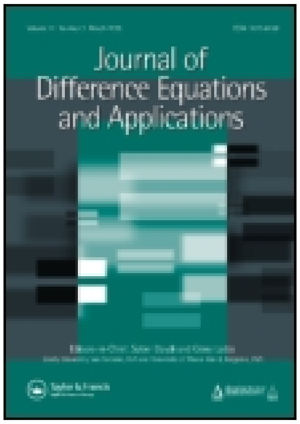

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A Sharkovsky Theorem for a Discrete System

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A class of discrete systems is described. It is shown that if one of these systems has a boundary consisting of periodic point of period n, then the system has a periodic point of period m for all $m <_s n$, where $<_s$ is the ordering of the positive integers given by Sharkovsky's Theorem.

Keywords: Sharkovsky's theorem; Discrete systems; Periodic point

1991 Mathematics Subject Classification: 37E05

INTRODUCTION

Over the years, several authors have suggested that the universe is discrete and that the models for describing it should be discrete. One such of the most recent authors is Wolfram [5]. Sharkovsky's theorem is a beautiful result from discrete dynamical systems. However, discrete in this case refers to time and not to the underlying space. This theorem is about continuous maps of the real line (or interval). In Ref. [5, page 955], Wolfram says that it is difficult to find extensions of Sharkovsky's theorem to other types of systems. This statement seems reasonable in that the proof of the theorem makes repeated use of the Intermediate Value Theorem. However, if one looks closely at this proof one sees that the main arguments are essentially all discrete. Thus it could be possible that in the theorem the hypothesis of continuous functions be replaced with an appropriate statement about discrete systems and still have the proof of this new theorem be essentially the same as the old one. This is what is done below.

First, we introduce the discrete systems. We take finite subsets of the positive integers as the underlying space and define substitution systems following Wolfram. Then we add more structure so that we have discrete analogues to continuous functions and closed intervals. Then basic results about deducing the existence of periodic orbits when "closed intervals" cover one another are proved. Finally, we give a proof of a Sharkovsky type theorem.

We note that in the next section it is shown that substitution systems can be described as vertex shifts on directed graphs. This means that they are subshifts of finite type. Thus the Sharkovsky theorem could be re-stated for a certain class of subshifts of finite type.

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CONTINUOUS SUBSTITUTION SYSTEMS

We will define our systems using finite subsets of the integers $\{1, \ldots, n\}$. Clearly, the dynamics induced by iterating any function from $\{1, \ldots, n\}$ to itself is rather simple. To get more interesting dynamics we change the idea of the underlying functions.

Define a *substitution system*, ϕ, to be a function from $\{1, \ldots, n\}$ to the set of non-empty subsets of $\{1, \ldots, n\}$ and let $i \to j$ if $j \in \phi(i)$. If we can get from i to k via a series of arrows, $i \to \cdots \to k$, then we say there is a *path* from i to k. If there is a path from an integer i back to itself we call the path a *periodic orbit*. If periodic orbit is not the repetition of a shorter periodic orbit, its length, the number of arrows, is called the *period* of the orbit.

Example 1 Let $\phi : \{1, 2\} \to \{\{1\}, \{2\}, \{1, 2\}\}$ be defined by $\phi(1) = \{1, 2\}$ and $\phi(2) = \{1, 2\}$. Then $1 \to 1$, $1 \to 2$, $2 \to 1$ and $2 \to 2$. In this case there are periodic orbit of all periods. An example of a periodic orbit of period 4 is $1 \to 1 \to 1 \to 2 \to 1$.

Example 2 Let $\eta : \{1, 2\} \to \{\{1\}, \{2\}, \{1, 2\}\}$ be defined by $\eta(1) = \{2\}$ and $\eta(2) = \{1, 2\}$. Then $1 \to 2$, $2 \to 1$ and $2 \to 2$. Again there are periodic orbits of all periods.

Example 3 Let $\gamma : \{1, 2, 3\} \to \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ be defined by $\gamma(1) = \{3\}$, $\gamma(2) = \{2, 3\}$ and $\gamma(3) = \{1\}$. Then $1 \to 3$, $2 \to 2$, $2 \to 3$ and $3 \to 1$. In this case there are only periodic orbits of periods 1 and 2. These are given by $2 \to 2$ and $1 \to 3 \to 1$ (or $3 \to 1 \to 3$).

The first three levels of the trees for η and γ are sketched below. These trees will be later referred to in the paper.

Trees for η Trees for γ

When determining the periods of periodic orbits, it is often easier to use the graph defined by the substitution system rather than working with the trees. The graphs of η and γ are given below.

Graph for η Graph for γ

In Ref. [5], Wolfram discusses substitution systems. Indeed our first two examples appear on page 82. Later all three of these examples will be seen to be examples of continuous substitution systems when viewed in the appropriate way.

Sharkovsky's theorem is a result about continuous maps of the interval. These maps are continuous and map closed intervals by stretching and folding. We extend the ideas of one-dimensionality, continuity, stretching and folding to substitution systems on the integers in the following natural way. First, the analogue of stretching has already been achieved by the fact that one integer can sent to a number of other integers.

The analogue of one-dimensionality is obtained by looking at sequences of integers, and continuity by the sequences having repeated or consecutive integers as consecutive terms.

More formally, let $a_1, \ldots a_n$ be a sequence of positive integers. We say that the sequence is *folded* if it satisfies the following four conditions for $1 \leq i \leq n-1$:

1. $|a_i - a_{i+1}|$ is either 0 or 1,
2. if $a_{i-1} < a_i$ then $a_i \leq a_{i+1}$,
3. if $a_{i-1} > a_i$ then $a_i \geq a_{i+1}$,
4. if $a_i \neq a_{i+1} = a_{i+2} = \cdots = a_{i+k} \neq a_{i+k+1}$ then $a_i = a_{i+k+1}$ if and only if k is even.

Thus $1, 2, 3, 4, 4, 3, 2, 2, 2, 1, 1, 2$ is an example of a folded sequence formed from the integers $1, 2, 3, 4$.

