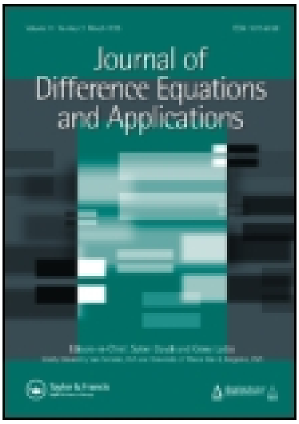

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A Sharkovsky theorem for vertex maps on trees

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A Sharkovsky theorem for vertex maps on trees

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Let T be a tree with n vertices. Let $f : T \to T$ be continuous and suppose that the n vertices form a periodic orbit under f. We show:

(1) a. If n is not a divisor of 2^k then f has a periodic point with period 2^k.
b. If $n = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then f has a periodic point with period $2^p r$ for any $r \geq q$.
c. The map f also has periodic orbits of any period m where m can be obtained from n by removing ones from the right of the binary expansion of n and changing them to zeroes.

(2) Conversely, given any n, there is a tree with n vertices and a map f such that the vertices form a periodic orbit and f has no other periods apart from the ones given above.

Keywords: tree maps; periods of orbits; Sharkovsky's theorem; binary expansion

1991 Mathematics Subject Classification: 37E05

1. Introduction

In this paper, we consider maps on trees. We will denote a map associated to a tree as a *vertex map*, if the vertices form exactly one periodic orbit. (Note that the trees we consider are trees in the combinatorial sense, i.e. where it is possible to remove a vertex and divide the tree into two connected components as opposed to the topological sense where this is not possible). There have been a number of papers that have studied maps on trees. Baldwin [6] began this study by considering certain types of trees and it has been continued by work of Alseda working in conjunction with Ye [1], Juher and Mumbru [2], Llibre and Misiurewicz [3], Guaschi et al. [5]. The study of vertex maps was started in [8]. We generalize Theorem 1 of that paper.

Given a periodic orbit on an interval, we can consider the periodic points to be the vertices on the convex hull of the orbit. In this way, we can consider the study of periodic orbits on the interval as a special case as the study of vertex maps. Sharkovsky's theorem [10] becomes a theorem about vertex maps in this special case when the tree is topologically an interval. In this paper, we prove a Sharkovsky-type theorem for vertex maps in general.

The theorem makes use of a process in which the number n is written in binary and then the rightmost one in its expansion is changed to a zero. The process is then repeated until it ends with the number zero. For example, 31 has binary expansion 11111. Applying

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the process to this number yields the following binary expansions 11110, 11100, 11000, 10000 and 00000, or in decimal notation 30, 28, 24, 16 and 0. We now state the result of this paper:

THEOREM 1. *Let T be a tree with n vertices. Let $f : T \to T$ be continuous and suppose that the n vertices form a periodic orbit under f. Then*

(1) (a) *If n is not a divisor of 2^k then f has a periodic point with period 2^k.*
(b) *If $n = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then f has a periodic point with period $2^p r$ for any $r \geq q$.*
(c) *The map f also has periodic orbits of any period m where m can be obtained from n by removing ones from the right of the binary expansion of n and changing them to zeroes.*

(2) *Conversely, given any n, there is a tree with n vertices and a vertex map f that has no other periods apart from the ones given above.*

We make the following observations. First, parts (1)(a) and (b) were proved in [8]. If we restrict to the periods greater than n then the set of periods given by Theorem 1 is exactly the same set of periods greater than n that are forced by the standard Sharkovsky theorem on the interval. The difference between Sharkovsky's theorem and Theorem 1 is in the set of forced periods that have period less than n. For example, in Sharkovsky's theorem, 31 forces all even integers less than 31, but Theorem 1 says that 31 forces only 24, 28 and 30 in addition to the powers of 2. The second part of the theorem shows that part (1) is best possible in that for any n there exists a vertex map with exactly the set of periods given by part (1).

2. Basics

Given a tree T with n vertices let $V_1, \ldots, V_n$ denote the vertices and $E_1, \ldots, E_{n-1}$ denote the edges. Let $[V_i, V_j]$ denote the shortest path in the tree that connects V_i to V_j.

