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Šarkovskii-minimal orbits

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1. My first paper (5), submitted for publication by my supervisor Philip Hall, dealt with iteration of a continuous map of an interval. This subject has recently become fashionable and many interesting results are discussed in the book of Collet and Eckmann (4). Several of these results hold for special types of map, for example those with negative Schwarzian derivative. However here, as in my original paper, we will be concerned with arbitrary continuous maps.

Let I be a compact interval and $f: I \to I$ a continuous map of this interval into itself. The sequence of iterates f^n is defined by

$$f^1 = f, \quad f^{n+1} = f \circ f^n \quad (n \geqslant 1).$$

A point $x \in I$ is a fixed point of f if $f(x) = x$ and a periodic point of period n if

$$f^n(x) = x, \quad f^k(x) \neq x \quad \text{for} \quad 1 \leqslant k < n.$$

The orbit $\{f^n(x)\}$ of x under f is then periodic. In (5) I showed that the sequence $\{f^n(x)\}$ converges for every $x \in I$ if and only if the map f has no periodic point of period 2. This implies, as a corollary, that if f has a periodic point of period $n > 1$ then it also has a periodic point of period 2.

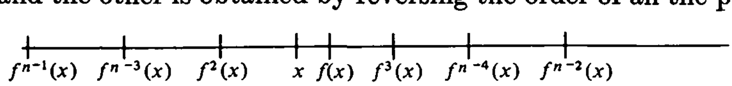
Šarkovskii (9) reproved this corollary and later (11) also the theorem. Meanwhile, however, in (10) he obtained a remarkable extension of the corollary. Let the positive integers be totally ordered in the following way:

$$3 \vdash 5 \vdash 7 \vdash 9 \vdash \ldots \vdash 2\cdot 3 \vdash 2\cdot 5 \vdash \ldots \vdash 2^2\cdot 3 \vdash 2^2\cdot 5 \vdash \ldots \vdash 2^3 \vdash 2^2 \vdash 2 \vdash 1.$$

Šarkovskii's theorem says that if a continuous map $f: I \to I$ has a periodic point of period n and if $n \vdash m$ then f also has a periodic point of period m.

This result is best possible in the sense that, for any positive integer n, there exists a continuous map f with a periodic point of period n but no periodic point of period m for any $m \vdash n$. Also, there exists a continuous map f which has periodic points of period 2^d, for every $d \geqslant 0$, but of no other periods.

An edited version in English of Šarkovskii's proof has been given by Štefan (12). Štefan also made an interesting observation concerning the distribution along the real line of points in an orbit of odd period $n > 1$, when there are no orbits of odd period m with $1 < m < n$. In fact there are just two possibilities. One of these is illustrated below and the other is obtained by reversing the order of all the points.



The purpose of the present paper is to extend this description from odd integers to arbitrary positive integers. We introduce the following definition:

A periodic orbit of f of period n is *Šarkovskii-minimal*, or *S-minimal*, if f does not have any periodic orbit of period m, where $m \vdash n$ and $1 < m < n$.

We wish to determine the possible distributions along the real line of all S-minimal orbits. On account of Štefan's result we need only consider even $n \geqslant 4$.

Recently a much simpler approach to Šarkovskii's theorem has been found, using directed graphs. The basic idea is due to Straffin (13), and the proof has been completed independently by Ho and Morris (7) and Burkart (3). Let $x_1 < \ldots < x_n$ be a periodic orbit of f of period n and set $I_j = [x_j, x_{j+1}]$ $(1 \leqslant j < n)$. The vertices of the directed graph are the intervals $I_1, \ldots, I_{n-1}$ and there is an arc $I_j \to I_k$ if I_k is contained in the interval $\{f(x_j), f(x_{j+1})\}$ with endpoints $f(x_j)$ and $f(x_{j+1})$. (We find it convenient to follow the definition of Ho and Morris, rather than that of Straffin.) Straffin showed that f also has a periodic orbit of period m if the digraph contains a primitive cycle $J_1 \to J_2 \to \ldots \to J_m \to J_1$ of length m. Here a cycle is said to be *primitive*, or non-repetitive according to Straffin, if it does not consist entirely of a cycle of smaller length described several times. This principle will be invoked without further comment in the present paper.

A cycle $J_1 \to J_2 \to \ldots \to J_n \to J_1$ of length n in this digraph will be said to be a *standard* n-cycle if J_1 has an endpoint c such that $f^{k-1}(c)$ is an endpoint of J_k for $1 \leqslant k \leqslant n$. A standard n-cycle need not be primitive. It is readily seen that any vertex I_j occurs at most twice among $J_1, \ldots, J_n$ and that at least one vertex occurs twice. Moreover a standard n-cycle always exists. These facts are implicitly contained in Straffin's proof of his theorem B.

