Šarkovskij ordering, as follows:

$$\begin{aligned}
3 \prec 5 \prec 7 \prec \cdots \prec\ & 2k-1 \prec 2k+1 \prec \dots \\
&\prec 2\cdot 3 \prec 2\cdot 5 \prec \cdots \prec 2(2k-1) \prec 2(2k+1) \prec \dots \\
&\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&\prec 2^l\cdot 3 \prec 2^l\cdot 5 \prec \cdots \prec 2^l(2k-1) \prec 2^l(2k+1) \prec \dots \\
&\prec 2^{l+1}\cdot 3 \prec 2^{l+1}\cdot 5 \prec \cdots \prec 2^{l+1}(2k-1) \prec 2^{l+1}(2k+1) \prec \dots \\
&\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&\ \dots \quad \dots \quad \dots \quad \dots \quad \cdots \prec 2^{n+1} \prec 2^n \prec \cdots \prec 8 \prec 4 \prec 2 \prec 1\,.
\end{aligned}$$

This ordering starts with the number 3 so, indeed, Theorem 2.2.2 follows from the following theorem:

Theorem 2.2.3 (Šarkovskij). *Let (X, f) be a dynamical system on an interval X in $\mathbb{R}$. If f has a periodic point with primitive period n, than f has periodic points with primitive period m for all $m \in \mathbb{N}$ with $n \prec m$.*

Proof. See Section 2.5. □

In the next two sections we shall show by examples that everything that is allowed by Šarovskij's Theorem is possible. To this end, we reformulate this theorem in a more convenient form. For every $n \in \mathbb{N}$, let

$$\mathbf{S}(n) := \{n\} \cup \{\, k \in \mathbb{N} \,\vdots\, n \prec k \,\}\,.$$

In addition, let

$$\mathbf{S}(2^\infty) := \{\, 2^m \,\vdots\, m \in \mathbb{Z}^+ \,\}\,,$$

where the symbol 2^∞ means something like 'an entity larger than all powers of 2'. The set $\mathbb{N} \cup \{2^\infty\}$ will be denoted by $\mathbb{N}_\infty$ and the sets of the form $\mathbf{S}(n)$ with $n \in \mathbb{N}_\infty$ will be called *Šarkovskij tails* of $\mathbb{N}$. Finally, for any dynamical system (X, f) we denote by $\mathbf{Per}(f)$ the set of all primitive periods of periodic orbits under f. Now Šarkovskij's Theorem can be formulated as follows:

Theorem 2.2.4 (Šarkovskij). *Let (X, f) be a dynamical system on an interval X in $\mathbb{R}$. If $\mathbf{Per}(f) \neq \emptyset$ then there exists $n \in \mathbb{N}_\infty$ such that $\mathbf{Per}(f) = \mathbf{S}(n)$.*

Proof. Observe that, if $\mathbf{Per}(f) \neq \emptyset$ then it has a first element in the Šarkovskij ordering. □

2.3 The truncated tent map

For every $\lambda \in [0; 1]$ we define *the tent map truncated by λ* as

$$T_\lambda(x) := \min\{\, T(x), \lambda \,\} \quad \text{for } x \in [0; 1]\,.$$

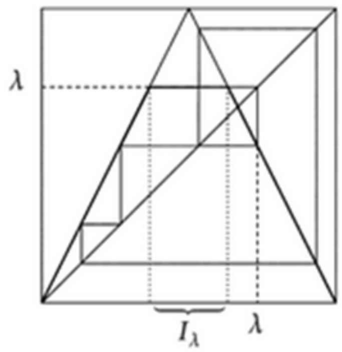

Fig. 2.8. The truncated tent map T_λ. In this example the point λ is both T_λ-periodic and T-periodic, with primitive periods 2 and 4, respectively.

Here T denotes the tent map defined in Example 0.4.2 ; see also 1.7.3. Below we shall repeatedly use the following trivial observation: if $x \in [0; 1]$ and x is not in the open interval $I_\lambda := (\frac{\lambda}{2}; 1 - \frac{\lambda}{2})$ where T is strictly greater than λ then $T(x) = T_\lambda(x)$. In fact,

$$T_\lambda(x) < T(x) \iff x \in I_\lambda \implies T_\lambda(x) = \lambda\,.$$

In general, points will not have the same orbits under T and T_λ. Obviously, a T-orbit is a T_λ-orbit, or a T_λ-orbit is a T-orbit, iff T and T_λ agree on that orbit, iff that orbit is included in the interval $[0; \lambda]$, iff that orbit has no points in I_λ. On the other hand, it is well possible that T and T_λ have periodic points in common which have different primitive periods, hence have different orbits. See Figure 2.8. Such a phenomenon can only occur if the periodic T-orbit includes a point x in I_λ, so that $T_\lambda(x) = \lambda$ and, in addition, the point λ is periodic under T_λ.

