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A SHARKOVSKY THEOREM FOR NON-LOCALLY CONNECTED SPACES

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Abstract. We extend Sharkovsky's Theorem to several new classes of spaces, which include some well-known examples of non-locally connected continua, such as the topologist's sine curve and the Warsaw circle. In some of these examples the theorem applies directly (with the same ordering), and in other examples the theorem requires an altered partial ordering on the integers. In the latter case, we describe all possible sets of periods for functions on such spaces, which are based on multiples of Sharkovsky's order.

1. **Introduction.** Sharkovsky [18] has proven a remarkable result about self-maps of an interval, and which periodic orders imply the existence of other orders. He introduces a total ordering of the positive integers ($\preceq$) that starts with the odd integers in ascending order (excluding 1), followed by 2 times the odd integers, then 2^2 times the odds, and 2^i times the odds. The end of the ordering is the powers of 2 in descending order:

$$3 \preceq 5 \preceq 7 \preceq 9 \preceq \cdots \preceq 2{\cdot}3 \preceq 2{\cdot}5 \preceq \cdots \preceq 2^2{\cdot}3 \preceq 2^2{\cdot}5 \preceq \cdots \preceq 2^n \preceq \cdots \preceq 2^2 \preceq 2 \preceq 1.$$

Definitions 1.1. Let f be a map from a space to itself. A point x has *period* k if $f^k(x) = x$. If k is the smallest such positive integer, we say that x has *order* k, or *least period* k. The period set of f, $\mathrm{Per}(f)$, is the set of all least periods (orders) for the function f. A *tail* of the Sharkovsky order is a set S of positive integers such that if $n \in S$, then $m \in S$ whenever $n \preceq m$.

Theorem 1.2 (Sharkovsky)**.** *Let I denote an interval.*

A: For every continuous map $f : I \to I$, $\mathrm{Per}(f)$ is a tail of the Sharkovsky order.
B: Every non-empty tail of the Sharkovsky order occurs as $\mathrm{Per}(f)$ for some continuous map $f : I \to I$.

We show in Lemma 7.1 that the maps f in Theorem 1.2B can be taken to fix the endpoints of the interval. This will be useful in some of our later results.

Although it is sometimes assumed that I is a closed interval, this is not necessary, and the theorem is true for open (or half open) intervals as well [9, 18]. In the case where I is not closed, I no longer has the fixed point property. However,

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Theorem 1.2A is still true, as any map without a fixed point has no periodic points. Theorem 1.2B is also true as stated, but can be strengthened by removing 'non-empty' from the statement. A proof of Sharkovsky's Theorem can be found in many articles or books on dynamical systems, see for example [3, 11, 8, 9, 18].

Definition 1.3. A space X is called a *Sharkovsky space* if Theorem 1.2A is true when I is replaced by X.

Sharkovsky's Theorem has been generalized in various ways. Schirmer proves that any connected linear order space is a Sharkovsky space [17]. While Schirmer also proves that Theorem 1.2B also holds for connected linear spaces if they contain an arc, Baldwin shows that Theorem 1.2B does not hold for all connected linear spaces [5]. It is interesting to note that Baldwin shows that the only way Theorem 1.2B can fail for a connected linear space is if the space does not admit any map with a certain period. While chainable (or arc-like) continua are generally not Sharkovsky spaces, Minc and Transue show that hereditarily decomposable chainable continua are Sharkovsky spaces [14].

While most spaces are not Sharkovsky spaces, many people have studied the possible period sets for maps on other spaces, and have classified what periods imply the existence of other periods, and more generally, what period sets are possible. We note that such work has been done for spaces such as n-ods, trees, circles, and others, see for example [6, 8, 15, 19, 4, 13, 1, 7, 2, 10, 12, 20, 21].

In this paper, we prove that the topologist's sine curve and the Warsaw circle are both Sharkovsky spaces, as are other examples of non-locally connected continua based on those spaces. These spaces also satisfy Theorem 1.2B, and most have the fixed point property. We also discuss examples that are not Sharkovsky spaces, and we describe the possible period sets for functions on these spaces. For these spaces, the possible period sets are usually based on combinations of multiples of tails of the Sharkovsky order.

2. Non-locally connected spaces.

Definition 2.1. A space X is *locally connected* if for every point $x \in X$, and for every neighborhood U containing x, there is a connected neighborhood V with $x \in V \subset U$.

Many common spaces are locally connected, for example: arcs, graphs, Euclidean n-space, and manifolds. While all of these examples are locally simply connected, there also many examples that are not locally simply connected, including the Hawaiian earring, the Sierpinski curve, and the Menger curve.

We will discuss various non-locally connected spaces, in relation to Sharkovsky's Theorem. Perhaps the simplest example of a space that is not locally connected is that of a convergent sequence, with its limit point, since no neighborhood of the limit point is connected. While this space is not connected, there are examples of connected, non-locally connected spaces such as the topologist's sine curve, and the Warsaw circle, which we discuss below.

Example 2.2 (Topologist's sine curve)**.** Let C be the graph of $\sin(1/x),\ x \in (0,1]$, and let A be the limit arc $\{0\} \times [-1,1]$. The topologist's sine curve is the space $X = C \cup A$. Note that X is compact and has two path components C and A which are intervals (C is half-open, while A is closed). See Figure 1.