Suppose θ is a permutation on $1, \ldots, n$. Then, we define L_θ, the connect-the-dot map, to be the continuous map from T to itself given by L_θ maps $[V_k, V_l]$ linearly onto $[V_{\theta(k)}, V_{\theta(l)}]$ for each edge $[V_k, V_l]$ in the tree.

We define the *Markov Graph* associated to the tree T and L_θ in the following way: the vertices of the Markov graph correspond to the edges of the tree (we will abuse notation and also denote these vertices by $E_1, \ldots, E_{n-1}$, the context should make it clear whether E_k refers to the edge in the tree or the corresponding vertex in the Markov graph); we draw a directed edge from vertex E_i to E_j if and only if $L_\theta(E_i) \supseteq E_j$. We will denote this Markov graph by $G(\theta)$.

We will denote the symmetric group on n letters by S_n. The next result is standard (see [4] or [9], for example, for a formal proof).

LEMMA 1. *Let $\theta \in S_n$. Suppose that $E_{k_0} \to E_{k_1} \to \cdots E_{k_m} \to E_{k_0}$ is a loop in $G(\theta)$. Then there exists a periodic point x of L_θ with $L_\theta^{m+1}(x) = x$ such that $L_\theta^r(x) \in E_{k_r}$ for $r = 0, 1, \ldots, m$. Conversely, if x is a periodic point in L_θ of period $m+1$ and if x is not a vertex then there exists a loop $E_{k_0} \to E_{k_1} \to \cdots E_{k_m} \to E_{k_0}$ in $G(\theta)$ such that $L_\theta^r(x) \in E_{k_r}$ for $r = 0, 1, \ldots, m$.*

As in [8], we can extend the first half of this lemma to continuous maps.

LEMMA 2. *Suppose that f is a continuous map on a tree with n vertices $V_1, V_2, \ldots, V_n$. Let $\theta \in S_n$ be such that $f(V_i) = V_{\theta(i)}$ for each $1 \leq i \leq n$. Suppose that $E_{k_0} \to E_{k_1} \to \cdots E_{k_m} \to E_{k_0}$ is a loop in $G(\theta)$. Then, there exists a periodic point x of f with $f^{m+1}(x) = x$ such that $f^r(x) \in E_{k_r}$ for $r = 0, 1, \ldots, m$.*

If θ consists of a single cycle and if the loop $E_{k_0} \to E_{k_1} \to \cdots E_{k_m} \to E_{k_0}$ in the above statement is not a repetition of a shorter loop then it can be checked that the period of x is $m+1$. In such a case, we will say the that there is a *non-repetitive* loop of length $m+1$.

To prove the first part of Theorem 1, it is enough to prove the following:

LEMMA 3. *Let T be a tree and f a continuous map from T to itself. Suppose that the tree has vertices $V_1, V_2, \ldots, V_n$. Let $\theta \in S_n$ be such that $f(V_i) = V_{\theta(i)}$ for each vertex. Suppose that θ consists of a single cycle of length n.*

(1) *If n is not a divisor of 2^k then $G(\theta)$ has a non-repetitive loop of length 2^k.*
(2) *If $n = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then $G(\theta)$ has a non-repetitive loop of length $2^p r$ for any $r \geq q$.*
(3) *$G(\theta)$ also has non-repetitive loops of any length m where m can be obtained from n by removing ones from the right of the binary expansion of n and changing them to zeroes.*

We now choose an orientation for each of the edges and define the *oriented Markov Graph* associated to the tree T and L_θ to be the Markov graph defined above with the addition of plus or minus signs to each of the directed edges to keep track of orientation. More formally, suppose there is a directed edge from vertex E_i to vertex E_j. We assign a sign of $+$ or $-$ to this directed edge according to whether L_θ maps E_i onto E_j in an orientation preserving or orientation reversing way.

Let T be a fixed tree with n vertices. Assigned to each $\theta \in S_n$ is an *oriented transition matrix*, denoted $M(\theta)$ and defined by

$$M(\theta)_{i,j} = \begin{cases} 1, & \text{if there is a positive edge from } E_j \text{ to } E_i, \\ -1, & \text{if there is a negative edge from } E_j \text{ to } E_i, \\ 0, & \text{otherwise.} \end{cases}$$

In [8], the following results were proved.

