\mathcal{V}_t$ of the space C_t has a star-finite open refinement $\mathcal{U}_t$. The union $\bigcup_{t\in T} \mathcal{U}_t$ is a star-finite open refinement of the cover $\mathcal{U}$. ■

Let us observe that the assumption of local finiteness in (ii) and (iii) above cannot be omitted; indeed, for every T_1-space X the family of all one-point subsets of X is a star-finite closed refinement of any cover of X.

The last theorem yields

5.3.11. COROLLARY. *Every Lindelöf space is strongly paracompact.* ■

5.3.12. EXAMPLE. It follows from the above corollary and Example 3.8.15 that the Cartesian product of two strongly paracompact spaces is not necessarily strongly paracompact. ■

Historical and bibliographic notes

Weakly paracompact spaces were introduced by Arens and Dugundji in [1950]; their paper contains Theorems 5.3.1 and 5.3.2 (another proof of the latter is suggested in the hint to Problem 3.12.23(a)). Theorem 5.3.3 was proved independently by Michael in [1955] and Nagami in [1955]; Theorem 5.3.7 was proved by Worrell in [1966].

Strongly paracompact spaces were defined by Dowker in [1947]. Theorem 5.3.10 was proved by Smirnov in [1956]; Corollary 5.3.11 appeared earlier in Morita's paper [1948] (Kaplan proved in [1947] that every separable metrizable space is strongly paracompact).

Exercises

5.3.A. (a) (Traylor [1964], Hodel [1969]) Prove that if a space X is locally separable and every open cover of X has a point-countable open refinement, then every open cover of X has a star-countable open refinement. Deduce that every locally separable, weakly paracompact regular space is strongly paracompact.

(b) (A. H. Stone [1962]) Show that if every open cover of a space X has a point-countable open refinement and for every $x \in X$ there exists a neighbourhood U of x such that $w(U) \leq \mathfrak{m}$, then $X = \bigoplus_{s\in S} X_s$, where $w(X_s) \leq \mathfrak{m}$ for every $s \in S$ (cf. Exercise 4.4.F(c)).

(c) (Charlesworth [1976]) Show that if a topological space X has a base $\mathcal{B}$ with the property that for every point $x \in X$ there exists a neighbourhood U such that the family

$\{V \in \mathcal{B} : U \cap V \neq \emptyset\}$ has cardinality $\leq \mathfrak{m}$, then $X = \bigoplus_{s\in S} X_s$, where $w(X_s) \leq \mathfrak{m}$ for every $s \in S$.

Hint. Consider those sets in $\mathcal{B}$ that meet at most $\mathfrak{m}$ members of $\mathcal{B}$.

5.3.B. (a) Check that the Niemytzki plane, the square of the Sorgenfrey line, the space $N^{\aleph_1}$ and the space of all countable ordinals are not weakly paracompact.

Hint. Apply Exercise 5.3.A(a).

(b) Show that the Cartesian product $A(\aleph_0) \times A(\mathfrak{c})$ is hereditarily weakly paracompact.

(c) Verify that the space X in Example 5.3.4 is perfect but not countably paracompact.

5.3.C. (a) Note that weak paracompactness is an additive property but is not finitely multiplicative.

(b) (Čoban [1970]) Show that weak paracompactness is hereditary with respect to F_σ-sets.

Remark. The limit of an inverse sequence of weakly paracompact spaces need not be weakly paracompact (cf. Problem 5.5.4(c)).

5.3.D (Michael [1955], Nagami [1955]). (a) Show that for every countable point-finite open cover $\{W_i\}_{i=1}^\infty$ of a normal space X there exists a locally finite open cover $\{V_i\}_{i=1}^\infty$ of X such that $V_i \subset W_i$ for $i = 1, 2 \ldots$

Hint. Apply either Theorem 1.5.18 and Lemma 5.2.4 or Exercise 5.1.J(d).

(b) Show that every point-finite open cover of a collectionwise normal space has a locally finite open refinement.

Hint. Observe that the cover $\{W_i\}_{i=1}^\infty$ defined in the proof of Theorem 5.3.3 is point-finite and apply part (a).

5.3.E (Worrell [1966]). Prove Theorem 5.1.33 by modifying the proof of Theorem 5.3.7.

Hint. Let the covers $\mathcal{G}_1, \mathcal{G}_2, \ldots$ in the proof of Theorem 5.3.7 be locally finite and take a closed cover $\{F_i\}_{i=1}^\infty$ of X such that $F_i \subset f^{-1}(V_i)$, where $V_i = \bigcup \mathcal{V}_i$. For $i = 1, 2, \ldots$ consider the open cover $\mathcal{V}_i \cup \{Y \setminus f(F_i)\}$ of the space Y and apply Theorem 1.5.18. Observe that every point-finite closure-preserving (see Exercise 5.1.G) family of closed sets is locally finite.

5.3.F. (a) Observe that strong paracompactness is an additive property and is hereditary with respect to closed sets.

(b) (Nagata [1957]) Prove that the Cartesian product of the open unit interval (0,1) and the Baire space $B(\aleph_1)$ is not strongly paracompact. Deduce that the limit of an inverse sequence of strongly paracompact spaces need not be strongly paracompact (cf. Problem 5.5.4(c)).

(c) Show that the Baire space $B(\mathfrak{m})$ is strongly paracompact for every $\mathfrak{m} \geq \aleph_0$ (cf. Exercise 7.2.E and Example 7.3.14).

Hint. The union of any family of $1/i$-balls is closed-and-open.

(d) Observe that (b), (c) and Exercise 5.3.H(b) below imply that – even in the class of metrizable spaces – strong paracompactness is not hereditary with respect to F_σ-sets and is not finitely multiplicative.

(e) Give an example of a metrizable space which can be represented as the union of two closed strongly paracompact subspaces and yet is not strongly paracompact.

Remark. Yasui proved in [1967] that if a regular space X has a locally finite closed cover consisting of strongly paracompact subspaces whose boundaries in X all have the local

Lindelöf property (i.e. every point has a neighbourhood with Lindelöf closure), then the space X is strongly paracompact.

(f) (Morita [1954]) Show that every strongly paracompact metrizable space of weight $\mathfrak{m} \geq \aleph_0$ is embeddable in the Cartesian product $I^{\aleph_0} \times B(\mathfrak{m})$.

5.3.G. (a) Show that strong paracompactness is equivalent to paracompactness in the realm of locally compact spaces.

(b) Give an example of a non-normal weakly paracompact locally compact space.

Hint. See Exercise 5.3.B(b).

5.3.H. (a) (Hanai [1956]) Show that weak paracompactness and strong paracompactness are inverse invariants of perfect mappings. Note that the class of weakly paracompact spaces is perfect.

(b) (Begle [1949] for strong paracompactness) Note that the Cartesian product $X \times Y$ of a weakly (strongly) paracompact space X and a compact space Y is weakly (strongly) paracompact.

(c) (Ponomarev [1962]) Observe that strong paracompactness is not an invariant of perfect mappings.

Hint. Apply Exercise 5.3.F(e) and Theorem 3.7.22.

(d) (Ponomarev [1962a]) Prove that strong paracompactness is an invariant of open perfect mappings.

Hint. Let $f: X \to Y$ be an open perfect mapping of X onto Y; verify that if $\{U_s\}_{s\in S}$ is a star-finite open cover of X, then the sets

$$U(s_1, s_2, \ldots, s_k) = \bigcap_{i=1}^{k} f(U_{s_i}) \cap (Y \setminus f(X \setminus \bigcup_{i=1}^{k} U_{s_i})),$$

where $s_1, s_2, \ldots, s_k \in S$ and $k = 1, 2, \ldots$, form a star-finite open cover of Y.

(e) Show that if there exists an open mapping $f: X \to Y$ of a paracompact space X onto a Hausdorff space Y such that all fibers $f^{-1}(y)$ are compact, then the space Y is weakly paracompact.

5.4. Metrization theorems II

In this section the notions of paracompactness, collectionwise normality and weak paracompactness are applied to establish further topological characterizations of the class of metrizable spaces.

A sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of covers of a topological space X is called a *development for the space* X if all covers $\mathcal{W}_i$ are open, and for every point $x \in X$ and any neighbourhood U of x there exists a natural number i such that $\mathrm{St}\,(x, \mathcal{W}_i) \subset U$. One easily observes that a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers of a topological space X is a development for X if and only if for every $x \in X$ any family $\{W_i\}_{i=1}^{\infty}$ such that $x \in W_i \in \mathcal{W}_i$ for $i = 1, 2, \ldots$ is a base for X at the point x.

5.4.1. BING'S METRIZATION CRITERION. *A topological space is metrizable if and only if it is collectionwise normal and has a development.*

PROOF. Let X be a metrizable space and ρ a metric on the space X. For every discrete family $\{F_s\}_{s\in S}$ of closed subsets of X, the family $\{U_s\}_{s\in S}$, where $U_s = \{x \in X : \rho(x, F_s) < \rho(x, \bigcup_{s'\neq s} F_{s'})\}$, satisfies conditions in Theorem 5.1.17, so that X is collectionwise normal. One can easily check that the sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$, where $\mathcal{W}_i = \{B(x, 1/i)\}_{x\in X}$, is a development for X.

Consider now a collectionwise normal space X which has a development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ To begin, we shall show that any open cover $\{U_s\}_{s\in S}$ of the space X has a σ-locally finite open refinement (which, by virtue of Theorem 5.1.11, means that X is paracompact).

Take a well-ordering relation $<$ on the set S and let

$$F_{s,i} = X \setminus [\mathrm{St}\,(X \setminus U_s, \mathcal{W}_i) \cup \bigcup_{s'<s} U_{s'}] \subset U_s.$$

The closed sets $F_{s,i}$, where $s \in S$ and $i = 1, 2, \ldots$, form a cover of the space X. Indeed, taking for any $x \in X$ the smallest element $s(x)$ in S such that $x \in U_{s(x)}$, and a natural number $i(x)$ such that $\mathrm{St}\,(x, \mathcal{W}_{i(x)}) \subset U_{s(x)}$ we have $x \in F_{s(x),i(x)}$. Since for a fixed i the neighbourhood $U_{s(x)} \cap \mathrm{St}\,(x, \mathcal{W}_i)$ of the point x meets only one member of the family $\mathcal{F}_i = \{F_{s,i}\}_{s\in S}$, namely the set $F_{s(x),i}$, the family $\mathcal{F}_i$ is discrete. By the collectionwise normality of X, there exist open sets $U_{s,i}$ such that $F_{s,i} \subset U_{s,i} \subset U_s$ for $s \in S$ and $i = 1, 2, \ldots$ and that for every i the family $\{U_{s,i}\}_{s\in S}$ is discrete. Hence $\{U_{s,i}\}_{i=1,s\in S}^{\infty}$ is a σ-locally finite open refinement of $\{U_s\}_{s\in S}$.

Now, for $i = 1, 2, \ldots$, let $\mathcal{B}_i$ be a σ-locally finite open refinement of the cover $\mathcal{W}_i$. One readily sees that $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a base for the space X, so that X is metrizable by the Nagata-Smirnov metrization theorem. ■

Let us observe that, when proving the invariance of metrizability under perfect mappings (see Theorem 4.4.15), we first defined a development for the space X and then, using the fact that Y is paracompact (established earlier in Lemma 4.4.13), we defined a σ-locally finite base for Y exactly as in the proof of Bing's metrization criterion.

We shall now show how, by a strengthening of the concept of a development, one can obtain a necessary and sufficient condition for metrizability of T_0-spaces.

A sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of covers of a topological space X is called a *strong development for the space* X if all covers $\mathcal{W}_i$ are open and for every point $x \in X$ and any neighbourhood U of x there exist a neighbourhood V of the point x and a natural number i such that $\mathrm{St}\,(V, \mathcal{W}_i) \subset U$. Clearly, every strong development is a development.

5.4.2. THE MOORE METRIZATION THEOREM. *A topological space is metrizable if and only if it is a T_0-space and has a strong development.*

PROOF. One can easily check that, for any metric ρ on a space X, the sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$, where $\mathcal{W}_i = \{B(x, 1/i)\}_{x\in X}$, is a strong development for X.

Consider now a T_0-space X which has a strong development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ By virtue of Bing's metrization criterion, it suffices to show that X is collectionwise normal.

To begin, we shall show that X is a T_1-space. Take a pair x, y of distinct points of X; since X is a T_0-space, there exists an open set U containing exactly one of the points x and y; we can assume that $x \in U$ and $y \notin U$. Let us take a natural number i such that $\mathrm{St}\,(x, \mathcal{W}_i) \subset U$ and a set $W \in \mathcal{W}_i$ that contains y; since $x \notin W$, we see that X is a T_1-space.

Now, for any discrete family $\{F_s\}_{s\in S}$ of closed subsets of X we shall define a family $\{U_s\}_{s\in S}$ of open sets such that $F_s \subset U_s$ for every $s \in S$ and $U_s \cap U_{s'} = \emptyset$ whenever $s \neq s'$; by virtue of Theorem 5.1.17, this will conclude the proof.

For every $s \in S$ and each $x \in F_s$ there exists a neighbourhood V_x of x and a natural number $i(x)$ such that $\mathrm{St}\,(V_x, \mathcal{W}_{i(x)}) \subset X \setminus \bigcup_{s'\neq s} F_{s'}$, i.e., $V_x \cap \mathrm{St}\,(\bigcup_{s'\neq s} F_{s'}, \mathcal{W}_{i(x)}) = \emptyset$. From the last equality it follows that the sets

$$W_{s,i} = \bigcup\{V_x : x \in F_s \quad\text{and}\quad i(x) = i\}, \quad\text{where}\quad s \in S \quad\text{and}\quad i = 1, 2, \ldots,$$

satisfy $W_{s,i} \cap \mathrm{St}\,(\bigcup_{s'\neq s} F_{s'}, \mathcal{W}_i) = \emptyset$. Hence $\mathrm{St}\,(F_s, \mathcal{W}_i) \cap \bigcup_{s'\neq s} W_{s',i} = \emptyset$ and

$$(1) \qquad F_s \cap \overline{\bigcup_{s'\neq s} W_{s',i}} = \emptyset \quad\text{for}\quad s \in S \quad\text{and}\quad i = 1, 2, \ldots$$

For every $s \in S$ let $U_s = \bigcup_{i=1}^{\infty} G_{s,i}$, where $G_{s,i} = W_{s,i} \setminus \bigcup_{j\leq i} \overline{\bigcup_{s'\neq s} W_{s',j}}$; the sets U_s are open, and – by (1) and the obvious inclusion $F_s \subset \bigcup_{i=1}^{\infty} W_{s,i}$ – we have $F_s \subset U_s$ for every $s \in S$. Since $G_{s,i} \cap G_{s',j} = \emptyset$ for $s \neq s'$ and $i, j = 1, 2, \ldots$, $U_s \cap U_{s'} = \emptyset$ whenever $s \neq s'$.■

The next two metrization theorems will be formulated in terms of special bases.

We say that a base $\mathcal{B}$ for a topological space X is *point-regular** if for every point $x \in X$ and any neighbourhood U of x the set of all members of $\mathcal{B}$ that contain x and meet $X \setminus U$ is finite. One easily observes that a base $\mathcal{B}$ for a space X is point-regular if and only if for every $x \in X$ any family consisting of $\aleph_0$ members of $\mathcal{B}$ which all contain x is a base for X at the point x. It is easy to check that if $\mathcal{B}$ is a point-regular base for a space X, then $|\mathcal{B}| \leq \aleph_0 \cdot d(X)$.

We say that a base $\mathcal{B}$ for a topological space X is *regular* if for every point $x \in X$ and any neighbourhood U of x there exists a neighbourhood $V \subset U$ of the point x such that the set of all members of $\mathcal{B}$ that meet both V and $X \setminus U$ is finite. Clearly, every regular base is point-regular.

The proofs of Theorems 5.4.6 and 5.4.8 are based on a few lemmas that we are now going to prove; to simplify the statements of the lemmas, for a family $\mathcal{A}$ of sets we shall denote by $\mathcal{A}^m$ the subfamily of $\mathcal{A}$ consisting of all maximal elements (i.e., of sets $A \in \mathcal{A}$ such that if $A \subset A' \in \mathcal{A}$, then $A = A'$), and for a topological space X we shall denote by $\mathcal{I}(X)$ the family of all open one-point subsets of X.

5.4.3. LEMMA. *If $\mathcal{B}$ is a point-regular (regular) base for a space X, then the family $\mathcal{B}^m \subset \mathcal{B}$ is a point-finite (locally finite) cover of X.*

PROOF. First, we shall show that $\bigcup \mathcal{B}^m = X$. For every $x \in X$ there exists a $U_0 \in \mathcal{B}$ that contains x; assuming that x is not contained in any member of $\mathcal{B}^m$, we can define an infinite sequence $U_0 \subset U_1 \subset U_2 \subset \ldots$ of members of $\mathcal{B}$ such that $U_i \neq U_{i+1}$ for $i = 1, 2, \ldots$, i.e., an infinite subfamily $\{U_i\}_{i=1}^{\infty}$ of $\mathcal{B}$ whose members contain x and meet $X \setminus U_0$, which is impossible. Hence, $x \in \bigcup \mathcal{B}^m$ and $\bigcup \mathcal{B}^m = X$.

Now, we shall show that if the base $\mathcal{B}$ is point-regular (regular), then the cover $\mathcal{B}^m$ is point-finite (locally finite). Take a point $x \in X$ and a set $U \in \mathcal{B}^m$ which contains x. The set of all members of $\mathcal{B}$ that meet both $X \setminus U$ and $\{x\}$ (and a neighbourhood V of x) is finite. However, every $U' \in \mathcal{B}^m \setminus \{U\}$ that contains x (meets V) also meets $X \setminus U$, so that

* The term *uniform base* is also used.

only finitely many members of $\mathcal{B}^m$ contain x (meet V), which shows that $\mathcal{B}^m$ is a point-finite (locally finite) cover of X.■

5.4.4. LEMMA. *If $\mathcal{B}$ is a base for a T_1-space, then for every point-finite cover $\mathcal{B}' \subset \mathcal{B}$ the family $\mathcal{B}'' = (\mathcal{B} \setminus \mathcal{B}') \cup \mathcal{I}(X)$ is a base for X. Moreover, if the base $\mathcal{B}$ is point-regular (regular), then the base $\mathcal{B}''$ also is point-regular (regular).*

PROOF. Let x be a point of X and U a neighbourhood of x. If x is an isolated point, then $\{x\} \in \mathcal{I}(X)$ and $x \in \{x\} \subset U$; if the point x is not isolated, then the intersection $\bigcap\{W \in \mathcal{B}' : x \in W\}$ contains a point $y \neq x$, and any neighbourhood $V \in \mathcal{B}$ of the point x satisfying $V \subset U \setminus \{y\}$ belongs to $\mathcal{B} \setminus \mathcal{B}'$. The second part of the lemma is obvious.■

5.4.5. LEMMA. *If $\mathcal{B}$ is a point-regular (regular) base for a T_1-space X, then letting*

$$(2) \qquad \mathcal{B}_1 = \mathcal{B}^m \quad \textit{and} \quad \mathcal{B}_i = [(\mathcal{B} \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_j) \cup \mathcal{I}(X)]^m \quad \textit{for } i = 2,3,\ldots$$

we define a sequence $\mathcal{B}_1, \mathcal{B}_2, \ldots$ of point-finite (locally finite) open covers of X such that $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$.

PROOF. Lemmas 5.4.3 and 5.4.4 imply that $\mathcal{B}_1, \mathcal{B}_2, \ldots$ is a sequence of point-finite (locally finite) open covers of X. The base $\mathcal{B}$ being point-regular, for every $U \in \mathcal{B}$ only finitely many sets $U' \in \mathcal{B}$ satisfy the inclusion $U \subset U'$, so that we have $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ by virtue of (2).■

5.4.6. THE ARHANGEL'SKIĬ METRIZATION THEOREM. *A topological space is metrizable if and only if it is a T_1-space and has a regular base.*

PROOF. To begin, let us check that every metrizable space X has a regular base. Let ρ be a metric on the space X and $\mathcal{B}_i$ a locally finite open refinement of the cover $\{B(x, 1/4i)\}_{x \in X}$ (see Theorem 4.4.1); clearly, $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a base for X. For every point $x \in X$ and any neighbourhood U of x, there exists a natural number i such that $B(x, 1/i) \subset U$. Let $V_0 = B(x, 1/2i)$, and for $j = 1, 2, \ldots, i$ let V_j be a neighbourhood of x that meets only finitely many members of $\mathcal{B}_j$. One easily checks that the set of all members of $\mathcal{B}$ that meet both $V = \bigcap_{j=0}^{i} V_j$ and $X \setminus U$ is finite.

Consider now a T_1-space X which has a regular base $\mathcal{B}$. Since, as one readily verifies, the sets U and V in the definition of a regular base satisfy the inclusion $\overline{V} \subset U$, it follows that X is a regular space. To conclude the proof it suffices to apply Lemma 5.4.5 and the Nagata-Smirnov metrization theorem.■

In order to explain the relation between developments and point-regular bases, our next lemma is formulated in a more general way than is needed in the proof of Theorem 5.4.8.

5.4.7. LEMMA. *For every Hausdorff space X the following conditions are equivalent:*

(i) *The space X has a point-regular base.*

(ii) *The space X is weakly paracompact and has a development.*

(iii) *The space X has a development consisting of point-finite covers.*

PROOF. We shall show first that (i)⇒(ii). Let $\mathcal{U}$ be an open cover of a Hausdorff space X which has a point-regular base $\mathcal{B}$. The family $\mathcal{B}_0$ of all members of $\mathcal{B}$ that are contained in a member of $\mathcal{U}$ clearly form a point-regular base for the space X, so that – by Lemma 5.4.3 – the family $\mathcal{B}_0^m$ is a point-finite open refinement of $\mathcal{U}$. Hence X is a weakly paracompact space.

It remains to prove that the space X has a development. Let us consider the sequence $\mathcal{B}_1, \mathcal{B}_2, \ldots$ of open covers of X defined in (2); by virtue of Lemma 5.4.5 we have $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$. For a point $x \in X$ and any neighbourhood U of x, only finitely many members of $\mathcal{B}$, say sets $U_1, U_2, \ldots, U_k$, contain x and meet $X \setminus U$. Since $\mathcal{B}_i \cap \mathcal{B}_j \subset \mathcal{I}(X)$ whenever $i \neq j$, and the sets $U_1, U_2, \ldots, U_k$ contain more than one point, there exists a natural number i such that $U_j \notin \mathcal{B}_i$ for $j = 1, 2, \ldots, k$. Hence we have $\mathrm{St}(x, \mathcal{B}_i) \subset U$, which shows that $\mathcal{B}_1, \mathcal{B}_2, \ldots$ is a development for X.

The implication (ii)$\Rightarrow$(iii) is obvious. We shall show that (iii)$\Rightarrow$(i). Suppose that $\mathcal{W}_1, \mathcal{W}_2, \ldots$ is a development for a space X consisting of point-finite covers; without loss of generality we can assume that $\mathcal{W}_j$ is a refinement of $\mathcal{W}_i$ whenever $j \geq i$. Clearly, the family $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{W}_i$ is a base for the space X. This base is point-regular, because for every $x \in X$ and any neighbourhood U of x there exists a natural number i such that $\mathrm{St}(x, \mathcal{W}_j) \subset U$ for $j \geq i$ and in each of the covers $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{i-1}$ there are only finitely many members that contain x. ■

5.4.8. ALEXANDROFF'S METRIZATION CRITERION. *A topological space is metrizable if and only if it is collectionwise normal and has a point-regular base.*

PROOF. Every metrizable space is collectionwise normal and, by 5.4.6, has a point-regular base.

Every collectionwise normal space which has a point-regular base is metrizable by virtue of Lemma 5.4.7 and Bing's metrization criterion. ■

Let us observe that, in the last proof, instead of using Bing's metrization criterion one could apply Lemma 5.4.7, the Michael-Nagami theorem and the Nagata-Smirnov metrization theorem.

We conclude this section with the chronologically earliest metrization theorem.

5.4.9. THE ALEXANDROFF-URYSOHN METRIZATION THEOREM. *A topological space is metrizable if and only if it is a T_0-space and has a development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ such that for every natural number i and any two sets $W_1, W_2 \in \mathcal{W}_{i+1}$ with non-empty intersection there exists a set $W \in \mathcal{W}_i$ such that $W_1 \cup W_2 \subset W$.*

PROOF. One can easily check that for any metric ρ on a space X the sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$, where $\mathcal{W}_i = \{B(x, 1/2^i)\}_{x \in X}$, is a development for X that has the required property.

Consider now a T_0-space X and a development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ for X which has the property under consideration; we shall show that $\mathcal{W}_1, \mathcal{W}_2, \ldots$ is a strong development for X which, by virtue of the Moore metrization theorem, will conclude the proof. Let x be a point of X and U a neighbourhood of x. Take a natural number i such that $\mathrm{St}(x, \mathcal{W}_i) \subset U$ and a $V \in \mathcal{W}_{i+1}$ such that $x \in V$. For every $W \in \mathcal{W}_{i+1}$ that meets V there exists a $W' \in \mathcal{W}_i$ such that $V \cup W \subset W'$; since $x \in V$, we have $V \cup W \subset W' \subset \mathrm{St}(x, \mathcal{W}_i) \subset U$, i.e., $\mathrm{St}(V, \mathcal{W}_{i+1}) \subset U$. ■

5.4.10. COROLLARY. *A topological space is metrizable if and only if it is a T_0-space and has a development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ such that $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$* ■

Historical and bibliographic notes

Bing established his metrization criterion in [1951]. The notion of a development was introduced by Chittenden and Pitcher in [1919]; the existence of a development is one of R. L. Moore's axioms for abstract spaces (cf. notes in Section 1.1). Developments also appear in Alexandroff and Urysohn's paper [1923].

Regular spaces which have a development are called *Moore spaces.* Since Jones' paper [1937], where it was shown that, under the assumption of the inequality $2^{\aleph_0} < 2^{\aleph_1}$, every separable normal Moore space is metrizable, a great deal of work has been devoted to the problem whether every normal Moore space is metrizable (the assumption of normality is necessary: the Niemytzki plane is a Moore space). J. H. Silver, in collaboration with Tall, proved that the existence of a non-metrizable separable normal Moore space is consistent with the axioms of set theory (see Tall's thesis [1969]). From the results of Jones and Silver and Tall it follows that the existence of a non-metrizable separable normal Moore space is independent of the axioms of set theory. The above fact and Heath's results in [1964] imply that the existence of a non-metrizable separable normal space with a point-regular base also is independent of the axioms of set theory (cf. Exercise 5.4.B). Fleissner proved in [1982] that the existence of a non-metrizable normal Moore space follows from the continuum hypothesis. Results of Nyikos [1980] and Fleissner [1982] imply that the non-existence of such a space is related to large cardinal numbers axioms (for details see Fleissner [1984]).

The Moore metrization theorem has a long history. In the present form it was proved independently by A. H. Stone in [1960] and Arhangel'skiĭ in [1961a], where the notion of a strong development was introduced. In a slightly different form the theorem was given by R. L. Moore in [1935] (see also Jones [1966]) and was rediscovered by Morita in [1951]. The A. H. Frink metrization theorem established in [1937] (see Exercise 5.4.C) may be considered as yet another form of the same theorem. Let us also mention that Kuratowski in [1933] announced a metrization theorem due to Aronszajn that is quite close to Theorem 5.4.2.

The notion of a point-regular base and Theorem 5.4.8 appeared in Alexandroff's paper [1960]; the same paper contains the proof of the equivalence of conditions (i) and (iii) in Lemma 5.4.7 (the equivalence of conditions (i) and (ii) was established by Heath in [1964]). Theorem 5.4.6 was proved by Arhangel'skiĭ in [1960]; in the same paper the notion of a regular base was introduced and our proof of Theorem 5.4.8 was given.

Roughly speaking, the notions of a strong development and of a regular base are obtained from the notions of a development and of a point-regular base by replacing points by open sets; such a modification yields conditions equivalent to metrizability. Another strengthening of developments and point-regular bases that yields conditions equivalent to metrizability is discussed in Exercise 5.4.E.

Theorem 5.4.9 was proved by Alexandroff and Urysohn in [1923a]; Corollary 5.4.10 (called by some authors the Alexandroff-Urysohn theorem) appeared in Tukey's book [1940] – this is a weak form of the theorem on inducing uniformities by metrics (see Theorem 8.1.21): it states only that if the topology of a space X is induced by a uniformity $\mathcal{U}$ with a countable base, then the space X is metrizable while Theorem 8.1.21 states more, namely that there is a metric on the space X that induces the uniformity $\mathcal{U}$.

Many more metrization theorems are known; some of them are stated in Exercises below. There are also many ways to deduce them from each other. Clearly, the proof of the first theorem established has to contain a construction of a metric. In this book we started with the Nagata-Smirnov theorem; one could also start with Corollary 5.4.10 (the construction of a metric is sketched in Exercise 5.4.H(a)) – such an ordering of metrization theorems is presented in Nagata's book [1968] (cf. Exercise 5.4.I). Prior to the discovery of the Nagata-Smirnov theorem and Theorem 8.1.21, Chittenden's theorem (formulated here in Exercise 5.4.G), which reduces the existence of a metric to the existence of a function ρ with weaker properties, was usually applied as the first link. Although Chittenden's characterization of metrizability is not purely topological, it was an important step in the study of metrizability.

Exercises

5.4.A (Smirnov [1951a]). Note that every paracompact locally metrizable space (i.e., a space in which every point has a metrizable neighbourhood) is metrizable.

Hint. See Theorem 4.4.19.

5.4.B (Heath [1964]). Let X be the subset of the plane defined by the condition $y \geq 0$, i.e., the closed upper half-plane. Generate a topology on X letting all points above the x-axis be isolated and taking as a base at a point $(x, 0)$ the family of all segments starting at $(x, 0)$ which form with the x-axis an angle of 90° if x is rational and an angle of 45° if x is irrational. Prove that X is a completely regular non-normal space with a point-regular base. Show that X is weakly paracompact and Čech-complete but is not countably paracompact. Note that X can be represented as the union of two open metrizable subspaces.

Hint. When proving that X is not normal and that X is not countably paracompact, apply the Baire category theorem.

5.4.C (A. H. Frink [1937]). Show that a T_0-space X is metrizable if and only if it has a neighbourhood system $\{\mathcal{B}(x)\}_{x \in X}$, where $\mathcal{B}(x) = \{B_i(x)\}_{i=1}^{\infty}$, such that for every $x \in X$ and any natural number i there exists a j such that $B_j(y) \subset B_i(x)$ whenever $B_j(x) \cap B_j(y) \neq \emptyset$.

Hint. See the notes in this section.

5.4.D. (a) (Morita [1955]) Prove that a T_0-space X is metrizable if and only if it has a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally finite closed covers such that for every point $x \in X$ and any neighbourhood U of x there exists a natural number i such that $\mathrm{St}(x, \mathcal{F}_i) \subset U$.

Hint. One can assume that $\mathcal{F}_{i+1}$ is a refinement of $\mathcal{F}_i$ for $i = 1, 2, \ldots$ Check that the sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$, where $\mathcal{W}_i = \{\mathrm{Int}\,\mathrm{St}(x, \mathcal{F}_i)\}_{x \in X}$, is a strong development for X.

(b) Prove the Hanai-Morita-Stone theorem by applying the characterization of metrizability established in part (a).

5.4.E. (a) (Jones [1958]) Show that a T_2-space X is metrizable if and only if it has a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers such that for every compact set $Z \subset X$ and any open set U containing Z there exists a natural number i such that $\mathrm{St}(Z, \mathcal{W}_i) \subset U$.

Hint. Apply Theorem 5.4.2.

(b) (Arhangel'skiĭ [1965] (announcement [1963])) Show that a T_2-space X is metrizable if and only if it has a base $\mathcal{B}$ such that for every compact set $Z \subset X$ and any open set U containing Z the set of all members of $\mathcal{B}$ that meet both Z and $X \setminus U$ is finite.

Hint. Apply Theorem 5.4.6.

5.4.F (Nagata [1957a]). Prove that a T_0-space X is metrizable if and only if for every $x \in X$ there exists a sequence $U_1(x), U_2(x), \ldots$ of neighbourhoods of x and a sequence $A_1(x), A_2(x), \ldots$ of subsets of X satisfying the conditions:

(1) *For every $x \in X$ and any neighbourhood U of x there exists an i such that $A_i(x) \subset U$.*

(2) *If $U_i(x) \cap U_i(y) \neq \emptyset$, then $y \in A_i(x)$.*

(3) *If $y \in U_i(x)$, then $U_i(y) \subset A_i(x)$.*

Hint. One can assume that $U_j(x) \subset U_i(x)$ for $j \geq i$ and $x \in X$. Verify that the sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$, where $\mathcal{W}_i = \{U_i(x)\}_{x \in X}$, is a strong development for X.

5.4.G (Chittenden [1917]). Prove that a T_0-space X is metrizable if and only if there exists a function ρ defined on the set $X \times X$, assuming non-negative real values, satisfying for every $A \subset X$ the equivalence

$$\inf_{a \in A} \rho(x, a) = 0 \quad \text{if and only if} \quad x \in \overline{A},$$

and subject to conditions (M1) and (M2) in the definition of a metric as well as to the condition.

(M3′) *There exists a function f from the set of all non-negative real numbers to itself such that $\lim f(t_n) = 0$ whenever $\lim t_n = 0$ and that for all $x, y, z \in X$ the inequalities $\rho(x,y) \leq \epsilon$ and $\rho(y,z) \leq \epsilon$ imply the inequality $\rho(x,z) \leq f(\epsilon)$.*

5.4.H. (a) (Tukey [1940]) Prove that for every sequence $\mathcal{W}_0, \mathcal{W}_1, \ldots$ of covers of a set X, where $\mathcal{W}_0 = \{X\}$ and $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$, there exists a pseudometric ρ on the set X such that for every $i \geq 1$ and each $x \in X$

$$B(x, 1/2^i) \subset \mathrm{St}\,(x, \mathcal{W}_i) \subset \{y \in X : \rho(x,y) \leq 1/2^i\}.$$

Observe that if X is a topological space and the covers $\mathcal{W}_i$ are open, then ρ is a continuous mapping of $X \times X$ to R.

Applying Remark 4.4.2, deduce Lemma 5.1.16 from the above result.

Hint. For any pair x, y of points of X consider all finite sequences $x_0, x_1, \ldots, x_k$ of points of X such that $x_0 = x$ and $x_k = y$, and define $\rho(x,y)$ as the greatest lower bound of the numbers

$$1/2^{i_1} + 1/2^{i_2} + \ldots + 1/2^{i_k}, \quad \text{where} \quad x_j \in \mathrm{St}\,(x_{j-1}, \mathcal{W}_{i_j}) \quad \text{for} \quad j = 1, 2, \ldots, k.$$

Prove by induction with respect to k that for every sequence $x_0, x_1, \ldots, x_k$ of points of X if $x_j \in \mathrm{St}\,(x_{j-1}, \mathcal{W}_{i_j})$ for $j = 1, 2, \ldots, k$ and $1/2^{i_1} + 1/2^{i_2} + \ldots + 1/2^{i_k} < 1/2^i$, then $x_k \in \mathrm{St}\,(x_0, \mathcal{W}_i)$ (cf. the proof of Theorem 8.1.10).

(b) (Tukey [1940]) Verify that if $\mathcal{W}_0, \mathcal{W}_1, \ldots$, where $\mathcal{W}_0 = \{X\}$ and $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$, is a development for a T_0-space X, then the pseudometric ρ in part (a) is a metric on the space X (cf. Corollary 5.4.10).

(c) (Tukey [1940], Morita [1962] and [1964]) Show that for every open cover $\{U_s\}_{s \in S}$ of a topological space X the following conditions are equivalent:

(1) *The cover $\{U_s\}_{s \in S}$ is normal.*

(2) *There exist a pseudometric ρ on the set X, such that $\rho: X \times X \to R$ is continuous, and a cover $\{V_s\}_{s \in S}$ of the set X, which is open with respect to the topology induced by ρ, such that $V_s \subset U_s$ for every $s \in S$.*

(3) *There exist a continuous mapping $f: X \to Y$ of X onto a metrizable space Y and an open cover $\{W_s\}_{s\in S}$ of the space Y such that $f^{-1}(W_s) \subset U_s$ for every $s \in S$.*

(4) *The cover $\{U_s\}_{s\in S}$ has a locally finite refinement consisting of functionally open sets.*

(5) *The cover $\{U_s\}_{s\in S}$ has a σ-locally finite refinement consisting of functionally open sets.*

Hint. Apply Exercise 5.1.J.

(d) (A. H. Stone [1948]) Verify that an open cover of a normal space is normal if and only if it has a locally finite open refinement.

5.4.I (Nagata [1950]). Deduce from Corollary 5.4.10 that every regular space which has a σ-locally finite base is metrizable.

Hint. Let X be a regular space and $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ a base for X, where $\mathcal{B}_i = \{U_s\}_{s\in S_i}$ is a locally finite cover of X. For every i denote by $\mathcal{T}_i$ the family of all finite subsets of S_i and apply Lemma 4.4.5 to obtain for each $s \in S_i$ closed sets $F_{s,1}, F_{s,2}, \ldots$ such that $U_s = \bigcup_{j=1}^{\infty} F_{s,j}$, then for $T \in \mathcal{T}_i$ and $j = 1, 2, \ldots$ let $W_{T,j} = \bigcap_{s\in T} U_s \cap (X \setminus \bigcup_{s\notin T} F_{s,j})$. Show that for $i, j = 1, 2, \ldots$ the family $\mathcal{W}_{i,j} = \{W_{T,j}\}_{T\in\mathcal{T}_i}$ is a locally finite open cover of X and that the covers $\mathcal{W}_{i,j}$, when arranged into a sequence, form a development for the space X. Apply Theorem 1.5.18, Lemmas 5.1.13 and 5.1.15 and Remark 5.1.14 to obtain a development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ such that $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$ (one could also apply Theorems 5.1.11 and 5.1.12).

5.5. Problems

Collectionwise normality

5.5.1. (a) (McAuley [1958]) Prove that for every T_1-space X the following conditions are equivalent:

(1) *The space X is hereditarily collectionwise normal.*

(2) *Every open subspace of X is collectionwise normal.*

(3) *For every family $\{F_s\}_{s\in S}$ of subsets of X which is discrete in the union $F = \bigcup_{s\in S} F_s$ and consists of sets closed in F, there exists a family $\{U_s\}_{s\in S}$ of pairwise disjoint open subsets of X such that $F_s \subset U_s$ for every $s \in S$.*

(b) (Šediva [1959]) Show that collectionwise normality is hereditary with respect to F_σ-sets. Note that every perfectly normal collectionwise normal space is hereditarily collectionwise normal.

Hint. Cf. Problem 2.7.2(a).

(c) Prove that a T_1-space X is collectionwise normal if and only if, for each cardinal number $\mathfrak{m} \geq \aleph_0$, every continuous mapping $f: A \to J(\mathfrak{m})$ of a closed subspace A of X to the hedgehog $J(\mathfrak{m})$ is continuously extendable over X.

Hint. Let $f: A \to J(\mathfrak{m})$ be defined on a closed subspace A of a collectionwise normal space X. Consider the mapping $g: J(\mathfrak{m}) \to I$ defined by letting $g([(x,s)]) = x$ for every $x \in I$ and $s \in S$, where $[(x,s)]$ and S are as in Example 4.1.5, and take a continuous extension $G: X \to I$ of the composition $gf: A \to I$. Note that the sets $F_s = f^{-1}(\{[(x,s)] : 0 < x \leq 1\})$ form a discrete family of closed subsets of $G^{-1}((0,1])$. Applying (b), take a family $\{U_s\}_{s\in S}$ of pairwise disjoint open subsets of X such that $F_s \subset U_s$ for every $s \in S$. Extend the function $h: A \cup (X \setminus \bigcup_{s\in S} U_s) \to I$ defined by the conditions $h|A = gf$ and $h(X \setminus \bigcup_{s\in S} U_s) \subset \{0\}$ to a continuous function $H: X \to I$; let $F(x) = [(H(x),s)]$ for $x \in U_s$ and $F(x) = [(0,s)]$ for $x \in X \setminus \bigcup_{s\in S} U_s$.

Operation X_M

5.5.2. (a) Check that for every subspace M of a hereditarily normal space X the space X_M is hereditarily normal.

(b) Check that for every subspace M of a hereditarily collectionwise normal space X the space X_M is hereditarily collectionwise normal.

(c) (Furdzik [1968]) Show that for a subspace M of a perfectly normal space X the space X_M is perfectly normal if and only if M is a G_δ-set in X.

(d) Show that for a subspace M of a metrizable space X the space X_M is metrizable if and only if M is a G_δ-set in X.

Around Bing's and Michael's examples

5.5.3. (a) (Bing [1951]) Note that the space X in Example 5.1.23 is hereditarily normal but is not perfectly normal.

Consider the set $Z = (M \cup \{0\}) \cup \bigcup_{i=1}^{\infty}(X \times \{1/i\})$, and generate a topology on Z taking as a base at a point $(x, 0)$ the sets $\{(x,0)\} \cup \bigcup_{i=k}^{\infty}(U \times \{1/i\})$, where U is a neighbourhood of the point x in the space X and $k = 1, 2, \ldots$, and letting all the remaining points be isolated. Verify that the space Z is perfectly normal but is not collectionwise normal. Check that Z has a σ-discrete network and deduce that every open cover of the space Z has a closed σ-discrete refinement.

Remark. A topological space X with the property that every open cover of X has a closed σ-discrete refinement is called ***subparacompact***. The class of subparacompact spaces has been quite extensively studied. It was introduced by McAuley in [1958] under the name of ***F_σ-screenable*** spaces. Then, Arhangel'skiĭ defined in [1966b] the class of σ-paracompact spaces: a space X is ***σ-paracompact*** if for every open cover $\mathcal{U}$ of X there exists a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers of X with the property that for every point $x \in X$ there exists a natural number i and a set $U \in \mathcal{U}$ such that $\mathrm{St}(x, \mathcal{W}_i) \subset U$ (thus, every topological space which has a development is σ-paracompact). It was proved by Burke in [1969] that F_σ-screenable and σ-paracompact spaces coincide; he also introduced the term "subparacompact" and gave other characterizations of subparacompactness (in particular, he proved that a topological space X is subparacompact if and only if every open cover of X has a closed σ-locally finite refinement). Burke's paper [1984] is a survey of recent results on subparacompact spaces, weakly paracompact spaces and other classes of spaces defined by "covering properties".

(b) (Michael [1955]) Show that the space X in Example 5.1.23 and the space Z in part (a) are not weakly paracompact.

Hint. Apply Problem 2.7.11(b) to show that only countably many members of each point-finite open cover of X meet the set M.

(c) (Michael [1955]) Let S be the subspace of the space X in Example 5.1.23 consisting of all points of $D^{2^{\mathfrak{c}}}$ which have at most finitely many coordinates distinct from zero. Verify that the space $X_0 = S \cup M \subset X$ is normal and weakly paracompact but is not collectionwise normal. Give an example of a perfectly normal space with such properties.

5.5.4. (a) (Michael [1963]) Modifying Example 5.1.32, define a hereditarily paracompact space X which has the Lindelöf property and a separable metrizable space Y such that the Cartesian product $X \times Y$ is not normal (cf. Problem 5.5.5 and Exercise 5.1.F(c)).

Hint (Kuratowski and Sierpiński [1926]). Applying transfinite induction, prove that for every set X of cardinality $\mathfrak{c}$ and any family $\mathcal{C}$ of cardinality $\mathfrak{c}$, consisting of subsets of X which have all cardinality $\mathfrak{c}$, there exists a set $A \subset X$ such that $|A| = \mathfrak{c}$ and $A \cap C \neq \emptyset \neq (X \setminus A) \cap C$ for every $C \in \mathcal{C}$. Deduce, making use of Problem 4.5.5, that the real line contains a subset A such that $|A| = \mathfrak{c}$ and every compact set contained either in A or in $R \setminus A$ is countable (any set $A \subset R$ with the last property is called a *Bernstein set*; sets of cardinality $\mathfrak{c}$ with no uncountable compact subsets were first defined by Bernstein in [1908]).

(b) (Michael [1963]) Modifying Example 5.1.32, define a separable Lindelöf space X and a separable metrizable space Y such that the Cartesian product $X \times Y$ is not normal.

Hint. Define a separable Lindelöf space that contains the space X in (a) as a closed subspace.

(c) Define an inverse sequence of Lindelöf spaces whose limit is neither normal nor weakly paracompact.

Hint. Decompose R into disjoint sets $A_1, A_2, \ldots$ of cardinality $\mathfrak{c}$ such that for $i = 1, 2, \ldots$ every compact set contained in $R \setminus A_i$ is countable.

Paracompactness of Cartesian products

5.5.5. (a) (Michael [1953]) Prove that the Cartesian product $X \times Y$ of a perfectly normal paracompact space X and a metrizable space Y is paracompact.

Hint. Show that the Cartesian product $X \times Y$ satisfies condition (ii) in Theorem 5.1.11. Apply Problem 4.5.16(a) and the fact that the family $\mathcal{B}$ defined in the proof of Theorem 5.1.28 is σ-locally finite in X.

(b) (Michael [1953]) Prove that the Cartesian product $X \times Y$ of a paracompact space X and a regular σ-compact space Y is paracompact.

(c) (Morita [1953]) Prove that the Cartesian product $X \times Y$ of a paracompact space X and a locally compact paracompact space Y is paracompact.

(d) (Willard [1971]) Note that the Cartesian product $X \times Y$ of a Lindelöf space X and a space Y which contains a dense σ-compact set (in particular, of a Lindelöf space X and a separable space Y) is paracompact if and only if it has the Lindelöf property.

Hint. See Theorem 5.1.25.

(e) (Tamano [1962]) Deduce from the Tamano theorem that for every topological space X the following conditions are equivalent:

(1) *For every paracompact space Y the Cartesian product $X \times Y$ is paracompact.*

(2) *For every paracompact space Y the Cartesian product $X \times Y$ is normal.*

Remark. There is no internal characterization of the class of spaces whose Cartesian product with every paracompact space is paracompact (cf., however, Katuta [1971]); Telgársky's paper [1971] contains new contributions and a discussion of earlier results in that area (cf. Problem 5.5.9(b)).

5.5.6. (a) (A. H. Stone [1948]) Show that for every family $\{X_s\}_{s\in S}$ of metrizable spaces the following conditions are equivalent:

(1) *The Cartesian product* $\prod_{s\in S} X_s$ *is paracompact.*
(2) *The Cartesian product* $\prod_{s\in S} X_s$ *is collectionwise normal.*
(3) *The Cartesian product* $\prod_{s\in S} X_s$ *is normal.*
(4) *The family* $\{X_s\}_{s\in S}$ *contains at most countably many non-compact spaces.*

Hint. Apply Exercise 2.3.E(a).

Remark. As noted in the remark to Problem 2.7.16, there exists a non-normal Cartesian product $\prod_{i=1}^{\infty} X_i$ such that all finite Cartesian products $X_1 \times X_2 \times \ldots \times X_i$ are hereditarily paracompact (cf. Problem 5.5.19).

(b) (Nagami [1972]) Observe that for every family $\{X_s\}_{s\in S}$ of metrizable spaces the conditions in part (a) are equivalent to

(5) *The Cartesian product* $\prod_{s\in S} X_s$ *is countably paracompact.*

Hint. Apply Exercise 5.2.C(c).

Paracompact spaces with G_δ-diagonals

5.5.7. (a) (Okuyama [1964], Borges [1966]) Show that if X is a paracompact space and the diagonal Δ is a G_δ-set in $X \times X$, then there exists a one-to-one continuous mapping of X onto a metrizable space.

Hint. Define a countable family $\{\mathcal{V}_i\}_{i=1}^{\infty}$ of open covers of the space X such that $\mathcal{V}_{i+1}$ is a star refinement of $\mathcal{V}_i$ for $i = 1, 2, \ldots$ and for every pair x, y of distinct points of X there exists a natural number i with the property that $y \notin \mathrm{St}\,(x, \mathcal{V}_i)$; then apply Corollary 5.4.10.

One can also take a closed G_δ-set $F \subset X \times \beta X$ satisfying $F \cap (X \times X) = \Delta$, define a continuous function $f: X \times \beta X \to I$ such that $F = f^{-1}(0)$ and let $\rho(x,y) = \sup\{|f(x,z) - f(y,z)| : z \in \beta X\}$.

(b) (Okuyama [1964], Borges [1966]) Prove that a paracompact space X such that the diagonal Δ is a G_δ-set in $X \times X$ is metrizable if and only if there exists a perfect mapping of X onto a metrizable space.

Hint. Apply Theorem 3.7.27.

(c) Note that (b) solves Exercise 4.2.B.

(d) Give an example of a paracompact space X such that there is no perfect mapping of X onto a metrizable space.

Hint. Apply (b) or Theorem 3.7.9.

(e) (Nagata [1969]) Observe that a topological space X can be mapped by a perfect mapping onto a metrizable space if and only if X is homeomorphic to a closed subspace of the Cartesian product $Y \times Z$ of a metrizable space Y and a compact space Z.

Remark. The class of spaces for which there exists a perfect mapping onto a metrizable space has been extensively studied; a discussion of results in that area can be found in Arhangel'skiĭ [1966b], Morita [1971] and Gruenhage [1984] (cf. Problems 5.5.9(a) and (d)).

Paracompactness and Čech-complete spaces

5.5.8. (a) (Pasynkov [1967]; for metrizable X, Michael [1959]) Prove that if $f: X \to Y$ is an open mapping of a Čech-complete space X onto a paracompact space Y, then there exists a closed G_δ-set $A \subset X$ such that the restriction $f|A: A \to Y$ is a perfect mapping of A onto Y.

Hint. Consider the extension $F: \beta X \to \beta Y$ of the mapping f and the perfect mapping $g = F_Y: Z \to Y$, where $Z = F^{-1}(Y)$. To begin, show that, for every open subset G of the space Z such that $f(X \cap G) = Y$, there exists an open set $H \subset Z$ such that $f(X \cap H) = Y$ and $\overline{H} \subset G$, where the bar denotes the closure in Z. To that end choose, for every $y \in Y$, an open set $V(y) \subset Z$ such that $f^{-1}(y) \cap V(y) \neq \emptyset$ and $\overline{V(y)} \subset G$, then take a locally finite open refinement $\{U_s\}_{s \in S}$ of the cover $\{f(X \cap V(y))\}_{y \in Y}$ of the space Y; for every $s \in S$ choose a $y(s) \in Y$ such that $U_y \subset f[X \cap V(y(s))]$, and let $H = \bigcup_{s \in S}[V(y(s)) \cap g^{-1}(U_s)]$.

Represent X as $\bigcap_{i=1}^{\infty} G_i$, where G_i is open in Z, define inductively a sequence $H_0 = Z, H_1, H_2, \ldots$ of open subsets of Z such that $\overline{H}_i \subset G_i \cap H_{i-1}$ and $f(X \cap H_i) = Y$ for $i = 1, 2, \ldots$ and let $A = \bigcap_{i=1}^{\infty} H_i = \bigcap_{i=1}^{\infty} \overline{H}_i$.

(b) (Pasynkov [1967]) Show that if there exists an open mapping $f: X \to Y$ of a locally Čech-complete space X onto a paracompact space Y, then Y is Čech-complete.

Hint. Every locally Čech-complete space is a continuous image of a Čech-complete space under an open mapping.

(c) (Arhangel'skiĭ [1961], Frolík [1961]) Deduce from part (b) that every locally Čech-complete paracompact space is Čech-complete.

(d) (Michael [1959]; for metrizable Y, Hausdorff [1934]; for separable metrizable X and Y, Sierpiński [1930]) Prove that if a paracompact space Y is a continuous image of a completely metrizable space X under an open mapping, then Y is completely metrizable.

Remark. It was proved by Michael in [1959] that if there exists an open mapping $f: X \to Y$ of a metrizable space X onto a paracompact space Y such that all fibers $f^{-1}(y)$ are complete with respect to a metric on the space X, then the space Y is metrizable.

5.5.9. (a) (Frolík [1960]) Prove that a topological space X is paracompact and Čech-complete if and only if there exists a perfect mapping of X onto a completely metrizable space.

Hint. If X is paracompact and Čech-complete, then there exists a continuous function $f: X \times \beta X \to I$ such that $\Delta \subset f^{-1}(0) \subset X \times X$. Consider the pseudometric ρ on the set X defined by letting $\rho(x, y) = \sup\{|f(x,z) - f(y,z)| : z \in \beta X\}$ and the space $Y = X/\rho$ (see Exercise 4.2.I).

(b) (Frolík [1960]) Deduce from part (a) and Theorem 3.7.9 that the Cartesian product of countably many Čech-complete paracompact spaces is a Čech-complete paracompact space (cf. Exercise 3.9.F(a)).

(c) Observe that (a) and Problem 5.5.7(b) solve Exercise 5.1.I.

(d) (Šostak [1974]) Show that a topological space X of weight $\leq \mathfrak{m} \geq \aleph_0$ is paracompact and Čech-complete if and only if X is homeomorphic to a closed subset of the Cartesian product $[J(\mathfrak{m})]^{\aleph_0} \times I^{\mathfrak{m}}$.

Hint. Apply Problem 5.5.7(e) and Exercise 4.4.B.

Paracompactness and realcompact spaces

5.5.10 (Katětov [1951]). (a) Prove that a paracompact space X is realcompact if and only if every closed discrete subspace of X is realcompact (cf. Problem 8.5.13(h)).

Hint (Mrówka [1964]). Take a point $x_0 \in \beta X \setminus X$ and a σ-discrete closed cover $\mathcal{F}$ of the space X such that $x_0 \notin \overline{F}$ for each $F \in \mathcal{F}$, where the bar denotes the closure in βX; let $F_i = \bigcup \mathcal{F}_i$, where $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$ and the families $\mathcal{F}_i$ are discrete. If $x_0 \notin \overline{F}_i$ for $i = 1, 2, \ldots$

one can easily define a continuous function $h: X \to I$ such that $h(x_0) = 0$ and $h(x) > 0$ for $x \in X$; thus one can assume that $x_0 \in \overline{F}_{i_0}$. Consider the equivalence relation E on the space $Y = F_{i_0} \cup \{x_0\}$ determined by the decomposition $\mathcal{E} = \mathcal{F}_{i_0} \cup \{x_0\}$, the quotient space $Z = Y/E$ and the natural quotient mapping $q: Y \to Z$. Observe that Z is a Tychonoff space satisfying $\beta Z = \beta T$ for $T = q(F_{i_0})$ and apply realcompactness of T.

(b) Note that if the cardinality of every closed discrete subspace of a paracompact space X is a non-measurable cardinal number, then X is realcompact (cf. Problem 8.5.13(h)).

Hint. Apply Exercise 3.11.D(a).

(c) Note that if there exists a continuous mapping $f: X \to Y$ of a realcompact space X onto a paracompact space Y, then Y is realcompact.

Compact-covering mappings

5.5.11. A continuous mapping $f: X \to Y$ is called *compact-covering* if for every compact $B \subset Y$ there exists a compact $A \subset X$ such that $f(A) = B$.

(a) (Arhangel'skiĭ [1964]) Show that every compact-covering mapping with values in a k-space is a quotient mapping.

(b) (Michael [1964a]) Show that if $f: X \to Y$ is a closed mapping of a space X to a space Y of pointwise countable type, then for every continuous function $g: X \to R$ and any $y \in Y$ the set $g(\operatorname{Fr} f^{-1}(y))$ is bounded. Deduce that if X is a normal space, then the sets $\operatorname{Fr} f^{-1}(y)$ are countably compact (cf. Lemma 4.4.16 and Exercise 4.4.I).

Hint. Every point $y \in Y$ has a family of neighbourhoods $\{V_i\}_{i=1}^{\infty}$ such that for every sequence $y_1, y_2, \ldots$, where $y_i \in V_i$ for $i = 1, 2, \ldots$ and $y_i \neq y_j$ whenever $i \neq j$, the set $\{y_1, y_2, \ldots\}$ has an accumulation point.

(c) (Michael [1964a]; announcement for metrizable X, Arhangel'skiĭ [1964]) Prove that every closed mapping $f: X \to Y$ of a paracompact space X onto an arbitrary space Y is compact-covering.

Hint. See the proof of Theorem 4.4.17.

(d) Show that in part (c) the assumption of paracompactness of X cannot be weakened to the assumption that X is collectionwise normal.

(e) (Arhangel'skiĭ [1966a] (announcement [1964]); for metrizable X, Bourbaki [1958]) Prove that every open mapping $f: X \to Y$ of a Čech-complete space X onto an arbitrary space Y is compact-covering.

Hint. See Problem 5.5.8(a).

(f) (Michael [1959]) Give an example of an open mapping of a separable metrizable space onto the closed unit interval I which is not compact-covering.

Hint. Remove in an appropriate way one point from each vertical line in the square I^2.

Irreducible mappings

5.5.12 (Lašnev [1965]). Prove that for every closed mapping $f: X \to Y$ of a paracompact space X onto a Fréchet space Y there exists a closed subspace $X_0 \subset X$ such that $f(X_0) = Y$ and $f|X_0: X_0 \to Y$ is irreducible (cf. Exercise 3.1.C).

Hint. Consider the closed set $B = A \cup f^{-1}(Y^{\mathrm{d}}) \subset X$, where A is obtained by choosing one point in the fiber $f^{-1}(y)$ of every isolated point $y \in Y$, and the family $\mathcal{F}$ of all closed subsets $F \subset B$ such that $f(F) = Y$ ordered by the relation $\supset$. Assume that for a linearly ordered subfamily $\{F_s\}_{s\in S}$ of $\mathcal{F}$ there is a point $y_0 \in Y$ such that $f^{-1}(y_0) \cap \bigcap_{s\in S} F_s = \emptyset$. Take a non-trivial sequence $y_1, y_2, \ldots$ convering to y_0 and applying Problem 5.5.11(b) observe that the set $f^{-1}(y_0) \cap \overline{\bigcup_{i=1}^{\infty} f^{-1}(y_i)}$ is compact; deduce a contradiction. Then apply the Kuratowski-Zorn lemma.

Paracompactness of function spaces

5.5.13. (a) (O'Meara [1971]) Prove that if X is a separable metrizable space and Y is a metrizable space, then the space Y^X with the compact-open topology is perfectly normal and hereditarily paracompact (cf. Exercise 3.8.D).

Hint. Let $\{x_1, x_2, \ldots\}$ be a countable dense subset of X and let $\mathcal{B}$ be a σ-discrete base for Y. Take a metric ρ on the space X and for every finite sequence $i_1, i_2, \ldots, i_k$ of natural numbers, every sequence $B_1, B_2, \ldots, B_k$ of members of $\mathcal{B}$ and any natural number m let

$$U(i_1, i_2, \ldots, i_k;\ B_1, B_2, \ldots, B_k) = \bigcap_{j=1}^{k} M(\{x_{i_j}\}, B_j) \tag{1}$$

and

$$M_m(i_1, i_2, \ldots, i_k;\ B_1, B_2, \ldots, B_k) = \bigcap_{j=1}^{k} M(B(x_{i_j}, 1/m), B_j). \tag{2}$$

Verify that the sets in (1) form a σ-discrete open cover of Y^X and the sets in (2) form a network for Y^X.

(b) Prove that if X is a separable merizable space and Y is a metrizable space, then the space Y^X with the topology of pointwise convergence is perfectly normal and hereditarily paracompact (cf. Exercise 3.8.D).

Hint. Analyse the proof of (a).

F_σ-sets in countably paracompact spaces

5.5.14 (Zenor [1976]). (a) Show that the property of being a countably paracompact normal space is hereditary with respect to F_σ-sets.

(b) Prove that if all F_σ-sets in the Cartesian product $X \times Y$ are countably paracompact, then X is normal or all countable discrete subspaces of Y are closed.

Hint. Assume that $\{y_1, y_2, \ldots\}$ is a countable discrete subspace of Y that has an accumulation point y_0. Take a closed set $F \subset X$, an open set $W \subset X$ that contains F, and consider the subspace $A = \bigcup_{i=0}^{\infty} F_i$ of $X \times Y$, where $F_0 = F \times \{y_0\}$ and $F_i = X \times \{y_i\}$ for $i \geq 1$. The open cover $\{U_i\}_{i=0}^{\infty}$ of A, where $U_0 = (W \times Y) \cap A$ and $U_i = F_i$ for $i \geq 1$, has a locally finite open refinement $\{V_i\}_{i=0}^{\infty}$ such that $V_i \subset U_i$ for $i = 0, 1, \ldots$ Check that the sequence $W_1, W_2, \ldots$, where $W_i = X \setminus \overline{p(V_i)}$ and $p: X \times Y \to X$ is the projection, satisfies the conditions in Lemma 1.5.15.

(c) Deduce from part (b) that countable paracompactness is not hereditary with respect to F_σ-sets.

Countable paracompactness in normal spaces

5.5.15 (Mansfield [1957]). Prove that for every normal space X the following conditions are equivalent:

(1) *The space X is countably paracompact.*
(2) *Every countable open cover of the space X has a locally finite closed refinement.*
(3) *Every countable open cover of the space X has a σ-locally finite closed refinement.*
(4) *Every countable open cover of the space X has a σ-discrete closed refinement.*
(5) *Every countable open cover of the space X has a countable open star refinement.*
(6) *Every countable open cover of the space X has an open star refinement.*

Extending locally finite families of sets

5.5.16 (Mansfield [1957]). Show that a Hausdorff space X is countably paracompact if and only if for every countable locally finite family $\{F_i\}_{i=1}^{\infty}$ of closed subsets of X there exists a locally finite family $\{V_i\}_{i=1}^{\infty}$ of open sets such that $F_i \subset V_i$ for $i = 1, 2, \ldots$

Hint. Let $\mathcal{S}$ be the family of all finite sets of natural numbers. For every $x \in X$ take a neighbourhood $U(x)$ of x and a set $S(x) \in \mathcal{S}$ such that $F_i \cap U(x) = \emptyset$ whenever $i \notin S(x)$. Let $U_S = \bigcup\{U(x) : S(x) = S\}$, take a locally finite open cover $\{W_S\}_{S \in \mathcal{S}}$ of X such that $W_S \subset U_S$ for every $S \in \mathcal{S}$ and define $V_i = \bigcup_{i \in S} W_S$.

5.5.17. (a) (Dowker [1956]) Prove that a normal space X is collectionwise normal and countably paracompact if and only if for every locally finite family $\{F_s\}_{s \in S}$ of closed subsets of X there exists a locally finite family $\{V_s\}_{s \in S}$ of open sets such that $F_s \subset V_s$ for every $s \in S$.

Hint (Katětov [1958]). To begin, observe that, under the additional assmumption that no point of X belongs to more than i sets F_s, the existence of $\{V_s\}_{s \in S}$ follows from collectionwise normality of X (apply induction with respect to i). Then, returning to the general case, denote by W_i the subset of X consisting of all points which belong to at most i sets F_s and consider a locally finite open cover $\{U_i\}_{i=1}^{\infty}$ of X such that $\overline{U}_i \subset W_i$ for $i = 1, 2, \ldots$ For every i take a locally finite family $\{V_{s,i}\}_{s \in S}$ of open subsets of $\overline{U}_i$ such that $F_s \cap \overline{U}_i \subset V_{s,i}$ and let $V_s = \bigcup_{i=1}^{\infty}(V_{s,i} \cap U_i)$.

(b) (Smith and Krajewski [1971]) Prove that for every locally finite family $\{F_s\}_{s \in S}$ of closed subsets of a weakly paracompact space X there exists a point-finite family $\{V_s\}_{s \in S}$ of open sets such that $F_s \subset V_s$ for every $s \in S$.

Hint. See the hint to Problem 5.5.16.

Normality and related properties in Cartesian products IV (see Problems 2.7.16, 3.12.15, 3.12.20, 4.5.15, 4.5.16, 5.5.5, 5.5.6 and Exercise 2.3.E)

5.5.18. (a) (Morita, cited in Ishii [1966]) Prove that if the Cartesian product $X \times Y$ of a normal space X and a metrizable space Y is countably paracompact, then $X \times Y$ is normal.

Hint (M. E. Rudin and Starbird [1975]). Take a metric on the space Y and consider a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of locally finite open covers of Y such that $\delta(U) < 1/i$ for $U \in \mathcal{U}_i$ and $\mathcal{U}_{i+1}$ is a refinement of $\mathcal{U}_i$. Consider a closed set $F \subset X \times Y$ and an open set $W \subset X \times Y$

that contains F; for every $U \in \mathcal{U}_i$ take the set

$$A(U) = \overline{p[(X \times \overline{U}) \cap F]} \cap \overline{p[(X \times \overline{U}) \setminus W]},$$

where p denotes the projection of $X \times Y$ onto X. Check that the sets $A_i = \bigcup\{A(U) \times \overline{U} : U \in \mathcal{U}_i\}$ are closed, form a decreasing sequence and satisfy $\bigcap_{i=1}^{\infty} A_i = \emptyset$. Applying Theorem 5.2.1, take a decreasing sequence $V_1, V_2, \ldots$ of open subsets of $X \times Y$ such that $A_i \subset V_i$ for $i = 1, 2, \ldots$ and $\bigcap_{i=1}^{\infty} \overline{V}_i = \emptyset$. Verify that for every $U \in \mathcal{U}_i$ the sets

$$B(U) = \overline{p[(X \times \overline{U}) \cap F] \setminus p[(X \times U) \cap V_i]} \quad \text{and} \quad C(U) = \overline{p[(X \times \overline{U}) \setminus W]}$$

are closed and disjoint. Take an open $W(U) \subset X$ such that $C(U) \cap \overline{W(U)} = \emptyset$ and $B(U) \subset W(U)$; check that the sets $W_i = \bigcup\{W(U) \times U : U \in \mathcal{U}_i\}$ satisfy the conditions in Lemma 1.5.15.

(b) (M. E. Rudin and Starbird [1975]) Prove that if the Cartesian product $X \times Y$ of a normal space X and a non-discrete metrizable space Y is normal, then $X \times Y$ is countably paracompact.

Hint (Bešlagić [1986]). Take a metric on the space Y and consider a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of locally finite families of open subsets of Y such that $\delta(U) < 1/j$ for $U \in \mathcal{U}_j$, the union $\mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{U}_j$ is a base for Y and no discrete member of $\mathcal{B}$ contains more than one point. For every non-discrete $U \in \mathcal{B}$ define points $a(U), b(U) \in U$ in such a way that all the points $a(U), b(U)$ are distinct. Consider an increasing sequence $W_1 \subset W_2 \subset \ldots$ of open subsets of $X \times Y$ satisfying $\bigcup_{i=1}^{\infty} W_i = X \times Y$ and for every $U \in \mathcal{B}$ and $i = 1, 2, \ldots$ denote by $W_i(U)$ the union of all open sets $W \subset X$ such that $W \times \overline{U} \subset W_i$; then for every $U \in \mathcal{B}$ satisfying $|U| > 1$ let $A(U) = \{a(U)\} \times C(U)$ and $B(U) = \{b(U)\} \times C(U)$, where $C(U) = X \setminus \bigcup_{i=1}^{\infty} W_i(U)$, and for every $U \in \mathcal{B}$ satisfying $|U| = 1$ let $A(U) = B(U) = \emptyset$. Show that $A = \bigcup\{A(U) : U \in \mathcal{B}\}$ and $B = \bigcup\{B(U) : U \in \mathcal{B}\}$ are disjoint closed subsets of $X \times Y$, take open sets $G, H \subset X \times Y$ such that $A \subset G$, $B \subset H$ and $\overline{G} \cap \overline{H} = \emptyset$, observe that for every $U \in \mathcal{B}$ the family $\{W_i(U)\}_{i=1}^{\infty} \cup \{W(U)\}$, where $W(U) = p[G \cap (X \times U)] \cap p[H \cap (X \times U)]$ and p denotes the projection of $X \times Y$ onto X, is an open cover of X, apply Remark 5.2.9 to show that X is countably paracompact, and consider an open cover $\{V_i(U)\}_{i=1}^{\infty} \cup \{V(U)\}$ such that $\overline{V_i(U)} \subset W_i(U)$ for $i = 1, 2, \ldots$ and $\overline{V(U)} \subset W(U)$. To conclude, check that the sets $V_{i,j} = \bigcup\{V_i(U) \times U : U \in \mathcal{U}_j\}$ form an open cover of $X \times Y$ and that $\overline{V}_{i,j} \subset W_i$ for $i, j = 1, 2, \ldots$, let $F_i = \overline{V}_{1,i} \cup \overline{V}_{2,i-1} \cup \ldots \cup \overline{V}_{i,1}$ and apply Theorem 5.2.1.