Notice that folded sequences consist of increasing and decreasing subsequences. Whenever a term is repeated, the orientation switches. Thus each term in the sequence belongs to an increasing or decreasing component. In the above example, the first 4 is part of an increasing component and the second 4 is part of a decreasing component. (If a folded sequence consists entirely of repetitions of one term the increasing and decreasing components are not well-defined, but once an orientation is chosen for the first term the orientations of the rest of the sequence is forced—alternating between increasing and decreasing for each consecutive term.)

Definition 1 *A function ϕ from $\{1, 2, \ldots, n\}$ to finite sequences of integers is called a continuous substitution system on $1, 2, \ldots, n$ if*

(1) for each $1 \leq i \leq n$, $\phi(i)$ is a folded sequence formed from $1, 2, \ldots, n$, and
(2) the concatenation of the sequences $\phi(1)\phi(2)\ldots\phi(n)$ is also a folded sequence.

Let the *reverse* of the sequence be the sequence written in reverse order i.e. $\text{reverse}\big(\{a_k\}_{k=1}^m\big) = \{a_{m+1-k}\}_{k=1}^m$

Definition 2 *We extend the definition of ϕ to any folded sequence of integers obtained from $1, 2, \ldots, n$ in the following way. If i is part of an increasing component of a folded sequence, let $\phi(i, +) = \phi(i)$. If i is part of a decreasing sequence, let $\phi(i, -) =$ reverse$(\phi(i))$. Suppose that $a_1, \ldots, a_k$ is a folded sequence, then we define $\phi(a_1, \ldots, a_k)$ to be the concatenation $\phi(a_1 \pm)\ldots\phi(a_k \pm)$, where $\pm$ is chosen to be either plus or minus depending whether the integers are part of an increasing or decreasing sequence.*

Example 4 Let ϕ denote the system given by $\phi(1) = 2, 1$, $\phi(2) = 1, 2, 3$ and $\phi(3) = 3, 3, 3, 2$.
Then $\phi(1, 2, 3, 3, 2, 1) = 2, 1, 1, 2, 3, 3, 3, 3, 2, 2, 3, 3, 3, 3, 2, 1, 1, 2$.

Notice that the trees for η and γ that follow the first three examples have folded sequences at all three levels. As the following lemma shows, it is always true that continuous substitution systems send folded sequences to folded sequences.

Theorem 1 *Let ϕ and η be continuous substitution systems on $1, 2, \ldots, n$. Then the composition $\phi \circ \eta$ is also a continuous substitution system on $1, 2, \ldots, n$.*

Proof *It is enough to show that if ϕ is a continuous substitution system and if $\{a_k\}_{k=1}^m$ is a folded sequence then $\phi\big(\{a_k\}_{k=1}^m\big)$ is a folded sequence. Given i such that $1 \leq i \leq m$, we will consider $\phi(a_i, \pm)\phi(a_{i+1}, \pm)$. If a_i and a_{i+1} are part of an increasing component,* *then* $a_{i+1} = a_i + 1$. *So* $\phi(a_i,+)\phi(a_{i+1},+) = \phi(a_i,+)\phi(a_i+1,+)$ *is a folded sequence from the definition of continuous substitution system.*

If a_i *and* a_{i+1} *are part of a decreasing component, then* $a_{i+1} = a_i - 1$. *In this case,* $\phi(a_i,-)\phi(a_{i+1},-) = \phi(a_i,-)\phi(a_i-1,-)$ *is the reverse of* $\phi(a_i,+)$ *concatenated with the reverse of* $\phi(a_i-1,+)$, *but this is just the reverse of* $\phi(a_i-1,+)\phi(a_i,+)$. *Again from the definition of a continuous substitution system,* $\phi(a_i-1,+)\phi(a_i,+)$ *is a folded sequence and so its reverse must also be a folded sequence.*

If a_i *and* a_{i+1} *have different orientations then* $a_{i+1} = a_i$ *and so* $\phi(a_i,-)\phi(a_{i+1},-)$ *is either* $\phi(a_i,+)\,(reverse(\phi(a_i,+)))$ *or* $(reverse(\phi(a_i,+)))\phi(a_i,+)$. *In either case, we obtain a folded sequence.* □

If we let ϕ^k denote the composition of ϕ with itself k times, we get the following as an immediate consequence of the previous lemma.

COROLLARY 2 *Let* ϕ *be a continuous substitution system on* $1, 2, \ldots, n$. *Then for all positive integers, k, the composition* ϕ^k *is also a continuous substitution system on* $1, 2, \ldots, n$.

DEFINITION 3 *Let* ϕ *be continuous substitution system on* $1, 2, \ldots, n$. *For any i,* $1 \le i \le n$ *if* $j \in \phi(i)$ *is part of an increasing sequence we will say the arrow going from i to j is positive. If* $j \in \phi(i)$ *is part of a decreasing sequence we will say the arrow going from i to j is negative. Given a periodic orbit, we will say the orbit is negative if the number of negative arrows in the orbit is odd, otherwise the orbit is positive.*

Returning to γ defined in Example 3, the arrow from 1 to 3 is positive, the arrow from 3 to 1 is negative, so the periodic orbit $1 \to 3 \to 1$ is negative.

We conclude this section with an example that shows the set of periods associated to a continuous substitution system can be different from the set of periods given by continuous maps of the interval.

Example 5 Let δ on $\{1, \ldots, 4\}$ be given by $\delta(1) = 2$, $\delta(2) = 3, 4, 4$, $\delta(3) = 3$ and $\delta(4) = 2, 1$. There is a loop from 2 to itself given by $2 \to 4 \to 2$. There is also a loop from 2 to itself given by $2 \to 4 \to 1 \to 2$. So δ has periodic points of periods 2 and 3. We can form a periodic point of period k for any odd $k > 1$ by going around the loop of length 3 once and the loop of length 2 another $(k-3)/2$ times. For example, a periodic point of period 7 would be given by $2 \to 4 \to 1 \to 2 \to 4 \to 2 \to 4 \to 2$. We can form a periodic point of period k for any even $k > 6$ by going around the loop of length 3 twice and the loop of length 2 another $(k-6)/2$ times. It is straightforward to check that the only orbit that is not formed as a concatenation of these two loops is the fixed point $3 \to 3$. Thus the set of periods associated to the periodic points of δ is $\{z \in \mathbb{Z} | z > 0,\ \ z \neq 4,\ \ z \neq 6\}$. Notice that there are two arrows from 2 to 4, one positive, and the other negative. So if $k \notin \{1, 4, 6\}$, we can find both positive and negative periodic points with period k. The fixed point is a negative fixed point.