LEMMA 4. *Let $\theta \in S_n$. Then, the ijth entry of $[M(\theta)]^k$ equals the number of orientation preserving paths in $G(\theta)$ from E_i to E_j of length k minus the number of orientation reversing paths of length k.*

THEOREM 2. *Let $\theta \in S_n$. If $\theta(i) \neq i$ for $1 \leq i \leq n$ then* $\mathrm{Trace}(M(\theta)) = -1$.

THEOREM 3. *Suppose that $\alpha, \beta \in S_n$. Then, $M(\alpha)M(\beta) = M(\alpha\beta)$.*

3. Proof of Lemma 3

As noted above to prove the first part of Theorem 1, it is sufficient to prove Lemma 3. The proof of part 1 of Lemma 3 was given in [8]. We repeat it here because it is short and aids the exposition.

3.1. Proof of Lemma 3 part (1)

Proof. By hypothesis $\theta \in S_n$ where n does not divide 2^k. Note that θ^{2^j} does not fix any of $1, \dots, n$ for $0 \leq j \leq k$, so by Theorem 2, we know that $\mathrm{Trace}(M(\theta^{2^j})) = -1$. Theorem 3 then tells us that $\mathrm{Trace}(M(\theta^{2^j})) = \mathrm{Trace}((M(\theta))^{2^j}) = -1$ for $0 \leq j \leq k$. Lemma 4 shows that there must be at least one orientation-reversing path from a vertex E to itself in $G(\theta)$ of length 2^j for $0 \leq j \leq k$. This loop must be non-repetitive because any repetitive loop of length 2^j would have to be an even repetition of a shorter loop and thus have positive orientation. □

3.2. Some technical lemmas

Let $\theta \in S_n$. We only need to consider the cases when n is not a power of 2. In these cases, n does not divide 2^k for any $k \geq 0$ and so the first part of Lemma 3 tells that there must be a non-repetitive loop of length 2^k. The proof above shows that we can strengthen this statement a little more and say that there must be an orientation reversing non-repetitive loop of length 2^k for any $k \geq 0$. The rest of the proof of Lemma 3 is obtained by looking at these loops and seeing when they must intersect other loops.

It is here that the proof of Theorem 1 differs from the proof of Sharkovsky's Theorem. We will say that f *fixes* an edge E, if $E \subseteq f(E)$. When the tree is topologically an interval, it is the case that if f^k fixes more than one edge then f^k contains a 2-horseshoe and so must f^k must have periodic orbits of all periods. For trees in general, it is possible for f^k to have many fixed edges but still not contain a horseshoe. For more information about this and the proof of Sharkovsky's theorem see [7].

If an edge E is fixed by L_θ^k then the associated matrix $M(\theta)^k$ will have either $+1$ or -1 in the diagonal entry corresponding to E. The sign depends on whether E is mapped onto itself in an orientation preserving or reversing way.

Lemma 5. *Let $\theta \in S_n$ where n is not a power of 2. For a positive integer m, consider the non-repetitive loops of length 2^j in $G(\theta)$ for $0 \leq j \leq m$. If these loops are not disjoint then $G(\theta)$ has non-repetitive loops of length $k2^m$ for any $k \geq 1$.*

Proof. If $k = 1$, there is nothing to prove. So, we consider $k > 1$. Suppose non-repetitive loops of length 2^t and 2^s intersect in a vertex E in $G(\theta)$. Then, we can obtain a loop of length $k2^m$ by going 2^{m-t} times around the loop of length 2^t and $(k-1)2^{m-s}$ times around the loop of length 2^s. We now show how a non-repetitive loop of length $k2^m$ can be constructed from this loop.

We say a loop from E to itself is *prime*, if it is not the concatenation of shorter loops from E to itself. Thus, the loop of length $k2^m$ constructed above can be considered as the concatenation of prime loops. Notice that taking this concatenation and re-ordering the prime loops results in a loop of length $k2^m$ from E to itself that also belongs to $G(\theta)$.