(c) (M. E. Rudin and Starbird [1975]) Prove that the Cartesian product $X \times Y \times Z$ of a normal space X, a non-empty metrizable space Y, and a non-empty compact space Z is normal if and only if the Cartesian products $X \times Y$ and $X \times Z$ are normal.

Hint. Apply (b) and (a).

(d) (M. E. Rudin and Starbird [1975]) Prove that for the Cartesian product $X \times Y$ of a paracompact space X and a metrizable space Y the following conditions are equivalent:

(1) *The Cartesian product $X \times Y$ is paracompact.*

(2) *The Cartesian product $X \times Y$ is collectionwise normal.*

(3) *The Cartesian product $X \times Y$ is normal.*

(4) *The Cartesian product $X \times Y$ is countably paracompact.*

Hint. To show that (4)$\Rightarrow$(1) observe that the Cartesian product $X \times I^{\mathfrak{m}}$, where $\mathfrak{m} = w(X \times Y)$, is normal and apply parts (a) and (c) and Theorem 5.1.39.

(e) (Alas [1971]) Prove that if the space X is collectionwise normal and countably paracompact, then for every $\mathfrak{m} \geq \aleph_0$ the Cartesian product $X \times A(\mathfrak{m})$ is normal.

Hint. Apply Problem 5.5.17(a).

(f) (Alas [1971]) Show that if the Cartesian product $X \times A(\mathfrak{m})$, where $\mathfrak{m} = |X|$, is normal, then the space X is collectionwise normal and countably paracompact.

(g) (Starbird [1974]) Prove that if the Cartesian product $X \times Y$ of a collectionwise normal space X and a metrizable space Y is normal, then it is also collectionwise normal. Deduce that the Cartesian product $X \times Y$ of a perfectly normal collectionwise normal space X and a metrizable space Y is collectionwise normal (cf. Problems 4.5.16(b) and 5.5.5(a)).

Hint. One can assume that Y is a non-discrete space. Apply (b) and Exercise 5.2.G(b) to show that the Cartesian product $(X \times Y) \times A(\mathfrak{m})$, where $\mathfrak{m} = |X \times Y|$, is countably paracompact, observe that from (e) it follows that the Cartesian product $X \times A(\mathfrak{m})$ is normal, deduce from (a) that the Cartesian product $(X \times Y) \times A(\mathfrak{m})$ is normal, and apply (f).

(h) (Starbird [1974]) Prove that if the Cartesian product $X \times Y$ of a collectionwise normal space X and a compact space Y is normal, then it is also collectionwise normal.

Hint. One can assume that Y is a non-discrete space. Consider a discrete family of closed subsets of $X \times Y$ and arrange its members into a transfinite sequence $F_1, F_2, \ldots, F_\alpha, \ldots$, $\alpha < \xi$. Observe that the family $\{p(F_\alpha) : \alpha < \xi\}$, where p denotes the projection of $X \times Y$ onto X, is locally finite, apply Exercise 5.2.H and Problem 5.5.17(a) to obtain a locally finite family $\{V_\alpha : \alpha < \xi\}$ of open subsets of X such that $p(F_\alpha) \subset V_\alpha$ for $\alpha < \xi$, and define by transfinite induction open subsets U_α of $X \times Y$ in such a way that $F_\alpha \subset U_\alpha \subset \overline{U}_\alpha \subset V_\alpha \times Y$ and $\overline{U}_\alpha \cap \bigcup\{\overline{U}_\beta : \beta < \alpha\} = \emptyset = \overline{U}_\alpha \cap \bigcup\{F_\gamma : \gamma > \alpha\}$ for $\alpha < \xi$.

5.5.19. (a) (Zenor [1971], Nagami [1972]) Prove that a countable Cartesian product $X = \prod_{i=1}^{\infty} X_i$, where $|X_i| > 1$, is normal if and only if all finite Cartesian products $X_1 \times X_2 \times \ldots \times X_i$ are normal and X is countably paracompact.

Hint. To prove that normality of X implies countable paracompactness of X, consider a decreasing sequence $F_1 \supset F_2 \supset \ldots$ of closed subsets of X satisfying $\bigcap_{i=1}^{\infty} F_i = \emptyset$, choose in every X_i a pair x_i, y_i of distinct points and show that the sets

$$A = \bigcup_{i=1}^{\infty}(\overline{q_i(F_i)} \times \{(x_{i+1}, x_{i+2}, \ldots)\}) \quad \text{and} \quad B = \bigcup_{i=1}^{\infty}(\overline{q_i(F_i)} \times \{(y_{i+1}, y_{i+2}, \ldots)\}),$$

where $q_i: X \to X_1 \times X_2 \times \ldots \times X_i$ is the projection, are closed in X and disjoint; then consider open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$, let $W_i = q_i^{-1}(q_i(U) \cap q_i(V))$ and apply Theorem 5.2.1. To prove that countable paracompactness of X and normality of the Cartesian products $X(i) = X_1 \times X_2 \times \ldots \times X_i$ imply normality of X, consider a two-element open cover $\mathcal{U} = \{U_1, U_2\}$ of X, for $k = 1, 2$ define $U_k(i)$ as the union of all open sets $U \subset X(i)$ such that $q_i^{-1}(U) \subset U_k$ and check that $q_i^{-1}(U_k(i)) \subset q_{i+1}^{-1}(U_k(i+1))$ and $\bigcup_{i=1}^{\infty} q_i^{-1}(U_k(i)) = U_k$. Then observe that the sets $F_i = X \setminus q_i^{-1}(U_1(i) \cup U_2(i))$ are decreasing and have an empty intersection, take a decreasing sequence $W_1 \supset W_2 \supset \ldots$ of open subsets of X such that $F_i \subset W_i$ for $i = 1, 2, \ldots$ and $\bigcap_{i=1}^{\infty} \overline{W}_i = \emptyset$, note that $A(i) = X(i) \setminus q_i(W_i) \subset U_1(i) \cup U_2(i)$ and find open subsets $G_1(i), G_2(i)$ of $X(i)$ such that $A(i) \subset G_1(i) \cup G_2(i)$ and

$\overline{G_k(i)} \subset U_k(i)$ for $k = 1,2$. To conclude, show that $\bigcup_{i=1}^{\infty} q_i^{-1}(A(i)) = X$, consider the open cover $\bigcup_{i=1}^{\infty}\{q_i^{-1}(G_1(i)), q_i^{-1}(G_2(i))\}$ of X and apply Lemma 1.5.15.

(b) (Nagami [1972]) Prove that a countable Cartesian product $X = \prod_{i=1}^{\infty} X_i$, where $|X_i| > 1$, is collectionwise normal if and only if all finite Cartesian products $X_1 \times X_2 \times \ldots \times X_i$ are collectionwise normal and X is countably paracompact.

Hint. Apply Problems 5.5.18(e) and (f), Exercise 5.2.G(b) and part (a).

(c) (Nagami [1972]; announcement Zenor [1971a]) Prove that a countable Cartesian product $X = \prod_{i=1}^{\infty} X_i$ is paracompact (has the Lindelöf property) if and only if all finite Cartesian products $X_1 \times X_2 \times \ldots \times X_i$ are paracompact (have the Lindelöf property) and X is countably paracompact.

Hint. To prove that X is paracompact (has the Lindelöf property), consider an arbitrary open cover $\mathcal{U}$ of X and follow the second part of the hint to (a).

Semicontinuous functions III (see Problems 1.7.14-1.7.16, 2.7.4 and 3.12.23(g))

5.5.20. (a) (Dowker [1951], Katětov [1951a]; for paracompact spaces, Dieudonné [1944]) Prove that a T_1-space X is normal and countably paracompact if and only if for every pair f, g of real-valued functions defined on X, where f is upper semicontinuous, g is lower semicontinuous and $f(x) < g(x)$ for every $x \in X$, there exists a continuous function $h: X \to R$ such that $f(x) < h(x) < g(x)$ for every $x \in X$.

Hint. When defining h consider the countable open cover of X consisting of all sets $U_r = \{x \in X : f(x) < r < g(x)\}$, where r is a rational number.

(b) (Michael [1956]) Apply part (a) to give a solution of Problem 1.7.15(d).

Hint. Let $h_1: X \to R$ be a continuous function such that $f(x) \le h_1(x) \le g(x)$ for every $x \in X$, $k: U \to R$, where $U = \{x \in X : f(x) < g(x)\}$, a continuous function such that $f(x) < k(x) < g(x)$ for every $x \in U$, and $p: X \to I$ a continuous function such that $p^{-1}(0) = X \setminus U$. Consider the function $h = h_1 + h_2$, where

$$h_2(x) = p(x) \cdot (k(x) - h_1(x)) \cdot (1 + |k(x) - h_1(x)|)^{-1} \quad \text{if} \quad x \in U$$

and $h_2(x) = 0$ if $x \in X \setminus U$.

The Borsuk homotopy extension theorem

5.5.21. Let X and Y be topological spaces and let f_0 and f_1 be continuous mappings of X to Y. If there exists a continuous mapping $F: X \times I \to Y$ such that $f(x, i) = f_i(x)$ for $i = 0, 1$, then we say that the mappings f_0 and f_1 are *homotopic*; the mapping F is called a *homotopy* between f_0 and f_1.

(a) (Dowker [1951]; for metrizable X, Borsuk [1937]) Prove that for a countably paracompact normal space X, a closed subspace M of X and a pair f_0, f_1 of homotopic continuous mappings of M to the n-sphere S^n, if f_0 is continuously extendable over X, then f_1 also is continuously extendable over X and – moreover – for every extension of f_0 one can find an extension of f_1 homotopic to it (this is the *Borsuk homotopy extension theorem*).

Hint (Dowker, cited in Hurewicz and Wallman [1941]). Let $F: M \times I \to S^n$ be a homotopy between f_0 and f_1 and let $f_0^*: X \to S^n$ be an extension of f_0. Observe that the combination of F and f_0^* (the latter being considered as a mapping of $X \times \{0\}$ to S^n) is extendable – by virtue of Theorem 2.1.8 and Exercise 3.2.C – to a mapping $G: U \to S^n$, where $U \subset X \times I$ is an open set containing $(X \times \{0\}) \cup (M \times I)$, and apply Theorem 3.1.16 to obtain an open set $V \subset X$ such that $M \times I \subset V \times I \subset U$. Consider the mapping $F: X \times I \to S^n$ defined by letting $F(x,t) = G(x, tg(x))$, where $g: X \to I$ is a continuous function satisfying the conditions $g(X \setminus V) \subset \{0\}$ and $g(M) \subset \{1\}$.

(b) (Morita [1975], Starbird [1975]) Prove that the Borsuk homotopy extension theorem holds under the assumption that X is a normal space.

Hint (M. E. Rudin [1984]). First, note that to establish the theorem it suffices to prove that every continuous function from $(X \times \{0\}) \cup (M \times I)$ to I is continuously extendable over $X \times I$; then observe that it suffices to show that every continuous function from $M \times I$ to I is continuously extendable over $X \times I$. Thus, by Exercise 2.1.J, it is enough to show that any disjoint functionally closed subsets A, B of $M \times I$ can be separated by a continuous function $g: X \times I \to I$. To that end, consider a continuous function $f: M \times I \to I$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$, arrange in a sequence $(G_1, H_1), (G_2, H_2), \ldots$ all pairs (G_i, H_i) of finite unions of members of a countable base for I satisfying the inclusion $\overline{G_i} \subset H_i$, and apply Exercise 3.1.D(b) to show that the sets $W_i = \{x \in M : f_x^{-1}(0) \subset G_i \text{ and } f_x^{-1}(1) \cap \overline{H}_i = \emptyset\}$, where $f_x(t) = f(x,t)$, are functionally open in M. Observe that $\{W_i\}_{i=1}^{\infty}$ is a cover of M and find a cover $\{U_i\}_{i=0}^{\infty}$ of X consisting of functionally open sets such that $U_0 \cap M = \emptyset$ and $U_i \cap M = W_i$ for $i = 1, 2, \ldots$ Apply Lemma 5.2.4 to obtain a locally finite cover $\{V_i\}_{i=0}^{\infty}$ of X consisting of functionally open sets such that $V_i \subset U_i$ for $i = 0, 1, \ldots$, consider continuous functions $g_i: X \to I$ satisfying $g_i^{-1}((0,1]) = V_i$ for $i = 0, 1, \ldots$ and $\sum_{i=0}^{\infty} g_i(x) = 1$ for $x \in X$, and define g by letting $g(x,t) = \sum_{i=1}^{\infty} g_i(x) \cdot h_i(t)$, where $h_i: I \to I$ is for $i = 1, 2, \ldots$ a continuous function satisfying $h_i(\overline{G_i}) \subset \{0\}$ and $h_i(I \setminus H_i) \subset \{1\}$.

Linearly ordered spaces IV (see Problems 1.7.4, 2.7.5, 3.12.3, 3.12.4, 3.12.12(f), 6.3.2 and 8.5.13(j))

5.2.22. (a) (Bennett [1971]) Let $\mathcal{A}$ be a connected family of convex subsets of a set X linearly ordered by the relation $<$; show that in $\bigcup \mathcal{A}$ there exist sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, finite or infinite, such that

(1) $$x_1 = y_1, \quad x_i < x_{i+1} \quad \text{and} \quad y_{i+1} < y_i \quad \text{for} \quad i = 1, 2, \ldots$$

(2) $$x_{i+1} \notin \operatorname{St}(x_i, \mathcal{A}) \quad \text{and} \quad y_{i+1} \notin \operatorname{St}(y_i, \mathcal{A}) \quad \text{for} \quad i = 1, 2, \ldots$$

(3) $$\operatorname{St}(x_i, \mathcal{A}) \cap \operatorname{St}(x_{i+1}, \mathcal{A}) \neq \emptyset \neq \operatorname{St}(y_i, \mathcal{A}) \cap \operatorname{St}(y_{i+1}, \mathcal{A}) \quad \text{for} \quad i = 1, 2, \ldots$$

(4) $$\bigcup_{i=1}^{\infty} \operatorname{St}(x_i, \mathcal{A}) \cup \bigcup_{i=1}^{\infty} \operatorname{St}(y_i, \mathcal{A}) = \bigcup \mathcal{A}.$$

(b) Deduce from (a) that for every open cover $\mathcal{U}$ of a linearly ordered space the following conditions are equivalent:

(1) *The cover $\mathcal{U}$ has a star-finite open refinement.*

(2) *The cover $\mathcal{U}$ has a point-countable open refinement.*

(c) (Ball [1954], Gillman and Henriksen [1954]) Show that every linearly ordered space is countably paracompact.

(d) (Gillman and Henriksen [1954]; Fedorčuk [1966]) Show that for every linearly ordered space X the following conditions are equivalent:

(1) *The space X is strongly paracompact.*

(2) *The space X is paracompact.*

(3) *The space X is weakly paracompact.*

(4) *Every open cover of the space X has a point-countable open refinement.*

(e) (Gillman and Henriksen [1954]) In a linearly ordered set X, besides gaps defined in the Introduction (which are sometimes called *interior gaps*) we shall consider *end-gaps*: if X has no largest element then the pair $(X, \emptyset)$ is the *right end-gap* of X, and if X has no smallest element then the pair $(\emptyset, X)$ is the *left end-gap* of X. A gap (A, B) in X is a *left Q-gap* if either $A = \emptyset$ or there exists an increasing transfinite sequence $x_0, x_1, \ldots x_\xi, \ldots,\ \xi < \alpha$ of elements of A which is cofinal in A such that for every limit ordinal number $\lambda < \alpha$ the set $\{x_\xi : \xi < \lambda\}$ has no least upper bound in X; a *right Q-gap* in X is defined analogously. By a *Q-gap* in a linearly ordered set X we mean a gap in X which is both a left Q-gap and a right Q-gap.

Prove that a linearly ordered space X is paracompact if and only if every gap in X is a Q-gap.

Hint. Applying (a), show that a connected cover $\mathcal{U}$ of a linearly ordered space X consisting of intervals has a point-countable open refinement if and only if end-gaps of X (if there are any) are Q-gaps.

(f) (Engelking and Lutzer [1977]) For any ordinal number α let $W(\alpha)$ denote the set of all ordinal numbers less than α with the topology induced by the natural order $<$. A *stationary* subset of $W(\alpha)$ is a set $S \subset W(\alpha)$ such that $S \cap C \neq \emptyset$ for every closed set C cofinal in $W(\alpha)$.

Prove that a linearly ordered space X fails to be paracompact if and only if for some limit ordinal number α, such that ω_0 is not cofinal with α, the space X contains a closed subspace homeomorphic to a stationary subset of $W(\alpha)$.

Hint. If X is not paracompact then X has a gap (A, B) which is not a Q-gap; one can assume that this gap is not a left Q-gap. Consider an increasing transfinite sequence $x_0, x_1, \ldots, x_\xi, \ldots,\ \xi < \alpha$ of elements of A which is cofinal in A and show that X contains a closed subspace homeomorphic to the subspace S of $W(\alpha)$ consisting of all limit numbers λ such that the set $\{x_\xi : \xi < \lambda\}$ has a least upper bound in X.

To show that a stationary subset S of $W(\alpha)$ is not paracompact, assume that there exists a locally finite open cover $\mathcal{U}$ of S such that no member of $\mathcal{U}$ is cofinal in S and consider the set $C \subset W(\alpha) \setminus S$ consisting of all points of $W(\alpha)$ each neighbourhood of which meets infinitely many members of $\mathcal{U}$.

(g) (Lutzer [1971]) Show that every perfectly normal linearly ordered space is paracompact. Observe that the lexicographically ordered square (see problem 3.12.3(d)) is a hereditarily paracompact linearly ordered space which is not perfectly normal.

Hint (Engelking and Lutzer [1977]). Apply part (f).

(h) Observe, using an appropriate modification of (b), that every linearly ordered space is hereditarily countably paracompact. Note that if a linearly ordered space hereditarily satisfies one of the conditions in (d), it also hereditarily satisfies all the remaining conditions.

(i) (Mansfield [1957a]) Show that every linearly ordered space is collectionwise normal (see part (j) below).

Hint. See Problem 1.7.4(c).

(j) (Steen [1970]) Show that every linearly ordered space is hereditarily collectionwise normal.

Hint. See the hint to Problem 2.7.5(c).

(k) (Lutzer [1969]) Prove that a linearly ordered space X is metrizable if and only if the diagonal Δ is a G_δ-set in the Cartesian product $X \times X$.

Hint (Lutzer [1970]). Define a development for X and apply 5.4.1 and part (i).

(l) (Lutzer [1969]) Deduce from (k) that there is no linear order on the Sorgenfrey line which induces the topology of that space.

Every pseudocompact weakly paracompact space is compact

5.5.23 (Scott [1979], Watson [1981]). Prove that every pseudocompact weakly paracompact space is compact.

Hint. Let $\mathcal{U}$ be a point-finite open cover of a pseudocompact space X. Consider a maximal family $\mathcal{V}$ of open subsets of X with the property that $|\{V \in \mathcal{V} : U \cap V \neq \emptyset\}| \leq 1$ for every $U \in \mathcal{U}$ and $|\{U \in \mathcal{U} : V \cap U \neq \emptyset\}| < \aleph_0$ for every $V \in \mathcal{V}$ and apply Exercises 3.10.F(e) and 3.9.I to show that $X = \overline{\bigcup\{U \in \mathcal{U} : U \cap \bigcup \mathcal{V} \neq \emptyset\}}$.

Chapter 6
Connected spaces

The property of connectedness that we are going to study in this chapter is of an entirely different character than the topological properties studied in the preceding chapters. In particular connectedness does not imply any of those properties and is not implied by any of them. Roughly speaking, a connected space consists of a single piece, as opposed to spaces consisting of many pieces far from each other, such as discrete spaces.

In Section 6.1 we define connectedness and study the behaviour of connected spaces under various operations; it turns out that the class of connected spaces is closed under Cartesian products and, under the additional assumption of compactness, also under the operation of taking the limit of an inverse system. Then we introduce the notions of a component and of a quasi-component, and show that these notions coincide in the realm of compact spaces. We conclude the section with the Sierpiński theorem, stating that no connected compact space can be represented as a countable union of pairwise disjoint closed sets, and with a short study of monotone mappings – mappings with connected fibers.

Section 6.2 is devoted to a study of four classes of topological spaces which are highly disconnected, viz., hereditarily disconnected spaces, zero-dimensional spaces, strongly zero-dimensional spaces and extremally disconnected spaces. These four classes form a decreasing sequence. The classes of zero-dimensional and strongly zero-dimensional spaces are related to the concept of the dimension of a space. In dimension theory one defines for each non-empty space X, in a rather broad class of spaces, the dimension of X which is either a non-negative integer or the "infinite number" ∞. This can be done in several ways, but always the dimension of a space is, in a sense, a measure of its connectedness. The names of zero-dimensional and strongly zero-dimensional spaces are due to the fact that those spaces have dimension zero with respect to two different concepts of dimension. These two classes are studied in Section 6.2 quite independently of dimension theory; however, the study of these classes is an introduction to dimension theory developed in Chapter 7. After discussing relationships among the first three classes and investigating their behaviour under operations, we define two classes of mappings – light mappings and zero-dimensonal mappings, and we prove that every perfect mapping can be uniquely represented as the composition of a monotone mapping and a zero-dimensional mapping. The final part of the section is devoted to extremally disconnected spaces. That class of spaces is quite specialized and at first sight may seem a little artificial; however, it plays an important role in the theory of Boolean algebras and in some problems of functional analysis.

6.1. Connected spaces

We say that a topological space X is *connected* if X cannot be represented in the form $X_1 \oplus X_2$, where X_1 and X_2 are non-empty subspaces of X. We start with a set of conditions characterizing connectedness.

6.1.1. THEOREM. *For every topological space X the following conditions are equivalent:*
(i) *The space X is connected.*
(ii) *The empty set and the whole space are the only closed-and-open subsets of the space X.*
(iii) *If $X = X_1 \cup X_2$ and the sets X_1 and X_2 are separated, then one of them is empty.*
(iv) *Every continuous mapping $f : X \to D$ of the space X to the two-point discrete space $D = \{0,1\}$ is constant, i.e., either $f(X) \subset \{0\}$ or $f(X) \subset \{1\}$.*

PROOF. To show that (i)$\Rightarrow$(ii) it suffices to observe that by Proposition 2.2.4 for a closed-and-open set $X_1 \subset X$ such that $\emptyset \neq X_1 \neq X$ and for $X_2 = X \setminus X_1$ we would have $X = X_1 \oplus X_2$ and $X_1 \neq \emptyset \neq X_2$.

We shall show that (ii)$\Rightarrow$(iii). Suppose that $X = X_1 \cup X_2$, where $X_1 \cap \overline{X}_2 = \emptyset = \overline{X}_1 \cap X_2$. Thus we have $\overline{X}_1 \subset X \setminus X_2 \subset X_1$ and $\overline{X}_1 = X_1$; similarly, $\overline{X}_2 = X_2$. Since the sets X_1 and X_2 are closed and disjoint, they are also open, and by virtue of (ii) one of them is empty.

To establish the implication (iii)$\Rightarrow$(iv) it suffices to note that if there exists a continuous mapping $f : X \to D$ such that $X_1 = f^{-1}(0) \neq \emptyset$ and $X_2 = f^{-1}(1) \neq \emptyset$, then – by the obvious equalities $X = X_1 \cup X_2$ and $X_1 \cap \overline{X}_2 = \emptyset = \overline{X}_1 \cap X_2$ – the space X does not satisfy (iii).

The implication (iv)$\Rightarrow$(i) follows from the fact that if (i) does not hold, i.e., if $X = X_1 \oplus X_2$, where $X_1 \neq \emptyset \neq X_2$, then the conditions $f(x) = 0$ for $x \in X_1$ and $f(x) = 1$ for $x \in X_2$ define a continuous function $f : X \to D$ such that $f(X) = D$.■

6.1.2. COROLLARY. *A space X is connected if and only if it cannot be represented as the union $X_1 \cup X_2$ of two non-empty disjoint closed subsets.*■

One easily sees that the last corollary holds with "closed" replaced by "open", i.e., a space is connected if and only if it cannot be represented as the union of two non-empty disjoint open subsets.

The equivalence of conditions (i) and (iv) in Theorem 6.1.1 yields

6.1.3. THEOREM. *Connectedness is an invariant of continuous mappings.*■

6.1.4. COROLLARY. *Every connected Tychonoff space containing at least two points has cardinality not less than* $\mathfrak{c}$.

PROOF. Let X be a connected Tychonoff space and let x_1, x_2 be a pair of distinct points of X. By the definition of Tychonoff spaces there exists a continuous function $f : X \to I$ such that $f(x_1) = 0$ and $f(x_2) = 1$. From Theorem 6.1.3 it follows that $f(X) = I$ and thus $|X| \geq \mathfrak{c}$.■

6.1.5. EXAMPLES. It follows from 2.2.8 that a discrete space containing at last two points, as well as the Sorgenfrey line, are not connected. It follows also from 2.2.8 that the real line is connected; by virtue of Theorem 6.1.3 this implies that all intervals on the real line are connected. The empty space and one-point spaces are obviously connected. One readily sees that the two-point space $F = \{0,1\}$ with the topology consisting of the empty set, the set $\{0\}$ and the whole space, also is connected. ■

6.1.6. EXAMPLE. We shall now describe a countable connected Hausdorff space. From Corollary 6.1.4 it follows that there is no countable connected Tychonoff space, and – by virtue of Theorem 1.5.17 – there is no such space which is regular.

Denote by X the set of all points (r_1, r_2) of the plane R^2, where r_1 and r_2 are rational numbers and $r_2 \geq 0$; clearly $|X| = \aleph_0$. For every $x = (r_1, r_2) \in X$ and for $i = 1, 2, \ldots$ let

$$U_i(x) = \{x\} \cup \{(r,0) : |r - (r_1 - r_2/\sqrt{3})| < 1/i\} \cup \{(r,0) : |r - (r_1 + r_2/\sqrt{3})| < 1/i\}.$$

If the point x is situated above the x-axis, then the set $U_i(x)$ consists of x and all rational points on the x-axis whose distance from a vertex of the equilateral triangle, with one vertex at x and the other two on the x-axis, is less than $1/i$. If the point x is situated on the x-axis, then the set $U_i(x)$ consists of all rational points of the x-axis whose distance from x is less than $1/i$. One easily verifies that the collection $\{\mathcal{B}(x)\}_{x \in X}$, where $\mathcal{B}(x) = \{U_i(x)\}_{i=1}^{\infty}$, has properties (BP1)–(BP4), so that it generates a topology of a Hausdorff space on X.

The closure of the set $U_i(x)$ with respect to that topology consists of all points of X whose distance from the line passing through the point x and forming an angle of 60° with the x-axis is not larger than $\sqrt{3}/2i$, as well as of all points of X whose distance from the line passing through the point x and forming an angle of 120° with the x-axis is not larger than $\sqrt{3}/2i$. Hence, for any points $x_1, x_2 \in X$ and any natural numbers i_1, i_2 we have $\overline{U_{i_1}(x_1)} \cap \overline{U_{i_2}(x_2)} \neq \emptyset$ which shows that the space X cannot be represented as the union of two non-empty, disjoint, closed-and-open subsets, i.e., that the space X is connected. ■

Now, we are going to study operations on connected spaces.

Obviously, a subspace of a connected space generally is not connected. Let us discuss briefly connected subspaces of arbitrary topological spaces. We start with a simple characterization.

6.1.7. THEOREM. *A subspace C of a topological space X is connected if and only if for every pair X_1, X_2 of separated subsets of X such that $C = X_1 \cup X_2$ we have either $X_1 = \emptyset$ or $X_2 = \emptyset$.*

PROOF. Suppose that C is connected and $C = X_1 \cup X_2$, where $X_1 \cap \overline{X_2} = \emptyset = \overline{X_1} \cap X_2$. Clearly, the sets X_1 and X_2 are separated in the space C, so that by Theorem 6.1.1 one of them is empty.

On the other hand, if C is not connected, then there exists a pair X_1, X_2 of non-empty disjoint closed subsets of C such that $C = X_1 \cup X_2$. Clearly, the sets X_1 and X_2 are separated in the space X, so that the condition in the theorem does not hold. ■

6.1.8. COROLLARY. *If a subspace C of a topological space X is connected, then for every pair X_1, X_2 of separated subsets of X such that $C \subset X_1 \cup X_2$ we have either $C \subset X_1$ or $C \subset X_2$.*

PROOF. The sets $C \cap X_1$ and $C \cap X_2$ are separated in X and their union is equal to C; by the last theorem one of them must be empty, so that C is contained in the other. ■

6.1.9. THEOREM. *Let $\{C_s\}_{s \in S}$ be a family of connected subspaces of a topological space X. If there exists an $s_0 \in S$ such that the set C_{s_0} is not separated from any of the sets C_s, then the union $\bigcup_{s \in S} C_s$ is connected.*

PROOF. Suppose that $C = \bigcup_{s\in S} C_s = X_1 \cup X_2$, where X_1 and X_2 are separated subsets of X. By virtue of Corollary 6.1.8 we have either $C_{s_0} \subset X_1$ or $C_{s_0} \subset X_2$, let $C_{s_0} \subset X_1$. Since any of the sets C_s is contained either in X_1 or in X_2 and none is separated from C_{s_0}, we have $C_s \subset X_1$ for every $s \in S$. Hence $C \subset X_1$ and $X_2 = \emptyset$.■

6.1.10. COROLLARY. *If the family $\{C_s\}_{s\in S}$ of connected subspaces of a topological space has a non-empty intersection, then the union $\bigcup_{s\in S} C_s$ is connected.*■

6.1.11. COROLLARY. *If a subspace C of X is connected, then every subspace A of X which satisfies $C \subset A \subset \overline{C}$ also is connected.*

PROOF. Since the set C is not separated from the set $\{x\}$ for any $x \in A$, the family $\{C\} \cup \{\{x\}\}_{x\in A}$ satisfies the condition in Theorem 6.1.9.■

6.1.12. COROLLARY. *If a topological space X contains a connected dense subspace, then X itself is connected.*■

6.1.13. COROLLARY. *If any two points of a topological space X can be joined by a connected subspace of X, then the space X is connected.*

PROOF. Let x_0 be a fixed point of the space X and for every $x \in X$ let C_x denote a connected subspace of X joining x_0 and x. The family $\{C_x\}_{x\in X}$ satisfies the assumptions of Corollary 6.1.10 and $\bigcup_{x\in X} C_x = X$.■

Let us note that Corollaries 6.1.12 and 3.6.5 yield

6.1.14. THEOREM. *The Čech-Stone compactification βX of a Tychonoff space X is connected if and only if the space X is connected.*■

From the definition of connectedness it follows immediately that no non-trivial sum of topological spaces is connected.

6.1.15. THEOREM. *The Cartesian product $\prod_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is connected if and only if all spaces X_s are connected.*

PROOF. If the Cartesian product $X = \prod_{s\in S} X_s$ is connected and non-empty, then all spaces X_s are connected by Theorem 6.1.3, because the projection $p_s : X \to X_s$ is a continuous mapping of X onto X_s.

We shall now prove that Cartesian products of connected spaces are connected. To begin, let us consider the Cartesian product $X \times Y$ of two connected spaces. Any two points (x_1, y_1) and (x_2, y_2) of the space $X \times Y$ can be joined by the set $(X \times \{y_1\}) \cup (\{x_2\} \times Y)$, which is connected as the union of two connected sets with non-empty intersection; hence, the space $X \times Y$ is connected by Corollary 6.1.13.

By a straightforward inductive argument one can readily show that the Cartesian product of finitely many connected spaces is connected.

Let us now consider a family $\{X_s\}_{s\in S}$ of non-empty connected spaces and let us choose for every $s \in S$ a point $a_s \in X_s$. Denote by $\mathcal{T}$ the family of all finite subsets of the set S and for every $T \in \mathcal{T}$ let

$$C_T = \prod_{s\in S} A_s, \quad \text{where } A_s = \{a_s\} \text{ if } s \notin T \text{ and } A_s = X_s \text{ for } s \in T.$$

By the finite case of our theorem, the family $\{C_T\}_{T\in\mathcal{T}}$ consists of connected spaces. As $a = \{a_s\} \in \bigcap_{T\in\mathcal{T}} C_T \neq \emptyset$, it follows from Corollary 6.1.10 that the union $C = \bigcup_{T\in\mathcal{T}} C_T$ is connected. But C is a dense subspace of $\prod_{s\in S} X_s$, so that to conclude the proof it suffices to apply Corollary 6.1.12. ■

6.1.16. COROLLARY. *Euclidean n-space R^n, the Tychonoff cube $I^{\mathfrak{m}}$ and the Alexandroff cube $F^{\mathfrak{m}}$ are all connected.* ■

Obviously, quotient spaces of connected spaces are always connected.

We shall now give an example to show that the limit of an inverse sequence of connected spaces need not be connected.

6.1.17. EXAMPLE. Let X be the plane R^2 without the open interval with end-points $x_1 = (-1,0)$ and $x_2 = (1,0)$. For $i = 1,2,\ldots$ denote by X_i the intersection of X with the region inside the rectangle with vertices at $(-1-1/i, 1/i)$, $(-1-1/i, -1/i)$, $(1+1/i, -1/i)$ and $(1+1/i, 1/i)$. The spaces X_i are connected and $X_i \subset X_j$ whenever $j \leq i$; for any i, j satisfying $j \leq i$ let π^i_j be the embedding of X_i in X_j. By Example 2.5.4 the family $\{X_i, \pi^i_j\}$ is an inverse sequence of connected spaces and the limit of this inverse sequence consists of the points x_1 and x_2, i.e., is not connected. ■

The spaces in Example 6.1.17 are not compact; Theorem 6.1.20 below asserts that the limit of an inverse system of compact connected spaces is connected.

A topological space X is called a *continuum* if X is both connected and compact. Theorems 3.2.4 and 6.1.15 imply that the Cartesian product of continua is a continuum; similarly, Theorems 3.1.10 and 6.1.3 imply that a continuous image of a continuum is a continuum, provided it is a Hausdorff space.

6.1.18. THEOREM. *Let $\{C_s\}_{s\in S}$ be a family of subspaces of a topological space X each of which is a continuum. If for any $s_1, s_2 \in S$ there exists an $s_3 \in S$ such that $C_{s_3} \subset C_{s_1} \cap C_{s_2}$, i.e., if the family $\{C_s\}_{s\in S}$ of continua is directed by $\supset$, then the intersection $\bigcap_{s\in S} C_s$ is a continuum.*

PROOF. We can assume that X is a compact space. Indeed, let us fix an $s_0 \in S$ and for every s in S choose an $s' \in S$ such that $C_{s'} \subset C_{s_0} \cap C_s$; replacing the space X by C_{s_0} and the continua C_s by the continua $C_{s'}$ we reduce the problem to the case when X is a continuum.

Suppose that $C = \bigcap_{s\in S} C_s = X_1 \cup X_2$, where X_1 and X_2 are disjoint closed subsets of C. Since C is closed in X, the sets X_1 and X_2 are closed in X, and by Theorem 3.1.9 there exist disjoint open sets $U_1, U_2 \subset X$ such that $X_1 \subset U_1$ and $X_2 \subset U_2$. By virtue of Corollary 3.1.5 for a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ we have $\bigcap_{i=1}^k C_{s_i} \subset U_1 \cup U_2$. The family $\{C_s\}_{s\in S}$ being directed by $\supset$, there exists an $s \in S$ with the property that $C_s \subset \bigcap_{i=1}^k C_{s_i}$. Thus $C \subset C_s \subset U_1 \cup U_2$ and, by Corollary 6.1.8, either $C \subset C_s \subset U_1$ or $C \subset C_s \subset U_2$ which implies that $X_2 = \emptyset$ or $X_1 = \emptyset$. Hence the space C is connected; its compactness is obvious. ■

6.1.19. COROLLARY. *The intersection $\bigcap_{i=1}^\infty X_i$ of a decreasing sequence $X_1 \supset X_2 \supset \ldots$ of continua is a continuum.* ■

6.1.20. THEOREM. *The limit of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of continua is a continuum.*

PROOF. Let $X = \prod_{\sigma\in\Sigma} X_\sigma$ and for each $\rho \in \Sigma$ let $Z_\rho = \{\{x_\sigma\} \in X : \pi^\rho_\tau(x_\rho) = x_\tau$ for $\tau \leq \rho\}$ and $\Sigma_\rho = \Sigma \setminus \{\sigma \in \Sigma : \sigma \leq \rho\}$. The space Z_ρ is a continuum; indeed, it is the image of the continuum $\prod_{\sigma\in\Sigma_\rho} X_\sigma$ under the continuous mapping which assigns to the point $\{x_\sigma\} \in \prod_{\sigma\in\Sigma_\rho} X_\sigma$ the point $\{z_\sigma\} \in Z_\rho \subset X$, where $z_\sigma = x_\sigma$ for $\sigma \in \Sigma_\rho$ and $z_\sigma = \pi^\rho_\sigma(x_\rho)$ for $\sigma \leq \rho$. Since $Z_{\rho_1} \subset Z_{\rho_2}$ whenever $\rho_1 \leq \rho_2$ and since Σ is a directed set, by virtue of Theorem 6.1.18 the intersection $\bigcap_{\rho\in\Sigma} Z_\rho$ is a continuum. As this intersection coincides with $\varprojlim \mathbf{S}$, the proof is concluded.∎

The function space Y^X generally is not connected for connected spaces X and Y. The study of conditions for connectedness of function spaces leads to important results; it requires, however, special methods which are not developed in this book. For instance, one proves that for a subspace X of R^n the space $(S^{n-1})^X$ with the compact-open topology is connected if and only if the space $S^n \setminus X$ is connected (here we consider the n-sphere S^n as the compactification of Euclidean n-space R^n obtained by adjoining to R^n a "point in infinity"; for a compact subspace X of R^n, the space $S^n \setminus X$ is connected if and only if the space $R^n \setminus X$ is connected). Since for two homeomorphic subspaces X_1, X_2 of R^n the function spaces $(S^{n-1})^{X_1}$ and $(S^{n-1})^{X_2}$ are homeomorphic, it follows that $S^n \setminus X_1$ is connected if and only if $S^n \setminus X_2$ is connected. This is an important and deep result generalizing the classical Jordan theorem which says that for every subspace X of the plane R^2 homeomorphic to the circle S^1, the complement $R^2 \setminus X$ is not connected.

The *component of a point* x in a topological space X is the union of all connected subspaces of X which contain the point x. From Corollaries 6.1.10 and 6.1.11 it follows that components are connected closed subsets of X. The components of two distinct points of a topological space X either coincide or are disjoint, so that all components constitute a decomposition of the space X into connected pairwise disjoint closed subsets, which are called the *components of the space* X.

The following simple result will be applied in Section 6.2.

6.1.21. THEOREM. *The component of a point $x = \{x_s\}$ in the Cartesian product $\prod_{s\in S} X_s$ coincides with the Cartesian product $\prod_{s\in S} C_s$, where C_s is the component of the point x_s in the space X_s.*

PROOF. Denote the component of x in $\prod_{s\in S} X_s$ by C. Since for every $s \in S$ the projection $p_s(C)$ is connected, we have $p_s(C) \subset C_s$. Hence, $C \subset \prod_{s\in S} C_s$; the equality $C = \prod_{s\in S} C_s$ follows from Theorem 6.1.15.∎

The *quasi-component of a point* x in a topological space X is the intersection of all closed-and-open subsets of X which contain the point x. Quasi-components are closed subsets of X. The quasi-components of two distinct points of a topological space X either coincide or are disjoint, so that all quasi-components constitute a decomposition of the space X into pairwise disjoint closed subsets, which are called the *quasi-components of the space* X.

6.1.22. THEOREM. *The component C of a point x in a topological space X is contained in the quasi-component Q of the point x.*

PROOF. Let F be a closed-and-open subset of X that contains x. The sets F and $X \setminus F$, and – *a fortiori* – the sets $C \cap F$ and $C \setminus F$, are separated; as $C \cap F \neq \emptyset$ it follows from Theorem 6.1.7 that $C \setminus F = \emptyset$, i.e., that $C \subset F$. Thus we have $C \subset Q$.∎

6.1.23. THEOREM. *In a compact space X the component of a point $x \in X$ coincides with the quasi-component of the point x.*

PROOF. By virtue of the last theorem, it suffices to show that the quasi-component Q of the point x is connected.

Suppose that two disjoint closed subsets X_1, X_2 of the space Q satisfy the conditions $Q = X_1 \cup X_2$ and $x \in X_1$. The sets X_1 and X_2 being closed in X, by normality of compact spaces there exist open subsets U, V of X such that

$$(1) \qquad X_1 \subset U, \quad X_2 \subset V \quad \text{and} \quad U \cap V = \emptyset.$$

Hence, we have $Q \subset U \cup V$ and, by Corollary 3.1.5, there exist closed-and-open sets $F_1, F_2, \ldots, F_k$ such that $Q \subset F = \bigcap_{i=1}^{k} F_i \subset U \cup V$; clearly, the set F is closed-and-open. Since

$$\overline{U \cap F} \subset \overline{U} \cap F = \overline{U} \cap (U \cup V) \cap F = U \cap F,$$

the intersection $U \cap F$ also is closed-and-open. As $x \in U \cap F$, we have $Q \subset U \cap F$ and

$$(2) \qquad X_2 \subset Q \subset U \cap F \subset U.$$

From (1) and (2) it follows that $X_2 = \emptyset$, which shows that the set Q is connected.■

6.1.24. EXAMPLE. We shall now describe a space in which components and quasi-components are different from each other.

Let X be the subspace of the plane consisting of the closed intervals $I_i = [0,1] \times \{1/i\}$, where $i = 1, 2, \ldots$, and the points $p_0 = (0,0)$ and $p_1 = (1,0)$. The segments I_i and the points p_0, p_1 are the components of the space X. We shall show that the quasi-component Q of the point p_0 coincides with the set $\{p_0, p_1\}$.

Every closed-and-open set F containing p_0 contains almost all terms of the sequence $\{(0, 1/i)\}$; the segments I_i being connected, almost all of them are contained in F. Hence, the set F contains almost all terms of the sequence $\{(1, 1/i)\}$, and the limit p_1 of this sequence also belongs to F. Thus $\{p_0, p_1\} \subset Q$, and since no other point of X can belong to Q, we have $Q = \{p_0, p_1\}$.

Replacing points p_0 and p_1 by intervals $[0, 1/2) \times \{0\}$ and $(1/2, 1] \times \{0\}$, we obtain a locally compact space with similar properties.■

It turns out that a continuum cannot be decomposed into countably many pairwise disjoint non-empty closed subsets (cf. Exercise 6.1.G); the proof of this fact will be preceded by two lemmas.

6.1.25. LEMMA. *If A is a closed subspace of a continuum X such that $\emptyset \neq A \neq X$, then for every component C of the space A we have $C \cap \operatorname{Fr} A \neq \emptyset$.*

PROOF. From Theorem 6.1.23 it follows that $C = \bigcap \mathcal{K}$, where $\mathcal{K}$ is the family of all closed-and-open subsets of A containing a point $x_0 \in C$. Suppose that $C \cap \operatorname{Fr} A = \emptyset$. Since the family $\mathcal{K}$ is closed with respect to finite intersections and the set $\operatorname{Fr} A$ is compact, there exists a $K \in \mathcal{K}$ such that $K \cap \operatorname{Fr} A = \emptyset$. Take an open set $U \subset X$ satisfying $U \cap A = K$. The equality $K \cap \operatorname{Fr} A = \emptyset$ implies that $K = U \cap \operatorname{Int} A$, i.e., that the set K is open in X. But the set K is also closed in X and contains x_0, so that $K = X$. This implies that $\operatorname{Fr} A = \emptyset$, which is impossible.■

6.1.26. LEMMA. *If a continuum X is covered by pairwise disjoint closed sets $X_1, X_2, \ldots$ of which at least two are non-empty, then for every i there exists a continuum $C \subset X$ such that $C \cap X_i = \emptyset$ and at least two sets in the sequence $C \cap X_1, C \cap X_2, \ldots$ are non-empty.*

PROOF. If $X_i = \emptyset$ we let $C = X$; thus we can assume that $X_i \neq \emptyset$. Take a $j \neq i$ such that $X_j \neq \emptyset$ and any disjoint open sets $U, V \subset X$ satisfying $X_i \subset U$ and $X_j \subset V$. Let x be a point of X_j and C the component of x in the subspace $\overline{V}$. Clearly, C is a continuum, $C \cap X_i = \emptyset$ and $C \cap X_j \neq \emptyset$. Since $C \cap \operatorname{Fr} \overline{V} \neq \emptyset$, by virtue of the previous lemma, and since $X_j \subset \operatorname{Int} \overline{V}$, there exists a $k \neq j$ such that $C \cap X_k \neq \emptyset$.■

6.1.27. THE SIERPIŃSKI THEOREM. *If a continuum X has a countable cover $\{X_i\}_{i=1}^{\infty}$ by pairwise disjoint closed subsets, then at most one of the sets X_i is non-empty.*

PROOF. Let $X = \bigcup_{i=1}^{\infty} X_i$, where the sets X_i are closed and $X_i \cap X_j = \emptyset$ whenever $i \neq j$; assume that at least two of the sets X_i are non-empty. From Lemma 6.1.26 it follows that there exists a decreasing sequence $C_1 \supset C_2 \supset \ldots$ of continua contained in X such that

$$(3) \qquad C_i \cap X_i = \emptyset \quad \text{and} \quad C_i \neq \emptyset \quad \text{for} \quad i = 1, 2, \ldots$$

The first part of (3) implies that $(\bigcap_{i=1}^{\infty} C_i) \cap (\bigcup_{i=1}^{\infty} X_i) = \emptyset$, i.e., that $\bigcap_{i=1}^{\infty} C_i = \emptyset$, and yet from the second part of (3) and compactness of X it follows that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.■

The concept of connectedness leads to a new class of mappings. We say that a continuous mapping $f : X \to Y$ is *monotone* if all fibers $f^{-1}(y)$ are connected. The name "monotone mapping" is due to the fact that a continuous function from the real line to itself has connected fibers if and only if it is either nondecreasing or nonincreasing (cf. Exercise 6.1.H).

6.1.28. THEOREM. *If $f : X \to Y$ is a monotone quotient mapping, then for every connected subset C of the space Y which is either closed or open, the inverse image $f^{-1}(C)$ is connected.*

PROOF. By virtue of Proposition 2.4.15, it suffices to prove that if Y is connected, then X also is connected. Consider a decomposition $X = X_1 \cup X_2$ of X into two disjoint closed-and-open sets. The fibers of f being connected, we have $X_i = f^{-1}(Y_i)$ for $i = 1, 2$, where $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$. Since f is a quotient mapping, both sets Y_i are closed-and-open. Hence, by the connectedness of Y one of the sets Y_i is empty. This clearly implies that one of the sets X_i is empty, so that the space X is connected.■

6.1.29. THEOREM. *If $f : X \to Y$ is a monotone mapping of X onto Y which is either closed or open, then for every connected subset C of the space Y the inverse image $f^{-1}(C)$ is connected.*

PROOF. By virtue of Proposition 2.1.4, it suffices to show that if Y is connected, then X also is connected; this, however, follows immediately from the last theorem.■

Historical and bibliographic notes

The present definition of connectedness was introduced by Jordan in 1893 for the class of compact subsets of the plane; generalization to abstract spaces is due to Riesz [1907], Lennes [1911] and Hausdorff [1914]. The condition in Exercise 6.1.D, equivalent to connectedness in the realm of compact metric spaces, was introduced by Cantor in 1883. A systematic study of connectedness was originated by Hausdorff in [1914] and by Knaster and Kuratowski in [1921]. Hausdorff's book [1914] contains Corollaries 6.1.10-6.1.13, definitions of a component and of a quasi-component, Theorem 6.1.22, and Theorem 6.1.23 for compact metric spaces (generalization of the latter to arbitrary compact spaces was given by Šura-Bura in [1941]). Knaster and Kuratowski's paper [1921] contains Theorems 6.1.7 and 6.1.9. The first example of a countable connected Hausdorff space was given by Urysohn in [1925]; our Example 6.1.6 is taken from Bing's paper [1953]; it is much simpler than the original one. Theorem 6.1.15 for finite Cartesian products was proved by van Dantzig in [1930], Theorem 6.1.20 is suggested as an exercise in Eilenberg and Steenrod's book [1952]; as stated in Zoretti's paper [1905], Corollary 6.1.19 (in a slightly different form) was proved by P. Painlevé for continua in the plane. The fact that for a subspace X of R^n the space $(S^{n-1})^X$ is connected if and only if the space $S^n \setminus X$ is connected was proved independently by Alexandroff in [1932] and by Borsuk in [1932] under the assumption of compactness of X (for the topology on $(S^{n-1})^X$ induced by the metric defined by formula (7) in Section 4.2); in full generality this fact was proved by Kuratowski in [1959]. Theorem 6.1.27 was proved by Sierpiński in [1918]. Monotone mappings defined on continua were first studied by R. L. Moore in [1925] (in terms of upper semicontinuous decompositions). The class of monotone mappings was introduced by Whyburn in [1934].

Exercises

6.1.A. Describe all connected subspaces of the real line.

Hint. Observe that every connected subspace of the real line is convex.

6.1.B. Check that for every sequence $C_1, C_2, \ldots$ of connected subspaces of a topological space such that $C_i \cap C_{i+1} \neq \emptyset$ for $i = 1, 2, \ldots$, the union $\bigcup_{i=1}^{\infty} C_i$ is connected.

6.1.C. Show that if X is a connected space, and for a connected subspace A of X we have $X \setminus A = U \cup V$, where U, V are open in $X \setminus A$ and disjoint, then the sets $A \cup U$ and $A \cup V$ are connected.

6.1.D. (a) Verify that if a space X with the topology induced by a metric ρ is connected, then for every pair x, y of points of X and any $\epsilon > 0$ there exists a finite sequence $x_1, x_2, \ldots, x_k$ of points of X such that $x_1 = x$, $x_k = y$ and $\rho(x_i, x_{i+1}) < \epsilon$ for $i = 1, 2, \ldots, k-1$.

(b) Show that every compact metric space (X, ρ) satisfying the condition in part (a) is connected and note that the assumption of compactness is essential.

6.1.E. Prove that in the realm of connected spaces strong paracompactness is equivalent to the Lindelöf property. Deduce that the hedgehog $J(\mathfrak{m})$ is not strongly paracompact if $\mathfrak{m} > \aleph_0$.

6.1.F. (a) Give an example of a continuum which is not a continuous image of a connected metrizable space.

Hint. Applying Corollary 3.6.15, show that βR is not a continuous image of a connected sequential space.

(b) Show that for every $\mathfrak{m} \geq \aleph_0$ the Tychonoff cube $I^{\mathfrak{m}}$ is a continuous image of a connected metrizable space.

6.1.G. Give an example of a connected subspace of the plane which can be decomposed into countably many pairwise disjoint nonempty closed subsets.

6.1.H. Prove that a continuous mapping $f : R \to R$ is monotone if and only if either $f(x) \leq f(y)$ whenever $x \leq y$ or $f(x) \geq f(y)$ whenever $x \leq y$.

6.1.I. Observe that Theorem 6.1.29 holds for monotone hereditarily quotient mappings and note that it does not hold for monotone quotient mappings.

Hint. Modify Example 2.4.17.

6.2. Various kinds of disconnectedness

A topological space X is called *hereditarily disconnected* if X does not contain any connected subsets of cardinality larger than one. Hence, a space X is hereditarily disconnected if and only if the component of any point $x \in X$ consists of the point x alone. Since the components of a space are closed, every hereditarily disconnected space is a T_1-space. Let us note that in the context of disconnectedness the term "hereditarily" has a meaning slightly different from that accepted throughout this book; in this context the standard meaning of that term would lead to the empty class of spaces.

A topological space X is called *zero-dimensional* if X is a non-empty T_1-space and has a base consisting of open-and-closed sets. Clearly, every zero-dimensional space is a Tychonoff space.

6.2.1. THEOREM. *Every zero-dimensional space is hereditarily disconnected.*

PROOF. Let A be a subset of a zero-dimensional space X such that $|A| > 1$. Take a pair x_1, x_2 of distinct points of A and an open-and-closed subset U of the space X such that $x_1 \in U \subset X \setminus \{x_2\}$. Since the sets $A \setminus U$ and $A \cap U$ are separated and non-empty, the set A is not connected. ■

A cover of a topological space consisting of functionally open (closed) sets will be called in the sequel a *functionally open (closed)* cover.

A topological space X is called *strongly zero-dimensional* if X is a non-empty Tychonoff space and every finite functionally open cover $\{U_i\}_{i=1}^k$ of the space X has a finite open refinement $\{V_i\}_{i=1}^m$ such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$. Clearly, the refinement $\{V_i\}_{i=1}^m$ consists of open-and-closed sets and thus is a functionally open cover of X.

6.2.2. LEMMA. *For every pair A, B of completely separated subsets of a strongly zero-dimensional space X there exists an open-and-closed set $U \subset X$ such that $A \subset U \subset X \setminus B$.*

PROOF. Let $f : X \to I$ be a continuous function satisfying

$$f(A) \subset \{0\} \quad \text{and} \quad f(B) \subset \{1\}.$$

The sets $f^{-1}((0,1])$ and $f^{-1}([0,1))$ constitute a functionally open cover of the space X; take a refinement $\mathcal{V}$ of this cover which consists of pairwise disjoint open sets. The set $U = \bigcup\{V \in \mathcal{V} : A \cap V \neq \emptyset\}$ is open-and-closed and we clearly have $A \subset U \subset X \setminus B$.■

6.2.3. LEMMA. *If for every pair A, B of completely separated subsets of a topological (normal) space X there exists an open-and-closed set $U \subset X$ such that $A \subset U \subset X \setminus B$, then every finite functionally open (open) cover $\{U_i\}_{i=1}^k$ of the space X has a finite open refinement $\{V_i\}_{i=1}^k$ such that $V_i \subset U_i$ for $i = 1, 2, \ldots, k$ and $V_i \cap V_j = \emptyset$ whenever $i \neq j$.*

PROOF. We apply induction with respect to k. For $k = 1$ the lemma is valid; assume that it is valid for every $k < m > 1$ and consider a functionally open (open) cover $\{U_i\}_{i=1}^m$ of the space X. By the inductive hypothesis there exists a cover $\{W_1, W_2, \ldots, W_{m-1}\}$ of the space X consisting of pairwise disjoint open-and-closed sets satisfying $W_i \subset U_i$ for $i < m-1$ and $W_{m-1} \subset U_{m-1} \cup U_m$.

The sets $W_{m-1} \setminus U_{m-1}$ and $W_{m-1} \setminus U_m$ are disjoint and functionally closed (closed), so that by Theorem 1.5.14 (Theorem 1.5.11) they are completely separated. Hence, there exists an open-and-closed set $U \subset X$ such that $W_{m-1} \setminus U_{m-1} \subset U$ and $U \subset X \setminus (W_{m-1} \setminus U_m) = (X \setminus W_{m-1}) \cup U_m$; clearly, $W_{m-1} \setminus U \subset U_{m-1}$ and $W_{m-1} \cap U \subset U_m$. One readily verifies that the family $\{V_i\}_{i=1}^m$, where

$$V_i = W_i \quad \text{for } i < m-1, \quad V_{m-1} = W_{m-1} \setminus U, \quad \text{and} \quad V_m = W_{m-1} \cap U,$$

is an open cover of the space X such that $V_i \subset U_i$ for $i = 1, 2, \ldots, m$ and $V_i \cap V_j = \emptyset$ whenever $i \neq j$.■

Lemmas 6.2.2 and 6.2.3 yield

6.2.4. THEOREM. *A non-empty Tychonoff space X is strongly zero-dimensional if and only if for every pair A, B of completely separated subsets of the space X there exists an open-and-closed set $U \subset X$ such that $A \subset U \subset X \setminus B$.*■

The last theorem and Lemma 6.2.3 imply

6.2.5. THEOREM. *A non-empty normal space X is strongly zero-dimensional if and only if every finite open cover $\{U_i\}_{i=1}^k$ of the space X has a finite open refinement $\{V_i\}_{i=1}^m$ such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$.*■

Let us observe that Lemma 6.2.2 immediately yields

6.2.6. THEOREM. *Every strongly zero-dimensional space is zero-dimensional.*■

6.2.7. THEOREM. *Every zero-dimensional Lindelöf space is strongly zero-dimensional.*

PROOF. It suffices to show that for every pair A, B of disjoint closed subsets of a zero-dimensional Lindelöf space X there exists an open-and-closed set $U \subset X$ such that $A \subset U \subset X \setminus B$. For every $x \in X$ choose an open-and-closed set $W_x \subset X$ which contains x and satisfies

$$A \cap W_x = \emptyset \quad \text{or} \quad B \cap W_x = \emptyset.$$

Let $\{W_{x_i}\}_{i=1}^\infty$ be a countable subcover of the cover $\{W_x\}_{x \in X}$ of the space X. The sets

$$U_i = W_{x_i} \setminus \bigcup_{j<i} W_{x_j}, \quad \text{where} \quad i = 1, 2, \ldots,$$

are open-and-closed and pairwise disjoint, and the family $\{U_i\}_{i=1}^{\infty}$ is a cover of the space X. The set $U = \bigcup\{U_i : A \cap U_i \neq \emptyset\}$ has the required properties.■

6.2.8. COROLLARY. *Every countable non-empty regular space X is strongly zero-dimensional.*

PROOF. Since X is a Lindelöf space, it suffices to show that all open-and-closed sets constitute a base for the space X. To that end, consider a point $x \in X$ and a neighbourhood V of the point x. Applying Theorem 1.5.17, take a continuous function $f : X \to I$ such that $f(x) = 0$ and $f(X \setminus V) \subset \{1\}$; the space X being countable, there exists an $r \in I \setminus f(X)$. Obviously, the inverse image $U = f^{-1}([0,r)) = f^{-1}([0,r])$ is an open-and-closed set which contains x and is contained in V.■

6.2.9. THEOREM. *Every non-empty hereditarily disconnected locally compact space is zero-dimensional.*

PROOF. Let X be a non-empty hereditarily disconnected locally compact space. Consider a point $x \in X$ and a neighbourhood V of x. Take a neighbourhood W of the point x such that $W \subset V$ and the closure $\overline{W}$ is compact. Theorem 6.1.23 implies that the set $\{x\} \subset \overline{W}$ coincides with the intersection of the family $\mathcal{K}$ consisting of all open-and-closed subsets of the space $\overline{W}$ that contain the point x. Hence, by Corollary 3.1.5, there exists a finite number of sets $F_1, F_2, \ldots, F_k \in \mathcal{K}$ such that $x \in U = F_1 \cap F_2 \cap \ldots \cap F_k \subset W \subset V$. The set U is closed in X, because it is closed in $\overline{W}$; it is also open in X, because it is open in W.■

From Theorems 6.2.1, 6.2.9, 6.2.6, 6.2.7 and 5.1.27 (cf. Theorem 6.2.13 below) we obtain:

6.2.10. THEOREM. *Hereditary disconnectedness, zero-dimensionality and strong zero-dimensionality are equivalent in the realm of non-empty locally compact paracompact spaces.*■

6.2.11. THEOREM. *Hereditary disconnectedness is a hereditary property and zero-dimensionality is hereditary with respect to non-empty sets.*

If X is a strongly zero-dimensional space and M is a non-empty subspace of X with the property that every continuous function $f : M \to I$ is continuously extendable over X, then the space M also is strongly zero-dimensional.

In particular, *in normal spaces strong zero-dimensionality is hereditary with respect to non-empty closed sets.*

PROOF. The first part of the theorem is obvious. The second part follows from Theorem 6.2.4, because, under the assumption on M, every pair of completely separated subsets of the space M is completely separated in X.■

As noted in the last paragraph of Example 6.2.20 below, strong zero-dimensionality is not a hereditary property (cf. Problem 7.4.C).

6.2.12. THEOREM. *The Čech-Stone compactification βX of a Tychonoff space X is strongly zero-dimensional if and only if the space X is strongly zero-dimensional.*

PROOF. By virtue of the last theorem it suffices to show that if X is strongly zero-dimensional, then so is βX. Let A, B be a pair of completely separated subsets of βX and let $f : \beta X \to I$ be a continuous function satisfying $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. The sets

$A_1 = X \cap f^{-1}([0,1/3))$ and $B_1 = X \cap f^{-1}((2/3,1])$ are completely separated subsets of X, so that by virtue of Theorem 6.2.4 there exists an open-and-closed set $U \subset X$ such that $A_1 \subset U$ and $B_1 \subset X \setminus U$. Corollary 3.6.5 implies that $\overline{U}$ is an open-and-closed subset of βX, and from Theorem 1.3.6 it follows that $A \subset \overline{A}_1$ and $B \subset \overline{B}_1$. Since $\overline{B}_1 \cap \overline{U} = \emptyset$, we have $A \subset \overline{A}_1 \subset \overline{U} \subset \beta X \setminus \overline{B}_1 \subset \beta X \setminus B$; hence, βX is strongly zero-dimensional by Theorem 6.2.4. ■

In the final part of Example 6.2.20 we shall show that the counterparts of Theorem 6.2.12 for hereditary disconnectedness and zero-dimensionality do not hold.

Theorem 6.2.11, Proposition 2.2.1, and the definition of the topology on a sum of spaces yield

6.2.13. THEOREM. *The sum $\bigoplus_{s\in S} X_s$, where $S \neq \emptyset$ and $X_s \neq \emptyset$ for $s \in S$, is hereditarily disconnected (zero-dimensional, strongly zero-dimensional) if and only if all spaces X_s are hereditarily disconnected (zero-dimensional, strongly zero-dimensional).* ■

6.2.14. THEOREM. *The Cartesian product $\prod_{s\in S} X_s$, where $S \neq \emptyset$ and $X_s \neq \emptyset$ for $s \in S$, is hereditarily disconnected (zero-dimensional) if and only if all spaces X_s are herditarily disconnected (zero-dimensional).*

PROOF. Each of the spaces X_s is homeomorphic to a subspace of $\prod_{s\in S} X_s$, so that, by Theorem 6.2.11, it suffices to show that the Cartesian product $\prod_{s\in S} X_s$ of hereditarily disconnected (zero-dimensional) spaces is hereditarily disconnected (zero-dimensional). For hereditary disconnectedness this is a consequence of Theorem 6.1.21. For zero-dimensionality this follows from the fact that the sets of the form $\prod_{s\in S} W_s$, where W_s is an open-and-closed subset of X_s and the set $\{s \in S : W_s \neq X_s\}$ is finite, constitute a base for $\prod_{s\in S} X_s$ and are open-and-closed. ■

6.2.15. COROLLARY. *The limit of an inverse system of hereditarily disconnected (zero-dimensional) spaces is hereditarily disconnected (zero-dimensional or empty).* ■

The Cartesian product of two strongly zero-dimensional spaces need not be strongly zero-dimensional; the example is too difficult to be discussed here. Similarly, the limit of an inverse sequence of strongly zero-dimensional spaces need not be strongly zero-dimensional or empty (see Problem 6.3.25).

6.2.16. THEOREM. *The Cantor cube $D^{\mathfrak{m}}$ is universal for all zero-dimensional spaces of weight $\mathfrak{m} \geq \aleph_0$.*

PROOF. By virtue of the last theorem it suffices to prove that every zero-dimensional space X of weight $\mathfrak{m}$ is embeddable in $D^{\mathfrak{m}}$.

It follows from Theorem 1.1.15 that there exists a base $\{U_s\}_{s\in S}$ for the space X consisting of open-and-closed sets and such that $|S| = \mathfrak{m}$. For every $s \in S$ define a mapping $f_s : X \to D$ by letting

$$f_s(x) = \begin{cases} 1 & \text{for } x \in U_s, \\ 0 & \text{for } x \in X \setminus U_s. \end{cases}$$

By the diagonal theorem, the mapping $f = \triangle_{s\in S} f_s$ is a homeomorphic embedding of X in the Cantor cube $D^{\mathfrak{m}}$. ■

6.2.17. COROLLARY. *Every zero-dimensional space X of weight $\mathfrak{m}$ has a zero-dimensional compactification of weight $\mathfrak{m}$.*

PROOF. We can assume that $\mathfrak{m} \geq \aleph_0$; then, the closure of a subspace of $D^{\mathfrak{m}}$ homeomorphic to X is the required compactification. ∎

By virtue of Theorem 3.2.2 (see also Exercise 3.2.B) there exists a mapping f of a closed subspace X of the Cantor set $D^{\aleph_0}$ onto the closed interval I. From Theorem 3.2.11 it follows that the quotient space $X/E(f)$ is homeomorphic to the interval I. Hence, a quotient space of a hereditarily disconnected (zero-dimensional, strongly zero-dimensional) space is not necessarily hereditarily disconnected (zero-dimensional, strongly zero-dimensional).

6.2.18. EXAMPLES. Every non-empty discrete space is strongly zero-dimensional.

The space W of all ordinal numbers $\leq \omega_1$ (see Example 3.1.27) is zero-dimensional; since this is a compact space, it is strongly zero-dimensional. By virtue of Theorem 6.2.11 and Example 3.1.27, the space W_0 of all countable ordinal numbers also is strongly zero-dimensional.

As noted in 2.2.8, the Sorgenfrey line K is zero-dimensional; since this is a Lindelöf space (see Example 3.8.14), it is strongly zero-dimensional.

From 6.1.5 it follows that a zero-dimensional subspace of the real line does not contain any interval. One easily checks that every non-empty subspace of the real line which does not contain any interval is zero-dimensional. Hence, a non-empty subspace of the real line is zero-dimensional or – equivalently (by Theorems 6.2.7 and 3.8.1) – strongly zero-dimensional if and only if it does not contain any interval. In particular, the set of rational numbers and the set of irrational numbers are zero-dimensional. Theorems 6.2.7 and 6.2.14 imply that the subspace of Euclidean n-space R^n (of the n-cube I^n, of the Hilbert cube $I^{\aleph_0}$) consisting of all points with rational coordinates is strongly zero-dimensional. Similarly, the subspace of Euclidean n-space R^n (of the n-cube I^n, of the Hilbert cube $I^{\aleph_0}$) consisting of all points with irrational coordinates is strongly zero-dimensional.