THE INTERMEDIATE VALUE THEOREM AND POSITIVE ORBITS

In the proof of Sharkovsky's theorem we need to use the Intermediate Value Theorem repeatedly. There is an obvious analogue of the intermediate value theorem to folded sequences whose proof is immediate.

THEOREM 3 *Let $\{a_k\}_{k=1}^m$ be a folded sequence. For any integer, r, satisfying min$(a_1, a_m) \leq r \leq \max(a_1, a_m)$ there is an integer t such that $1 \leq t \leq m$ and $a_t = r$.*

However, the particular fact that we need in the proof of Sharkovsky's theorem is that if a and b are real numbers with $f(a) < a$ and $f(b) > b$ then there is a fixed point between a and b, i.e. a number c between a and b with $f(c) = c$. Unfortunately, continuous substitution systems do not have to obey this as the following example shows.

Example 6 Let ϕ be defined by $\phi(1) = 2$ and $\phi(2) = 1$. The sequence 2, 1 is folded and so is ϕ continuous. However, the system only has periodic points of period 2. In particular, there is no fixed point.

Let the sequence $\{a + k\}_{k=0}^t$ be denoted by $[a, a + t]$, where $a > 0$ and $t \geq 0$ are integers. These subsequences $[a, b]$ play the analogous role to closed subintervals in the study of continuous maps of the interval. (Though it should be noted that we do allow the possibility of $a = b$.) We will refer to the subsequence $[a, b]$ as a *closed* subsequence.

Given a continuous substitution system ϕ, we will say a closed subsequence $[a, b]$ *covers* a closed subsequence $[c, d]$ if for each $e \in [c, d]$ there exists a $u \in [a, b]$ with $e \in \phi(u)$. Thus for each element in $[c, d]$ there is an arrow that ends here that started at some element of $[a, b]$. If $[a, b]$ covers $[c, d]$ and $[a, b]$ does not cover any $[e, f]$ where $[c, d] \subsetneq [e, f]$ we say that $[a, b]$ covers $[c, d]$ *exactly*.

In what follows, we will distinguish two ways in which $[a, b]$ can cover $[c, d]$. First we give the definition in the case when $c < d$. We know that both c and d must appear in the folded sequence $\phi([a, b])$, if there is an appearance of c to the left of an appearance of d we say $[a, b]$ covers $[c, d]$ *positively*, if the there is an appearance of c to the right of an appearance of d we say $[a, b]$ covers $[c, d]$ *negatively*. If $c = d$, and if there is a positive arrow from an element of $[a, b]$ to c we say $[a, b]$ covers $[c, c]$ *positively*. If there is a negative arrow from an element of $[a, b]$ to c we say $[a, b]$ covers $[c, c]$ *negatively*. Thus if $[a, b]$ covers $[c, d]$ positively (negatively) if for each element in $[c, d]$ there is a positive (negative) arrow that ends there that started at some element of $[a, b]$. Note that it is possible for $[a, b]$ to cover $[c, d]$ both positively and negatively.

The example above shows that if $[a, b]$ covers itself negatively then the continuous substitution system does not have to have a fixed point. However, the following lemma shows that if $[a, b]$ covers itself positively then the continuous substitution system is forced to have a fixed point.

LEMMA 1 *Let ϕ be a continuous substitution system and let $[a, b]$ be a closed subsequence. If $[a, b]$ covers itself positively then there is a positive fixed point $c \in [a, b]$.*

Proof *Notice that if $a = b$ then there is nothing to prove.*

Let $\{p_k\}_{k=1}^m$ denote the folded sequence $\phi(a)\ldots\phi(b)$. Since $[a, b]$ covers itself there must be at least one a and one b in this sequence. Since the covering is positive, an appearance of a must be to the left of an appearance of b. It is clear that there has to be a subsequence $\{p_k\}_{k=r}^s$ such that $p_r = a$, $p_s = b$ and such that $a < p_k < b$ if $r < k < s$. There are well-defined u and v in $[a, b]$ with $p_r \in \phi(u)$ and $p_s \in \phi(v)$. Clearly there are positive arrows from u to a and from v to b. So if either $a = u$ or $b = v$ we are done. If this is not the case then $[u,v]$ must cover itself positively and we have $a < u \leq v < b$. Once again we can repeat the argument to show that if neither u nor v are positive fixed points then there must be some x and y satisfying $u < x \leq y < v$ and such that $[x,y]$ covers itself positively. However, since *there are only a finite number of points in* $[a, b]$*, this argument can only be repeated a finite number of times before finding a positive fixed point.* □

Definition 4 *A substitution system* ϕ_1 *defined on* $[a, b]$ *is called a pruning of* ϕ *if*

1. $\phi_1(u)$ *is a folded sequence for each* $u \in [a, b]$,
2. $\phi_1([a, b])$ *is a folded sequence, and*
3. $\phi_1(u) \subseteq \phi(u)$ *for each* $u \in [a, b]$.

We make some observations about this definition. First, in the third part $\phi_1(u)$ and $\phi(u)$ are sequences of integers. When we write $\phi_1(u) \subseteq \phi(u)$ we are dropping the ordering and thinking of $\phi_1(u)$ and $\phi(u)$ as sets. All these statements say that every arrow that leaves u under $\phi_1(u)$ also appears as an arrow leaving u under $\phi(u)$. It should also be noted that though the domain of ϕ_1 is $[a, b]$ the folded sequences $\phi_1(u)$ will, in general, be sequences formed from $\{1, \ldots, n\}$.