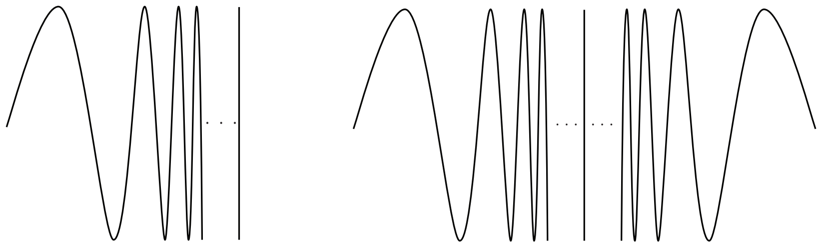

FIGURE 1. The topologist's sine curve (left) and the doubled sine curve (right).

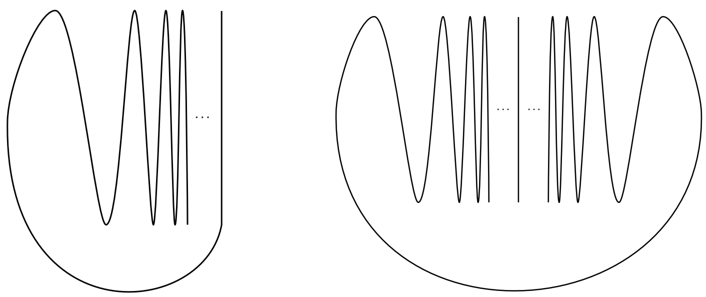

FIGURE 2. The Warsaw circle (left) and a doubled Warsaw circle (right).

Example 2.3 (Warsaw circle)**.** The Warsaw circle is a topologist's sine curve together with an arc connecting endpoints of the two path components. See Figure 2.

We note here that for the Warsaw circle it is important that the arc that connects the two path components of the topologist's sine curve connects to one of the endpoints of the limit arc A instead of an interior point of A; otherwise we get a different space that retracts onto a subspace triod, which clearly does not satisfy Sharkovsky's theorem.

We divide the remainder of the paper into sections about arc-like, circle-like, and star-like continua. We recall the definition of $\mathcal{P}$-like (for more information see [16]): If $\mathcal{P}$ is a collection of continua, a continuum X is $\mathcal{P}$-like if for every $\varepsilon > 0$ there is a surjective ε-map $f : X \to P$, for some $P \in \mathcal{P}$. A map f is an ε-map if $\operatorname{diam}(f^{-1}(p)) < \varepsilon$ for every $p \in P$. Note that the collections of arc-like, circle-like, and star-like continua are not disjoint; for example, the buckethandle continuum is both arc-like and circle-like, and Example 3.2 is both arc-like and triod-like (star-like). We will also discuss a few non-compact examples, and include them in the section with similar spaces.

3. **Arc-like continua.** In this section we discuss certain arc-like continua as Sharkovsky spaces. Arc-like continua are sometimes referred to as snake-like or chainable continua. While the examples we present are hereditarily decomposable, and thus are considered in [14], we include them here because of our simple direct proof, which applies to other examples in later sections. Additionally, we discuss some non-compact examples where our methods apply (recall that continua are compact by definition).

Theorem 3.1. *The topologist's sine curve is a Sharkovsky space.*

Proof. Denote the topologist's sine curve by $X = C \cup A$ as in Example 2.2, where C is the $\sin(1/x)$ curve, and A is the limit arc. Let $x \in X$ be a point of order n, and let $m \succeq n$. We will show that there is a point y of order m.

If $f(A) \subset C$, then by continuity f must also map C into C, and $f(X)$ is a compact connected subset of C, thus an interval I. Thus f maps I into I, and any periodic point of f must be in I; in particular $x \in I$. Therefore, by Sharkovsky's Theorem, there must be a point $y \in I$ of order m.

If $f(C) \subset A$, then f also maps A into A, and any periodic point must be in A. Again Sharkovsky's Theorem gives a point $y \in A$ of order m.

The remaining case is where f maps A into A and C into C. Since both A and C are intervals, Sharkovsky applies, and the existence of the point x of order n implies the existence of a point y of order m (in the same path component as x). $\square$

We can now discuss many other examples of Sharkovsky spaces based on this example. For many of these examples, the proof is a fairly straightforward extension of the proof for the topologist's sine curve; the main idea is generally to consider where the path components of the space map to, and show that a periodic point must lie in a path component that maps to itself.

There are, however, a few more interesting examples that we will discuss. In particular, the following example shows some difficulties that also arise in examples in later sections.

Example 3.2 (A doubled topologist's sine curve)**.** Let $X = X_1 \cup X_2$ be the union of two topologist's sine curves, where $A_1 = A_2$. We can also write $X = C_1 \cup A \cup C_2$. See Figure 1.

Theorem 3.3. *The doubled topologist's sine curve is a Sharkovsky space.*

Proof. Let $X = C_1 \cup A \cup C_2$ be the doubled sine curve as above. Similar to the case of the topologist's sine curve, by considering the images of path components, most of the cases reduce to the known cases of periodic points contained in either an interval or topologist's sine curve that maps to itself. The interesting case is where f maps A to A, but maps C_1 into C_2, and C_2 into C_1. If the periodic point x lies in A, then Sharkovsky applies. If $x \in C_i$, then x must have even order $2n$.

Then x has order n as a periodic point for the map f^2, which maps each path component to itself. Then f^2 has a periodic point y of order m, for every $m \succeq n$ (in the same component C_i as x). Then y must have order $2m$ for f.