Theorem 6.2.14 implies that the Baire space $B(\mathfrak{m})$ is zero-dimensional for every $\mathfrak{m} \geq \aleph_0$. In Example 7.3.14 we shall show that the space $B(\mathfrak{m})$ is strongly zero-dimensional. ∎

We shall now describe a hereditarily disconnected separable metric space which is not zero-dimensional.

6.2.19. ERDÖS' EXAMPLE. Let X be the subspace of Hilbert space H, defined in Example 4.1.7, consisting of all infinite sequence $\{r_i\}$ of rational numbers.

The space X is hereditarily disconnected. Indeed, for any pair $x = \{r_i'\}$ and $y = \{r_i''\}$ of distinct points of X there exists a natural number i_0 such that $r_{i_0}' \neq r_{i_0}''$; one can obviously assume that $r_{i_0}' < r_{i_0}''$. Now, for any irrational number t such that $r_{i_0}' < t < r_{i_0}''$, the set

$$W = \{\{r_i\} \in X : r_{i_0} < t\}$$

is open-and-closed, contains x and does not contain y. Hence, every connected subspace of X contains at most one point.

Let $x_0 = (0, 0, \ldots)$ be the point of X with all coordinates equal to zero and let $V = B(x_0, 1) = \{\{r_i\} \in X : \sum_{i=1}^{\infty} r_i^2 < 1\}$. We shall show that for any neighbourhood U of the point x_0 which is contained in the neighbourhood V of x_0 we have $\mathrm{Fr}\, U \neq \emptyset$, i.e., that no such U is open-and-closed. This will imply that the space X is not zero-dimensional.

We shall define inductively a sequence $a_1, a_2, \ldots$ of rational numbers satisfying the conditions

$$(1) \qquad x_k = (a_1, a_2, \ldots, a_k, 0, 0, \ldots) \in U \quad \text{and} \quad \rho(x_k, X \setminus U) \leq 1/k.$$

Obviously, (1) is satisfied for $k = 1$ if we let $a_1 = 0$. Suppose that the rational numbers $a_1, a_2, \ldots, a_{m-1}$ are already defined and conditions (1) are satisfied for all $k \leq m - 1$. The sequence

$$x_i^m = (a_1, a_2, \ldots, a_{m-1}, i/m, 0, 0, \ldots)$$

is an element of X for $i = 0, 1, \ldots, m$. Since $x_0^m \in U$ and $x_m^m \notin U$, there exists an $i < m$ such that $x_i^m \in U$ and $x_{i+1}^m \notin U$. One easily sees that conditions (1) are satisfied for $k = m$, if we let $a_m = i/m$. Thus the sequence $a_1, a_2, \ldots$ is defined. From (1) it follows that $\sum_{i=1}^k a_i^2 < 1$ for $k = 1, 2, \ldots$, so that the point $a = \{a_i\}$ is an element of X; it also follows from (1) that $a \in \overline{U} \cap \overline{X \setminus U} = \mathrm{Fr}\, U$. ■

We shall now describe a zero-dimensional normal space which is not strongly zero-dimensional.

6.2.20. DOWKER'S EXAMPLE. Let Q denote the set of all rational numbers in the interval I; by letting

$$xEy \quad \text{whenever} \quad |x - y| \in Q$$

we define an equivalence relation on the set I. Clearly, each equivalence class of E is countable and dense in I. Hence, the family of all equivalence classes has cardinality $\mathfrak{c}$; let us choose a subfamily of cardinality $\aleph_1$ which does not contain the equivalence class Q and let us arrange the members of this subfamily into a transfinite sequence $Q_0, Q_1, \ldots, Q_\alpha, \ldots,\ \alpha < \omega_1$.

By virtue of 6.2.18, for every $\alpha < \omega_1$ the set $S_\alpha = I \setminus \bigcup_{\gamma \geq \alpha} Q_\gamma$ is zero-dimensional. Let W be the space of all ordinal numbers $\leq \omega_1$ and let $W_0 = W \setminus \{\omega_1\}$. For every $\alpha < \omega_1$ the set $X_\alpha = \{\gamma \in W : \gamma \leq \alpha\}$ is closed-and-open in W. Consider the Cartesian product $W \times I$ and its subspaces

$$Y_\alpha = \bigcup_{\gamma \leq \alpha} (\{\gamma\} \times S_\gamma), \quad Y = \bigcup_{\gamma < \omega_1} (\{\gamma\} \times S_\gamma) \quad \text{and} \quad Y^* = Y \cup (\{\omega_1\} \times I).$$

The space $Y_\alpha = Y \cap (X_\alpha \times I)$ is closed-and-open in Y; as $Y_\alpha \subset X_\alpha \times S_\alpha$, the space Y_α is zero-dimensional by virtue of Theorems 6.2.11 and 6.2.14. Hence, the space Y also is zero-dimensional. Since the Cartesian product $X_\alpha \times I$ is a second-countable compact space, it is metrizable by Theorem 4.2.8; the subspace $Y_\alpha \subset X_\alpha \times I$ also is metrizable and, *a fortiori*, normal.

We shall show that the spaces Y and Y^* also are normal. Let A, B be a pair of disjoint closed subsets of Y^*. The sets $A_1 = A \cap (Y^* \setminus Y)$ and $B_1 = B \cap (Y^* \setminus Y)$ are compact, so that by Theorem 3.1.6 there exist open sets $U, V \subset Y^*$ such that

$$(2) \qquad A_1 \subset U, \quad B_1 \subset V \quad \text{and} \quad U \cap V = \emptyset.$$

Applying Corollary 3.1.5 and using compactness of $W \times I$ we easily verify that for some $\alpha < \omega_1$ the inclusions

$$A \cap (Y^* \setminus Y_\alpha) \subset U \quad \text{and} \quad B \cap (Y^* \setminus Y_\alpha) \subset V \tag{3}$$

hold. Formulas (2) and (3), along with normality of the space Y_α, imply that the sets A and B can be separated by open subsets of Y^*, i.e., that the space Y^* is normal.

Now, let A, B be a pair of disjoint closed subsets of Y. We shall show that the closures of A and B in Y^* are disjoint; this will yield the normality of Y and, by Corollary 3.6.4, the equality $\beta Y^* = \beta Y$.

Suppose that there exists a point $(\omega_1, x) \in \overline{A} \cap \overline{B}$ and take an $\alpha < \omega_1$ such that $x \in S_\gamma$ for all $\gamma \geq \alpha$. We can readily define by induction two sequences $\alpha_1, \alpha_1, \ldots$ and $\beta_1, \beta_2, \ldots$ of countable ordinal numbers larger than α and two sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ of elements of I such that

$$\alpha_i < \beta_i < \alpha_{i+1},\ |x - x_i| < 1/i,\ |x - y_i| < 1/i,\ (\alpha_i, x_i) \in A \text{ and } (\beta_i, y_i) \in B \text{ for } i = 1, 2, \ldots;$$

arguing as in Example 3.1.27 we infer that $A \cap B \neq \emptyset$, a contradiction.

Assume that the space Y is strongly zero-dimensional. By Theorem 6.2.4 there exists an open-and-closed set $U \subset Y$ such that $W_0 \times \{0\} \subset U$ and $W_0 \times \{1\} \subset Y \setminus U$; as proved above, $\overline{U} \cap \overline{(Y \setminus U)} = \emptyset$. Since $Y^* = \overline{U} \cup \overline{(Y \setminus U)}$, $(\omega_1, 0) \in \overline{U}$ and $(\omega_1, 1) \in \overline{Y \setminus U}$, our assumption contradicts the connectedness of $\{\omega_1\} \times I$. Hence, the space Y is not strongly zero-dimensional.

From the zero-dimensionality of Y, Corollary 6.2.17 and Theorem 6.2.7 it follows that a strongly zero-dimensional space may contain a non-empty normal subspace which is not strongly zero-dimensional. The space Y shows also that zero-dimensionality of a space does not imply that the Čech-Stone compactification of that space is zero-dimensional (cf. Theorem 6.2.12). In fact, by Theorems 6.2.12 and 6.2.7, the space βY cannot be zero-dimensional, because the space Y is not strongly zero-dimensional. ∎

The concepts introduced in this section lead to two new classes of mappings.

A continuous mapping $f : X \to Y$ is *light (zero-dimensional)* if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty). Clearly, every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes coincide. As far as the structure of fibers is concerned, light mappings and zero-dimensional mappings are the exact opposite of monotone mappings. As shown in Theorem 6.2.22 below, every perfect mapping can be represented as the composition of a monotone mapping and a zero-dimensional mapping.

6.2.21. LEMMA. *For every perfect mapping $f : X \to Y$ the equivalence relation E on the space X, determined by the decomposition of all fibers $f^{-1}(y)$ into components, is closed.*

PROOF. It suffices to show that the natural quotient mapping $g : X \to X/E = Z$ is closed. Let $C \in Z$ be a component of the fiber $f^{-1}(y)$ and let U be an open subset of X containing $g^{-1}(C) = C$. By virtue of Theorem 6.1.23, there exists closed-and-open subset F of $f^{-1}(y)$ such that $C \subset F \subset U$. The sets F and $f^{-1}(y) \setminus F$ being compact and disjoint, by Theorem 3.1.6 there exist disjoint open sets $W_1, W_2 \subset X$ such that $F \subset W_1 \subset U$ and

$f^{-1}(y) \setminus F \subset W_2$. From the closedness of f it follows that there exists an open set $W \subset Y$ satisfying $C \subset f^{-1}(y) \subset f^{-1}(W) \subset W_1 \cup W_2$. The set $V = W_1 \cap f^{-1}(W)$ satisfies the equality $g^{-1}(g(V)) = V$, so that $g(V)$ is a neighbourhood of the point $C \in Z$. Since $g^{-1}(g(V)) \subset U$, the mapping g is closed.■

6.2.22. THEOREM. *Every perfect mapping $f : X \to Y$ can be represented as the composition $f = hg$, where $g : X \to Z$ is a monotone perfect mapping and $h : Z \to Y$ is a zero-dimensional perfect mapping.*

PROOF. Let E be the equivalence relation defined in the above lemma and let $g : X \to X/E = Z$ be the natural quotient mapping. For every equivalence class $[x] \in Z$ denote by $h([x])$ the point $f(x) \in Y$; as $hg = f$, the mapping $h : Z \to Y$ is continuous by virtue of Proposition 2.4.2.

From the above lemma it follows that g is a perfect mapping, so that – by virtue of Theorem 3.7.20 – Z is a Hausdorff space. Proposition 2.1.3 implies that h is a perfect mapping.

The mapping g is monotone by virtue of the definition. It remains to prove that h is zero-dimensional; the fibers of h being compact, it suffices to check that for any $y \in Y$ the fiber $h^{-1}(y)$ is hereditarily disconnected. Let C be a connected subset of $h^{-1}(y)$. By virtue of Theorem 6.1.29, the set $g^{-1}(C) \subset f^{-1}(y)$ is connected and thus is contained in a component of $f^{-1}(y)$; hence the set $gg^{-1}(C) = C$ contains at most one point.■

Observe that the space Z in the last theorem is a continuous image of the space X under a perfect mapping, as well as an inverse image of the space Y under a perfect mapping. Hence, many topological properties of X and Y are shared by Z. For instance, if either X or Y is compact, locally compact or paracompact, then so is the space Z (see Theorems 3.7.21, 3.7.24, 5.1.33 and 5.1.35), and if X is metrizable, then the space Z also is metrizable (see Theorem 4.4.15).

Following the pattern of the proof of Theorem 6.2.22, one can define, for an arbitrary continuous mapping $f : X \to Y$ to a T_1-space Y, a space Z, a monotone quotient mapping $g : X \to Z$, and a light mapping $h : Z \to Y$ such that $f = hg$. However, if f is not perfect, then – even for nice spaces X and Y – the space Z generally is not a Hausdorff space (see Exercise 6.2.E). The next theorem shows that the factorization of f is unique.

6.2.23. THEOREM. *If a continuous mapping $f : X \to Y$ is represented for $i = 1,2$ as the composition $h_i g_i$, where $g_i : X \to Z_i$ is a monotone quotient mapping and $h_i : Z_i \to Y$ is a light mapping, then there exists a homeomorphism $h : Z_1 \to Z_2$ such that the following diagram is commutative.*

$$\begin{tikzcd}
 & Z_1 \arrow[dr, "h_1"] \arrow[dd, "h"] & \\
X \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & & Y \\
 & Z_2 \arrow[ur, "h_2"'] &
\end{tikzcd}$$

PROOF. For every $z \in Z_1$ let $h(z) = g_2 g_1^{-1}(z)$; since $g_1^{-1}(z)$ is a connected subset of $f^{-1}h_1(z) = g_2^{-1}h_2^{-1}h_1(z)$, the set $h(z)$ is a one-point set, because it is a connected subset of $h_2^{-1}h_1(z)$. The equality $hg_1 = g_2$ implies that h is a continuous mapping of Z_1 to Z_2. In a similar way, letting $h'(z) = g_1 g_2^{-1}(z)$ for every $z \in Z_2$, we define a continuous mapping h' of

Z_2 to Z_1. Since $hh' = \mathrm{id}_{Z_2}$ and $h'h = \mathrm{id}_{Z_1}$, the mapping h is a homeomorphism. The reader can easily check that $h_2h = h_1$. ∎

As every mapping of a compact space to a point is perfect, Lemma 6.2.21 together with Theorems 6.1.29 and 6.2.9 yield the following theorem.

6.2.24. THEOREM. *For every compact space X, the decomposition of X into components, or – equivalently – into quasi-components, determines a closed equivalence relation E on the space X; the quotient space X/E is compact and zero-dimensional.* ∎

The final part of this section is devoted to a class of spaces which are still more disconnected.

A topological space X is called *extremally disconnected* if X is a Hausdorff space and for every open set $U \subset X$ the closure $\overline{U}$ is open in X. The space of all rational numbers is an example of a strongly zero-dimensional space which is not extremally disconnected. Clearly, every extremally disconnected space is hereditarily disconnected (cf. Problem 6.3.18).

We have also the following theorem.

6.2.25. THEOREM. *Every non-empty extremally disconnected regular space X is strongly zero-dimensional.*

PROOF. By regularity, for every point $x \in X$ and any neighbourhood V of x there exists a neighbourhood W of x such that $x \in W \subset \overline{W} \subset V$; by extremal disconnectedness of X, the set $U = \overline{W}$ is open-and-closed, which implies that X is a Tychonoff space. Consider now a pair A, B of completely separated subsets of the space X and a continuous function $f : X \to I$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$. The set $U = \overline{f^{-1}([0,1/2))}$ is open-and-closed and satisfies $A \subset U \subset X \setminus B$, so that X is strongly zero-dimensional by Theorem 6.2.4. ∎

6.2.26. THEOREM. *A Hausdorff space X is extremally disconnected if and only if for every pair U, V of disjoint open subsets of X we have $\overline{U} \cap \overline{V} = \emptyset$.*

PROOF. Let U, V be a pair of disjoint open subsets of an extremally disconnected space X. Since $U \cap V = \emptyset$, we have $\overline{U} \cap V = \emptyset$, and – the set $\overline{U}$ being open – we also have $\overline{U} \cap \overline{V} = \emptyset$.

Now, assume that for every pair U, V of disjoint open subsets of a Hausdorff space X we have $\overline{U} \cap \overline{V} = \emptyset$. Let U be an open subset of X. The open sets U and $X \setminus \overline{U}$ are disjoint, so that $\overline{U} \cap \overline{X \setminus \overline{U}} = \emptyset$. Hence, $\overline{U} \subset X \setminus \overline{X \setminus \overline{U}} = \operatorname{Int} \overline{U}$, i.e., the set $\overline{U}$ is open and the space X is extremally disconnected. ∎

6.2.27. THEOREM. *The Čech-Stone compactification βX of a Tychonoff space X is extremally disconnected if and only if the space X is extremally disconnected.*

PROOF. Let X be an extremally disconnected Tychonoff space and U an open subset of βX. The closure $X \cap \overline{U \cap X}$ of the set $U \cap X$ in the space X is a closed-and-open set. By virtue of Theorem 1.3.6 we have

$$\overline{U} = \overline{U \cap X} \subset \overline{X \cap \overline{U}} = \overline{X \cap \overline{U \cap X}} \subset \overline{\overline{U \cap X}} = \overline{U};$$

hence, by Corollary 3.6.5, the set $\overline{U} = \overline{X \cap \overline{U \cap X}}$ is closed-and-open in βX.

Now, consider a Tychonoff space X such that the Čech-Stone compactification βX is extremally disconnected, and an open set $U \subset X$. Take an open set $W \subset \beta X$ satisfying

$W \cap X = U$. By Theorem 1.3.6 we have $\overline{U} = \overline{W \cap X} = \overline{W}$, so that the closure $\overline{U} \cap X = \overline{W} \cap X$ of the set U in the space X is closed-and-open. ∎

6.2.28. COROLLARY. *For every* $\mathfrak{m} \geq \aleph_0$ *the Čech-Stone compactification* $\beta D(\mathfrak{m})$ *of the discrete space* $D(\mathfrak{m})$ *is extremally disconnected.* ∎

6.2.29. COROLLARY. *The space* βN *is extremally disconnected.* ∎

6.2.30. THEOREM. *The sum* $\bigoplus_{s \in S} X_s$ *is extremally disconnected if and only if all spaces* X_s *are extremally disconnected.* ∎

As shown by the example of the Cantor set $D^{\aleph_0}$, the Cartesian product of extremally disconnected spaces generally is not extremally disconnected (cf. Problem 6.3.21).

6.2.31. EXAMPLE. We shall show that the space $\beta N \setminus N$ is not extremally disconnected; hence, extremal disconnectedness is not hereditary with respect to closed sets (cf. Exercise 6.2.G(c)).

Let $X \subset \beta N$ be the non-normal Tychonoff space defined in Example 3.6.19. In the space X there exist two disjoint closed sets A, B which cannot be separated by disjoint open sets; clearly, one can assume that $A, B \subset X \setminus N$. Let

$$W_A = \bigcup\{U_t : x_t \in A\} \quad \text{and} \quad W_B = \bigcup\{U_t : x_t \in B\},$$

where $U_t \subset \beta N \setminus N$ and $x_t \in X$ are the open sets and the points considered in Example 3.6.19.

Assume that the space $\beta N \setminus N$ is extremally disconnected. Since W_A and W_B are disjoint open subsets of $\beta N \setminus N$, their closures in $\beta N \setminus N$ are disjoint. The space $\beta N \setminus N$ being closed in βN, we have $\overline{W}_A \cap \overline{W}_B = \emptyset$, where the bar denotes the closure in βN. By normality of βN, there exist open sets $U, V \subset \beta N$ such that $A \subset \overline{W}_A \subset U, B \subset \overline{W}_B \subset V$ and $U \cap V = \emptyset$. The sets $X \cap U$ and $X \cap V$ are disjoint open subsets of X which contain the sets A and B respectively. The contradiction shows that the space $\beta N \setminus N$ is not extremally disconnected. ∎

Let us note that from Example 2.5.4 and 6.2.31 it follows that the limit of an inverse sequence of extremally disconnected spaces need not be extremally disconnected.

Historical and bibliographic notes

Hereditarily disconnected spaces were introduced by Hausdorff in [1914]. The spaces of this class are sometimes called totally disconnected; however, at present the term *totally disconnected* is usually applied to a space X such that the quasi-component of any point $x \in X$ consists of the point x alone (this class of spaces was introduced by Sierpiński in [1921]). One easily checks that every zero-dimensional space is totally disconnected, and that every totally disconnected space is hereditarily disconnected. A related class is that of *punctiform*, or *discontinuous*, spaces, defined as spaces which do not contain any continuum of cardinality larger than one (this class of spaces was introduced by Janiszewski in [1912]). Clearly, every hereditarily disconnected space is punctiform. The space in Example 6.2.19 is totally disconnected; examples of hereditarily disconnected spaces whih are not totally disconnected, as well as examples of connected punctiform spaces can be found in Problems 6.3.23 and 6.3.24.

Zero-dimensional spaces were defined by Sierpiński in [1921], before the dimension theory was originated; on the other hand, strongly zero-dimensional spaces were first studied in the context of dimension theory, and our theorems on that class of spaces are special cases of theorems in dimension theory (see notes to Section 7.1). Theorems 6.2.5, 6.2.7, 6.2.9 and 6.2.16 were proved by Vedenissoff in [1939]. An example of a strongly zero-dimensional normal space X such that the Cartesian product $X \times X$ is normal but is not strongly zero-dimensional was given by Wage in [1977] (see also Wage [1978]). In his original construction Wage applied the continuum hypothesis; Przymusiński (cited in Wage [1977]) showed how the continuum hypothesis can be avoided by a modification of Wage's construction (for details see Przymusiński [1979]). Example 6.2.19 was described in Erdős' paper [1940] (the first example of a separable metric space with similar properties was defined by Sierpiński in [1921]; Sierpiński's space is, moreover, completely metrizable); Example 6.2.20 was described in Dowker's paper [1955].

Light mappings and zero-dimensional mappings were introduced by Stoïlow in [1928]. Theorem 6.2.22 was proved by Eilenberg in [1934] and by Whyburn in [1934] under the assumption that X and Y are compact metric spaces; for arbitrary metric spaces it was proved by Vaĭnšteĭn in [1947a], and in the present form, by Ponomarev in [1959a]. Theorem 6.2.23 was noted by Whyburn in [1942].

Extremally disconnected spaces were defined by M. H. Stone in [1937a]. The proof that $\beta N \setminus N$ is not extremally disconnected, given in Example 6.2.31, is taken from Gillman and Jerison [1960]; the fact itself was proved (using a different language) by Hausdorff in [1936] (cf. the remark to Exercise 3.6.A(b)).

Exercises

6.2.A. (a) (Mazurkiewicz [1917]) Prove that every G_δ-set which is both dense and co-dense in a separable zero-dimensional completely metrizable space is homeomorphic to the space of irrational numbers.

Hint. Applying the hint to Exercise 4.3.G, show that a set with the above properties is homeomorphic to $B(\aleph_0)$.

(b) (Alexandroff and Urysohn [1928]) Show that every separable zero-dimensional completely metrizable space which does not contain any non-empty compact open subset is homeomorphic to the space of irrational numbers.

(c) (Brouwer [1910]) Prove that every dense in itself zero-dimensional compact metrizable space is homeomorphic to the Cantor set.

Hint. Modify the construction in the hint to Exercise 4.3.G in such a way that the sets $F_{i_1 i_2 \ldots i_k}$ are defined for $i_1 \leq m_1, i_2 \leq m_2, \ldots, i_k \leq m_k$, where $m_1, m_2, \ldots$ is a sequence of powers of the number 2.

(d) (Sierpiński [1920a] (announcement [1915])) Prove that every dense in itself countable metrizable space is homeomorphic to the space of rational numbers.

Hint. By virtue of Exercise 4.3.H(d) it suffices to show that every dense in itself countable metrizable space X is homeomorphic to a dense subspace of the space P of irrational numbers. Embed X in P, remove in an appropriate way $\aleph_0$ points from $\overline{X} \setminus X$, and apply part (a).

One can also use Exercise 4.3.H(e) and apply (c).

(e) Note that every dense in itself separable metrizable space contains a dense subspace homeomorphic to the space of rational numbers. Show that every dense in itself separable completely metrizable space contains a dense subspace homeomorphic to the space of irrational numbers.

6.2.B (implicitly, Sierpiński [1928]). Prove that if for a non-empty closed subset A of a separable metrizable space X the subspace $X \setminus A$ is zero-dimensional, then A is a retract of the space X.

Hint. Consider a totally bounded metric ρ on the space X represent the subspace $X \setminus A$ as the union of a sequence $F_1, F_2, \ldots$ of pairwise disjoint open-and-closed subsets of X such that $\lim \delta(F_i) = 0$, then for $i = 1, 2, \ldots$ choose a point $x_i \in A$ such that $\rho(x_i, F_i) < \rho(A, F_i) + 1/i$ and check that the mapping $r : X \to A$ defined by letting $r(x) = x_i$ if $x \in F_i$ and $r(x) = x$ if $x \in A$ is a retraction.

6.2.C. (a) Show that a non-empty compact space is zero-dimensional if and only if it is homeomorphic to the limit of an inverse system of finite discrete spaces.

(b) (Terasawa [1972]) Show that a non-empty Tychonoff space X is strongly zero-dimensional if and only if every functionally closed subset of X can be represented as a countable intersection of open-and-closed sets.

6.2.D. (a) (Dowker [1955]) Show that to the space Y in Example 6.2.20 one point can be adjoined in such a way that either the space obtained is normal but not zero-dimensional or it is normal and strongly zero-dimensional.

Hint. Adjoin the point $(\omega_1, 0)$; identify the set $Y^* \setminus Y \subset Y^*$ to a point.

(b) Prove that if by adjoining a compact zero-dimensional space to a strongly zero-dimensional space one obtains a Tychonoff space, then the space obtained also is strongly zero-dimensional.

Hint. Apply Exercise 3.2.J(a).

(c) Prove that if by adjoining a strongly zero-dimensional space to a strongly zero-dimensional space one obtains a normal space in which the adjoined space is closed, then the space obtained also is strongly zero-dimensional.

6.2.E. (a) Give an example of a continuous mapping $f : X \to Y$, where X and Y are separable metrizable spaces, such that all fibers $f^{-1}(y)$ are compact, and yet in the factorization $f = hg$, where $g : X \to Z$ is a monotone quotient mapping and $h : Z \to Y$ is a light mapping, the space Z is not a Hausdorff space.

Hint. Take as X the space defined in Example 6.1.24.

(b) Give an example of a closed mapping $f : X \to Y$, where X and Y are separable metrizable spaces, such that in the factorization $f = hg$, where $g : X \to Z$ is a monotone quotient mapping and $h : Z \to Y$ is a light mapping, the space Z is not a Hausdorff space.

(c) (Michael [1964]) Show that if the spaces X and Y in Theorem 6.2.22 are completely regular, then the space Z also is completely regular.

6.2.F (Kuratowski [1928]). Show that any decomposition of a compact space into countably many continua determines a closed equivalence relation.

Hint. Apply Theorems 6.1.27 and 6.2.24.

6.2.G. (a) Check that in an extremally disconnected space there are no non-trivial convergent sequences and deduce that every extremally disconnected sequential space is discrete.

(b) Prove that every infinite closed subset of a compact extremally disconnected space contains a subspace homeomorphic to βN (cf. part (f) and Exercise 3.6.G(d)).

Hint. See the proof of Theorem 3.6.14.

Remark. As shown by Balcar and Franek in [1982], if X and Y are extremally disconnected compact spaces and $w(X) \leq w(Y)$, then X is embeddable in Y. In the same paper it is shown that every infinite extremally disconnected compact space X satisfies the equality $|X| = 2^{w(X)}$ and can be continuously mapped onto the Cantor cube $D^{w(X)}$.

(c) Show that extremal disconnectedness is hereditary with respect to both open subsets and dense subsets.

(d) Note that every extremally disconnected hereditarily normal space is hereditarily extremally disconnected. Give an example of such a space which is not discrete.

(e) (Gillman and Jerison [1960]) Show that a compact space X is extremally disconnected if and only if $X = \beta Y$ for every dense subspace Y of the space X.

(f) (Gillman and Jerison [1960]) Prove that a Hausdorff space X is extremally disconnected if and only if for each open set $M \subset X$ every continuous function $f : M \to I$ is continuously extendable over X. Deduce that every extremally disconnected completely regular space is an F-space (see Exercise 3.6.G(d)).

Hint. Apply Exercise 2.1.J.

6.2.H. (a) Show that zero-dimensionality and strong zero-dimensionality are not invariants either of perfect mappings or of open mappings (even in the realm of separable metrizable spaces), but are invariants of closed-and-open mappings.

Hint. See Exercises 3.2.B, 4.2.D(a) and 1.5.L(b).

(b) Show that extremal disconnectedness is not an invariant of perfect mappings, but is an invariant of open mappings.

(c) Observe that hereditary disconnectedness is not an invariant either of perfect mappings or of open mappings.

Remark. E. Pol and R. Pol gave in [1974] an example of a perfect open mapping of a hereditarily disconnected separable metrizable space onto a non-trivial connected space. Some results on invariance of hereditary disconnectedness under closed-and-open mappings satisfying additional conditions can be found in Kombarov's paper [1971].

6.3. Problems

A characterization of connectedness

6.3.1. Show that a space X is connected if and only if for every open cover $\{U_s\}_{s\in S}$ of X and every pair x_1, x_2 of points of X there exists a finite sequence $s_1, s_2, \ldots, s_k$ of elements of S such that $x_1 \in U_{s_1}, x_2 \in U_{s_k}$ and $U_{s_i} \cap U_{s_j} \neq \emptyset$ if and only if $|i - j| \leq 1$.

Linearly ordered spaces V (see Problems 1.7.4, 2.7.5, 3.12.3, 3.12.4, 3.12.12(f), 5.5.22 and 8.5.13(j))

6.3.2. (a) Show that a space X with the topology induced by a linear order $<$ is connected if and only if X is continuously ordered by $<$.

(b) Show that every separable linearly ordered continuum X is homeomorphic to the closed interval I.

Hint. Choose a countable set $S = \{s_1, s_2, \ldots\}$ which is dense in X and does not contain either the smallest or the largest element of X; arrange the rational numbers between 0 and 1 in a sequence $r_1, r_2, \ldots$ such that $r_i < r_j$ if and only if $s_i < s_j$. Check that letting $f(s_i) = r_i$ one defines a continuous mapping $f : S \to I$, and apply Theorem 3.2.1.

(c) Prove that every separable linearly ordered metrizable space X is embeddable in the real line.

Hint. To begin, note that one can assume that X is compact (see the hint to Problem 3.12.3(b)). Observe that there are only countably many jumps in X, replace every jump by I and apply (b).

(d) Give an example of a non-metrizable separable linearly ordered space.

(e) (Herrlich [1965]) Prove that every hereditarily disconnected linearly ordered space is strongly zero-dimensional.

(f) (Herrlich [1965]; for separable spaces, Lynn [1962]) Prove that for every strongly zero-dimensional metrizable space X there exists a linear order $<$ on the space X which induces the original topology of X.

Hint. To begin, show that for every $\epsilon > 0$ there exists a cover $\mathcal{U}$ of X consisting of pairwise disjoint open-and-closed sets of diameter less than ϵ (one can either use Theorem 7.3.15 or apply Theorem 4.4.1 to the cover of X by all $\epsilon/4$-balls and make use of property (4) of families $\mathcal{V}_i$ defined in the proof of that theorem); such a cover $\mathcal{U}$, linearly ordered by a relation $<$ with respect to which $\mathcal{U}$ has a smallest and a largest element and which induces on $\mathcal{U}$ the discrete topology, will be called an ϵ-chain in X.

Define by induction for $i = 1, 2, \ldots$ a $1/2^i$-chain $\mathcal{U}_i$ in X and two mappings f_i, g_i of $\mathcal{U}_i$ to X such that

(1) For every $U \in \mathcal{U}_i$ we have $f_i(U) \in U$ and $g_i(U) \in U$.

(2) If $f_i(U) = g_i(U) = x$, then $U = \{x\}$.

(3) The chain $\mathcal{U}_{i+1}$ is the union $\bigcup\{\mathcal{U}_{i+1}(U) : U \in \mathcal{U}_i\}$, ordered in the natural way, where $\mathcal{U}_{i+1}(U)$ is a $1/2^{i+1}$-chain in U whose smallest and largest element contain respectively $f_i(U)$ and $g_i(U)$.

(4) If $f_i(U) \in V \in \mathcal{U}_{i+1}$, then $f_{i+1}(V) = f_i(U)$.

(5) If $g_i(U) \in V \in \mathcal{U}_{i+1}$, then $g_{i+1}(V) = g_i(U)$.

For $x, y \in X$ let $x < y$ if there exists an i such that $x \in U \in \mathcal{U}_i, y \in U' \in \mathcal{U}_i$ and $U < U'$.

Remark. A topological characterization of those metrizable spaces whose topology can be induced by a linear order was given by Purisch in [1977].

Local connectedness

6.3.3. We say that a topological space X is *locally connected* if for every $x \in X$ and any neighbourhood U of the point x there exists a connected set $C \subset U$ such that $x \in \operatorname{Int} C$.

(a) Show that a space X is locally connected if and only if the components of all open subspaces of X are open.

Remark. Let us note that an open cover of a locally connected paracompact (in fact, separable metrizable) space does not necessarily have a locally finite open refinement consisting of connected sets (see Filippov [1974]); on the other hand, every open cover of a locally connected paracompact space has a σ-discrete open refinement consisting of connected sets (see Przymusiński [1973]).

(b) Note that in a locally connected space components coincide with quasi-components.

(c) Observe that local connectedness is not an invariant of continuous mappings, even in the realm of separable metrizable spaces.

(d) (Whyburn [1952]) Show that local connectedness is an invariant of quotient mappings.

6.3.4. (a) Note that local connectedness is hereditary with respect to open sets and give an example of a closed set on the plane which is not locally connected.

(b) Check that the sum $\bigoplus_{s\in S} X_s$ is locally connected if and only if all spaces X_s are locally connected.

(c) Show that the Cartesian product $\prod_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is locally connected if and only if all spaces X_s are locally connected and there exists a finite set $S_0 \subset S$ such that X_s is connected for $s \in S \setminus S_0$.

6.3.5. (a) (Wallace [1951] for normal spaces) Prove that for every Tychonoff (normal) space X a functionally open (open) set $U \subset \beta X$ is connected if and only if the intersection $X \cap U$ is connected.

Hint. Apply Lemma 7.1.13 and Exercise 2.1.B(c).

(b) (Banaschewski [1956]) Show that if the Čech-Stone compactification βX of a Tychonoff space X is locally connected, then X is pseudocompact.

Hint (de Groot and McDowell [1967]). Observe that the compactification of the real line described in Example 3.5.15 is not locally connected and apply Problem 6.3.3(d).

(c) (Henriksen and Isbell [1957]) Show that if a pseudocompact space X is locally connected, then every regular space containing X as a dense subspace also is locally connected.

Hint (de Groot and McDowell [1967]). Using the equivalence of conditions (i) and (ii) in Theorem 3.10.22, show that if open subsets $U, V \subset X$ satisfy the inclusion $\overline{V} \subset U$, then only finitely many components of U meet the set V.

(d) (Banaschewski [1956], Henriksen and Isbell [1957]) Prove that the Čech-Stone compactification βX of a Tychonoff space X is locally connected if and only if the space X is locally connected and pseudocompact.

6.3.6. (a) (Henriksen and Isbell [1957]) Show that for every regular space X the following conditions are equivalent:

(1) *Every finite open cover of the space X has a finite refinement consisting of connected sets.*

(2) *Every finite open cover of the space X has a finite open refinement consisting of connected sets.*

(3) *The space X is locally connected and countably compact.*

(b) Observe that if X is a normal space, then conditions (1)–(3) in (a) are equivalent to the following condition:

(4) *Every finite open cover of βX has a finite refinement consisting of continua.*

(c) Show that a compact space X is locally connected if and only if every open cover of the space X has a finite refinement consisting of continua.

6.3.7. Show that a T_1-space X is both connected and locally connected if and only if for every open cover $\{U_s\}_{s\in S}$ of X and every pair x_1, x_2 of points of X there exists a finite sequence $s_1, s_2, \ldots, s_k$ of elements of S and a sequence $V_1, V_2, \ldots, V_k$ of connected open subsets of X such that $x_1 \in V_1, x_2 \in V_k, V_i \subset U_{s_i}$ for $i = 1, 2, \ldots, k$ and $V_i \cap V_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

A topological characterization of the closed interval

6.3.8. We say that a point x of a connected space X is a *cut point* of X if $X \setminus \{x\} = U \cup V$, where the sets U and V are open, disjoint and non-empty, i.e., if the one point set $\{x\}$ is closed and the space $X \setminus \{x\}$ is not connected; if x is not a cut point of X, we say that it is a *noncut* point.

(a) Note that if a point x is a cut point of a continuum X and $X \setminus \{x\} = U \cup V$, where the sets U and V are open, disjoint and non-empty, then the sets $U \cup \{x\}$ and $V \cup \{x\}$ are continua (cf. Exercise 6.1.C).

(b) (Wallace [1942]; for metrizable continua, R. L. Moore [1920]) Show that every continuum X of cardinality > 1 contains at last two noncut points.

Hint. Prove that for every $x \in X$ there exists a noncut point $y \in X \setminus \{x\}$. To that end, order the family $\mathcal{C}$ of all proper subcontinua of X that contain the point x letting $C_1 \leq C_2$ if $C_1 \subset \operatorname{Int} C_2$; then take a maximal linearly ordered subfamily $\mathcal{C}_0 \subset \mathcal{C}$ containing $\{x\}$ and consider the union $\bigcup \mathcal{C}_0$.

(c) (Sierpiński [1916] and [1917], Straszewicz [1918], R. L. Moore [1920]; for plane continua, Lennes [1911]) Prove that a separable continuum X that contains exactly two noncut points is homeomorphic to the closed interval I.

Hint. Let a, b be the noncut points of X; applying (b), show that for every $x \in X \setminus \{a, b\}$ we have $X \setminus \{x\} = A(x) \cup B(x)$, where $A(x)$ and $B(x)$ are open sets, $A(x) \cap B(x) = \emptyset$, $a \in A(x)$, and $b \in B(x)$. For distinct $x, y \in X \setminus \{a, b\}$ let $x < y$ if $A(x) \subset A(y)$, and for every $x \in X \setminus \{a, b\}$ let $a < x < b$. Check that $<$ is a linear order on X which induces the original topology of X. Apply Problem 6.3.2(b).

Remark. The argument in the hint shows that in every continuum X that contains exactly two noncut points there exists a linear order which induces the original topology of X.

Pathwise connectedness and local pathwise connectedness

6.3.9. A space X is *pathwise connected* if for every pair x_1, x_2 of points of X there exists a continuos mapping $f : I \to X$ of the closed unit interval I to the space X satisfying $f(0) = x_1$ and $f(1) = x_2$.

(a) Verify that every pathwise connected space is connected and give an example of a continuum in the plane which is not pathwise connected.

(b) Observe that pathwise connectedness is an invariant of continuous mappings.

(c) Show that the Cartesian product $\prod_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is pathwise connected if and only if all spaces X_s are pathwise connected.

6.3.10. A space X is *locally pathwise connected* if for every $x \in X$ and any neighbourhood U of the point x there exists a neighbourhood V of x such that for any $y \in V$ there exists a continuous mapping $f : I \to U$ satisfying $f(0) = x$ and $f(1) = y$.

(a) Verify that every locally pathwise connected space is locally connected and give an example of a locally connected subspace of the plane which is not locally pathwise connected (cf. Problem 6.3.11). Show that every connected and locally pathwise connected space is pathwise connected.

(b) Observe that local pathwise connectedness is an invariant of quotient mappings but is not an invariant of continuous mappings, even in the realm of separable metrizable spaces.

(c) Note that local pathwise connectednss is hereditary with respect to open sets.

(d) Show that the Cartesian product $\prod_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is locally pathwise connected if and only if all spaces X_s are locally pathwise connected and there exists a finite set $S_0 \subset S$ such that X_s is pathwise connected for $s \in S \setminus S_0$.

6.3.11 (Menger [1929], R. L. Moore [1932] (announcement [1927]); for compact spaces, Mazurkiewicz [1913], [1913a] and [1920], R. L. Moore [1916]). Prove that for any pair x_1, x_2 of distinct points of a connected open subspace V of a locally connected and completely metrizable space X there exists in V a subspace homeomorphic to the closed unit interval I that contains x_1 and x_2.

Deduce that every locally connected metrizable continuum is pathwise connected and locally pathwise connected.

Hint. For $m = 1, 2, \ldots$ define a finite sequence $V_1^m, V_2^m, \ldots, V_{k_m}^m$ of connected open subsets of V such that

(1) $x_1 \in V_1^m \setminus (V_2^m \cup \ldots \cup V_{k_m}^m)$, $x_2 \in V_{k_m}^m \setminus (V_1^m \cup \ldots \cup V_{k_m-1}^m)$ and $\delta(V_i^m) \leq 1/m$ for $i = 1, 2, \ldots, k_m$.

(2) $V_i^m \cap V_j^m \neq \emptyset$ if and only if $|i - j| \leq 1$.

(3) For every $i \leq k_m$, where $m > 1$, there exists a $j(i) \leq k_{m-1}$ such that $\overline{V_i^m} \subset V_{j(i)}^{m-1}$ and that $i_1 \leq i_2$ implies $j(i_1) \leq j(i_2)$.

Applying Problem 6.3.8(c) show that the intersection $\bigcap_{m=1}^{\infty} \bigcup_{i=1}^{k_m} \overline{V_i^m}$ is homeomorphic to the interval I.

One can also define directly a homeomorphic embedding $h : I \to V$ such that $x_1, x_2 \in h(I)$. To that end, for $m = 1, 2, \ldots$ divide the interval I into k_m consecutive closed intervals $I_1^m, I_2^m, \ldots, I_{k_m}^m$ with pairwise disjoint interiors in such a way that $I_i^m \subset I_{j(i)}^{m-1}$ for $i = 1, 2, \ldots, k_m$, where $m > 1$, and $\delta(I_i^m) = \delta(I_{i'}^m)$ whenever $j(i) = j(i')$. Then define h by letting $h(t)$ to be the unique point in the intersection $\bigcap_{m=1}^{\infty} \bigcup \{\overline{V_i^m} : t \in I_i^m\}$.

6.3.12. (a) Show that a Hausdorff space X is pathwise connected if and only if for every pair x_1, x_2 of distinct points of X there exists a homeomorphic embedding $h : I \to X$ of the closed unit interval I in the space X satisfying $h(0) = x_1$ and $h(1) = x_2$ (i.e., if X is *arcwise connected*).

Hint. Apply Problem 6.3.11.

(b) Show that a Hausdorff space X is locally pathwise connected if and only if for every $x \in X$ and any neighbourhood U of the point x there exists a neighbourhood V of x such that for any $y \in V \setminus \{x\}$ there exists a homeomorphic embedding $h : I \to U$ satisfying $h(0) = x$ and $h(1) = y$ (i.e., if X is *locally arcwise connected*).

6.3.13. Let (X,ρ) be a locally connected compact metric space. Prove that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every pair x, y of points of X satisfying $0 < \rho(x,y) < \delta$ there exists a homeomorphic embedding $h : I \to X$ such that $h(0) = x$, $h(1) = y$ and $\delta(h(I)) < \epsilon$.

Hint. Apply Problem 6.3.11.

6.3.14 (Hahn [1914], Mazurkiewicz [1913], [1913a] and [1920a], Sierpiński [1920b]). Prove that for every metrizable continuum X the following conditions are equivalent:

(1) *The space X is a continuous image of the closed interval I.*

(2) *For every metric ρ on the space X and any $\epsilon > 0$ there exist connected closed sets $F_1, F_2, \ldots, F_k$ such that $X = F_1 \cup F_2 \cup \ldots \cup F_k$ and $\delta(F_i) \leq \epsilon$ for $i = 1, 2, \ldots, k$.*

(3) *The space X is locally connected.*

Hint. In the proof of the implication (3)⇒(1) apply Theorem 3.2.2 and Problems 6.3.11 and 6.3.13.

Quasi-components of Cartesian products

6.3.15 (Kuratowski [1961]). Prove that the quasi-component of a point $x = \{x_s\}$ in the Cartesian product $\prod_{s\in S} X_s$ coincides with the Cartesian product $\prod_{s\in S} Q_s$, where Q_s is the quasi-component of the point x_s in the space X_s.

Inverse systems III (see Problems 2.7.19 and 3.12.13)

6.3.16. (a) (Gentry [1969], E. Pol [1973]) Let $\{\phi, f_{\sigma'}\}$ be a mapping of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of compact spaces to an inverse system $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\rho'}, \Sigma'\}$ of T_1-spaces. Prove that if all mappings $f_{\sigma'}$ are monotone, then the limit mapping $f = \varprojlim \{\phi, f_{\sigma'}\}$ also is monotone.

Deduce that if in an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of compact spaces all bonding mappings π^σ_ρ are monotone, then the projections $\pi_\sigma : \varprojlim \mathbf{S} \to X_\sigma$ also are monotone.

(b) (E. Pol [1973]) Give an example of inverse sequences $\mathbf{S} = \{X_i, \pi^i_j\}$ and $\mathbf{S}' = \{Y_i, \tilde{\pi}^i_j\}$ of subspaces of the plane and of a mapping $\{\mathrm{id}_N, f_i\}$ of $\mathbf{S}$ to $\mathbf{S}'$, where $\pi^i_j, \tilde{\pi}^i_j$ and f_i are monotone closed mappings onto, such that $\varprojlim \{\mathrm{id}_N, f_i\}$ is a closed mapping onto, and yet is not monotone.

Give an example of an inverse sequence $\mathbf{S} = \{X_i, \pi^i_j\}$ of connected subspaces of the plane, where all bonding mappings π^i_j are monotone and onto, such that the limit $\varprojlim \mathbf{S}$ is not connected. Deduce that there exists an inverse sequence $\mathbf{S} = \{X_i, \pi^i_j\}$ of connected spaces, where all bonding mappings π^i_j are monotone and onto, such that some of the projections $\pi_i : \varprojlim \mathbf{S} \to X_i$ are not monotone.

(c) (E. Pol [1973]) Prove that if all bonding mappings π^i_j of an inverse sequence $\mathbf{S} = \{X_i, \pi^i_j\}$ of connected spaces are hereditarily quotient and monotone, then $\varprojlim \mathbf{S}$ is a connected space.

Deduce that if all bonding mappings π^i_j of an inverse sequence $\mathbf{S} = \{X_i, \pi^i_j\}$ are hereditarily quotient and monotone, then the projections $\pi_i : \varprojlim \mathbf{S} \to X_i$ also are monotone.

Urysohn spaces and semiregular spaces III (see Problems 1.7.7–1.7.9 and 2.7.6)

6.3.17. Show that applying the operation described in Problem 1.7.8(b) to the space X in Example 6.1.6 one obtains a countable semiregular space no pair of distinct points of which have neighbourhoods with disjoint closures (cf. Problem 1.7.8(d)).

Extremal disconnectedness and axioms of separation

6.3.18. Give an example of a connected T_1-space X such that for every open set $U \subset X$ the closure $\overline{U}$ is open in X. Check that every extremally disconnected space is an Urysohn space and that every extremally disconnected semiregular space is a Tychonoff space. Give an example of an extremally disconnected space which is not semiregular and an example of an extremally disconnected Tychonoff space which is not normal.

Projective spaces and projective resolutions (absolutes)

6.3.19 (Gleason [1958], Rainwater [1959]). We say that a compact space P is *projective in the class of compact spaces* if for every pair X, Y of compact spaces, every continuous mapping g of X onto Y and any continuous mapping $h : P \to Y$ there exists a continuous mapping $f : P \to X$ such that $gf = h$.

(a) Show that a space is projective in the class of compact spaces if and only if it is a retract of the Čech-Stone compactification of a discrete space.

Hint. Every compact space is a continuous image of the Čech-Stone compactification of a discrete space.

(b) Prove that every compact space X is a continuous image of a space $A(X)$, projective in the class of compact spaces, under an irreducible mapping π_X.

Check that if a space P and a mapping f have the properties of $A(X)$ and π_X, then there exists a homeomorphism $h : A(X) \to P$ such that $fh = \pi_X$. The pair $(A(X), \pi_X)$ is called the *projective resolution* or the *absolute* of the space X.

Hint. Verify that if a continuous mapping j of a Hausdorff space Y_0 into itself is not the identity mapping, then there exists a proper closed subset $Y_1 \subset Y_0$ such that $Y_1 \cup j^{-1}(Y_1) = Y_0$. Then consider a continuous mapping of a projective space Y onto X and apply Exercise 3.1.C(a).

(c) Show that every irreducible closed mapping of a Hausdorff space onto an extremally disconnected space is a homeomorphism.

(d) Prove that a compact space is projective in the class of compact spaces if and only if it is extremally disconnected.

Hint. Check that extremal disconnectedness is an invariant of retractions.

6.3.20 (Flachsmeyer [1963], Iliadis [1963], Ponomarev [1963]; for paracompact spaces, Ponomarev [1962]). We say that a regular space P is *projective in the class of regular spaces* if for every pair X, Y of regular spaces, every perfect mapping g of X onto Y and any perfect mapping $h : P \to Y$ there exists a continuous mapping $f : P \to X$ such that $gf = h$.

(a) Show that every extremally disconnected regular space X is projective in the class of regular spaces.

Hint. Consider the set $S = \{(x,p) \in X \times P : g(x) = h(p)\} \subset X \times P$ and note, applying Theorem 3.7.9 and Proposition 3.7.5, that the restrictions to S of the projections of $X \times P$

onto X and P are perfect. Then apply Exercise 3.1.C(a) and Problem 6.3.19(c).

(b) Let $\mathcal{O}(X)$ be the family of all open subsets of a regular space X. Consider the family $A(X)$ of all ultrafilters in $\mathcal{O}(X)$ which converge to points of X, and generate on $A(X)$ a topology taking as a base the family of all sets $U_* = \{\mathcal{F} \in A(X) : U \in \mathcal{F}\}$, where $U \in \mathcal{O}(X)$.

Note that $A(X) \setminus U_* = (X \setminus \overline{U})_*$ and show that $A(X)$ is an extremally disconnected regular space, and that the mapping $\pi_X : A(X) \to X$, which assigns to an ultrafilter $\mathcal{F} \in A(X)$ the limit of $\mathcal{F}$, is irreducible and perfect.

Check that if a space P and a mapping f have the properties of $A(X)$ and π_X, then there exists a homeomorphism $h : A(X) \to P$ such that $fh = \pi_X$. The pair $(A(X), \pi_X)$ is called the *projective resolution* or the *absolute* of the space X.

(c) Observe that for every Tychonoff space X the pair $(\beta A(X), \Pi)$, where $\Pi : \beta A(X) \to \beta X$ denotes the continuous extension of the mapping $\pi_X : A(X) \to X$, is the projective resolution of βX.

(d) Show that every space P projective in the class of regular spaces is extremally disconnected.

Hint. Consider $A(P)$.

Remark. Further information on absolutes can be found in Woods' survey [1978].

Extremal disconnectedness and Cartesian products.

6.3.21. (a) (Henriksen cited in Gillman and Jerison [1960]) Show that the Cartesian product $\beta N \times \beta N$ is not extremally disconnected and deduce that the Cartesian product $\beta N \times \beta N$ is not homeomorphic to the Čech-Stone compactification of $N \times N$ (cf. Exercise 3.6.D(b) and Problem 3.12.21(c)).

Hint. Consider the open set $\{(i,i) : i = 1, 2, \ldots\} \subset \beta N \times \beta N$.

(b) (Henriksen and P. C. Curtis cited in Gillman and Jerison [1960]) Prove that the Cartesian product $X \times Y$ of infinite pseudocompact spaces is not extremally disconnected.

Hint. Define two sequences $U_1, U_2, \ldots$ and $V_1, V_2, \ldots$ of non-empty pairwise disjoint open subsets of X and Y respectively, and consider the sets $U = \bigcup\{U_i \times V_j : i < j\}$ and $V = \bigcup\{U_i \times V_j : i > j\}$.

(c) (Gubbi [1985]) Show that if the Cartesian product $X \times X$ is extremally disconnected, then the space X is discrete.

Hint. Assume that there is a non-isolated point x in X and consider a maximal family of disjoint open subsets of $X \setminus \{x\}$ whose closures do not contain x.

Remark. The existence of non-discrete spaces X, Y such that the Cartesian product $X \times Y$ is extremally disconnected is equivalent to the assumption that not all cardinal numbers are non-measurable (see Szymański [1985]).

Spaces of closed subsets IV (see Problems 2.7.20, 3.12.27, 4.5.23, 8.5.13(i) and 8.5.16)

6.3.22. (a) (Vietoris [1923], Michael [1951]) Prove that if X is a T_1-space, then the exponential space 2^X, and the space $\mathcal{Z}(X)$ of all non-empty compact closed subsets of X, are connected if and only if X is connected. Give an example of a connected and separable bounded metric space (X, ρ) such that the space $(2^X, \rho_H)$ is not connected.

Hint. Apply Problem 2.7.20(b).

(b) (Michael [1951]) Show that if X is a normal space, then the subspace of 2^X consisting of all non-empty closed connected subsets of X is closed. Note that the assumption of normality is essential.

Hint. Use the long segment.

(c) (Michael [1951]) Show that if X is a T_1-space, then the space $\mathcal{Z}(X)$ is locally connected if and only if X is locally connected. Check that 2^R is not locally connected.

(d) (Michael [1951]) Check that if X is a T_1-space, then the space $\mathcal{Z}(X)$ is zero-dimensional if and only if X is zero-dimensional.

(e) Show that if X is a normal space, then the space 2^X is zero-dimensional if and only if X is strongly zero-dimensional.

(f) (Segal [1959] for metrizable continua) Let $\mathcal{T}(X)$ be the subspace of 2^X consisting of all non-empty continua contained in X. Show that if $\mathrm{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ is an inverse system of T_2-spaces, then $\tilde{\tilde{\mathrm{S}}} = \{\mathcal{T}(X_\sigma), \tilde{\tilde{\pi}}^\sigma_\rho, \Sigma\}$, where $\tilde{\tilde{\pi}}^\sigma_\rho : \mathcal{T}(X_\sigma) \to \mathcal{T}(X_\rho)$ is defined by letting $\tilde{\tilde{\pi}}^\sigma_\rho(C) = \pi^\sigma_\rho(C)$, is an inverse system and that the limit $\varprojlim \tilde{\tilde{\mathrm{S}}}$ is homeomorphic to $\mathcal{T}(\varprojlim \mathrm{S})$.

Hint. See Problem 3.12.27(f).

The Knaster-Kuratowski fan

6.3.23 (Knaster and Kuratowski [1921]). Let C be the Cantor set on the interval $[0,1]\times\{0\} \subset R^2$; denote by Q the set of all end-points of intervals removed from $[0,1]\times\{0\}$ in the process of constructing the Cantor set as described in Example 3.1.28, and let $P = C \setminus Q$. Join every point $c \in C$ to the point $q = (1/2, 1/2) \in R^2$ by a segment L_c and denote by F_c the set of all points $(x,y) \in L_c$, where y is rational if $c \in Q$ and y is irrational if $c \in P$. The subspace $F = \bigcup_{c\in C} F_c$ of the plane is called the *Knaster-Kuratowski fan.*

(a) Prove that the space F is connected and punctiform.

Hint. Let $r_1, r_2, \ldots$ be the sequence of all rational numbers in the interval $[0, 1/2]$ and let $P_i \subset R^2$ be the horizontal line $y = r_i$. Suppose that $F = (F\cap A) \cup (F \cap B)$, where the sets A and B are closed in R^2, $F \cap A \cap B = \emptyset$ and $q \in A$; consider the sets $K_i = \{c \in C : A \cap B \cap L_c \cap P_i \neq \emptyset\}$. Check that $\overline{K_i} = K_i$ for $i = 1,2,\ldots$ and $\bigcup_{i=1}^\infty K_i \subset P$. Show that $F_c \cap B = \emptyset$ for every point $c \in P \setminus \bigcup_{i=1}^\infty K_i$ and – applying the Baire category theorem – deduce that $F \cap B = \emptyset$.

(b) Prove that the space $F \setminus \{q\}$ is hereditarily disconnected, but is not totally disconnected.

Around Erdös' example

6.3.24. (a) Let X be the space discussed in Example 6.2.19. Observe that by assigning to a point $x = \{x_i\} \in X$ the same point $x = \{x_i\} \in Q^{\aleph_0}$ one defines a continuous mapping and deduce that there exists a one-to-one continuous mapping $f : X \to C$ of X to the Cantor set C such that $f(x_0) = 0$. Define a mapping $g : X \to I^2$ by letting

$$g(x) = \left(\frac{f(x)}{\max(1, \|x\|)}, \frac{\|x\|}{1+\|x\|}\right), \quad \text{where} \quad \|x\| = \sqrt{\sum_{i=1}^\infty x_i^2} \quad \text{for} \quad x = \{x_i\};$$

show that the space $X_1 = g(X) \cup \{(0,1/2)\} \subset I^2$ is hereditarily disconnected but is not totally disconnected and that the space $X_2 = g(X) \cup \{(0,1)\} \subset I^2$ is connected and punctiform.

Hint. Note that there exists a one-to-one continuous mapping of $g(C)$ onto $f(X) \subset C$ and that if for a closed-and-open set $A \subset X$, the set $\{\|x\| : x \in A\}$ is bounded, then $A = \emptyset$.

(b) (Roberts [1956]) Prove that the mapping g in (a) is a homeomorphic embedding.

Hint. Prove that a sequence $x^1, x^2, \ldots$ in the space X, where $x^k = \{x_i^k\}$, converges to a point $x = \{x_i\}$ if and only if the sequence $\|x^k\|$ converges to $\|x\|$ and the sequence $x_i^1, x_i^2, \ldots$ converges to x_i for $i = 1, 2 \ldots$

(c) Give examples of completely metrizable spaces $Y, Y_1, Y_2 \subset I^2$ such that Y is totally disconnected, but not zero-dimensional, Y_1 is hereditarily disconnected, but is not totally disconnected, and Y_2 is connected and punctiform.

Hint. Consider the subspace of Hilbert space H consisting of all sequences of irrational numbers; follow the pattern of (a) and (b).

Remark. The first example of a separable metric space which is connected and punctiform was given by Sierpiński in [1920]; such a space which, moreover, is completely metrizable was described by Mazurkiewicz in [1920]. The first example of a separable metric space which is hereditarily disconnected, but is not totally disconnected was given by Sierpiński in [1921]; that space is, moreover, completely metrizable.

An inverse sequence of strongly zero-dimensional spaces whose limit is not strongly zero-dimensional

6.3.25. Show that the limit of an inverse sequence of strongly zero-dimensional spaces need not be strongly zero-dimensional or empty.

Hint. Applying Problem 3.12.23(c), define in $D^{\aleph_1}$ a sequence of subspaces $X_1 \supset X_2 \supset \ldots$ such that $\beta X_i = D^{\aleph_1}$ for $i = 1, 2, \ldots$ and $\bigcap_{i=1}^{\infty} X_i$ is homeomorphic to the space Y in Example 6.2.20.

Chapter 7
Dimension of topological spaces

In this chapter, generalizing the notion of the dimension number of Euclidean spaces, we shall assign to certain topological spaces a non-negative integer – the dimension of the space; moreover, to the empty space the number -1 will be assigned, and to "infinite dimensional" spaces we shall assign the symbol ∞. Such an assignment will be established in three different ways; to a topological space X we shall assign three numbers $\operatorname{ind} X$, $\operatorname{Ind} X$ and $\dim X$, all of which have characteristics of dimension. The natural domains for a study of ind, Ind and dim are regular spaces, normal spaces and Tychonoff spaces respectively; when considered in larger classes of spaces, the dimension functions develop pathological properties.

Dimension theory is not a part of a classical course in general topology. The main reason is that the theory, first developed for compact metrizable spaces and then extended to separable metrizable spaces, only partially generalizes to topological spaces. Moreover, the fundamental theorem of dimension theory for separable metrizable spaces, asserting that ind, Ind and dim coincide, does not hold either in the class of all metrizable spaces or in the class of all compact spaces. Thus, for general spaces we have three dimension theories instead of one, all of which are a little poorer and less harmonious than the dimension theory of separable metrizable spaces. However, all these theories are now sufficiently developed to be included in a book on general topology. The reader will see that they contain many interesting theorems and shed light on classical dimension theory.

The chapter is strongly related to Section 6.2, where – as the reader will now see – we developed a part of dimension theory: the theory of dimension zero. Relations to other chapters are weaker. Although we use many earlier results, the problems considered and the methods developed here have their own particular character. The next chapter is quite independent of the present one. The present chapter is chiefly meant for readers familiar with the classical dimension theory of separable metrizable spaces; it can be skipped by readers interested only in fundamental notions and results of general topology.

Section 7.1 opens with the definitions of ind, Ind and dim and with some immediate consequences of those definitions. Then we prove two auxiliary theorems on swelling and shrinking of covers which are crucial in the theory of the covering dimension dim; in particular, we deduce from those theorems that in the class of normal spaces the definition of the covering dimension can be stated in a simpler way. We conclude the section by establishing that $\operatorname{Ind} \beta X = \operatorname{Ind} X$ and $\dim \beta X = \dim X$, but generally $\operatorname{ind} \beta X \neq \operatorname{ind} X$.

In Section 7.2 we study the covering dimension dim. We begin with two "sum theorems" which state that if a normal space can be represented as the union of closed subspaces of dimension $\leq n$ which form a family that is either countable or locally finite, then the space also has dimension $\leq n$. A lemma needed for the locally finite sum theorem yields an important characterization of covering dimension in normal spaces, to be used in the next section. We also prove in Section 7.2 that $\dim X \le \operatorname{ind} X$ for every Lindelöf space X and $\dim X \le \operatorname{Ind} X$ for every normal space X. The final part of the section is devoted to the characterization of dim in terms of partitions.

The last section is devoted to the dimension theory of metrizable spaces. We start by characterizing the covering dimension of metric spaces in terms of special sequences of covers and then prove the fundamental Katětov-Morita theorem asserting that $\operatorname{Ind} X = \dim X$ for every metrizable space X. The latter theorem, along with results in Section 7.2, implies that $\operatorname{ind} X = \operatorname{Ind} X = \dim X$ for every separable metrizable space X. Next, we establish two characterizations of dimension Ind in metrizable spaces, one in terms of special bases and another one in terms of decompositions into subspaces of dimension zero. Those characterizations yield a sum theorem and a theorem on the dimension of Cartesian products. In the proofs of all theorems mentioned above, the Katětov-Morita theorem is applied; a way to avoid this rather difficult result is outlined in Exercise 7.3.C(a). The section concludes with the theorem on the dimension of Euclidean spaces stating that $\operatorname{ind} R^n = \operatorname{Ind} R^n = \dim R^n = n$. That theorem, the proof of which requires a deeper insight into the structure of Euclidean spaces, is deduced here from he Brouwer fixed-point theorem.

In order to make the book self-contained, the proof of the Brouwer fixed-point theorem is given in the Appendix to Section 7.3.

7.1. Definition and basic properties of dimensions ind, Ind, and dim

Let X be a regular space and let n denote a non-negative integer; we say that:

(MU1) $\operatorname{ind} X = -1$ *if and only if* $X = \emptyset$.

(MU2) $\operatorname{ind} X \le n$ *if for every point* $x \in X$ *and any neighbourhood* V *of the point* x *there exists an open set* $U \subset X$ *such that* $x \in U \subset V$ *and* $\operatorname{ind} \operatorname{Fr} U \le n-1$.

(MU3) $\operatorname{ind} X = n$ *if* $\operatorname{ind} X \le n$ *and the equality* $\operatorname{ind} X \le n-1$ *does not hold.*

(MU4) $\operatorname{ind} X = \infty$ *if the inequality* $\operatorname{ind} X \le n$ *does not hold for any* n.

Conditions (MU1)–(MU4) assign to every regular space X a number $\operatorname{ind} X$ which is an integer ≥ -1 or the "infinite number" ∞. The number $\operatorname{ind} X$ is called the *Menger-Urysohn dimension*, or the *small inductive dimension*, of the space X. One can easily verify that if two spaces X and Y are homeomorphic, then $\operatorname{ind} X = \operatorname{ind} Y$.

By virtue of regularity of X one can assume that the set U in (MU2) satisfies the stronger condition $\overline{U} \subset V$. Thus, the inequality $\operatorname{ind} X \le n \ge 0$ means that every point $x \in X$ can be separated from any closed set F which does not contain x by a set of dimension $\le n-1$.

From the definition of the Menger-Urysohn dimension it follows directly that $\operatorname{ind} X \le n \ge 0$ if and only if there exists a base $\mathcal{B}$ for the space X such that $\operatorname{ind} \operatorname{Fr} U \le n-1$ for every $U \in \mathcal{B}$. In particular, $\operatorname{ind} X = 0$ if and only if the space X is zero-dimensional in the sense of Section 6.2.

In order to simplify statements of some results proved in the sequel, we shall assume that the formulas $n \le \infty$ and $n + \infty = \infty + n = \infty$ hold for every integer n.

Since regularity is a hereditary property, if the dimension ind is defined for a space X, then it also is defined for every subspace $M \subset X$.

7.1.1. THEOREM. *For every subspace M of a regular space X we have* $\operatorname{ind} M \leq \operatorname{ind} X$.

PROOF. Clearly, the inequality holds if $\operatorname{ind} X = -1$ and if $\operatorname{ind} X = \infty$. Assume that the inequality is proved for all spaces X with $\operatorname{ind} X \leq n-1$. Consider a regular space X with $\operatorname{ind} X = n$, a subspace $M \subset X$, a point $x \in M$ and a neighbourhood V of the point x in M. Let V_1 be an open subset of X satisfying $V = M \cap V_1$. As $\operatorname{ind} X \leq n$, there exists an open set $U_1 \subset X$ such that

$$x \in U_1 \subset V_1 \quad \text{and} \quad \operatorname{ind} \operatorname{Fr} U_1 \leq n-1. \tag{1}$$

The set $U = M \cap U_1$ is a neighbourhood of x in M and the boundary $\operatorname{Fr} U = M \cap \overline{M \cap U_1} \cap \overline{M \setminus U_1}$ of U in M is a subspace of the boundary of U_1 in X. Hence, by (1) and the inductive assumption, we have $\operatorname{ind} M \leq n$, and the proof is concluded. ■

Let X be a normal space and let n denote a non-negative integer; we say that:

(BČ1) $\operatorname{Ind} X = -1$ *if and only if* $X = \emptyset$.
(BČ2) $\operatorname{Ind} X \leq n$ *if for every closed set* $A \subset X$ *and any open set* $V \subset X$ *that contains* A *there exists an open set* $U \subset X$ *such that* $A \subset U \subset V$ *and* $\operatorname{Ind} \operatorname{Fr} U \leq n-1$.
(BČ3) $\operatorname{Ind} X = n$ *if* $\operatorname{Ind} X \leq n$ *and the inequality* $\operatorname{Ind} X \leq n-1$ *does not hold.*
(BČ4) $\operatorname{Ind} X = \infty$ *if the inequality* $\operatorname{Ind} X \leq n$ *does not hold for any* n.

Conditions (BČ1)–(BČ4) assign to every normal space X a number $\operatorname{Ind} X$ which is an integer ≥ -1 or the "infinite number" ∞. The number $\operatorname{Ind} X$ is called the *Brouwer-Čech dimension*, or the *large inductive dimension*, of the space X. One can easily verify that if two spaces X and Y are homeomorphic, then $\operatorname{Ind} X = \operatorname{Ind} Y$.

By virtue of normality of X one can assume that the set U in (BČ2) satisfies the stronger condition $\overline{U} \subset V$. Thus, the inequality $\operatorname{Ind} X \leq n \geq 0$ means that every pair A, B of disjoint closed subsets of X can be separated by a set of dimension $\leq n-1$.

Theorem 6.2.4 implies that for a normal space X we have $\operatorname{Ind} X = 0$ if and only if the space X is strongly zero-dimensional in the sense of Section 6.2.

The reader can easily prove by induction the following theorem which justifies the names of the small and the large inductive dimensions.