Since the original loops of length 2^t and 2^s were non-repetitive these loops must contain at least two prime loops between them. This means that the loop of length $k2^m$ constructed above must contain at least two distinct prime loops since $k > 1$. Let P denote a prime loop contained within the loop of length $k2^m$, and suppose that it appears i times. Construct a new loop in $G(\theta)$ by re-arranging the prime loops within the loop of length $k2^m$. Start with the i copies of P followed by the other prime loops in any order. This results in a non-repetitive loop of length $k2^m$ that belongs to $G(\theta)$. □

We will start removing edges from the tree T. (When we remove an edge we leave the vertices that consist of the endpoints of the edges.) Suppose a tree has s vertices. Removing an edge from this tree results in two trees. Consider the tree with the least number of vertices. If s is even this number can be at most $s/2$, if s is odd this number can be at most $(s-1)/2$. We define $h : \mathbb{N} \to \mathbb{N}$ by $h(s) = s/2$, if s is even and $h(s) = (s-1)/2$, if s is odd. We will let h^n denote the nth iterate of h and h^0 denote the identity map.

Let $\theta \in S_n$. Consider $M(\theta)^k$, where k is not an integer multiple of n. If $M(\theta)^k$ has $+1$ as an entry on the main diagonal then we will call the edge that corresponds to this entry as an *expanding edge under* L_θ^k. Note that expanding edges are just the fixed edges under L_θ^k for which L_θ^k preserves the orientation. If E is an expanding edge under L_θ^k, we have $E \subseteq L_\theta^k(E)$. Since k is not a multiple of n, the map L_θ^k does not have any of the vertices as fixed points. Since L_θ^k is orientation preserving on E, we see that L_θ^k cannot map one of the vertices of E to the other vertex of E. This means that applying $L_\theta^{2^j}$ to an expanding edge gives a path which contains the expanding edge as an interior edge, so removing an expanding edge never gives an isolated vertex.

The following example illustrates some of the above ideas.

Example 1. Figure 1 shows a tree with 6 vertices labeled 1 through 6 and edges E_1 through E_5. Let $\theta = (1, 2, 3, 4, 5, 6)$. Then, the Figure 2 shows the associated Markov graph for L_θ. If we choose left to right orientation on the edges, we get

$$M(\theta) = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

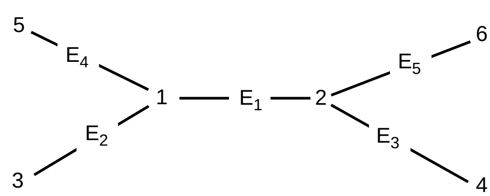

Figure 1. Tree with vertex permutation $\theta = (1,2,3,4,5,6)$.

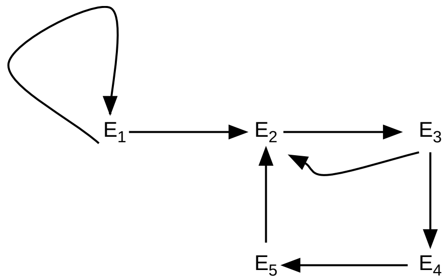

Figure 2. Markov graph $M(\theta)$.

and

$$M(\theta)^2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Thus, edge E_1 is expanding under L_{θ^2}.

We now consider the case when the loops of length 2^k for $0 \leq k \leq j$ are disjoint. The matrix $M(\theta)^{2^j}$ will have at least 2^k entries of $+1$ along the main diagonal corresponding to the loop(s) of length 2^k for $0 \leq k < j$. Since $\sum_{k=0}^{j-1} 2^k = 2^j - 1$, we must have at least $2^j - 1$ entries of $+1$ down the main diagonal. This means that there must be at least $2^j - 1$ expanding edges under $L_\theta^{2^j}$. The following is easily proved by induction.

Lemma 6. *Let $\theta \in S_n$ where n is not a power of 2. Suppose the non-repetitive loops in $G(\theta)$ of length 2^k for $0 \leq k < j$ are disjoint. Suppose the expanding edges under $L_\theta^{2^j}$ are removed from the tree T. Then the result is at least 2^j trees. The tree with the least number of vertices will have at most $h^j(n)$ vertices.*

Lemma 7. *Let $\theta \in S_n$ where n is not a power of 2. Suppose the non-repetitive loops in $G(\theta)$ of length 2^k for $0 \leq k < j$ are disjoint. Suppose that $h^{j-1}(n)$ is odd. Then, $G(\theta)$ contains non-repetitive loops of length $k2^j h^j(n) + l2^{j-1}$ for any integers $k > 0$ and $l \geq 0$.*

Proof. As above there are at least $2^j - 1$ edges in T that are expanding under $L_\theta^{2^j}$. Each one of these edges corresponds to a non-repetitive loop of length 2^k for $0 \leq k < j$ and so each corresponds to a possibly repetitive loop of length 2^{j-1}. We consider two cases.