By considering Sharkovsky's ordering, this proves everything except that a point of even order implies a fixed point. Note however, that in this case $f(A) = A$, and thus has a fixed point (so that this space has the fixed point property). $\square$

Example 3.4 (A line of topologist's sine curves)**.** Let S be a consecutive sequence of integers (finite, infinite, or bi-infinite). For i in S, let $X_i = C_i \cup A_i$ be a topologist's sine curve, with C_i being the $\sin(1/x)$ curve, and A_i being the limit arc. Let $X = \bigcup X_i$, where $A_i \subset C_{i+1}$ for each i. We call the space X a line of topologist's sine curves. See Figure 3.

Theorem 3.5. *Any line of topologist's sine curves is a Sharkovsky space. It has the fixed point property if and only if it is a finite line of sine curves.*

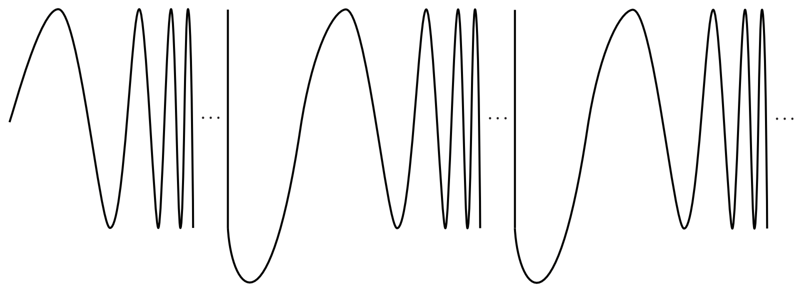

FIGURE 3. A line of topologist's sine curves.

Proof. This comes down to the fact that for any map $f : X \to X$, f must be weakly increasing on path components of X, i.e. if we define n_i so that $f(C_i) \subset C_{n_i}$, then for $i < j$ we have $n_i \leq n_j$. To see this, note that C_i limits on $A_i \subset C_{i+1}$, and that $f(A_i)$ is a compact subset of $f(C_{i+1}) = C_{n_{i+1}}$. Thus $f(C_i)$ limits on $C_{n_{i+1}}$, and therefore we can see that $n_i \in \{n_{i+1}, n_{i+1} - 1\}$.

With this property of being weakly increasing on path components, we see that the only way that a point can be periodic is if it lies in a path component that maps to itself. Each path component is an interval, so Sharkovsky's Theorem applies.

Note that X has the fixed point property if and only if it is the union of finitely many sine curves. Even for an infinite line, Sharkovsky's Theorem still holds, so that any map without a fixed point (such as a translation) cannot have any periodic points. Also note that the infinite line of topologist's sine curves is an infinite cover of the Warsaw circle. □

4. **Archetypal.** We now discuss a class of spaces that are bijective images of arcs, but not necessarily arc-like (chainable), or even Hausdorff. While these spaces may have certain strange properties, they are similar enough to arcs that they are still Sharkovsky spaces (Theorem 4.4).

Definition 4.1. A space X is *archetypal* if X is a uniquely arcwise connected T_1 space and there is a continuous bijection from an arc to X. The arc may be open, closed, or half-open.

Conjecture 4.2. *The only archetypal spaces that are arc-like are arcs.*

An example of an archetypal space that is not an arc is the Warsaw circle, since it is the bijective image of a half-open interval. An example of an archetypal space that is the image of an open arc is shown in Figure 4, on the left. This example is the union of two Warsaw circles sharing a common limit arc. The circle S^1 is not archetypal: although there is a continuous bijection from $[0, 1)$ to S^1, it is not uniquely arcwise connected. Note that archetypal spaces need not be compact.

Example 4.3. We construct an archetypal space X that is not Hausdorff. We will write X as a union of three arcs, and then describe the topology. The first arc A is a topologist's sine curve, without the limit arc. The second arc B is the arc in the Warsaw circle that connects A to its limit arc, although B does not include any of the limit arc. As it will be useful later in defining the topology, let x be the endpoint of B that lies in the limit arc of A. Note that $A \cup B$ is just an open arc, but $A \cup B \cup \{x\}$ is not locally connected at x. The final arc C is a closed arc with

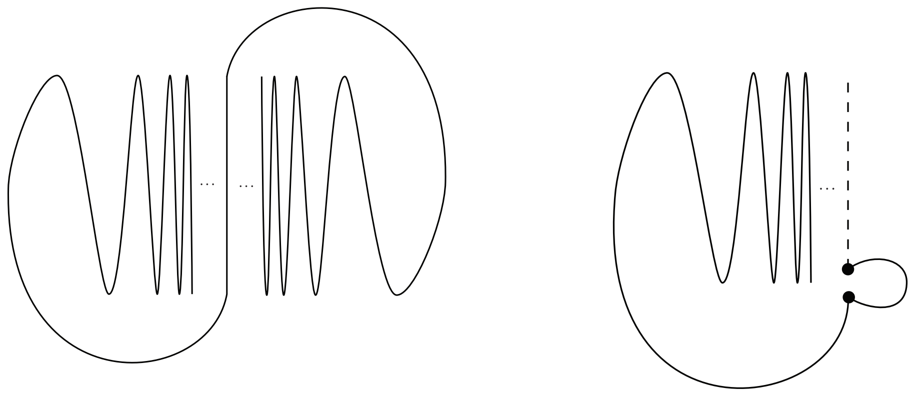

FIGURE 4. Two archetypal spaces. Example 4.3, shown on the right, is a schematic for a non-Hausdorff archetypal space, and is described below.

endpoints y, z. The space X as a set is $A \cup B \cup C$, where y, z are roughly identified with x, as described in the next paragraph. See Figure 4, where y is the upper dot, and z is the lower dot.