7.1.2. THEOREM. *For every normal space X we have* $\operatorname{ind} X \leq \operatorname{Ind} X$. ■

Since normality is not a hereditary property (see Example 2.3.36), it may happen that the dimension Ind is defined for a space X, and yet is not defined for a subspace $M \subset X$. However, Theorem 2.1.6 implies that if $\operatorname{Ind} X$ is defined, then $\operatorname{Ind} M$ is defined for every closed subspace $M \subset X$. The proof of the next theorem, analogous to the proof of Theorem 7.1.1, is left to the reader.

7.1.3. THEOREM. *For every closed subspace M of a normal space X we have* $\operatorname{Ind} M \leq \operatorname{Ind} X$. ■

It follows from the final part of Example 6.2.20 that the last theorem does not hold when the assumption that M is a closed subspace of X is replaced by the weaker assumption that M is a normal space, i.e., that $\operatorname{Ind} M$ is defined.

In the definition of the covering dimension the concept of the order of a cover will be applied. By the *order* of a family $\mathcal{A}$ of subsets of a set X we mean the largest integer n such that the family $\mathcal{A}$ contains $n+1$ sets with non-empty intersection, or the "infinite number" ∞, if no such integer exists. Thus, if the order of a family $\mathcal{A} = \{A_s\}_{s\in S}$ equals n, then for any $n+2$ distinct indices $s_1, s_2, \ldots, s_{n+2} \in S$ we have $A_{s_1} \cap A_{s_2} \cap \ldots \cap A_{s_{n+2}} = \emptyset$. In particular, a family of order -1 consists of the empty set alone, and a family of order 0 consists of pairwise disjoint sets, not all of which are empty. The order of a family $\mathcal{A}$ is denoted by $\operatorname{ord}\mathcal{A}$.

Let X be a Tychonoff space and let n denote an integer ≥ -1; we say that:

(ČL1) $\dim X \leq n$ *if every finite functionally open cover of the space X has a finite functionally open refinement of order* $\leq n$.

(ČL2) $\dim X = n$ *if* $\dim X \leq n$ *and the inequality* $\dim X \leq n-1$ *does not hold.*

(ČL3) $\dim X = \infty$ *if the inequality* $\dim X \leq n$ *does not hold for any n.*

Conditions (ČL1)–(ČL3) assign to every Tychonoff space X a number $\dim X$ which is an integer ≥ -1 or the "infinite number" ∞. The number $\dim X$ is called the *Čech-Lebesgue dimension*, or the *covering dimension*, of the space X. Clearly, if two spaces X and Y are homeomorphic, then $\dim X = \dim Y$.

From the definition of the covering dimension it follows immediately that $\dim X = -1$ if and only if $X = \emptyset$ and $\dim X = 0$ if and only if the space X is strongly zero-dimensional.

Since the property of being a Tychonoff space is hereditary, if the dimension dim is defined for a space X, then it is also defined for every subspace $M \subset X$. To prove a counterpart of Theorems 7.1.1 and 7.1.3 for the covering dimension, we need some properties of functionally closed and functionally open sets. The two theorems on functionally closed and functionally open sets we are now going to prove will often be applied in the sequel; in a normal space, analogous theorems (which are formulated in parentheses) hold for all closed and all open sets.

A *swelling* of a family $\{A_s\}_{s\in S}$ of subsets of a space X is a family $\{B_s\}_{s\in S}$ of subsets of X such that $A_s \subset B_s$ for every $s \in S$ and for every finite set $\{s_1, s_2, \ldots, s_m\} \subset S$ we have

$$A_{s_1} \cap A_{s_2} \cap \ldots \cap A_{s_m} = \emptyset \quad \text{if and only if} \quad B_{s_1} \cap B_{s_2} \cap \ldots \cap B_{s_m} = \emptyset.$$

7.1.4. THEOREM. *Every finite family $\{F_i\}_{i=1}^k$ of functionally closed (closed) subsets of a topological (normal) space X has a swelling $\{U_i\}_{i=1}^k$ consisting of functionally open sets. If, moreover, a family $\{V_i\}_{i=1}^k$ of functionally open (open) subsets of X satisfying $F_i \subset V_i$ for $i = 1, 2, \ldots, k$ is given, then the swelling can be defined in such a way that $\overline{U}_i \subset V_i$ for $i = 1, 2, \ldots, k$.*

PROOF. The union S_1 of all intersections $F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_m}$ satisfying $F_1 \cap F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_m} = \emptyset$ is a functionally closed (closed) set disjoint from F_1. Hence, by Theorem 1.5.14 (by Urysohn's lemma) there exists a continuous function $f_1 \colon X \to I$ such that

$$f_1(F_1) \subset \{0\} \quad \text{and} \quad f_1(S_1) \subset \{1\}.$$

One easily sees that the family $\{K_1, F_2, F_3, \ldots, F_k\}$, where $K_1 = f_1^{-1}([0, 1/2])$, is a swelling of the family $\{F_i\}_{i=1}^k$.

Now, assume that for $i = 1, 2, \ldots, j$ a continuous function $f_i \colon X \to I$ is defined in such a way that $f_i(F_i) \subset \{0\}$ and the family $\{K_1, K_2, \ldots, K_j, F_{j+1}, \ldots, F_k\}$, where $K_i = f_i^{-1}([0, 1/2])$, is a swelling of the family $\{F_i\}_{i=1}^k$. The union S_{j+1} of all intersections of members of the family $\{K_1, K_2, \ldots, K_j, F_{j+1}, \ldots, F_k\}$ which are disjoint from F_{j+1} is a functionally closed (closed) set disjoint from F_{j+1}. Hence, there exists a continuous function $f_{j+1}: X \to I$ such that

$$f_{j+1}(F_{j+1}) \subset \{0\} \quad \text{and} \quad f_{j+1}(S_{j+1}) \subset \{1\}.$$

One easily sees that the family $\{K_1, K_2, \ldots, K_{j+1}, F_{j+2}, \ldots, F_k\}$, where $K_{j+1} = f_{j+1}^{-1}([0, 1/2])$, is a swelling of the family $\{F_k\}_{i=1}^k$.

Thus, one can assume that for every $i \leq k$ a continuous function $f_i: X \to I$ is defined in such a way that $f_i(F_i) \subset \{0\}$ and $\{K_i\}_{i=1}^k$, where $K_i = f_i^{-1}([0, 1/2])$, is a swelling of $\{F_i\}_{i=1}^k$. One readily sees that the family $\{U_i\}_{i=1}^k$, where $U_i = f_i^{-1}([0, 1/2))$, satisfies the first part of the theorem.

To obtain the second part it suffices to replace the sets U_i by the sets $g_i^{-1}([0, 1/2))$, where $g_i: X \to I$ is a continuous function satisfying

$$g_i(F_i) \subset \{0\} \quad \text{and} \quad g_i(X \setminus (U_i \cap V_i)) \subset \{1\}. \blacksquare$$

A *shrinking* of a cover $\{A_s\}_{s \in S}$ of a space X is a cover $\{B_s\}_{s \in S}$ of the space X such that $B_s \subset A_s$ for every $s \in S$. Clearly, the order of any shrinking of a cover $\mathcal{A}$ is not larger than ord $\mathcal{A}$.

7.1.5. THEOREM. *Every finite functionally open (open) cover $\{U_i\}_{i=1}^k$ of a topological (normal) space X has shrinkings $\{F_i\}_{i=1}^k$ and $\{W_i\}_{i=1}^k$, functionally closed and functionally open respectively, such that $F_i \subset W_i \subset \overline{W}_i \subset U_i$ for $i = 1, 2, \ldots, k$.*

PROOF. The family $\{X \setminus U_i\}_{i=1}^k$ consists of functionally closed (closed) sets with empty intersection; by virtue of Theorem 7.1.4 it has a swelling $\{V_i\}_{i=1}^k$ consisting of functionally open sets. The family $\{F_i\}_{i=1}^k$, where $F_i = X \setminus V_i$, is a functionally closed shrinking of $\{U_i\}_{i=1}^\infty$. The existence of the sets W_i follows from Theorem 1.5.14 (from Urysohn's lemma). ■

Let us observe that the normal space version of the last theorem follows also from Theoem 1.5.18 and Urysohn's lemma.

From theorems on swelling and shrinking we shall deduce some useful characterizations of the covering dimension.

7.1.6. THEOREM. *For every Tychonoff space X the following conditions are equivalent:*

(i) *The space X satisfies the inequality* $\dim X \leq n$.

(ii) *Every finite functionally open cover of the space X has a functionally open shrinking of order $\leq n$.*

(iii) *Every finite functionally open cover of the space X has a functionally closed shrinking of order $\leq n$.*

(iv) *Every finite functionally open cover of the space X has a finite functionally closed refinement of order $\leq n$.*

PROOF. Let $\{U_i\}_{i=1}^k$ be a functionally open cover of a Tychonoff space X satisfying $\dim X \leq n$. Take a finite functionally open refinement $\{V_j\}_{j=1}^m$ of order $\leq n$ and for every $j \leq m$ choose an $i(j) \leq k$ such that $V_j \subset U_{i(j)}$. The family $\{W_i\}_{i=1}^k$, where $W_i = \bigcup\{V_j : i(j) = i\}$ is a functionally open shrinking of the cover $\{U_i\}_{i=1}^k$ and has order $\leq n$, so that (i)⇒(ii). The implication (ii)⇒(iii) follows from Theorem 7.1.5, the implication (iii)⇒(iv) is obvious, and the implication (iv)⇒(i) is a consequence of Theorem 7.1.4. ■

In the class of normal spaces, all conditions in the last theorem can be stated in a simpler way; Theorems 7.1.4-7.1.6 yield

7.1.7. THEOREM. *For every normal space X the following conditions are equivalent:*
(i) *The space X satisfies the inequality* $\dim X \leq n$.
(ii) *Every finite open cover of the space X has a finite open refinement of order $\leq n$.*
(iii) *Every finite open cover of the space X has an open shrinking of order $\leq n$.*
(iv) *Every finite open cover of the space X has a closed shrinking of order $\leq n$.*
(v) *Every finite open cover of the space X has a finite closed refinement of order $\leq n$.* ■

7.1.8. THEOREM. *If a subspace M of a Tychonoff space X has the property that every continuous function $f: M \to I$ is continuously extendable over X, then we have* $\dim M \leq \dim X$.

In particular, *for every closed subspace M of a normal space X we have* $\dim M \leq \dim X$.

PROOF. Clearly, the inequality holds if $\dim X = \infty$. Assume that $\dim X = n$ and let M be a subspace of X satisfying the condition in the theorem. Consider a functionally open cover $\{U_i\}_{i=1}^k$ of the space M. Applying Theorem 7.1.5, take a functionally closed shrinking $\{F_i\}_{i=1}^k$ of the cover under consideration. By Theorem 1.5.14 for $i = 1, 2, \ldots, k$ there exists a continuous function $f_i: M \to I$ such that

$$(2) \qquad f_i(M \setminus U_i) \subset \{0\} \quad \text{and} \quad f_i(F_i) \subset \{1\}.$$

Let $\bar{f}_i: X \to I$ be an extension of f_i over X. The family $\mathcal{A}$ consisting of the sets $\bar{f}_1^{-1}((1/2, 1])$, $\bar{f}_2^{-1}((1/2, 1]), \ldots, \bar{f}_k^{-1}((1/2, 1])$, $\bigcap_{i=1}^k \bar{f}_i^{-1}([0, 1))$ is a finite functionally open cover of the space X. From (2) it follows that

$$(3) \qquad M \cap \bar{f}_i^{-1}((1/2, 1]) \subset U_i \quad \text{for} \quad i = 1, 2, \ldots, k \quad \text{and} \quad M \cap \bigcap_{i=1}^{k} \bar{f}_i^{-1}([0, 1)) = \emptyset.$$

As $\dim X \leq n$, the cover $\mathcal{A}$ has a finite functionally open refinement $\mathcal{B}$ such that $\operatorname{ord} \mathcal{B} \leq n$. It follows from (3) that the intersections of members of $\mathcal{B}$ with M constitute a finite functionally open refinement of the cover $\{U_i\}_{i=1}^k$ of the space M; clearly, the order of this refinement is $\leq n$ so that $\dim M \leq n$. ■

It follows from the final part of Example 6.2.20 that the last theorem does not hold for an arbitrary $M \subset X$.

The reader certainly observed that several theorems in this section generalize to higher dimensions theorems proved in Section 6.2 for dimension zero: Theorems 7.1.1, 7.1.3 and 7.1.8 generalize Theorem 6.2.11, Theorem 7.1.2 is a generalization of Theorem 6.2.6, and Theorem 7.1.7 generalizes Theorem 6.2.5. Our next theorem is a generalization of Theorem 6.2.13; the proof is left to the reader.

7.1.9. THEOREM. *The sum $X = \bigoplus_{s \in S} X_s$ satisfies the condition* $\operatorname{ind} X \leq n$ ($\operatorname{Ind} X \leq n$, $\dim X \leq n$) *if and only if* $\operatorname{ind} X_s \leq n$ ($\operatorname{Ind} X_s \leq n$, $\dim X_s \leq n$) *for every $s \in S$.* ■

Using the dimension functions ind, Ind and dim, one can reformulate Theorems 6.2.5, 6.2.7 and 6.2.10 as follows:

7.1.10. THEOREM. *For every normal space X the conditions* $\operatorname{Ind} X = 0$ *and* $\dim X = 0$ *are equivalent.* ■

7.1.11. THEOREM. *For every Lindelöf space X the conditions* $\operatorname{ind} X = 0$, $\operatorname{Ind} X = 0$ *and* $\dim X = 0$ *are equivalent.* ∎

7.1.12. THEOREM. *For every non-empty locally compact paracompact space X the conditions* $\operatorname{ind} X = 0$, $\operatorname{Ind} X = 0$ *and* $\dim X = 0$ *are equivalent to hereditary disconnectedness of X.* ∎

Let us observe that Theorems 7.1.10 and 7.1.11 cannot be generalized to higher dimensions. In fact, there exists a compact space X such that $\operatorname{ind} X = \operatorname{Ind} X = 2$, and yet $\dim X = 1$ (see Problem 7.4.5). There also exists a compact space X such that $\operatorname{ind} X = \dim X = 2$, and yet $\operatorname{Ind} X = 3$; however, the construction of the latter space is very complicated and will not be described in this book (cf. Theorem 7.2.9).

We conclude this section with two theorems stating that $\operatorname{Ind} \beta X = \operatorname{Ind} X$ and $\dim \beta X = \dim X$, thus generalizing Theorem 6.2.12. In the proofs of both, an operator extending open subsets of X to sets open in βX is used; basic properties of this operator, denoted by Ex, are established in two lemmas. Until the end of the proof of Theorem 7.2.17, the bar denotes the closure in βX; the closure of a set A in the space X is written as $X \cap \overline{A}$.

For every open subset U of a Tychonoff space X the set

$$\operatorname{Ex} U = \beta X \setminus \overline{(X \setminus U)}$$

is open in βX. Since

$$X \cap \operatorname{Ex} U = X \setminus \overline{(X \setminus U)} = X \setminus [X \cap \overline{(X \setminus U)}] = U,$$

and since for every open set $V \subset \beta X$ such that $X \cap V = U$ we have

$$\overline{X \setminus U} = \overline{X \setminus (X \cap V)} = \overline{X \setminus V} \subset \overline{\beta X \setminus V} = \beta X \setminus V,$$

i.e.,

$$V \subset \beta X \setminus \overline{(X \setminus U)} = \operatorname{Ex} U,$$

the set $\operatorname{Ex} U$ is the largest open subset of βX whose intersection with X is equal to U.

Clearly, we have $\overline{\operatorname{Ex} U} = \overline{U}$ and $\operatorname{Ex} U \subset \operatorname{Ex} V$ whenever $U \subset V$.

7.1.13. LEMMA. *For every pair U, V of open subsets of a Tychonoff space X we have*

$$\operatorname{Ex}(U \cap V) = \operatorname{Ex} U \cap \operatorname{Ex} V \quad \textit{and} \quad \operatorname{Ex}(U \cup V) \supset \operatorname{Ex} U \cup \operatorname{Ex} V.$$

If, moreover, the sets U, V are functionally open or the space X is normal, then

$$\operatorname{Ex}(U \cup V) = \operatorname{Ex} U \cup \operatorname{Ex} V.$$

PROOF. The first two formulas in the theorem follow directly from the definition of the operator Ex. Hence, it suffices to prove that if the sets U, V are functionally open or the space X is normal, then

$$\operatorname{Ex}(U \cup V) \subset \operatorname{Ex} U \cup \operatorname{Ex} V. \tag{4}$$

Letting $A = X \setminus U$ and $B = X \setminus V$ one replaces (4) by the equivalent inclusion

$$\overline{A} \cap \overline{B} \subset \overline{A \cap B}. \tag{5}$$

Take a point $x \in \overline{A} \cap \overline{B}$ and a neighbourhood $G \subset \beta X$ of the point x. Let F be a functionally closed subset of βX such that $x \in \operatorname{Int} F \subset F \subset G$. Thus, we have

$$x \in \overline{A} \cap \operatorname{Int} F \subset \overline{A \cap F} \quad \text{and} \quad x \in \overline{B} \cap \operatorname{Int} F \subset \overline{B \cap F};$$

therefore, by Corollary 3.6.2, the intersections $A \cap F$ and $B \cap F$ are not completely separated subsets of X. Since $A \cap F$ and $B \cap F$ are functionally closed sets or closed subsets of a normal space, we have

$$\emptyset \neq (A \cap F) \cap (B \cap F) = (A \cap B) \cap F \subset (A \cap B) \cap G.$$

The set G being an arbitrary neighbourhood of x, this implies that $x \in \overline{A \cap B}$. Hence the inclusion (5) is established.■

7.1.14. LEMMA. *For every open subset U of a normal space X we have*

$$\operatorname{Fr} \operatorname{Ex} U = \overline{\operatorname{Fr} U},$$

where $\operatorname{Fr} \operatorname{Ex} U$ *is the boundary of the set* $\operatorname{Ex} U$ *in the space* βX *and* $\operatorname{Fr} U$ *is the boundary of the set* U *in the space* X.

PROOF. Let us note first that

$$\beta X \setminus \operatorname{Ex}(X \setminus \overline{U}) = \overline{X \setminus (X \setminus \overline{U})} = \overline{\overline{U}} = \overline{U}. \tag{6}$$

Applying (6) and the last formula in Lemma 7.1.13, we have $\operatorname{Fr} \operatorname{Ex} U = \overline{\operatorname{Ex} U} \setminus \operatorname{Ex} U = \overline{U} \setminus \operatorname{Ex} U = \beta X \setminus [\operatorname{Ex}(X \setminus \overline{U}) \cup \operatorname{Ex} U] = \beta X \setminus \operatorname{Ex}[(X \setminus \overline{U}) \cup U] = \overline{X \setminus [(X \setminus \overline{U}) \cup U]} = \overline{\operatorname{Fr} U}$.■

7.1.15. THEOREM. *For every normal space X we have* $\operatorname{Ind} \beta X = \operatorname{Ind} X$.

PROOF. To begin, we shall prove that $\operatorname{Ind} X \leq \operatorname{Ind} \beta X$. Clearly, the inequality holds if $\operatorname{Ind} \beta X = -1$ and if $\operatorname{Ind} \beta X = \infty$. Assume that the inequality is proved for all normal spaces whose Čech-Stone compactifications have the large inductive dimension $\leq n-1$. Consider a normal space X such that $\operatorname{Ind} \beta X \leq n$, a closed set $A \subset X$ and an open set $V \subset X$ that contains A. By Corollary 3.6.4 the closures of A and $X \setminus V$ in βX are disjoint, so that there exists an open set $W \subset \beta X$ such that

$$\overline{A} \subset W \subset \beta X \setminus \overline{X \setminus V} \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr} W \leq n-1. \tag{7}$$

The intersection $U = X \cap W$ is open in X and satisfies

$$A \subset U = X \cap W \subset X \setminus \overline{X \setminus V} = V.$$

The boundary $\operatorname{Fr} U$ of the set U in the space X is contained in $X \cap \operatorname{Fr} W$, so that $\overline{\operatorname{Fr} U} \subset \operatorname{Fr} W$ and $\operatorname{Ind} \overline{\operatorname{Fr} U} \leq n-1$ by virtue of Theorem 7.1.3 and the second part of (7). Applying Corollary 3.6.8 and the inductive assumption we infer that $\operatorname{Ind} \operatorname{Fr} U \leq n-1$. Hence $\operatorname{Ind} X \leq n$ and the inequality $\operatorname{Ind} X \leq \operatorname{Ind} \beta X$ is established.

Now, we shall prove that $\operatorname{Ind} \beta X \leq \operatorname{Ind} X$. Clearly, the inequality holds if $\operatorname{Ind} X = -1$ and if $\operatorname{Ind} X = \infty$. Assume that the inequality is proved for all normal spaces with large inductive dimension $\leq n-1$ and consider a normal space X such that $\operatorname{Ind} X \leq n$. Let A be a closed subset of βX and $V \subset \beta X$ an open set that contains A. Take open sets $G, H \subset \beta X$ satisfying

$$A \subset G \subset \overline{G} \subset H \subset \overline{H} \subset V; \tag{8}$$

as $\operatorname{Ind} X \leq n$, there exists an open set $U \subset X$ such that

$$X \cap \overline{G} \subset U \subset X \cap H \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr} U \leq n-1, \tag{9}$$

where $\operatorname{Fr} U$ is the boundary of the set U in the space X.

Formulas (8) and (9) imply that

$$A \subset G \subset \operatorname{Ex}(X \cap G) \subset \operatorname{Ex} U \subset \operatorname{Ex}(X \cap H) \subset \overline{H} \subset V.$$

Applying the last formula, the second part of (9), Lemma 7.1.14, Corollary 3.6.8 and the inductive assumption, we infer that

$$A \subset \operatorname{Ex} U \subset V \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr} \operatorname{Ex} U \leq n-1.$$

Hence $\operatorname{Ind} \beta X \leq n$ and the inequality $\operatorname{Ind} \beta X \leq \operatorname{Ind} X$ is established.∎

The last theorem, along with Corollary 3.6.7 yields

7.1.16. COROLLARY. *For every normal space X and a dense normal subspace $M \subset X$ which has the property that every continuous function $f: M \to I$ is continuously extendable over X we have* $\operatorname{Ind} M = \operatorname{Ind} X$.

In other words, *for every normal space Y and any normal subspace $T \subset \beta Y$ which contains Y we have* $\operatorname{Ind} Y = \operatorname{Ind} T$.∎

7.1.17. THEOREM. *For every Tychonoff space X we have* $\dim \beta X = \dim X$.

PROOF. Theorems 3.6.1 and 7.1.8 imply that $\dim X \leq \dim \beta X$, so that it suffices to prove that for any integer $n \geq -1$ if $\dim X \leq n$, then $\dim \beta X \leq n$.

Let X be a Tychonoff space such that $\dim X \leq n$; consider a finite open cover $\{U_i\}_{i=1}^k$ of the space βX. By virtue of Theorem 7.1.5 there exists a functionally open shrinking $\{W_i\}_{i=1}^k$ of $\{U_i\}_{i=1}^k$ such that

$$\overline{W}_i \subset U_i \quad \text{for} \quad i = 1, 2, \ldots, k. \tag{10}$$

The cover $\{X \cap W_i\}_{i=1}^k$ of the space X has a functionally open shrinking $\{V_i\}_{i=1}^k$ of order $\leq n$. Formula (10) implies that

$$\operatorname{Ex} V_i \subset \overline{V}_i \subset \overline{W}_i \subset U_i.$$

Hence, by Lemma 7.1.13, the family $\{\operatorname{Ex} V_i\}_{i=1}^k$ is an open shrinking of the cover $\{U_i\}_{i=1}^k$ of βX. By the same lemma the order of that shrinking is $\leq n$, so that we have $\dim \beta X \leq n$.∎

The last theorem, along with Corollary 3.6.7, yields

7.1.18. COROLLARY. *For every Tychonoff space X and a dense subspace $M \subset X$ which has the property that every continuous function $f: M \to I$ is continuously extendable over X we have* $\dim M = \dim X$.

In other words, *for every Tychonoff space Y and any subspace $T \subset \beta Y$ which contains Y we have* $\dim Y = \dim T$.■

Let us remind the reader, in connection with Theorems 7.1.15 and 7.1.17, that for the space Y in Example 6.2.20 we have $\operatorname{ind} Y = 0$, and yet $\operatorname{ind} \beta Y > 0$.

7.1.19. EXAMPLES. For every point x in the real line R or in the circle S^1 and for any neighbourhood V of the point x there exists a neighbourhood U of x such that $U \subset V$ and the boundary $\operatorname{Fr} U$ is a two-point set; hence, we have $\operatorname{ind} R \leq 1$ and $\operatorname{ind} S^1 \leq 1$. Since $\operatorname{Ind} I > 0$ by 6.2.18, Theorem 7.1.1 implies that $\operatorname{ind} R = \operatorname{ind} S^1 = \operatorname{ind} I = 1$.

For every point x in Euclidean n-space R^n or in the n-sphere S^n and for any neighbourhood V of the point x there exists a neighbourhood U of x such that $U \subset V$ and the boundary $\operatorname{Fr} U$ is homeomorphic to S^{n-1}. Hence, by an easy induction, we have $\operatorname{ind} R^n \leq n$, $\operatorname{ind} S^n \leq n$ and $\operatorname{ind} I^n \leq n$ for any natural number n.

The small inductive dimension of R^n, S^n and I^n, as well as the large inductive dimension and the covering dimension of these spaces, is indeed equal to n. The proof of this fact is much more difficult than the above evaluations; it will be given in Section 7.3.■

7.1.20. EXAMPLE. Let Y and Y^* be the spaces defined in Example 6.2.20. We know already that $\operatorname{ind} Y = 0$, $\operatorname{Ind} Y > 0$ and $\dim Y > 0$. Corollaries 7.1.16 and 7.1.18 yield the equalities $\operatorname{Ind} Y = \operatorname{Ind} Y^*$ and $\dim Y = \dim Y^*$. We shall now show that $\operatorname{Ind} Y^* \leq 1$.

Let $A \subset Y^*$ be a closed set and let $V \subset Y^*$ be an open set that contains A. In the interval I take an open set G which is a finite union of disjoint open intervals and satisfies

$$A \cap (\{\omega_1\} \times I) \subset \{\omega_1\} \times G \subset V \cap (\{\omega_1\} \times I).$$

Applying second-countability of $I \setminus G$ and G, we verify that for some $\alpha < \omega_1$ the inclusions

$$A \setminus Y_\alpha \subset Y^* \cap [(W \setminus X_\alpha) \times G] \subset V \tag{11}$$

hold. Since Y_α is a second-countable metrizable space satisfying $\operatorname{ind} Y_\alpha = 0$, by Theorems 3.8.1 and 7.1.11 there exists a set H, open-and-closed in Y_α, and, *a fortiori*, in Y^*, such that

$$A \cap Y_\alpha \subset H \subset V. \tag{12}$$

By virtue of (11) and (12), the set $U = H \cup \{Y^* \cap [(W \setminus X_\alpha) \times G]\}$ satisfies $A \subset U \subset V$. Since $\operatorname{Fr} U \subset (W \setminus X_\alpha) \times \operatorname{Fr} G$ and $\operatorname{Fr} G$ is a finite set, by Theorem 7.1.9 we have $\operatorname{Ind} \operatorname{Fr} U \leq 0$; hence $\operatorname{Ind} Y^* \leq 1$.

In a similar way one can prove that $\dim Y^* \leq 1$; however, this will also be a consequence of Theorem 7.2.8 below.

Let us note that the inequality $\operatorname{Ind} Y^* \leq 1$, along with Theorems 7.1.1 and 7.1.2, yields $\operatorname{ind} Y^* = 1$. Thus, finally we have

$$\operatorname{ind} Y = 0, \quad \operatorname{ind} Y^* = 1, \quad \operatorname{Ind} Y = \operatorname{Ind} Y^* = 1, \quad \dim Y = \dim Y^* = 1. ■$$

Historical and bibliographic notes

An inductive definition of dimension was outlined by Poincaré in [1912]. The first precise definition of a dimension function was formulated by Brouwer in [1913]; Brouwer's function coincides with the dimension Ind in the class of locally connected completely metrizable spaces. In Brouwer's paper, however, the dimension function is only an auxiliary tool in proving that the spaces R^m and R^n are not homeomorphic whenever $m \neq n$ (the fist correct proof of this important fact was given by Brouwer in [1911]).

Dimension theory was originated by Menger and Urysohn. The definition of the dimension ind was formulated by Urysohn in [1922] and by Menger in [1923]. The definition of the dimension Ind was formulated by Čech in [1931]. The covering dimension dim was defined (by condition (ii) in Theorem 7.1.7) in Čech's paper [1933]. Lebesgue's paper [1911] contains a theorem asserting that $\dim I^n = n$ (as a corollary Lebesgue obtains the fact that the spaces R^m and R^n are not homeomorphic whenever $m \neq n$); the outline of the proof, however, contains a gap that was only filled in Lebesgue's paper [1921]. The definition of the covering dimension given by Čech proved appropriate only in the class of normal spaces. The modification of this definition, obtained by considering functionally open covers was made by Katětov in [1950a]; another class of covers leading to the same dimension function was considered by Smirnov in [1956b]. The first exposition of the dimension theory of Tychonoff spaces is given in Gillman and Jerison's book [1960].

Theorem 7.1.8 was proved by Katětov in [1950a] (for normal spaces, by Čech in [1933]). The first example of a compact space X such that $\operatorname{ind} X \neq \operatorname{Ind} X$ was given by Filippov in [1969] (a normal space with that property was described by Smirnov in [1951b]); further examples, simplified, but still very complicated, were constructed by Filippov in [1970], Pasynkov and Lifanov in [1970] and Pasynkov in [1970]. The operator Ex was introduced by Šanin in [1943]. The non-trivial inclusion in Lemma 7.1.13 was established (for normal spaces) by Smirnov in [1951] and Wallace in [1951]; Lemma 7.1.14 is due to Smirnov [1951b] (cf. Exercise 7.1.H). Theorem 7.1.15 was proved by Vedenissoff in [1941] and Theorem 7.1.17 was proved by Wallman in [1938] for normal spaces (it was observed by Katětov in [1950a] that the theorem can be generalized to Tychonoff spaces). In connection with Theorems 7.1.3, 7.1.8 and the final part of Example 6.2.20 (see also Problem 7.4.12), one should note that E. Pol and R. Pol defined in [1977] a hereditarily normal space X such that $\dim X = 0$ and yet X contains a subspace A with $\dim A > 0$. The same authors defined in [1979] a hereditarily normal space X such that $\dim X = 0$, and yet X contains for every natural number n a subspace A_n with $\dim A_n = \operatorname{Ind} A_n = n$; under the assumption that there exists a Souslin space (see the remark to Problem 2.7.9(f)), such a space was defined by Filippov in [1973].

The reader wishing to obtain a better understanding of the genesis and geometrical significance of theorems in this chapter is advised to consult Hurewicz and Wallman [1941] or Engelking [1978]; each of these books contains a comprehensive course in the dimension theory of separable metrizable spaces.

Exercises

7.1.A (Erdős [1940]). Check that the small inductive dimension of the space X in Example 6.2.19 is equal to one.

Hint. It suffices to show that for every natural number i the point x_0 has a neighbourhood $V_i \subset X$ such that $\delta(V_i) \leq 1/i$ and $\operatorname{ind} \operatorname{Fr} V_i = 0$. To that end, observe that for every r the set $\{x \in X : \rho(x_0, x) = r\}$ is homeomorphic to a subspace of the Hilbert cube consisting of points all of whose coordinates are rational.

7.1.B. Prove that every locally finite functionally open cover $\{U_s\}_{s\in S}$ of a topological space X has shrinkings $\{F_s\}_{s\in S}$ and $\{W_s\}_{s\in S}$, functionally closed and functionally open respectively, such that $F_s \subset W_s \subset \overline{W}_s \subset U_s$ for every $s \in S$.

Hint. Take functions $f_s : X \to I$ satisfying $U_s = f_s^{-1}((0,1])$ and consider the function $f: X \to I$ defined by letting $f(x) = \sup_{s\in S} f_s(x)$.

One can also apply Exercise 5.1.J(b).

7.1.C. Prove that if X is a normal space satisfying $\operatorname{Ind} X \leq n \geq 0$ and if by adjoining a strongly zero-dimensional space to X one obtains a normal space Y in which the adjoined space is closed, then $\operatorname{Ind} Y \leq n$.

7.1.D. Prove that if X is a normal space satisfying $\dim X \leq n \geq 0$ and if by adjoining a strongly zero-dimensional space to X one obtains a normal space Y in which the adjoined space is closed, then $\dim Y \leq n$.

7.1.E. Show that if the spaces X and Y are compact and $\dim Y = 0$, then $\dim(X \times Y) = \dim X$ (see Problem 7.4.10).

7.1.F (implicitly, Freudenthal [1937]). Prove that if $\mathbf{S} = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$ is an inverse system of compact spaces such that $\dim X_\sigma \leq n$ for every $\sigma \in \Sigma$, then the limit $X = \varprojlim \mathbf{S}$ satisfies the inequality $\dim X \leq n$.

7.1.G. (a) (Morita [1950a]) Prove that if for a family $\{F_s\}_{s\in S}$ of closed subsets of a normal space X there exists a locally finite family $\{V_s\}_{s\in S}$ of open subsets of X satisfying $F_s \subset V_s$ for every $s \in S$, then the family $\{F_s\}_{s\in S}$ has a swelling $\{U_s\}_{s\in S}$ consisting of open sets and such that $\overline{U}_s \subset V_s$ for every $s \in S$.

Hint. Apply transfinite induction.

(b) Modify the proof of Theorem 5.1.18 to show directly that every locally finite family of closed subsets of a paracompact space has a swelling consisting of open sets.

(c) For every locally finite family of closed subsets of a metrizable space define with the aid of a metric on that space a swelling consisting of open sets.

Hint. Let $\{F_s\}_{s\in S}$ be a locally finite family of closed subsets of a metrizable space X. For every $x \in X$ take an $\epsilon_x > 0$ such that if $F_s \cap B(x, 2\epsilon_x) \neq \emptyset$ then $x \in F_s$, and let $U_s = \bigcup_{x\in F_s} B(x, \epsilon_x)$.

7.1.H (Skljarenko [1962]). Prove that Lemma 7.1.14 holds under the assumption that X is a Tychonoff space.

Hint (van Douwen [1981]). It suffices to show that $\overline{\operatorname{Ex} U} \setminus \overline{\operatorname{Fr} U} \subset \operatorname{Ex} U$. Consider a point $z \in \overline{\operatorname{Ex} U} \setminus \overline{\operatorname{Fr} U}$, take a continuous function $f: \beta X \to I$ such that $f(z) = 1$ and $f(\overline{\operatorname{Fr} U}) \subset \{0\}$, and define $g: X \to [-1, 1]$ by the formulas $g(x) = f(x)$ for $x \in \overline{U}$, $g(x) = -f(x)$ for $x \in X \setminus U$. Show that for the extension $G: \beta X \to [-1, 1]$ of g we have $G(z) = 1$ and deduce that $z \in \operatorname{Ex} U$.

7.2. Further properties of the dimension dim

We begin with two theorems that will be often applied in this and in the following section; they are called the *sum theorems.*

7.2.1. THE COUNTABLE SUM THEOREM. *If a normal space X has a countable closed cover $\{F_j\}_{j=1}^{\infty}$ such that* $\dim F_j \leq n$ *for* $j = 1, 2, \ldots$, *then* $\dim X \leq n$.

PROOF. Let $\{U_i\}_{i=1}^{k}$ be a finite open cover of the space X. We shall inductively define a sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of open covers of X, where $\mathcal{U}_j = \{U_{j,i}\}_{i=1}^{k}$, satisfying the conditions:

(1) $$\overline{U}_{j,i} \subset U_{j-1,i} \text{ if } j \geq 1 \quad \text{and} \quad U_{0,i} \subset U_i.$$

(2) $$\operatorname{ord}(\{\overline{U}_{j,i} \cap F_j\}_{i=1}^{k}) \leq n, \quad \text{where} \quad F_0 = \emptyset.$$

Both conditions are satisfied for $j = 0$ if we let $U_{0,i} = U_i$ for $i = 1, 2, \ldots, k$. Assume that the covers $\mathcal{U}_j$ satisfying (1) and (2) are defined for all $j < m \geq 1$. By virtue of Theorem 7.1.7, the open cover $\{F_m \cap U_{m-1,i}\}_{i=1}^{k}$ of the subspace $F_m \subset X$ has an open shrinking $\{V_i\}_{i=1}^{k}$ of order $\leq n$. One readily observes that the family $\{W_i\}_{i=1}^{k}$, where $W_i = (U_{m-1,i} \setminus F_m) \cup V_i \subset U_{m-1,i}$, is an open cover of X and that $\operatorname{ord}(\{W_i \cap F_m\}_{i=1}^{k}) \leq n$. By virtue of Theorem 7.1.5, there exists an open shrinking $\mathcal{U}_m = \{U_{m,i}\}_{i=1}^{k}$ of the latter cover such that $\overline{U}_{m,i} \subset W_i$ for $i = 1, 2, \ldots, k$. Clearly, the cover $\mathcal{U}_m$ satisfies (1) and (2) for $j = m$, so that the construction of the sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ is concluded.

For every $x \in X$ there exists an $i(x) \leq k$ such that x belongs to infinitely many sets $U_{j,i(x)}$; hence, by (1), we have $x \in \bigcap_{j=1}^{\infty} U_{j,i(x)}$. Applying (1) and (2) we see that the family $\{\bigcap_{j=1}^{\infty} \overline{U}_{j,i}\}_{i=1}^{k}$ is a closed shrinking of the cover $\{U_i\}_{i=1}^{k}$ and has order $\leq n$. Therefore, we have $\dim X \leq n$ by virtue of Theorem 7.1.7.■

The locally finite sum theorem will be deduced from a lemma. The lemma is formulated in a more general way, so that it also yields another theorem which is fundamental for the dimension theory of metrizable spaces.

7.2.2. LEMMA. *Let $\mathcal{U} = \{U_s\}_{s \in S}$ be an open cover of a normal space X. If the space X has a locally finite closed cover $\mathcal{F}$ each member of which has covering dimension $\leq n$ and meets only finitely many sets U_s, then the cover $\mathcal{U}$ has an open shrinking of order $\leq n$.*

PROOF. Let us adjoin to the cover $\mathcal{F}$ the set $F_0 = \emptyset$ and let us arrange the members of this cover into a transfinite sequence $F_0, F_1, \ldots, F_\alpha, \ldots$, $\alpha \leq \xi$ of type $\xi + 1$. Applying transfinite induction, we shall define a transfinite sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_\alpha, \ldots$, $\alpha \leq \xi$ of open covers of X, where $\mathcal{U}_\alpha = \{U_{\alpha,s}\}_{s \in S}$, satisfying the conditions:

(3) $$U_{\alpha,s} \subset U_{\beta,s} \text{ if } \beta < \alpha \quad \text{and} \quad U_{0,s} \subset U_s.$$

(4) $$\operatorname{ord}(\{U_{\alpha,s} \cap F_\alpha\}_{s \in S}) \leq n.$$

(5) $$U_{\beta,s} \setminus U_{\alpha,s} \subset \bigcup_{\beta \leq \gamma \leq \alpha} F_\gamma \quad \text{whenever} \quad \beta < \alpha.$$

Conditions (3)–(5) are satisfied for $\alpha = 0$ if we let $U_{0,s} = U_s$ for every $s \in S$. Assume that the covers $\mathcal{U}_\alpha$ satisfying (3)–(5) are defined for all $\alpha < \alpha_0 \geq 1$.

To begin, we shall show that the family of all sets

$$U'_{\alpha_0,s} = \bigcap_{\alpha<\alpha_0} U_{\alpha,s},$$

where $s \in S$, is an open cover of the space X. This is clear if $\alpha_0 = \alpha_1 + 1$, because then we have $U'_{\alpha_0,s} = U_{\alpha_1,s}$; hence, we can assume that α_0 is a limit number.

Every point $x \in X$ has a neighbourhood U that meets only finitely many members of $\mathcal{F}$; thus there exists a $\beta < \alpha_0$ such that for any γ satisfying $\beta \leq \gamma < \alpha_0$ we have $U \cap F_\gamma = \emptyset$. As $\mathcal{U}_\beta$ is a cover of X, there exists an $s \in S$ such that $x \in U_{\beta,s}$. It follows from (5) that $U \cap U_{\beta,s} \subset U_{\alpha,s}$ whenever $\beta \leq \alpha < \alpha_0$, so that $x \in U \cap U_{\beta,s} \subset U'_{\alpha_0,s}$; hence the sets $U'_{\alpha_0,s}$ are open and cover the space X.

The open cover $\{F_{\alpha_0} \cap U'_{\alpha_0,s}\}_{s\in S}$ of the subspace $F_{\alpha_0} \subset X$ has an open shrinking $\{V_s\}_{s\in S}$ of order $\leq n$, because – by (3) and the assumptions in the lemma – only finitely many members of that cover are non-empty. The family $\mathcal{U}_{\alpha_0} = \{U_{\alpha_0,s}\}_{s\in S}$, where $U_{\alpha_0,s} = (U'_{\alpha_0,s} \setminus F_{\alpha_0}) \cup V_s$, is an open cover of X satisfying conditions (3)–(5) for $\alpha = \alpha_0$, so that the construction of the sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_\alpha, \ldots,\ \alpha \leq \xi$ is concluded. Now, it follows from (4) that $\operatorname{ord} \mathcal{U}_\xi \leq n$; since $\mathcal{U}_\xi$ is a shrinking of $\mathcal{U}$ by virtue of (3), the lemma is established. ■

Lemma 7.2.2, along with Theorem 7.1.7, yields

7.2.3. THE LOCALLY FINITE SUM THEOREM. *If a normal space X has a locally finite closed cover $\{F_s\}_{s\in S}$ such that $\dim F_s \leq n$ for every $s \in S$, then $\dim X \leq n$.* ■

Let us observe that Theorems 7.2.1 and 7.2.3 imply the σ-locally finite sum theorem: every normal space X which has a σ-locally finite closed cover $\{F_s\}_{s\in S}$ such that $\dim F_s \leq n$ for every $s \in S$ satisfies $\dim X \leq n$.

7.2.4. THE DOWKER THEOREM. *For every normal space X the following conditions are equivalent:*

(i) *The space X satisfies the inequality* $\dim X \leq n$.
(ii) *Every locally finite open cover of the space X has an open shrinking of order $\leq n$.*
(iii) *Every locally finite open cover of the space X has an open refinement of order $\leq n$.*

PROOF. We shall show that (i)⇒(ii). Let X be a normal space satisfying $\dim X \leq n$ and let $\mathcal{U} = \{U_s\}_{s\in S}$ be a locally finite open cover of the space X. Denote by $\mathcal{T}$ the family of all non-empty finite subsets of S and for every $T \in \mathcal{T}$ consider the closed set

$$F_T = \bigcap_{s\in T} \overline{U}_s \cap \bigcap_{s\notin T} (X \setminus U_s);$$

by virtue of Theorem 7.1.8 we have $\dim F_T \leq n$. The family $\mathcal{F} = \{F_T\}_{T\in\mathcal{T}}$ is a closed cover of X each member of which meets only finitely many sets U_s. The cover $\mathcal{F}$ is locally finite. Indeed, for every point $x \in X$ there exists a neighbourhood $U(x)$ of the point x and a finite set $S(x) \subset S$ such that $U(x) \cap U_s = \emptyset$ whenever $s \notin S(x)$; we clearly have $U(x) \cap \overline{U}_s = \emptyset$ whenever $s \notin S(x)$, so that if $U(x) \cap F_T \neq \emptyset$, then $T \subset S(x)$, and this shows that $\mathcal{F}$ is locally finite. Lemma 7.2.2 implies that the cover $\mathcal{U}$ has an open shrinking of order $\leq n$.

The implication (ii)⇒(iii) is obvious; the proof of the implication (iii)⇒(i) is left to the reader (cf. the proof of Theorem 7.1.6). ■

One readily verifies that if the space X is paracompact, then in the last theorem one can replace the words "locally finite open cover" by the words "open cover". In the case of a paracompact space one also can instead of open shrinkings (refinements) take closed shrinkings (locally finite closed refinements) of order $\leq n$; this is an easy consequence of Remark 5.1.19 and Exercise 7.1.G(a).

We are now going to study relations between the dimension dim and the inductive dimensions ind and Ind. We shall prove two theorems establishing inequalities between different dimensions of a space (cf. Theorem 7.1.2). We start with a lemma generalizing Theorem 6.2.7.

7.2.5. LEMMA. *Let X be a Lindelöf space and $\mathcal{B}$ a base for the space X. For every pair A, B of disjoint closed subsets of the space X there exist an open set $W \subset X$ and a countable family $\{W_i\}_{i=1}^{\infty}$ of members of $\mathcal{B}$ such that*

$$A \subset W \subset \overline{W} \subset X \setminus B \quad \textit{and} \quad \operatorname{Fr} W \subset \bigcup_{i=1}^{\infty} \operatorname{Fr} W_i.$$

PROOF. For every $x \in X$ choose a neighbourhood $W_x \in \mathcal{B}$ of the point x such that $A \cap \overline{W}_x = \emptyset$ or $B \cap \overline{W}_x = \emptyset$ and take a countable subcover $\mathcal{W} = \{W_i\}_{i=1}^{\infty}$ of the open cover $\{W_x\}_{x \in X}$ of the space X. Denote by $\{U_i\}_{i=1}^{\infty}$ the family of all members of $\mathcal{W}$ whose closures meet the set A and by $\{V_i\}_{i=1}^{\infty}$ the family of all the remaining members of $\mathcal{W}$. Hence,

$$A \subset \bigcup_{i=1}^{\infty} U_i, \quad B \subset \bigcup_{i=1}^{\infty} V_i \quad \text{and} \quad \overline{U}_i \cap B = \emptyset = \overline{V}_i \cap A \quad \text{for} \quad i = 1, 2, \ldots \tag{6}$$

Let

$$G_i = U_i \setminus \bigcup_{j<i} \overline{V}_j \quad \text{and} \quad H_i = V_i \setminus \bigcup_{j \leq i} \overline{U}_j \quad \text{for} \quad i = 1, 2, \ldots \tag{7}$$

Formulas (6) and (7) imply that the open sets $W = \bigcup_{i=1}^{\infty} G_i$ and $V = \bigcup_{i=1}^{\infty} H_i$ satisfy the conditions $A \subset W, B \subset V$ and $W \cap V = \emptyset$; therefore we have $\overline{W} \cap V = \emptyset$ and $\overline{W} \subset X \setminus V \subset X \setminus B$. Since $\operatorname{Fr} W \subset X \setminus (W \cup V)$, to conclude the proof it suffices to check that

$$X \setminus (W \cup V) \subset \bigcup_{i=1}^{\infty} \operatorname{Fr} U_i \cup \bigcup_{i=1}^{\infty} \operatorname{Fr} V_i = \bigcup_{i=1}^{\infty} \operatorname{Fr} W_i.$$

Take a point $x \in X \setminus (W \cup V)$ and denote by F the first element of the sequence $\overline{U}_1, \overline{V}_1, \overline{U}_2, \overline{V}_2, \ldots$ that contains x. If $F = \overline{U}_i$, then $x \in \operatorname{Fr} U_i = \overline{U}_i \setminus U_i$, because $x \notin G_i$ and $x \notin \overline{V}_j$ for $j < i$. If, on the other hand, $F = \overline{V}_i$, then $x \in \operatorname{Fr} V_i = \overline{V}_i \setminus V_i$, because $x \notin H_i$ and $x \notin \overline{U}_j$ for $j \leq i$. Hence, in both cases $x \in \bigcup_{i=1}^{\infty} \operatorname{Fr} U_i \cup \bigcup_{i=1}^{\infty} \operatorname{Fr} V_i$. ■

7.2.6. LEMMA. *If for every pair A, B of disjoint closed subsets of a normal space X there exists an open set $W \subset X$ such that*

$$A \subset W \subset \overline{W} \subset X \setminus B \quad \textit{and} \quad \dim \operatorname{Fr} W \leq n - 1,$$

then $\dim X \leq n$.

PROOF. Let $\{U_i\}_{i=1}^k$ be a finite open cover of the space X and let $\{A_i\}_{i=1}^k$ be a closed shrinking of $\{U_i\}_{i=1}^k$. For $i = 1, 2, \ldots, k$ take an open set $W_i \subset X$ such that $A_i \subset W_i \subset \overline{W}_i \subset U_i$ and $\dim \operatorname{Fr} W_i \leq n-1$. Since $F = \bigcup_{i=1}^k \operatorname{Fr} W_i$ is a normal space, $\dim F \leq n-1$ by virtue of Theorem 7.2.1. From Theorems 7.1.7 and 7.1.4 it follows that there exists a family $\{V_i\}_{i=1}^k$ of open subsets of X such that $\overline{V}_i \subset U_i$ for $i = 1, 2, \ldots, k$,

$$F \subset V = \bigcup_{i=1}^{k} V_i \quad \text{and} \quad \operatorname{ord}(\{\overline{V}_i\}_{i=1}^k) \leq n-1. \tag{8}$$

The sets $\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_k, F_1, F_2, \ldots, F_k$, where

$$F_i = \overline{W}_i \setminus (V \cup \bigcup_{j<i} W_j),$$

constitute a closed refinement of $\{U_i\}_{i=1}^k$; hence, by Theorem 7.1.7, to conclude the proof it suffices to show that this refinement has order $\leq n$. The last inequality follows immediately from the second part of (8) and from the fact that for $j < i \leq k$ we have

$$F_j \cap F_i \subset \overline{W}_j \cap [\overline{W}_i \setminus (V \cup W_j)] \subset (X \setminus V) \cap \operatorname{Fr} W_j = \emptyset. \blacksquare$$

From 7.2.5, 7.2.6 and 7.2.1, by applying induction with respect to $\operatorname{ind} X$, we obtain (cf. Exercise 7.2.F)

7.2.7. THEOREM. *For every Lindelöf space X we have* $\dim X \leq \operatorname{ind} X$. $\blacksquare$

From Lemma 7.2.6, by applying induction with respect to $\operatorname{Ind} X$, we obtain

7.2.8. THEOREM. *For every normal space X we have* $\dim X \leq \operatorname{Ind} X$. $\blacksquare$

Let us also note that Lemma 7.2.5, along with Theorems 7.1.11, 7.2.1 and 7.1.2, yields

7.2.9. THEOREM. *For every Lindelöf space X the conditions* $\operatorname{ind} X = 1$ *and* $\operatorname{Ind} X = 1$ *are equivalent.* $\blacksquare$

7.2.10. EXAMPLE. From 7.1.19, 7.2.7, 7.2.8 and 7.1.11 it follows that $\operatorname{Ind} R = \operatorname{Ind} S^1 = \operatorname{Ind} I = 1$ and $\dim R = \dim S^1 = \dim I = 1$; similarly from 7.1.19 and 7.2.7 it follows that $\dim R^n \leq n$, $\dim S^n \leq n$ and $\dim I^n \leq n$ for any natural number n. $\blacksquare$

7.2.11. EXAMPLE. For every pair n, m of integers satisfying $0 \leq m \leq n \geq 1$ denote by Q_m^n the subspace of R^n consisting of all points which have exactly m rational coordinates. From 6.2.18 it follows that $\dim Q_n^n = 0$; we shall show that $\dim Q_m^n = 0$ for each $m < n$.

For any choice of m distinct natural numbers $i_1, i_2, \ldots, i_m$ not larger than n and any choice of m rational numbers $r_1, r_2, \ldots, r_m$, the space $\prod_{i=1}^n R_i$, where $R_{i_k} = \{r_k\}$ for $k = 1, 2, \ldots, m$ and $R_i = R$ for $i \neq i_k$, is a closed subspace of R^n; hence the space $Q_m^n \cap \prod_{i=1}^n R_i$ is a closed subspace of Q_m^n. Since the space $Q_m^n \cap \prod_{i=1}^n R_i$ is homeomorphic to the subspace of R^{n-m} consisting of all points with irrational coordinates, it follows from 6.2.18 and 7.1.11 that $\dim (Q_m^n \cap \prod_{i=1}^n R_i) = 0$. The countable sum theorem implies that $\dim Q_m^n = 0$, because the family of all sets of the form $Q_m^n \cap \prod_{i=1}^n R_i$ is a countable closed cover of Q_m^n. $\blacksquare$

We conclude this section with a characterization of the covering dimension in terms of partitions. We start with a definition and two lemmas.

Let X be a topological space and A, B a pair of disjoint subsets of the space X; a closed set $L \subset X$ is a *partition* between A and B if there exist open sets $U, V \subset X$ such that

$$(9) \qquad A \subset U, \quad B \subset V, \quad U \cap V = \emptyset \quad \text{and} \quad X \setminus L = U \cup V.$$

7.2.12. LEMMA. *If a partition L between subsets A and B of a topological space X is functionally closed, then the sets U and V in* (9) *are functionally open.*

PROOF. Let $f: X \to R$ be a continuous function satisfying $f^{-1}(0) = L$. By virtue of Theorem 2.1.13 the formula

$$g(x) = \begin{cases} f(x), & \text{if } x \in U \cup L, \\ 0, & \text{if } x \in V \cup L, \end{cases}$$

defines a continuous function $g: X \to R$. Since $g^{-1}(R \setminus \{0\}) = U$, the set U is functionally open. By the symmetry of our assumptions, the set V also is functionally open. ■

7.2.13. LEMMA. *If every $(n+2)$-element functionally open (open) cover $\{U_i\}_{i=1}^{n+2}$ of a Tychonoff (normal) space X has a functionally open (open) shrinking $\{W_i\}_{i=1}^{n+2}$ such that the intersection $\bigcap_{i=1}^{n+2} W_i$ is empty, then* $\dim X \leq n$.

PROOF. We shall show that every Tychonoff (normal) space X such that $\dim X > n$ has an $(n+2)$-element functionally open (open) cover $\{U_i\}_{i=1}^{n+2}$, each functionally open (open) shrinking $\{W_i\}_{i=1}^{n+2}$ of which satisfies the condition $\bigcap_{i=1}^{n+2} W_i \neq \emptyset$. As $\dim X > n$, there exists a finite functionally open (open) cover $\mathcal{V}$ of the space X which has no functionally open (open) shrinking of order $\leq n$. Moreover, one can assume – replacing, if necessary, $\mathcal{V}$ by a suitable shrinking – that the cover $\mathcal{V}$ is a swelling of each of its functionally open (open) shrinkings. In fact, if the members $V_1, V_2, \ldots, V_k$ of $\mathcal{V}$ have non-empty intersection and $\mathcal{V}$ has a functionally open (open) shrinking, the corresponding members of which have empty intersection, then we replace $\mathcal{V}$ by that shrinking; after finitely many steps we obtain a cover with the required property. As $\operatorname{ord} \mathcal{V} > n$, we can assume that $\mathcal{V} = \{V_i\}_{i=1}^{m}$, where $\bigcap_{i=1}^{n+2} V_i \neq \emptyset$.

Now, consider the $(n+2)$-element functionally open (open) cover $\{U_i\}_{i=1}^{n+2}$ of the space X, where $U_i = V_i$ for $i = 1, 2, \ldots, n+1$ and $U_{n+2} = \bigcup_{i=n+2}^{m} V_i$. Let $\{W_i\}_{i=1}^{n+2}$ be a functionally open (open) shrinking of $\{U_i\}_{i=1}^{n+2}$. The cover

$$\{W_1, W_2, \ldots, W_{n+1}, W_{n+2} \cap V_{n+2}, \ldots, W_{n+2} \cap V_m\}$$

is a functionally open (open) shrinking of $\mathcal{V}$, so that $\mathcal{V}$ is a swelling of that cover and $\bigcap_{i=1}^{n+2} W_i \supset \bigcap_{i=1}^{n+1} W_i \cap (W_{n+2} \cap V_{n+2}) \neq \emptyset$. ■

It follows from Theorems 7.1.6 and 7.1.7 that the condition in the above lemma is also necessary for the inequality $\dim X \leq n$. Hence, we incidentally obtain a characterization of dimension. This characterization, and the dual characterization which is easily obtained by applying De Morgan's laws, are stated in the following corollary.

7.2.14. COROLLARY. *A Tychonoff (normal) space X satisfies the condition* $\dim X \leq n$ *if and only if every $(n+2)$-element functionally open (open) cover $\{U_i\}_{i=1}^{n+2}$ of the space X has a functionally open (open) shrinking $\{W_i\}_{i=1}^{n+2}$ such that the intersection $\bigcap_{i=1}^{n+2} W_i$ is empty or – equivalently – if and only if for every $(n+2)$-element family $\{B_i\}_{i=1}^{n+2}$ of functionally* *closed (closed) subsets of the space X satisfying $\bigcap_{i=1}^{n+2} B_i = \emptyset$ there exists a functionally closed (closed) cover $\{F_i\}_{i=1}^{n+2}$ of the space X such that $\bigcap_{i=1}^{n+2} F_i = \emptyset$ and $B_i \subset F_i$ for $i = 1, 2, \ldots, n+2$.* ■

7.2.15. THEOREM ON PARTITIONS. *A Tychonoff (normal) space X satisfies the condition $\dim X \leq n \geq 0$ if and only if for every sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint functionally closed (closed) subsets of X there exist functionally closed (closed) sets $L_1, L_2, \ldots, L_{n+1}$ such that L_i is a partition between A_i and B_i and that $L_1 \cap L_2 \cap \ldots \cap L_{n+1} = \emptyset$.*

PROOF. We first prove that the condition in the theorem is necessary for the inequality $\dim X \leq n$. Let X be a Tychonoff (normal) space such that $\dim X \leq n \geq 0$ and let $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ be a sequence of $n+1$ pairs of disjoint functionally closed (closed) subsets of X. The family $\{B_i\}_{i=1}^{n+2}$, where $B_{n+2} = \bigcup_{i=1}^{n+1} A_i$, consists of functionally closed (closed) subsets of X satisfying $\bigcap_{i=1}^{n+2} B_i = \emptyset$; hence, by Corollary 7.2.14 and Theorem 7.1.4, there exists a functionally open (open) cover $\{U_i\}_{i=1}^{n+2}$ of the space X such that $\bigcap_{i=1}^{n+2} U_i = \emptyset$ and $B_i \subset U_i$ for $i = 1, 2, \ldots, n+2$. Since $A_i \subset B_{n+2} \subset U_{n+2}$ for $i = 1, 2, \ldots, n+1$, we can assume – replacing if necessary U_i by $U_i \setminus A_i$ – that

$$A_i \subset X \setminus U_i \quad \text{for} \quad i = 1, 2, \ldots, n+1. \tag{10}$$

Let $\{F_i\}_{i=1}^{n+2}$ be a functionally closed (closed) shrinking of the cover $\{U_i\}_{i=1}^{n+2}$; we can assume – replacing if necessary F_i by $F_i \cup B_i$ – that

$$B_i \subset F_i \quad \text{for} \quad i = 1, 2, \ldots, n+1. \tag{11}$$

By Theorem 1.5.14 there exists for $i = 1, 2, \ldots, n+1$ a continuous function $f_i \colon X \to I$ such that $f_i(X \setminus U_i) \subset \{0\}$ and $f_i(F_i) \subset \{1\}$. By (10) and (11) the functionally closed set $L_i = f^{-1}(1/2) \subset U_i \setminus F_i$ is a partition between A_i and B_i for $i = 1, 2, \ldots, n+1$. Since

$$\bigcap_{i=1}^{n+1} (U_i \setminus F_i) \subset \bigcap_{i=1}^{n+1} (U_i \setminus \bigcup_{i=1}^{n+1} F_i) = (X \setminus \bigcup_{i=1}^{n+1} F_i) \cap \bigcap_{i=1}^{n+1} U_i \subset F_{n+2} \cap \bigcap_{i=1}^{n+1} U_i \subset \bigcap_{i=1}^{n+2} U_i = \emptyset,$$

we have $\bigcap_{i=1}^{n+1} L_i = \emptyset$ which shows that the condition in the theorem is necessary for the inequality $\dim X \leq n$.

Now, consider a Tychonoff (normal) space X that satisfies the condition in the theorem. Let $\{U_i\}_{i=1}^{n+2}$ be an $(n+2)$-element functionally open (open) cover of the space X. Take a functionally closed (closed) shrinking $\{B_i\}_{i=1}^{n+2}$ of the cover $\{U_i\}_{i=1}^{n+2}$ and let $A_i = X \setminus U_i$ for $i = 1, 2, \ldots, n+1$. The sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ consists of $n+1$ pairs of disjoint functionally closed (closed) subsets of X, so that for $i = 1, 2, \ldots, n+1$ there exist sets V_i, W_i, functionally open by Lemma 7.2.12 (open), satisfying

$$A_i \subset V_i, \quad B_i \subset W_i, \quad V_i \cap W_i = \emptyset \quad \text{for} \quad i = 1, 2, \ldots, n+1 \tag{12}$$

and such that $\bigcap_{i=1}^{n+1} [X \setminus (V_i \cup W_i)] = X \setminus \bigcup_{i=1}^{n+1} (V_i \cup W_i) = \emptyset$, i.e.,

$$\bigcup_{i=1}^{n+1} V_i \cup \bigcup_{i=1}^{n+1} W_i = X. \tag{13}$$

From (13), (12) and the inclusion $B_{n+2} \subset U_{n+2}$ it follows that

$$\Big[U_{n+2} \cap \bigcup_{i=1}^{n+1} V_i\Big] \cup \bigcup_{i=1}^{n+1} W_i = \Big[U_{n+2} \cup \bigcup_{i=1}^{n+1} W_i\Big] \cap \Big[\bigcup_{i=1}^{n+1} V_i \cup \bigcup_{i=1}^{n+1} W_i\Big] \supset \bigcup_{i=1}^{n+2} B_i = X;$$

hence, the family $\{W_i\}_{i=1}^{n+2}$, where

$$W_{n+2} = U_{n+2} \cap \bigcup_{i=1}^{n+1} V_i,$$

is a functionally open (open) shrinking of the cover $\{U_i\}_{i=1}^{n+2}$. Moreover, it follows from (12) that

$$\bigcap_{i=1}^{n+2} W_i = \bigcap_{i=1}^{n+1} W_i \cap \Big[U_{n+2} \cap \bigcup_{i=1}^{n+1} V_i\Big] \subset \bigcap_{i=1}^{n+1} W_i \cap \bigcup_{i=1}^{n+1} V_i = \emptyset,$$

so that $\dim X \leq n$ by virtue of Lemma 7.2.13.∎

Historical and bibliographic notes

Theorem 7.2.1 was proved by Čech in [1933]; the present proof is due to Chaber (cited in Engelking [1973]). The countable sum theorem for the dimension ind was proved by Menger in [1924] and by Urysohn in [1926] (announcement [1922]) for compact metrizable spaces, and was generalized to separable metrizable spaces by Tumarkin in [1926] (announcement in [1925]) and by Hurewicz in [1927]. Lemma 7.2.2 was proved by Katětov in [1952]; Theorem 7.2.3 was obtained independently by Morita in [1950a] and Katětov in [1952]. Theorem 7.2.4 was proved by Dowker in [1947]. Theorem 7.2.7 was proved by Menger in [1924] and by Urysohn in [1926] (announcement [1922]) for compact metrizable spaces, and was generalized to separable metrizable spaces by Hurewicz in [1927b]; for compact spaces the inequality $\dim X \leq \operatorname{ind} X$ was established by Alexandroff in [1941], the generalization to Lindelöf spaces is due to Morita [1950] and Smirnov [1951b]. Theorems 7.2.8 and 7.2.9 are both due to Vedenissoff, the former was proved in [1939] and the latter in [1941]. Theorem 7.2.15 was established by Eilenberg and Otto in [1938] for separable metrizable spaces; generalization to normal spaces appeared in Hemmingsen's paper [1946] (which contains also Corollary 7.2.14). Our proof of that theorem is taken from Holsztyński's paper [1966]. The importance of Theorem 7.2.15 consists in the fact that – via the theorem on extending mappings into S^n (see Problem 7.4.13) – it leads to a homological characterization of dimension.

The countable sum theorem for the dimension Ind holds in perfectly normal spaces (Čech [1932] (announcement [1931])) and in hereditarily paracompact spaces (Dowker [1955]), the locally finite sum theorem for dimension Ind also holds in the above two classes (Kimura [1963]); proofs can be found in Engelking [1978]. In the class of compact spaces even the finite sum theorem for the dimension Ind does not hold (see Problem 7.4.5).

The dimension ind does not satisfy the finite sum theorem either in compact spaces (see Problem 7.4.5; a simpler example of a normal space which does not satisfy that theorem is given in Exercise 7.2.C) or in metrizable spaces. The last fact is – as observed by van Douwen in [1973] and by Przymusiński in [1974] – a simple consequence of the existence of a metrizable space X such that $\operatorname{ind} X \neq \operatorname{Ind} X$ (see the notes to Section 7.3).

The small induction dimension ind – with which dimension theory was launched – proved later to be relatively less important when compared with dimensions dim and Ind.

Exercises

7.2.A. Let X be a Tychonoff space and let M be a subspace of X which satisfies $\dim M \leq n$ and has the property that every continuous function $f: M \to I$ is continuously extendable over X. Show that for every functionally open cover $\{U_i\}_{i=1}^k$ of the space X there exists a family $\{V_i\}_{i=1}^k$ of functionally open sets and a functionally closed set F such that

$$V_i \subset U_i \quad \text{for} \quad i = 1, 2, \ldots, k, \quad M \subset F \subset \bigcup_{i=1}^{k} V_i \quad \text{and} \quad \operatorname{ord}(\{V_i\}_{i=1}^k) \leq n.$$

7.2.B (Katětov [1950a]). Show that if a Tychonoff space X has a countable closed cover $\{M_j\}_{j=1}^\infty$ such that $\dim M_j \leq n$ for $j = 1, 2, \ldots$ and for each j every continuous function $f: M_j \to I$ is continuously extendable over X, then $\dim X \leq n$.

Hint. Modify the proof of Theorem 7.2.1 and apply Exercise 7.2.A.

One can also apply Theorem 7.1.17 and the fact that every σ-compact space is normal.

7.2.C (Przymusiński [1974]). Let Y and Y^* be the spaces defined in Example 6.2.20. Denote by X the space obtained by identifying to points closed subsets $W \times \{0\}$ and $W \times \{1\}$ of the subspace $Y \cup (\{\omega_1\} \times \{0, 1\})$ of Y^*. Show that X is a normal space, $\operatorname{ind} X = 1$ and yet X can be represented as the union $X = X_1 \cup X_2$, where X_1, X_2 are closed and $\operatorname{ind} X_1 = 0 = \operatorname{ind} X_2$ (cf. Problem 7.4.5).

7.2.D. (a) (Dowker [1947]) Observe that a normal space X satisfies the inequality $\dim X \leq n$ if and only if every star-finite open cover of the space X has a star-finite open refinement of order $\leq n$.

(b) (Pasynkov [1965]) Show that a Tychonoff space X satisfies the inequality $\dim X \leq n$ if and only if every locally finite functionally open cover of the space X has a locally finite functionally open refinement of order $\leq n$.