In the first case, suppose that one of these expanding edges E_i ceases to be expanding under $L_\theta^{2^j r}$ for $0 \leq r \leq h^j(n)$. Let s denote the smallest positive integer such E_i is not expanding under $L_\theta^{2^j s}$. This means that the ith entry on the main diagonal of $M(\theta)^{2^j r}$ equals 1 for $1 \leq r < s$ but is not equal to 1 when $r = s$. This implies that in addition to the positive (repetitive) loop of length $2^{j-1}2s$ that this edge belongs to there must be another negative loop of length $2^j s$. Since s is minimal this loop must be non-repetitive. We can obtain a non-repetitive loop of length $k2^j h^j(n) + l2^{j-1}$ by first going around the negative loop k times and going $2k(h^j(n) - s) + l$ times around the positive loop of length 2^{j-1}.

In the second case, we consider what happens when all the expanding edges remain expanding under $L_\theta^{2^j r}$ for $0 \leq r \leq h^j(n)$.

As in the previous lemma, removing the expanding edges results in at least 2^j trees. We will let the tree with the least number of vertices be denoted by T^*. The previous lemma shows that T^* has at most $h^j(n)$ vertices.

We now add back the expanding edges and consider T^* within T. The only edges that connect a vertex in T^* to a vertex not in T^* are expanding edges and each one of these edges corresponds to a non-repetitive loop of length 2^k for $0 \leq k < j$ and so each corresponds to a possibly repetitive loop of length 2^{j-1}. It is enough to complete the proof, if we can show that one of these edges is also on a distinct loop of length $2^j h^j(n)$.

Suppose that there are e of these expanding edges connecting T^* to the rest of T. We will denote these edges by $E_1^* \ldots E_e^*$ and the vertices that both belong to T^* and an expanding edge by $V_1^*, \ldots, V_e^*$. Since, by hypothesis, $h^{j-1}(n)$ is odd we know that $2^j h^j(n) < n$ and so $L_\theta^{2^j w}(V_i^*) \neq V_i^*$ for $0 \leq w \leq h^j(n)$. Since T^* contains at most $h^j(n)$ vertices, it must be the case that $L_\theta^{2^j w}$ will map any vertex in T^* to some vertex that does not lie in T^* for at least one value of w satisfying $0 \leq w \leq h^j(n)$. Let k_i for $1 \leq i \leq e$ denote the smallest positive integer such that $L_\theta^{2^j k_i}(V_i^*) \notin T^*$. Notice that $L_\theta^{2^j k_i}(E_i^*)$ must contain another of the expanding edges that connect T^* to T because $L_\theta^{2^j k_i}(E_i^*)$ is expanding on E_i and the only paths to the other vertices of T are through the other expanding edges. Thus, in $G(\theta)$, there is a path from E_i^* to one of the other E^* edges of length k_i. Since this statement is true for $1 \leq i \leq e$, we must be able to find a loop $E_{q_0}^* \to E_{q_1}^* \to \cdots E_{q_m}^* \to E_{q_0}^*$, where the arrows denote paths of length $2^j k_{q_0}, 2^j k_{q_1} \ldots 2^j k_{q_m}$ in $G(\theta)$. We will assume that $E_{q_0}^* \to E_{q_1}^* \to \cdots E_{q_m}^* \to E_{q_0}^*$ is the loop of least length that can be constructed in this way, i.e. such that $k_{q_0} + k_{q_1} + \cdots + k_{q_m}$ is a minimum.

