PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 3, July 1987

MINIMAL PERIODIC ORBITS AND TOPOLOGICAL ENTROPY OF INTERVAL MAPS

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ABSTRACT. For any two integers $m \geq 0$ and $n \geq 1$, we construct continuous functions from $[0,1]$ into itself which have exactly one minimal periodic orbit of least period $2^m(2n+1)$, but with topological entropy equal to ∞.

Introduction. Let I denote the unit interval $[0,1]$ and let $g \in C^0(I,I)$. If g has a periodic point of least period $2^m(2n+1)$, where $m \geq 0$ and $n \geq 1$ are integers, then it is well known [**3**] that the topological entropy of g is greater than or equal to $(\log \lambda_n)/2^m$, where λ_n is the (unique) positive zero of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$. The converse is false, but known counterexamples are rather complicated [**4**, p. 407]. The purpose of this note is to indicate how to use an easy and well-known method to construct examples which are simpler but with stronger properties than those given in [**4**, p. 407] except that our examples are not piecewise monotone. As a consequence of our construction, we also obtain the well-known example g_∞ as described in [**8**, p. 14] which has exactly one periodic orbit of least period 2^m for every $m \geq 0$ and no other periodic orbits. It is worth mentioning that the set of all periodic points of the example g_∞ described in [**8**, p. 14] is not closed. This is in contrast to the fact [**7**] that if the set of all periodic points of a continuous function in $C^0(I,I)$ is closed, then this function can only have periodic points of periods some powers of 2.

The construction. For every continuous function g in $C^0(I,I)$, let $G\colon [0,3] \to [0,3]$ be the continuous function defined by (i) $G(x) = g(x) + 2$ for $0 \leq x \leq 1$; (ii) $G(x) = x - 2$ for $2 \leq x \leq 3$; and (iii) G is linear on $[1,2]$. Then it is clear that $G^2|I = g$. Now let $\tilde{g}$ be the scaled-down copy of G on I. That is, $\tilde{g}(x) = [g(3x)+2]/3$ for $0 \leq x \leq 1/3$; $\tilde{g}(x) = [2 + g(1)](2/3 - x)$ for $1/3 \leq x \leq 2/3$; and $\tilde{g}(x) = x - 2/3$ for $2/3 \leq x \leq 1$. It follows from [**1**] that the topological entropy of $\tilde{g}$ is greater than or equal to one half of that of g. This function $\tilde{g}$ is called the renormalized square root of g on I. For every continuous function g_0 in $C^0(I,I)$, we define the sequence $\langle g_m \rangle$ $(m \geq 1)$ inductively by letting g_m be the renormalized square root of g_{m-1} on I. This sequence $\langle g_m \rangle$ $(m \geq 1)$ is called the sequence of successive renormalized square roots of g_0 on I.

For every positive integer k, choose $2k+2$ real numbers $a_{k,i}$ with $0 = a_{k,0} < a_{k,1} < a_{k,2} < \cdots < a_{k,2k+1} = 1$. Let p_k be the continuous function in $C^0(I,I)$ defined by (i) $p_k(a_{k,i}) = 0$ for all even i; (ii) $p_k(a_{k,i}) = 1$ for all odd i; and (iii) p_k is linear on each interval $[a_{k,i}, a_{k,i+1}]$, $0 \leq i \leq 2k$. Let q_k be the continuous function

Received by the editors April 20, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26A18; Secondary 54C70, 58F20.

Key words and phrases. Least period, periodic points, periodic orbits, minimal periodic orbits, topological entropy.

0002-9939/87 $1.00 + $.25 per page

from the interval $[1/(k+1), 1/k]$ onto itself which is the scaled-down copy of p_k on $[1/(k+1), 1/k]$. That is,

$$q_k(x) = 1/(k+1) + p_k(k(k+1)(x - 1/(k+1)))/[k(k+1)].$$

Finally, let $f_0 \in C^0(I,I)$ be the continuous function defined by $f_0(0) = 0$ and $f_0(x) = q_k(x)$ on $[1/(k+1), 1/k]$ for each positive integer k and let $\langle f_m \rangle$ $(m \geq 1)$ be the sequence of successive renormalized square roots of f_0 on I. Now we can state the following theorem whose proof is easy and omitted. (For the definition of minimal periodic orbits, see [**2** or **5**].)