Hint (R. Pol, cited in Engelking [1973]). Take a locally finite functionally open cover $\{U_s\}_{s\in S}$ of the space X; consider a functionally closed shrinking $\{F_s\}_{s\in S}$ of $\{U_s\}_{s\in S}$ (cf. Exercise 7.1.B) and continuous functions $f_s: X \to I$ such that $f_s(F_s) \subset \{1\}$ and $f_s(X \setminus U_s) \subset \{0\}$. Define a pseudometric ρ on the set X by letting $\rho(x, y) = \sum_{s\in S} |f_s(x) - f_s(y)|$ and consider the mapping $f: X \to Y = X/\rho$ (see Exercise 4.2.I). Take the extension $F: \beta X \to \beta Y$ of the mapping f and apply Theorems 5.1.3, 5.1.35, 7.1.17 and 7.2.4.

To define f one can also use Exercise 5.1.J(b).

7.2.E. Observe that every strongly zero-dimensional paracompact space is strongly paracompact.

7.2.F (Morita [1950a]). Strengthen Theorem 7.2.7 by showing that the inequality $\dim X \leq \operatorname{ind} X$ holds for every strongly paracompact space X.

Hint. Modify Lemma 7.2.5, apply Lemmas 5.3.8, 5.3.9 and Theorem 7.1.9.

7.2.G (A. H. Stone [1962a]). Prove that every strongly zero-dimensional completely metrizable space X of weight $\mathfrak{m} > \aleph_0$ such that $w(U) = \mathfrak{m}$ for every non-empty open set $U \subset X$ is homeomorphic to the Baire space $B(\mathfrak{m})$ (cf. Exercises 6.2.A(b) and 4.3.G).

Hint. Modify the construction in the hint to Exercise 4.3.G; apply Exercise 4.1.H(a) and Theorem 7.2.4.

7.3. Dimension of metrizable spaces

We start with a theorem characterizing the dimension dim of metrizable spaces in terms of special sequences of covers. The characterizations in that theorem will be applied in the proof of the Katětov-Morita theorem, stating that $\operatorname{Ind} X = \dim X$ for every metrizable space X, which is one of the most important theorems in dimension theory. Let us note that conditions (ii) and (iii) in Theorem 7.3.1 are stated in terms of metrics; purely topological characterizations of a similar type are formulated in Exercise 7.3.A below.

7.3.1. THEOREM. *For every metrizable space X the following conditions are equivalent:*

(i) *The space X satisfies the inequality* $\dim X \leq n$.

(ii) *For every metric ρ on the space X there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of locally finite open covers of the space X such that* $\operatorname{ord} \mathcal{U}_i \leq n$, *$\delta(U) < 1/i$ for $U \in \mathcal{U}_i$, and for each $U \in \mathcal{U}_{i+1}$ there exists a $V \in \mathcal{U}_i$ that contains $\overline{U}$.*

(iii) *There exists a metric ρ on the space X and a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers of the space X such that* $\operatorname{ord} \mathcal{W}_i \leq n$, *$\delta(W) < 1/i$ for $W \in \mathcal{W}_i$, and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$.*

PROOF. We shall show first that (i)$\Rightarrow$(ii). The sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ will be defined by induction. Assume that $k = 1$ or $k > 1$ and the covers $\mathcal{U}_i$ are already defined for $i < k$. For every $x \in X$ take a neighbourhood U_x of the point x such that $\delta(U_x) < 1/k$ and $\overline{U}_x$ is contained in a member of $\mathcal{U}_{k-1}$, if $k > 1$. It follows from Theorems 5.1.3 and 7.2.4 that the cover $\{U_x\}_{x \in X}$ of the space X has a locally finite open refinement $\mathcal{U}_k$ of order $\leq n$. The sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ obtained in this way satisfies (ii).

As the implication (ii)$\Rightarrow$(iii) is obvious, it remains to show that (iii)$\Rightarrow$(i).

For every i take a mapping f_i^{i+1} of $\mathcal{W}_{i+1}$ to $\mathcal{W}_i$ such that $W \subset f_i^{i+1}(W)$ for each $W \in \mathcal{W}_{i+1}$; let $f_i^k = f_i^{i+1} f_{i+1}^{i+2} \ldots f_{k-1}^k$ for $i < k$ and let $f_i^i(W) = W$ for $W \in \mathcal{W}_i$. Obviously

$$(1) \qquad W \subset f_i^k(W) \quad \text{for every} \quad W \in \mathcal{W}_k \quad \text{and} \quad i \leq k.$$

Let $\{H_j\}_{j=1}^l$ be a finite open cover of the space X. The sets $X_1, X_2, \ldots$, where

$$(2) \qquad X_k = \bigcup \{W \in \mathcal{W}_k : \text{ there exists a } j \leq l \text{ such that } \operatorname{St}(W, \mathcal{W}_k) \subset H_j\},$$

form an open cover of the space X. Consider the subfamilies

$$\mathcal{U}_k = \{U \in \mathcal{W}_k : \ U \cap X_k \neq \emptyset\} \quad \text{and} \quad \mathcal{V}_k = \{V \in \mathcal{U}_k : \ V \cap (\bigcup_{j<k} X_j) = \emptyset\}$$

of the cover $\mathcal{W}_k$, where $k = 1, 2, \ldots$ For every $U \in \mathcal{U}_k$ we have $f_k^k(U) \cap X_k = U \cap X_k \neq \emptyset$; denote by $i(U)$ the smallest integer $i \leq k$ such that $f_i^k(U) \cap X_i \neq \emptyset$. Since $\mathcal{V}_1 = \mathcal{U}_1$ and $f_{i(U)}^k(U) \subset f_{i(U)-1}^k(U) \subset \ldots \subset f_1^k(U)$ whenever $i(U) > 1$, we have

$$(3) \qquad f_{i(U)}^k(U) \in \mathcal{V}_{i(U)}.$$

For every $V \in \mathcal{V}_i$ take the open set

$$(4) \qquad V^* = \bigcup_{k=i}^{\infty} \bigcup \{U \cap X_k : U \in \mathcal{U}_k,\ f_i^k(U) = V \text{ and } i(U) = i\}.$$

As $V \cap X_i \neq \emptyset$, by (2) there exist a $W \in \mathcal{W}_i$ such that $V \cap W \neq \emptyset$ and a $j(V) \leq l$ satisfying $V \subset \operatorname{St}(W, \mathcal{W}_i) \subset H_{j(V)}$. From (1) we have $V^* \subset V$, so that $V^* \subset H_{j(V)}$. Since $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ whenever $i \neq j$, for every $V \in \mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ the set V^* and the integer $j(V)$ are well-defined.

To conclude the proof it suffices to show that the family $\{V_j\}_{j=1}^{l}$, where $V_j = \bigcup\{V^* : V \in \mathcal{V} \text{ and } j(V) = j\} \subset H_j$, is a cover of X and has order $\leq n$, or – equivalently – that $\mathcal{V}^* = \{V^* : V \in \mathcal{V}\}$ is a cover of X and has order $\leq n$.

For any $x \in X$ take an integer k such that

$$(5) \qquad x \in X_k \setminus \bigcup_{j<k} X_j$$

and a set $U \in \mathcal{W}_k$ which contains the point x; since $U \cap X_k \neq \emptyset$, $U \in \mathcal{U}_k$. It follows from (3) and (4) that $x \in U \cap X_k \subset (f_{i(U)}^k(U))^* \in \mathcal{V}^*$, so that $\mathcal{V}^*$ is a cover of the space X.

Consider a non-empty intersection $V_1^* \cap V_2^* \cap \ldots \cap V_h^*$, where $V_i \in \mathcal{V}_{m_i}$ and $V_i \neq V_j$ whenever $i \neq j$, and take a point x in this intersection. It follows from the definition of $\mathcal{V}_{m_i}$ that the integer k in (5) satisfies $m_i \leq k$ for $i = 1, 2, \ldots, h$. By (4) there exist sets $U_1, U_2, \ldots, U_h$ such that $U_i \in \mathcal{U}_{k_i}$, $f_{m_i}^{k_i}(U_i) = V_i$, $i(U_i) = m_i$ and $x \in U_i \cap X_{k_i}$. Since $x \in X_{k_i}$ it follows from (5) that $k \leq k_i$. The sets $W_1, W_2, \ldots, W_h$, where $W_i = f_k^{k_i}(U_i) \in \mathcal{W}_k$, all contain the point x, so that – as $\operatorname{ord} \mathcal{W}_k \leq n$ – it suffices to show that $W_i \neq W_j$ whenever $i \neq j$. Let us observe that the sets W_i belong to $\mathcal{U}_k$ and that $i(W_i) = i(U_i) = m_i$ by the definition of $i(U_i)$; hence, $W_i \neq W_j$ whenever $m_i \neq m_j$. When $m_i = m_j$, we also have $W_i \neq W_j$, because then

$$f_{m_i}^k(W_i) = f_{m_i}^{k_i}(U_i) = V_i \neq V_j = f_{m_j}^{k_j}(U_j) = f_{m_j}^k(W_j). \blacksquare$$

7.3.2. THE KATĚTOV–MORITA THEOREM. *For every metrizable space X we have* $\operatorname{Ind} X = \dim X$.

PROOF. By virtue of Theorem 7.2.8 it suffices to show that $\operatorname{Ind} X \leq \dim X$; obviously, we can suppose that $\dim X < \infty$. We shall apply induction with respect to $\dim X$. If $\dim X = -1$, we have $X = \emptyset$ and $\operatorname{Ind} X \leq \dim X$. Assume that our inequality holds for all metrizable spaces with covering dimension $\leq n-1$ and consider a metrizable space X such that $\dim X = n$, a closed set $A \subset X$ and an open set $V \subset X$ that contains A.

Let σ be an arbitrary metric on the space X and let $f: X \to I$ be a continuous function satisfying $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$, where $B = X \setminus V$. The formula $\rho(x,y) = \sigma(x,y) + |f(x) - f(y)|$ defines a metric on the space X, and by Theorem 7.3.1 there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of locally finite open covers of the space X such that $\operatorname{ord} \mathcal{U}_i \leq n$, $\delta(U) < 1/i$ for $U \in \mathcal{U}_i$, where δ denotes the diameter with respect to ρ, and for any $U \in \mathcal{U}_{i+1}$ there exists a $V \in \mathcal{U}_i$ that contains $\overline{U}$.

Let $K_0 = A$, $M_0 = B$, and for $i \geq 1$ let $K_i = X \setminus H_i$, and $M_i = X \setminus G_i$, where

$$G_i = \bigcup\{U \in \mathcal{U}_i : \overline{U} \cap M_{i-1} = \emptyset\} \quad \text{and} \quad H_i = \bigcup\{U \in \mathcal{U}_i : \overline{U} \cap M_{i-1} \neq \emptyset\}.$$

In this way two sequences $K_0, K_1, \ldots$ and $M_0, M_1, \ldots$ are defined. Let us observe that

(6) $$\text{if } U \in \mathcal{U}_i \text{ and } \overline{U} \cap M_{i-1} \neq \emptyset, \text{ then } \overline{U} \cap K_{i-1} = \emptyset.$$

The validity of (6) for $i = 1$ follows from the definition of ρ, because no set of diameter less than 1 meets both A and B. If $U \in \mathcal{U}_i$ with $i > 1$ and $\overline{U} \cap M_{i-1} \neq \emptyset$, then for any $V \in \mathcal{U}_{i-1}$ that contains $\overline{U}$ we have $V \cap M_{i-1} \neq \emptyset$, so that V is not contained in G_{i-1}; this implies that $V \subset H_{i-1}$ which gives the equality $\overline{U} \cap K_{i-1} = \emptyset$.

From the local finiteness of $\mathcal{U}_i$, the definitions of G_i and H_i, and from (6) it follows that $\overline{G}_i \cap M_{i-1} = \emptyset = \overline{H}_i \cap K_{i-1}$ for $i = 1, 2, \ldots$ and this implies that $K_{i-1} \subset X \setminus \overline{H}_i = \operatorname{Int} K_i$ and $M_{i-1} \subset X \setminus \overline{G}_i = \operatorname{Int} M_i$; moreover, as $G_i \cup H_i = X$, we have $K_i \cap M_i = \emptyset$. Hence, the sets $K = \bigcup_{i=0}^{\infty} K_i$ and $M = \bigcup_{i=0}^{\infty} M_i$ are open, disjoint and contain A and B respectively. Thus we have $A \subset K \subset V$ and $\operatorname{Fr} K \subset L = X \setminus (K \cup M)$. To conclude the proof it suffices to show that $\dim L \leq n-1$, because – by Theorem 7.1.3 and the inductive assumption – this implies that $\operatorname{Ind} \operatorname{Fr} K \leq \operatorname{Ind} L \leq \dim L \leq n-1$.

Since $K \cup M = \bigcup_{i=1}^{\infty}(K_i \cup M_i)$, we have $L = \bigcap_{i=1}^{\infty} L_i$, where $L_i = X \setminus (K_i \cup M_i) = G_i \cap H_i$. For $i = 1, 2, \ldots$ the family

$$\mathcal{W}_i = \{U \cap L : \ U \in \mathcal{U}_i \text{ and } \overline{U} \cap M_{i-1} \neq \emptyset\}$$

is an open cover the space $L \subset H_i$ and $\operatorname{ord} \mathcal{W}_i \leq n-1$, because any point $x \in L \subset L_i \subset G_i$ belongs to at least one $U \in \mathcal{U}_i$ satisfying $\overline{U} \cap M_{i-1} = \emptyset$. If $U \in \mathcal{U}_{i+1}$ and $\overline{U} \cap M_i \neq \emptyset$, then for any $V \in \mathcal{U}_i$ that contains $\overline{U}$ we have $V \cap M_i \neq \emptyset$, so that V is not contained in G_i which implies that $\overline{V} \cap M_{i-1} \neq \emptyset$, i.e., that $V \cap L \in \mathcal{W}_i$. Hence $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ and, as obviously $\delta(W) < 1/i$ for $W \in \mathcal{W}_i$, $\dim L \leq n-1$ by virtue of Theorem 7.3.1.■

Theorems 7.1.2, 7.2.7, 7.3.2 and Corollary 4.1.16 yield

7.3.3. THEOREM. *For every separable metrizable space X we have* $\operatorname{ind} X = \operatorname{Ind} X = \dim X$.■

In connection with Theorems 7.3.2 and 7.3.3, the question arises whether the small inductive dimension of a metrizable space can differ from the large inductive dimension and the covering dimension of that space. It turns out that there exists a completely metrizable space X such that $\operatorname{ind} X = 0$, and yet $\operatorname{Ind} X = \dim X = 1$. However, the definition of that space and computation of its dimensions ind and Ind are quite difficult and will not be discussed in this book.

Since every subspace of a space X which satisfies condition (iii) in Theorem 7.3.1 also satisfies that condition, Theorems 7.3.1 and 7.3.2 yield

7.3.4. THEOREM. *For every subspace M of a metrizable space X we have* $\operatorname{Ind} M \leq \operatorname{Ind} X$.■

Let us observe that from Theorems 7.2.1, 7.2.3 and 7.3.2 we obtain

7.3.5. THE σ-LOCALLY FINITE SUM THEOREM. *If a metrizable space X has a σ-locally finite closed cover $\{F_s\}_{s \in S}$ such that* $\operatorname{Ind} F_s \leq n$ *for every $s \in S$, then* $\operatorname{Ind} X \leq n$.■

We shall now prove a theorem characterizing the dimension Ind of metrizable spaces in terms of special bases and in terms of decompositions into sets of smaller dimension.

7.3.6. LEMMA. *If a metrizable space X has a σ-locally finite base $\mathcal{B}$ consisting of open-and-closed sets, then* $\operatorname{Ind} X \leq 0$.

PROOF. Let $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$, where the families $\mathcal{B}_i$ are locally finite. Consider a closed set $A \subset X$ and an open set $V \subset X$ that contains A; for $i = 1, 2, \ldots$ let

$$(7) \qquad V_{2i} = \bigcup\{U \in \mathcal{B}_i : U \subset V\} \quad \text{and} \quad V_{2i+1} = \bigcup\{U \in \mathcal{B}_i : U \cap A = \emptyset\}.$$

The above sets are open-and-closed and satisfy the conditions

$$\bigcup_{i=1}^{\infty} V_{2i} = V \quad \text{and} \quad \bigcup_{i=1}^{\infty} V_{2i+1} = X \setminus A,$$

so that $\{V_i\}_{i=1}^{\infty}$ is a cover of the space X. The family $\{U_i\}_{i=1}^{\infty}$, where $U_i = V_i \setminus \bigcup_{j<i} V_j$, is a cover of X consisting of pairwise disjoint open-and-closed sets; it follows from (7) that the set $U = \bigcup\{U_i : U_i \subset V\}$ is open-and-closed and satisfies the inclusions $A \subset U \subset V$. ∎

7.3.7. LEMMA. *Let X be a metrizable space and Z a subspace of X. If $\operatorname{Ind} Z \le 0$, then for every closed set $A \subset X$ and any open set $V \subset X$ that contains A there exists an open set $U \subset X$ such that $A \subset U \subset \overline{U} \subset V$ and $Z \cap \operatorname{Fr} U = \emptyset$.*

PROOF. Take open sets $W_1, W_2 \subset X$ satisfying the inclusions $A \subset W_1 \subset \overline{W}_1 \subset W_2 \subset \overline{W}_2 \subset V$. As $\operatorname{Ind} Z = \emptyset$, there exists a set U_0, open-and-closed in Z, such that $Z \cap \overline{W}_1 \subset U_0 \subset Z \cap W_2$. Clearly, we have

$$Z \setminus U_0 \subset Z \setminus (Z \cap W_1) = Z \setminus W_1 \subset X \setminus W_1 \quad \text{and} \quad \overline{U}_0 \subset \overline{W}_2.$$

One readily verifies that the sets $A \cup U_0$ and $[(X \setminus V) \cup (Z \setminus U_0)]$ are separated. By virtue of Corollary 4.1.13 and Theorem 2.1.7, there exists an open set $U \subset X$ such that $A \cup U_0 \subset U$ and $\overline{U} \cap [(X \setminus V) \cup (Z \setminus U_0)] = \emptyset$; clearly, the set U has the required properties. ∎

7.3.8. THEOREM. *For every metrizable space X and any integer $n \ge 0$ the following conditions are equivalent:*

(i) *The space X satisfies the inequality $\operatorname{Ind} X \le n$.*

(ii) *The space X has a σ-locally finite base $\mathcal{B}$ such that $\operatorname{Ind} \operatorname{Fr} U \le n-1$ for every $U \in \mathcal{B}$.*

(iii) *$X = Y \cup Z$, where $\operatorname{Ind} Y \le n-1$ and $\operatorname{Ind} Z \le 0$.*

PROOF. We shall show that (i)⇒(ii). Let X be a metrizable space with $\operatorname{Ind} X \le n$ and let ρ be an arbitrary metric on the space X. It follows from Theorem 5.1.3 that for $i = 1, 2, \ldots$ the space X has a locally finite open cover $\mathcal{V}_i = \{V_s\}_{s \in S_i}$ consisting of sets of diameter less than $1/i$. By virtue of Theorem 1.5.18, for every i there exists a closed shrinking $\{F_s\}_{s \in S_i}$ of the cover $\mathcal{V}_i$. As $\operatorname{Ind} X \le n$, for every $s \in S_i$ there exists an open set $U_s \subset X$ such that

$$F_s \subset U_s \subset V_s \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr} U_s \le n-1.$$

The family $\mathcal{B}_i = \{U_s\}_{s \in S_i}$ is a locally finite open cover of the space X consisting of sets of diameter less than $1/i$; hence, $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a σ-locally finite base for the space X.

To prove the implication (ii)⇒(iii) observe that by virtue of Theorem 7.3.5 the subspace $Y = \bigcup\{\operatorname{Fr} U : U \in \mathcal{B}\} \subset X$ satisfies the inequality $\operatorname{Ind} Y \le n-1$ and that the subspace $Z = X \setminus Y$ has a σ-locally finite base $\{Z \cap U : U \in \mathcal{B}\}$ consisting of open-and-closed sets, so that $\operatorname{Ind} Z \le 0$ by Lemma 7.3.6.

The implication (iii)⇒(i) follows readily from Lemma 7.3.7 and Theorem 7.1.3. ∎

The last theorem yields

7.3.9. THE DECOMPOSITION THEOREM. *A metrizable space X satisfies the inequality* $\operatorname{Ind} X \leq n \geq 0$ *if and only if* $X = Z_1 \cup Z_2 \cup \ldots \cup Z_{n+1}$, *where* $\operatorname{Ind} Z_i \leq 0$ *for* $i = 1, 2, \ldots, n+1$. ∎

The decomposition theorem yields in turn

7.3.10. THE ADDITION THEOREM. *For every pair X, Y of subspaces of a metrizable space we have*

$$\operatorname{Ind}(X \cup Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y + 1. \blacksquare$$

Our next theorem is a generalization of Lemma 7.3.7.

7.3.11. THE SEPARATION THEOREM. *Let X be a metrizable space and M a subspace of X. If* $\operatorname{Ind} M \leq n \geq 0$, *then for every closed set $A \subset X$ and any open set $V \subset X$ that contains A there exists an open set $U \subset X$ such that $A \subset U \subset \overline{U} \subset V$ and* $\operatorname{Ind}(M \cap \operatorname{Fr} U) \leq n-1$.

PROOF. By Theorem 7.3.8, we have $M = Y \cup Z$, where $\operatorname{Ind} Y \leq n-1$ and $\operatorname{Ind} Z \leq 0$. Now, the open set U satisfying Lemma 7.3.7 also satisfies our theorem, because $M \cap \operatorname{Fr} U \subset M \setminus Z \subset Y$ and thus $\operatorname{Ind}(M \cap \operatorname{Fr} U) \leq n-1$ by Theorem 7.1.3. ∎

From the separation theorem we obtain the following corollary which, by the Katětov-Morita theorem, is a particular case of Theorem 7.2.15.

7.3.12. COROLLARY. *For every sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of a metrizable space X with* $\operatorname{Ind} X \leq n \geq 0$ *there exist open sets $U_1, U_2, \ldots, U_{n+1}$ such that $A_i \subset U_i \subset \overline{U}_i \subset X \setminus B_i$ for $i = 1, 2, \ldots, n+1$ and* $\operatorname{Fr} U_1 \cap \operatorname{Fr} U_2 \cap \ldots \cap \operatorname{Fr} U_{n+1} = \emptyset$. ∎

7.3.13. EXAMPLE. From either 7.3.3 and 7.1.19 or 7.3.3 and 7.2.10 it follows that $\operatorname{Ind} R^n \leq n$, $\operatorname{Ind} S^n \leq n$ and $\operatorname{Ind} I^n \leq n$ for any natural number n.

The decomposition of R^n into $n+1$ zero-dimensional set (see Theorem 7.3.9) is given – according to Example 7.2.11 – by the formula

$$R^n = Q_0^n \cup Q_1^n \cup \ldots \cup Q_n^n.$$

The last formula also yields another proof of the inequality $\operatorname{Ind} R^n \leq n$. ∎

7.3.14. EXAMPLE. We shall show that the Baire space $B(\mathfrak{m})$, defined in Example 4.2.12, satisfies the equality $\operatorname{Ind} B(\mathfrak{m}) = 0$ for every $\mathfrak{m} \geq \aleph_0$.

Consider a pair of points $x = \{x_i\}$ and $y = \{y_i\}$ of $B(\mathfrak{m})$ and a real number r satisfying $0 < r \leq 1$. If the intersection $B(x, r) \cap B(y, r)$ is non-empty, then there exists a point $z = \{z_i\} \in B(\mathfrak{m})$ such that $x_1 = z_1 = y_1$, $x_2 = z_2 = y_2, \ldots, x_k = z_k = y_k$, where k is the integer satisfying $1/(k+1) < r \leq 1/k$, so that we have $B(x, r) = B(y, r)$. Hence, in $B(\mathfrak{m})$ two r-balls either are disjoint or coincide. In particular, for every natural number i the family of all $1/i$-balls is a cover of $B(\mathfrak{m})$ consisting of pairwise disjoint open-and-closed sets; hence, by virtue of Lemma 7.3.6, we have $\operatorname{Ind} B(\mathfrak{m}) = 0$. ∎

7.3.15. THEOREM. *The Baire space $B(\mathfrak{m})$ is universal for all metrizable spaces X such that* $\operatorname{Ind} X = 0$ *and* $w(X) = \mathfrak{m} \geq \aleph_0$.

PROOF. By the last example, it suffices to show that every metrizable space X such that $\operatorname{Ind} X = 0$ and $w(X) = \mathfrak{m}$ is embeddable in $B(\mathfrak{m})$.

Let ρ be an arbitrary metric on the space X. From Theorem 7.1.10 and the implication (i)$\Rightarrow$(ii) in Theorem 7.3.1 it follows that for $i = 1, 2, \ldots$ there exists a cover $\mathcal{V}_i$ of X consisting of pairwise disjoint open-and-closed sets of diameter less than $1/i$ (such cover can be also defined directly: according to Theorem 7.3.8 there exists a σ-locally finite base $\mathcal{B}$ for the space X which consists of open-and-closed sets; the family $\mathcal{B}_i = \{U \in \mathcal{B} : \delta(U) < 1/i\}$ can be represented as the union of locally finite families $\mathcal{B}_{i,1}, \mathcal{B}_{i,2}, \ldots$; arrange the members of $\mathcal{B}_i$ into a transfinite sequence $U_1, U_2, \ldots, U_\alpha, \ldots$, $\alpha < \xi$ taking first all members of $\mathcal{B}_{i,1}$, then all members of $\mathcal{B}_{i,2}$, etc.; clearly, for every $\alpha < \xi$ the family $\{U_\beta\}_{\beta<\alpha}$ is locally finite, so that the family $\mathcal{V}_i = \{V_\alpha\}_{\alpha<\xi}$, where $V_\alpha = U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$, is a cover of X consisting of pairwise disjoint open-and-closed sets of diameter less than $1/i$). By virtue of Theorem 1.1.14, the cover $\mathcal{V}_i$ has a subcover $\{V_s\}_{s \in S_i}$, where $|S_i| \leq \mathfrak{m}$.

Without loss of generality, we can assume that the set S_i is a subset of the discrete space X_i of cardinality $\mathfrak{m}$ in the definition of $B(\mathfrak{m})$ as $\prod_{i=1}^{\infty} X_i$. By assigning to any point $x \in X$ the index $s \in S_i$ such that $x \in V_s$, we define a continuous mapping $f_i: X \to X_i$. One readily sees that the family $\{f_i\}_{i=1}^{\infty}$ separates points and closed sets, so that by Theorem 2.3.20 the diagonal $F = \triangle_{i=1}^{\infty} f_i: X \to B(\mathfrak{m})$ is a homeomorphic embedding. ■

Since the Cartesian product of $\aleph_0$ copies of $B(\mathfrak{m})$ is homeomorphic to $B(\mathfrak{m})$, Theorems 7.3.4 and 7.3.15 yield

7.3.16. THEOREM. *The Cartesian product* $X = \prod_{i=1}^{\infty} X_i$ *of metrizable spaces satisfies the condition* $\operatorname{Ind} X = 0$ *if and only if* $\operatorname{Ind} X_i = 0$ *for* $i = 1, 2, \ldots$ ■

We shall prove one more theorem on the dimension of the Cartesian product of metrizable spaces.

7.3.17. THEOREM. *For every pair* X, Y *of metrizable spaces of which at least one is non-empty we have*

$$\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y.$$

PROOF. The inequality is obvious if $\operatorname{Ind} X = \infty$ or $\operatorname{Ind} Y = \infty$, so that we can suppose that both $\operatorname{Ind} X$ and $\operatorname{Ind} Y$ are finite. We shall apply induction with respect to the number $\operatorname{Ind} X + \operatorname{Ind} Y$. If $\operatorname{Ind} X + \operatorname{Ind} Y = -1$, then either $X = \emptyset$ or $Y = \emptyset$ and our inequality holds. Assume that the inequality is proved for every pair of merizable spaces the sum of the large inductive dimensions of which is not larger than $k - 1 \geq -1$ and consider metrizable spaces X and Y such that $\operatorname{Ind} X = m \geq 0$, $\operatorname{Ind} Y = n \geq 0$, and $m + n = k$. By Theorem 7.3.8 the space X has a base $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$, where the families $\mathcal{C}_i$ are locally finite and $\operatorname{Ind} \operatorname{Fr} U \leq m - 1$ for every $U \in \mathcal{C}$. Similarly, the space Y has a base $\mathcal{D} = \bigcup_{j=1}^{\infty} \mathcal{D}_j$, where the families $\mathcal{D}_j$ are locally finite and $\operatorname{Ind} \operatorname{Fr} V \leq n - 1$ for every $V \in \mathcal{D}$. The family

$$\mathcal{B}_{i,j} = \{U \times V : U \in \mathcal{C}_i \text{ and } V \in \mathcal{D}_j\}$$

consists of open subsets of $X \times Y$ and is locally finite. As

$$\operatorname{Fr}(U \times V) \subset (X \times \operatorname{Fr} V) \cup (\operatorname{Fr} U \times Y),$$

for every $(U \times V) \in \mathcal{B}_{i,j}$ we have $\operatorname{Ind} \operatorname{Fr}(U \times V) \leq k-1$ by virtue of Theorem 7.3.5 and by the inductive assumption. The family $\mathcal{B} = \bigcup_{i,j=1}^{\infty} \mathcal{B}_{i,j}$ is a base for $X \times Y$, so that $\operatorname{Ind}(X \times Y) \leq k = m + n$ by Theorem 7.3.8.■

Let us note that the inequality in the last theorem cannot be replaced by an equality: there exist separable metrizable spaces X and Y such that $\operatorname{Ind}(X \times Y) < \operatorname{Ind} X + \operatorname{Ind} Y$ (see Exercise 7.3.G); there exist even compact metrizable spaces with similar property, but they are much more complicated.

Let us also observe that, by virtue of the Katětov-Morita theorem, all theorems in this section beginning with Theorem 7.3.4 also hold for the covering dimension dim.

We conclude this section by showing that the dimensions ind, Ind and dim of Euclidean n-space R^n (and also of the n-sphere S^n and the n-cube I^n) are equal to n. This fact justifies the definitions of our three dimension functions, because any dimension function assigning to R^n a number distinct from n would contradict the intuitive notion of dimension and thus would not be acceptable. The spaces R^n, S^n and I^n are our first examples of spaces of dimension larger than one; so far we have not shown that such spaces exist.

The proof that the space R^n has topological dimension n requires a deeper insight into the structure of this space; by the nature of things, some combinatorial arguments must appear in it. To preserve the uniformity of arguments used in this book, which are all of point-set character, the equality $\operatorname{ind} R^n = \operatorname{Ind} R^n = \dim R^n = n$ will be deduced (rather easily) from the Brouwer fixed-point theorem (cf. Exercise 7.3.J). The proof of the Brouwer fixed-point theorem also requires some combinatorial or algebraic arguments but the result itself is certainly known to most readers. To make the book complete, we give an elementary combinatorial proof of the Brouwer fixed-point theorem in the appendix to this section.

7.3.18. THE BROUWER FIXED-POINT THEOREM. *For every natural number n and for every continuous mapping $f: I^n \to I^n$ there exists a point $x \in I^n$ such that $f(x) = x$.*

7.3.19. THEOREM. *For every natural number n we have* $\operatorname{ind} R^n = \operatorname{Ind} R^n = \dim R^n = n$.

PROOF. By virtue of Theorems 7.3.3 and 7.1.3 and by Example 7.3.13, it suffices to show that $\operatorname{Ind} I^n > n-1$. Assume that $\operatorname{Ind} I^n \leq n-1$. For $i = 1, 2, \ldots, n$ let

$$A_i = \{\{x_j\} \in I^n : \ x_i = 0\} \quad \text{and} \quad B_i = \{\{x_j\} \in I^n : \ x_i = 1\}.$$

By Corollary 7.3.12 there exist open sets $U_1, U_2, \ldots, U_n$ such that $A_i \subset U_i \subset \overline{U}_i \subset I^n \setminus B_i$ for $i = 1, 2, \ldots, n$ and

$$\operatorname{Fr} U_1 \cap \operatorname{Fr} U_2 \cap \ldots \cap \operatorname{Fr} U_n = \emptyset. \tag{8}$$

It follows from Proposition 2.1.13 that for $i = 1, 2, \ldots, n$ the formulas

$$f_i(x) = \begin{cases} \dfrac{1}{2}\dfrac{\rho(x, \operatorname{Fr} U_i)}{\rho(x, \operatorname{Fr} U_i) + \rho(x, A_i)} + \dfrac{1}{2} & \text{for } x \in \overline{U}_i, \\[2ex] -\dfrac{1}{2}\dfrac{\rho(x, \operatorname{Fr} U_i)}{\rho(x, \operatorname{Fr} U_i) + \rho(x, B_i)} + \dfrac{1}{2} & \text{for } x \in I^n \setminus U_i, \end{cases}$$

where ρ is the natural metric on I^n, define a continuous function $f_i: I^n \to I$. Clearly, we have

$$f_i^{-1}(1/2) = \operatorname{Fr} U_i, \quad f_i(A_i) = \{1\} \quad \text{and} \quad f_i(B_i) = \{0\}. \tag{9}$$

The formulas (8) and (9) imply that the diagonal $f = f_1 \triangle f_2 \triangle \ldots \triangle f_n : I^n \to I^n$ does not assume the value $a = (1/2, 1/2, \ldots, 1/2) \in I^n$. The composition $g: I^n \to I^n$ of f and the projection of $I^n \setminus \{a\}$ from the point a onto the boundary of I^n, i.e., onto the set $B = \bigcup_{i=1}^n (A_i \cup B_i)$, satisfies $g(I^n) \subset B$; by the second and the third parts of (9), we have $g(A_i) \subset B_i$ and $g(B_i) \subset A_i$. The last three inclusions show that $g(x) \neq x$ for every $x \in I^n$ which contradicts the Brouwer fixed-point theorem. Hence, $\operatorname{Ind} I^n = n$.■

7.3.20. COROLLARY. *For every natural number n we have* $\operatorname{ind} S^n = \operatorname{Ind} S^n = \dim S^n = n$ *and* $\operatorname{ind} I^n = \operatorname{Ind} I^n = \dim I^n = n$.■

7.3.21. COROLLARY. *For the Hilbert cube $I^{\aleph_0}$ we have* $\operatorname{ind} I^{\aleph_0} = \operatorname{Ind} I^{\aleph_0} = \dim I^{\aleph_0} = \infty$.■

Historical and bibliographic notes

The equivalence of conditions (i) and (ii) in Theorem 7.3.1 was established by Dowker and Hurewicz in [1956], the equivalence of conditions (i) and (iii) was proved by Vopěnka in [1959]; our proof is a slight adaptation of an argument in Nagami and Roberts' paper [1967]. Let us observe that in (ii) and (iii) the assumption that every cover refines the preceding one is essential: Sitnikov defined in [1953] (see Example 1.10.23 in Engelking [1978]) a separable metric space of dimension two which for any natural number i has an open cover of order one consisting of sets of diameter less than $1/i$ (cf. Exercise 7.3.A(c)).

Theorem 7.3.2 was proved by Katětov in [1952] and by Morita in [1954]; our proof, consisting in combining Theorem 7.3.1 with a simplified argument from Dowker and Hurewicz's paper [1956], was published in Engelking [1973] (another proof, eschewing Theorem 7.3.1, is outlined in Exercise 7.3.C(b)). The equality of ind and Ind for compact metrizable spaces was announced by Brouwer in [1924] (with the comment that this fact also is known to Urysohn); the proof was given by Menger in [1924] (implicitly) and by Urysohn in [1926]. For separable metrizable spaces, the equality of ind and Ind was established by Tumarkin in [1926] (announcement in [1925]) and by Hurewicz in [1927]. The equality of ind and dim for compact metrizable spaces was proved by Urysohn in [1926] (in [1922] he announced the inequality $\dim X \leq \operatorname{ind} X$); it was generalized to separable metrizable spaces by Hurewicz in [1927b]. An example of a completely metrizable space X such that $\operatorname{ind} X = 0$ and $\operatorname{Ind} X = 1$ was outlined by Roy in [1962]; a detailed discussion of that example is contained in Roy's paper [1968]. A simpler (but still difficult) example was given by Kulesza in [1989].

Theorem 7.3.4 was established by Čech in [1932] (announcement in [1931]). The history of sum theorems is discussed in the notes to the previous section. For compact metrizable spaces and the dimension ind, the equivalence of conditions (i) and (iii) in Theorem 7.3.8, as well as Theorems 7.3.9 and 7.3.10, were established by Urysohn in [1926] (announcement [1922]). Generalization to separable metrizable spaces is due to Tumarkin [1926] (announcement in [1925]) and to Hurewicz [1927]. Theorem 7.3.11 was proved for compact metrizable spaces and the dimension ind by Menger in [1924] and was generalized to separable metrizable spaces by Hurewicz in [1927]. For arbitrary metrizable spaces, Theorems 7.3.9-7.3.11 and 7.3.15-7.3.17 were established by Katětov in [1952] and by Morita in [1954]; Theorem 7.3.8 was proved by Morita in [1954]. For separable metrizable spaces and the dimension ind, Theorem 7.3.15 was proved by Sierpiński in [1921], Theorem 7.3.16 by Kuratowski in

[1933], and Theorem 7.3.17 by Menger in [1928]. An example of two-dimensional compact metrizable spaces X and Y such that the Cartesian product $X \times Y$ is three-dimensional was given by Pontrjagin in [1930]. Brouwer's paper [1913] contains the proof of the equality $\operatorname{Ind} R^n = \dim R^n = n$ (cf. the notes to Section 7.1). The equality $\operatorname{ind} R^n = n$ was proved (without using Brouwer's result of which it is an easy consequence) by Menger in [1924] and by Urysohn in [1925b] (announcement in [1922]).

The Brouwer fixed-point theorem was established by Brouwer in [1912]. The proof given in the appendix, as well as Theorem 4 there is due to Knaster, Kuratowski and Mazurkiewicz [1929]; Sperner's lemma was proved by Sperner in [1928].

Exercises

7.3.A. (a) (Nagami and Roberts [1967]) Prove that a metrizable space X satisfies the inequality $\dim X \leq n$ if and only if X has a strong development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ such that $\operatorname{ord} \mathcal{W}_i \leq n$ and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$

Hint. Modify the proof of Theorem 7.3.1.

(b) (Nagata [1956]) Prove that a metrizable space X satisfies the inequality $\dim X \leq n$ if and only if X has a development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ such that $\operatorname{ord} \mathcal{W}_i \leq n$ and $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$

Hint. Cf. the proof of Theorem 5.4.9.

(c) Observe that a compact metrizable space X satisfies the inequality $\dim X \leq n$ if and only if for every metric ρ on the space X and for any $\epsilon > 0$ there exists an open cover of the space X which has order $\leq n$ and consists of sets of diameter less than ϵ, or – equivalently – if and only if there exists a metric ρ on the space X which has the above property.

(d) Check that in (a) the assumption that $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ cannot be omitted.

7.3.B (Morita [1950a]). Prove that for every metrizable space X that can be represented as the union of a σ-locally finite family of strongly paracompact closed subspaces we have $\operatorname{ind} X = \operatorname{Ind} X = \dim X$.

Hint. Apply Exercise 7.2.F.

7.3.C. (a) Prove Theorems 7.3.4, 7.3.5, 7.3.8-7.3.11, 7.3.15-7.3.17 and 7.3.19 without using the Katětov-Morita theorem, and applying only Theorem 7.1.10 and Theorem 7.2.1 for $n = 0$.

Hint (Morita [1954]; for separable metrizable spaces, Hurewicz [1927]). First of all observe, applying Theorem 4.4.1, that in metrizable spaces the countable sum theorem implies the σ-locally finite sum theorem. Then, denoting by $\Sigma(n)$ the countable sum theorem for dimension n and by $\Delta(n)$ Theorem 7.3.8 for dimension n, check that in the proof of $\Delta(n)$ only $\Sigma(n-1)$ is applied and show that $\Sigma(0)$ and $\Delta(n)$ imply $\Sigma(n)$. When proving the latter implication, observe that the sets $X_j = F_j \setminus \bigcup_{i<j} F_i$, where $\{F_j\}_{j=1}^{\infty}$ is a closed cover of X, are F_σ-sets and cover the space X.

(b) Prove the Katětov-Morita theorem without using Theorem 7.3.1.

Hint (Przymusiński [1974]). Let $\operatorname{ds}(X, \rho) \leq n$ denote that the space X has a sequence of covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ with the properties stated in condition (ii) in Theorem 7.3.1. Observe first, modifying the proof of Theorem 7.3.2, that if $\operatorname{ds}(X, \rho) \leq n \geq 0$, then for every closed set $A \subset X$ and any open set $V \subset X$ satisfying $\rho(A, X \setminus V) > 0$ there exists an open set $U \subset X$

such that $A \subset U \subset V$ and $\operatorname{ds}(\operatorname{Fr} U, \rho_{\operatorname{Fr} U}) \le n-1$; then show, using the above observation, that the inequality $\operatorname{ds}(X,\rho) \le n \ge 0$ implies condition (ii) in Theorem 7.3.8.

7.3.D (Nagami [1957]). Show that if a metrizable space X can be represented as the union of a transfinite sequence $K_1, K_2, \ldots, K_\alpha, \ldots$, $\alpha \le \xi$ of subspaces of X, where $\operatorname{Ind} K_\alpha \le n$ and the set $F_\alpha = \bigcup_{\beta<\alpha} K_\beta$ is closed for every $\alpha < \xi$, then $\operatorname{Ind} X \le n$.

Hint. Take a metric on the space X and check that for $i = 1, 2, \ldots$ the family $\{F_{i,\alpha}\}_{\alpha<\xi}$, where $F_{i,\alpha} = K_\alpha \setminus B(F_\alpha, 1/i)$, is discrete and that $X = \bigcup_{i=1}^\infty \bigcup_{\alpha<\xi} F_{i,\alpha}$.

7.3.E. Check that in Theorem 7.3.8 one can replace σ-local finiteness of $\mathcal{B}$ by σ-discreteness.

7.3.F (Hausdorff [1934], de Groot [1956]). A metric ρ on a set X is called *non-Archimedean* (or an *ultrametric*) if

$$\rho(x,z) \le \max[\rho(x,y), \rho(y,z)]$$

for all $x, y, z \in X$.

Show that for a non-empty metrizable space X we have $\operatorname{Ind} X = 0$ if and only if there exists a non-Archimedean metric on the space X.

Hint. The metric on $B(\mathfrak{m})$ is non-Archimedean.

7.3.G. Show that the inequality in Theorem 7.3.17 cannot be replaced by an equality.

Hint. Apply Exercise 7.1.A.

7.3.H (A. H. Stone [1962a]). Prove that a completely metrizable space X such that $\operatorname{Ind} X = 0$ and $w(X) = \mathfrak{m}$ is homeomorphic to a closed subspace of the Baire space $B(\mathfrak{m})$. Deduce that every G_δ-subspace of $B(\mathfrak{m})$ is homeomorphic to a closed subspace of $B(\mathfrak{m})$.

7.3.I (Nagami [1959]). Prove that if $\mathbf{S} = \{X_i, \pi_j^i\}$ is an inverse sequence of metrizable spaces satisfying $\dim X_i \le n$ for $i = 1, 2, \ldots$, then the limit $X = \varprojlim \mathbf{S}$ also satisfies $\dim X \le n$.

Hint. Apply condition (iii) in Theorem 7.3.1.

7.3.J. Note that the equality $\dim R^n = n$ follows from Theorem 4 in the Appendix.

Appendix: Proof of the Brouwer fixed-point theorem

Let $x = \{x_i\}$ and $y = \{y_i\}$ be points of Euclidean n-space R^n. The sum $x+y$ of the points x and y and the product λx of the point x by a real number λ are defined by the formulas

$$x + y = z = \{z_i\}, \text{ where } z_i = x_i + y_i, \text{ and } \lambda x = t = \{t_i\}, \text{ where } t_i = \lambda x_i.$$

The point of R^n which has all coordinates equal to 0 will be denoted by the symbol 0; the distance $\rho(0,x)$ will be denoted by $|x|$. One can readily verify that

$$|x-y| = \rho(x,y), \quad |\lambda x| = |\lambda||x| \quad \text{and} \quad |x+y| \le |x| + |y|;$$

the last inequality is a reformulation of the triangle inequality in R^n.

Points $x_0, x_1, \ldots, x_k \in R^n$ are called *linearly independent* if for any sequence $\lambda_0, \lambda_1, \ldots, \lambda_k$ of real numbers the conditions

$$\lambda_0 x_0 + \lambda_1 x_1 + \ldots + \lambda_k x_k = 0 \quad \text{and} \quad \lambda_0 + \lambda_1 + \ldots + \lambda_k = 0$$

imply that $\lambda_j = 0$ for $j = 0, 1, \ldots, k$ (note that this is formally different from the notion of the linear independence in a linear space).

Consider $m + 1$ linearly independent points $a_0, a_1, \ldots, a_m$ in R^n; the subset of R^n consisting of all points

$$x = \lambda_0 a_0 + \lambda_1 a_1 + .. + \lambda_m a_m, \tag{1}$$

where

$$\lambda_0 + \lambda_1 + \ldots + \lambda_m = 1 \quad \text{and} \quad \lambda_j \geq 0 \quad \text{for} \quad j = 0, 1, \ldots, m, \tag{2}$$

is called the *m-dimensional simplex* spanned by the points $a_0, a_1, \ldots, a_m$ and is denoted by $a_0 a_1 \ldots a_m$. Clearly, the simplex $a_0 a_1 \ldots a_m$ does not depend on the ordering of points $a_0, a_1, \ldots, a_m$, it depends on the set $\{a_0, a_1, \ldots, a_m\}$ only.

Consider a simplex $a_0 a_1 \ldots a_m \subset R^n$; for any $k + 1$ distinct non-negative integers $j_0, j_1, \ldots, j_k$ not larger than m the points $a_{j_0}, a_{j_1}, \ldots, a_{j_k}$ are linearly independent, so that the k-dimensional simplex $a_{j_0} a_{j_1} \ldots a_{j_k}$ is well-defined. Every simplex of that form is called a *k-dimensional face* of the simplex $a_0 a_1 \ldots a_m$; 0-dimensional faces $a_0, a_1, \ldots, a_m$ are also called *vertices* of $a_0 a_1 \ldots a_m$. The simplex $a_0 a_1 \ldots a_m$ also is considered as its own face. One can readily check that the k-dimensional face $a_{j_0} a_{j_1} \ldots a_{j_k}$ consists of all points of the form (1) satisfying (2) and such that

$$\lambda_j = 0 \quad \text{whenever} \quad j \neq j_k \quad \text{for} \quad i = 0, 1, \ldots, k.$$

It follows from the linear independence of vertices, that every point $x \in a_0 a_1 \ldots a_m \subset R^n$ can be represented in the form (1), under conditions (2), in a unique way. The coefficients $\lambda_0, \lambda_1, \ldots, \lambda_m$ in (1) are called the *barycentric coordinates* of the point x; the barycentric coordinates of x will also be denoted by $\lambda_0(x), \lambda_1(x), \ldots, \lambda_m(x)$.

THEOREM 1. *For any $m+1$ linearly independent points $a_0, a_1, \ldots, a_m$ in R^n the simplex $S = a_0 a_1 \ldots a_m$ is a compact subspace of R^n and the barycentric coordinates $\lambda_0, \lambda_1, \ldots, \lambda_m$ are continuous functions from S to I.*

PROOF. Let $\delta_i^j = 0$ whenever $i \neq j$ and let $\delta_i^i = 1$. The points $d_0, d_1, \ldots, d_m$ in R^{m+1}, where $d_j = \{\delta_i^{j+1}\}$, are linearly independent, so that the simplex $T = d_0 d_1 \ldots d_m \subset R^{m+1}$ is well-defined. The barycentric coordinates of points of T coincide with their coordinates in R^{m+1}, and thus, by (2), T is a closed and bounded subset of R^{m+1}. Theorem 3.2.8 implies that T is compact.

It follows from Proposition 2.3.6 that the formula

$$f(x) = x_1 a_0 + x_2 a_1 + \ldots + x_{m+1} a_m, \quad \text{where} \quad x = \{x_i\} \in T \subset R^{m+1},$$

defines a continuous mapping $f: T \to S$. By linear independence of the points $a_0, a_1, \ldots, a_m$, the mapping f is one-to-one. Since $f(d_j) = a_j$ for $j = 0, 1, \ldots, m$, we have $f(T) = S$, and Theorem 3.1.13 implies that f is a homeomorphism of T onto S (this could also be deduced from elementary theorems of linear algebra). Hence, S is a compact subspace of R^n and the barycentric coordinates λ_j are continuous for $j = 0, 1, \ldots, m$, because $\lambda_j(x) = p_{j+1} f^{-1}(x)$, where p_i denotes the projection of R^{m+1} onto the ith axis. ∎

The continuity of barycentric coordinates immediately yields

COROLLARY. *Any two m-dimensional simplexes are homeomorphic.*■

A *simplicial subdivision* of a simplex $S \subset R^n$ is a family $\mathcal{P} = \{S_i\}_{i=1}^k$ of simplexes in R^n satisfying the following three conditions:

(3) The family $\mathcal{P}$ is a cover of S.
(4) For any $i, j \le k$ the intersection $S_i \cap S_j$ either is empty or is a face of both S_i and S_j.
(5) For $i = 1, 2, \ldots, k$ all faces of S_i are in $\mathcal{P}$.

The *mesh* of a simplicial subdivision $\{S_i\}_{i=1}^k$ of a simplex S is defined as the largest of the numbers $\delta(S_1), \delta(S_2), \ldots, \delta(S_k)$.

For every simplex $S = a_0 a_1 \ldots a_m \subset R^n$ the point

$$b(S) = \frac{1}{m+1} a_0 + \frac{1}{m+1} a_1 + \ \ldots \ + \frac{1}{m+1} a_m$$

is called the *barycenter* of the simplex S; clearly $b(S) \in S$ and $b(S)$ does not belong to any $(m-1)$-dimensional face of S.

THEOREM 2. *Let $S = a_0 a_1 \ldots a_m$ be a simplex. For every decreasing sequence $S_0 \supset S_1 \supset \ldots \supset S_k$ of distinct faces of the simplex S the points $b(S_0), b(S_1), \ldots, b(S_k)$ are linearly independent. The family $\mathcal{P}$ of all simplexes of the form $b(S_0)b(S_1)\ldots b(S_k)$ is a simplicial subdivision of the simplex S; every $(m-1)$-dimensional simplex $T \in \mathcal{P}$ is a face of one or two m-dimensional simplexes of $\mathcal{P}$, depending on whether or not the simplex T is contained in an $(m-1)$-dimensional face of S.*

PROOF. Every decreasing sequence of distinct faces of S can be completed to a sequence $S_0 \supset S_1 \supset \ldots \supset S_m$, consisting of $m+1$ faces of S, such that

$$S_0 = a_{i_0} a_{i_1} \ldots a_{i_m}, \quad S_1 = a_{i_1} a_{i_2} \ \ldots \ a_{i_m}, \ \ldots, \quad S_m = a_{i_m}, \tag{6}$$

where $i_0, i_1, \ldots, i_m$ is a permutation of $0, 1, \ldots, m$. To prove the first part of the theorem it suffices to show that the points $b(S_0), b(S_1), \ldots, b(S_m)$ are linearly independent.

Consider a linear combination

$$\mu_0 b(S_0) + \mu_1 b(S_1) + \ldots + \mu_m b(S_m); \tag{7}$$

applying the definition of the barycenter, we can represent (7) as a linear combination of points $a_{i_0}, a_{i_1}, \ldots, a_{i_m}$, namely it equals

$$\lambda_{i_0} a_{i_0} + \lambda_{i_1} a_{i_1} + \ldots + \lambda_{i_m} a_{i_m}, \tag{8}$$

where

$$\lambda_{i_j} = \sum_{k=0}^{j} \frac{1}{(m+1)-k} \mu_k \quad \text{for} \quad j = 0, 1, \ldots, m \tag{9}$$

and

$$\sum_{j=0}^{m} \lambda_{i_j} = \sum_{j=0}^{m} \sum_{k=0}^{j} \frac{1}{(m+1)-k} \mu_k = \sum_{i=0}^{m} \sum_{j=i}^{m} \frac{1}{(m+1)-i} \mu_i = \sum_{i=0}^{m} \mu_i. \tag{10}$$

Now, if the linear combination (7) is equal to 0 and $\mu_0+\mu_1+\ldots+\mu_m=0$, then, as the points $a_{i_0}, a_{i_1}, \ldots, a_{i_m}$ are linearly independent, it follows from (10) that $\lambda_{i_j}=0$ for $j=0,1,\ldots,m$; applying (9) we see that $\mu_0=0, \mu_1=0, \ldots, \mu_m=0$. Hence, the points $b(S_0), b(S_1), \ldots, b(S_m)$ are linearly independent and the simplex $b(S_0)b(S_1)\ldots b(S_m)$ is well-defined.

Every simplex in $\mathcal{P}$ is a face of an m-dimensional simplex of the form $b(S_0)b(S_1)\ldots b(S_m)$. From formulas (7)–(10) it follows that the simplex $b(S_0)b(S_1)\ldots b(S_m)$ is a subset of S; we shall show that it coincides with the set

$$\{x \in S:\ \lambda_{i_0}(x) \leq \lambda_{i_1}(x) \leq \ldots \leq \lambda_{i_m}(x)\}. \tag{11}$$

By virtue of (9) it suffices to show that every point x in the set (11) can be represented in the form (7) with $\mu_0+\mu_1+\ldots+\mu_m=1$ and $\mu_j \geq 0$ for $j=0,1,\ldots,m$. The reader can easily verify that such a representation is obtained if the coefficients μ_j are defined by the formulas

$$\mu_0=(m+1)\lambda_{i_0}(x) \quad \text{and} \quad \mu_j=[(m+1)-j](\lambda_{i_j}(x)-\lambda_{i_{j-1}}(x)) \tag{12}$$

for $j=1,2,\ldots,m$.

From (12) it follows that the faces of the simplex $b(S_0)b(S_1)\ldots b(S_m)$ are described by adding to the conditions defining the set (11) a number of conditions of the form $\lambda_{i_j}(x)=\lambda_{i_{j-1}}(x)$, where $1 \leq j \leq m$, and possibly also the condition $\lambda_{i_0}(x)=0$. As the intersection of a face determined by such a set of conditions with a face of another simplex $b(S_0')b(S_1')\ldots b(S_m')$, corresponding to a different permutation $i_0', i_1', \ldots, i_m'$ of $0,1,\ldots,m$, is still determined by a similar set of conditions or is empty, the family $\mathcal{P}$ satisfies (4). Condition (3) also is satisfied, because any point of S belongs to a set of the form (11). Condition (5) follows from the definition of $\mathcal{P}$. Hence, the family $\mathcal{P}$ is a simplicial subdivision of the simplex S.

Now, consider an $(m-1)$-dimensional simplex $T=b(S_0)b(S_1)\ldots b(S_{m-1}) \in \mathcal{P}$. If the simplex T is contained in an $(m-1)$-dimensional face of S, i.e., if $S_0 \neq S$, then T is a face of exactly one m-dimensional simplex of $\mathcal{P}$, namely of the simplex $b(S)b(S_0)\ldots b(S_{m-1})$. On the other hand, if T is not contained in any $(m-1)$-dimensional face of S, i.e., if $S_0=S$, then either there exists a $j \leq m-1$ such that the simplex S_j is obtained from S_{j-1} by dropping two vertices or S_{m-1} is a one-dimensional simplex; one can readily verify that in both cases T is a face of exactly two m-dimensional simplexes of $\mathcal{P}$. ■

The simplicial subdivision of a simplex S defined in Theorem 2 is called the *barycentric subdivision* of S. We shall now define for every natural number l the *l-th barycentric subdivision* of a simplex S: the barycentric subdivision of S is the first barycentric subdivision of S; if the jth barycentric subdivision $\{S_i\}_{i=1}^k$ of S is already defined, then we define the $(j+1)$th barycentric subdivision of S as the union $\bigcup_{i=1}^k \mathcal{P}_i$, where $\mathcal{P}_i$ is the barycentric subdivision of S_i. Applying the fact that for each face S' of S_i and each member S'' of $\mathcal{P}_i$ the intersection $S' \cap S''$ either is empty or is a face of S'' and a member of the barycentric subdivision of S', the reader can easily verify that the union $\bigcup_{i=1}^k \mathcal{P}_i$ is a simplicial subdivision of S.

Consider a simplex $S=a_0a_1\ldots a_m \subset R^n$, a point $x=\lambda_0a_0+\lambda_1a_1+\ldots+\lambda_ma_m$, where $\lambda_0+\lambda_1+\ldots+\lambda_m=1$ and $\lambda_j \geq 0$ for $j=0,1,\ldots,m$, i.e., a point $x \in S$, and an arbitrary point $y \in R^n$; we shall show that

$$\rho(x,y) \leq \max_{j \leq m} \rho(a_j,y). \tag{13}$$

In fact, we have

$$\rho(x,y) = |x-y| = \left|\sum_{j=0}^{m} \lambda_j a_j - \sum_{j=0}^{m} \lambda_j y\right| = \left|\sum_{j=0}^{m} \lambda_j (a_j - y)\right|$$
$$\leq \sum_{j=0}^{m} \lambda_j |a_j - y| \leq \max_{j\leq m} |a_j - y| \sum_{j=0}^{m} \lambda_j = \max_{j\leq m} \rho(a_j, y).$$

LEMMA 1. *The diameter of a simplex $a_0 a_1 \ldots a_m \subset R^n$ is equal to the diameter of the set $\{a_0, a_1, \ldots, a_m\}$ of its vertices.*

PROOF. Let $x, y \in a_0 a_1 \ldots a_m$. By virtue of (13) we have $\rho(x,y) \leq \max_{j\leq m} \rho(a_j, y)$; applying (13) again we infer that $\rho(a_j, y) \leq \max_{i\leq m} \rho(a_j, a_i)$. Thus $\rho(x,y) \leq \max_{j,i\leq m} \rho(a_j, a_i)$, i.e., $\delta(a_0 a_1 \ldots a_m) = \delta(\{a_0, a_1, \ldots, a_m\})$. ■

LEMMA 2. *The mesh of the barycentric subdivision of an m-dimensional simplex $S = a_0 a_1 \ldots a_m$ is not larger than $[m/(m+1)]\delta(S)$.*

PROOF. By virtue of Lemma 1 it suffices to show that the distance between any two points of the form

$$b(S_j) = \frac{1}{j+1}(a_{i_0} + a_{i_1} + \ldots + a_{i_j}) \quad \text{and} \quad b(S_k) = \frac{1}{k+1}(a_{i_0} + a_{i_1} + \ldots + a_{i_k}),$$

where $k < j \leq m$ and $i_0, i_1, \ldots, i_m$ is an arbitrary permutation of $0, 1, \ldots, m$, is not larger than $[m/(m+1)]\delta(S)$. It follows from (13) that $\rho(b(S_k), b(S_j)) \leq \rho(a_{i_l}, b(S_j))$ for some $l \leq k < j$. Hence

$$\begin{aligned}\rho(b(S_k), b(S_j)) &\leq \rho(a_{i_l}, b(S_j)) = |b(S_j) - a_{i_l}| \\ &= \left|\frac{1}{j+1}(a_{i_0} + a_{i_1} + \ldots + a_{i_j}) - a_{i_l}\right| \\ &= \frac{1}{j+1}\left|\sum_{h=0}^{j}(a_{i_h} - a_{i_l})\right| \leq \frac{1}{j+1}\sum_{h=0}^{j}|a_{i_h} - a_{i_l}| \\ &\leq \frac{j}{j+1}\delta(S) \leq \frac{m}{m+1}\delta(S). \blacksquare\end{aligned}$$

The last lemma yields

THEOREM 3. *For every simplex S and any $\epsilon > 0$ there exists a natural number l such that the mesh of the l-th barycentric subdivision of S is less than ϵ.* ■

SPERNER'S LEMMA. *Let $\mathcal{P}$ be the l-th barycentric subdivision of an m-dimensional simplex $a_0 a_1 \ldots a_m$, and let V be the set of all vertices of simplexes in $\mathcal{P}$. If a function h defined on V and taking values in the set $\{0, 1, \ldots, m\}$ satisfies the condition*

$$h(v) \in \{i_0, i_1, \ldots, i_k\} \quad \textit{whenever} \quad v \in a_{i_0} a_{i_1} \ldots a_{i_k},$$

then the number of simplexes in $\mathcal{P}$ on vertices of which h assumes all values from 0 to m is an odd number.

PROOF. We apply induction with respect to m. If $m = 0$, the lemma holds, because $\mathcal{P} = \{a_0\}$ and $h(a_0) = 0$. Assume that the lemma is established for $m = n - 1$; consider an n-dimensional simplex $S = a_0a_1 \ldots a_n$, the lth barycentric subdivision $\mathcal{P}$ of S, and a function h satisfying the condition in the lemma. Let $\mathcal{P}'$ denote the family of all $(n-1)$-dimensional simplexes of $\mathcal{P}$ on vertices on which h assumes all values from 0 to $n - 1$; clearly, the only $(n-1)$-dimensional face of S that contains simplexes of $\mathcal{P}'$ is the face $a_0a_1 \ldots a_{n-1}$ and – by the inductive assumption – the number of simplexes of $\mathcal{P}'$ contained in $a_0a_1 \ldots a_{n-1}$ is an odd number; let us denote it by a.

Arrange all n-dimensional simplexes of $\mathcal{P}$ into a sequence $S_1, S_2, \ldots, S_t$. For every $j \leq t$ denote by b_j the number of faces of S_j belonging to $\mathcal{P}'$ and by N_j denote the set of values of the function h on vertices of S_j. The reader can easily verify that

(14) If $N_j = \{0, 1, \ldots, n\}$, then $b_j = 1$.
(15) If $N_j = \{0, 1, \ldots, n-1\}$, then $b_j = 2$.
(16) If the inclusion $\{0, 1, \ldots, n-1\} \subset N_j$ does not hold, then $b_j = 0$.

Denote by c the number of simplexes in $\mathcal{P}$ on vertices of which h assumes all values from 0 to n. By (14)–(16), there exists an integer d_1 such that

$$c - (b_1 + b_2 + \ldots + b_t) = 2d_1. \tag{17}$$

If we assign to every simplex S_j its b_j faces belonging to $\mathcal{P}'$, then any simplex T in $\mathcal{P}'$ will be assigned, as easily follows from the last part of Theorem 2, to one or two simplexes S_j, depending on whether or not the simplex T is contained in an $(n-1)$-dimensional face of S. Hence, there exists an integer d_2 such that

$$a - (b_1 + b_2 + \ldots + b_t) = 2d_2. \tag{18}$$

Formulas (17) and (18) imply that $c - a = 2(d_1 - d_2)$, so that c is an odd number.■

THEOREM 4. *Let $\{F_i\}_{i=0}^m$ be a family of closed subsets of a simplex $S = a_0a_1 \ldots a_m$. If for each face $a_{i_0}a_{i_1} \ldots a_{i_k}$ of S we have*

$$a_{i_0}a_{i_1} \ldots a_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \ldots \cup F_{i_k},$$

then $F_0 \cap F_1 \cap \ldots \cap F_m \neq \emptyset$.

PROOF. Suppose that $F_1 \cap F_2 \cap \ldots \cap F_m = \emptyset$. The family $\{U_i\}_{i=0}^m$, where $U_i = S \setminus F_i$, is an open cover of S, so that by compactness of S and Theorem 4.3.31 there exists an $\epsilon > 0$ such that every subset of S of diameter less than ϵ is contained in a set U_i, i.e., is disjoint from a set F_i.

By Theorem 3 there exists a natural number l such that the mesh of the lth barycentric subdivision $\mathcal{P}$ of the simplex S is less than ϵ. Denote by V the set of all vertices of simplexes in $\mathcal{P}$. For every $v \in V$ the intersection of all faces of S that contain V is a face of S, i.e., is of the form $a_{i_0}a_{i_1} \ldots a_{i_k}$; since $v \in a_{i_0}a_{i_1} \ldots a_{i_k}$, by the assumption of our theorem there exists a $j \leq k$ such that $v \in F_{i_j}$. Letting $h(v) = i_j$ we define on V a function h which satisfies the assumption of Sperner's lemma, so that there exists at least one simplex $T = v_0v_1 \ldots v_m \in \mathcal{P}$ such that $h(v_i) = i$ for $i = 0, 1, \ldots, m$. Hence for $i = 0, 1, \ldots, m$ we have $v_i \in F_i$, which implies that $T \cap F_i \neq \emptyset$ for $i = 0, 1, \ldots, m$; this is a contradiction, because $\delta(T) < \epsilon$.■

PROOF OF THE BROUWER FIXED-POINT THEOREM. The n-cube I^n is contained in the n-dimensional simplex $T = a_0a_1\ldots a_n \subset R^n$, where $a_0 = 0, a_j = \{n\delta_i^j\}$ for $j = 1,2,\ldots,n$, $\delta_i^j = 0$ whenever $i \neq j$, and $\delta_i^i = 1$. If there existed a continuous mapping $f: I^n \to I^n$ such that $f(x) \neq x$ for all $x \in I^n$, then for a continuous extension $\overline{f}: T \to I^n$ of f, which exists by virtue of 2.1.8 and 2.3.6, we would have $\overline{f}(x) \neq x$ for all $x \in T$. Hence, it suffices to prove that for every continuous mapping $\overline{f}: T \to T$ there exists a point $x \in T$ such that $\overline{f}(x) = x$. Let us observe that the possibility of replacing I^n with T also follows from the fact that I^n and T are homeomorphic (see Exercise 3.2.C).

For every point $x \in T$ we have

$$x = \lambda_0(x)a_0 + \lambda_1(x)a_1 + \ldots + \lambda_n(x)a_n, \tag{19}$$

where

$$\lambda_0(x) + \lambda_1(x) + \ldots + \lambda_n(x) = 1 \tag{20}$$

and $\lambda_j(x) \geq 0$ for $j = 0,1,\ldots,n$.

The image of the point $x \in T$ under the mapping $\overline{f}$ can be written as

$$\overline{f}(x) = \lambda_0(\overline{f}(x))a_0 + \lambda_1(\overline{f}(x))a_1 + \ldots + \lambda_n(\overline{f}(x))a_n, \tag{21}$$

where

$$\lambda_0(\overline{f}(x)) + \lambda_1(\overline{f}(x)) + \ldots + \lambda_n(\overline{f}(x)) = 1 \tag{22}$$

and $\lambda_j(\overline{f}(x)) \geq 0$ for $j = 0,1,\ldots,n$.