The digraph associated with an S-minimal orbit of odd period $n > 1$ is illustrated in Fig. 1. If x is the midpoint of the orbit then $J_1 = \{x, f(x)\}$ and $J_k = \{f^{k-2}(x), f^k(x)\}$ for $1 < k < n$. A standard n-cycle cannot omit both J_1 and J_2. Therefore it must contain J_1, and in fact is the uniquely determined cycle

$$J_1 \to J_2 \to \ldots \to J_{n-1} \to J_1 \to J_1.$$

With any digraph there is associated an adjacency matrix. If the digraph has $n-1$ vertices $I_1, \ldots, I_{n-1}$ the $(n-1) \times (n-1)$ adjacency matrix $A = (a_{jk})$ is defined by $a_{jk} = 1$ if there is an arc $I_j \to I_k$ and $= 0$ otherwise. By relabelling the vertices, the rows and columns of the adjacency matrix are subjected to the same permutation. In spite of this lack of uniqueness we will still refer to *the* adjacency matrix of a digraph. The characteristic polynomial of a digraph is the uniquely determined polynomial $\det(\lambda I - A)$. The relationship between the characteristic polynomial of a digraph and its cycle structure is discussed in Cvetković, Doob and Sachs (6), especially theorems 1·2 and 3·1.

For example, the digraph in Fig. 1 has the adjacency matrix

$$H_{n-1} = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 1 & \ldots & 1 & 0 \end{pmatrix},$$

and the characteristic polynomial

$$\psi_n(\lambda) = (\lambda^n - 2\lambda^{n-2} - 1)/(\lambda + 1).$$

We further set ourselves the task of determining the adjacency matrix and characteristic polynomial of the digraph associated with an S-minimal orbit of even period $n \geqslant 4$.

To conclude these introductory remarks we draw attention also to a paper of Block (1). Block's results are connected with the structure of S-minimal orbits of period a power of 2, but he does not formulate the problem in these terms and his results for this case are less explicit than those obtained here.

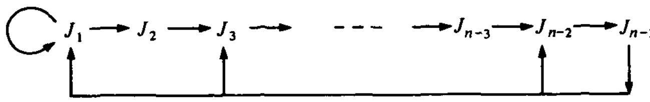


FIGURE 1. Digraph of an S-minimal orbit of odd period n.

2. The definition of the digraph associated with a periodic orbit $x_1 < \ldots < x_n$ of a continuous map f extends immediately to the case where the set $\{x_1, \ldots, x_n\}$ is not a single periodic orbit, but a union of disjoint periodic orbits. This has the advantage that digraphs with the same vertices, but different arcs, are associated to the iterates of f.

LEMMA 1. *Let f and g be continuous maps of a compact interval I into itself and let the set $\{x_1, \ldots, x_n\}$, where $x_1 < \ldots < x_n$, be a union of disjoint periodic orbits for f and also for g. Put $I_j = [x_j, x_{j+1}]$, and let $A(f)$, $A(g)$ be the adjacency matrices of the associated digraphs with the vertices $I_1, \ldots, I_{n-1}$ in their natural order. Then $\{x_1, \ldots, x_n\}$ is a union of disjoint periodic orbits also for the composite map $f \circ g$ and*

$$A(f \circ g) \equiv A(g)A(f) \pmod 2.$$

Proof. Let $e_1, \ldots, e_n$ be the standard basis for the n-dimensional vector space V over the field of 2 elements. Then $e_1 + e_2, e_2 + e_3, \ldots, e_{n-1} + e_n$ form a basis for the subspace V_0 of all vectors with coordinate sum zero. Any permutation π of $e_1, \ldots, e_n$ admits a unique extension to a linear transformation P of V. Moreover P maps V_0 into itself. The matrix $\bar{A}$ of this linear transformation relative to the basis $e_1 + e_2, \ldots, e_{n-1} + e_n$ is determined in the following way. If π maps e_i into e_j and e_{i+1} into e_k then P maps $e_i + e_{i+1}$ into

$$e_j + e_k = \sum_{s=l}^{m-1} (e_s + e_{s+1}),$$

where $l = \min(j, k)$ and $m = \max(j, k)$.

Under the bijection $x_i \to e_i$ the map f determines a unique permutation $\pi = \pi(f)$. From what has been said, the adjacency matrix $A(f)$ is just the matrix $\bar{A}$ determined by this permutation π. Since $f = f_2 \circ f_1$ implies $\pi(f) = \pi(f_2) \circ \pi(f_1)$ and $\bar{A} = \bar{A}_1 \bar{A}_2$, the result follows. (The inversion of order results from retaining both the analyst's definition of functional composition and the graph-theorist's definition of adjacency matrix.)

COROLLARY. *If h is the least common multiple of the periods of the orbits of f in $\{x_1, \ldots, x_n\}$ then*

$$A(f)^h \equiv I \pmod 2.$$

In particular, $A(f)$ is a non-singular matrix.

The following elementary result is surely known, but I am unable to supply a reference in full generality.

Lemma 2. *Let $f: I \to I$ be a continuous map and $x \in I$. If x is a periodic point of f with period n then, for any positive integer h, x is a periodic point of f^h with period $n/(h,n)$. Conversely, if x is a periodic point of f^h with period m then x is a periodic point of f with period hm/d, where d divides h and is relatively prime to m.*

We will also require the next result, which is contained in (9) of Štefan (12).

Lemma 3. *Let $f: I \to I$ be a continuous map of a compact interval into itself which does not have an orbit of period 3. If $x_1 < \ldots < x_n$ is an orbit of period $n > 3$ and if $f(x_j) > x_j$, resp. $f(x_j) < x_j$, for some j such that $1 < j < n$ then $f(x) > x$ for $x \in [x_1, x_j]$, resp. $f(x) < x$ for $x \in [x_j, x_n]$.*

As a first step towards the description of S-minimal orbits we now prove

Lemma 4. *Let $f: I \to I$ be a continuous map of a compact interval into itself with an orbit $x_1 < \ldots < x_n$ of even period $n \geq 4$. If f does not have an orbit of odd period m, where $1 < m < n$, then $x_1 < \ldots < x_{n/2}$ and $x_{n/2+1} < \ldots < x_n$ are orbits of f^2 with period $n/2$.*

Proof. Let U be the subset of points x_j for which $f(x_j) > x_j$ and D the subset of points x_j for which $f(x_j) < x_j$. We show first that $f(U) \subset D$.