THEOREM 1. *Let the sequence $\langle f_m \rangle$ $(m \geq 0)$ be defined as above. Then $\langle f_m \rangle$ is a uniformly convergent sequence in $C^0(I,I)$ with the following two properties*:

(1) *For every integer $m \geq 0$, f_m has infinitely many minimal periodic orbits of least period $2^m \cdot 3$ and the topological entropy of f_m is ∞.*

(2) *If f is the uniform limit of the sequence $\langle f_m \rangle$, then f is exactly the same as the function g_∞ described in* [**8**, *p.* 14] *with zero topological entropy* [**6**].

In the above theorem, every function f_k has minimal periodic orbits of least period $2^k \cdot 3$. In the following, we will use these functions f_k to construct, for any two integers $m \geq 0$ and $n \geq 1$, continuous functions $F_{m,n}$ in $C^0(I,I)$ which have exactly one minimal periodic orbit of least period $2^m(2n+1)$, but with topological entropy equal to ∞.

For every positive integer n, let u_n be any continuous function from $[2/3, 1]$ into itself with exactly one minimal periodic orbit [**9**] (see [**2**, **5**] also) of least period $2n+1$ and let $F_{0,n}$ be the continuous function in $C^0(I,I)$ defined by (i) $F_{0,n}(x) = f_k(3x)/3$ for $0 \leq x \leq 1/3$, where k is any positive integer and f_k is defined as in Theorem 1; (ii) $F_{0,n}(x) = u_n(x)$ for $2/3 \leq x \leq 1$; and (iii) $F_{0,n}$ is linear on $[1/3, 2/3]$. It is clear that $F_{0,n}$ has exactly one minimal periodic orbit of least period $2n+1$ and its topological entropy is ∞. For any fixed integer $n > 0$, let $\langle F_{m,n} \rangle$ $(m \geq 1)$ be the sequence of successive renormalized square roots of $F_{0,n}$ on I.

Now we can state the following theorem whose proof is again easy and omitted.

THEOREM 2. *For every positive integer n, let the sequence $\langle F_{m,n} \rangle$ $(m \geq 0)$ be defined as above. Then for every fixed $n > 0$, $\langle F_{m,n} \rangle$ $(m \geq 0)$ is a uniformly convergent sequence in $C^0(I,I)$ with the following two properties*:

(1) *For every integer $m \geq 0$, the function $F_{m,n}$ has exactly one minimal periodic orbit of least period $2^m(2n+1)$ and the topological entropy of $F_{m,n}$ is ∞.*

(2) *If F_n is the uniform limit of the sequence $\langle F_{m,n} \rangle$, then $F_n = f$, where f is defined as in Theorem 1.*

REMARK. We can also construct functions $G_{m,n}$ in $C^0(I,I)$ with the properties as stated in part (1) of Theorem 2 as follows: For any two integers $m \geq 0$ and $n \geq 1$, let $v_{m,n}$ be any continuous function from $[2/3, 1]$ into itself which has exactly one minimal periodic orbit [**2**, **5**] of least period $2^m(2n+1)$. Let $G_{m,n}$ be the continuous function in $C^0(I,I)$ defined by (i) $G_{m,n}(x) = f_k(3x)/3$ for $0 \leq x \leq 1/3$, where $k > m$ is any integer and f_k is defined as in Theorem 1; (ii) $G_{m,n}(x) = v_{m,n}(x)$ for $2/3 \leq x \leq 1$; and (iii) $G_{m,n}$ is linear on $[1/3, 2/3]$. Then it is easy to see that $G_{m,n}$ has exactly one minimal periodic orbit of least period $2^m(2n+1)$ and the topological entropy of $G_{m,n}$ is ∞.

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