For $i = 0,1,\ldots,n$ the set

$$F_i = \{x \in T:\ \lambda_i(\overline{f}(x)) \leq \lambda_i(x)\} \tag{23}$$

is closed, because λ_i and $\overline{f}$ are continuous. We shall show that the family $\{F_i\}_{i=0}^n$ satisfies the assumptions of Theorem 4. Let $a_{i_0}a_{i_1}\ldots a_{i_k}$ be a face of T; consider a point $x \in a_{i_0}a_{i_1}\ldots a_{i_k}$. We have

$$\lambda_{i_0}(x) + \lambda_{i_1}(x) + \ldots + \lambda_{i_k}(x) = 1,$$

so that by (22)

$$\lambda_{i_0}(\overline{f}(x)) + \lambda_{i_1}(\overline{f}(x)) + \ldots + \lambda_{i_k}(\overline{f}(x)) \leq \lambda_{i_0}(x) + \lambda_{i_1}(x) + \ldots + \lambda_{i_k}(x);$$

hence, $\lambda_{i_j}(\overline{f}(x)) \leq \lambda_{i_j}(x)$ for at least one $j \leq k$, which means that $x \in F_{i_j}$. Thus we have shown that

$$a_{i_0}a_{i_1}\ldots a_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \ldots \cup F_{i_k}.$$

By virtue of Theorem 4 there exists a point $x \in F_0 \cap F_1 \cap \ldots \cap F_n$; it follows from (23) that

$$\lambda_0(\overline{f}(x)) \leq \lambda_0(x), \quad \lambda_1(\overline{f}(x)) \leq \lambda_1(x), \quad \ldots, \quad \lambda_n(\overline{f}(x)) \leq \lambda_n(x).$$

However, by virtue of (20) and (22), the strict inequality $\lambda_j(\overline{f}(x)) < \lambda_j(x)$ cannot hold for any j, so that

$$\lambda_0(\overline{f}(x)) = \lambda_0(x), \quad \lambda_1(\overline{f}(x)) = \lambda_1(x), \quad \ldots, \quad \lambda_n(\overline{f}(x)) = \lambda_n(x),$$

and by (19) and (21) we have $\overline{f}(x) = x$.■

7.4. Problems

The addition theorems

7.4.1 (Urysohn [1925b]). Let X be a hereditarily normal space and M a subspace of X. Show that $\operatorname{ind} M \le n \ge 0$ if and only if for every point $x \in M$ and any neighbourhood V of the point x in the space X there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\operatorname{ind}(M \cap \operatorname{Fr} U) \le n-1$.

Deduce that for every pair X, Y of subspaces of a hereditarily normal space we have

$$\operatorname{ind}(X \cup Y) \le \operatorname{ind} X + \operatorname{ind} Y + 1.$$

Hint. Apply induction with respect to $\operatorname{ind} X + \operatorname{ind} Y$.

7.4.2 (Smirnov [1951b]). Let X be a hereditarily normal space and M a subspace of X. Show that $\operatorname{Ind} M \le n \ge 0$ if and only if for every closed set $A \subset X$ and any open set $V \subset X$ that contains A there exists an open set $U \subset X$ such that $A \subset U \subset V$ and $\operatorname{Ind}(M \cap \operatorname{Fr} U) \le n-1$.

Deduce that for every pair X, Y of subspaces of a hereditarily normal space we have

$$\operatorname{Ind}(X \cup Y) \le \operatorname{Ind} X + \operatorname{Ind} Y + 1.$$

7.4.3 (Čech [1932a]). Let X be a hereditarily normal space and M a subspace of X. Show that every open cover of M has a swelling consisting of open subsets of X.

Hint. Applying induction with respect to k, show that for every finite family $\{V_i\}_{i=1}^k$ of open subsets of M satisfying $\bigcap_{i=1}^k V_i = \emptyset$ there exists a family $\{U_i\}_{i=1}^k$ of open subsets of X such that $V_i \subset U_i$ for $i = 1, 2, \ldots, k$ and $\bigcap_{i=1}^k U_i = \emptyset$.

7.4.4 (Smirnov [1951b]). Let X be a hereditarily normal space and M a subspace of X. Show that $\dim M \le n$ if and only if for every finite family $\{U_i\}_{i=1}^k$ of open subsets of X satisfying $M \subset \bigcup_{i=1}^k U_i$ there exists a family $\{V_i\}_{i=1}^k$ of open subsets of X such that $V_i \subset U_i$ for $i = 1, 2, \ldots, k$, $M \subset \bigcup_{i=1}^k V_i$ and $\operatorname{ord}(\{V_i\}_{i=1}^k) \le n$.

Deduce that for every pair X, Y of subspaces of a hereditarily normal space we have

$$\dim(X \cup Y) \le \dim X + \dim Y + 1.$$

Hint. In the proof of the first part apply Problem 7.4.3.

A compact space X such that $\operatorname{ind} X \ne \dim X$

7.4.5 (Lokucievskiĭ [1949]). Let V denote the long segment (see Problem 3.12.19) and let $f: D^{\aleph_0} \to I$ be the function defined in Exercise 3.2.B. In the Cartesian product $V \times D^{\aleph_0}$ consider the equivalence relation E defined by letting $(x_1, c_1)E(x_2, c_2)$ whenever $(x_1, c_1) = (x_2, c_2)$ or $x_1 = x_2 = \omega_1$ and $f(c_1) = f(c_2)$. Prove that for the quotient space $X_1 = (V \times D^{\aleph_0})/E$ we have $\operatorname{ind} X_1 = \operatorname{Ind} X_1 = \dim X_1 = 1$.

Let A_1 be the subspace of X_1 consisting of all points whose inverse images under the natural quotient mapping $q: V \times D^{\aleph_0} \to X_1$ are two-point sets; let X_2 be another copy of the space X_1, disjoint from X_1, and let A_2 be the subset of X_2 which corresponds to $A_1 \subset X_1$.

Applying Exercise 4.3.H(a), match X_1 with X_2 along the segment $q(\{\omega_1\} \times D^{\aleph_0})$ in such a way that no point of A_1 is matched with a point of A_2. Check that for the compact space X obtained in this way we have $\operatorname{ind} X = \operatorname{Ind} X = 2$ and $\dim X = 1$.

Note that the countable sum theorem, and even the finite sum theorem, does not hold for dimensions ind and Ind in the realm of compact spaces.

Remark. The first example of a compact space X such that $\operatorname{ind} X \neq \dim X$ was given by Lunc in [1949].

A Tychonoff space Z and a closed subspace M of Z such that $\dim Z < \dim M$

7.4.6 (Smirnov [1956b]). Using the space Y in Example 6.2.20 define a Tychonoff space Z such that $\dim Z = 0$, and yet $\dim M > 0$ for a closed subspace $M \subset Z$.

Hint. Let cY be a zero-dimensional compactification of Y and let W denote the space of all ordinal numbers $\leq \omega_1$. Apply Problem 3.12.20(c) to show that for the space $Z = (W \times cY) \setminus [\{\omega_1\} \times (cY \setminus c(Y))]$ we have $\beta Z = W \times cY$.

A normal space Z such that $\operatorname{ind} Z = 0$ and $\dim Z = \infty$

7.4.7 (Smirnov [1958]). Modifying the construction of the space Y in Exercise 6.2.20 define a normal space Z such that $\operatorname{ind} Z = 0$ and $\dim Z = \infty$.

Hint. Replace the interval by the Hilbert cube and the sets S_α by the powers $S_\alpha^{\aleph_0}$. Apply Corollary 7.1.18.

Polyhedra

7.4.8. A space P is a *polyhedron* if there exists a simplex S and a family $\mathcal{P}$ of faces of S such that $P = \bigcup \mathcal{P}$. All vertices of S which belong to P are called the *vertices* of the polyhedron P; the *dimension* of the polyhedron P is the maximum of dimensions of simplexes in $\mathcal{P}$.

Observe that the definitions of a vertex and of the dimension of a polyhedron do not depend on the choice of the simplex S and the family $\mathcal{P}$. Note that every polyhedron is a compact space.

Let P be a polyhedron and let $a_0, a_1, \ldots, a_m$ be all the vertices of P. Show that every point $x \in P$ can be uniquely represented in the form

$$x = \lambda_0(x)a_0 + \lambda_1(x)a_1 + \ldots + \lambda_m(x)a_m,$$

where $\sum_{i=0}^m \lambda_i(x) = 1$ and $\lambda_j(x) \geq 0$ for $j = 0, 1, \ldots, m$. Check that $\lambda_0, \lambda_1, \ldots, \lambda_m$ are continuous functions from P to I. The subset

$$G_j = \{x \in P : \lambda_j(x) > 0\}$$

of the polyhedron P is called the *star of the vertex* a_j. Show that for a sequence $a_{j_0}, a_{j_1}, \ldots, a_{j_k}$ of vertices of P the intersection $G_{j_0} \cap G_{j_1} \cap \ldots \cap G_{j_k}$ is non-empty if and only if $a_{j_0} a_{j_1} \ldots a_{j_k} \subset P$.

Prove that for every polyhedron P the dimensions $\operatorname{ind} P$, $\operatorname{Ind} P$ and $\dim P$ coincide the with dimension of P.

A characterization of the dimension dim in terms of $\mathcal{A}$-mappings

7.4.9 (implicitly, Kuratowski [1933a]; for compact metrizable spaces Alexandroff [1928]). Let X and Y be topological spaces and let $\mathcal{A}$ be a cover of the space X. A continuous mapping $f: X \to Y$ is an *$\mathcal{A}$-mapping* if there exists an open cover $\{U_s\}_{s\in S}$ of the space Y such that $\{f^{-1}(U_s)\}_{s\in S}$ is a refinement of $\mathcal{A}$.

(a) Prove that a Tychonoff (normal) space X satisfies the inequality $\dim X \leq n$ if and only if for every finite functionally open (open) cover $\mathcal{A}$ of the space X there exists an $\mathcal{A}$-mapping of X to a polyhedron of dimension $\leq n$.

Hint. Let $\mathcal{V} = \{V_i\}_{i=0}^m$ be a functionally open refinement of $\mathcal{A}$ such that $\operatorname{ord} \mathcal{V} \leq n$ and let $\{F_i\}_{i=0}^m$ be a functionally closed shrinking of $\mathcal{V}$. Take an m-dimensional simplex $S = a_0 a_1 \ldots a_m$, a family $\{f_i\}_{i=0}^m$ of continuous functions, where $f_i: X \to I$ satisfies $f_i(X \setminus V_i) \subset \{0\}$ and $f_i(F_i) \subset \{1\}$, let

$$\lambda_i(x) = \frac{f_i(x)}{f_0(x) + f_1(x) + \ldots + f_m(x)},$$

and consider the mapping $f: X \to S$ defined by the formula $f(x) = \lambda_0(x)a_0 + \lambda_1(x)a_1 + \ldots + \lambda_m(x)a_m$.

(b) Show that in the characterization of the dimension dim given in part (a) the words "to a polyhedron" can be replaced by the words "onto a polyhedron".

Hint. Show that for every subset A of a polyhedron $P = \bigcup \mathcal{P}$ there exists a subfamily $\mathcal{P}_0$ of $\mathcal{P}$ and a continuous mapping $g: A \to P$ such that $g(A) = P_0 = \bigcup \mathcal{P}_0$ and $g(A \cap T) \subset T$ for every $T \in \mathcal{P}$.

Covering dimension of Cartesian products

7.4.10 (Hemmingsen [1946]). Prove that for every pair X, Y of compact spaces of which at least one is non-empty we have

$$\dim (X \times Y) \leq \dim X + \dim Y$$

(cf. Problem 7.4.12(b)).

Hint. Show that every finite open cover of $X \times Y$ has a refinement of the form $\{U \times V : U \in \mathcal{A}, V \in \mathcal{B}\}$, where $\mathcal{A}$ and $\mathcal{B}$ are finite open covers of X and Y respectively. Apply Problem 7.4.9(a).

Dimension of subspaces

7.4.11 (Dowker [1955]). Let X be a normal space and M a closed subspace of X such that $\dim M \leq n$. Show that $\dim X \leq n$ if and only if every closed subspace $F \subset X$ disjoint from M satisfies $\dim F \leq n$.

7.4.12 (Čech [1933], Smirnov [1951c]). (a) Let X be a normal space satisfying the inequality $\dim X \leq n$. Prove that for every subspace $M \subset X$ which is an F_σ-set, or – more generally – is normally placed in X, we have $\dim M \leq n$.

Deduce that for every subspace M of a perfectly normal space X we have $\dim M \leq \dim X$.

Remark. Čech proved in [1932] (announcement in [1931]) that for every subspace M of a perfectly normal space X we have $\operatorname{Ind} M \leq \operatorname{Ind} X$; for a proof see Nagami [1970] or Engelking [1978].

(b) Show that the inequality in Problem 7.4.10 remains valid for any pair X, Y of Tychonoff spaces such that the Cartesian product $X \times Y$ is a Lindelöf space.

A characterization of the dimension dim in terms of mappings into spheres

7.4.13. (a) (Hemmingsen [1946], Alexandroff [1947], Dowker [1947]; for compact spaces, Alexandroff [1940], Morita [1940]; for compact metrizable spaces Alexandroff [1932] and Hurewicz [1935]) Prove that a normal space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if every continuous mapping $f: A \to S^n$ defined on a closed subspace A of X is continuously extendable over X.

Hint. Use the characterization of the dimension dim established in Theorem 7.2.15. Instead of S^n take the boundary S_I^n of the $(n+1)$-cube I^{n+1} in R^{n+1}. Apply Problem 5.5.21(b) and the fact that for every $x_0 \in I^{n+1} \setminus S_I^n$ the boundary S_I^n is a retract of $I^{n+1} \setminus \{x_0\}$.

One can also consider first the case of a compact space X, apply Problem 5.5.21(a) and extend the characterization to normal spaces by applying Theorem 7.1.17.

(b) (Smirnov [1955b]) Prove that a Tychonoff space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if every continuous mapping $f: A \to S^n$ defined on a closed subspace A of X, and such that there exists a continuous mapping $F: X \to B^{n+1}$ satisfying $F(x) = f(x)$ for $x \in A$, is continuously extendable over X.

(c) Show that if A is a closed subspace of a normal space X and if every closed subspace $F \subset X$ disjoint from A satisfies $\dim F \leq n \geq 0$, then every continuous mapping $f: A \to S^n$ is continuously extendable over X.

(d) Let X_1 and X_2 be closed subspaces of a metrizable space X such that $X_1 \cup X_2 = X$ and let $f_1: X_1 \to S^n$ and $f_2: X_2 \to S^n$ be continuous mappings. Show that if the set $D = \{x \in X_1 \cap X_2 : f_1(x) \neq f_2(x)\}$ satisfies the inequality $\dim D \leq n - 1$, then both f_1 and f_2 are continuously extendable over X.

Hint. Apply Problem 5.5.21.

The Mardešić factorization theorem

7.4.14 (Mardešić [1960]). Prove that for every continuous mapping $f: X \to Y$ of a compact space X satisfying $\dim X \leq n$ to a compact space Y satisfying $w(Y) \leq \mathfrak{m}$ there exists a compact space Z and continuous mappings $g: X \to Z$ and $h: Z \to Y$ such that $\dim Z \leq n$, $w(Z) \leq \mathfrak{m}$, $g(X) = Z$ and $f = hg$ (this is the *Mardešić factorization theorem*).

Hint (Arhangel'skiĭ [1967]). Take a base $\mathcal{B}$ for the space Y satisfying $|\mathcal{B}| \leq \mathfrak{m}$ and consider the collection $\{\mathcal{U}_s\}_{s \in S}$ of all finite covers of Y consisting of members of $\mathcal{B}$. Let $\mathbf{V}_0 = \{\mathcal{V}_s\}_{s \in S}$, where $\mathcal{V}_s = \{f^{-1}(U) : U \in \mathcal{U}_s\}$, and define a sequence $\mathbf{V}_1, \mathbf{V}_2, \ldots$ of collections of open covers of X, where $\mathbf{V}_{i+1}$ is obtained by taking for every pair $\mathcal{V}, \mathcal{V}' \in \mathbf{V}_i$ a finite open cover of order $\leq n$ which is a star refinement of both $\mathcal{V}$ and $\mathcal{V}'$. Check that by letting xEy whenever $y \in \bigcap_{i=1}^{\infty} \bigcap \{\operatorname{St}(x, \mathcal{V}) : \mathcal{V} \in \mathbf{V}_i\}$ we define a closed equivalence relation E on the space X. Verify that the quotient space $Z = X/E$, the natural quotient mapping g and the mapping h defined by the equality $h([x]) = f(x)$ satisfy the required conditions.

Compactifications preserving dimension and weight

7.4.15 (Skljarenko [1958]; for separable merizable spaces, Hurewicz [1927b] and [1930]). Prove that for every Tychonoff space X satisfying $\dim X \le n$ and $w(X) \le \mathfrak{m}$ there exists a compactification cX such that $\dim cX \le n$ and $w(cX) \le \mathfrak{m}$.

Hint (Mardešić [1960]). Extend a homeomorphic embedding $i: X \to I^{\mathfrak{m}}$ over βX and apply Problem 7.4.14.

One can also apply Problem 7.4.16.

Universal spaces of dimension n and weight $\mathfrak{m}$

7.4.16 (Pasynkov [1964], Zarelua [1964]). Prove that for every natural number n and any cardinal number $\mathfrak{m} \ge \aleph_0$ there exists a compact space which is universal for all Tychonoff spaces X such that $\dim X \le n$ and $w(X) \le \mathfrak{m}$.

Hint (Pasynkov [1964]). Let $\mathcal{X}$ be the family of all subspaces X of $I^{\mathfrak{m}}$ with $\dim X \le n$. Let $Y = \bigoplus_{X \in \mathcal{X}} X$ and $i = \nabla_{X \in \mathcal{X}} i_X : Y \to I^{\mathfrak{m}}$, where $i_X: X \to I^{\mathfrak{m}}$ is the embedding of X in $I^{\mathfrak{m}}$; consider the extension $f: \beta Y \to I^{\mathfrak{m}}$ of the mapping i and apply Problem 7.4.14.

Remark. It was shown by Nöbeling in [1931] that for every $n \ge 0$ the space $N_n^{2n+1} = \bigcup_{m=0}^{n} Q_m^{2n+1}$, i.e., the subspace of R^{2n+1} consisting of all points with at most n rational coordinates, is universal for all separable metrizable spaces of dimension $\le n$. This implies in particular that every separable metrizable space of dimension $\le n$ is embeddable in R^{2n+1}; the last fact was also proved by Lefschetz in [1931], as well as by Pontrjagin and Tolstowa in [1931]. Proofs can be found in Hurewicz and Wallman's book [1941] and in Engelking's book [1978].

Enlarging n-dimensional sets to n-dimensional G_δ-sets

7.4.17 (Katětov [1952], Morita [1954]; for separable metrizable spaces, Tumarkin [1926] (announcement [1925])). Show that for every subspace M of a metrizable space X there exists a G_δ-set $M^* \subset X$ such that $M \subset M^*$ and $\operatorname{Ind} M^* = \operatorname{Ind} M$.

Deduce that every metrizable space X is homeomorphic to a dense subspace of a completely metrizable space X^* such that $\operatorname{Ind} X^* = \operatorname{Ind} X$ and $w(X^*) = w(X)$.

Hint. Consider first the case of a subspace M satisfying the equality $\operatorname{Ind} M = 0$. Modify the first part of the proof of Theorem 7.3.8 and apply Lemma 7.3.7 (one can also apply Theorems 7.3.15 and 4.3.21). To obtain the general case apply Theorem 7.3.9. In the proof of the second part apply Theorems 4.3.19 and 4.3.23.

n-dimensional subspaces of R^n

7.4.18 (Menger [1924], Urysohn [1925b] (announcement [1922])). Prove that a subspace M of Euclidean n-space R^n has dimension n if and only if $\operatorname{Int} M \ne \emptyset$.

Deduce that for every non-empty open set $U \subset R^n$ such that $\overline{U} \ne R^n$ the boundary $\operatorname{Fr} U$ has dimension $n-1$.

Hint. Apply Problem 4.5.2.

Mappings which raise and mappings which lower dimension

7.4.19. (a) (Engelking [1973]) Observe that a metrizable space X satisfies the inequality $\operatorname{Ind} X \leq n \geq 0$ if and only if X has a σ-locally finite network $\mathcal{N}$ such that $\operatorname{Ind} \operatorname{Fr} M \leq n-1$ for every $M \in \mathcal{N}$.

(b) (Morita [1955]; for separable metrizable spaces, Hurewicz [1927a]) Prove that if $f: X \to Y$ is a closed mapping of a metrizable space X onto a metrizable space Y and $|f^{-1}(y)| \leq k+1 \geq 1$ for every $y \in Y$, then $\operatorname{Ind} Y \leq \operatorname{Ind} X + k$.

Hint (Engelking [1973]). Apply induction with respect to $\operatorname{Ind} Y + k$. Take a σ-locally finite base $\mathcal{B}$ for the space X such that $\operatorname{Ind} \operatorname{Fr} U \leq \operatorname{Ind} X - 1$ for every $U \in \mathcal{B}$, consider the family $\{f(U) : U \in \mathcal{B}\}$, use Lemma 3.10.11 and part (a).

7.4.20. (a) (Hurewicz and Wallman [1941] for separable metrizable spaces) Prove that if a metrizable space X has a closed cover $\mathcal{K}$ such that $\operatorname{Ind} K \leq n \geq 0$ for every $K \in \mathcal{K}$ and an open σ-locally finite cover $\mathcal{U}$ such that for every $K \in \mathcal{K}$ and any open $V \subset X$ that contains K there exists a $U \in \mathcal{U}$ satisfying $K \subset U \subset V$ and $\operatorname{Ind} \operatorname{Fr} U \leq n-1$, then $\operatorname{Ind} X \leq n$.

Hint. Prove that every continuous mapping $f: A \to S^n$ defined on a closed subspace A of the space X has a continuous extension $F: X \to S^n$ (cf. Problem 7.4.13(a)). To begin, applying Problem 7.4.13(c), for every $K \in \mathcal{K}$ define a $U_K \in \mathcal{U}$ such that $K \subset U_K$, $\operatorname{Ind} \operatorname{Fr} U_K \leq n-1$ and f is continuously extendable over $A \cup \overline{U}_K$. Arrange all members of the family $\{U_K : K \in \mathcal{K}\}$ into a transfinite sequence $U_0, U_1, \ldots, U_\alpha, \ldots,\ \alpha < \xi$ such that the family $\{U_\beta\}_{\beta<\alpha}$ is locally finite for every $\alpha < \xi$; let $A_0 = A \cup \overline{U}_0$ and $A_\alpha = \overline{U}_\alpha$ for $\alpha > 0$. Applying Problem 7.4.13(d), define by transfinite induction a transfinite sequence of mappings $F_0, F_1, \ldots, F_\alpha, \ldots,\ \alpha < \xi$, where $F_\alpha : (\bigcup_{\beta \leq \alpha} A_\beta) \to S^n$, such that $F_\alpha | (\bigcup_{\beta \leq \gamma} A_\beta) = F_\gamma$ for $\gamma \leq \alpha$ and $F_0|A = f$; let $F = \nabla_{\alpha<\xi}(F_\alpha | U_\alpha)$.

(b) (Morita [1956], Nagami [1957]; for separable metrizable spaces, Hurewicz and Wallman [1941]; for compact metrizable spaces, Hurewicz [1927a]) Prove that if $f: X \to Y$ is a closed mapping of a metrizable space X to a metrizable space Y and $\operatorname{Ind} f^{-1}(y) \leq k \geq 0$ for every $y \in Y$, then $\operatorname{Ind} X \leq \operatorname{Ind} Y + k$.

Hint. Fix k and apply induction with respect to $\operatorname{Ind} Y$. Take a σ-locally finite base $\mathcal{B}$ for the space Y such that $\operatorname{Ind} \operatorname{Fr} U \leq \operatorname{Ind} Y - 1$ for every $U \in \mathcal{B}$, consider the family $\{f^{-1}(U) : U \in \mathcal{B}\}$ and use (a).

Remark. A survey of results on mappings which raise and on mappings which lower dimension, as well as an exhaustive bibliography of the subject, can be found in Lelek's paper [1971] (see also Nagami [1970] and Engelking [1978]). Filippov's paper [1972] describes a general method which simplifies many proofs in that area.

Around the Brouwer fixed-point theorem

7.4.21. (a) Show that the Brouwer fixed-point theorem implies Sperner's lemma and the two following facts:

(1) *The sphere S^n is not a retract of the ball B^{n+1}.*

(2) *The identity mapping $\mathrm{id}_{S^n}: S^n \to S^n$ is not homotopic to the constant mapping of S^n to a point $x \in S^n$.*

Show that each of the above two facts, as well as the theorem that $\dim I^n = n$, implies the Brouwer fixed-point theorem.

(b) Prove that for every continuous mapping $f: I^{\aleph_0} \to I^{\aleph_0}$ there exists a point $x \in I^{\aleph_0}$ such that $f(x) = x$.

Borel sets III (see Problems 1.7.5, 4.5.7 and 4.5.8)

7.4.22 (Lebesgue [1905]). Prove that for every countable ordinal number α there exists in the Hilbert cube $I^{\aleph_0}$ a set M_α of the multiplicative class α which is not of the additive class α, as well as a set A_α of the additive class α which is not of the multiplicative class α.

Hint (Engelking, Holsztyński and Sikorski [1966]). Define inductively the sets M_α and $A_\alpha = I^{\aleph_0} \setminus M_\alpha$ taking as M_0 an arbitrary one-point subset of $I^{\aleph_0}$ and for $\alpha > 0$ letting

$$M_\alpha = \begin{cases} A_\beta^{\aleph_0} \subset (I^{\aleph_0})^{\aleph_0} = I^{\aleph_0}, & \text{if } \alpha = \beta + 1, \\ \prod_{\beta<\alpha} A_\beta \subset (I^{\aleph_0})^{\aleph_0} = I^{\aleph_0}, & \text{if } \alpha \text{ is a limit number.} \end{cases}$$

Show that for every metrizable space X and any set $B \subset X$ of the multiplicative (the additive) class α there exists a mapping $f: X \to I^{\aleph_0}$ such that $f^{-1}(M_\alpha) = B$ (such that $f^{-1}(A_\alpha) = B$). Apply Problem 7.4.21(b).

Chapter 8

Uniform spaces and proximity spaces

The concepts of a uniform space and of a proximity space can be considered either as axiomatizations of some geometric notions, close to but quite independent of the concept of a topological space, or as convenient tools for an investigation of topological spaces. Uniformities, when introduced by Weil, were considered as such tools, suitable, in contrast to metrics, for studying topological spaces with no countability assumptions. Proximities can also be applied as topological tools; they are particularly efficient in studying compactifications. Bourbaki, who studies the theory of uniform spaces at great length in his book, emphasises its character as an independent theory which is, however, strongly related to the theory of topological spaces. The relation between the two theories consists in the fact that to uniform spaces and uniformly continuous mappings one can assign, in a standard way, topological spaces and continuous mappings. The passage from uniform spaces to topological spaces can be split into two parts, with proximity spaces in the middle.

The theory of uniform spaces shows striking analogies with the theory of metric spaces, but the realm of its applicability is much broader. In particular, on every topological group there are three natural uniformities which are applied in the theory of topological groups.

In Section 8.1 we introduce the concepts of a uniformity and of a uniform space, and we show how a topology is induced by a uniformity. Then we define two notions often applied in the theory of uniform spaces: uniform covers and uniform pseudometrics. From the important result that for any uniformity $\mathcal{U}$ there are many pseudometrics uniform with respect to $\mathcal{U}$, we infer that a space with a topology induced by a uniformity is a Tychonoff space; later on we show that the topology of every Tychonoff space can be induced by a uniformity. Next we consider the three uniformities on a topological group, and establish a necessary and sufficient condition that a uniformity be induced by a metric. We conclude the section with a discussion of uniformly continuous mappings and of uniform isomorphisms.

In Section 8.2 three operations on uniform spaces are examined; we consider subspaces, Cartesian products and functions spaces. In contrast to metric spaces, the Cartesian product of any family of uniform spaces is again a uniform space; it is not necessary to restrict the cardinality of the family. By introducing a uniformity on function spaces, we can define the concept of an equicontinuous family of mappings and prove counterparts of the classical Ascoli theorem.

Section 8.3 is devoted to totally bounded uniform spaces and complete uniform spaces. The analogy between uniform spaces and metric spaces is distinctly visible there. Many theoems in that section are generalizations – almost identically stated and proved – of theorems in Section 4.3. In the final part of the section, we define the completion of a uniform space and discuss properties of compact uniform spaces.

In Section 8.4 proximities and proximity spaces are studied. We show that every proximity induces a topology and we study interrelations between uniformities and proximities. The most important result in that section is the Smirnov theorem which establishes a one-to-one correspondence between proximities on a Tychonoff space X and compactifications of the space X.

8.1. Uniformities and uniform spaces

We begin by introducing some notation. Let X be a set and let A and B be subsets of $X \times X$, i.e., relations on the set X. The inverse relation of A will be denoted by $-A$, and the composition of A and B will be denoted by $A + B$; thus we have

$$-A = \{(x, y) : (y, x) \in A\}$$

and

$$A + B = \{(x, z) : \text{there exists a } y \in X \text{ such that } (x, y) \in A \text{ and } (y, z) \in B\}.$$

One readily sees that the composition of relations is associative, i.e., that $(A + B) + C = A + (B + C)$. On the other hand, as shown by simple examples, the composition of relations is not commutative, i.e., generally $A + B \neq B + A$. For a relation $A \subset X \times X$ and a natural number n the relation $nA \subset X \times X$ is defined inductively by the formulas

$$1A = A \quad \text{and} \quad nA = (n-1)A + A.$$

It follows from associativity of composition that $mA + nA = nA + mA = (m + n)A$.

As defined in Section 2.3, the diagonal of the Cartesian product $X \times X$ is the set $\Delta = \{(x, x) : x \in X\}$. Every set $V \subset X \times X$ that contains Δ and satisfies the condition $V = -V$ is called an *entourage of the diagonal*; the family of all entourages of the diagonal $\Delta \subset X \times X$ will be denoted by $\mathcal{D}_X$. If for a pair x, y of points of X and a $V \in \mathcal{D}_X$ we have $(x, y) \in V$, we say that the *distance between x and y is less than V* and we write $|x - y| < V$; otherwise we write $|x - y| \geq V$. If for every pair x, y of points of a set $A \subset X$ and a $V \in \mathcal{D}_X$ we have $|x - y| < V$, i.e., if $A \times A \subset V$, we say that the *diameter of A is less than V* and we write $\delta(A) < V$. One readily checks that for any $x, y, z \in X$ and $V, V_1, V_2 \in \mathcal{D}_X$ the following conditions hold:

(1) $|x - x| < V$.
(2) $|x - y| < V$ *if and only if* $|y - x| < V$.
(3) *If* $|x - y| < V_1$ *and* $|y - z| < V_2$, *then* $|x - z| < V_1 + V_2$.

Let x_0 be a point of X and let $V \in \mathcal{D}_X$; the set $B(x_0, V) = \{x \in X : |x_0 - x| < V\}$ is called the *ball with centre x_0 and radius V* or, briefly, the *V-ball about x_0*. It follows immediately from (3) that the diameter of a V-ball is less than $2V$. For a set $A \subset X$ and a $V \in \mathcal{D}_X$, by the *V-ball about A* we mean the set $B(A, V) = \bigcup_{x \in A} B(x, V)$.

A *uniformity* on a set X is a subfamily $\mathcal{U}$ of $\mathcal{D}_X$ which satisfies the following conditions:

(U1) *If* $V \in \mathcal{U}$ *and* $V \subset W \in \mathcal{D}_X$, *then* $W \in \mathcal{U}$.
(U2) *If* $V_1, V_2 \in \mathcal{U}$, *then* $V_1 \cap V_2 \in \mathcal{U}$.
(U3) *For every* $V \in \mathcal{U}$ *there exists a* $W \in \mathcal{U}$ *such that* $2W \subset V$.
(U4) $\bigcap \mathcal{U} = \Delta$.

A family $\mathcal{B} \subset \mathcal{U}$ is called a *base for the uniformity* $\mathcal{U}$ if for every $V \in \mathcal{U}$ there exists a $W \in \mathcal{B}$ such that $W \subset V$. The smallest cardinal number of the form $|\mathcal{B}|$, where $\mathcal{B}$ is a base for $\mathcal{U}$, is called the *weight of the uniformity* $\mathcal{U}$ and is denoted by $w(\mathcal{U})$.

Any base $\mathcal{B}$ for a uniformity on a set X has the following properties:

(BU1) *For any $V_1, V_2 \in \mathcal{B}$ there exists a $V \in \mathcal{B}$ such that $V \subset V_1 \cap V_2$.*
(BU2) *For every $V \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ such that $2W \subset V$.*
(BU3) $\bigcap \mathcal{B} = \Delta$.

Observe that any entourage of the diagonal $V \in \mathcal{D}_X$ yields a cover $\mathcal{C}(V) = \{B(x,V)\}_{x\in X}$ of the set X. Let $\mathcal{U}$ be a uniformity on a set X; any cover of the set X which has a refinement of the form $\mathcal{C}(V)$, where $V \in \mathcal{U}$, is called *uniform with respect to* $\mathcal{U}$. The collection $\mathbf{C}$ of all covers of a set X which are uniform with respect to a uniformity $\mathcal{U}$ on the set X has the following properties:

(UC1) *If $\mathcal{A} \in \mathbf{C}$ and $\mathcal{A}$ is a refinement of a cover $\mathcal{B}$ of the set X, then $\mathcal{B} \in \mathbf{C}$.*
(UC2) *For any $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{C}$ there exists an $\mathcal{A} \in \mathbf{C}$ which is a refinement of both $\mathcal{A}_1$ and $\mathcal{A}_2$.*
(UC3) *For every $\mathcal{A} \in \mathbf{C}$ there exists a $\mathcal{B} \in \mathbf{C}$ which is a star refinement of $\mathcal{A}$.*
(UC4) *For every pair x, y of distinct points of X there exists an $\mathcal{A} \in \mathbf{C}$ such that no member of $\mathcal{A}$ contains both x and y.*

Property (UC1) is obvious, property (UC2) is a consequence of (U2) and of the equality $B(x, V \cap V_2) = B(x,V_1) \cap B(x,V_2)$, and property (UC4) follows from (U4) and (U3). It remains to establish (UC3); to that end it suffices to show that for every $\mathcal{A} = \mathcal{C}(V) \in \mathbf{C}$ the cover $\mathcal{B} = \mathcal{C}(W)$, where $W \in \mathcal{U}$ satisfies the inclusion $4W \subset V$, is a star refinement of $\mathcal{A}$. Let us take a member $B(x,W)$ of $\mathcal{B}$ and consider a $y \in X$ such that $B(x,W) \cap B(y,W) \neq \emptyset$. The last inequality implies that $|x-y| < 2W$, so that for every $z \in B(y,W)$ we have $|x-z| < 3W \subset 4W \subset V$, i.e., $z \in B(x,V)$. Hence $\mathrm{St}\,(B(x,W),\mathcal{B}) \subset B(x,V) \in \mathcal{A}$ which show that $\mathcal{B}$ is a star refinement of $\mathcal{A}$.

A *uniform space* is a pair $(X,\mathcal{U})$ consisting of a set X and a uniformity $\mathcal{U}$ on the set X. The *weight of a uniform space* $(X,\mathcal{U})$ is defined as the weight of the uniformity $\mathcal{U}$.

Every uniformity $\mathcal{U}$ on a set X induces a topology $\mathcal{O}$ on X; hence, every uniform space $(X,\mathcal{U})$ defines a topological space $(X,\mathcal{O})$. More exactly we have

8.1.1. THEOREM. *For every uniformity $\mathcal{U}$ on a set X the family $\mathcal{O} = \{G \subset X :$ for every $x \in G$ there exists a $V \in \mathcal{U}$ such that $B(x,V) \subset G\}$ is a topology on the set X and the topological space $(X,\mathcal{O})$ is a T_1-space.*

The topology $\mathcal{O}$ is called the *topology induced by the uniformity* $\mathcal{U}$.

PROOF. It follows directly from the definition of the family $\mathcal{O}$ that it satisfies conditions (O1) and (O3). We shall show that condition (O2) also is satisfied. Consider two sets $G_1, G_2 \in \mathcal{O}$ and a point $x \in G_1 \cap G_2$. By the definition of $\mathcal{O}$ there exist $V_1, V_2 \in \mathcal{U}$ such that $B(x,V_1) \subset G_1$ and $B(x,V_2) \subset G_2$. From (U2) it follows that $V = V_1 \cap V_2 \in \mathcal{U}$, and since $B(x,V) = B(x,V_1) \cap B(x,V_2) \subset G_1 \cap G_2$, we have $G_1 \cap G_2 \in \mathcal{O}$.

It remains to check that $(X,\mathcal{O})$ is a T_1-space. Clearly, it suffices to show that for every $x \in X$ the set $G = X \setminus \{x\}$ is open. For any $y \in G$ we have $x \neq y$, so that by (U4) there exists a $V \in \mathcal{U}$ satisfying $|x-y| \geq V$. Since $B(y,V) \subset G$, the set G is open. ■

As in the case of metrics, one can ask when for a topological space $(X, \mathcal{O})$ there exists a uniformity $\mathcal{U}$ on the set X such that the topology induced by $\mathcal{U}$ coincides with the original topology $\mathcal{O}$. In Theorem 8.1.20 below we shall show that such a uniformity exists if and only if $(X, \mathcal{O})$ is a Tychonoff space. This is the reason why Tychonoff spaces are called *uniformizable spaces* by some authors.

If X is a topological space and a uniformity $\mathcal{U}$ on the set X induces the original topology of X, then we say that $\mathcal{U}$ is a *uniformity on the space* X; generally, there exist many uniformities on a given Tychonoff space (see Example 8.1.7, and Exercises 8.1.B and 8.1.C).

8.1.2. PROPOSITION. *The interior of a set $A \subset X$ with respect to the topology induced by a uniformity $\mathcal{U}$ on the set X coincides with the set $B = \{x \in X :$ there exists a $V \in \mathcal{U}$ such that $B(x, V) \subset A\}$.*

PROOF. Since every open set $G \subset A$ is contained in B, it suffices to show that B is an open set. For any $x \in B$ there exists a $V \in \mathcal{U}$ such that $B(x, V) \subset A$; let us take a $W \in \mathcal{U}$ satisfying $2W \subset V$. It follows from (3) that for every $y \in B(x, W)$ we have $B(y, W) \subset B(x, V) \subset A$; hence $B(x, W) \subset B$ which shows that B is an open set. ■

8.1.3. COROLLARY. *If the topology of a space X is induced by a uniformity $\mathcal{U}$, then for every $x \in X$ and any $V \in \mathcal{U}$ the set $\operatorname{Int} B(x, V)$ is a neighbourhood of x.*

PROOF. As $B(x, V) \subset B(x, V)$, the point x belongs to the interior of the ball $B(x, V)$. ■

8.1.4. COROLLARY. *If the topology of a space X is induced by a uniformity $\mathcal{U}$, then for every $x \in X$ and any $A \subset X$ we have $x \in \overline{A}$ if and only if $A \cap B(x, V) \neq \emptyset$ for every $V \in \mathcal{U}$.* ■

Let us observe that from the above corollaries it follows that a set X with the topology induced by a uniformity $\mathcal{U}$ is a regular space (cf. Corollary 8.1.13). Indeed, X is a T_1-space by Theorem 8.1.1, and for a point $x \in X$ and a closed set $F \subset X$ such that $x \notin F$ the neighbourhood $\operatorname{Int} B(x, W)$ of the point x, where $W \in \mathcal{U}$ satisfies $2W \subset V$ with a $V \in \mathcal{U}$ satisfying $F \cap B(x, V) = \emptyset$, has the property that $\overline{\operatorname{Int} B(x, W)} \cap F = \emptyset$, because $\overline{B(x, W)} \subset B(x, 2W)$ by Corollary 8.1.4.

8.1.5. COROLLARY. *If the topology of a space X is induced by a uniformity $\mathcal{U}$, then for every $A \subset X$ and any $V \in \mathcal{U}$ we have $\delta(\overline{A}) < 3V$ whenever $\delta(A) < V$.*

PROOF. By virtue of Corollary 8.1.4, for any $x, y \in \overline{A}$ there exist $x', y' \in A$ such that $x' \in B(x, V)$ and $y' \in B(y, V)$. Hence we have $|x - y| < V + V + V = 3V$. ■

8.1.6. EXAMPLE. For any set X, the family $\mathcal{U} = \mathcal{D}_X$ is a uniformity on X; it is called the *discrete uniformity* on X, and the space $(X, \mathcal{U})$ is called a *discrete uniform space.* The one-element family $\mathcal{B} = \{\Delta\}$ is a base for $\mathcal{U}$, so that $w(\mathcal{U}) = 1$. Since $B(x, \Delta) = \{x\}$, every cover of the set X is uniform with respect to $\mathcal{U}$ and every subset A of X is open with respect to the topology induced by $\mathcal{U}$, i.e., the discrete uniformity induces the discrete topology. ■

It follows from the above example that the weight of a topological space $(X, \mathcal{O})$, where the topology $\mathcal{O}$ is induced by a uniformity $\mathcal{U}$, can be larger than the weight of $\mathcal{U}$. On the other hand, one readily verifies that the character of the space $(X, \mathcal{O})$ is less than or equal to the weight of $\mathcal{U}$.

8.1.7. EXAMPLE. Let X be an arbitrary set; for every finite sequence $x_1, x_2, \ldots, x_k$ of elements of the set X define

$$V(x_1, x_2, \ldots, x_k) = X \times X \setminus \bigcup_{i=1}^{k} \{[(X \times \{x_i\}) \cup (\{x_i\} \times X)] \setminus \Delta\}.$$

The sets $V(x_1, x_2, \ldots, x_k)$ are entourages of the diagonal. Consider the family $\mathcal{U} \subset \mathcal{D}_X$ consisting of all entourages of the diagonal which contain a set $V(x_1, x_2, \ldots, x_k)$. Hence, $\mathcal{U}$ consists of all members of $\mathcal{D}_X$ which are obtained by removing from $X \times X$ a set that can be covered by a finite union of sets $(X \times \{x\}) \cup (\{x\} \times X)$. We shall show that $\mathcal{U}$ is a uniformity on the set X.

The fact that $\mathcal{U}$ satisfies conditions (U1) and (U2) follows from the definition. Since, as one can easily check, $2V(x_1, x_2, \ldots, x_k) = V(x_1, x_2, \ldots, x_k)$, condition (U3) also is satisfied. Finally, condition (U4) is satisfied, because $(x, y) \notin V(x)$ whenever $x \neq y$.

As $B(x, V(x)) = \{x\}$, the uniformity $\mathcal{U}$ induces the discrete topology on X. If the set X is infinite, then $\mathcal{U} \neq \mathcal{D}_X$, so that, as follows from a comparison with Example 8.1.6, distinct uniformities may induce the same topology. One readily sees that if $|X| \geq \aleph_0$, then $w(\mathcal{U}) = |X|$. ∎

8.1.8. EXAMPLE. Let $I = [0, 1]$ be the closed unit interval. The reader can easily check that the family $\mathcal{U}$ of all entourages of the diagonal $\Delta \subset I \times I$ which contain an open subset of $I \times I$ containing Δ is a uniformity on I; clearly, $\mathcal{U}$ induces the natural topology on the interval I and $w(\mathcal{U}) = \aleph_0$. ∎

Let $\mathcal{U}$ be a uniformity on a set X; the Tychonoff topology on the Cartesian product $X \times X$, where X has the topology induced by $\mathcal{U}$, is called the *topology induced by the uniformity $\mathcal{U}$ on the set $X \times X$*. The space $X \times X$ with that topology plays an important role in the theory of uniform spaces.

Consider a uniform space $(X, \mathcal{U})$ and a pseudometric ρ on the set X; we say that the pseudometric ρ is *uniform with respect to $\mathcal{U}$* if for every $\epsilon > 0$ there exists a $V \in \mathcal{U}$ such that $\rho(x, y) < \epsilon$ whenever $|x - y| < V$.

8.1.9. PROPOSITION. *If a pseudometric ρ on a set X is uniform with respect to a uniformity $\mathcal{U}$ on X, then ρ is a continuous function from the set $X \times X$ with the topology induced by the uniformity $\mathcal{U}$ to the real line.*

PROOF. Let (x_0, y_0) be a point of $X \times X$; take an $\epsilon > 0$ and a $V \in \mathcal{U}$ such that

$$\rho(x, y) < \epsilon/2 \quad \text{whenever} \quad |x - y| < V.$$

Since by Corollary 8.1.3 the set $\operatorname{Int} B(x_0, V) \times \operatorname{Int} B(y_0, V)$ is a neighbourhood of the point (x_0, y_0), it suffices to show that

$$|\rho(x_0, y_0) - \rho(x, y)| < \epsilon \quad \text{for every} \quad (x, y) \in B(x_0, V) \times B(y_0, V).$$

However, if $(x, y) \in B(x_0, V) \times B(y_0, V)$, then $|x_0 - x| < V$ and $|y_0 - y| < V$, and – by virtue of the triangle inequality – we have

$$|\rho(x_0, y_0) - \rho(x, y)| \leq \rho(x_0, x) + \rho(y_0, y) < \epsilon/2 + \epsilon/2 = \epsilon. ∎$$

Our next theorem is one of the most important results of the theory of uniform spaces; it states – roughly speaking – that for any uniformity $\mathcal{U}$ there are many pseudometrics uniform with respect to $\mathcal{U}$. This fact will be often applied in the sequel. In particular, we shall deduce from it that any space with the topology induced by a uniformity is a Tychonoff space.

8.1.10. THEOREM. *For every sequence $V_0, V_1, \ldots$ of members of a uniformity $\mathcal{U}$ on a set X, where*

$$V_0 = X \times X \quad \textit{and} \quad 3V_{i+1} \subset V_i \quad \textit{for} \quad i = 1, 2, \ldots,$$

there exists a pseudometric ρ on the set X such that for every $i \geq 1$

$$\{(x,y) : \rho(x,y) < 1/2^i\} \subset V_i \subset \{(x,y) : \rho(x,y) \leq 1/2^i\}.$$

PROOF. For any pair x, y of points of X consider all finite sequences $x_0, x_1, \ldots, x_k$ of points of X such that $x_0 = x$ and $x_k = y$, and define $\rho(x,y)$ as the greatest lower bound of the numbers

$$1/2^{i_1} + 1/2^{i_2} + \ldots + 1/2^{i_k}, \quad \text{where} \quad (x_{j-1}, x_j) \in V_{i_j} \quad \text{for} \quad j = 1, 2, \ldots, k.$$

One easily sees that ρ is a pseudometric on X. From the definition of ρ it follows that $V_i \subset \{(x,y) : \rho(x,y) \leq 1/2^i\}$. It remains to show that if $\rho(x,y) < 1/2^i$, then $(x,y) \in V_i$, i.e., that for every sequence $x_0, x_1, \ldots, x_k$ of points of X, where $(x_{j-1}, x_j) \in V_{i_j}$ for $j = 1, 2, \ldots, k$,

$$\text{(4)} \qquad \text{if} \quad 1/2^{i_1} + 1/2^{i_2} + \ldots + 1/2^{i_k} < 1/2^i, \quad \text{then} \quad (x_0, x_k) \in V_i.$$

We shall prove (4) by induction with respect to k. For $k = 1$ we have $1/2^{i_1} < 1/2^i$, i.e., $i < i_1$, and $(x_0, x_k) \in V_{i_1} \subset V_i$. Assume that $m > 1$ and that (4) holds for all $k < m$. Consider a sequence $x_0, x_1, \ldots, x_m$ such that $(x_{j-1}, x_j) \in V_{i_j}$ for $j = 1, 2, \ldots, m$ and

$$\text{(5)} \qquad 1/2^{i_1} + 1/2^{i_2} + \ldots + 1/2^{i_m} < 1/2^i.$$

Obviously, either $1/2^{i_1} < 1/2^{i+1}$ or $1/2^{i_m} < 1/2^{i+1}$, and by the symmetry of assumptions we can assume that the first inequality holds. Let n be the largest integer $\leq m - 1$ such that

$$\text{(6)} \qquad 1/2^{i_1} + 1/2^{i_2} + \ldots + 1/2^{i_n} < 1/2^{i+1}.$$

If $n < m - 1$, we have $1/2^{i_1} + 1/2^{i_2} + \ldots + 1/2^{i_{n+1}} \geq 1/2^{i+1}$ and from (5) it follows that

$$\text{(7)} \qquad 1/2^{i_{n+2}} + 1/2^{i_{n+3}} + \ldots + 1/2^{i_m} < 1/2^{i+1}.$$

By the inductive assumption, (6) and (7) imply that $(x_0, x_n) \in V_{i+1}$ and $(x_{n+1}, x_m) \in V_{i+1}$; by (5) we have $1/2^{i_{n+1}} < 1/2^i$, i.e., $i + 1 \leq i_{n+1}$, and $(x_n, x_{n+1}) \in V_{i_{n+1}} \subset V_{i+1}$, so that $(x_0, x_m) \in 3V_{i+1} \subset V_i$.

If $n = m - 1$, by the inductive assumption and (6) we have $(x_0, x_{m-1}) \in V_{i+1}$ and – as (5) implies that $1/2^{i_m} < 1/2^i$ – $(x_{m-1}, x_m) \in V_{i_m} \subset V_{i+1}$, so that $(x_0, x_m) \in 2V_{i+1} \subset V_i$.

Thus the proof of (4), and of the theorem, is concluded. ■

8.1.11. COROLLARY. *For every uniformity $\mathcal{U}$ on a set X and any $V \in \mathcal{U}$ there exists a pseudometric ρ on the set X which is uniform with respect to $\mathcal{U}$ and satisfies the condition*

$$\{(x,y) : \rho(x,y) < 1\} \subset V.$$

PROOF. It follows from (U3) that there exists a sequence $V_0, V_1, \ldots$ of members of $\mathcal{U}$ such that $V_0 = X \times X$, $V_1 = V$ and $3V_{i+1} \subset V_i$ for $i = 1, 2, \ldots$ One readily sees that the pseudometric $\rho = 2\rho_0$, where ρ_0 is a pseudometric satisfying Theorem 8.1.10, has the required property.■

8.1.12. COROLLARY. *For every uniformity $\mathcal{U}$ on a set X the family of all members of $\mathcal{U}$ which are open with respect to the topology induced by $\mathcal{U}$ on $X \times X$, and the family of all members of $\mathcal{U}$ which are closed with respect to that topology, are both bases for $\mathcal{U}$.*

PROOF. Let V be a member of $\mathcal{U}$ and let ρ be the pseudometric defined in the proof of Corollary 8.1.11. By virtue of (U1) the sets

$$W = \{(x,y) : \rho(x,y) < 1\} \subset V \quad \text{and} \quad U = \{(x,y) : \rho(x,y) \leq 1/2\} \subset V$$

are members of $\mathcal{U}$. It follows from Proposition 8.1.9 that the former is open and the latter is closed.■

8.1.13. COROLLARY. *For every uniformity $\mathcal{U}$ on a set X, the set X with the topology induced by $\mathcal{U}$ is a Tychonoff space.*

PROOF. For every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a $V \in \mathcal{U}$ satisfying $F \cap B(x, V) = \emptyset$. The function $f : X \to I$ defined by the equality $f(y) = \min(1, \rho(x,y))$, where ρ is the pseudometric in Corollary 8.1.11, is continuous, vanishes at x and is equal to one on F.■

Let $(X, \mathcal{U})$ be a uniform space; we shall show that the family P of all pseudometrics on the set X which are uniform with respect to $\mathcal{U}$ has the following properties:

(UP1) *If $\rho_1, \rho_2 \in P$, then* $\max(\rho_1, \rho_2) \in P$.
(UP2) *For every pair x, y of distinct points of X there exists a $\rho \in P$ such that $\rho(x,y) > 0$.*

To check property (UP1), consider $\rho_1, \rho_2 \in P$, $\rho = \max(\rho_1, \rho_2)$, and an arbitrary $\epsilon > 0$. By the definition of a pseudometric uniform with respect to $\mathcal{U}$ there exist $V_1, V_2 \in \mathcal{U}$ such that $\rho_1(x,y) < \epsilon$ whenever $|x - y| < V_1$ and $\rho_2(x,y) < \epsilon$ whenever $|x - y| < V_2$. Since $V = V_1 \cap V_2 \in \mathcal{U}$ and $\rho(x,y) < \epsilon$ whenever $|x - y| < V$, we have $\rho \in P$.

Property (UP2) follows from (U4) and Corollary 8.1.11.

When defining a uniformity on a set, often it is more convenient not to describe the family $\mathcal{U}$ of entourages of the diagonal directly. We shall now give three methods of generating uniformities; they consist in the definition of a base, of a family of uniform covers, or of a family of uniform pseudometrics.

8.1.14. PROPOSITION. *Suppose we are given a set X and a family $\mathcal{B} \subset \mathcal{D}_X$ of entourages of the diagonal which has properties* (BU1)–(BU3). *The family $\mathcal{U}$ consisting of all members of $\mathcal{D}_X$ which contain a member of $\mathcal{B}$ is a uniformity on the set X. The family $\mathcal{B}$ is a base for $\mathcal{U}$.*

If, moreover, X is a topological space and the family $\mathcal{B}$ consists of open subsets of the Cartesian product $X \times X$, and if for every $x \in X$ and any neighbourhood G of x there exists a $V \in \mathcal{B}$ such that $B(x, V) \subset G$, then $\mathcal{U}$ is a uniformity on the space X.

The uniformity $\mathcal{U}$ is called the *uniformity generated by the base $\mathcal{B}$*.■

8.1.15. EXAMPLE. The reader can easily verify that the method of generating a uniformity by defining a base was applied in Examples 8.1.7 and 8.1.8.■

8.1.16. PROPOSITION. *Suppose we are given a set X and a collection $\mathbf{C}$ of covers of X which has properties* (UC1)–(UC4). *The family $\mathcal{B} \subset \mathcal{D}_X$ of all entourages of the diagonal which are of the form $\bigcup\{H \times H : H \in \mathcal{A}\}$, where $\mathcal{A} \in \mathbf{C}$, is a base for a uniformity $\mathcal{U}$ on the set X. The collection $\mathbf{C}$ is the collection of all covers of X which are uniform with respect to $\mathcal{U}$.*

If, moreover, X is a topological space and the collection $\mathbf{C}$ consists of open covers of X, and if for every $x \in X$ and any neighbourhood G of x there exists an $\mathcal{A} \in \mathbf{C}$ such that $\mathrm{St}(x, \mathcal{A}) \subset G$, *then $\mathcal{U}$ is a uniformity on the space X.*

The uniformity $\mathcal{U}$ is called the *uniformity generated by the collection $\mathbf{C}$ of uniform covers.*

PROOF. For every cover $\mathcal{A}$ of the set X let

$$V(\mathcal{A}) = \bigcup\{H \times H : H \in \mathcal{A}\}.$$

As $V(\mathcal{A}) \in \mathcal{D}_X$, it suffices to check that the family $\mathcal{B} = \{V(\mathcal{A}) : \mathcal{A} \in \mathbf{C}\}$ has properties (BU1)–(BU3). However, those properties follow directly from properties (UC2)–(UC4) of $\mathbf{C}$, if one observes that $V(\mathcal{A}) \subset V(\mathcal{B})$ whenever $\mathcal{A}$ is a refinement of $\mathcal{B}$ and that $2V(\mathcal{A}) \subset V(\mathcal{B})$ whenever $\mathcal{A}$ is a star refinement of $\mathcal{B}$. The fact that $\mathbf{C}$ is the collection of all covers of X which are uniform with respect to $\mathcal{U}$ follows from (UC1) and the readily established equality $B(x, V(\mathcal{A})) = \mathrm{St}(x, \mathcal{A})$.

From the last equality we also obtain the second part of the theorem.■

8.1.17. EXAMPLE. A *group* is a set G such that for any pair x, y of elements of G an element $xy \in G$, the *product* of x and y, is defined and the following three conditions are satisfied:

(G1) $(xy)z = x(yz)$ *for all* $x, y, z \in G$.
(G2) *There exists an $e \in G$ such that $xe = ex = x$ for every $x \in G$.*
(G3) *For every $x \in G$ there exists an $x^{-1} \in G$ such that $xx^{-1} = e$.*

The element e is called the *identity* of G, and x^{-1} is called the *inverse element* of x. One readily verifies that there is only one identity in G and that for every $x \in G$ there exists exactly one inverse element. If in a group G besides conditions (G1)–(G3) also the condition

(G4) $xy = yx$ *for all* $x, y \in G$

is satisfied, then we say that G is an *Abelian* or a *commutative* group.

A *topological group* is a group G which is in the same time a T_1-space such that the following two conditions are satisfied:

(TG1) *The formula $f(x, y) = xy$ defines a continuous mapping $f : G \times G \to G$.*
(TG2) *The formula $f(x) = x^{-1}$ defines a continuous mapping $f : G \to G$.*

Let G be a group and let A, B be subsets of G; we define

$$A^{-1} = \{x^{-1} : x \in A\} \quad \text{and} \quad AB = \{xy : x \in A \text{ and } y \in B\}.$$

For a subset $A \subset G$ and an element $x \in G$, instead of $\{x\}A$ and $A\{x\}$ we write xA and Ax. One readily checks that if A is an open subset of a topological group G, then the set A^{-1}

also is open. Similarly, the set AB is open if at least one of the sets A and B is open. In particular, for every open set $H \subset G$ the sets xH and Hx are open.

Now, let G be a topological group and let $\mathcal{B} = \mathcal{B}(e)$ be a base for G at the point e. Every member H of $\mathcal{B}$ determines three covers of G:

$$\mathcal{C}_l(H) = \{xH\}_{x \in G}, \quad \mathcal{C}_r(H) = \{Hx\}_{x \in G}, \quad \text{and} \quad \mathcal{C}(H) = \{xHy\}_{x,y \in G}.$$

Denote by $\mathbf{C}_l, \mathbf{C}_r$, and $\mathbf{C}$ respectively the collection of all covers of G which have a refinement of the form $\mathcal{C}_l(H), \mathcal{C}_r(H)$, or $\mathcal{C}(H)$, where $H \in \mathcal{B}$. Each of the collections $\mathbf{C}_l$, $\mathbf{C}_r$ and $\mathbf{C}$ has properties (UC1)–(UC4), and thus generates a uniformity on the set G. Moreover, it turns out that the topology induced by each of those uniformities coincides with the original topology of G. In all three cases the proofs are similar to each other, so that we shall discuss the case of the collection $\mathbf{C}_l$ only. Let us observe, however, that if G is an Abelian group then the collection $\mathbf{C}_l, \mathbf{C}_r$ and $\mathbf{C}$ coincide, and thus generate the same uniformity on the set G.

Consider a topological group G and the above defined collection $\mathbf{C}_l$ of covers of G. It follows from the definition that $\mathbf{C}_l$ has property (UC1). Since for any $H_1, H_2 \in \mathcal{B}$ there exists an $H \in \mathcal{B}$ such that $H \subset H_1 \cap H_2$, the collection $\mathbf{C}_l$ has also property (UC2).

To prove that $\mathbf{C}_l$ has property (UC3) it suffices to show that

(8) For every $H \in \mathcal{B}$ there exists $H_1 \in \mathcal{B}$ such that $\mathrm{St}\,(xH_1, \mathcal{C}_l(H_1)) \subset xH$ for all $x \in G$.

Since the formula $f(x_1, x_2, x_3) = x_1 x_2^{-1} x_3$ defines a continuous mapping $f : G \times G \times G \to G$ and since $f(e,e,e) = e$, for every $H \in \mathcal{B}$ there exists an $H_1 \in \mathcal{B}$ such that $f(H_1 \times H_1 \times H_1) = H_1 H_1^{-1} H_1 \subset H$; we shall show that H_1 satisfies (8). In fact, if for a fixed $x \in G$ we have $xH_1 \cap x_1 H_1 \neq \emptyset$, then $xh_0 = x_1 h_1$, i.e., $x_1 = xh_0 h_1^{-1}$ for some $h_0, h_1 \in H_1$; hence, for every element $x_1 h$ of $x_1 H_1$ we have $x_1 h = xh_0 h_1^{-1} h \in xH_1 H_1^{-1} H_1 \subset xH$, which shows that (8) holds.

For every pair x, y of distinct points of G, we have $x^{-1}y \neq e$. As G is a T_1-space, there exists an $H \in \mathcal{B}$ such that $x^{-1}y \notin H$. We shall show that no member of the cover $\mathcal{C}_l(H_1)$, where $H_1 \in \mathcal{B}$ satisfies $H_1^{-1}H_1 \subset H$, contains both x and y. Indeed, from the equalities $x = x_0 h_1$ and $y = x_0 h_2$, where $h_1, h_2 \in H_1$, it follows that $x^{-1}y = h_1^{-1}h_2 \in H_1^{-1}H_1 \subset H$ which contradicts the choice of the neighbourhood H_1. Thus $\mathbf{C}_l$ has also property (UC4).

Now, we shall check that the topology induced by the uniformity generated by $\mathbf{C}_l$ coincides with the original topology of G. Since the collection $\mathbf{C}_l$ consists of open covers of G it suffices to observe that for every $x \in G$ and any neighbourhood U of x there exists an $H \in \mathcal{B}$ such that $xH \subset U$, and to apply (8). In particular, we obtain the important corollary that every topological group is a Tychonoff space. ■

8.1.18. PROPOSITION. *Suppose we are given a set X and a family P of pseudometrics on the set X which has properties* (UP1)–(UP2). *The family $\mathcal{B} \subset \mathcal{D}_X$ of all entourages of the diagonal which are of the form $\{(x,y) : \rho(x,y) < 1/2^i\}$, where $\rho \in P$ and $i = 1, 2, \ldots$, is a base for a uniformity $\mathcal{U}$ on the set X. Every pseudometric $\rho \in P$ is uniform with respect to $\mathcal{U}$.*

If, moreover, X is a topological space and all pseudometrics of the family P are continuous functions from $X \times X$ to the real line, and if for every $x \in X$ and every non-empty closed set $A \subset X$ such that $x \notin A$ there exists a $\rho \in P$ such that $\inf\{\rho(x,a) : a \in A\} > 0$, then $\mathcal{U}$ is a uniformity on the space X.

The uniformity $\mathcal{U}$ is called the *uniformity generated by the family P of uniform pseudometrics.* ■

8.1.19. EXAMPLE. Let X be a Tychonoff space; denote by $C(X)$ the family of all continuous real-valued functions defined on X and by $C^*(X)$ the subfamily of $C(X)$ consisting of all bounded functions. For every finite sequence $f_1, f_2, \ldots, f_k$ of elements of $C(X)$ the formula

$$\rho_{f_1,f_2,\ldots,f_k}(x,y) = \max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|, \ldots, |f_k(x) - f_k(y)|\} \tag{9}$$

defines a pseudometric on the set X. Let P denote the family of all pseudometrics $\rho_{f_1,f_2,\ldots,f_k}$, where $f_1, f_2, \ldots, f_k \in C(X)$, and let P^* denote the subfamily of P consisting of all pseudometrics $\rho_{f_1,f_2,\ldots,f_k}$, where $f_1, f_2, \ldots, f_k \in C^*(X)$.

The families P and P^* both have properties (UP1)–(UP2) so that they generate uniformities $\mathcal{C}$ and $\mathcal{C}^*$ on the set X. We shall show that the topologies induced by $\mathcal{C}$ and $\mathcal{C}^*$ coincide with the original topology of X.

It follows from (9) that all pseudometrics in the family P are continuous functions from $X \times X$ to the real line. Consider an $x \in X$ and a non-empty closed set $A \subset X$ such that $x \notin A$. Since X is a Tychonoff space, there exists a function $f \in C^*(X) \subset C(X)$ satisfying $f(x) = 0$ and $f(A) \subset \{1\}$. The pseudometric $\rho_f \in P^* \subset P$ clearly satisfies the equality $\inf\{\rho_f(x,a) : a \in A\} = 1$, so that, by virtue of the second part of Proposition 8.1.18, $\mathcal{C}$ and $\mathcal{C}^*$ are uniformities on the space X. ■

In connection with the above described methods of generating uniformities, we call the reader's attention to the following simple fact: if $\mathcal{U}_0$ is a uniformity on a set X, then for a base $\mathcal{B}$ for the uniformity $\mathcal{U}_0$ (the collection $\mathbf{C}$ of all covers of X uniform with respect to $\mathcal{U}_0$, the family P of all pseudometrics on X uniform with respect to $\mathcal{U}_0$), the uniformity $\mathcal{U}$ on X generated by the base $\mathcal{B}$ (the collection $\mathbf{C}$, the family P) according to Proposition 8.1.14 (Proposition 8.1.16, Proposition 8.1.18), coincides with the original uniformity $\mathcal{U}_0$.

Example 8.1.19 and Corollary 8.1.13 yield

8.1.20. THEOREM. *The topology of a space X can be induced by a uniformity on the set X if and only if X is a Tychonoff space.* ■

Let X be a set and let ρ be a metric on the set X. Since the family $\{\rho\}$ consisting of the single pseudometric ρ has properties (UP1)–(UP2), it generates a uniformity $\mathcal{U}$ on the set X. Moreover, by virtue of Corollaries 4.2.6 and 4.1.11, the topologies induced on X by the metric ρ and by the uniformity $\mathcal{U}$ coincide. The uniformity $\mathcal{U}$ is called the *uniformity induced by the metric ρ*. A uniform space $(X, \mathcal{U})$ is *metrizable* if there exists a metric ρ on the set X such that the uniformity induced by ρ coincides with the original uniformity $\mathcal{U}$. Obviously, the question arises whether there exist internal characterizations of metrizable uniform spaces. Our next theorem contains such a characterizations.

8.1.21. THEOREM. *A uniformity $\mathcal{U}$ on a set X is induced by a metric on the set X if and only if $w(\mathcal{U}) \leq \aleph_0$.*

PROOF. Clearly, every uniformity induced by a metric has weight $\leq \aleph_0$.

Consider a uniformity $\mathcal{U}$, on a set X, which has a countable base $\{U_i\}_{i=1}^{\infty}$. There exists a sequence $V_0, V_1, \ldots$ of members of $\mathcal{U}$ such that

$$V_0 = X \times X, \quad 3V_{i+1} \subset V_i \quad \text{and} \quad V_i \subset U_i \quad \text{for} \quad i = 1, 2, \ldots$$

The last inclusion implies that $\bigcap_{i=1}^{\infty} V_i = \Delta$ so that the pseudometric ρ in Theorem 8.1.10, corresponding to our sequence $V_0, V_1, \ldots$ of members of $\mathcal{U}$, is a metric. The double inclusion in Theorem 8.1.10 immediately implies that ρ induces the original uniformity $\mathcal{U}$.■

Let us observe, in connection with the last theorem, that – as shown in Example 8.1.7 – it can happen that a metrizable topology on a set X is induced by a uniformity $\mathcal{U}$ on X which cannot be induced by a metric on X.

Let us also note that 8.1.21 and 8.1.18 imply Lemma 4.4.6; similarly, 8.1.21 and 8.1.16 imply Corollary 5.4.10.

We conclude this section with a brief discussion of uniformly continuous mappings. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces; a mapping f of the set X to the set Y is called *uniformly continuous with respect to the uniformities $\mathcal{U}$ and $\mathcal{V}$* if for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that for all $x, x' \in X$ we have $|f(x) - f(x')| < V$ whenever $|x - x'| < U$. It follows immediately from the definition that f is a continuous mapping of the space X with the topology induced by $\mathcal{U}$ to the space Y with the topology induced by $\mathcal{V}$. The fact that f is a uniformly continuous mapping with respect to the uniformities $\mathcal{U}$ and $\mathcal{V}$, on X and Y respectively, will be written in symbols as $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$.

The reader can easily verify that if $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ and $g : (Y, \mathcal{V}) \to (Z, \mathcal{W})$, then $gf : (X, \mathcal{U}) \to (Z, \mathcal{W})$, i.e., that the composition of uniformly continuous mappings is uniformly continuous.

As in the case of topological spaces, one can formulate criteria for uniform continuity of a mapping in terms of various methods of generating uniformities.