Assume on the contrary that $f^2(x_j) > f(x_j) > x_j$ for some j. If h is the least positive integer such that $f^{h+1}(x_j) = x_1$ then $1 < h < n$. If h is odd then, since $f^h(f(x_j)) < f(x_j)$ and $f^h(x_1) > x_1$, the map f^h has a fixed point between x_1 and $f(x_j)$. This point is also a fixed point of f, since f does not have a periodic point of odd period strictly between 1 and n. Since $f^2(x_j) > f(x_j)$ it follows from Lemma 3 that f has a point of period 3, which is contrary to hypothesis. If h is even then $h+1$ is odd, $h+1 < n$ and $f^{h+1}(x_j) = x_1 < x_j$, which leads to a contradiction in the same way.

Thus $f(U) \subset D$, and similarly $f(D) \subset U$. It follows that U and D contain the same number of elements and $f(U) = D$, $f(D) = U$. But if $x_j \in U$ then $x_i \in U$ for $1 \leq i \leq j$, by Lemma 3. Hence $U = \{x_1, \ldots, x_{n/2}\}$ and $D = \{x_{n/2+1}, \ldots, x_n\}$. The result follows.

Lemma 5. *Let $f: I \to I$ be a continuous map of a compact interval into itself with an S-minimal orbit $x_1 < \ldots < x_n$ of period $n = 2q$, for some odd integer $q > 3$. Then*

(i) *$x_1 < \ldots < x_q$ and $x_{q+1} < \ldots < x_{2q}$ are S-minimal orbits of f^2 with period q;*

(ii) *f maps one of the two midpoints $x_{(q+1)/2}$, $x_{(3q+1)/2}$ onto the other.*

Proof. The assertion (i) follows directly from Lemma 4 and Lemma 2. It has some immediate consequences for the digraph associated with the orbit of period n. If we again set $I_j = [x_j, x_{j+1}]$, then $I_j \to I_j$ if and only if $j = q$. Also, $I_j \to I_q$ only if $j = q$.

If $J_1 \to J_2 \to \ldots \to J_n \to J_1$ is a standard n-cycle for f then I_q does not appear in this cycle, and
$$J_1 \to J_3 \to \ldots \to J_{n-1} \to J_1, \quad J_2 \to J_4 \to \ldots \to J_n \to J_2,$$
are standard q-cycles for f^2. Thus their structure is already known. In particular, $\{J_1, J_3, \ldots, J_{n-1}\}$ and $\{J_2, J_4, \ldots, J_n\}$ coincide with $\{I_1, \ldots, I_{q-1}\}$ and $\{I_{q+1}, \ldots, I_{2q-1}\}$, apart from order, with one adjacent repetition in each cycle. We may assume the notation chosen so that $J_{n-1} = J_1$. If $J_{2g} = J_{2h}$ $(1 \leq g < h \leq q)$ then either $g = 1$ and $h = q$ or $h = g+1$. In the latter case we must have $h = q$. For if $h < q$ then
$$J_1 \to J_2 \to \ldots \to J_{2g} \to J_{2g+3} \to \ldots \to J_{n-1},$$
is a primitive cycle for f of length $n' = n-4$. Since $2 < n' < n$ and $n' \equiv 2 \pmod 4$, this is impossible. Thus, by starting from J_n if necessary, we may always assume that a standard n-cycle has the form
$$J_1 \to J_2 \to \ldots \to J_{n-2} \to J_1 \to J_2 \to J_1,$$

where the vertices $J_1, \ldots, J_{n-2}$ are distinct. The repeated vertices J_1 and J_2 are, in some order, the intervals $\{x_r, f^2(x_r)\}$ and $\{x_{q+r}, f^2(x_{q+r})\}$, where $r = (q+1)/2$. If there were an arc $J_1 \to J_k$ for some $k > 2$ then k must be even and by omitting intermediate vertices we could obtain primitive cycles of lengths $n - k$ and $n-k+2$, which is impossible unless $k = n-2$. But also this case cannot occur, since the structure of an S-minimal orbit of period q shows that $J_1 \to J_2$ and $J_1 \to J_{n-2}$ imply also $J_1 \to J_4$. Consequently f maps the endpoints of J_1 onto the endpoints of J_2. Since $n \neq 4$, this implies $f(x_r) = x_{q+r}$ or $f(x_{q+r}) = x_r$, according as x_r or x_{q+r} is in J_1. ∎

It follows that a standard n-cycle is uniquely determined. Moreover, by taking the vertices of the digraph in the order $J_1, J_3, \ldots, J_{n-3}, J_2, J_4, \ldots, J_{n-2}, I_q$ the adjacency matrix assumes the canonical form

$$\begin{pmatrix} 0 & E_{q-1} & 0 \\ H_{q-1} & 0 & 0 \\ * & * & 1 \end{pmatrix},$$

where E_k denotes the $k \times k$ unit matrix, and hence the characteristic polynomial of the digraph is $(\lambda - 1)\,\psi_q(\lambda^2)$.