The topology of X at any point other than y, z is the standard topology of an arc; that is, a basis element is any open set in $X - \{y, z\} = A \cup B \cup \operatorname{int}(C)$. Define a basis element containing y to be the union of an open set of $C - \{z\}$ containing y together with an open set of A that is a neighborhood of x in the plane. Define a basis element containing z to be the union of an open set of $C - \{y\}$ containing z together with an open set of $A \cup B$ that is a neighborhood of x.

There is a bijection from a half open arc onto X, that maps the endpoint to y, then proceeds along C to z, and then follows B and then A. The space X is T_1; the only possible difficulty lies in considering the points y, z, and there are open sets around each that miss the other point. The points y, z show that X is not Hausdorff, since neighborhoods of the two points must intersect in A (in a neighborhood of x).

The space X is clearly arc connected, being the bijective image of an arc. It remains to show that X is uniquely arc connected. First note that $X - \{y, z\}$ is the disjoint union of two open arcs, with the standard topology. Notice that in small neighborhoods of y, z, the path components containing these points are exactly the neighborhoods in the arc $B \cup C$ (which is homeomorphic to a standard arc). Thus any arc in the space X must correspond to the image under the bijection described in the previous paragraph, and we see that X is uniquely arcwise connected.

Thus X is an archetypal space that is not Hausdorff.

Theorem 4.4. *Archetypal spaces are Sharkovsky spaces.*

This theorem generalizes the result in [20] that the Warsaw circle is a Sharkovsky space. The proof relies on showing that a map of the Warsaw circle lifts to a map of the interval. Our definition of an archetypal space gives the necessary conditions to prove the generalized theorem.

Proof of Theorem 4.4. Let X be archetypal, with continuous bijection $p : I \to X$, and let $f : X \to X$ be a continuous map. Since p is a bijection, $\tilde{f} = p^{-1} \circ f \circ p : I \to I$ is a function, and we claim that $\tilde{f}$ is continuous. Then since p is a bijection, the periods of f and $\tilde{f}$ are the same, and since I is a Sharkovsky space, so is X.

To see that $\tilde{f}$ is continuous, consider a point $t \in I$, and let U be a metric ball in I containing $\tilde{f}(t) = (p^{-1} \circ f \circ p)(t)$. Let A be the set of endpoints of U, i.e. $A = \partial U$ and $|A| \leq 2$. Since X is T_1, points are closed, and thus $X - p(A)$ is an open set in X containing $(f \circ p)(t)$. Let $W \subset I$ be the path component of $(f \circ p)^{-1}(X - p(A))$ containing t. The set W is open in I since $f \circ p$ is continuous, and I is locally path connected. Now consider $p\tilde{f}(W) = (f \circ p)(W) \subset X - p(A)$. Since W is path connected, we see that $(p \circ \tilde{f})(W)$ is contained in the path component of $X - p(A)$ containing $(f \circ p)(t)$.

This path component is just $p(U)$: clearly the path component contains $p(U)$, and it cannot contain any other point x since X is uniquely arcwise connected and p defines an arc from $p(t)$ to x that goes through $p(A)$. Thus $(p \circ \tilde{f})(W) \subset p(U)$, so $\tilde{f}(W) \subset U$, and therefore $\tilde{f}$ is a continuous map on the interval I. □

Corollary 4.5. *The Warsaw circle is a Sharkovsky space.*

Note that while the Warsaw circle is the bijective image of a half-open arc, it does not retract to a half-open arc, but only to a closed arc. In fact, the Warsaw circle does satisfy Theorem 1.2B, and it has the fixed point property, even though the half-open arc does not.

Note that the proof that the Warsaw circle is a Sharkovsky space is substantially different than the proof for the previous examples, such as the topologist's sine curve. The main difference is that the Warsaw circle only has one path component, which limits on itself, so that it is not an arc; on the other hand, in the previous examples, every path component was an arc.

5. **Circle-like continua.** We will now discuss various examples of non-locally connected circle-like continua. A primary example of these is the Warsaw circle, which was discussed in the last section. While some of the spaces we discuss are Sharkovsky spaces, others are not, and for these we discuss the possible period sets for maps on these spaces. We note that the results for circle maps are quite different than Sharkovsky's Theorem [19, 4, 15].

Example 5.1 (A doubled Warsaw circle)**.** Let X be the space obtained by taking the double topologist's sine curve from Example 3.2, and joining the endpoints of C_i by an arc. We can write $X = C \cup A$, where C is an open interval, with each 'end' limiting on the closed arc A as a topologist's sine curve. See Figure 2.

Theorem 5.2. *The doubled Warsaw circle is a Sharkovsky space.*

Proof. This is actually much simpler than either the double topologist's sine curve or the Warsaw circle. Either A maps to C (in which case C also maps to C), or C maps to A (and A also maps to A), or each path component maps to itself. Any periodic point must then lie in a path component that maps to itself and Sharkovsky's Theorem applies. Note that this space has the fixed point property. □

All of the above examples satisfy Theorem 1.2 as stated, and all have the fixed point property, except for infinite lines of topologist's sine curves. Also, these all retract onto an interval, and thus we can see that Theorem 1.2B is also satisfied for these spaces. Note that we can drop the requirement 'non-empty' for the infinite line of sine curves, as there is a shift map with no periodic points.