8.1.22. PROPOSITION. *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be uniform spaces and f a mapping of X to Y. The following conditions are equivalent:*

(i) *The mapping f is uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}$.*

(ii) *There exist bases $\mathcal{B}$ and $\mathcal{C}$ for $\mathcal{U}$ and $\mathcal{V}$ respectively, such that for every $V \in \mathcal{C}$ there exists a $U \in \mathcal{B}$ satisfying $U \subset (f \times f)^{-1}(V)$.*

(iii) *For every cover $\mathcal{A}$ of the set Y which is uniform with respect to $\mathcal{V}$ the cover $\{f^{-1}(A) : A \in \mathcal{A}\}$ of the set X is uniform with respect to $\mathcal{U}$.*

(iv) *For every pseudometric ρ on the set Y which is uniform with respct to $\mathcal{V}$ the pseudometric σ on the set X defined by the formula $\sigma(x, y) = \rho(f(x), f(y))$ is uniform with respect to $\mathcal{U}$.*■

A one-to-one mapping f of a set X onto a set Y is a *uniform isomorphism with respect to the uniformities $\mathcal{U}$ and $\mathcal{V}$* on the sets X and Y respectively, if f is uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}$ and the inverse mapping f^{-1} is uniformly continuous with respect to $\mathcal{V}$ and $\mathcal{U}$. A uniform isomorphism is a homeomorphism of the induced topological spaces.

We say that two uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are *uniformly isomorphic* if there exists a uniform isomorphism of $(X, \mathcal{U})$ onto $(Y, \mathcal{V})$. The study of *uniform properties* or *uniform invariants*, i.e., of invariants of uniform isomorphisms, is the subject of the theory of uniform spaces. Clearly, every topological invariant is an uniform invariant.

Historical and bibliographic notes

The concept of a uniform space was introduced and studied by Weil in [1938]. In [1936] Weil announced his results on uniform spaces using a different but equivalent concept of uniformity defined in terms of collections of covers. It should be added that in the years 1936 and 1937 the notion of "uniformity" has been also investigated by other mathematicians (see L. W. Cohen [1937] and [1939], Graves [1937] and Kurepa [1936] (cf. Kurepa [1976])) but none of them has developd a satisfactory theory. The first systematic exposition of the theory of uniform spaces was given by Bourbaki in [1940]. Our definition of a uniformity differs slightly from Weil's definition: by an entourage of the diagonal he understands any set $V \subset X \times X$ that contains Δ (i.e., he does not assume the symmetry condition $V = -V$) and by a uniformity he understands a family $\mathcal{U}$ of entourages of the diagonal which satisfies conditions (U1)–(U4) and together with any V contains the set $-V$. The restriction to symmetric entourages slightly simplifies our exposition.

Weil's booklet [1938] contains Theorems 8.1.1, 8.1.20, 8.1.21 and Example 8.1.17, as well as the definition of a uniformly continuous mapping and of a uniform isomorphism (Theorem 8.1.1 and Corollary 8.1.13 were announced in [1936], Example 8.1.17 was announced in [1936a]). In his proof of Theorem 1.8.20 Weil extends an unpublished argument of Pontrjagin who proved that every topological group is a Tychonoff space. The counterpart of Theorem 8.1.21 for topological groups (see Exercise 8.1.G) also was known earlier: it was established by Birkhoff in [1936] and by Kakutani in [1936]. Theorem 8.1.21 can be regarded as the uniform space version of the Alexandroff-Urysohn metrization theorem proved by those authors in [1923] (cf. notes on Theorem 5.4.9).

Tukey gave in [1940] an exposition of the theory of uniform spaces in terms of collections of covers; his axioms (close to conditions (UC1)–(UC4)) are more handy than the original Weil's axioms from [1936]. Tukey's booklet contains Theorem 8.1.10 formulated in terms of covers (see Exercise 5.4.H(a)). Nowadays the "cover" language seems to be more widely used than the "entourage of the diagonal" language. Isbell's book [1964], which contains an important development of the theory of uniform spaces, is written in terms of covers.

From Theorem 8.1.10 it follows that uniform spaces can be also described in terms of pseudometrics (see Problem 8.5.5); such a description was given by Bourbaki in [1948]. The "pseudometric" language is used in Gillman and Jerison's book [1960]. We follow that book when discussing uniformities C and C^* in Examples 8.1.19, 8.3.4, 8.3.18 and 8.3.19.

Exercises

8.1.A. (a) Let (X, ρ) and (Y, σ) be metric spaces and let $\mathcal{U}$ and $\mathcal{V}$ denote the uniformities induced by ρ and σ on X and Y respectively. Show that a mapping f of X to Y is uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}$ if and only if f is uniformly continuous with respect to ρ and σ (cf. Problem 8.5.19(a)).

(b) Note that two metrics ρ_1 and ρ_2 on a set X are uniformly equivalent (see Exercise 4.1.B(b)) if and only if they induce the same uniformity.

8.1.B. If $\mathcal{U}_1$ and $\mathcal{U}_2$ are two uniformities on a set X and $\mathcal{U}_2 \subset \mathcal{U}_1$, then we say that the uniformity $\mathcal{U}_1$ is *finer* than the uniformity $\mathcal{U}_2$ or that $\mathcal{U}_2$ is *coarser* than $\mathcal{U}_1$.

(a) Verify that if a uniformity $\mathcal{U}_1$ on a set X is finer than a uniformity $\mathcal{U}_2$, then the topology induced by $\mathcal{U}_1$ is finer than the topology induced by $\mathcal{U}_2$.

(b) Show that every family $\{\mathcal{U}_s\}_{s \in S}$ of uniformities on a set X has a least upper bound, i.e., that there exists a uniformity $\mathcal{U}$ on X which is coarser than any uniformity on X that is finer than all uniformities $\mathcal{U}_s$. Verify that if the uniformity $\mathcal{U}_s$ induces the topology $\mathcal{O}_s$, then the topology induced by the least upper bound of the family $\{\mathcal{U}_s\}_{s\in S}$ is the least upper bound of the family $\{\mathcal{O}_s\}_{s\in S}$ of topologies on the set X. Note that if each of the uniformities $\mathcal{U}_s$ induces the same topology $\mathcal{O}$ on X, then the least upper bound of the family $\{\mathcal{U}_s\}_{s\in S}$ also induces the topology $\mathcal{O}$.

(c) Give an example of two uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ on a set X of cardinality $\aleph_0$ such that there is no uniformity on X coarser than both $\mathcal{U}_1$ and $\mathcal{U}_2$.

8.1.C. The least upper bound of all uniformities on a Tychonoff space X, i.e., the finest uniformity on the space X, is called the *universal uniformity* on the space X (see Exercise 8.1.B(b); cf. Problem 8.5.8). We say that a uniform space $(X, \mathcal{U})$ is *fine*, if $\mathcal{U}$ is the universal uniformity on the space X with the topology induced by the uniformity $\mathcal{U}$.

(a) Show that every continuous mapping of a Tychonoff space X to a Tychonoff space Y is uniformly continuous with respect to the universal uniformity on the space X and any uniformity on the space Y. Observe that the above property characterizes the universal uniformity on X.

(b) Show that for every Tychonoff space X the collection of all normal covers of X has properties (UC1)–(UC4); observe that the uniformity generated by this collection is the universal uniformity on X.

(c) Show that for every Tychonoff space X the family of all pseudometrics on the set X which are continuous functions from the Cartesian product $X \times X$ to the real line has properties (UP1)–(UP2); observe that the uniformity generated by this family is the universal uniformity on X.

8.1.D. Verify that for every Tychonoff space X the uniformity $\mathcal{C}$ (the uniformity $\mathcal{C}^*$) is the coarsest uniformity in the family of all uniformities $\mathcal{U}$ on the space X such that every continuous function $f : X \to R$ (every continuous function $f : X \to I$) is uniformly continuous with respect to $\mathcal{U}$ and the uniformity induced on the real line (the closed unit interval) by the natural metric.

8.1.E. Let X be a topological space, x a point of X and F a closed subset of X such that $x \notin F$. Show that for every uniformity $\mathcal{U}$ on the space X there exists a mapping $f : X \to I$, uniformly continuous with respect to $\mathcal{U}$ and the uniformity on I induced by the natural metric, such that $f(x) = 0$ and $f(F) \subset \{1\}$.

8.1.F. Suppose we are given a set X, a family $\{(Y_s, \mathcal{U}_s)\}_{s\in S}$ of uniform spaces and a family of mappings $\{f_s\}_{s\in S}$, where f_s is a mapping of X to Y_s. Show that in the class of all uniformities on the set X that make all the f_s's uniformly continuous there exists a coarsest uniformity $\mathcal{U}$. Define a base for the uniformity $\mathcal{U}$ and prove that the topology induced on the set X by $\mathcal{U}$ coincides with the topology generated by the family of mappings $\{f_s\}_{s\in S}$, where Y_s has the topology induced by $\mathcal{U}_s$. The uniformity $\mathcal{U}$ is called the *uniformity generated by the family of mappings* $\{f_s\}_{s\in S}$.

8.1.G. (a) (Birkhoff [1936], Kakutani [1936]) Show that a topological group is metrizable if and only if it is first-countable.

(b) Deduce (a) from Bing's metrization criterion.

8.1.H (Tukey [1940]). Show that for every Tychonoff space X the collection of all covers of X which have a finite normal refinement has properties (UC1)–(UC4); observe that the uniformity generated by this collection is the uniformity $\mathcal{C}^*$.

Hint (Isbell [1964]). When checking property (UC3), verify that if $\mathcal{B}$ is a star refinement of a finite cover $\mathcal{A}$, and for every pair $\mathcal{A}_0, \mathcal{A}_1$ of subfamilies of $\mathcal{A}$ one defines $\mathcal{B}(\mathcal{A}_0, \mathcal{A}_1)$ as the family of all $B \in \mathcal{B}$ such that $\mathcal{A}_0 = \{A \in \mathcal{A} : B \subset A\}$ and $\mathcal{A}_1 = \{A \in \mathcal{A} : \operatorname{St}(B, \mathcal{B}) \subset A\}$, then the sets $\bigcup \mathcal{B}(\mathcal{A}_0, \mathcal{A}_1)$ form a finite star refinement of $\mathcal{A}$. In the proof of the last part apply Corollary 8.1.11.

8.1.I. (a) (Shirota [1952]) Show that for every Tychonoff space X the collection of all covers of X which have a countable normal refinement has properties (UC1)–(UC4).

Hint (Kucia [1973]). When checking property (UC3) verify that if $\mathcal{B}$ is a star refinement of a countable cover $\mathcal{A} = \{A_i\}_{i=1}^{\infty}$ and for every natural number k and each subfamily $\mathcal{A}_0$ of $\{A_i\}_{i=1}^{k}$ one defines $\mathcal{B}_k(\mathcal{A}_0)$ as the family of all $B \in \mathcal{B}$ such that $\mathcal{A}_0 = \{A_i : B \subset A_i \text{ and } i \le k\}$ and $\operatorname{St}(B, \mathcal{B}) \subset A_k$, then the sets $\mathcal{B}_k(\mathcal{A}_0)$ form a countable barycentric refinement of $\mathcal{A}$.

(b) Show that the uniformity generated by the collection of all covers of a Tychonoff space X which have a countable normal refinement is finer than the uniformity $\mathcal{C}$ and generally differs from $\mathcal{C}$ (cf. Example 8.3.19 and Exercise 8.3.F).

Hint. Every cover uniform with respect to $\mathcal{C}$ has a refinement of finite order.

8.2. Operations on uniform spaces

Suppose we are given a uniform space $(X, \mathcal{U})$ and a set $M \subset X$. It is easy to see that the family $\mathcal{U}_M = \{(M \times M) \cap V : V \in \mathcal{U}\} \subset \mathcal{D}_M$ satisfies conditions (U1)–(U4), i.e., that $(M, \mathcal{U}_M)$ is a uniform space. The uniform space $(M, \mathcal{U}_M)$ is called a *subspace of the uniform space* $(X, \mathcal{U})$. One readily verifies that the topology induced on M by the uniformity $\mathcal{U}_M$ coincides with the topology of the subspace of X, where X has the topology induced by $\mathcal{U}$.

If the uniformity $\mathcal{U}$ is induced by a metric ρ on the set X, then the uniformity $\mathcal{U}_M$ coincides with the uniformity induced by the metric ρ_M on the set M.

For every uniform space $(X, \mathcal{U})$ and any set $M \subset X$, the formula $i_M(x) = x$ defines a uniformly continuous mapping $i_M : (M, \mathcal{U}_M) \to (X, \mathcal{U})$; the mapping i_M is called the *embedding of the subspace* $(M, \mathcal{U}_M)$ *in the space* $(X, \mathcal{U})$.

We pass to Cartesian products of uniform spaces. Let $\{(X_s, \mathcal{U}_s)\}_{s \in S}$ be a family of uniform spaces. The family $\mathcal{B}$ of all entourages of the diagonal $\Delta \subset (\prod_{s \in S} X_s) \times (\prod_{s \in S} X_s)$ which are of the form

$$\{(\{x_s\}, \{y_s\}) : |x_{s_i} - y_{s_i}| < V_i \text{ for } i = 1, 2, \ldots, k\},$$

where $s_1, s_2, \ldots, s_k \in S$ and $V_i \in \mathcal{U}_{s_i}$ for $i = 1, 2, \ldots, k$, has properties (BU1)–(BU3), so that – by virtue of Proposition 8.1.14 – the family $\mathcal{B}$ generates a uniformity on the set $\prod_{s \in S} X_s$; this uniformity is called the *Cartesian product of the uniformities* $\{\mathcal{U}_s\}_{s \in S}$ and is denoted by $\prod_{s \in S} \mathcal{U}_s$. The Cartesian product of a finite family of uniformities $\{\mathcal{U}_i\}_{i=1}^{k}$ is denoted by $\mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_k$. If all the uniformities $\mathcal{U}_s$ are equal to each other, i.e., if $X_s = X$ and $\mathcal{U}_s = \mathcal{U}$ for $s \in S$, then the Cartesian product $\prod_{s \in S} \mathcal{U}_s$ is also denoted by $\mathcal{U}^{\mathfrak{m}}$, where $\mathfrak{m} = |S|$. The uniform space $(\prod_{s \in S} X_s, \prod_{s \in S} \mathcal{U}_s)$ is called the *Cartesian product of the uniform spaces*

$\{(X_s, \mathcal{U}_s)\}_{s\in S}$. The reader can easily verify that the topology induced on $\prod_{s\in S} X_s$ by the uniformity $\prod_{s\in S} \mathcal{U}_s$ coincides with the Tychonoff topology of the Cartesian product $\prod_{s\in S} X_s$, where X_s has the topology induced by $\mathcal{U}_s$.

If for $i = 1, 2, \ldots$ the uniformity $\mathcal{U}_i$ on a set X_i is induced by a metric ρ_i on the space X_i bounded by 1, then the uniformity $\prod_{i=1}^{\infty} \mathcal{U}_i$ on the set $\prod_{i=1}^{\infty} X_i$ coincides with the uniformity induced by the metric ρ defined by formula (2) in Section 4.2.

One readily sees that if $(\prod_{s\in S} X_s, \prod_{s\in S} \mathcal{U}_s)$ is the Cartesian product of the uniform spaces $\{(X_s, \mathcal{U}_s)\}_{s\in S}$, then for every $s \in S$ the projection p_s of $\prod_{s\in S} X_s$ onto X_s, defined by the formula $p_s(\{x_s\}) = x_s$, is uniformly continuous with respect to $\prod_{s\in S} \mathcal{U}_s$ and $\mathcal{U}_s$.

We have the following uniform counterpart of Proposition 2.3.6:

8.2.1. PROPOSITION. *Let $(X, \mathcal{U})$ be a uniform space, $\{(Y_s, \mathcal{V}_s)\}_{s\in S}$ a family of uniform spaces and f a mapping of the set X to the Cartesian product $\prod_{s\in S} Y_s$. The mapping f is uniformly continuous with respect to $\mathcal{U}$ and $\prod_{s\in S} \mathcal{V}_s$ if and only if the composition $p_s f$ is uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}_s$ for every $s \in S$.*■

8.2.2. EXAMPLE. Consider the interval I with the uniformity $\mathcal{U}$ induced by the natural metric on I. The uniformity $\mathcal{U}^{\mathfrak{m}}$ induces on the Cartesian product $I^{\mathfrak{m}}$, where $\mathfrak{m} \geq \aleph_0$, the topology of the Tychonoff cube of weight $\mathfrak{m}$. One readily sees that $w(\mathcal{U}^{\mathfrak{m}}) \leq \mathfrak{m}$, and from Corollary 2.3.25 it follows that $w(\mathcal{U}^{\mathfrak{m}}) = \mathfrak{m}$.

Every Tychonoff space X of weight $\leq \mathfrak{m}$ can be regarded as a subspace of $I^{\mathfrak{m}}$. Since the uniformity $\mathcal{U}^{\mathfrak{m}}$ induces on $I^{\mathfrak{m}}$ the topology of the Tychonoff cube, the uniformity $(\mathcal{U}^{\mathfrak{m}})_X$ induces the original topology on X. Hence, the topology of every Tychonoff space of weight $\leq \mathfrak{m}$ can be induced by a uniformity of weight $\leq \mathfrak{m}$.■

8.2.3. THEOREM. *Every uniform space is uniformly isomorphic to a subspace of the Cartesian product of a family of metrizable uniform spaces.*

PROOF. Consider a uniform space $(X, \mathcal{U})$. By virtue of (U3), for every $V \in \mathcal{U}$ there exists a sequence $V_0, V_1, \ldots$ of members of $\mathcal{U}$ such that

$$V_0 = X \times X, \quad V_1 = V \quad \text{and} \quad 3V_{i+1} \subset V_i \quad \text{for} \quad i = 1, 2, \ldots \tag{1}$$

Applying Theorem 8.1.10, take a pseudometric ρ_V on the set X such that for every $i \geq 1$

$$\{(x, y) : \rho_V(x, y) < 1/2^i\} \subset V_i \subset \{(x, y) : \rho_V(x, y) \leq 1/2^i\}. \tag{2}$$

By letting $xE_V y$ whenever $\rho_V(x, y) = 0$ we define an equivalence relation E_V on the set X (cf. Exercise 4.2.I); let X_V be the set of all equivalence classes of E_V. It follows from the triangle inequality that for all $x, x', y, y' \in X$ such that $xE_V x'$ and $yE_V y'$ we have

$$\rho_V(x, y) = \rho_V(x', y').$$

Therefore, by letting $\overline{\rho}_V([x], [y]) = \rho_V(x, y)$ for all $[x], [y] \in X_V$ we define a metric $\overline{\rho}_V$ on the set X_V. Let $\mathcal{U}_V$ be the uniformity on the set X_V induced by the metric $\overline{\rho}_V$. It follows from (2), that letting $f_V(x) = [x]$, we define a uniformly continuous mapping f_V of $(X, \mathcal{U})$ to $(X_V, \mathcal{U}_V)$.

Proposition 8.2.1 implies that the diagonal $\triangle_{V \in \mathcal{U}} f_V$ is a uniformly continuous mapping of $(X, \mathcal{U})$ to the Cartesian product $(\prod_{V \in \mathcal{U}} X_V, \prod_{V \in \mathcal{U}} \mathcal{U}_V)$. We shall show that the restriction $f = (\triangle_{V \in \mathcal{U}} f_V) | X$ is a uniform isomorphism.

From (U4), (1) and (2) it follows that for every pair x, y of distinct points of X there exists a $V \in \mathcal{U}$ such that $\rho_V(x, y) > 0$; hence, we have $f(x) \neq f(y)$ which shows that f is a one-to-one mapping.

It remains to prove that f^{-1} is uniformly continuous with respect to $(\prod_{V \in \mathcal{U}} \mathcal{U}_V)_{f(X)}$ and $\mathcal{U}$, i.e., to show that for every $V_0 \in \mathcal{U}$ there exists a $W \in \prod_{V \in \mathcal{U}} \mathcal{U}_V$ such that $|x - y| < V_0$ whenever $|f(x) - f(y)| < W$. However, it follows from (1) and (2) that

$$W = \{(\{x_V\}, \{y_V\}) : \rho_{V_0}(x_{V_0}, y_{V_0}) < 1/2\}$$

has the required property. ■

8.2.4. REMARK. Let us note that if in the proof of the above theorem we considered only V's belonging to a base for the uniformity $\mathcal{U}$, then the mapping f would also be a uniform isomorphism. Hence, every uniform space of weight $\mathfrak{m}$ is uniformly isomorphic to a subspace of the Cartesian product of $\mathfrak{m}$ metrizable uniform spaces.

Let us also observe that there is no universal space for all uniform spaces of weight $\leq \mathfrak{m}$, i.e., there is no uniform space $(X, \mathcal{U})$ of weight $\leq \mathfrak{m}$ that, for every uniform space $(Y, \mathcal{V})$ of weight $\leq \mathfrak{m}$, contains a subspace uniformly isomorphic to $(Y, \mathcal{V})$. Indeed, it follows from Example 8.1.6 that the weight of every discrete uniform space equals 1, and since the cardinality of the set X can be arbitrarily large, there is no uniform space containing subspaces uniformly isomorphic to all discrete uniform spaces.

In the remaining part of this section we shall discuss function spaces.

Let X be a topological space and let $(Y, \mathcal{U})$ be a uniform space. We shall denote by Y^X the set of all continuous mappings of the space X to the space Y, where Y is equipped with the topology induced by $\mathcal{U}$. For every $V \in \mathcal{U}$ denote by $\hat{V}$ the entourage of the diagonal $\Delta \subset Y^X \times Y^X$ defined by the formula

$$\hat{V} = \{(f, g) : |f(x) - g(x)| < V \quad \text{for every} \quad x \in X\}.$$

From readily established formulas

$$\hat{U} \cap \hat{V} = \widehat{U \cap V} \quad \text{and} \quad \hat{U} + \hat{V} \subset \widehat{U + V}$$

it follows that the family $\{\hat{V} : V \in \mathcal{U}\}$ has properties (BU1)–(BU3); the uniformity on the set Y^X generated by this family will be called the *uniformity of uniform convergence* induced by $\mathcal{U}$ and will be denoted by $\hat{\mathcal{U}}$.

If the uniformity $\mathcal{U}$ is induced by a bounded metric ρ on Y, then $w(\hat{\mathcal{U}}) \leq \aleph_0$, so that – by Theorem 8.1.21 – the uniformity $\hat{\mathcal{U}}$ is induced by a metric on Y^X. One readily verifies that the metric $\hat{\rho}$ defined by formula (7) in Section 4.2 induces the uniformity $\hat{\mathcal{U}}$. Hence, it follows from Example 4.2.14 that for two uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ on Y which induce the same topology, the topologies on Y^X induced by $\hat{\mathcal{U}}_1$ and $\hat{\mathcal{U}}_2$ can be different. It turns out, however, that for a compact space X – as in the case of metric spaces – the topology on Y^X is independent of the choice of a particular uniformity $\mathcal{U}$ on the space Y, because the topology induced by $\hat{\mathcal{U}}$ coincides with the compact-open topology on Y^X. This fact is a corollary to Theorem 8.2.6 proved below; to formulate the theorem we have to introduce another uniformity on Y^X.

For a Hausdorff space X and a uniform space $(Y, \mathcal{U})$ we shall denote by $\hat{\mathcal{U}}|\mathcal{Z}(X)$ the uniformity on Y^X generated by the base consisting of all finite intersections of the sets of the form

$$\hat{V}|Z = \{(f,g) : |f(x) - g(x)| < V \quad \text{for every} \quad x \in Z\}, \tag{3}$$

where $V \in \mathcal{U}$, $Z \in \mathcal{Z}(X)$ and $\mathcal{Z}(X)$ is the family of all compact subsets of X (the reader can easily check that the family of all finite intersections of the sets in (3) has properties (BU1)–(BU3)). The uniformity $\hat{\mathcal{U}}|\mathcal{Z}(X)$ will be called the *uniformity of uniform convergence on compacta* induced by $\mathcal{U}$.

8.2.5. LEMMA. *If $\mathcal{U}$ is a uniformity on a space X, then for every compact set $Z \subset X$ and any open set G containing Z there exists a $V \in \mathcal{U}$ such that $B(Z, V) \subset G$.*

PROOF. For every $x \in Z$ choose a $V_x \in \mathcal{U}$ such that $B(x, 2V_x) \subset G$. By Corollary 8.1.3, the family $\{Z \cap \operatorname{Int} B(x, V_x)\}_{x \in Z}$ is an open cover of the set Z, so that there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset Z$ such that

$$Z \subset \operatorname{Int} B(x_1, V_{x_1}) \cup \operatorname{Int} B(x_2, V_{x_2}) \cup \ldots \cup \operatorname{Int} B(x_k, V_{x_k}); \tag{4}$$

let $V = V_{x_1} \cap V_{x_2} \cap \ldots \cap V_{x_k}$. It follows from (4) that for any $x \in Z$ there exists an $i \leq k$ such that $|x - x_i| < V_{x_i}$. For every $x' \in B(x, V) \subset B(x, V_{x_i})$ we have $x' \in B(x_i, 2V_{x_i}) \subset G$, so that $B(Z, V) \subset G$.■

8.2.6. THEOREM. *For every Hausdorff space X and any uniform space $(Y, \mathcal{U})$ the topology on Y^X induced by the uniformity $\hat{\mathcal{U}}|\mathcal{Z}(X)$ of uniform convergence on compacta coincides with the compact-open topology on Y^X, where Y has the topology induced by $\mathcal{U}$.*

PROOF. Denote by $\mathcal{O}_1$ the topology on Y^X induced by the uniformity $\hat{\mathcal{U}}|\mathcal{Z}(X)$ and by $\mathcal{O}_2$ the compact-open topology. First we shall prove that $\mathcal{O}_2 \subset \mathcal{O}_1$. Clearly, it suffices to show that all sets $M(Z, G)$, where $Z \in \mathcal{Z}(X)$ and G is an open subset of Y, belong to $\mathcal{O}_1$. Consider a $Z \in \mathcal{Z}(X)$, an open set $G \subset Y$ and an $f \in M(Z, G)$. Since Y is a Tychonoff space, $f(Z)$ is a compact subspace of G. Applying Lemma 8.2.5, take a $V \in \mathcal{U}$ such that $B(f(Z), V) \subset G$. We clearly have $B(f, \hat{V}|Z) \subset M(Z, G)$, and f being an arbitrary element of $M(Z, G)$, this implies that $M(Z, G) \in \mathcal{O}_1$.

We shall now prove that $\mathcal{O}_1 \subset \mathcal{O}_2$. Clearly, it suffices to show that for any $Z \in \mathcal{Z}(X), V \in \mathcal{U}$ and $f \in Y^X$ there exist compact subsets $Z_1, Z_2, \ldots, Z_k$ of X and open subsets $G_1, G_2, \ldots, G_k$ of Y such that

$$f \in \bigcap_{i=1}^{k} M(Z_i, G_i) \subset B(f, \hat{V}|Z).$$

By Corollary 8.1.12 there exists an entourage $W \in \mathcal{U}$ of the diagonal $\Delta \subset Y \times Y$ which is closed with respect to the topology induced by $\mathcal{U}$ on $Y \times Y$ and satisfies the inclusion $3W \subset V$.

It follows from the compactness of $f(Z)$ that there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset Z$ such that $f(Z) \subset \bigcup_{i=1}^k B(f(x_i), W)$. We shall show that the sets

$$Z_i = Z \cap f^{-1}(B(f(x_i), W)) \quad \text{and} \quad G_i = \operatorname{Int} B(f(x_i), 2W)$$

have the required properties.

To begin, let us observe that from the closedness of W in $Y \times Y$ follows the closedness of the balls $B(f(x_i), W)$ in Y and the compactness of the sets Z_i. Moreover, $f \in \bigcap_{i=1}^k M(Z_i, G_i)$. Let us now consider a mapping $g \in \bigcap_{i=1}^k M(Z_i, G_i)$. For every $x \in Z$ there exists an $i \leq k$ such that $x \in Z_i$; we clearly have

$$g(x) \in B(f(x_i), 2W) \quad \text{and} \quad f(x) \in B(f(x_i), W).$$

Hence, $|f(x) - g(x)| < 3W \subset V$ for every $x \in Z$, and this shows that $g \in B(f, \hat{V}|Z)$. ∎

8.2.7. COROLLARY. *For every compact space X and any uniform space $(Y, \mathcal{U})$ the topology on Y^X induced by the uniformity $\hat{\mathcal{U}}$ of uniform convergence coincides with the compact-open topology on Y^X, and depends only on the topology induced on Y by the uniformity $\mathcal{U}$.* ∎

Let us observe that Theorem 8.2.6 implies Theorem 3.4.15 in the case of a Hausdorff space X. In fact, for every Tychonoff space Y there exists a uniformity $\mathcal{U}$ on Y which induces the original topology of the space Y; by Theorem 8.2.6, the compact-open topology on Y^X is induced by a uniformity on Y^X so that the space Y^X with the compact-open topology is a Tychonoff space.

For mappings of a topological space X to a Tychonoff space Y one can define the notion of equicontinuity with respect to a uniformity $\mathcal{U}$ on the space Y: we say that a family F of mappings of X to Y is *equicontinuous with respect to a uniformity $\mathcal{U}$* on the space Y if for every $x \in X$ and any $V \in \mathcal{U}$ there exists a neighbourhood G of the point x such that $|f(x) - f(x')| < V$ whenever $f \in F$ and $x' \in G$. With equicontinuity thus defined, counterparts of the classical Ascoli theorem hold. Before stating those counterparts we shall state and prove two lemmas to show how equicontinuity is related to the notion of even continuity introduced in Section 3.4.

Let us remind the reader that a family of mappings $F \subset Y^X$ is evenly continuous if for every $x \in X$, every $y \in Y$ and any neighbourhood H of y there exist a neighbourhood G of x and a neighbourhood H' of y such that the conditions $f \in F$ and $f(x) \in H'$ imply the inclusion $f(G) \subset H$.

8.2.8. LEMMA. *Let X be a topological space, Y a Tychonoff space and $\mathcal{U}$ a uniformity on the space Y. If a family $F \subset Y^X$ of mappings of X to Y is equicontinuous with respect to $\mathcal{U}$, then the family F is evenly continuous.*

PROOF. Let $F \subset Y^X$ be equicontinuous with respect to $\mathcal{U}$; consider $x \in X$, $y \in Y$ and a neighbourhood H of y. Take a $V \in \mathcal{U}$ satisfying $B(y, 2V) \subset H$ and a neighbourhood G of the point x such that $|f(x) - f(x')| < V$ whenever $f \in F$ and $x' \in G$. Let $H' = \operatorname{Int} B(y, V)$ and consider an $f \in F$ such that $f(x) \in H'$. Since $f(x) \in H'$, $|y - f(x)| < V$, so that for every $x' \in G$ we have $|y - f(x')| < 2V$, i.e., $f(x') \in B(y, 2V) \subset H$. Hence $f(G) \subset H$ which shows that the family F is evenly continuous. ∎

8.2.9. LEMMA. *Let X be a topological space, Y a Tychonoff space and $\mathcal{U}$ a uniformity on the space Y. If a family $F \subset Y^X$ of mappings of X to Y is evenly continuous and for every $x \in X$ the set $\{f(x) : f \in F\} \subset Y$ has a compact closure, then the family F is equicontinuous with respect to the uniformity $\mathcal{U}$.*

PROOF. Assume that $F \subset Y^X$ is evenly continuous and the sets $A(x) = \overline{\{f(x) : f \in F\}}$ are compact; consider a point $x \in X$ and a $V \in \mathcal{U}$. Take a $W \in \mathcal{U}$ such that $2W \subset V$ and for every $y \in A(x)$ and the neighbourhood $H(y) = \operatorname{Int} B(y, W)$ of y choose a neighbourhood $G(y)$ of x and a neighbourhood $H'(y)$ of y such that the conditions $f \in F$ and $f(x) \in H'(y)$ imply the inclusion $f(G(y)) \subset H(y)$. The set $A(x)$ being compact, there exists a finite set $\{y_1, y_2, \ldots, y_k\} \subset A(x)$ such that $A(x) \subset \bigcup_{i=1}^k H'(y_i)$.

Let f be a function in F and x' a point in the neighbourhood $G = \bigcap_{i=1}^k G(y_i)$ of the point x. There exists an $i \leq k$ such that $f(x) \in H'(y_i)$, so that we have $f(G(y_i)) \subset H(y_i)$. Since x and x' belong to $G(y_i)$, we have $f(x), f(x') \in H(y_i)$; thus $|f(x) - f(x')| < V$, which shows that the family F is equicontinuous with respect to $\mathcal{U}$.■

The above lemmas and Theorem 3.4.20 yield

8.2.10. THE ASCOLI THEOREM. *Let X be a k-space, Y a Tychonoff space and $\mathcal{U}$ a uniformity on the space Y. A closed subset F of the space Y^X with the compact-open topology is compact if and only if F is equicontinuous with respct to $\mathcal{U}$ and the set $\{f(x) : f \in F\} \subset Y$ has a compact closure for every $x \in X$.*■

The following variant of the Ascoli theorem is a counterpart of Theorem 3.4.21; it is a consequence of Lemmas 8.2.8, 8.2.9 and Theorem 3.4.21. In the statement of the theorem we use the symbol $F|Z$, where $F \subset Y^X$ and $Z \subset X$, to denote the family of restrictions $\{f|Z : f \in F\} \subset Y^Z$.

8.2.11. THEOREM. *Let X be a k-space, Y a Tychonoff space and $\mathcal{U}$ a uniformity on the space Y. A closed subset F of the space Y^X with the compact-open topology is compact if and only if for each compact $Z \subset X$ the family $F|Z$ is equicontinuous with respect to $\mathcal{U}_Z$ and the set $\{f(x) : f \in F\} \subset Y$ has a compact closure for every $x \in X$.*■

Historical and bibliographic notes

Subspaces and Cartesian products of uniform spaces were defined by Weil in [1938]; Theorem 8.2.3 also is proved there. The uniformity of uniform convergence on compacta was introduced and Theorem 8.2.6 was proved in Arens' paper [1946]. The uniformity of uniform convergence was introduced and Theorem 8.2.10 was proved for a locally compact X in Bourbaki's book [1949]; generalization of Theorem 8.2.10 to k-spaces was obtained by Bagley and Yang in [1966]. Theorem 8.2.11 was proved by Kelley in [1955]; it is implicitly stated as an exercise by Bourbaki in [1949].

Exercises

8.2.A. (a) Let $\{(X_s, \mathcal{U}_s)\}_{s\in S}$ be a family of uniform spaces. Show that the uniformity $\prod_{s\in S} \mathcal{U}_s$ on the set $\prod_{s\in S} X_s$ coincides with the uniformity generated by the family of mappings $\{p_s\}_{s\in S}$, where p_s is the projection of $\prod_{s\in S} X_s$ onto X_s (see Exercise 8.1.F).

(b) Show that the uniform space $(X, \mathcal{U})$ is uniformly isomorphic to the Cartesian product $(\prod_{s\in S} X_s, \prod_{s\in S} \mathcal{U}_s)$ if and only if there exists a family $\{p_s\}_{s\in S}$ of uniformly continuous mappings, where $p_s : (X, \mathcal{U}) \to (X_s, \mathcal{U}_s)$, satisfying the uniform counterparts of conditions (1) and (2) in Exercise 2.3.H.

8.2.B. (a) Let $\{(X_s, \mathcal{U}_s)\}_{s\in S}$ be a family of uniform spaces such that $X_s \cap X_{s'} = \emptyset$ for $s \neq s'$. Check that the family $\mathcal{U} = \{\bigcup_{s\in S} V_s : V_s \in \mathcal{U}_s\}$ is a uniformity on the set $X = \bigcup_{s\in S} X_s$ and that for every $s \in S$ the formula $i_s(x) = x$ defines a uniform isomorphism of $(X_s, \mathcal{U}_s)$ onto the subspace $(X_s, \mathcal{U}_{X_s})$ of $(X, \mathcal{U})$. Observe that the topology induced by $\mathcal{U}$ on X coincides with the topology of the sum $\bigoplus_{s\in S} X_s$, where X_s has the topology induced by $\mathcal{U}_s$. The uniform space $(\bigcup_{s\in S} X_s, \mathcal{U})$ is called the *sum of the uniform spaces* $\{(X_s, \mathcal{U}_s)\}_{s\in S}$. Define the sum of an arbitrary family $\{(X_s, \mathcal{U}_s)\}_{s\in S}$ of uniform spaces in such a way that the uniform counterpart of Exercise 2.2.F holds.

(b) Let $(X, \mathcal{U})$ be a uniform space and E an equivalence relation on the set X. Observe that the finest uniformity $\mathcal{V}$ on the set X/E which makes the natural mapping of X to X/E uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}$, induces on the set X/E a topology which generally is different from the quotient topology on X/E, where X is taken with the topology induced by $\mathcal{U}$. Check that for the uniformity $\mathcal{V}$, the uniform counterpart of Proposition 2.4.2 holds.

(c) An *inverse system of uniform spaces* is a family $\mathbf{S} = \{(X_\sigma, \mathcal{U}_\sigma), \pi^\sigma_\rho, \Sigma\}$, where $\mathbf{S}' = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ is an inverse system of topological spaces and for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$ the mapping π^σ_ρ is uniformly continuous with respect to $\mathcal{U}_\sigma$ and $\mathcal{U}_\rho$. The subspace $(X, \mathcal{U})$ of the Cartesian product $(\prod_{\sigma\in\Sigma} X_\sigma, \prod_{\sigma\in\Sigma} \mathcal{U}_\sigma)$, where $X = \varprojlim \mathbf{S}'$ and $\mathcal{U} = (\prod_{\sigma\in\Sigma} \mathcal{U}_\sigma)_X$, is called the *limit of the inverse system* $\mathbf{S}$. Check that for $(X, \mathcal{U})$ and the mappings $\pi_\sigma : (X, \mathcal{U}) \to (X_\sigma, \mathcal{U}_\sigma)$ the uniform counterpart of Exercise 2.5.F holds.

8.2.C. Let $(X, \mathcal{U})$ be a uniform space. Verify that the topology induced by the uniformity $\mathcal{U}$ on the set $X \times X$ coincides with the topology induced by the Cartesian product $\mathcal{U} \times \mathcal{U}$. Show that a pseudometric ρ on a set X is uniform with respect to a uniformity $\mathcal{U}$ on the set X if and only if ρ is a uniformly continuous mapping of $(X \times X,\ \mathcal{U} \times \mathcal{U})$ to the real line with the uniformity induced by the natural metric.

8.2.D. Let X be a topological space and $(Y, \mathcal{V})$ a uniform space. Applying 2.6.11, 3.4.1 and 2.6.12, show that the topology induced on Y^X by the uniformity of uniform convergence is finer than the compact-open topology.

8.2.E. Note that Lemma 8.2.5 implies Theorem 3.2.10 under the additional assumption that all spaces X_s are Tychonoff spaces.

8.3. Totally bounded and complete uniform spaces. Compactness in uniform spaces

Let $(X, \mathcal{U})$ be a uniform space, V a member of the uniformity $\mathcal{U}$, and A a subset of X; we say that A is *V-dense* in $(X, \mathcal{U})$ if for every $x \in X$ there exists an $x' \in A$ such that $|x - x'| < V$.

A uniform space $(X, \mathcal{U})$ is *totally bounded* if for every $V \in \mathcal{U}$ there exists a finite set $A \subset X$ which is V-dense in $(X, \mathcal{U})$; a uniformity $\mathcal{U}$ on a set X is *totally bounded* if the space $(X, \mathcal{U})$ is totally bounded.

One readily verifies that if there exists a uniformly continuous mapping f of a totally bounded uniform space $(X, \mathcal{U})$ to a uniform space $(Y, \mathcal{V})$ such that $f(X) = Y$, then the space $(Y, \mathcal{V})$ also is totally bounded; in particular, total boundedness is a uniform invariant.

8.3.1. PROPOSITION. *If the uniformity $\mathcal{U}$ on a set X is induced by a metric ρ, then the uniform space $(X, \mathcal{U})$ is totally bounded if and only if the metric space (X, ρ) is totally bounded.* ∎

The above definition yields a class of uniform spaces; the question arises whether there exists an internal characterization of topological spaces whose topology can be induced by a totally bounded uniformity. Such a characterization exists but is not interesting: we shall show in Example 8.3.4 that the topology of any Tychonoff space can be induced by a totally bounded uniformity.

We shall now state two theorems concerning operations on totally bounded uniform spaces. The proof of the first one is obtained from the proof of Theorem 4.3.2 by taking a $V \in \mathcal{U}$ instead of an $\epsilon > 0$ and by replacing $\epsilon/2$ by a $W \in \mathcal{U}$ such that $2W \subset V$. To prove the second theorem it suffices to consider the base for the uniformity $\prod_{s \in S} \mathcal{U}_s$ described in the previous section, observe that each space $(X_s, \mathcal{U}_s)$ is uniformly isomorphic to a subspace of the Cartesian product $(\prod_{s \in S} X_s, \prod_{s \in S} \mathcal{U}_s)$ and then follow the pattern of the proof of Theorem 4.3.3.

8.3.2. THEOREM. *If $(X, \mathcal{U})$ is a totally bounded uniform space, then for every subset M of X the space $(M, \mathcal{U}_M)$ is totally bounded.*

If $(X, \mathcal{U})$ is an arbitrary uniform space and for a subset M of X the space $(M, \mathcal{U}_M)$ is totally bounded, then the space $(\overline{M}, \mathcal{U}_M)$, where $\overline{M}$ is the closure of M with respect to the topology induced by $\mathcal{U}$, also is totally bounded. ∎

8.3.3. THEOREM. *Let $\{(X_s, \mathcal{U}_s)\}_{s \in S}$ be a family of non-empty uniform spaces. The Cartesian product $(\prod_{s \in S} X_s, \prod_{s \in S} \mathcal{U}_s)$ is totally bounded if and only if all spaces $(X_s, \mathcal{U}_s)$ are totally bounded.* ∎

8.3.4. EXAMPLE. Let X be a Tychonoff space and $\mathcal{C}^*$ the uniformity on the space X defined in Example 8.1.19; we shall show that the uniform space $(X, \mathcal{C}^*)$ is totally bounded.

It suffices to prove that for every finite sequence $f_1, f_2, \ldots, f_k$ of elements of $C^*(X)$ and any $\epsilon > 0$ one can define a finite set $A \subset X$ such that for every $x \in X$ there exists an $x' \in A$ satisfying

$$\rho_{f_1, f_2, \ldots, f_k}(x, x') = \max\{|f_1(x) - f_1(x')|, |f_2(x) - f_2(x')|, \ldots, |f_k(x) - f_k(x')|\} < \epsilon.$$

Take a bounded closed interval $J \subset R$ which contains all the sets $f_1(X), f_2(X), \ldots, f_k(X)$ and consider a cover $\{A_i\}_{i=1}^m$ of the interval J by sets of diameter less than ϵ. The sets of the form

$$(1) \qquad f_1^{-1}(A_{i_1}) \cap f_2^{-1}(A_{i_2}) \cap \ldots \cap f_k^{-1}(A_{i_k}),$$

where $1 \leq i_j \leq m$ for $j \leq k$, constitute a cover of the space X and the diameter of each of them with respect to the pseudometric $\rho_{f_1, f_2, \ldots, f_k}$ is less than ϵ. Choosing one point from every non-empty set of the form (1) we obtain a finite set A with the required property.

The reader can easily verify that the real line R with the uniformity $\mathcal{C}$ defined in Example 8.1.19 is not totally bounded.■

Let $(X, \mathcal{U})$ be a uniform space and $\mathcal{F}$ a family of subsets of X; we say that $\mathcal{F}$ *contains arbitrarily small sets* if for every $V \in \mathcal{U}$ there exists an $F \in \mathcal{F}$ such that $\delta(F) < V$. It follows from (U4), that if $\mathcal{F}$ contains arbitrarily small sets, then the intersection $\bigcap \mathcal{F}$ contains at most one point.

A uniform space $(X, \mathcal{U})$ is *complete* if every family $\mathcal{F}$ of closed (with respect to the topology induced by $\mathcal{U}$) subsets of X which has the finite intersection property and contains arbitrarily small sets has non-empty intersection; a uniformity $\mathcal{U}$ on a set X is *complete* if the space $(X, \mathcal{U})$ is complete.

One readily verifies that completeness is a uniform invariant, but is not an invariant of uniformly continuous mappings (cf. Exercises 4.3.B(a) and 8.1.A(a)).

Theorem 4.3.10 yields

8.3.5. PROPOSITION. *If the uniformity $\mathcal{U}$ on a set X is induced by a metric ρ, then the uniform space $(X, \mathcal{U})$ is complete if and only if the metric space (X, ρ) is complete.*■

8.3.6. THEOREM. *If $(X, \mathcal{U})$ is a complete uniform space, then for every subset M of X which is closed with respect to the topology induced by $\mathcal{U}$ the uniform space $(M, \mathcal{U}_M)$ is complete.*

If $(X, \mathcal{U})$ is an arbitrary uniform space and for a subset M of X the uniform space $(M, \mathcal{U}_M)$ is complete, then M is closed in X with respect to the topology induced by $\mathcal{U}$.

PROOF. The fact that if $(X, \mathcal{U})$ is complete, and M is closed in X with respect to the topology induced by $\mathcal{U}$, then the space $(M, \mathcal{U}_M)$ is complete follows readily from the definition of completeness.

Assume that the space $(M, \mathcal{U}_M)$ is complete and consider a point $x \in \overline{M}$. Let $\mathcal{F}$ be the family of all sets of the form $B(x, V) \cap M$, where V is a member of $\mathcal{U}$ closed with respect to the topology induced by $\mathcal{U}$ on $X \times X$. It follows from Corollaries 8.1.4 and 8.1.12 that the family $\mathcal{F}$ has the finite intersection property. One easily sees that $\mathcal{F}$ consists of subsets of M closed with respect to the topology induced by $\mathcal{U}_M$ and that $\mathcal{F}$ contains arbitrarily small sets. As $\bigcap \mathcal{F} \subset \{x\}$, it follows from completeness of $(M, \mathcal{U}_M)$ that $x \in M$.■

Theorem 4.3.14, Proposition 8.3.5 and the simple observation that an isometry onto is a uniform isomorphism of the induced uniform spaces yield

8.3.7. LEMMA. *For every metrizable uniform space $(X, \mathcal{U})$ there exists a complete metrizable uniform space $(Y, \mathcal{V})$ such that for some $M \subset Y$ the space $(X, \mathcal{U})$ is uniformly isomorphic to the space $(M, \mathcal{V}_M)$.*■

From Theorems 8.2.3, 8.3.6 and the above lemma we obtain

8.3.8. THEOREM. *Every complete uniform space is uniformly isomorphic to a closed subspace of the Cartesian product of a family of complete metrizable uniform spaces.*■

8.3.9. THEOREM. *Let $\{(X_s, \mathcal{U}_s)\}_{s\in S}$ be a family of non-empty uniform spaces. The Cartesian product $(\prod_{s\in S} X_s, \prod_{s\in S} \mathcal{U}_s)$ is complete if and only if all spaces $(X_s, \mathcal{U}_s)$ are complete.*

PROOF. If the Cartesian product $(\prod_{s\in S} X_s, \prod_{s\in S} \mathcal{U}_s)$ is complete, then every space $(X_s, \mathcal{U}_s)$ is complete, because it is uniformly isomorphic to a closed subspace of that Cartesian product.

The proof of completeness of a Cartesian product of complete uniform spaces is similar to the proof of the Tychonoff theorem; one only has to observe that if the family $\mathcal{F}_0$ considered in that proof contains arbitrarily small sets, then the families $\mathcal{F}_s$ there defined also contain arbitrarily small sets. ■

Theorems 8.3.9, 8.3.6 and 3.11.3 imply that the topology of any realcompact space X can be induced by a complete uniformity on the set X. One proves (see Problem 8.5.13(h)) that if the topology of a space X can be induced by a complete uniformity and the cardinal number $|X|$ is non-measurable, then X is a realcompact space. One also proves (see Problem 8.5.13(a)) that the class of topological spaces whose topology can be induced by a complete uniformity coincides with the class of closed subspaces of Cartesian products of metrizable spaces. Let us observe that from the latter result it follows that the topology of any metrizable space can be induced by a complete uniformity. It turns out that all paracompact spaces have the same property (see Problems 8.5.13(b) and (d)). Hence, every paracompact space (and, in particular, every metrizable space) of non-measurable cardinality is realcompact (cf. Problem 5.5.10(b)).

8.3.10. THEOREM. *If $(X, \mathcal{U})$ is a uniform space and $(Y, \mathcal{V})$ a complete uniform space, then every uniformly continuous mapping $f : (A, \mathcal{U}_A) \to (Y, \mathcal{V})$, where A is a subset of X dense with respect to the topology induced by $\mathcal{U}$, is extendable to a uniformly continuous mapping $F : (X, \mathcal{U}) \to (Y, \mathcal{V})$.*

PROOF. For every $x \in X$ the family $\{\overline{f(B(x, U) \cap A)}\}_{U\in\mathcal{U}}$ has the finite intersection property and contains arbitrarily small sets. Hence, by virtue of completeness of the space $(Y, \mathcal{V})$, the formula

$$F(x) \in \bigcap_{U\in\mathcal{U}} \overline{f(B(x, U) \cap A)} \tag{2}$$

defines a mapping F of X to Y; one readily sees that for every $x \in A$ we have $F(x) = f(x)$.

We shall show that the mapping F is uniformly continuous. For any $W \in \mathcal{V}$ take a $V \in \mathcal{V}$ such that $6V \subset W$ and a $U \in \mathcal{U}$, open with respect to the topology induced by $\mathcal{U}$ on $X \times X$, such that for all $a, a' \in A$ if $|a - a'| < 2U$, then $|f(a) - f(a')| < V$. From Corollary 8.1.5 it follows that for every $M \subset X$ we have

$$\delta(\overline{f(M \cap A)}) < 3V \quad \text{whenever} \quad \delta(M) < 2U. \tag{3}$$

Now, consider a pair of points $x, y \in X$ such that $|x - y| < U$ and let $B_1 = B(x, U)$ and $B_2 = B(y, U)$. The intersection $B_1 \cap B_2$ is non-empty and open in X so that there exists a point $a \in A \cap B_1 \cap B_2$. We have $f(a) \in \overline{f(B_1 \cap A)} \cap \overline{f(B_2 \cap A)}$ and (3) implies that $\delta(\overline{f(B_1 \cap A)} \cup \overline{f(B_2 \cap A)}) < 6V$; hence, by (2), $|F(x) - F(y)| < 6V \subset W$. ■

The last theorem, along with Theorem 1.5.4, yields

8.3.11. COROLLARY. *If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are complete uniform spaces then every uniform isomorphism of $(A,\mathcal{U}_A)$ onto $(B,\mathcal{V}_B)$, where A and B are dense subsets of X and Y respectively, is extendable to a uniform isomorphism of $(X,\mathcal{U})$ onto $(Y,\mathcal{V})$.* ■

8.3.12. THEOREM. *For every uniform space $(X,\mathcal{U})$ there exists exactly one (up to a uniform isomorphism) complete uniform space $(\tilde{X},\tilde{\mathcal{U}})$ such that for a dense subset A of $\tilde{X}$ the space $(X,\mathcal{U})$ is uniformly isomorphic to $(A,\tilde{\mathcal{U}}_A)$. Moreover, we have $w(\tilde{\mathcal{U}}) = w(\mathcal{U})$ and if $(X,\mathcal{U})$ is a totally bounded space, then $(\tilde{X},\tilde{\mathcal{U}})$ also is totally bounded.*

PROOF. The existence of the space $(\tilde{X},\tilde{\mathcal{U}})$ follows from 8.2.3, 8.3.7, 8.3.9 and 8.3.6; the uniqueness of $(\tilde{X},\tilde{\mathcal{U}})$ is a consequence of Corollary 8.3.11.

The equality $w(\tilde{\mathcal{U}}) = w(\mathcal{U})$ is a consequence of Remark 8.2.4 and of the readily observed fact that if $w(\mathcal{U}) < \aleph_0$, then $\mathcal{U}$ is a discrete uniformity. Finally, by the second part of Theorem 8.3.2, if $(X,\mathcal{U})$ is a totally bounded space, then $(\tilde{X},\tilde{\mathcal{U}})$ also is totally bounded. ■

The space $(\tilde{X},\tilde{\mathcal{U}})$ satisfying the conditions in Theorem 8.3.12 is called the *completion of the uniform space* $(X,\mathcal{U})$.

We conclude this section with an investigation of uniformities on compact spaces.

8.3.13. THEOREM. *For every compact space X there exists exactly one uniformity $\mathcal{U}$ on the set X that induces the original topology of X. All entourages of the diagonal $\Delta \subset X \times X$ which are open in the Cartesian product $X \times X$ form a base for the uniformity $\mathcal{U}$.*

PROOF. The existence of a uniformity $\mathcal{U}$ on the space X follows from Theorem 8.1.20. We shall show that all entourages of the diagonal $\Delta \subset X \times X$ which are open in $X \times X$ form a base for $\mathcal{U}$. Clearly, this will imply the uniquenss of $\mathcal{U}$.

It follows from Corollary 8.1.12 that every $V \in \mathcal{U}$ contains an entourage of the diagonal which is open in $X \times X$. Consider now an open entourage W of the diagonal $\Delta \subset X \times X$. By (U4) and Corollary 8.1.12 we have $\Delta = \bigcap\{\overline{V} : V \in \mathcal{U}\} \subset W$; applying compactness of $X \times X$ and Corollary 3.1.5, we infer that there exists a finite family $\{V_1, V_2, \dots, V_k\} \subset \mathcal{U}$ such that $\Delta \subset V_1 \cap V_2 \cap \dots \cap V_k \subset \overline{V}_1 \cap \overline{V}_2 \cap \dots \cap \overline{V}_k \subset W$. Hence, by (U1) and (U2), we have $W \in \mathcal{U}$. ■

The proof of the following theorem, which parallels the proof of Theorem 4.3.27, is left to the reader (cf. Problem 8.5.10).

8.3.14. THEOREM. *Every uniformity on a compact space is totally bounded.* ■

From the definition of completeness immediately follows

8.3.15. THEOREM. *Every uniformity on a compact space is complete.* ■

A uniform space $(X,\mathcal{U})$ is called *compact* if the set X with the topology induced by $\mathcal{U}$ is a compact space.

8.3.16. THEOREM. *A uniform space $(X,\mathcal{U})$ is compact if and only if it is both totally bounded and complete.*

PROOF. By virtue of Theorems 8.3.14 and 8.3.15, it suffices to prove that every uniform space $(X,\mathcal{U})$ which is both totally bounded and complete is compact. It follows from Theorem 8.3.8 that $(X,\mathcal{U})$ is uniformly isomorphic to a closed subspace of the Cartesian product

$(\prod_{s\in S} X_s, \prod_{s\in S} \mathcal{U}_s)$, where the spaces $(X_s, \mathcal{U}_s)$ are complete and metrizable. Without loss of generality, one can assume that $X = \overline{X} \subset \prod_{s\in S} X_s$, $\mathcal{U} = (\prod_{s\in S} \mathcal{U}_s)_X$ and $\overline{p_s(X)} = X_s$ for every $s \in S$. Since total boundedness is an invariant of uniformly continuous mappings, the spaces $(p_s(X), (\mathcal{U}_s)_{p_s(X)})$ are totally bounded, and the second part of Theorem 8.3.2 implies that the spaces $(X_s, \mathcal{U}_s)$ also are totally bounded. Hence, by 8.3.1, 8.3.5 and 4.3.29, the spaces $(X_s, \mathcal{U}_s)$ are compact; applying the Tychonoff theorem we infer that the space $(X, \mathcal{U})$ is compact. ■

8.3.17. COROLLARY. *The completion of a uniform space $(X, \mathcal{U})$ is compact if and only if $(X, \mathcal{U})$ is a totally bounded space.* ■

8.3.18. EXAMPLE. Let X be a Tychonoff space; as shown in Example 8.3.4, the uniformity $\mathcal{C}^*$ on X is totally bounded. Hence the completion $(\tilde{X}, \tilde{\mathcal{C}}^*)$ is compact and the space $\tilde{X}$ is a compactification of X. We shall prove that $\tilde{X}$ is the Čech-Stone compactification of the space X.

It suffices to show that every continuous function $f : X \to I$ is continuously extendable over $\tilde{X}$. Since the unique uniformity $\mathcal{V}$ on the interval I which induces the natural toplogy of I is complete, by Theorem 8.3.10 it suffices to show that f is a uniformly continuous mapping of $(X, \mathcal{C}^*)$ to $(I, \mathcal{V})$ (cf. Exercise 8.1.D).

The family of all entourages of the diagonal $\Delta \subset I \times I$ which are of the form $V_i = \{(t,z) \in I \times I : \rho(t,z) < 1/2^i\}$, where $i = 1, 2, \ldots$ and ρ is the natural metric on I, is a base for the uniformity $\mathcal{V}$ on I. Since $f \in C^*(X)$, the sets

$$W_i = \{(x,y) \in X \times X : \ \rho_f(x,y) < 1/2^i\}$$

belong to $\mathcal{C}^*$ for $i = 1, 2, \ldots$ Now, we have $|f(x) - f(y)| < V_i$ whenever $|x - y| < W_i$, so that f is uniformly continuous. ■

8.3.19. EXAMPLE. Let X be a Tychonoff space and $\mathcal{C}$ the uniformity on the space X defined in Example 8.1.19; we shall prove that the space $\tilde{X}$, where $(\tilde{X}, \tilde{\mathcal{C}})$ is the completion of the space $(X, \mathcal{C})$, is the Hewitt realcompactification of X.

For every $f \in C = C(X)$ let $(R_f, \mathcal{U}_f)$ be the real line with the uniformity induced by the natural metric on R. By virtue of 8.3.5 and 8.3.9, the Cartesian product $(\prod_{f\in C} R_f, \prod_{f\in C} \mathcal{U}_f)$ is complete. One can easily verify that the restriction $F = (\triangle_{f\in C} f)|X$ is a uniform isomorphism of $(X, \mathcal{C})$ onto a subspace of that Cartesian product. From the uniqueness of completion and from Theorem 8.3.6 it follows that $\tilde{X} = \overline{F(X)} \subset \prod_{f\in C} R_f$, so that $\tilde{X}$ is a realcompact space by Theorem 3.11.3. Since every function $f \in C(X)$ is continuously extendable over $\prod_{f\in C} R_f$ and, *a fortiori*, over $\tilde{X}$, the space $\tilde{X}$ is the Hewitt realcompactification of X. ■

Completeness of uniform spaces can be characterized in terms of nets and filters. Let $(X, \mathcal{U})$ be a uniform space and $\{x_\sigma, \ \sigma \in \Sigma\}$ a net in X; we say that $\{x_\sigma, \ \sigma \in \Sigma\}$ is a *Cauchy net* in $(X, \mathcal{U})$ if for every $V \in \mathcal{U}$ there exists a $\sigma_0 \in \Sigma$ such that $|x_\sigma - x_{\sigma_0}| < V$ whenever $\sigma \geq \sigma_0$. Similarly, a filter $\mathcal{F}$ in the family of all subsets of X is a *Cauchy filter* in $(X, \mathcal{U})$ if for every $V \in \mathcal{U}$ there exists an $F \in \mathcal{F}$ such that $\delta(F) < V$. The reader can easily verify that the family of Cauchy nets and the family of Cauchy filters correspond to one another under the one-to-one correspondence between nets and filters established in Section 1.6.

8.3.20. THEOREM. *A uniform space $(X, \mathcal{U})$ is complete if and only if every Cauchy net in $(X, \mathcal{U})$ is convergent to a point of X.*

PROOF. The proof follows the pattern of the proof of Theorem 3.1.23; one need only to note that the assignement of families of closed sets with the finite intersection property to nets, described in that proof, assigns to Cauchy nets the families containing arbitrarily small sets, and that the assignement of nets to families of closed sets with the finite intersection property behaves similarly. ■

The proof of the filter counterpart of the above theorem is left to the reader.

8.3.21. THEOREM. *A uniform space $(X, \mathcal{U})$ is complete if and only if every Cauchy filter in $(X, \mathcal{U})$ is convergent to a point of X.* ■

Historical and bibliographic notes

The notion of a totally bounded uniform space was introduced by Weil in [1936]. Theorems 8.3.3 and 8.3.9 were proved by Bourbaki in [1940]. The notion of a complete uniform space was also introduced by Weil in [1936]; in [1938] Weil proved Theorems 8.3.6, 8.3.8, 8.3.10, 8.3.12, 8.3.13 and 8.3.16 (Theorems 8.3.12, 8.3.13 and 8.3.16 were announced in [1936]).

Exercises

8.3.A. (a) Note that every uniformity $\mathcal{U}_1$ on a set X which is coarser than a totally bounded uniformity $\mathcal{U}_2$ on X is itself totally bounded.

(b) Note that every uniformity $\mathcal{U}_1$ on a space X which is finer than a complete uniformity $\mathcal{U}_2$ on the space X is itself complete. Observe that generally this is not true for uniformities on a set.

8.3.B. Let $\mathcal{U}$ be a uniformity on a set X and let A be a subset of X dense with respect to the topology induced by $\mathcal{U}$. Show that if for every family $\mathcal{F}$ of subsets of A which has the finite intersection property and contains arbitrarily small sets the intersection $\bigcap\{\overline{F} : F \in \mathcal{F}\}$, where $\overline{F}$ denotes the closure of F in the space X, is non-empty, then the space $(X, \mathcal{U})$ is complete.

8.3.C. (a) Show that for every topological space X and any complete uniform space $(Y, \mathcal{U})$ the uniformity of uniform convergence induced by $\mathcal{U}$ on Y^X is complete.

(b) Show that for every k-space X and any complete uniform space $(Y, \mathcal{U})$ the uniformity of uniform convergence on compacta induced by $\mathcal{U}$ on Y^X is complete. Note that the assumption that X is a k-space is essential.

Hint. See Exercise 4.3.F(b).

8.3.D. (a) Verify that the uniformity $\mathcal{U}$ generated by a collection C of uniform covers of a set X is totally bounded if and only if for every $\mathcal{A} \in$ C there exists a finite refinement $\mathcal{B} \in$ C.

(b) Verify that the uniformity $\mathcal{U}$ generated by a collection C of uniform covers of a set X is complete if and only if every family $\mathcal{F}$ of closed (with respect to the topology induced by $\mathcal{U}$) subsets of X which has the finite intersection property, and for every $\mathcal{A} \in \mathrm{C}$ contains a set F contained in a member of $\mathcal{A}$, has non-empty intersection.

(c) Verify that the uniformity $\mathcal{U}$ generated by a family P of uniform pseudometrics on a set X is totally bounded if and only if for every $\rho \in P$ and every $\epsilon > 0$ one can find a finite set $A \subset X$ such that for every $x \in X$ there exists an $x' \in A$ satisfying $\rho(x, x') < \epsilon$.

(d) Verify that the uniformity $\mathcal{U}$ generated by a family P of uniform pseudometrics on a set X is complete if and only if every family $\mathcal{F}$ of closed (with respect to the topology induced by $\mathcal{U}$) subsets of X which has the finite intersection property, and for every $\rho \in P$ and every $\epsilon > 0$ contains a set F of diameter with respect to ρ less than ϵ, has non-empty intersection.

8.3.E. (a) Prove that a complete uniform space $(Y, \mathcal{V})$ is the completion of a uniform space $(X, \mathcal{U})$ if and only if there exists a uniformly continuous mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ such that $\overline{f(X)} = Y$ and for every pseudometric ρ on the set X which is uniform with respect to $\mathcal{U}$ there exists a pseudometric $\overline{\rho}$ on the set Y which is uniform with respect to $\mathcal{V}$ such that $\overline{\rho}(f(x), f(y)) = \rho(x, y)$ for $x, y \in X$.

(b) Let $\{(X_s, \mathcal{U}_s)\}_{s \in S}$ be a family of uniform spaces and let $\{\tilde{X}_s, \tilde{\mathcal{U}}_s)$ be the completion of the space $(X_s, \mathcal{U}_s)$. Show that the Cartesian product $(\prod_{s \in S} \tilde{X}_s, \prod_{s \in S} \tilde{\mathcal{U}}_s)$ is the completion of the Cartesian product $(\prod_{s \in S} X_s, \prod_{s \in S} \mathcal{U}_s)$.

8.3.F (Shirota [1952]). Let X be a Tychonoff space and $\mathcal{U}$ the uniformity generated by the collection of all covers of X which have a countable normal refinement (see Exercise 8.1.I(a)). Prove that the space $\tilde{X}$, where $(\tilde{X}, \tilde{\mathcal{U}})$ is the completion of the space $(X, \mathcal{U})$, is the Hewitt realcompactification of X (cf. Example 8.3.19 and Exercise 8.1.I(b)).

Hint. Observe that the family P of all pseudometrics ρ on the set X such that the mapping $\rho : X \times X \to R$ is continuous and the space X/ρ defined in Exercise 4.2.I is separable has properties (UP1)–(UP2). Show that the uniformity generated by the family P of uniform pseudometrics coincides with $\mathcal{U}$ and apply Exercises 8.3.E(b) and 8.1.I(b) and Theorem 8.2.3.

8.3.G. Show that if $(X, \mathcal{U})$ is a compact uniform space, then every open cover of the space X with the topology induced by $\mathcal{U}$ is uniform with respect to $\mathcal{U}$.

Hint. Apply Exercise 5.1.A(b), Proposition 8.1.16, and Theorem 8.3.13.

A direct proof can be obtained by modifying the proof of Lemma 8.2.5.

8.3.H. Observe that Exercise 4.2.B can be solved by applying Theorems 8.1.21 and 8.3.13.

8.3.I. (a) Verify that if a space X with the topology induced by a uniformity $\mathcal{U}$ is connected, then for every pair x, y of points of X and any $V \in \mathcal{U}$ there exists a finite sequence $x_1, x_2, \ldots, x_k$ of points of X such that $x_1 = x, x_k = y$ and $|x_i - x_{i+1}| < V$ for $i = 1, 2, \ldots, k-1$.

(b) Show that every compact uniform space $(X, \mathcal{U})$ satisfying the condition in part (a) is connected and note that the assumption of compactness is essential.

8.4. Proximities and proximity spaces

Let X be a set and δ a relation on the family of all subsets of X. We shall write $A\delta B$ if the sets $A, B \subset X$ are δ-related, otherwise we shall write $A\overline{\delta}B$. A relation δ on the family of all subsets of a set X is called a *proximity* on the set X if δ satisfies the following conditions:

(P1) *$A\delta B$ if and only if $B\delta A$.*
(P2) *$A\delta(B\cup C)$ if and only if $A\delta B$ or $A\delta C$.*
(P3) *$\{x\}\delta\{y\}$ if and only if $x=y$.*
(P4) *$\emptyset\bar{\delta}X$.*
(P5) *If $A\bar{\delta}B$, then there exist $C,D\subset X$ such that $A\bar{\delta}C$, $B\bar{\delta}D$ and $C\cup D=X$.*

A *proximity space* is a pair (X,δ) consisting of a set X and a proximity δ on the set X; two subsets A and B of the set X are *close* with respect to δ if $A\delta B$, otherwise they are *remote* with respect to δ.

Conditions (P1)–(P5) imply the following properties of proximities:

(1) *If $A\delta B$ and $B\subset C$, then $A\delta C$.*
(2) *If $A\cap B\neq\emptyset$, then $A\delta B$.*
(3) *$\emptyset\bar{\delta}A$ for every $A\subset X$.*

To establish (1) it suffices to note that if $B\subset C$, then $B\cup C=C$, so that $A\delta B$ implies $A\delta C$ by virtue of (P2). Property (2) follows from (P3), (P1) and (1); in fact, if $x\in A\cap B$, then $\{x\}\delta\{x\}$, $\{x\}\delta A$, $A\delta\{x\}$, and $A\delta B$. Finally, (3) follows from (P4) and (1).