Now, for $0 \leq i \leq m$, consider the vertices $V_{q_i}^*$ that belongs to both $E_{q_i}^*$ and to the tree T^*. For $0 \leq w < k_{q_i}$, we have $L_\theta^{2^j w}(V_{q_i}^*) \in T^*$. Since we are choosing a loop of minimum length, it must be the case that $\{L_\theta^{2^j w}(V_{q_i}^*) | 0 \leq w < k_{q_i}\}$ and $\{L_\theta^{2^j w}(V_{q_u}^*) | 0 \leq w < k_{q_u}\}$ are disjoint, if $i \neq u$. Since T^* has at most $h^j(n)$ vertices, we must have $k_{q_0} + k_{q_1} + \cdots + k_{q_m} = z \leq h^j(n)$ and so the loop from $E_{q_0}^*$ to itself has length $2^j z$. We can now obtain a non-repetitive loop of length $k2^j h^j(n) + l2^{j-1}$ by going k times around the loop of length $2^j z$ and $k2(h^j(n) - z) + l$ times around the loop of length 2^{j-1}. □

Our final observation is that if $h^{j-1}(n)$ is even then $2h^j(n) = h^{j-1}(n)$ and so $2^j h^j(n) = 2^{j-1} h^{j-1}(n)$. This observation and the previous lemma give us:

LEMMA 8. *Let $\theta \in S_n$ where n is not a power of 2. Suppose the non-repetitive loops in $G(\theta)$ of length 2^k for $0 \leq k < j$ are disjoint. Then, $G(\theta)$ contains non-repetitive loops of length $2^i h^i(n)$ for $1 \leq i \leq j$.*

3.3. *Proof of part (2) of Lemma 3*

Proof. Since $n = 2^p q$ where $q > 1$ is odd part (1) of Lemma 3 tells us that $G(\theta)$ must have non-repetitive loops of length 2^j for $1 \leq j \leq p$. If these loops are not disjoint then Lemma 5 completes the proof. If they are disjoint then $h^p(n) = q$ is odd and $2h^{p+1}(n) < q$. Lemma 7 tells us that there must be non-repetitive loops of length $k2^{p+1}h^{p+1}(n) + l2^p$. Taking $k = 1$ and $l = r - 2h^{p+1}(n)$ gives the required result. □

3.4. Proof of part (3) of Lemma 3

Proof. The process of taking the binary expansion of n and then removing ones from the right results in a set of positive even integers that lie between $n/2$ and n. Notice that this set of integers is exactly $\{2\hat{h}^i(n) | 1 \leq i \leq \log_2(n)\}$. Consider the non-repetitive loops of length 2^j for $1 \leq j \leq p$, we know they exist by part (1) of Lemma 3. If these are all disjoint then Lemma 8 completes the proof. If they are not all disjoint let t denote the largest positive integer such that non-repetitive loops of length 2^j for $1 \leq j \leq t$ are disjoint. Lemma 8 shows that we have non-repetitive loops of length $2\hat{h}^i(n)$ for $1 \leq i \leq t+1$. Lemma 5 shows that we have non-repetitive loops of length $k2^{t+1}$ for any $k \geq 1$, so we must have non-repetitive loops of length $2\hat{h}^i(n)$ for $i > t$. □

4. Proof of part (2) of Theorem 1

To complete the proof of Theorem 1, we need to show that for any n, there is a tree and a vertex map that has exactly the periods given by part (1) of the theorem.

The required family and result are contained in [8]. We simply re-state these facts here.

We define a family of trees, T_n, recursively on the number of vertices n. We will label the vertices by $1,2,\ldots,n$. If $n = 1$, there is just one possible tree and labeling, a single vertex labeled 1. Suppose that the tree and labeling have been defined for 2^m vertices labeled $1,\ldots,2^m$. For k satisfying $2^m < k \leq 2^{m+1}$ define a new edge and vertex k such that the new edge goes from vertex k to vertex $k - 2^m$.

Given the tree T_n we choose the map to be L_θ where θ is the permutation $(123,\ldots,n)$. (Figure 3 shows T_{20} and its associated Markov graph.)

The following is Theorem 6 in [8]. Notice that the periods are exactly the periods given in Theorem 1.

THEOREM 4. *Let T_n be as defined above.*

If $n = 2^m$ then T_n has periodic points with periods 2^k for each k satisfying $2^k \leq n$ and there are no others.

