Intermediate Value Theorem.

Let f : X → Y be a continuum map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between f (a) and f (b), then there exists a point x ∈ X such that f (c) = r.

Proof. Suppose f , X, and Y are as hypothesized. The sets A = f (X) ∩ (−∞,r) and B = f (X) ∩ (r, +∞) are disjoint (since (−∞,r) and (r, +∞) are disjoint) and nonempty since f (a) is in one of these sets and f (b) is in the other. Each is open in f (X) under the subspace topology. ASSUME there is no point c ∈ X such that f (c) = r. Then f (X) = A ∪ B and A and B form a separation of f (X). But since X is connected and f is continuous then f (X) is connected by Theorem 23.5, a CONTRADICTION. So the assumption that there is no such c ∈ X is false and hence f (c) = r for some c ∈ X.

Theorem 24.1. If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L. Proof. Recall that a subspace Y of L is convex if for every pair of points a, b ∈ Y with a < b, then entire interval [a, b] = {x ∈ L | a ≤ x ≤ b} lies in Y .

Let Y be convex. ASSUME that Y has a separation and that Y is the union of disjoint nonempty sets A and B, each of which is open in Y. Choose a ∈ A and b ∈ B. WLOG, say a < b. Since Y is convex then [a, b] ⊂ Y . Hence [a, b] is the union of the disjoint sets A0 = A ∩ [a, b] and B0 = B ∩ [a, b], each of which is open in [a, b] in the subspace topology on [a, b] (since A and B are open in Y) which is the same as the order topology (by Theorem 16.4). Since a ∈ A0 and b ∈ B0, then A0 6= ∅ 6= B0 and so A0 and B0 form a separation of [a, b].

Proof (continued). Let c = supA0. We now show in to cases that c 6∈ A0 and c 6∈ B0, which CONTRADICTS the fact that [a, b] = A0 ∪ B0 (since A0 ⊂ [a, b] then b is an upper bound for A0 and so a ≤ c ≤ b and so c ∈ [a, b] = A0 ∪ B0). From this contradiction, it follows that Y is connected. Case 1. Suppose c ∈ B0. Then c 6= a (since a ∈ A and A ∩ B = ∅). So either c = b or a < c < b. In either case, since B0 is open in [a, b] then there is some interval of the form (d, c] ⊂ B0. If c = b we have a contradiction since this implies that d is an upper bound of A0, but d < c. If c < b we note that (c, d] ∩ A0 = ∅ since c is an upper bound of A0. Then (with d as above where (d, c] ⊂ B0) we have that (d, b] = (d, c] ∪ (c, b] does not intersect A0. Again, d is a smaller upper bound of A0 than c, a CONTRADICTION. We conclude that c 6∈ B0.

Proof (continued).

Case 2. Suppose c ∈ A0. Then c 6= b since b ∈ B. So either c = a of a < c < b. Because A0 is open in [a, b], there must be some interval of the form [c, e) contained in A0. By property (2) of the linear continuum L, there is z ∈ L such that c < z < e. Then z ∈ A0, CONTRADICTING the fact that c is an upper bound of A0. We conclude that c 6∈ A0. We have shown that if Y is a convex subset of L then Y is connected. Notice that intervals and rays are convex sets and so are connected.