Every proximity δ on a set X induces a topology $\mathcal{O}$ on X; hence every proximity space (X,δ) defines a topological space $(X,\mathcal{O})$. More exactly, as we shall prove, the formula

$$\overline{A}=\{x\in X:\ \{x\}\delta A\} \tag{4}$$

defines a closure operator on the set X. We start with a lemma.

8.4.1. LEMMA. *For every proximity δ on a set X and any sets $A,B\subset X$*

$$\text{if}\quad B\bar{\delta}A,\quad\text{then}\quad B\bar{\delta}\bar{A}. \tag{5}$$

PROOF. If $B\bar{\delta}A$, then – by virtue of (P5) – there exist $C,D\subset X$ such that

$$B\bar{\delta}C,\qquad A\bar{\delta}D \tag{6}$$

and

$$C\cup D=X,\quad\text{i.e.,}\quad X\setminus D\subset C. \tag{7}$$

It follows from (1) and the second part of (6) that $\overline{A}\subset X\setminus D$; therefore we have $\overline{A}\subset C$ by (7) and $B\bar{\delta}\bar{A}$ by the first part of (6). ■

8.4.2. THEOREM. *For every proximity δ on a set X formula* (4) *defines a closure operator which satisfies conditions* (CO1)–(CO4). *The space X with the topology $\mathcal{O}$ generated by that closure operator is a T_1-space.*

The topology $\mathcal{O}$ is called the *topology induced by the proximity δ*.

PROOF. Conditions (CO1), (CO2) and (CO3) follow immediately from (3), (2) and (P2) respectively. To prove that condition (CO4) is satisfied it suffices to show that for every $A\subset X$ we have $\overline{(\overline{A})}\subset\overline{A}$, or – equivalently – that

$$\text{if}\quad x\notin\overline{A},\quad\text{then}\quad x\notin\overline{(\overline{A})}. \tag{8}$$

However, (8) follows from the lemma applied to the one-point set $B = \{x\}$.

The fact that $(X, \mathcal{O})$ is a T_1-space follows immediately from (P3). ■

We shall show later (see Theorem 8.4.9) that, just as in the case of uniformities, the topology of a space X can be induced by a proximity on the set X if and only if X is a Tychonoff space.

Let us observe that from Lemma 8.4.1, property (1) and condition (P1) it follows that for every proximity δ on a set X the closure operator defined in (4) has the following property

(9) *$A\delta B$ if and only if $\overline{A}\delta\overline{B}$.*

8.4.3. EXAMPLE. Let X be an arbitrary set. We shall verify that by letting $A\delta B$ if and only if $A \cap B \neq \emptyset$ we define a proximity on the set X. Indeed, one easily sees that δ satisfies conditions (P1)–(P4), and to check that δ also satisfies condition (P5) it suffices to note that for every pair A, B of remote, i.e., disjoint, subsets of X the sets $C = X \setminus A$ and $D = X \setminus B$ have the required properties. Obviously, the topology induced by δ is the discrete topology on X. The proximity δ is called the *discrete proximity* on X and the space (X, δ) is called a *discrete proximity space.* ■

8.4.4. EXAMPLE. Let X be a Tychonoff space. For any two sets $A, B \subset X$ let $A\delta B$ if and only if $A \neq \emptyset \neq B$ and there exists no continuous function $f : X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$, i.e., the sets A and B are not completely separated. We shall show that δ is a proximity on X.

It follows directly from the definition that δ satisfies conditions (P1), (P3) and (P4).

To prove that δ satisfies condition (P2), it suffices to show that if $A\overline{\delta}B$ and $A\overline{\delta}C$, then $A\overline{\delta}(B \cup C)$. Take continuous functions $f, g : X \to I$ such that

$$f(x) = g(x) = 0 \text{ for } x \in A, \quad f(x) = 1 \text{ for } x \in B, \quad g(x) = 1 \text{ for } x \in C.$$

The function $h = \max(f, g)$ vanishes on A and assumes the value one on the set $B \cup C$, so that $A\overline{\delta}(B \cup C)$.

Finally, δ satisfies condition (P5), because if $A\overline{\delta}B$, i.e., if there exists a continuous function $f : X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$ then the sets

$$C = f^{-1}([1/2, 1]) \quad \text{and} \quad D = f^{-1}([0, 1/2])$$

have the required properties.

From the definition of a Tychonoff space it follows that the proximity δ on X induces the original topology of the space X. ■

We shall now briefly discuss mappings of proximity spaces. Let (X, δ) and (Y, δ') be two proximity spaces; a mapping f of the set X to the set Y is called *proximally continuous with respect to the proximities δ and δ'* if for any sets $A, B \subset X$ close with respect to δ, the images $f(A), f(B) \subset Y$ are close with respect to δ'. Formula (4) and the equivalence of conditions (i) and (v) in Proposition 1.4.1 imply that f is a continuous mapping of the space X with the topology induced by δ to the space Y with the topology induced by δ'. Obviously, the composition of proximally continuous mappings is proximally continuous.

A one-to-one mapping f of a set X onto a set Y is a *proximal isomorphism with respect to proximities δ and δ'* on the sets X and Y respectively, if f is proximally continuous with respect to δ and δ' and the inverse mapping f^{-1} is proximally continuous with respect to δ' and δ. A proximal isomorphism is a homeomorphism of the induced topological spaces.

We say that two proximity spaces (X, δ) and (Y, δ') are *proximally isomorphic* if there exists a proximal isomorphism of (X, δ) onto (Y, δ'). The study of *proximal invariants*, i.e., of invariants of proximal isomorphisms, is the subject of the theory of proximity spaces. Clearly, every topological invariant is a proximal invariant.

Now, we are going to investigate relations between proximities and uniformities.

8.4.5. THEOREM. *For every uniformity $\mathcal{U}$ on a set X by letting for $A, B \subset X$*

$$A\delta B \quad \textit{whenever} \quad V \cap (A \times B) \neq \emptyset \quad \textit{for every} \quad V \in \mathcal{U}$$

we define a proximity on the set X. The topology induced by δ coincides with the topology induced by $\mathcal{U}$.

The proximity δ is called the *proximity induced by the uniformity $\mathcal{U}$.*

PROOF. To begin, let us note that $A\delta B$ if and only if for every $V \in \mathcal{U}$ there exist $x \in A$ and $y \in B$ such that $|x - y| < V$. Clearly, δ satisfies conditions (P1) and (P4), and from (U4) it follows that δ also satisfies (P3).

To prove that δ satisfies condition (P2) it suffices to show that if $A\bar{\delta}B$ and $A\bar{\delta}C$, then $A\bar{\delta}(B \cup C)$. Take $V_1, V_2 \in \mathcal{U}$ such that

$$|x - y| \geq V_1 \quad \text{and} \quad |x - z| \geq V_2 \quad \text{for all } x \in A,\ y \in B \text{ and } z \in C.$$

We clearly have $|x - t| \geq V = V_1 \cap V_2$ whenever $x \in A$ and $t \in B \cup C$; since $V \in \mathcal{U}$ by virtue of (U2), $A\bar{\delta}(B \cup C)$.

Finally, δ satisfies condition (P5), because if $A\bar{\delta}B$, i.e., if there exists a $V \in \mathcal{U}$ such that $|x - y| \geq V$ whenever $x \in A$ and $y \in B$, then the sets

$$C = X \setminus B(A, W) \quad \text{and} \quad D = X \setminus B(B, W),$$

where $W \in \mathcal{U}$ satisfies $2W \subset V$, have the required properties.

Corollary 8.1.4 implies that the topology on X induced by the proximity δ induced by a uniformity $\mathcal{U}$ on the set X coincides with the topology induced by $\mathcal{U}$. ■

One can readily verify that if a mapping f of a set X to a set Y is uniformly continuous with respect to uniformities $\mathcal{U}$ and $\mathcal{V}$ on the sets X and Y respectively, then f is proximally continuous with respect to the proximities δ and δ' induced by $\mathcal{U}$ and $\mathcal{V}$.

8.4.6. EXAMPLE. As established in Section 8.1, every metric ρ on a set X induces a uniformity $\mathcal{U}$ on the set X; by virtue of the last theorem, the uniformity $\mathcal{U}$ induces in turn a proximity δ on the set X. One can easily check that two sets $A, B \subset X$ are close with respect to δ if and only if $\rho(A, B) = 0$. Hence, if ρ is any metric on a set X and if we let $A\delta B$ whenever $\rho(A, B) = 0$, we define a proximity δ on the set X. The proximity δ is called the *proximity induced by the metric ρ.* ■

To discuss the passage from proximities to uniformities, it will be convenient to consider relations of strong inclusion. Let δ be a proximity on a set X; we say that a set A is *strongly* *contained* in a set B with respect to δ, and we write $A \ll B$, if $A\overline{\delta}(X \setminus B)$. Let us note that using the relation of strong inclusion we can rewrite (P5) in the following form:

(P5′) *If $A\overline{\delta}B$, then there exist $A_1, B_1 \subset X$ such that $A \ll A_1, B \ll B_1$ and $A_1 \cap B_1 = \emptyset$.*

We shall now show that the relation $\ll$ has the following properties (in (SI5) and (SI7) the topology induced by δ is being considered):

(SI1) *If $A \ll B$, then $X \setminus B \ll X \setminus A$.*

(SI2) *If $A \ll B$ then $A \subset B$.*

(SI3) *If $A_1 \subset A \ll B \subset B_1$, then $A_1 \ll B_1$.*

(SI4) *If $A_1 \ll B_1$ and $A_2 \ll B_2$, then $A_1 \cup A_2 \ll B_1 \cup B_2$.*

(SI5) *If $A \ll B$, then there exists an open set C such that $A \ll C \subset \overline{C} \ll B$.*

(SI6) $\emptyset \ll \emptyset$.

(SI7) *For every $x \in X$ and any neighbourhood A of x we have $\{x\} \ll A$.*

Property (SI1) follows from the definition of strong inclusion and from the identity $A = X \setminus (X \setminus A)$. Properties (SI2) and (SI3) follow from (2) and (1) respectively.

To establish (SI4) it suffices to observe that if $A_1 \ll B_1$ and $A_2 \ll B_2$, then, by virtue of (1), for $i = 1,2$ we have $A_i\overline{\delta}[(X \setminus B_1) \cap (X \setminus B_2)]$, so that $(A_1 \cup A_2)\overline{\delta}[X \setminus (B_1 \cup B_2)]$, i.e., $A_1 \cup A_2 \ll B_1 \cup B_2$, by virtue of (P1) and (P2).

The proof of (SI5) is a little more complicated. If $A \ll B$, then by (P5′) there exist $A_1, B_1 \subset X$ such that

$$A \ll A_1, \quad X \setminus B \ll B_1 \quad \text{and} \quad A_1 \cap B_1 = \emptyset.$$

By virtue of (SI1) we have $A_1 \subset X \setminus B_1 \ll B$, so that $A_1\overline{\delta}(X \setminus B)$. Since $A\overline{\delta}(X \setminus A_1)$, it follows from (5) that $A\overline{\delta}\,\overline{(X \setminus A_1)}$, i.e., $A\overline{\delta}(X \setminus \operatorname{Int} A_1)$. Letting $C = \operatorname{Int} A_1$ we have $A \ll C \subset A_1$; since $A_1\overline{\delta}(X \setminus B)$, we have $C\overline{\delta}(X \setminus B)$ which implies that $\overline{C}\,\overline{\delta}(X \setminus B)$, i.e., $\overline{C} \ll B$.

Properties (SI6) and (SI7) follow from (P4) and (4) respectively.

Let us note that from (SI1) and De Morgan's laws it follows that property (SI4) can be rewritten as

(SI4′) *If $A_1 \ll B_1$ and $A_2 \ll B_2$, then $A_1 \cap A_2 \ll B_1 \cap B_2$.*

More generally, we have

(SI4″) *If $A_i \ll B_i$ for $i = 1,2,\ldots,k$, then $\bigcup_{i=1}^k A_i \ll \bigcup_{i=1}^k B_i$ and $\bigcap_{i=1}^k A_i \ll \bigcap_{i=1}^k B_i$.*

Let us also note that from (SI5) it follows that a space X with the topology induced by a proximity on the set X is a regular space (cf. Theorem 8.4.9).

Let δ be a proximity on a set X. A finite cover $\{A_i\}_{i=1}^k$ of the set X is called *δ-uniform* if there exists a cover $\{B_i\}_{i=1}^k$ of the set X such that

$$B_i \ll A_i \quad \text{for} \quad i = 1,2,\ldots,k. \tag{10}$$

8.4.7. LEMMA. *For every proximity δ on a set X and $A, B \subset X$ we have $A\delta B$ if and only if every δ-uniform cover $\{A_i\}_{i=1}^k$ of the set X contains a set A_j such that $A \cap A_j \neq \emptyset \neq B \cap A_j$.*

PROOF. Consider $A, B \subset X$ such that $A\delta B$. Let $\mathcal{A} = \{A_i\}_{i=1}^k$ be a δ-uniform cover of the set X and let $\mathcal{B} = \{B_i\}_{i=1}^k$ be a cover of X which satisfies (10). From (SI4″) it follows that the sets

$$C = \operatorname{St}(A, \mathcal{B}) \quad \text{and} \quad D = \operatorname{St}(A, \mathcal{A}) \tag{11}$$

satisfy $C \ll D$. Since $A \subset C$, we have $A\bar{\delta}(X \setminus D)$, and this implies that

$$B \cap D \neq \emptyset; \tag{12}$$

indeed, otherwise we would have $B \subset X \setminus D$ and $A\bar{\delta}B$ which is impossible. The second part of (11) and (12) imply the existence of a $j \leq k$ such that $A \cap A_j \neq \emptyset \neq B \cap A_j$.

Now consider $A, B \subset X$ such that $A\bar{\delta}B$. By virtue of (P5′) there exist $B_1, B_2 \subset X$ satisfying $A \ll X \setminus B_1, B \ll X \setminus B_2$ and $(X \setminus B_1) \cap (X \setminus B_2) = \emptyset$. The sets $A_1 = X \setminus A$ and $A_2 = X \setminus B$ constitute a cover $\mathcal{A}$ of the set X, and by virtue of (SI1) we have $B_1 \ll A_1$ and $B_2 \ll A_2$. As $B_1 \cup B_2 = X$, $\mathcal{A}$ is a δ-uniform cover of X no member of which meets both A and B. ∎

8.4.8. THEOREM. *For every proximity δ on a set X the collection $\mathbf{C}$ of all covers of X which have a δ-uniform refinement has properties* (UC1)–(UC4). *The uniformity $\mathcal{U}$ on the set X generated by the collection $\mathbf{C}$ is totally bounded and induces the proximity δ. The topology induced by $\mathcal{U}$ coincides with the topology induced by δ.*

The uniformity $\mathcal{U}$ is called the *uniformity induced by the proximity δ*.

PROOF. Let $\mathbf{C}_0$ be the collection of all δ-uniform covers of the set X. As the collection $\mathbf{C}$ clearly has property (UC1), to prove the first part of the theorem it suffices to show that $\mathbf{C}_0$ has properties (UC2)–(UC4).

To begin, we shall show that $\mathbf{C}_0$ has property (UC2). Consider $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{C}_0$ and let $\mathcal{A}_1 = \{A_{1,i}\}_{i=1}^k$ and $\mathcal{A}_2 = \{A_{2,j}\}_{j=1}^m$. Take covers $\mathcal{B}_1 = \{B_{1,i}\}_{i=1}^k$ and $\mathcal{B}_2 = \{B_{2,j}\}_{j=1}^m$ of the set X such that

$$B_{1,i} \ll A_{1,i} \text{ for } i = 1, 2, \ldots, k \quad \text{and} \quad B_{2,j} \ll A_{2,j} \text{ for } j = 1, 2, \ldots, m. \tag{13}$$

Property (SI4′) together with (13) imply that the family $\mathcal{A}$ of all intersections $A_{1,i} \cap A_{2,j}$, where $i \leq k$ and $j \leq m$, is a δ-uniform cover of the set X. Since $\mathcal{A}$ is a refinement of both $\mathcal{A}_1$ and $\mathcal{A}_2$, the collection $\mathbf{C}_0$ has property (UC2).

The proof that $\mathbf{C}_0$ has property (UC3) is longer. First of all, let us observe that by Lemma 5.1.15 it suffices to show that for every $\mathcal{A} \in \mathbf{C}_0$ there exists a $\mathcal{B} \in \mathbf{C}_0$ which is a barycentric refinement of $\mathcal{A}$.

Let us consider first a two-element cover $\mathcal{A} = \{A_1, A_2\} \in \mathbf{C}_0$ and let $B_1, B_2 \subset X$ satisfy the conditions

$$B_1 \ll A_1, \quad B_2 \ll A_2 \quad \text{and} \quad B_1 \cup B_2 = X. \tag{14}$$

It follows from (14) that the family $\mathcal{B} = \{A_1 \setminus B_2, A_2 \setminus B_1, A_1 \cap A_2\}$ is a cover of the set X. As $(A_1 \setminus B_2) \cap (A_2 \setminus B_1) = \emptyset$, the cover $\mathcal{B}$ is a barycentric refinement of $\mathcal{A}$. By virtue of (SI5) there exist $C_1, C_2 \subset X$ such that

$$B_1 \ll C_1 \ll A_1 \quad \text{and} \quad B_2 \ll C_2 \ll A_2. \tag{15}$$

Properties (SI1) and (SI4′), along with formulas (14) and (15), imply that

$$C_1 \setminus C_2 \ll A_1 \setminus B_2, \quad C_2 \setminus C_1 \ll A_2 \setminus B_1, \quad C_1 \cap C_2 \ll A_1 \cap A_2,$$

and that

$$(C_1 \setminus C_2) \cup (C_2 \setminus C_1) \cup (C_1 \cap C_2) = X,$$

i.e., that $\mathcal{B} \in \mathcal{C}_0$. Hence every two-element cover $\mathcal{A} \in \mathbf{C}_0$ has a barycentric refinement $\mathcal{B} \in \mathbf{C}_0$.

Now, let us consider an arbitrary cover $\mathcal{A} = \{A_i\}_{i=1}^k \in \mathbf{C}_0$ and let $B_1, B_2, \dots, B_k \subset X$ satisfy the conditions

$$\text{(16)} \qquad B_i \ll A_i \quad \text{for } i = 1,2,\dots,k \quad \text{and} \quad B_1 \cup B_2 \cup \ldots \cup B_k = X.$$

It follows from properties (SI5) and (SI1) that $\mathcal{A}_i = \{A_i, X \setminus B_i\} \in \mathbf{C}_0$ for $i = 1,2,\dots,k$. By the part of the theorem already proved, for $i = 1,2,\dots,k$ there exists a barycentric refinement $\mathcal{B}_i \in \mathbf{C}_0$ of $\mathcal{A}_i$ and there exists a cover $\mathcal{B} \in \mathbf{C}_0$ which refines all the $\mathcal{B}_i$'s. For every $x \in X$ and any $i \leq k$ we have $\mathrm{St}(x, \mathcal{B}) \subset \mathrm{St}(x, \mathcal{B}_i)$, so that

$$\mathrm{St}(x, \mathcal{B}) \subset A_i \quad \text{or} \quad \mathrm{St}(x, \mathcal{B}) \subset X \setminus B_i.$$

Since by (16), the second part of the above alternative cannot hold for all i's, there exists an $i \leq k$ such that $\mathrm{St}(x, \mathcal{B}) \subset A_i$, i.e., $\mathcal{B}$ is a barycentric refinement of $\mathcal{A}$. Thus the proof of property (UC3) is concluded.

The fact that the family $\mathbf{C}_0$ has property (UC4), as well as the fact that the uniformity $\mathcal{U}$ generated by the collection $\mathbf{C}$ induces the proximity δ, follows immediately from Lemma 8.4.7. As the uniformity $\mathcal{U}$ induces the proximity δ, by Theorem 8.4.5 the topology induced by $\mathcal{U}$ coincides with the topology induced by δ.

By Proposition 8.1.16, the sets $V(\mathcal{A}) = \bigcup_{i=1}^k (A_i \times A_i)$, where $\mathcal{A} = \{A_i\}_{i=1}^k \in \mathbf{C}_0$, form a base for the uniformity $\mathcal{U}$. Choosing a point in every non-empty member of $\mathcal{A}$, we obtain a finite set which is $V(\mathcal{A})$-dense in $(X, \mathcal{U})$. Hence the uniformity $\mathcal{U}$ is totally bounded. ■

It turns out that if the proximity δ is induced by a totally bounded uniformity $\mathcal{U}$, then the uniformity induced by δ coincides with $\mathcal{U}$ (see Exercise 8.4.D). Hence, there is a one-to-one correspondence between proximities and totally bounded uniformities on a set.

Theorems 8.4.8 and 8.1.20 and Example 8.4.4 yield

8.4.9. THEOREM. *The topology of a space X can be induced by a proximity on the set X if and only if X is a Tychonoff space.* ■

If X is a topological space and a proximity δ on the set X induces the original topology of X, then we say that δ is a *proximity on the space* X. By the last theorem, for every Tychonoff space X there exists at least one proximity on the space X; generally, there exist more proximities on a given Tychonoff space. We shall conclude this section with a theorem establishing a one-to-one correspondence between proximities on a Tychonoff space X and compactifications of the space X. To begin, we shall prove that on a compact space there exists only one proximity.

8.4.10. THEOREM. *For every compact space X there exists exactly one proximity δ on the set X that induces the original topology of X, viz., the proximity δ defined by letting*

$$A\delta B \quad \text{whenever} \quad \overline{A} \cap \overline{B} \neq \emptyset. \tag{17}$$

PROOF. It follows from Example 8.4.4 that (17) defines a proximity on the space X.

Let δ' be a proximity on the space X. From (2) and (9) it follows that if $A\delta B$, then $A\delta' B$. Hence, to conclude the proof it suffices to show that if A and B are disjoint closed subsets of X, then $A\overline{\delta'}B$. For every $x \in A$ we have $\{x\}\overline{\delta'}B$, so that by (SI5) one can choose an open set V_x satisfying

$$x \in V_x \quad \text{and} \quad V_x\overline{\delta'}B. \tag{18}$$

By virtue of Theorem 3.1.3, there exists a finite set $\{x_1, x_2, \ldots, x_k\} \subset A$ such that

$$A \subset V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_k}. \tag{19}$$

Now, from (P2), (1), (18) and (19) it follows that $A\overline{\delta'}B$.■

8.4.11. LEMMA. *Let X be a Tychonoff space and let cX be a compactification of X. Letting for $A, B \subset X$*

$$A\delta(c)B \quad \text{whenever} \quad \overline{c(A)} \cap \overline{c(B)} \neq \emptyset,$$

where the closures are taken in cX, we define a proximity $\delta(c)$ on the space X.

For compactifications c_1X and c_2X of the space X we have $\delta(c_1) = \delta(c_2)$ if and only if c_1X and c_2X are equivalent.

PROOF. The verification that $\delta(c)$ is a proximity on the space X is left to the reader. The second part of the lemma follows readily from Theorem 3.5.5.■

8.4.12. LEMMA. *For every proximity δ on a Tychonoff space X there exists a compactification cX of the space X such that $\delta = \delta(c)$.*

PROOF. Let $\mathcal{U}$ be the uniformity induced by the proximity δ and let $(\tilde{X}, \tilde{\mathcal{U}})$ be the completion of the uniform space $(X, \mathcal{U})$. Since the uniformity $\mathcal{U}$ is totally bounded, the space $\tilde{X}$ is compact by virtue of Corollary 8.3.17. Consider a mapping c of X to $\tilde{X}$ which is uniformly continuous with respect to $\mathcal{U}$ and $\tilde{\mathcal{U}}$ and such that $c|X$ is a uniform isomorphism of $(X, \mathcal{U})$ onto $(A, \tilde{\mathcal{U}}_A)$, where $A = c(X)$ is a dense subspace of $\tilde{X}$. Clearly, $\tilde{X} = cX$ is a compactification of the space X.

Let δ' be the proximity induced on the set $\tilde{X}$ by the uniformity $\tilde{\mathcal{U}}$. By Theorem 8.4.8, the uniformity $\mathcal{U}$ induces the proximity δ and since $c|X$ is a uniform isomorphism, for $A, B \subset X$ we have

$$A\delta B \quad \text{if and only if} \quad c(A)\delta' c(B).$$

On the other hand, from compactness of $\tilde{X}$ and Theorem 8.4.10 we have

$$c(A)\delta' c(B) \quad \text{if and only if} \quad \overline{c(A)} \cap \overline{c(B)} \neq \emptyset,$$

so that $\delta = \delta(c)$.■

Lemmas 8.4.11 and 8.4.12 yield

8.4.13. THE SMIRNOV THEOREM. *By assigning to every compactification cX of a Tychonoff space X the proximity $\delta(c)$ on the space X we establish a one-to-one correspondence between all compactifications of X and all proximities on the space X.* ■

8.4.14. EXAMPLE. The reader can easily verify that the proximity defined in Example 8.4.4 on a Tychonoff space X corresponds to the Čech-Stone compactification of the space X. ■

Historical and bibliographic notes

Riesz in [1908] formulated a set of axioms to describe the notion of closeness of pair of sets; although not equivalent, his concept was similar to a proximity relation. However, neither Riesz nor any other mathematician of his time took interest in a closer study of that idea. The notion of proximity was introduced by Efremovič in [1951]; Theorem 8.4.2 was also proved there. Theorem 8.4.10 was proved in Efremovič's paper [1952]. An analysis of proximity spaces was carried out by Smirnov in [1952]; Theorems 8.4.5, 8.4.8, 8.4.9 and 8.4.13 were proved in that paper. Naimpally and Warrack's book [1970] is the most complete exposition of the theory of proximity spaces.

Exercises

8.4.A. (a) Show that every continuous mapping of a compact space X to a Tychonoff space Y is proximally continuous with respect to any proximities δ, δ' on the spaces X and Y respectively.

(b) (Efremovič [1952]) Let (X, δ) be a proximity space and let A, B be two subsets of X remote with respect to δ. Show that there exists a proximally continuous mapping f of X to the interval I, with the proximity induced by the natural metric on I, such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.

8.4.B. (a) Let δ and δ' be proximities on X and Y respectively, and let $\mathcal{U}$ and $\mathcal{U}'$ be the uniformities induced by δ and δ'. Verify that every mapping of X to Y which is proximally continuous with respect to δ and δ' is uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{U}'$.

Hint. Show that f is proximally continuous with respect to δ and δ' if and only if for every δ'-uniform cover $\{A_i\}_{i=1}^k$ of the set Y, the cover $\{f^{-1}(A_i)\}_{i=1}^k$ of the set X is δ-uniform.

(b) Deduce from (a) that every mapping f of a space X to a space Y which is proximally continuous with respect to proximities δ and δ' on the spaces X and Y respectively, is extendable to a continuous mappings $F : cX \to c'Y$, where cX and $c'Y$ are the compactifications of X and Y satisfying $\delta(c) = \delta$ and $\delta(c') = \delta'$.

8.4.C. If δ_1 and δ_2 are two proximities on a set X and for all $A, B \subset X$ the relation $A\delta_1 B$ implies the relation $A\delta_2 B$, then we say that the proximity δ_1 is *finer* than the proximity δ_2 or that δ_2 is *coarser* than δ_1 and we write $\delta_2 \le \delta_1$.

(a) Check that the relation $\le$ is an order in the family of all proximities on X.

(b) Verify that for compactifications c_1X and c_2X of a Tychonoff space X we have $c_1X \le c_2X$ if and only $\delta(c_1) \le \delta(c_2)$.

8.4.D (Smirnov [1952]). (a) Show that in the family of all uniformities on a set X which induce a fixed proximity δ there exists a coarsest uniformity.

Hint. This is the uniformity induced by the proximity δ.

(b) Verify that in the family of all uniformities on a set X which induce a fixed proximity δ there is only one totally bounded uniformity – the coarsest one. Deduce that for every Tychonoff space X there exists a one-to-one correspondence between proximities and totally bounded uniformities on the space X.

8.4.E. Give a direct description of the proximity on a locally compact space X which corresponds to the Alexandroff compactification of the space X.

8.5. Problems

A characterization of paracompactness in terms of entourages of the diagonal

8.5.1 (Kelley [1955]). Show that if an open cover $\{U_s\}_{s\in S}$ of a topological space X has a closed locally finite refinement $\{F_t\}_{t\in T}$, then there exists an entourage V of the diagonal $\Delta \subset X \times X$, open in the Cartesian product $X \times X$, such that the cover $\{B(x,V)\}_{x\in X}$ is a refinement of $\{U_s\}_{s\in S}$.

Hint. For every $t \in T$ choose an $s(t) \in S$ such that $F_t \subset U_{s(t)}$ and let $V_t = (U_{s(t)} \times U_{s(t)}) \cup [(X \setminus F_t) \times (X \setminus F_t)]$; check that $V = \bigcap_{t\in T} V_t$ has the required properties.

8.5.2 (Kelley [1955]). Prove that a regular space X is paracompact if and only if every open cover of the space X has a refinement of the form $\{B(x,V)\}_{x\in X}$, where V is an entourage of the diagonal $\Delta \subset X \times X$ which is open in the Cartesian product $X \times X$.

Hint. When proving that X is paracompact, apply condition (ii) in Theorem 5.1.12; take a refinement of the form $\{B(x,V)\}_{x\in X}$, for every $x \in X$ choose a neighbourhood W_x of the point x such that $W_x \times W_x \subset V$ and consider the cover $\{W_x\}_{x\in X}$.

p-adic uniformities

8.5.3. (a) Let Z be the set of all integers and let p be a prime number. Show that the family $\mathcal{B} = \{W_i\}_{i=1}^{\infty}$ of entourages of the diagonal $\Delta \subset Z \times Z$, where $W_i = \{(x,y) \in Z \times Z : x - y$ is divisible by $p^i\}$, has properties (BU1)–(BU3); the uniformity generated by the base $\mathcal{B}$ is called the *p-adic uniformity on Z* and is denoted by $\mathcal{W}_p$.

Show that if p_1 and p_2 are distinct prime numbers, then the uniformities $\mathcal{W}_{p_1}$ and $\mathcal{W}_{p_2}$, as well as the topologies on Z induced by those uniformities, are incomparable, i.e., none of them is finer than the other.

(b) Let p be a prime number; consider the Cartesian product $X_p = \prod_{i=0}^{\infty} X_i$, where $X_i = \{-p+1, -p+2, \ldots, -1, 0, 1, \ldots, p-1\}$ is a discrete space, and the unique uniformity $\mathcal{U}_p$ on the space X_p. Denote by Y the subset of X_p consisting of all sequences $\{x_i\}$ with only finitely many x_i's different from zero and for every $\{x_i\} \in Y$ let $f(\{x_i\}) = \sum_{i=0}^{\infty} x_i p^i$. Verify that f is a uniform isomorphism of $(Y, (\mathcal{U}_p)_Y)$ onto $(Z, \mathcal{W}_p)$ and deduce that $(X_p, \mathcal{U}_p)$ is the completion of the uniform space $(Z, \mathcal{W}_p)$. Applying Exercise 6.2.A(c), note that X_p is homemorphic to the Cantor set.

(c) Let Q be the set of all rational numbers and let p be a prime number. Show that the family $\mathcal{B} = \{V_i\}_{i=1}^{\infty}$ of entourages of the diagonal $\Delta \subset Q \times Q$, where $V_i = \{(x,y) \in Q \times Q$: there exist $k, m \in Z$ such that $x - y = (k/m)p^i$ and m is not divisible by $p\}$, has properties (BU1)–(BU3); the uniformity generated by the base $\mathcal{B}$ is called the *p-adic uniformity on Q* and is denoted by $\mathcal{V}_p$. Verify that $(\mathcal{V}_p)_Z = \mathcal{W}_p$, where $\mathcal{W}_p$ is the p-adic uniformity on Z. The space Q_p with the topology induced by $\tilde{\mathcal{V}}_p$, where $(Q_p, \tilde{\mathcal{V}}_p)$ is the completion of the uniform space $(Q, \mathcal{V}_p)$, is called the *space of p-adic numbers*. Prove that Q_p is homeomorphic to the sum of $\aleph_0$ copies of the Cantor set.

(d) Let p be a prime number; define $v_p(0) = 0$ and for every rational number $x \neq 0$ denote by $v_p(x)$ the natural number i such that $x = (k/m)p^i$, where k and m are not divisible by p. Show that the formula $\rho_p(x,y) = p^{-v_p(x-y)}$ defines a metric on the set Q of rational numbers and that this metric induces the p-adic uniformity on Q.

Topological groups

8.5.4. (a) Let $\{G_s\}_{s\in S}$ be a family of groups. Check that the Cartesian product $\prod_{s\in S} G_s$ is a group with respect to coordinatewise multiplication; this group is called the *Cartesian product of the groups $\{G_s\}_{s\in S}$*.

Show that the Cartesian product of topological groups $\{G_s\}_{s\in S}$ is a topological group with respect to the Tychonoff topology and that the uniformity on $\prod_{s\in S} G_s$ generated by either of the three collections of uniform covers defined in Example 8.1.17 is the Cartesian product of the corresponding uniformities on the groups G_s.

(b) A subset G_0 of a group G is a *subgroup* if $x^{-1} \in G_0$ and $xy \in G_0$ whenever $x, y \in G_0$. Show that a subgroup G_0 of topological group G is a topological group with respect to the subspace topology and that if $\mathcal{U}$ is the uniformity on G generated by either of the three collections of uniform covers defined in Example 8.1.17, then $\mathcal{U}_{G_0}$ coincides with the corresponding uniformity on the group G_0.

(c) (A. H. Stone [1948]) Note that $Z^{\aleph_1}$, where Z is the group of all integers with the usual addition is a non-normal topological group.

Hint. See Exercise 2.3.E(a).

(d) (Tukey [1940]) Prove that $D^{\aleph_1} \times \Sigma$, where $\Sigma \subset D^{\aleph_1}$ consists of all points with at most countably many coordinates different from zero, is a countably compact non-normal topological group.

Hint. See Example 3.10.17, Problem 3.12.24(c) and Theorem 5.1.38.

Remark. Comfort's paper [1984] is a survey of topological properties of topological groups.

Uniformities in terms of pseudometrics

8.5.5. Suppose we are given a set X and a family P of pseudometrics on the set X which has properties (UP1)–(UP2) and the following property

(UP3) *If σ is a pseudometric on the set X and for every $\epsilon > 0$ there exists a $\rho \in P$ and a $\delta > 0$ such that $\sigma(x,y) < \epsilon$ whenever $\rho(x,y) < \delta$, then $\sigma \in P$.*

Verify that P is the family of all pseudometrics on X uniform with respect to the uniformity $\mathcal{U}$ generated by the family P.

Note that there exists a one-to-one correspondence between uniformities on X and families of pseudometrics on X which have properties (UP1)–(UP3).

Extending uniform pseudometrics and uniformly continuous functions

8.5.6. (a) (Isbell [1959]) Let $(X, \mathcal{U})$ be a uniform space and let M be a subset of X. Prove that every bounded pseudometric ρ on the set M, which is uniform with respect to $\mathcal{U}_M$, is extendable to a bounded pseudometric σ on the set X which is uniform with respect to $\mathcal{U}$.

Hint. One can assume that ρ is bounded by $1/2$. For $i = 1, 2, \ldots$ take a $V_i \in \mathcal{U}$ satisfying $V_i \cap (M \times M) \subset \{(x, y) \in M \times M : \rho(x, y) < 1/2^i\}$ and take a pseudometric σ_i on the set X bounded by 1 and such that $\{(x, y) \in X \times X : \sigma_i(x, y) < 1/4\} \subset V_i$. Check that the formula $\sigma'(x, y) = 8\sum_{i=1}^{\infty} \frac{1}{2^i}\sigma_i(x, y)$ defines a pseudometric σ' on X which is uniform with respect to $\mathcal{U}$ and satisfies $\rho(x, y) \le \sigma'(x, y)$ for all $x, y \in M$. Let $\sigma(x, y) = \min(\sigma'(x, y), \sigma''(x, y))$, where $\sigma''(x, y) = \inf\{(\sigma'(x, a) + \rho(a, b) + \sigma'(b, y)) : a, b \in M\}$.

(b) (Katětov [1951a]) Let $(X, \mathcal{U})$ be a uniform space and let M be a subset of X. Prove that every bounded real-valued function f defined on M, uniformly continuous with respect to $\mathcal{U}_M$ and the uniformity $\mathcal{V}$ on R induced by the natural metric, is extendable to a bounded real-valued function F defined on X and uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}$.

Hint (Gantner [1969]). Consider first a function f which assumes only non-negative values and satisfies $f^{-1}(0) \neq \emptyset$. Applying (a), extend the pseudometric ρ on M, defined by the formula $\rho(x, y) = |f(x) - f(y)|$, to a pseudometric σ on X and let $F(x) = \sigma(x, f^{-1}(0))$.

(c) Observe that the assumption that ρ is bounded in (a), and the assumption that f is bounded in (b), are essential.

Coarser totally bounded uniformities and Samuel compactifications

8.5.7. (a) (Samuel [1948]) Show that for every uniformity $\mathcal{U}$ on a Tychonoff space X the uniformity $\mathcal{U}_0$, generated by the family of all mappings of X to I which are uniformly continuous with respect to $\mathcal{U}$ and the unique uniformity on I, is a totally bounded uniformity on the space X coarser than the uniformity $\mathcal{U}$. Verify that $\mathcal{U}_0$ is the finest uniformity on the space X which is totally bounded and coarser than $\mathcal{U}$. The space sX, where $(sX, \tilde{\mathcal{U}}_0)$ is the completion of the uniform space $(X, \mathcal{U}_0)$, is called the *Samuel compactification of the space X with respect to the uniformity $\mathcal{U}$.*

Prove that the Samuel compactification of X with respect to $\mathcal{U}$ contains the space $\tilde{X}$, where $(\tilde{X}, \tilde{\mathcal{U}})$ is the completion of the uniform space $(X, \mathcal{U})$ (cf. Problem 8.5.20).

(b) Prove that for every uniformity $\mathcal{U}$ on a Tychonoff space X the collection $\mathbf{C}$ of all covers of X which have a finite refinement uniform with respect to $\mathcal{U}$ has properties (UC1)–(UC4). Show that the uniformity generated by the collection $\mathbf{C}$ of uniform covers coincides with the uniformity $\mathcal{U}_0$ defined in (a).

Hint. See the hint to Exercise 8.1.H.

Remark. It is a natural question whether for every cardinal number $\mathfrak{m} \ge \aleph_0$ the collection $\mathbf{C}_\mathfrak{m}$ of all covers of X which have a refinement of cardinality $< \mathfrak{m}$ uniform with respect to $\mathcal{U}$ has property (UC3). Isbell in [1964] noted that it is so for $\mathfrak{m} = \aleph_1$, Kucia in [1973] observed that under the assumption of the generalized continuum hypothesis (i.e., the assumption that $2^{\mathfrak{m}} = \mathfrak{m}^+$ for each cardinal number $\mathfrak{m} \geq \aleph_0$) the answer is positive, and Pelant proved in [1975] that the positive answer for $\mathfrak{m} > \aleph_1$ is independent of the axioms of set theory.

Universal uniformities

8.5.8. (a) Let $\mathcal{U}$ be the universal uniformity on a Tychonoff space X. Show that the finest uniformity $\mathcal{U}_0$ on the space X which is totally bounded and coarser than $\mathcal{U}$ (see Problem 8.5.7) coincides with the uniformity C^* defined in Example 8.1.19. Deduce that the Samuel compactification of a Tychonoff space X with respect to the universal uniformity on X coincides with the Čech-Stone compactification of X.

(b) Let $\mathcal{U}$ be the universal uniformity on a Tychonoff space X and let $(\mu X, \tilde{\mathcal{U}})$ be the completion of the uniform space $(X, \mathcal{U})$. Prove that μX is a subspace of the Hewitt realcompactification υX of the space X, so that we have $X \subset \mu X \subset \upsilon X \subset \beta X$.

(c) Let $\mathcal{U}$ be the universal uniformity on a Tychonoff space X and let $(\mu X, \tilde{\mathcal{U}})$ be the completion of the uniform space $(X, \mathcal{U})$. Note that every pseudometric ρ on the set X which is a continuous mapping of $X \times X$ to R can be extended to a pseudometric $\tilde{\rho}$ on the set μX which is a continuous mapping of $\mu X \times \mu X$ to R.

Hint. Apply Exercise 8.3.E(a).

Spaces with a coarsest uniformity

8.5.9 (Samuel [1948]). Show that a coarsest uniformity on a Tychonoff space X exists if and only if the space X is locally compact.

Hint. Apply Problem 8.5.7 and Theorems 3.5.12, 8.3.10 and 8.3.16.

A characterization of pseudocompactness in terms of uniformities

8.5.10 (Doss [1947]). Prove that for every Tychonoff space X the following conditions are equivalent:

(1) *The space X is pseudocompact.*
(2) *Every uniformity on the space X is totally bounded.*
(3) *The uniformities C and C^* on the space X coincide.*

Hint. To prove that (1) implies (2) show that if (2) is not satisfied, then there exists an infinite discrete family of non-empty open subsets of X. The implication (3)$\Rightarrow$(1) follows from Exercise 3.11.C.

Spaces with a unique uniformity

8.5.11 (Doss [1949]). Prove that for every Tychonoff space X the following conditions are equivalent (cf. Problem 3.12.16(a)):

(1) *The space X has a unique (up to equivalence) compactification.*
(2) *There exists only one uniformity on the space X.*

(3) *There exists only one totally bounded uniformity on the space X.*

Hint. In the proof of the implication (1)⇒(2) use Problems 3.12.16(a) and 8.5.10.

Spaces with no complete uniformity

8.5.12 (Dieudonné [1939a]). Observe that if a non-compact Tychonoff space X has a unique compactification, then no uniformity on the space X is complete (cf. Problem 3.12.16(a)).

Dieudonné complete spaces

8.5.13. A topological space X is *Dieudonné complete* if there exists a complete uniformity on the space X; one easily sees that a topological space X is Dieudonné complete if and only if X is a Tychonoff space and the universal uniformity on the space X is complete.

(a) (Dieudonné [1939]) Note that Dieudonné completeness is hereditary with respect to closed subsets and is multiplicative.

Prove that for every topological space X the following conditions are equivalent:

(1) *The space X is Dieudonné complete.*

(2) *The space X is homeomorphic to a closed subspace of a Cartesian product of completely metrizable spaces.*

(3) *The space X is homeomorphic to a closed subspace of a Cartesian product of metrizable spaces.*

(4) *The space X is homeomorphic to the limit of an inverse system of completely metrizable spaces.*

(5) *The space X is homeomorphic to the limit of an inverse system of metrizable spaces.*

Deduce that realcompact spaces are Dieudonné complete (this can be also deduced from Example 8.3.19 or Problem 8.5.8(b)); observe that from Exercise 5.1.J(f) it follows that every paracompact space is Dieudonné complete (cf. parts (b) and (d) below).

Hint. To prove that (3)⇒(2) it suffices to show that every metrizable space X is homeomorphic to a closed subspace of a Cartesian product of completely metrizable spaces. To that end, consider a completely metrizable space Y that contains X and the Cartesian product $\prod_{x\in Y\setminus X} Y_x$, where $Y_x = Y\setminus\{x\}$.

(b) (Kac [1954], Frolík [1961b], Tamano [1962]) Prove that for every Tychonoff space X the following conditions are equivalent:

(1) *The space X is Dieudonné complete.*

(2) *For every point $x_0 \in \beta X\setminus X$ there exists a paracompact subspace T of βX such that $X\subset T\subset \beta X\setminus\{x_0\}$.*

(3) *For every point $x_0\in\beta X\setminus X$ there exists a locally finite functionally open cover $\mathcal{A}$ of the space X such that the point x_0 does not belong to the closure in βX of any member of $\mathcal{A}$.*

(4) *For every point $x_0\in\beta X\setminus X$ there exists a locally finite partition of unity $\{f_s\}_{s\in S}$ on the space X such that the point x_0 does not belong to the closure in βX of any set $f_s^{-1}((0,1])$.*

Deduce that every paracompact space is Dieudonné complete.

Hint. When proving that (1)⇒(2) note that there exists a pseudometric $\rho : X\times X\to R$ which cannot be continuously extended over $(X\cup\{x_0\})\times(X\cup\{x_0\})$. Consider the mapping

$f : X \to Y = X/\rho$ (see Exercise 4.2.I), take the extension $F : \beta X \to \beta\tilde{Y}$ of f, where $\tilde{Y}$ is the completion of the space Y, and the restriction $F_{\tilde{Y}} : F^{-1}(\tilde{Y}) \to \tilde{Y}$; show that the space $T = F^{-1}(\tilde{Y})$ has the required properties (apply Theorem 5.1.35). In the proof of the implication (3)$\Rightarrow$(4) apply Exercise 7.1.B. When proving that (4)$\Rightarrow$(1) consider the completion $\mu X \subset \beta X$ of X with respect to the universal uniformity on X and, assuming that there exists a point $x_0 \in \mu X \setminus X \subset \beta X \setminus X$, apply the partition of unity in (4) to define a pseudometric $\rho : X \times X \to R$ which cannot be continuously extended over $\mu X \times \mu X$.

(c) Observe that in the realm of Dieudonné complete spaces, compactness, countable compactness and pseudocompactness are equivalent.

(d) (Nagata [1950a]) Give a direct proof of the fact that on every paracompact space there exists a complete uniformity (cf. parts (a) and (b) above).

Hint. Verify that the collection of all open covers of the space X has properties (UC1)–(UC4). Assume that there exists a family $\mathcal{F}$ of closed subsets of X which has the finite intersection property and contains arbitrarily small sets, and yet satisfies $\bigcap \mathcal{F} = \emptyset$. For every $x \in X$ choose a neighbourhood U_x of the point x which is disjoint from a member of $\mathcal{F}$ and consider the cover $\{U_x\}_{x \in X}$ of the space X.

(e) (Kac [1954]) Prove that if X is a subspace of a Dieudonné complete space Y and for every point $x_0 \in Y \setminus X$ there exists a locally finite functionally open cover $\mathcal{A}$ of the space X such that the point x_0 does not belong to the closure in Y of any member of $\mathcal{A}$, then the space X is Dieudonné complete.

Hint. Assume that X is dense in Y, consider the extension $f : \beta X \to \beta Y$ of the embedding of X in Y and observe that $\beta X \setminus X = [f^{-1}(Y) \setminus X] \cup [f^{-1}(\beta Y) \setminus f^{-1}(Y)]$.

(f) (Dieudonné [1939]) Prove that if X is a subspace of a Dieudonné complete space Y and for every point $x_0 \in Y \setminus X$ there exists a countable family $\{W_i\}_{i=1}^{\infty}$ of open subsets of Y such that $x_0 \in \bigcap_{i=1}^{\infty} W_i \subset Y \setminus X$, then the space X is Dieudonné complete.

Deduce that Dieudonné completeness is hereditary with respect to F_σ-sets. Observe that Dieudonné completeness is not hereditary with respect to open sets.

(g) (Dieudonné [1939b], Kac [1954]) Let X be a first-countable Dieudonné complete space. Prove that if a topology $\mathcal{O}_2$ on the set X is finer than the original topology $\mathcal{O}_1$ and if $(X, \mathcal{O}_2)$ is a Tychonoff space, then $(X, \mathcal{O}_2)$ is Dieudonné complete.

Hint. Consider the extension of the identity mapping from $(X, \mathcal{O}_2)$ to $(X, \mathcal{O}_1)$ over the Čech-Stone compactifications.

(h) (Shirota [1952]) Prove that a Dieudonné complete space X is realcompact if and only if every discrete closed subspace of X is realcompact. Deduce that if the cardinality of every closed discrete subspace of a Dieudonné complete space X is a non-measurable cardinal number, then X is realcompact. Show that if the cardinality of every family of pairwise disjoint non-empty open subsets of a Dieudonné complete space X is a non-measurable cardinal number, then X is realcompact.

Hint. Apply (a) and Problem 5.5.10(a).

(i) (Zenor [1970]) Prove that the space $\mathcal{Z}(X)$ of all non-empty compact closed subsets of a space X is Dieudonné complete if and only if X is Dieudonné complete.

Hint. Apply Problem 3.12.27(f).

(j) (Ishii [1959]) Prove that a linearly ordered space is Dieudonné complete if and only if it is paracompact.

Hint (Engelking and Lutzer [1977]). By virtue of 5.5.22(f) it suffices to prove that no stationary subset S of the space $W(\alpha)$, where α is a limit ordinal number such that ω_0 is not cofinal with α, is Dieudonné complete. Since the family $\mathcal{F} = \{S \cap (x, \to)\}_{x \in S}$ has the finite intersection property and yet $\bigcap \mathcal{F} = \emptyset$, it is enough to show that for every open neighbourhood U of the diagonal $\Delta \subset S \times S$ there exists an $x(U) \in S$ such that $(x, y) \in U$ whenever $x, y \in S \cap (x(U), \to)$. Assuming the contrary define two transfinite sequences $x_0, x_1, \ldots, x_\xi, \ldots,\ \xi < \gamma$ and $y_0, y_1, \ldots, y_\xi, \ldots,\ \xi < \gamma$ cofinal in S such that $(x_\xi, y_\xi) \notin U$ and $x_\xi < y_\xi < x_{\xi'}$ for all $\xi < \xi' < \gamma$; consider the set $\overline{\{x_\xi : \xi < \gamma\}}$.

A direct construction of the completion

8.5.14 (Samuel [1948]). (a) We say that a Cauchy filter $\mathcal{F}$ in a uniform space $(X, \mathcal{U})$ is *minimal*, if for every Cauchy filter $\mathcal{F}'$ in $(X, \mathcal{U})$ that is contained in $\mathcal{F}$ we have $\mathcal{F}' = \mathcal{F}$.

(a) Show that any Cauchy filter in $(X, \mathcal{U})$ contains exactly one minimal Cauchy filter. Observe that for every point $x \in X$ the family of all subsets of X that contain a neighbourhood of x is a minimal Cauchy filter in $(X, \mathcal{U})$.

(b) Let $(X, \mathcal{U})$ be a uniform space. Denote by $\tilde{X}$ the collection of all minimal Cauchy filters in $(X, \mathcal{U})$ and for every $V \in \mathcal{U}$ define

$$\tilde{V} = \{(\mathcal{F}_1, \mathcal{F}_2) \in \tilde{X} \times \tilde{X} :\ \delta(F) < V \text{ for some } F \in \mathcal{F}_1 \cap \mathcal{F}_2\}.$$

Observe that the family $\{\tilde{V} : V \in \mathcal{U}\}$ of entourages of the diagonal $\Delta \subset \tilde{X} \times \tilde{X}$ has properties (BU1)–(BU3), so that it generates a uniformity $\tilde{\mathcal{U}}$ on the set $\tilde{X}$. Show that the uniform space $(\tilde{X}, \tilde{\mathcal{U}})$ is the completion of the space $(X, \mathcal{U})$.

Hint. Apply Exercise 8.3.B.

The completion of an Abelian group is an Abelian group

8.5.15. Let G be an Abelian topological group. Consider the uniform space $(G, \mathcal{U})$, where $\mathcal{U}$ is the uniformity on G defined in Example 8.1.17, and the completion $(\tilde{G}, \tilde{\mathcal{U}})$. Show that one can define an operation of multiplication in $\tilde{G}$, with respect to which $\tilde{G}$ is an Abelian topological group, in such a way that G becomes a subgroup of $\tilde{G}$.

Spaces of closed subsets V (see Problems 2.7.20, 3.12.27, 4.5.23, 6.3.22 and 8.5.13(i))

8.5.16. Let $(X, \mathcal{U})$ be a uniform space and 2^X the family of all non-empty subsets of X closed with respect to the topology induced by $\mathcal{U}$.

(a) Show that the family $\mathcal{B}$ of all sets

$$2^V = \{(A, A') \in 2^X \times 2^X :\ A \subset B(A', V) \text{ and } A' \subset B(A, V)\},$$

where $V \in \mathcal{U}$, has properties (BU1)–(BU3); the uniformity on the set 2^X generated by the base $\mathcal{B}$ is denoted by $2^{\mathcal{U}}$. Verify that $(X, \mathcal{U})$ is uniformly isomorphic to a subspace of the uniform space thus obtained which is closed with respect to the topology induced on 2^X by the uniformity $2^{\mathcal{U}}$.

(b) Verify that if a uniformity $\mathcal{U}$ on a set X is induced by a metric ρ on the set X, then the uniformity $2^{\mathcal{U}}_{\mathcal{M}}$ on the family $\mathcal{M}$ of all bounded, non-empty closed subsets of (X,ρ) coincides with the uniformity induced by the Hausdorff metric.

(c) (Michael [1951]) Show that for every uniformity $\mathcal{U}$ on a topological space X, the topology on $\mathcal{Z}(X)$ induced by the uniformity $2^{\mathcal{U}}_{\mathcal{Z}(X)}$ coincides with the Vietoris topology.

(d) Verify that if the uniform space $(X,\mathcal{U})$ is totally bounded, then the space $(2^X, 2^{\mathcal{U}})$ also is totally bounded.

(e) Give an example of a complete uniform space $(X,\mathcal{U})$ such that the space $(2^X, 2^{\mathcal{U}})$ is not complete.

Hint. Consider the uniformity on the real line generated by the base consisting of all sets of the form $\bigcup\{A \times A : A \in \mathcal{A}\}$, where $\mathcal{A}$ is a countable cover of the real line by pairwise disjoint sets.

(f) Show that if the uniform space $(X,\mathcal{U})$ is compact, then the space $(2^X, 2^{\mathcal{U}})$ also is compact.

Cardinal functions IV (see Problems 1.7.12, 1.7.13, 2.7.9–2.7.11, 3.12.4, 3.12.7–3.12.11, 3.12.12(h) and 3.12.12(j))

8.5.17. The smallest cardinal number $\mathfrak{m} \geq \aleph_0$ such that on a Tychonoff space X there exists a uniformity of weight $\leq \mathfrak{m}$ is called the ***uniform weight*** of the space X and is denoted by $u(X)$.

(a) (Schaerf [1968]) Show that for every Tychonoff space X we have $w(X) = c(X)u(X)$.

Hint (Juhász [1971a]). Apply Remark 8.2.4.

(b) Observe that for every Tychonoff space X we have $w(X) = e(X)u(X)$ and deduce that $w(X) = f(X)u(X)$ for every cardinal function f in the diagram in Problem 3.12.7(e), except for $f = k$.

(c) Give an example of a topological space X of weight $\mathfrak{m}$ and of uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ on the space X such that $w(\mathcal{U}_1) > \mathfrak{m} > w(\mathcal{U}_2)$.

Strong inclusions

8.5.18 (Alexandroff and Ponomarev [1959]). Let X be a T_1-space and let $\mathcal{O}$ and $\mathcal{C}$ denote the families of all open and all closed subsets of X respectively. A relation $\ll$ between members of $\mathcal{C}$ and members of $\mathcal{O}$ is called a ***strong inclusion on the space*** X if it has properties (SI1)–(SI7) formulated in Section 8.4.

Show that for every T_1-space X there exists a one-to-one correspondence between proximities and strong inclusions on the space X.

Proximally continuous mappings of metric spaces

8.5.19 (Efremovič [1952]). (a) Let (X,ρ) and (Y,σ) be metric spaces and let δ and δ' denote the proximities induced by ρ and σ on X and Y respectively. Show that a mapping f of X to Y is proximally continuous with respect to δ and δ' if and only if f is uniformly continuous with respect to ρ and σ (cf. Exercise 8.1.A(a)).

Hint (Isbell [1964]). If a mapping f is not uniformly continuous, then there exist an $\epsilon > 0$ and two sequences $x_1, x_2, \ldots$ and $x'_1, x'_2, \ldots$ of points of X such that $\lim \rho(x_i, x'_i) = 0$ and $\sigma(f(x_i), f(x'_i)) \geq \epsilon$ for $i = 1, 2, \ldots$ Define an infinite set M of natural numbers such that $\sigma(f(x_i), f(x'_j)) \geq \epsilon/4$ for all $i, j \in M$.

(b) Note that two metric ρ_1 and ρ_2 on a set X are uniformly equivalent (see Exercise 4.1.B(b)) if and only if they induce the same proximity.

Uniformities and proximities

8.5.20 (Smirnov [1952]). Show that if a uniformity $\mathcal{U}$ on a space X induces a proximity δ on the space X, then the proximity $\mathcal{U}_0$ defined in Problem 8.5.7, the finest uniformity on the space X which is totally bounded and coarser than $\mathcal{U}$, also induces the proximity δ, i.e., the uniformity $\mathcal{U}_0$ coincides with the uniformity induced by the proximity δ induced by the uniformity $\mathcal{U}$. Observe that the Samuel compactification of a Tychonoff space X with respect to a uniformity $\mathcal{U}$ is the compactificaction which corresponds to the proximity δ induced by the uniformity $\mathcal{U}$ (cf. Theorem 8.4.13).

8.5.21 (Smirnov [1952]). Let δ be a proximity on a set X. A cover $\mathcal{A}$ of the set X is called *weakly δ-uniform* if there exists a sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots$ of covers of X, where $\mathcal{A}_1 = \mathcal{A}$, satisfying the following two conditions:

(1) *$\mathcal{A}_{i+1}$ is a star refinement of $\mathcal{A}_i$ for $i = 1, 2, \ldots$*

(2) *For every pair A, B of subsets of X satisfying $A\delta B$ there exists for each i a set $A_i \in \mathcal{A}_i$ such that $A \cap A_i \neq \emptyset \neq B \cap A_i$.*

(a) Show that every δ-uniform cover of X is weakly δ-uniform.

(b) Prove that for every weakly δ-uniform cover $\mathcal{A}$ of X there exists a uniformity $\mathcal{U}$ on the set X which induces the proximity δ and contains the set $\bigcup\{A \times A : A \in \mathcal{A}\}$.

8.5.22 (Katětov [1959], Dowker [1961]). Verify that the collection $\mathbf{C}$ of all covers of the set $X = N \times N$, where N is the set of natural numbers, which have a refinement of the form $\{(x) \times A : x \in N, A \in \mathcal{A}\}$, where $\mathcal{A}$ is a finite cover of N by pairwise disjoint sets, has properties (UC1)–(UC4). Let $\mathcal{U}$ be the uniformity generated by $\mathbf{C}$ and let δ be the proximity on X induced by $\mathcal{U}$; check that $\Delta\delta[(N \times N) \setminus \Delta]$. Show that the covers $\{\{x\} \times N\}_{x \in N}$ and $\{N \times \{y\}\}_{y \in N}$ are both weakly δ-uniform and that the only common refinement of those covers is the cover consisting of all one-point subsets of X. Deduce that in the family of all uniformties on the set X which induce the proximity δ there is no finest uniformity.

8.5.23 (Smirnov [1952]). Prove that if the proximity δ on a set X is induced by a metric on X, then in the family of all uniformities on the set X which induce the proximity δ there exists a finest uniformity.

Hint. Show that if for an entourage V of the diagonal $\Delta \subset X \times X$ there exist two sequences $x_1, x_2, \ldots$ and $x'_1, x'_2, \ldots$ of points of X such that $|x_i - x'_i| \geq V$ for $i = 1, 2, \ldots$, then for any entourage W of the diagonal $\Delta \subset X \times X$ satisfying $4W \subset V$ there exists an infinite set M of natural numbers such that $|x_i - x'_j| \geq W$ for all $i, j \in M$.

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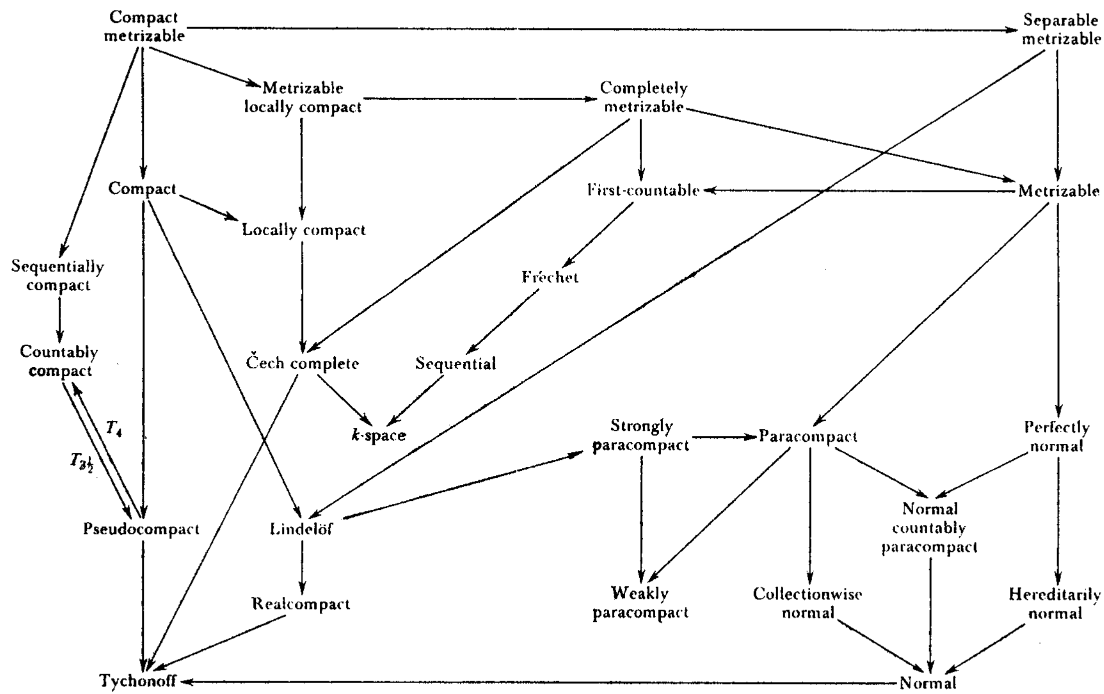

Relations between main classes of topological spaces

Invariants of operations

	Limit of an inverse system	Limit of an inverse sequence	Cartesian product	Countable Cartesian product	Finite Cartesian product	Sum	Finite sum	Open subspace	Closed subspace	Subspace
T_0-space	+	+	+	+	+	+	+	+	+	+
T_1-space	+	+	+	+	+	+	+	+	+	+
Hausdorff	+	+	+	+	+	+	+	+	+	+
Regular	+	+	+	+	+	+	+	+	+	+
Tychonoff	+	+	+	+	+	+	+	+	+	+
Normal	−	−	−	−	−	+	+	−	+	−
Hereditarily normal	−	−	−	−	−	+	+	+	+	+
Perfectly normal	−	+	−	−	−	+	+	+	+	+
Character $\leq \mathfrak{m} \geq \aleph_0$	−	+	−	+	+	+	+	+	+	+
Weight $\leq \mathfrak{m} \geq \aleph_0$	−	+	−	+	+	−	+	+	+	+
Density $\leq \mathfrak{m} \geq \aleph_0$	−	−	−	+	+	−	+	+	−	−
Fréchet	−	−	−	−	−	+	+	+	+	+
Sequential	−	−	−	−	−	+	+	+	+	−
k-space	−	−	−	−	−	+	+	+	+	−
Compact	+	+	+	+	+	−	+	−	+	−
Locally compact	−	−	−	−	+	+	+	+	+	−
Countably compact	−	−	−	−	−	−	+	−	+	−
Pseudocompact	−	−	−	−	−	−	+	−	−	−
Sequentially compact	−	+	−	+	+	−	+	−	+	−
Čech-complete	−	+	−	+	+	+	+	+	+	−
Lindelöf	−	−	−	−	−	−	+	−	+	−
Realcompact	+	+	+	+	+	?	+	−	+	−
Metrizable	−	+	−	+	+	+	+	+	+	+
Paracompact	−	−	−	−	−	+	+	−	+	−
Collectionwise normal	−	−	−	−	−	+	+	−	+	−
Countably paracompact	−	−	−	−	−	+	+	−	+	−
Weakly paracompact	−	−	−	−	−	+	+	−	+	−
Strongly paracompact	−	−	−	−	−	+	+	−	+	−
Connected	−	−	+	+	+	−	−	−	−	−
Hereditarily disconnected	+	+	+	+	+	+	+	+	+	+
Zero-dimensional	+	+	+	+	+	+	+	+	+	+
Strongly zero-dimensional	−	−	−	−	−	+	+	−	−	−
Extremally disconnected	−	−	−	−	−	+	+	+	−	−

Invariants and inverse invariants of mappings

	Invariants							Inverse invariants	
	Continuous	Quotient	Open	Closed	Closed-and-open	Perfect	Open perfect	Perfect	Open perfect
T_0-space	−	−	−	−	−				
T_1-space	−	−	−	+	+				
Hausdorff	−	−	−	−	−	+	+	+	+
Regular	−	−	−	−	+	+	+	+	+
Tychonoff	−	−	−	−	+	−	+	−	−
Normal	−	−	−	+	+	+	+	−	−
Hereditarily normal	−	−	−	+	+	+	+	−	−
Perfectly normal	−	−	−	+	+	+	+	−	−
Character $\leqslant \mathfrak{m}$	−	−	+	−	+	−	+	−	−
Weight $\leqslant \mathfrak{m}$	−	−	+	−	+	+	+	−	−
Density $\leqslant \mathfrak{m}$	+	+	+	+	+	+	+	−	−
Fréchet	−	−	+	+	+	+	+	−	−
Sequential	−	+	+	+	+	+	+	−	−
k-space	−	$+_2$	$+_2$	$+_2$	$+_2$	+	+	+	+
Compact	$+_2$	$+_2$	$+_2$	+	+	+	+	+	+
Locally compact	−	−	$+_2$	−	+	+	+	+	+
Countably compact	$+_2$	$+_2$	$+_2$	$+_2$	$+_2$	+	+	+	+
Pseudocompact	$+_{3\frac{1}{2}}$	$+_{3\frac{1}{2}}$	$+_{3\frac{1}{2}}$	$+_{3\frac{1}{2}}$	+	+	+	−	+
Sequentially compact	$+_2$	$+_2$	$+_2$	$+_2$	$+_2$	+	+	−	−
Čech-complete	−	−	−	−	?	$+_{3\frac{1}{2}}$	+	$+_{3\frac{1}{2}}$	$+_{3\frac{1}{2}}$
Lindelöf	$+_3$	$+_3$	$+_3$	+	+	+	+	+	+
Realcompact	−	−	−	−	?	−	+	$+_{3\frac{1}{2}}$	$+_{3\frac{1}{2}}$
Metrizable	−	−	−	−	+	+	+	−	−
Paracompact	−	−	−	+	+	+	+	+	+
Collectionwise normal	−	−	−	+	+	+	+	−	−
Countably paracompact	−	−	−	−	$+_2$	+	+	+	+
Weakly paracompact	−	−	−	$+_2$	$+_2$	+	+	+	+
Strongly paracompact	−	−	−	−	?	−	+	+	+
Connected	+	+	+	+	+	+	+	−	+
Hereditarily disconnected	−	−	−	−	−	−	−	−	−
Zero-dimensional	−	−	−	−	+	−	+	−	−
Strongly zero-dimensional	−	−	−	−	+	−	+	−	−
Extremally disconnected	−	−	+	−	+	−	+	−	−

List of special symbols

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