Lemma 2 *Let* ϕ *be a continuous substitution system. If* $[a, b]$ *covers* $[c, d]$ *positively and if* $[e, f] \subseteq [c, d]$ *then there is a pruning* ϕ_1 *defined on* $[u, v] \subseteq [a, b]$ *of* ϕ *such that under* ϕ_1 $[u, v]$ *covers* $[e, f]$ *exactly and positively.*

Proof Let $\{p_k\}_{k=1}^m$ *denote the folded sequence* $\phi(a)\ldots\phi(b)$*. Since* $[a, b]$ *covers* $[c, d]$ *there must be at least one* e *and one* f *in this sequence. Since the covering is positive, an appearance of* e *must be to the left of an appearance of* f*. It is clear that there has to be a subsequence* $\{p_k\}_{k=r}^s$ *such that* $p_r = e$, $p_s = f$ *and such that* $e < p_k < f$ *if* $r < k < s$*. Since* $\{p_k\}_{k=r}^s$ *comes from a substitution system there are well-defined* u *and* v *with* $p_r \in \phi(u)$ *and* $p_s \in \phi(v)$*. Let* $\phi_1(k) = \phi(k)$ *for* $u < k < v$*. Define* $\phi_1(u)$ *and* $\phi_1(v)$ *to be the appropriate subsets of* $\phi(u)$ *and* $\phi(v)$ *such that* $\phi_1(u), \ldots, \phi_1(v) = \{p_k\}_{k=r}^s$. □

An analogous proof shows the following.

Lemma 3 *Let* ϕ *be a continuous substitution system. If* $[a, b]$ *covers* $[c, d]$ *negatively and if* $[e, f] \subseteq [c, d]$ *then there is a pruning* ϕ_1 *defined on* $[u, v] \subseteq [a, b]$ *of* ϕ *such that under* $\phi_1([u, v])$ *covers* $[e, f]$ *exactly and negatively.*

Horseshoes play an important role in the combinatorial dynamics associated to maps of the interval. We define horseshoes for continuous substitution systems and then consider the periods of periodic orbits that a horseshoe must have.

Definition 5 *Let* ϕ *be a continuous substitution system. Let* $[a, b]$ *and* $[c, d]$ *be disjoint. If* $[a, b]$ *covers both* $[a, b]$ *and* $[e, d]$ *positively and if* $[c, d]$ *covers both* $[a, b]$ *and* $[c, d]$ *negatively we say that we have a horseshoe.*

Theorem 4 *Suppose that* ϕ*, a continuous substitution system, has a horseshoe then* ϕ *has positive periodic points of all periods except possibly 2.*

Proof Let $[a, b]$ *and* $[c, d]$ *be as in the definition. Since* $[a, b]$ *covers itself positively, Lemma 1 shows that* ϕ *has a fixed point. We now show how to find a periodic point of period* k *where* $k \geq 3$.

Since $[c, d]$ *covers* $[a, b]$ *negatively we can find, by Lemma 2, a pruning* ϕ_1 *and* $[c_1, d_1] \subseteq [c, d]$ *such that under* ϕ_1, $[c_1, d_1]$ *covers* $[a, b]$ *exactly and negatively. Similarly, we can find* ϕ_2 *and* $[c_2, d_2] \subseteq [c, d]$ *such that* $[c_2, d_2]$ *covers* $[c_1, d_1]$ *exactly and negatively. Now choose* $[a_3, b_3] \subseteq [a, b]$ *and a pruning* ϕ_3 *such that* $[a_3, b_3]$ *covers* $[c_2, d_2]$ *exactly and* *positively. Inductively define for* $j = 4, \ldots, k$, $[a_j, b_j] \subseteq [a, b]$ *and a pruning* ϕ_j *such that* $[a_j, b_j]$ *covers* $[a_{j-1}, b_{j-1}]$ *exactly and positively.*

Now the composition $\phi_1 \circ \phi_2 \circ \ldots \circ \phi_k$ *is well-defined and gives a positive covering of* $[a, b]$ *by* $[a_k, b_k] \subseteq [a, b]$ *and so by Lemma 1 has a positive fixed point. Since all the arrows given by the prunings exist in* ϕ*, it must be true that* ϕ^k*has a positive fixed point. The construction shows that the orbit of this point under* ϕ *is in* $[a, b]$ *for the first* $k-2$ *iterations and is in* $[c, d]$ *for the remaining 2 iterations. So this orbit can not be the repetition of some shorter orbit and thus* ϕ *must have a positive periodic point of period* k. □

Example 7 Let ϕ on {1,2} be defined by $\phi(1) = 1, 2$ and $\phi(2) = 2, 1$. Let $[a, b]$ be [1,1] and $[c, d]$ be $[2, 2]$. Then we have a horseshoe and $1 \rightarrow 1$ is a positive fixed point and $1 \rightarrow 2 \rightarrow 2 \rightarrow 1$ followed by $k-2$ arrows from 1 to itself gives a positive periodic point of period k for $k > 2$.

In the above example there is a point of period 2, namely $1 \rightarrow 2 \rightarrow 1$. This is a negative periodic orbit. The next example shows that it is possible to have a horseshoe that has no periodic points of period 2.

Example 8 Let ψ be defined by $\psi(1) = 1, 2, 3$, $\psi(2) = 4$, $\psi(3) = 4, 3, 2$ and $\psi(4) = 1$. Then if we let $[a, b] = [1, 2]$ and $[c, d] = [3, 4]$ it is clear that we have a horseshoe. It is straightforward to check that there are no points of period 2.

THE BOUNDARY FUNCTION

Before we start to investigate negative periodic orbits we need to look at the boundary function.