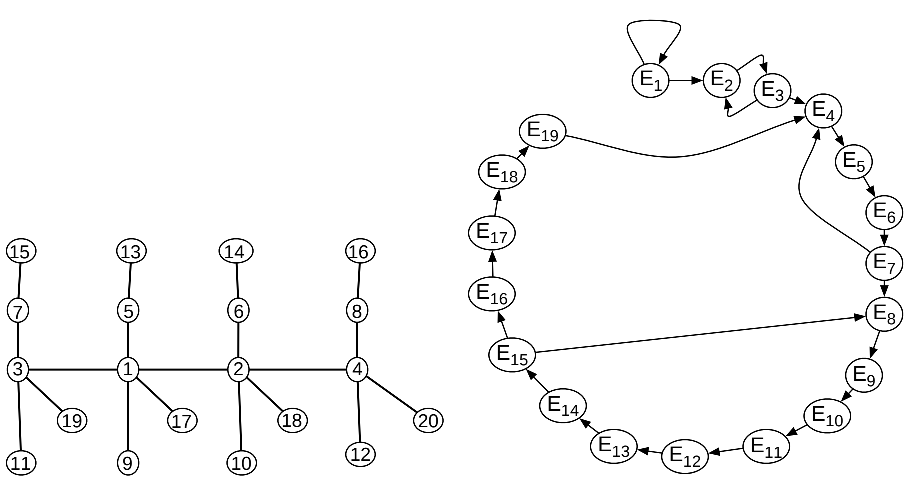

Figure 3. T_{20} and associated Markov graph.

If $n = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_j}$ *where* $r_i < r_{i+1}$ *and* $j > 1$ *then* T_n *has periodic points with periods* 2^k *for each k, periods* $n + v2^{r_1}$ *for any v, periods* $\sum_{t=m}^{j} 2^{r_t}$ *for* $m \geq 1$*, and there are no others.*

5. The partial order on the positive integers

We conclude the paper by showing that Theorem 1 gives a partial order on the positive integers and give a diagram showing this ordering (Figure 4).

We will say that an integer $a > 0$ *forces* integer $b > 0$, if

(1) $a = b$, or
(2) if $b = 2^k$ for some k and a is not a divisor of b, or
(3) if $a = 2^p q$, where $q > 1$ is odd and $p \geq 0$ and $b = 2^p r$ for some $r > q$, or

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$3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright 11 \triangleright 13 \triangleright 15 \triangleright$

$6 \triangleright 10 \triangleright 14 \triangleright 18 \triangleright 22 \triangleright 26 \triangleright 30 \triangleright$

$12 \triangleright 20 \triangleright 28 \triangleright 36 \triangleright 44 \triangleright 52 \triangleright 60$

$24 \triangleright 40 \triangleright 56 \triangleright 72 \triangleright 88 \triangleright 104 \triangleright 120$

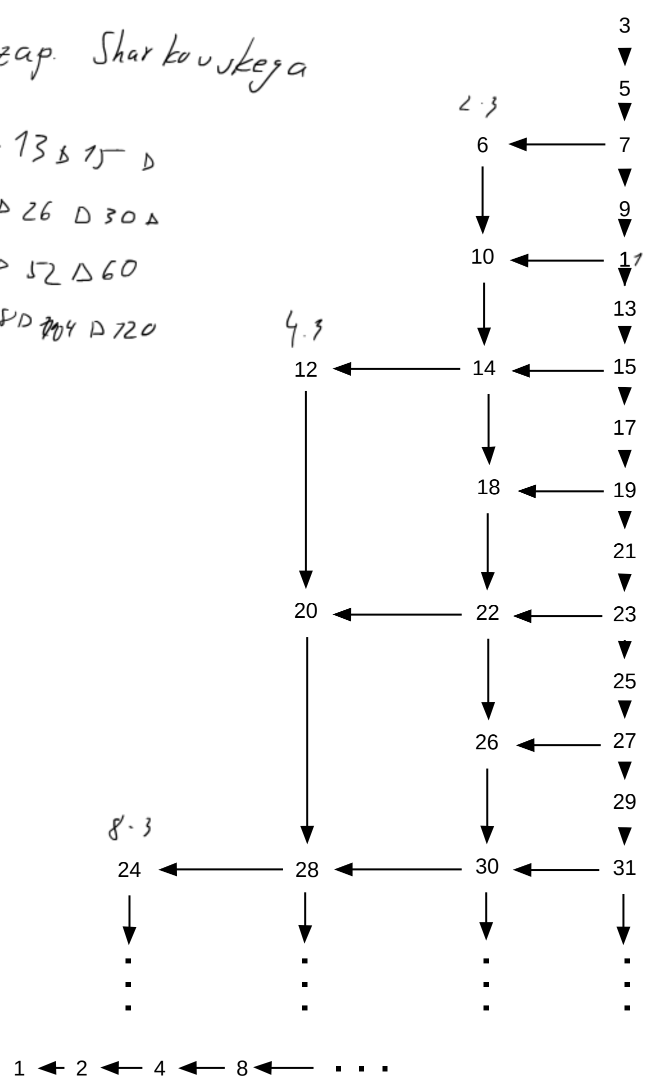

Figure 4. The partial order on the positive integers.