If $n = 6$, i.e., $q = 3$, the preceding argument breaks down at the point $n' = 2$. There are now two possibilities. A standard 6-cycle has one or other of the forms

$$J_1 \to J_2 \to J_3 \to J_4 \to J_1 \to J_2 \to J_1,$$
$$J_1 \to J_2 \to J_3 \to J_4 \to J_3 \to J_2 \to J_1,$$

where the vertices $J_1, \ldots, J_4$ are distinct. Moreover $\{J_1, J_3\}, \{J_2, J_4\}$, coincide in some order with $\{I_1, I_2\}, \{I_4, I_5\}$. The vertices of the digraph can be numbered so that the adjacency matrix has one or other of the forms

$$\begin{pmatrix} 0 & E_2 & 0 \\ H_2 & 0 & 0 \\ * & * & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & K_2^* & 0 \\ K_2 & 0 & 0 \\ * & * & 1 \end{pmatrix},$$

where

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The characteristic polynomial of the digraph is

$$(\lambda - 1)(\lambda^4 - \lambda^2 - 1), \quad \text{resp.} \quad (\lambda - 1)(\lambda^4 - 3\lambda^2 + 1).$$

These results may be verified directly, but they will also follow from Theorem 1 (iii) and the discussion following its proof.

Before stating our characterization of S-minimal orbits in the general case we introduce some notation. Let C_h denote the standard $h \times h$ circulant matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & \cdots\cdots & & \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

and let $C(T_1, \ldots, T_h)$ denote the circulant block matrix in which the element 1 in the jth row of C_h is replaced by the submatrix T_j for $j = 1, \ldots, h$. We will also denote a block triangular matrix, for example

$$\begin{pmatrix} A & 0 & 0 \\ * & B & 0 \\ * & * & C \end{pmatrix},$$

by $A\backslash B\backslash C$. Finally, any positive integer n can be uniquely expressed in the form $n = pq$, where $p = 2^d$ is a power of 2 and $q \geqslant 1$ is odd. Throughout the rest of the paper q, p and d will be defined in terms of n in this way.

THEOREM 1. *Let $f: I \to I$ be a continuous map of a compact interval into itself with an S-minimal orbit $x_1 < \ldots < x_n$ of even period n. Then $x_1 < \ldots < x_{n/2}$ and $x_{n/2+1} < \ldots < x_n$ are S-minimal orbits of f^2 with period $n/2$. Moreover, if $q > 3$ the p points x_{kq+r}, where $r = (q+1)/2$ and $1 \leqslant k \leqslant p$, may be ordered so that f maps each point, except the last, onto its successor.*

The adjacency matrix and characteristic polynomial of the associated digraph are determined as follows:

(i) *if $q = 1$ the adjacency matrix has the form*

$$F_p = C_{p/2}\backslash C_{p/4}\backslash \ldots \backslash C_2\backslash E_1,$$

and the characteristic polynomial is

$$\chi_p(\lambda) = (\lambda^{p/2}-1)(\lambda^{p/4}-1)\ldots(\lambda^2-1)(\lambda-1);$$

(ii) *if $q > 3$ the adjacency matrix has the form*

$$C(E_{q-1}, \ldots, E_{q-1}, H_{q-1})\backslash F_p,$$

where the unit matrix occurs $p-1$ times, and the characteristic polynomial is

$$\psi_q(\lambda^p)\,\chi_p(\lambda);$$

(iii) *if $q = 3$ the adjacency matrix has the form*

$$C(T_1, \ldots, T_{p-1}, T_p)\backslash F_p,$$

where $T_1, \ldots, T_p$ are non-singular 2×2 zero-one matrices such that the trace of the product $T = T_1 \ldots T_p$ is an odd integer. The characteristic polynomial is

$$(\lambda^{2p} - a\lambda^p + b)\,\chi_p(\lambda),$$

where $a =$ trace T and $b = \det T$.

Proof. By Lemma 4, $x_1 < \ldots < x_{n/2}$ and $x_{n/2+1} < \ldots < x_n$ are orbits of f^2 with period $n/2$ and, by Lemma 2, these orbits are S-minimal. By replacing n by $n/2, \ldots$ it follows that for any $m = 2^e$ $(1 \leqslant e \leqslant d)$

$$x_1 < \ldots < x_{n/m}; \quad x_{n/m+1} < \ldots < x_{2n/m}; \quad \ldots, x_{(m-1)n/m+1} < \ldots < x_n \tag{$*$}$$

are S-minimal orbits of f^m with period n/m. Moreover f permutes these orbits in some cyclic order. This implies that in the digraph the intervals $\{I_{n/m}, I_{3n/m}, \ldots, I_{(m-1)n/m}\}$ form, in some order, an $m/2$-cycle and that each of these vertices is joined to only one other among them. Furthermore, if $I_j \to I_{kn/m}$, where $k \in \{1, 3, \ldots, m-1\}$, then $j = in/m$ for some $i \in \{1, 2, \ldots, m-1\}$.

If $q = 1$, i.e. $n = 2^d$, this shows that the adjacency matrix can be given the form F_p by taking first the vertices I_j with $j \equiv 1 \pmod 2$, next those with $j \equiv 2 \pmod 4$, then those with $j \equiv 4 \pmod 8$, and so on.