DEFINITION 6 *Suppose that we are given a continuous substitution system,* ϕ *on* $\{1, \ldots, n\}$*. The boundary function,* $\partial\phi : \{1, \ldots, n+1\} \rightarrow \{1, \ldots, n+1\}$*, is defined as follows.*

Suppose that $\phi(i)$ *is a sequence of length m. We will denote the first and last elements in the sequence by* $\{\phi(i)\}_1$ *and* $\{\phi(i)\}_m$*, respectively. If* $\{\phi(i)\}_1$ *is part of an increasing component then* $\partial\phi(i) = \{\phi(i)\}_1$*, if* $\{\phi(i)\}_1$ *is part of a decreasing component then* $\partial\phi(i) = \{\phi(i)\}_1 + 1$*. If* $\{\phi(i)\}_m$ *is part of an increasing component then* $\partial\phi(i+1) = \{\phi(i)\}_m + 1$*, if* $\{\phi(i)\}_m$ *is part of a decreasing component then* $\partial\phi(i+1) = \{\phi(i)\}_m$*.*

LEMMA 4 *Given a continuous substitution system,* ϕ*, the boundary function is well-defined.*

Proof *Let i satisfy* $1 < i < n+1$*. We will show that* $\partial\phi(i)$ *is well-defined. Let* $\{\phi(i)\}_1$ *be as in the statement of the theorem and let* $\{\phi(i-1)\}_{\text{last}}$ *denote the last element of the sequence* $\{\phi(i-1)\}$*. If* $\{\phi(i-1)\}_{\text{last}}$ *and* $\{\phi(i)\}_1$ *have the same orientation then they are consecutive integers. If they have opposite orientations then they are equal. Thus it is straightforward to check the following.*

If both $\{\phi(i-1)\}_{\text{last}}$ *and* $\{\phi(i)\}_1$ *belong to increasing components then* $\partial\phi(i) = \{\phi(i-1)\}_{\text{last}} + 1 = \{\phi(i)\}_1$.

If both $\{\phi(i-1)\}_{\text{last}}$ *and* $\{\phi(i)\}_1$ *belong to decreasing components then* $\partial\phi(i) = \{\phi(i-1)\}_{\text{last}} = \{\phi(i)\}_1 + 1$.

If $\{\phi(i-1)\}_{\text{last}}$ *belongs to an increasing component and* $\{\phi(i)\}_1$ *belongs to a decreasing component then* $\partial\phi(i) = \{\phi(i-1)\}_{\text{last}} + 1 = \{\phi(i)\}_1 + 1$.

If $\{\phi(i-1)\}_{\text{last}}$ *belongs to a decreasing component and* $\{\phi(i)\}_1$ *belongs to an increasing component then* $\partial\phi(i) = \{\phi(i-1)\}_{\text{last}} = \{\phi(i)\}_1$. □

Referring back to γ given in the third example, we have $\partial\gamma(1) = 3$, $\partial\gamma(2) = 4$, $\partial\gamma(3) = 2$ and $\partial\gamma(4) = 1$. Notice also that $\partial\gamma^2(1) = 2$, $\partial\gamma^2(2) = 1$, $\partial\gamma^2(3) = 4$ and $\partial\gamma^2(4) = 3$. Thus $\partial\gamma^2 = (\partial\gamma)^2$. The following shows that it is true in general that the boundary operator commutes with composition of continuous substitution systems.

THEOREM 5 *Let* ϕ *and* η *be continuous substitution systems on* $1, 2, \ldots, n$. *Then* $\partial\phi \circ \partial\eta = \partial(\phi \circ \eta)$.

Proof *We will show that* $\partial\phi \circ \partial\eta(i) = \partial(\phi \circ \eta)(i)$. *Let* $\eta(i) = a_1, \ldots, a_k$ *and* $\phi(a_1) = b_1, \ldots, b_r$.

If both a_1 *and* b_1 *are parts of increasing components then* $\partial\eta(i) = a_1$ *and* $\partial\phi(a_1) = b_1$. *The composition* $\phi \circ \eta$ *sends i to a sequence that begins with* $b_1, \ldots, b_r$ *and since the composition of increasing components is increasing we have* $\partial(\phi \circ \eta)(i) = b_1$.

A similar argument works if a_1 *is part of an increasing sequence and* b_1 *is part of a decreasing sequence. In this case,* $\partial\eta(i) = a_1$ *and* $\partial(\phi \circ \eta)(i) = b_1 + 1$.
Suppose that a_1 *is part of a decreasing sequence, then* $\partial\eta(i) = a_1 + 1$ *and* $\phi \circ \eta$ *sends i to a sequence that begins with* $b_r, \ldots, b_1$. *Suppose that* b_r *in this sequence is part of an increasing component. Then* $\partial(\phi \circ \eta)(i) = b_r$. *Now* $\phi(a_1) = b_1, \ldots, b_r$ *and* b_r *in this sequence must be part of a decreasing component. This means that the first term of* $\phi(a_1 + 1)$ *is* $b_r - 1$ *or* b_r *depending on whether this term of* $\phi(a_1 + 1)$ *is decreasing or increasing, but notice that in either case we obtain* $\partial\phi(a_1 + 1) = b_r$.

A similar argument to the above works in the final case. □

The following is immediate.

COROLLARY 6 *Let* ϕ *be continuous substitution system. Then for all positive integers, k,* $\partial(\phi^k) = (\partial\phi)^k$.

NEGATIVE ORBITS

We now return to look at negative coverings and negative periodic points.

THEOREM 7 *Let* ϕ *be a continuous substitution system. If* $\partial\phi$ *does not have a fixed point then* ϕ *has a negative fixed point.*

Proof *Let* ϕ *be a continuous substitution system on* $1, 2, \ldots, n$. *First we will show that* ϕ *has a fixed point. Suppose for a contradiction that* $i \notin \phi(i)$ *for* $1 \leq i \leq n$. *Then all the terms in* $\phi(1)$ *must be greater than 1 and all the terms in* $\phi(n)$ *must be less than n. Let k denote the smallest integer with all the terms of* $\phi(k)$ *less than k. Then the last term in* $\phi(k-1)$ *must satisfy* $\{\phi(k-1)\}_{\text{last}} \geq k$ *and the first term in* $\phi(k)$ *must satisfy* $\{\phi(k-1)\}_1 \leq k - 1$. *Since we have a folded sequence these two terms have to be consecutive. So,* $\{\phi(k-1)\}_{\text{last}} = k$ *and* $\{\phi(k)\}_1 = k - 1$, *but this gives a contradiction as it forces* $\partial\phi(k) = k$. *Thus* ϕ *has a fixed point.*

Since $\partial\phi(1) > 1$ *the first k such that* $k \in \phi(k)$ *must be a negative fixed point.* □

LEMMA 5 *Let* ϕ *be a continuous substitution system and let* $[a, b]$ *be a closed subsequence. If* $[a, b]$ *covers itself negatively and if* $\partial\phi$ *does not have a fixed point in* $[a+1, b]$ *then* ϕ *has a negative fixed point.*