The examples that we will discuss for the remainder of this paper are not Sharkovsky spaces. Just as the earlier examples, all of these spaces retract to

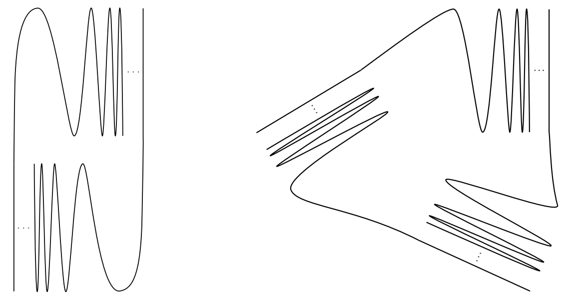

FIGURE 5. A double cover (left) and a 3-fold cover (right) of the Warsaw circle.

an arc, and thus Theorem 1.2B holds for these examples; additionally, some (but not all) have the fixed point property.

Example 5.3 (A double cover of the Warsaw circle)**.** This space is the union of two topologist's sine curves where $A_1 \subset C_2$ and $A_2 \subset C_1$. If each A_i is at the closed end of C_j $(i \neq j)$, then this connected space is a double cover of the Warsaw circle. See Figure 5.

The double cover is almost a Sharkovsky space, but there is one set of implications that do not hold – an even period does not always imply a fixed point.

Theorem 5.4. *Let f be a map of the double cover of the Warsaw circle to itself. Suppose f has a point of order n, and $m \preceq n$. If either n is odd or if $m \neq 1$, then f has a point of order m.*

Proof. This is very similar to the doubled topologist's sine curve. The only difficult case is if f permutes the path components C_1, C_2. In this case, f^2 maps each path component to itself, and just as in the proof for the doubled sine curve, by looking at the map f^2 we get all of Sharkovsky's Theorem, except for a point of even order implying the existence of a point of order 1. This part of Sharkovsky's Theorem is not in fact true for this space, as a simple rotation gives every point order 2, with no fixed point. Otherwise, Sharkovsky's Theorem holds as stated.

Note that any function that permutes the path components shows that this space does not have the fixed point property. However, we can prove that any map without a fixed point has a point of order 2.

If f maps X into one path component C_i, then $f(X)$ is a compact interval that maps to itself, giving a fixed point. If f maps into both path components, it can be seen that f must be surjective, and must satisfy $f(A_i) = A_j$. If f maps each C_i to itself, then we get fixed points in each A_i. If f permutes the path components, then $f^2(A_i) = A_i$, so that f^2 has a fixed point in each A_i, and thus f has a point of order 2 (in each A_i). □

This example gives rise to more questions than the simple "Is X a Sharkovsky space?" For any space X, we can define a partial order $\preceq_X$ on the positive integers by $n \preceq_X m$ if every map of X to itself that has a point of order n has a point of order m. We can then ask what this partial order $\preceq_X$ is. However, for many spaces this is not as informative as the standard Sharkovsky Theorem. In general, it is more informative to ask: What are all possible $\mathrm{Per}(f)$ for a given space X? If we know all possible sets of least periods, we can reconstruct the partial order $\preceq_X$, however the converse is not true.

We introduce some notation to deal with these questions. Write the Sharkovsky order as a relation from $\mathbb{N}$ to itself: $\mathcal{S} = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid n \preceq m\}$. A multiple of the Sharkovsky order is then $d \cdot \mathcal{S} = \{(dn, dm) \in \mathbb{N} \times \mathbb{N} \mid n \preceq m\}$. This multiple of Sharkovsky's order only has multiples of d as initial points. As such, we can extend the relation to include all other integers as initial points: $\overline{d \cdot \mathcal{S}} = d \cdot \mathcal{S} \cup \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid d \text{ does not divide } a\}$. We note that we could have extended $d \cdot \mathcal{S}$ to a total order, but that does not work as well in the following. Essentially, we will want the maximal set of implications, even if many of them are vacuous in this case.

In the following, it will be useful to note that there exist maps of the closed interval to itself with any possible $\mathrm{Per}(f)$ that fix the endpoints (see Lemma 7.1).

Example 5.5 (An n-fold cover of the Warsaw circle)**.** This space is the union of n topologist's sine curves $X_i = C_i \cup A_i$, where $A_i \subset C_{i+1}$ (indices taken mod n). If each A_i is at the closed end of C_{i+1}, then this connected space is an n-fold cover of the Warsaw circle. See Figure 5.

Some dynamical properties of the n-fold cover of the Warsaw circle have been studied in [21]. They show that if a function has a fixed point, then it satisfies Theorem 1.2 for that function; however if there are no fixed points then it does not. While the n-fold cover is not a Sharkovsky space, we prove which period sets are possible.

Theorem 5.6. *Let f be a map of the n-fold cover of the Warsaw circle to itself. Then* $\mathrm{Per}(f)$ *is a non-empty tail of $d \cdot \mathcal{S}$ for some $d|n$. Furthermore, every such tail is* $\mathrm{Per}(f)$ *for some f.*

Proof. As with the double cover, it can be seen that either f maps X into one component C_i, or f is surjective, in which case there is some fixed number k such that $f(C_i) = C_{i+k}$, and $f(A_i) = A_{i+k}$ for all i (indices taken mod n). Let d be the order of the induced map on components, so that $f^d(C_i) = C_i$; in other words, d is the order of k in $\mathbb{Z}_n$. We say that f has *type* d. Note that d divides n.

If $d = 1$, then we get the usual Sharkovsky ordering, as any periodic point is in an interval C_i that maps to itself. Since $f(A_i) = A_i$, f has a fixed point as well. Now consider the case where $d \neq 1$. For any periodic point x of f, d must divide the order of x, call it $d \cdot n$. Then x is a point of order n for the map f^d. So there is a point y of order $m \succeq n$ for f^d, which will have order $d \cdot m$ for f.