If $q > 1$ then (*) shows also, for $m = p$, that the remaining vertices I_j, where $q \nmid j$, fall into blocks

$$\{I_1, \ldots, I_{q-1}\}, \quad \{I_{q+1}, \ldots, I_{2q-1}\}, \quad \ldots$$

such that vertices in one block are joined to vertices in just one other block. We will call these $p(q-1)$ vertices I_j, where j is not divisible by q, 'top' vertices.

Consider first the case $q > 3$. We claim that a standard n-cycle is uniquely determined and consists of the $p(q-1)$ top vertices in some order, followed by the same first p vertices. These p vertices represent the intervals $\{x_{kq+r}, f^p(x_{kq+r})\}$ and f maps each point x_{kq+r}, except the last, onto its successor, as claimed in the statement of the theorem. For $p = 2$, i.e., $d = 1$, this claim was already established in the proof of Lemma 5. It is readily established in general by induction on d, using the same argument.

To obtain the adjacency matrix in the required form we order the vertices so that the top vertices come first, next the vertices I_{jq} with $j \equiv 1 \pmod 2$, then the vertices I_{jq} with $j \equiv 2 \pmod 4$, and so on. Moreover we order the top vertices in blocks

$$\{I_{kq+1}, \ldots, I_{(k+1)q-1}\},$$

so that the block containing the point x_{kq+r} is succeeded by the block containing its image x_{lq+r} under f. Finally we order the vertices within a block so that the first vertex is the interval $\{x_{kq+r}, f^p(x_{kq+r})\}$, next the interval $\{x_{kq+r}, f^{2p}(x_{kq+r})\}$, then the interval $\{f^p(x_{kq+r}), f^{3p}(x_{kq+r})\}$, and so on.

In the case $q = 3$ it only remains to determine the form of the adjacency matrix. From what has already been said it has the form $C(T_1, \ldots, T_p)\backslash F_p$, where $T_1, \ldots, T_p$ are 2×2 submatrices whose rows correspond to pairs $\{I_{3k+1}, I_{3k+2}\}$ of top vertices. Moreover each matrix T_j is non-singular, by the corollary to Lemma 1. The characteristic polynomial of $C(T_1, \ldots, T_p)$ is $\det(\lambda^p I - T)$, where $T = T_1 \ldots T_p$, and thus is equal to $\lambda^{2p} - a\lambda^p + b$, where $a = \operatorname{trace} T$ and $b = \det T$. To show that a is necessarily odd we use induction on d. The points $x_1 < \ldots < x_{n/2}$ and $x_{n/2+1} < \ldots < x_n$ form S-minimal orbits of period $n/2$ for f^2. It follows from Lemma 1 that the corresponding adjacency matrices are congruent, mod 2, to $C(T_1 T_2, \ldots, T_{p-1} T_p)\backslash F_{p/2}$ and $C(T_2 T_3, \ldots, T_p T_1)\backslash F_{p/2}$. Hence T has odd trace by the induction hypothesis. ∎

It is possible to normalize the matrices T_j. In fact by renumbering the vertices the submatrices T_j can be permuted cyclically and $C(T_1, \ldots, T_p)$ can be replaced by $C(U_1 T_1 U_2^*, U_2 T_2 U_3^*, \ldots, U_p T_p U_1^*)$ for any 2×2 permutation matrices $U_1, \ldots, U_p$. The matrix T_j cannot itself be a permutation matrix for every j, since then T would also be a permutation matrix, contrary to the fact that it has odd trace. It follows that we can always require $\det T_j = +1$ for $1 \leqslant j < p$ and $T_p = H_2$ or K_2. This canonical form makes it apparent that there is a path of length $2p$ which passes through all the $2p$ top vertices. (A standard n-cycle need not contain such a path.)

The results already mentioned in Cvetković, Doob and Sachs (6) imply that a is the number of different cycles of length p in the subgraph formed by the $2p$ top vertices. Also, $b = -1$ if this subgraph contains a cycle of length $2p$ passing through all the $2p$ top vertices and $b = +1$ otherwise.

3. It will now be shown that the inductive conditions for an S-minimal orbit which were shown to be necessary in Theorem 1 are also sufficient. Much use has been made of the principle that primitive cycles in the digraph imply the existence of periodic orbits. We begin by establishing a converse.

LEMMA 6. *Let $f: I \to I$ be a continuous map of a compact interval into itself with an orbit $x_1 < \ldots < x_n$ of period n, and let $\mathcal{G}$ be the associated digraph. Suppose also that f is strictly monotonic on each subinterval $I_j = [x_j, x_{j+1}]$ $(1 \leqslant j < n)$. If f has an orbit of period m in the open interval (x_1, x_n) then either $\mathcal{G}$ contains a primitive cycle of length m or m is even, $m > 2$, and $\mathcal{G}$ contains a primitive cycle of length $m/2$.*

Proof. If the point c has period m for f then, for each k, there is a unique vertex J_k of $\mathcal{G}$ such that $f^{k-1}(c) \in J_k$. Moreover $J_1 \to \ldots \to J_m \to J_1$ is a cycle of length m in $\mathcal{G}$, since f is monotonic on each interval J_k. This cycle is a multiple of a primitive cycle of length l, where l divides m and $1 \leqslant l \leqslant m$. The map f^l of a subinterval of J_1 onto J_1, determined by this primitive cycle, is strictly monotonic. If it is increasing the orbit of period m can 'close' only if $m = l$. If it is decreasing the orbit can close only if $m = l$ or if $l > 1$ and $m = 2l$.