Proof *Notice that if $a = b$ then there is nothing to prove.*

Let $\{p_k\}_{k=1}^m$ denote the folded sequence $\phi(a)\ldots\phi(b)$. Since $[a, b]$ covers itself there must be at least one a and one b in this sequence. Since the covering is negative an appearance of a must be to the right of an appearance of b. It is clear that there has to be a subsequence $\{p_k\}_{k=r}^s$ such that $p_r = b$, $p_s = a$ and such that $a < p_k < b$ if $r < k < s$. There are well-defined u and v in $[a, b]$ with $p_r \in \phi(u)$ and $p_s \in \phi(v)$. Thus there are negative arrows from u to b and from v to a. If either u or v are negative fixed points, we are done. If neither u nor v are negative fixed points then let $\phi_1(k) = \phi(k)$ for $u < k < v$ and define $\phi_1(u)$ and $\phi_1(v)$ to be the appropriate subsets of $\phi(u)$ and $\phi(v)$ such that $\phi_1(u), \ldots, \phi_1(v) = \{p_k\}_{k=r}^s$. All the terms in $\phi_1(u)$ must be greater than u and all the terms in $\phi_1(v)$ must be less than v. As above the smallest t such that $t \in [u, v]$ and such that $\phi_1(t)$ contains terms less than t must be a negative fixed point unless $\partial\phi_1(t) = t$. Since $\partial\phi_1(s) = \partial\phi(s)$ for $u < s < v$ and $\partial\phi$ does not have a fixed point in $[a + 1, b]$, $\partial\phi_1(t) \neq t$. □

Using this result we can modify the proof of Theorem 4 to obtain:

THEOREM 8 *Let ϕ be a continuous substitution system that has a horseshoe. If $(\partial\phi)^k$ does not have a fixed point then ϕ has a negative periodic point with period k.*

Proof *Let $[a, b]$ and $[c, d]$ be as in the definition of a horseshoe. Since $[c, d]$ covers itself negatively, Lemma 5 shows that ϕ has negative fixed point if $\partial\phi$ does not have a fixed point. We now show how to find a negative periodic point of period k where $k \geq 2$.*

Since $[c, d]$ covers $[a, b]$ negatively we can find, by Lemma 2, a pruning ϕ_1 and $[c_1, d_1] \subseteq [c, d]$ such that under ϕ_1, $[c_1, d_1]$ covers $[a, b]$ exactly and negatively. Similarly, we can find ϕ_2 and $[a_2, b_2] \subseteq [a, b]$ such that $[a_2, b_2]$ covers $[c_1, d_1]$ exactly and positively. Inductively define, for $j = 3, \ldots, k$, $[a_j, b_j] \subseteq [a, b]$ and a pruning ϕ_j such that $[a_j, b_j]$ covers $[a_{j-1}, b_{j-1}]$ exactly and positively.

Now the composition $\phi_1 \circ \phi_2 \circ \cdots \circ \phi_k$ is well-defined and gives a negative covering of $[a, b]$ by $[a_k, b_k] \subseteq [a, b]$ and so by Lemma 5 has a negative fixed point if $\partial(\phi_1 \circ \phi_2 \circ \ldots \circ \phi_k)$ doesn't have a fixed point in $[a_k + 1, b_k]$. Notice that $\partial(\phi_1 \circ \phi_2 \circ \cdots \circ \phi_k)(t) = (\partial\phi^k)(t)$ for $t \in [a_k + 1, b_k]$ and since $(\partial\phi)^k$ has no fixed points $\partial(\phi_1 \circ \phi_2 \circ \cdots \circ \phi_k)$ doesn't have a fixed point in $[a_k + 1, b_k]$. Since all the arrows given by the prunings exist in ϕ it must be true that ϕ^k has a negative fixed point. The construction shows that the orbit of this point under ϕ is in $[a, b]$ for the first $k - 1$ iterations and is in $[c, d]$ for the remaining iterations. So this orbit can not be the repetition of some shorter orbit and thus ϕ must have a negative periodic point of period k. □

LEMMA 6 *Let ϕ be a continuous substitution system on $\{1, \ldots, n\}$. If $\partial\phi$ equals the identity on $\{1, \ldots, n\}$ then for each k, $1 \leq k \leq n$, k is a positive fixed point.*

Proof *We know $\partial\theta(k) = k$ and $\partial(k + 1) = k + 1$. We consider various cases depending on the orientations of $\{\phi(k)\}_1$ and $\{\phi(k)\}_{\text{last}}$.*

If the orientation of $\{\phi(k)\}_1$ is positive then $\{\phi(k)\}_1 = \partial\theta(k) = k$ and we are done. If the orientation of $\{\phi(k)\}_{\text{last}}$ is positive then $\{\phi(k)\}_{\text{last}} + 1 = \partial\theta(k + 1) = k + 1$ and we are done. If the orientations of both $\{\phi(k)\}_1$and$\{\phi(k)\}_{\text{last}}$ are negative then $\{\phi(k)\}_1 = k - 1$ and $\{\phi(k)\}_{\text{last}} = k + 1$. Lemma 1 shows that there must be a $k \in \phi(k)$ with positive orientation. □

SYSTEMS WITH CYCLIC BOUNDARY

Given a continuous substitution system ϕ defined on $\{1,\ldots,n\}$ if its boundary function, $\partial\phi$, gives a cyclic permutation on $\{1,\ldots,n+1\}$ we will say that ϕ has *cyclic boundary*. In this section we will start by restricting attention to systems where the boundary gives a permutation and then we will study what happens when the permutation consists of one cycle.

Given any permutation, θ on $\{1,\ldots,n+1\}$, we can define a continuous substitution system S_θ in the following way. If $\theta(i) < \theta(i+1)$ let $S_\theta(i) = \theta(i), \theta(i)+1,\ldots,\theta(i+1)-1$. If $\theta(i+1) < \theta(i)$ let $S_\theta(i) = \theta(i)-1,\ldots,\theta(i+1)+1,\theta(i+1)$. It is straightforward to prove the following two lemmas.