Thus for any map f of type d, we see that Sharkovsky's Theorem is true for the partial ordering on the integers which is d times the original Sharkovsky order: $d \cdot \mathcal{S}$. Thus every map of type d satisfies Sharkovsky's Theorem with the extended partial ordering $\overline{d \cdot \mathcal{S}}$. Then for the n-fold cover X of the Warsaw circle we get that the maximal partial ordering ($\preceq_X$) for Sharkovsky's Theorem is defined by the relation $\mathcal{S}(n) = \bigcap_{d|n} \overline{d \cdot \mathcal{S}}$.

We note that this actually agrees with our result for the double cover of the Warsaw circle, although it may appear different at first. The number 2 is unique with respect to $\mathcal{S}$ in the sense that $2 \cdot \mathcal{S}$ is actually a subset of $\mathcal{S}$ (as a relation). In fact, except for 1, all of the numbers that do not show up in $2 \cdot \mathcal{S}$ (namely the odds) precede all the evens. Thus $\mathcal{S}(2) = \mathcal{S} \cap (\overline{2 \cdot \mathcal{S}}) = \mathcal{S} - \{(2n, 1) \mid n \in \mathbb{N}\}$.

As with the double cover, we note that the n-fold cover does not have the fixed point property, but that every map of type d has a point of order d (since $f^d(A_i) =$

A_i). This is related to the fact that all factors d of n are maximal elements in $\mathcal{S}(n)$ (or equivalently $\preceq_X$). So every self-map of the n-fold cover of the Warsaw circle has a point of period n (not necessarily least period).

Now we discuss the possible period sets $\mathrm{Per}(f)$ for the n-fold cover. We note that our partial order $\preceq_X$ defined by $\mathcal{S}(n) = \bigcap_{d|n} \overline{d \cdot \mathcal{S}}$ is not particularly informative here. It can easily be seen that for most integers this intersection will remove almost all information about periods of functions. For instance, we get no implications for a point of period $d|n$. However, the way we have written the partial order as an intersection of other partial orders is more informative, as will be seen.

If f maps into one path component, then the possible period sets are just tails of $\mathcal{S}$. Such maps must have a fixed point, since f is essentially the map from the compact set $\mathrm{im}(f)$ to itself. Each map f that does not map into one path component is surjective and has type d for some $d|n$, and $\mathrm{Per}(f)$ must be a tail of $d \cdot \mathcal{S}$. We show that we get all such non-empty tails.

Given any (non-empty) tail T of $\mathcal{S}$, by Lemma 7.1 there is a map $h : I \to I$ fixing the endpoints of I with $\mathrm{Per}(h) = T$. We will use h to construct a map f of type d with $\mathrm{Per}(f) = d \cdot T$. First take homeomorphisms $f_i : C_i \to C_{i+1}$ (indices mod n) that respect the limit structure on A_i and that are coherent, i.e. so that F defined by $F|_{C_i} = f_i$ is not only a homeomorphism of type n, but also so that $F^n = \mathrm{Id_X}$.

Now, choose intervals $I_i \subset C_i - A_{i-1}$ coherent with the homeomorphisms f_i, i.e. $f_i(I_i) = I_{i+1}$. Then define the map H_d to perform h on the first n/d intervals I_i,

$$H_d = \begin{cases} h & \text{on } I_i \quad \text{for } 1 \le i \le n/d \\ \mathrm{Id} & \text{otherwise} \end{cases},$$

and then we can define f by

$$f = F^{(n/d)} \circ H_d.$$

Notice that $f^d = H_1$, which is a map of type 1 that performs h on each I_i, and is the identity otherwise. Thus $\mathrm{Per}(f) = d \cdot \mathrm{Per}(h) = d \cdot T$, which is a (non-empty) tail of $d \cdot \mathcal{S}$. Note that we only consider non-empty tails since every map of type d has a point of order d.

Thus for maps f of the n-fold cover of the Warsaw circle, $\mathrm{Per}(f)$ is a non-empty tail of $d \cdot \mathcal{S}$ for some $d|n$, and every such tail is $\mathrm{Per}(f)$ for some f. □

6. **Star-like continua.** Our last set of examples are star-like continua. An n-star, or n-od, is just a union of n arcs each sharing one endpoint, and disjoint otherwise. Baldwin discusses Sharkovsky's Theorem for all n-ods, giving a complete characterization of the possible period sets [6]. We discuss non-locally connected n-ods, and note that our results are related to Baldwin's.

Example 6.1 (A topologist's n-od)**.** Let X_n be a union of n topologist's sine curves with a common limit arc, $X_n = A \cup \bigcup C_i$. Note that for $n = 2$ this is just the doubled sine curve of Example 3.2. Also, this space is planar for all n, which can be seen by letting C_i be the graph of $\sin(1/x) + ix$, for $x \in (0, 1]$.

Theorem 6.2. *Let f be a map of the topologist's n-od to itself. Then there is some partition K of n such that* $\mathrm{Per}(f)$ *is a union of non-empty tails of $d \cdot \mathcal{S}$, where d varies over all integers in $K \cup \{1\}$. Furthermore, every such union of tails occurs as* $\mathrm{Per}(f)$ *for some map f.*

Proof. First note that if A maps into any C_i, then X_n maps into a compact interval in C_i, and the standard Sharkovsky Theorem applies. Suppose now that A maps into A. If $\mathrm{im}(f)$ contains any point in C_i, then by connectivity we can see that $\mathrm{im}(f)$ must contain a tail of the sine curve C_i, and thus limits on all of A. So if $f(A) \neq A$, then $\mathrm{im}(f) \subset A$, and the standard theorem applies.