(4) if b can be obtained from a by removing ones from the right of the binary expansion of a.

THEOREM 5. *The relation a forces b gives a partial order on the set of positive integers.*

Proof. The reflexive property is immediate. To prove the antisymmetric and transitive properties, there are many cases to consider, most of which are straightforward. We begin proving the transitive property.

We must show that if a forces b and b forces c then a must force c. There are many cases to check, but the only non-trivial ones are if both parts (3) and (4) in the above ordering are used.

First we consider the case when a forces b using (3) and then b forces c by (4). Suppose $a = 2^p q$, where $q > 1$ is odd and $p \geq 0$ and $b = 2^p r$ for some $r > q$. Let $b_{n,} b_{n-1}, \ldots, b_0$ denote the binary expansion of b. Let $a_{n,} a_{n-1}, \ldots, a_0$ be the binary expansion of a, where we use the same number of digits in both expansions, so it is possible for a_n to be 0. Then, if $p \geq 1$, we have $b_i = a_i = 0$ for $0 \leq i \leq p-1$. Let m be the smallest integer such that $b_{n-m} \neq a_{n-m}$. Since $a < b$, we must have $b_{n-m} = 1$ and $a_{n-m} = 0$. The integer c is obtained from b by removing ones from the right. If b_{n-m} has not been removed then $c > a$ and since $b_i = a_i = c_i = 0$ for $0 \leq i \leq p-1$, we have that $c = 2^p s$ for $s > q$, and so a forces c by (3). If b_{n-m} has been removed, then since $a_i = b_i$ for $m < i \leq n$, c can also be obtained from a by removing ones from the right and so a forces c by 4.

We now consider the case when $a = 2^p q$, where $q > 1$ is odd, forces b using (4) and then b forces c by (3). Let $a_n a_{n-1}, \ldots, a_0$, $b_n b_{n-1}, \ldots, b_0$, $c_n c_{n-1}, \ldots, c_0$ be the binary expansions of a, b and c, respectively, where n is chosen to be smallest such that 2^n is greater than the maximum of a, b and c. Let w denote the smallest integer such that $b_w = 1$. Then, $w > p$ and $b_i = 0$ for $0 \leq i < w$ and $b_i = a_i$ for $w \leq i \leq n$. Now, b forces c using (3) so $c_i = b_i = 0$ for $0 \leq i < w$. Since $c > b$ and $b_i = a_i$ for $w \leq i \leq n$, we must have $c > a$. But this means that c can be written as $c = 2^p h$ where $h > q$ and so a forces c by (3).

To prove the antisymmetric property, we again need to consider many cases. However, these are trivial except to show that a if forces b using (3) then b cannot force a using (4), or that if a if forces b using (4) then b cannot force a using (3). The proof follows easily after observing that the powers of two in the prime factorizations. Notice that it a forces b using (3) the power of two in the prime factorization of b must be at least the power of two in the prime factorization of a, but if a forces b using (4) the power of two in the prime factorization of b must be greater than the power of two in the prime factorization of a. Thus, we cannot use a sequence of operations of type (3) and (4) to get from a to itself, because each time (4) is used the power of two increases and it doesn't decrease when we use (3). □

Thus, we can re-state the major result of the paper as:

THEOREM 6. *Let T be a tree with n vertices. Let $f : T \to T$ be continuous and suppose that the n vertices form a periodic orbit under f. If n forces m then f has a periodic point with period m.*

Finally, we include a diagram to illustrate this ordering. The vertical columns denote the positive odd integers, two times the positive odd integers and so on. The ordering among the columns is the same as in the standard Sharkovsky ordering. However, the difference between the tree ordering in this paper and Sharkovsky's ordering is given by the horizontal arrows which come from removing ones from the right.

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