LEMMA 7 *Let θ be a permutation on $\{1,\ldots,n+1\}$. Then $\partial S_\theta = \theta$*

LEMMA 8 *Let ϕ be continuous substitution system on $\{1,\ldots,n\}$ with boundary θ. If θ is a permutation on $\{1,\ldots,n+1\}$, then S_θ is a pruning of ϕ.*

The advantage of looking at $S_{\partial\phi}$ rather than ϕ is that all the arrows leaving a fixed $i \in [1,n]$ have the same orientation. Thus if $i \to j$ it is either positive or negative but cannot be both.

The following technical lemma gives conditions on a substitution system that force it to have a horseshoe. This result is important for the proof of the Sharkovsky theorem.

LEMMA 9 *Let θ be a permutation on $\{1,\ldots,n+1\}$ that contains no fixed points. Suppose that S_θ has more than one fixed point. Let $c_1 < c_2$ denote two fixed points. Assume that c_1 is a negative fixed point and that there is no fixed point c_3 with $c_1 < c_3 < c_2$. Suppose that there exists a $p \in \{1,\ldots,n+1\}$ and a positive integer k such that $c_1 < p < c_2+1$ and $\theta^k(p) \geq c_2+1$ then S_θ has a horseshoe.*

Proof *Since c_1 is a negative fixed point it follows that c_2 is a positive fixed point and that $\theta(c_1) > c_1$ and $\theta(c_2+1) > c_2+1$. If z satisfies $c_1 < z < c_2+1$ then $\theta(z) < z$. Let m denote the smallest positive integer such that $\theta^m(p) > c_2$. Then $\theta^{m-1}(p) \leq c_1$ and $c_1 < \theta^{m-2}(p) < c_2+1$. Thus under S_θ the sequence $[\theta^{m-1}(p), \theta^{m-2}(p)-1]$ covers itself and $[\theta^{m-2}(p), c_2]$ negatively and the sequence $[\theta^{m-2}(p), c_2]$ covers itself and $[\theta^{m-1}(p), \theta^{m-2}(p)-1]$ positively.* □

A SHARKOVSKY THEOREM

Sharkovsky first proved his famous theorem in Ref. [4] in 1962. Since then there have been various proofs given of this theorem and variations on it. Probably the simplest method of proof is a directed graph proof first given by Block *et al.* in Ref. [2] and Ho and Morris in Ref. [3]. The proof given below for substitution systems follows a variation of this proof in Ref. [1].

First we will define an ordering of the positive integers.

Let Sh_2 denote $\{2^k | k \geq 0\}$. Let $\mathrm{Sh}_{2^k 3}$ denote $\{2^k z | z > 1 \text{ is odd}\}$ for any $k \geq 0$. Clearly every positive integer belongs to one and only one of these sets.

We define the *Sharkovsky ordering* as ordering on the positive integers as follows:

1. $2^m >_s 2^n$ if $m > n$.
2. If $2^k a, 2^k b \in S_{2^k 3}$ then $2^k a <_s 2^k b$ if $a > b$.

3. If $m \in \mathrm{Sh}_{2^l 3}$ and $n \in Sh_{2^l 3}$ then $m <_s n$ if $k > l$.
4. If $m \in \mathrm{Sh}_2$ and $n \in \mathrm{Sh}_{2^l 3}$ then $m <_s n$.

Thus statements (1) and (2) describe the ordering within the Sh-sets and (3) and (4) describe how the Sh-sets are ordered.

We will now state the main result of the paper.

THEOREM 9 *Let ϕ be continuous substitution system on $\{1, \ldots, n-1\}$ with cyclic boundary. Then ϕ has a periodic point with period m for any $m <_s n$.*

Proof *Lemma 8 shows that it is enough if we can show that S_θ has a periodic point of period m where $\theta = \partial\phi$. We show this in the following lemmas.* □

LEMMA 10 *Let θ be a cycle of length n. If n is not a divisor of 2^k then S_θ has a periodic point of period 2^k.*

Proof *Since n is not a divisor of 2^k we know that $\partial S_{\theta^{2^k}} = (\partial S_\theta)^{2^k} = \theta^{2^k}$ contains no fixed points. Theorem 6 shows that $S_{\theta^{2^k}}$ has a negative fixed point.*

Thus, S_θ has a loop of length 2^k with negative orientation. Since the orientation is negative it cannot be the repetition of a shorter loop as any shorter loop would have to be repeated an even number of times. So S_θ must have a periodic point of period 2^k. □

Notice that this result proves the theorem for pairs of integers that satisfy either condition (1) or (4). The following shows that condition (3) is satisfied.

LEMMA 11 *Let θ be a cycle of length $2^k p$, where $p > 1$ is odd and $k \geq 0$. Then S_θ has a periodic orbit with period $2^k q$ for any $q \geq p$.*

Proof *Since $\partial S_{\theta^{2^k}} = (\partial S_\theta)^{2^k} = \theta^{2^k}$ contains no fixed points. Theorem 6 shows that $S_{\theta^{2^k}}$ has a negative fixed point. Thus S_θ has a loop from one vertex back to itself of length 2^k with negative orientation.*

Since $\theta^{2^k p}$ equals the identity, we know by Lemma 6 that under $S_{\theta^{2^k p}}$ every point is fixed with a positive orientation.

Thus one element of $\{1, \ldots, 2^k p - 1\}$ under S_θ has a loop of length $2^k p$ and a loop of length 2^k. Since the loop of length $2^k p$ has positive orientation, the loop of length 2^k has negative orientation, and p is odd, it follows that the longer loop is not a repetition of the shorter one. We can form a non-repeating loop of length $2^k q$, where $q \geq p$ by first going around the loop of length $2^k p$ once and then going around the loop of length 2^k $q - p$ times. So S_θ has a periodic orbit with period $2^k q$. □

The following result completes the proof of the theorem.