We may then assume that $f(A) = A$. The map f induces a map g on the path components, which we think of as a map on $Y = \{0, 1, \ldots, n\}$ where we consider $A = 0$ and $C_i = i$. Thus we are assuming $g(0) = 0$. Some points may map to 0, which is fixed; others may not be in the image. We are concerned with those points that are always in the image (of any iterate of g), since any periodic point of f will correspond to some such path component.

Let N be the number of such points (not counting 0), i.e. $N = \lim_m |g^m(Y)| - 1$. We note that equivalently, $N = |g^n(Y)| - 1$. By reordering, we may assume that for $1 \leq i \leq N$, $i \in g^m(Y)$ for all m. (If $N = 0$ this statement is vacuous.) Now if we restrict g to $Y(N) = \{1, \ldots, N\}$, we get a permutation in S_N. Let $K = \{k_1, \ldots, k_\ell\}$ be the cycle type of the permutation. We say that g (and also f) is of type K.

Suppose that f is of type K, and $k \in K$. Then there is a set of k of the path components C_i that f cyclicly permutes. Any periodic point in these path components must have period divisible by k. As in earlier examples, we get the implications on periodicity derived from the partial order $k \cdot \mathcal{S}$. Note that f^k need not have a fixed point in C_i, as the sine curve could be continually pushed toward the limit arc A. However, if this is the case then there are no periodic points in these path components, and we have an empty tail of $k \cdot \mathcal{S}$.

Then for a map f of type K, we see that $\mathrm{Per}(f)$ is a union of tails of $d \cdot \mathcal{S}$, where d ranges over $K \cup \{1\}$. (Recall that $f(A) = A$.) Note that the union of two tails of $d \cdot \mathcal{S}$ is still a tail of $d \cdot \mathcal{S}$. The possible types of f are all partitions of m, where $0 \leq m \leq n$. Since A is always fixed (as a set), we always include 1 as a possibility for d, whether or not 1 is in the type K. To simplify, we may then assume that K is a partition of n (instead of a partition of $m \leq n$).

Thus for each map f, there is some partition K of n such that $\mathrm{Per}(f)$ is a union of non-empty tails of $d \cdot \mathcal{S}$, where d varies over $K \cup \{1\}$. We may assume the tails are non-empty: since X_n has the fixed point property there is always a non-empty tail of $1 \cdot \mathcal{S}$, and if there were no non-empty tail for some $d \neq 1$ in the partition, we could consider a map of a different type K' where we replace d in the partition K by d copies of 1.

We will now show that all such $\mathrm{Per}(f)$ described in the last paragraph actually occur. For the most part, this is very similar to the last example of an n-fold cover of the Warsaw circle. Given a partition K of n, and an element $d \in K$, we can choose d path components of X_n to be cyclicly permuted by f, and design the map f to have the appropriate periods as before. For any other $d' \in K$, we repeat using distinct path components of X_n.

The only difficulty arises if $1 \notin K$, for example, if $K = \{n\}$. Then as before we can construct a map g that has any union of nonempty tails of $d \cdot \mathcal{S}$, for $d \in K$. The trick is to still get a tail T of $1 \cdot \mathcal{S}$ corresponding to the limit arc A. To do this, first note that our construction of g will give a neighborhood U of the limit arc A where g only permutes the path components by the coherent homeomorphisms f_{ij}. In other words, we want U to be disjoint from all the sets $I_i \subset C_i$ used to create the tails of $k \cdot \mathcal{S}$ for $k \in K$.

Now choose an interval $J \subset A$. Consider the preimage of J under a (suitably nice) projection map from X_n to A. For instance, with the standard embedding of the topologist's sine curve in the plane, take horizontal projection. Let J_i be the components of the preimage of J that are contained in U. By Lemma 7.1 there is a map $h : J \to J$ that fixes the endpoints of J, and has $\mathrm{Per}(h) = T$. To define f, precompose the function g with the map h on J and each J_i, where the map h must be performed coherently on each J_i. This will define a continuous map f, that has $T \subset \mathrm{Per}(f)$.

Unfortunately this may also give periodic points that were not desired in the $\sin(1/x)$ curves C_i. This can be corrected however, by pushing all of the sine curves C_i toward A, in the portion $C_i \cap U$. This is still continuous, and will avoid the creation of undesired periodic points. Thus all of the possible sets $\mathrm{Per}(f)$ are exactly as described above. □

We note here the relationship between our results for the topologist's n-od and Baldwin's results for the standard n-od [6]. Baldwin defines partial orders $\preceq_d$ for all integers d, and shows that for a map f of the n-od, $\mathrm{Per}(f)$ is a union of tails of $\preceq_d$, where d is allowed to vary over all positive integers less than n. We note that $d \cdot \mathcal{S}$ is a terminal segment of $\preceq_d$, so that all period sets for the topologist's n-od can be achieved as a period set for the standard n-od. However, the partial order $\preceq_d$ is a nontrivial extension of $d \cdot \mathcal{S}$, and allows for different periods. Additionally, there is no restriction in the case of the standard n-od that all the values of d used sum to n (or more precisely, some value $\leq n + 1$, since we are allowed a partition of n, union 1). In fact, Baldwin shows that you can achieve any possible period set fixing a neighborhood of the basepoint. This extra freedom is possible since the arcs of the n-od can map to more that just one other arc.