LEMMA 7. *Let $f: I \to I$ be a continuous map of a compact interval into itself with an S-minimal orbit $x_1 < \ldots < x_n$ of period n. Let g be a continuous map of the interval $J = [x_1, x_n]$ into itself such that $g(x_j) = f(x_j)$ for $1 \leqslant j \leqslant n$ and g is strictly monotonic on each subinterval $I_j = [x_j, x_{j+1}]$ $(1 \leqslant j < n)$. Then $x_1 < \ldots < x_n$ is also an S-minimal orbit of g.*

Proof. Since f and g have the same orbit of period n, the associated digraphs are the same. Suppose g had an orbit of period m, where $1 < m < n$ and $m \vdash n$. Then, by Lemma 6, the digraph contains a primitive cycle of length m or $m/2$. Since f does not have an orbit of period m the first possibility cannot occur. But the conditions $1 < m < n$ and $m \vdash n$ imply that m is not a power of 2. Hence $1 < m/2 < n$ and $m/2 \vdash n$. It follows that also the second possibility cannot occur.

THEOREM 2. *Let $x_1 < \ldots < x_n$ be any n points on the real line, where n is even, and let π be a cyclic permutation of $\{x_1, \ldots, x_n\}$ which interchanges $\{x_1, \ldots, x_{n/2}\}$ and $\{x_{n/2+1}, \ldots, x_n\}$. Moreover, if $q > 3$ let π map each point x_{kq+r}, where $r = (q+1)/2$ and $1 \leqslant k \leqslant p$, onto another such point, with one exception.*

Assume there exist continuous maps g of $[x_1, x_{n/2}]$ into itself and h of $[x_{n/2+1}, x_n]$ into itself such that $g(x_j) = \pi^2(x_j)$ for $1 \leqslant j \leqslant n/2$, $h(x_j) = \pi^2(x_j)$ for $n/2 < j \leqslant n$ and $x_1 < \ldots < x_{n/2}$, $x_{n/2+1} < \ldots < x_n$ are S-minimal orbits of g, h respectively.

Then there exists a continuous map f of $[x_1, x_n]$ into itself such that $f(x_j) = \pi(x_j)$ for $1 \leqslant j \leqslant n$ and $x_1 < \ldots < x_n$ is an S-minimal orbit of f. Indeed f has no orbit of period l, where $l \vdash n$.

Proof. Let f be any continuous map of $[x_1, x_n]$ into itself such that $f(x_j) = \pi(x_j)$ for $1 \leqslant j \leqslant n$ and f is strictly monotonic on each subinterval $I_j = [x_j, x_{j+1}]$. Then f has the orbit $x_1 < \ldots < x_n$ of period n and an associated digraph.

For any $m = 2^e$ $(1 \leqslant e \leqslant d)$ the orbits of π^m have period n/m and π permutes these orbits cyclically. It follows that π maps each S-minimal orbit

$$x_{kn/m+1} < \ldots < x_{(k+1)n/m}$$

of $g^{m/2}$ with period n/m onto an S-minimal orbit of $h^{m/2}$ of the same form, and vice versa. Each of these orbits is an orbit of f^m. Moreover it is an S-minimal orbit of f^m, regarded as a map of the smallest interval containing it, by Lemma 7. It follows that the adjacency matrix of the digraph has the same form as in Theorem 1, since this form was obtained by using only the information which we now possess.

Suppose f had an orbit of period l, where $l \vdash n$. Then the digraph contains a primitive cycle of length l or $l/2$. If $q = 1$ this is impossible, since the form of the adjacency matrix F_p shows that any primitive cycle has length 2^e $(1 \leqslant e < d)$. If $q = 3$ the corresponding adjacency matrix shows that any primitive cycle has length 2^e $(1 \leqslant e < d)$ or has length divisible by p. Again this contradicts $l \vdash n$. If $q > 3$ the corresponding adjacency matrix shows that any primitive cycle has length 2^e $(1 \leqslant e < d)$ or has length divisible by p. Moreover it cannot have length ps, where s is odd and $1 < s < q$. Once more this contradicts $l \vdash n$. ∎

Finally we show that the conditions on the adjacency matrix corresponding to an S-minimal orbit which were shown to be necessary in Theorem 1 are also sufficient. For $q = 1$ or $q > 3$ this already follows from Theorem 2 and Theorem 1 itself. Thus it remains to prove

THEOREM 3. *Let $x_1 < \ldots < x_n$ be any n points on the real line, where $n = 3p$ is even, and let $T_1, \ldots, T_p$ be non-singular 2×2 zero-one matrices such that the trace of the product $T = T_1 \ldots T_p$ is an odd integer. Then there exists a continuous map f of a compact interval into itself with $x_1 < \ldots < x_n$ as an S-minimal orbit and with corresponding adjacency matrix of the form $C(T_1, \ldots, T_p)\backslash F_p$.*