LEMMA 12 *Let θ be a cycle of length $2^k p$, where $p > 1$ is odd. Then S_θ has periodic points of all periods divisible by 2^{k+1}.*

Proof *Consider θ^{2^k}. This consists of 2^k cycles of length p. Given a cycle, $(a_1, a_2, \ldots, a_p)$, let its maximum be $\max\{a_1, \ldots, a_p\}$ and its minimum be $\min\{a_1, \ldots, a_p\}$.*

Now consider $S_{\theta^{2^k}}$. Since θ^{2^k} has no fixed points we know that $S_{\theta^{2^k}}$ must have at least one. Let c_1 denote the smallest fixed point of $S_{\theta^{2^k}}$. Consider the cycles of θ^{2^k} that have minimums less than or equal to c_1. From these cycles choose the one with the smallest maximum. Denote this cycle by $(b_1, \ldots, b_p)$ and its maximum and minimum by M and m.

If $S_{\theta^{2^k}}$ has more than one fixed point in $[m, M-1]$ then let c_1 and c_2 denote the first two fixed points. Since θ^{2^k} has no fixed points c_1 must be a negative fixed point and c_2 a positive one.

Let p satisfy $c_1 < p < c_2 + 1$. *If p belongs to the cycle* $(b_1, b_2, \ldots, b_p)$ *then some iterate of p under* θ^{2^k} *is* $M > c_2$ *and Lemma 9 shows that* $S_{\theta^{2^k}}$ *has a horseshoe.*

If p does not belong to the cycle $(b_1, b_2, \ldots, b_p)$ *then it must belong to some other cycle,* $(d_1, d_2, \ldots, d_p)$. *It is clear that* $\theta^{2^k}(p) \leq c_1$ *so the minimum of the cycle must be less than or equal to* c_1. *From the definition of the cycle* $(b_1, \ldots, b_p)$ *we know that the maximum of* $(d_1, d_2, \ldots, d_p)$ *is greater than M. So some iterate of p under* θ^{2^k} *is greater than M which is greater than* c_2 *and again Lemma 9 shows that* $S_{\theta^{2^k}}$ *has a horseshoe.*

Thus we have shown that either $S_{\theta^{2^k}}$ *has a horseshoe or* $S_{\theta^{2^k}}$ *restricted to* $[m, M-1]$ *has only one fixed point.*

If $S_{\theta^{2^k}}$ *has a horseshoe then Theorem 8 shows that* $S_{\theta^{2^k}}$ *has negative periodic periods of all period except for periods that are multiples of p. This means that* S_θ *has points of period* $2^k m$ *for any m that is not a multiple of p. However, Lemma 11 shows that* S_θ *has points of period* $2^k(rp)$ *for any* $r > 1$. *Thus* S_θ *has periodic points of periods* $2^k m$ *for every* $m \neq p$ *and so clearly has a point of period t where t is any multiple of* 2^{k+1}.

If $S_{\theta^{2k}}$ *restricted to* $[m, M-1]$ *has only one fixed point. Then we consider* $S_{\theta^{2^{k+1}}}$ *on* $[m, M-1]$. *There are two cases to consider depending on whether or not* c_1 *is a positive fixed point of* $S_{\theta^{2^{k+1}}}$. *If* c_1 *is a positive fixed point then since* $\theta^{2^k}(m) > m$ *there has to be a negative fixed point,* c_4, *for* $S_{\theta^{2^{k+1}}}$ *satisfying* $m \leq c_4 < c_1$. *Since the signs of the fixed points alternate we can assume there are no fixed points between* c_4 *and* c_1. *As above let p satisfy* $c_4 < p < c_1 + 1$. *If p belongs to the cycle* $(b_1, b_2, \ldots, b_p)$ *then some iterate of p under* $\theta^{2^{k+1}}$ *is* $M > c_1$ *and Lemma 9 shows that* $S_{\theta^{2^{k+1}}}$ *has a horseshoe.*

If p does not belong to the cycle $(b_1, b_2, \ldots, b_p)$ *then it must belong to some other cycle,* $(d_1, d_2, \ldots, d_p)$. *It is clear that the minimum of this cycle must be less than or equal to* c_1. *From the definition of the cycle* $(b_1, \ldots, b_p)$ *we know that the maximum of* $(d_1, d_2, \ldots, d_p)$ *is greater than M. So some iterate of p under* $\theta^{2^{k+1}}$ *is greater than M which is greater than* c_1 *and again Lemma 9 shows that* $S_{\theta^{2^{k+1}}}$ *has a horseshoe.*

The argument given above shows that if $S_{\theta^{2^{k+1}}}$ *has a horseshoe then* S_θ *has periodic points of periods* $2^{k+1}m$ *for every m that is not a multiple of p. However, Lemma 11 shows that* S_θ *has points of period* $2^k(rp)$ *for any* $r > 1$. *Thus* S_θ *has periodic points of periods* $2^{k+1}m$ *for every m.*

The final case to consider is when $S_{\theta^{2^k}}$ *restricted to* $[m, M-1]$ *has only one fixed point and when this point* c_1 *is not a positive fixed point of* $S_{\theta^{2^{k+1}}}$. *Since* c_1 *is a fixed point under* $S_{\theta^{2^k}}$ *it must be a positive fixed point of* $S^2_{\theta^{2^k}}$. *This means that for* S_θ *there must be a negative loop of length* 2^k *from* c_1 *to itself and a negative loop of length* 2^{k+1} *from* c_1 *to itself. Thus we can find a periodic point of period* $2^k m$ *for any positive* $m \geq 2$ *formed by going around the longer loop once and the shorter loop* $m-2$ *times.* □

References

[1] C. Bernhardt, A Proof of Sharkovsky's Theorem, *Journal of Difference Equations and Applications*, **9**(3–4) (2003), 373–379.

[2] L. Block, J. Guckenheimer, M. Misiurewicz and L.-S. Young, Periodic points and topological entropy for one dimensional maps. Global theory of dynamical systems, 18–34, Lecture Notes in Math. **819**, Springer Berlin, 1980.

[3] C.-W. Ho and C. Morris, A graph theoretic proof of Sharkovsky's theorem on the periodic points of continuous functions, *Pac. J. Math.*, **96** (1981), 361–370.

[4] A. N. Sharkovsky, Co-existence of the cycles of a continuous mapping of the line onto itself, *Ukrain. Math. Zh.*, **16**(1) (1964), 61–71.

[5] S. Wolfram, A New Kind of Science, Wolfram Media, Inc., 2002.