Example 6.3 (A topologist's ∞-od)**.** $X = A \cup \bigcup C_i$. This is similar to Example 6.1, but with infinitely many sine curves. This space is still planar; for instance take C_i to be the graph of $\sin(1/x) + x/i$.

This example can also be made compact by adding C_0 as the graph of $\sin(1/x)$.

Theorem 6.4. *For a map f of the topologist's ∞-od to itself,* $\mathrm{Per}(f)$ *can be any set of positive integers that contains 1.*

Proof. First note that X has the fixed point property. If $f(A) \subset A$, then there is clearly a fixed point, and if $f(A) \subset C_k$, then f maps the topologist's sine curve $A \cup C_k$ to itself, which also has the fixed point property.

Let S be a subset of $\mathbb{N}$ that contains 1. First note that $\mathbb{N}$ can be partitioned into sets N_i where each N_i has s_i elements, for every $s_i \in S$ (and N_0 will be infinite if S is finite). Then define the map f to be the identity on the limit arc A together with the sine curves C_k for $k \in N_0$, and for $i \neq 0$ define f to cyclicly permute the s_i sine curves C_k for $k \in N_i$. If the cyclic permutation is defined nicely, i.e. so that f restricts to coherent homeomorphisms that compose to the identity, then this gives points of all orders s_i, and fixed points, with no other periodic points. For an example of coherent homeomorphisms, consider the planar embedding described above, and use vertical projection between the corresponding sine curves. Note that if the ∞-od is taken to be compact, then the curve C_0 should be fixed by f.

While it may not be obvious at first, this result is similar to the n-od case. The answer for that case deals with $K \cup \{1\}$ for partitions of n. Here, we consider $S \cup \{1\}$, for any subset S of $\mathbb{N}$; and a set of positive integers could be considered a partition of infinity, in some sense. □

Note that the above result can also hold for the path connected, and even locally connected spaces. For instance, consider a 'standard' ∞-od as a cone over $\{1/n\}$, with or without the limit point 0 included. For an example of a locally connected space, simply make each arc of the ∞-od get smaller. Another way of expressing this last space is as the one-point compactification of a sequence of half-open arcs. Note that this space is planar and contractible, being a dendrite. The proof for these spaces is essentially the same as for the topologist's ∞-od, where the main difference is showing the fixed point property.

We also note that the same result also holds for a two-dimensional disc. It is easy to construct examples for the disk if for any n you can construct a map of the disk that fixes the boundary with $\mathrm{Per}(f) = \{1, n\}$. We leave the details to the reader.

A similar result holds for the Hawaiian earring H, which is the one-point compactification of a sequence of open arcs. However, there is a difference here, as the Hawaiian earring does not have the fixed point property since H retracts to a circle, which we can rotate without a fixed point. So while we can have any period set containing 1, we can also have any period set of a circle map of degree one (without a fixed point) – these are described by Misiurewicz in [15]. We note that any finite set of periods can be added to any period set of a degree one map of a circle, and there are possibly many more complicated things that can happen on the Hawaiian earring:

Question 6.5. *What are the possible period sets for maps of the Hawaiian earring that do not contain 1?*

In a similar way, we can also find self-maps of the Cantor set with any period set that contains 1. There are also many maps with no fixed point that have various period sets, and we believe:

Conjecture 6.6. *Given any subset S of the natural numbers, there is a continuous map f from the Cantor set to itself with* $\mathrm{Per}(f) = S$.

7. **Appendix.**

Lemma 7.1. *There are maps $f : I \to I$ that fix the endpoints of I with all possible period sets* $\mathrm{Per}(f)$.

Proof. Consider the one-parameter family of truncated tent maps $T_h : [0, 1] \to [0, 1]$, for $0 \le h \le 1$, as discussed by the authors of [3]:

$$T_h(x) = \min(h, 1 - 2|x - 1/2|) = \begin{cases} 2x & \text{if } x \in [0, h/2] \\ h & \text{if } x \in [h/2, 1 - h/2] \\ 2(1 - x) & \text{if } x \in [1 - h/2, 1]. \end{cases}$$

They note that given m there is a way to determine h so that $\mathrm{Per}(T_h)$ is exactly the tail of the Sharkovsky order beginning with m. While they do not give a precise formula, they describe h in terms of the orbits of size m of T_1 (of which there are less than 2^m): $h(m) = \min\{\max P \mid P \text{ is an } m\text{-cycle of } T_1\}$. They also give a value of h corresponding to 2^∞, so that $\mathrm{Per}(T_h) = \{2^n \mid n \in \mathbb{N} \cup \{0\}\}$.

These maps can be extended to maps $\widehat{T}_h : [0, 2] \to [0, 2]$ that fix the endpoints:

$$\widehat{T}_h(x) = \begin{cases} T_h(x) & \text{if } x \in [0, 1] \\ 2x - 2 & \text{if } x \in [1, 2]. \end{cases}$$

Clearly $\widehat{T}_h$ fixes 0 and 2, and 1 is not periodic since $(\widehat{T}_h)^n(1) = 0$. Any other periodic point must lie in $(0, 1)$, since for $x \in (1, 2)$ we have $\widehat{T}_h(x) < x$, which together with $\widehat{T}_h([0, 1]) \subset [0, 1]$ implies that $f^n(x) \neq x$. Thus the maps $\widehat{T}_h$ have the same periodic properties as T_h, but they also fix both endpoints of the interval. $\square$

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