Proof. We can assume that $d > 1$ and the result holds for smaller values of d. Let $V_1, \ldots, V_{p/2}$ be 2×2 zero-one matrices such that $V_j \equiv T_{2j-1}T_{2j} \pmod 2$ for $1 \leqslant j \leqslant p/2$. Then V_j is non-singular and the product $V_1 \ldots V_{p/2}$ has odd trace. Consequently there exists a continuous map g of $[x_1, x_{n/2}]$ into itself which has $x_1 < \ldots < x_{n/2}$ as an S-minimal orbit and for which the adjacency matrix of the corresponding digraph has the form $C(V_1, \ldots, V_{p/2})\backslash F_{p/2}$. Similarly, let $W_1, \ldots, W_{p/2}$ be 2×2 zero-one matrices such that $W_j \equiv T_{2j}T_{2j+1}$ for $1 \leqslant j < p/2$ and $W_{p/2} \equiv T_p T_1 \pmod 2$. Then there exists a continuous map h of $[x_{n/2+1}, x_n]$ into itself which has $x_{n/2+1} < \ldots < x_n$ as an S-minimal orbit and for which the adjacency matrix of the corresponding digraph has the form $C(W_1, \ldots, W_{p/2})\backslash F_{p/2}$.

We may assume that the two top vertices corresponding to the rows of each block V_j and W_j occur in their natural order I_{3k+1}, I_{3k+2}. We define a permutation π of $\{x_1, \ldots, x_n\}$ in the following way. Let I_{3i+1}, I_{3i+2} be the vertices corresponding to the first two rows of the adjacency matrix for g and I_{3j+1}, I_{3j+2} the vertices corresponding to the first two rows of the adjacency matrix for h. To define how π maps $\{x_{3i+1}, x_{3i+2}, x_{3i+3}\}$ onto $\{x_{3j+1}, x_{3j+2}, x_{3j+3}\}$ we first observe that if we take $n = 3$ in the proof of Lemma 1 then each non-singular 2×2 zero-one matrix $\tilde{A}$ arises from a unique permutation π (and hence the symmetric group on 3 letters is isomorphic to the two-dimensional general linear group over the field of 2 elements). If the permutation corresponding in this way to the non-singular matrix T_1 is $s \to s'$ $(s = 1, 2, 3)$ then we define

$$\pi(x_{3i+s}) = x_{3j+s'} \quad (s = 1, 2, 3).$$

Similarly, if I_{3k+1}, I_{3k+2} are the vertices corresponding to the third and fourth rows of the adjacency matrix for g and if the permutation corresponding to the non-singular matrix T_2 is $s' \to s''$ $(s' = 1, 2, 3)$ then we define

$$\pi(x_{3j+s'}) = x_{3k+s''} \quad (s' = 1, 2, 3).$$

Proceeding in this way, all points x_j are accounted for and we obtain a well-defined permutation π of $\{x_1, \ldots, x_n\}$. Evidently π interchanges $\{x_1, \ldots, x_{n/2}\}$ and $\{x_{n/2+1}, \ldots, x_n\}$.

In the same way V_1 uniquely determines a map of $\{x_{3i+1}, x_{3i+2}, x_{3i+3}\}$ onto $\{x_{3k+1}, x_{3k+2}, x_{3k+3}\}$, and this coincides with the map given by g. Proceeding in this way all points x_j with $1 \leqslant j \leqslant n/2$ are accounted for. Moreover, by the proof of Lemma 1, $\pi^2(x_j) = g(x_j)$ for $1 \leqslant j \leqslant n/2$. Similarly, $\pi^2(x_j) = h(x_j)$ for $n/2 < j \leqslant n$. It follows that π is a cyclic permutation of $\{x_1, \ldots, x_n\}$.

Define a continuous map f of $[x_1, x_n]$ into itself by setting $f(x_j) = \pi(x_j)$ for $1 \leqslant j \leqslant n$ and by requiring f to be strictly monotonic on each subinterval $[x_j, x_{j+1}]$. Then $x_1 < \ldots < x_n$ is an orbit of f of period n and the corresponding adjacency matrix has the form $C(T_1, \ldots, T_p)\backslash F_p$. By Theorem 2 and its proof, the orbit $x_1 < \ldots < x_n$ is S-minimal. ∎

There is another method of representing periodic orbits which is convenient in the construction of examples. Let f be a continuous map with an orbit $x_1 < \ldots < x_n$ of period n. If x is an arbitrary point of this orbit there is a unique integer a_i $(0 \leqslant a_i < n)$ such that $x_i = f^{a_i}(x)$. Then $\{a_1, \ldots, a_n\}$ is a permutation of $\{0, 1, \ldots, n-1\}$ and the orbit can be represented by the finite sequence $\mathbf{a} = (a_1, \ldots, a_n)$. Alternatively, we can regard the terms a_i as residue classes $(\operatorname{mod} n)$. If x' is another point of the same orbit and $x = f^j(x')$ then the orbit is also represented by the sequence $\mathbf{a}' = \mathbf{a} + \mathbf{j}$, where $\mathbf{j} = (j, \ldots, j)$. We choose to regard the n representations obtained in this way as distinct, although the orbit is the same.

Let A_n denote the set of all sequences $\mathbf{a}$ which represent S-minimal orbits of period n, for arbitrary continuous maps f. The results which have been established enable us to describe A_n completely.

Suppose first that $q = 1$. From any $\mathbf{a} \in A_n$ we obtain all others by the following operations. The two successive blocks of $n/2$ terms can be interchanged or not, in each block of $n/2$ terms the two successive blocks of $n/4$ terms can be interchanged or not, and so on until in each pair the two successive terms can be interchanged or not. These permutations form a group which is readily identified with the iterated wreath product $P_d = S_2 \sim S_2 \sim \ldots \sim S_2$, where there are d factors and S_n denotes the symmetric group on n letters. (For the definition and basic properties of wreath products, see Huppert (8), pp. 94–101.) By a theorem of Kaloujnine, P_d is the Sylow 2-subgroup of S_n for $n = 2^d$. Since P_d has cardinality $|P_d| = 2^{n-1}$ it follows that

$$|A_n| = 2^{n-1} \quad \text{if} \quad n = 2^d.$$

Suppose next that $q = 3$. Divide the terms of a sequence $\mathbf{a} \in A_n$ into $p = 2^d$ blocks of 3 successive terms and consider the permutation group P_d as acting on these blocks. By arbitrarily permuting the terms in each block we obtain the extended permutation group $S_3 \sim P_d$. From any $\mathbf{a} \in A_n$ we obtain all others by the operations of this group. It follows that $|A_n| = 2^{2p-1} \cdot 3^p$ if $n = 3p$, where $p = 2^d$.

If $q = 1$ or 3 then A_{2n} consists of all sequences $(2\mathbf{a}, 2\mathbf{a}' + \mathbf{1})$ and $(2\mathbf{a}' + \mathbf{1}, 2\mathbf{a})$, where $\mathbf{a}, \mathbf{a}' \in A_n$.

Suppose finally that $q > 3$. Again divide the terms of a sequence $\mathbf{a} \in A_n$ into $p = 2^d$ blocks of q successive terms and consider the permutation group P_d as acting on these blocks. By operating or not on each block with the operation σ which reverses the order of the terms we obtain the extended permutation group $P_{d+1} = S_2 \sim P_d$. The operation ζ which replaces $\mathbf{a}$ by $\mathbf{a}+\mathbf{1}$ commutes with all elements of the group P_{d+1} and generates a cyclic group Z_n of order n. From any $\mathbf{a} \in A_n$ we obtain all others by the operations of the group $Z_n \times P_{d+1}$. It follows that $|A_n| = 2^{2p-1} \cdot n$ if $n = pq$, where $p = 2^d$ and $q > 3$ is odd. To complete the description we need to know a representative of A_n. But A_q contains the sequence $(q-1, \ldots, 2, 0, 1, 3, \ldots, q-2)$ and if $\mathbf{a} \in A_n$ then $(2\mathbf{a}, 2\mathbf{a}+\mathbf{1}) \in A_{2n}$.

4. The preceding results provide information about arbitrary continuous maps of a compact interval into itself, since any such map has an S-minimal orbit of some period n. This is enhanced by the fact that n is not uniquely determined, on account of the requirement $1 < m < n$ in our definition of S-minimality. For example, a map with an S-minimal orbit of period 14 also has one of period 12 and one of period 8. However, these consequences cannot be fully developed here. The results also provide a basis for the construction of maps with special properties. We give just one example.

It is known that a continuous map $f: I \to I$ has zero topological entropy if and only if every periodic orbit of f has period a power of 2. If f has a periodic orbit of period $n = pq$, where $p = 2^d$ and $q > 1$ is odd, then, by (2), the topological entropy $h(f)$ is not less than $p^{-1} \log \lambda_q$, where λ_q is the largest positive zero of the polynomial $\lambda^q - 2\lambda^{q-2} - 1$ and, in particular, $\lambda_3 = (1+5^{\frac{1}{2}})/2$. This follows also from Theorem 1 above by invoking the principle that $h(f)$ is not less than the logarithm of the largest non-negative eigenvalue of an adjacency matrix. However, we now show how Theorem 1 can be used to construct maps with S-minimal orbits of period $n = 3p$ and with entropy much larger than this universal lower bound.

If in Theorem 3 we take $T_j = H_2$ $(j = 1, \ldots, p)$ then $T = H_2^p$. It is not difficult to show that T has trace τ_d, where τ_d is defined inductively by $\tau_1 = 3, \tau_{i+1} = \tau_i^2 - 2$. It follows that τ_d is indeed odd. The largest positive eigenvalue of T is λ_3^p, where $\lambda_3 = (1+5^{\frac{1}{2}})/2$ is the largest positive eigenvalue of H_2. Hence the largest positive eigenvalue of $C(T_1, \ldots, T_p)$ is λ_3. Thus we obtain maps f with S-minimal orbits of period $3 \cdot 2^d$ for any d and with entropy

$$h(f) \geqslant \log \lambda_3 = 0{\cdot}4812\ldots .$$

If d is large then in terms of the Šarkovskii ordering such maps are arbitrarily close to maps f with $h(f) = 0$.

I am grateful to I. M. Isaacs for discussion leading to the formulation and proof of Lemma 1, and for pointing out that it was appropriate to require $1 < m < n$ as well as $m \vdash n$ in the definition of an S-minimal orbit. Discussion with L. G. Kovács also helped to clarify some ideas. R. K. James observed that I appeared to have some wreath products. I hope Philip Hall will be pleased to see them turning up here.

Note added (January 1983). This paper was written for a volume to celebrate the 80th birthday of Philip Hall. Unfortunately he died on 30 December 1982, and the planned volume will not now appear.

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