The truncated tent map T_1 equals the non-truncated tent map T. In the Introduction we have seen that for every $n \in \mathbb{N}$ there are periodic points with period n under T_1, but it is not yet clear that there are also points with *primitive* period n. However, there are $2^3 = 8$ periodic points with period 3, among which the two invariant points. So there are two periodic orbits with primitive period 3. Then by Li and Yorke's theorem, all possible *primitive* periods occur under T_1, that is: $\mathbf{Per}(T_1) = \mathbb{N} = \mathbf{S}(3)$.

The truncated tent map T_0 is identically equal to 0, hence it has the unique invariant point 0. Obviously, there are no other periodic points, so $\mathbf{Per}(T_0) = \{1\} = \mathbf{S}(1)$.

This shows that the largest and the smallest Šarkovskij tails can be represented as the set of primitive periods for some truncated tent map T_λ ($\lambda = 1$ or 0, respectively). We shall show now that every other Šarkovskij tail can occur as $\mathbf{Per}(T_\lambda)$ for a suitable choice of λ.

Values of λ such that $\mathbf{Per}(T_\lambda) = \mathbf{S}(n)$ for $n \in \mathbb{N}\setminus\{1, 3\}$ are obtained as follows: Recall that for every $n \in \mathbb{N}$ the non-truncated tent map has at least one and at most 2^n periodic orbits with primitive period n. From each of these orbits with primitive period n choose the largest element. Let λ_n be the smallest element so obtained:

$$\lambda_n := \min\{\max B \,\vdots\, B \text{ a periodic } T\text{-orbit with primitive period } n\}\,.$$

It is easily seen that λ_n is the smallest real number λ with the property that the interval $[0; \lambda]$ includes a periodic T-orbit with primitive period n.

Lemma 2.3.1. *Let $n \in \mathbb{N}$ and let $1 \geq \lambda \geq \lambda_n$. Then the point λ_n is periodic under the truncated tent map T_λ with primitive period n. Consequently, for all $n \in \mathbb{N}$ we have $\mathbf{S}(n) \subseteq \mathbf{Per}(T_\lambda)$, for every $\lambda \geq \lambda_n$.*

Proof. Let A be the periodic T-orbit with primitive period n of which λ_n is the maximal element, that is, $\lambda_n = \max\{A\}$. As all values of T on A are in A, they are at most λ_n, hence not larger than λ. Consequently, T and T_λ agree on A, so A is also a T_λ-orbit, i.e., it is a periodic orbit with primitive period n under T_λ. Hence $n \in \mathbf{Per}(T_\lambda)$ and Šarkovskij's Theorem implies that $\mathbf{S}(n) \subseteq \mathbf{Per}(T_\lambda)$. Moreover, since $\lambda_n \in A$, this shows in particular that the point λ_n is periodic under T_λ with primitive period n. □

Example. By brute computational force, using that the graphs of T^2, T^3 and T^4 consist of 2, 4 and 8 'tents', respectively, it is not difficult to find the invariant points of T^2, T^3 and T^4. In addition to the invariant points 0 and $2/3$ for T, one finds periodic orbits with the following primitive periods:

Primitive period 2: $\frac{2}{5} \mapsto \frac{4}{5} \mapsto \frac{2}{5}$.
Primitive period 3: $\frac{2}{9} \mapsto \frac{4}{9} \mapsto \frac{8}{9} \mapsto \frac{2}{9}$ and $\frac{2}{7} \mapsto \frac{4}{7} \mapsto \frac{6}{7} \mapsto \frac{2}{7}$.
Primitive period 4: $\frac{2}{17} \mapsto \frac{4}{17} \mapsto \frac{8}{17} \mapsto \frac{16}{17} \mapsto \frac{2}{17}$, $\frac{6}{17} \mapsto \frac{12}{17} \mapsto \frac{10}{17} \mapsto \frac{14}{17} \mapsto \frac{6}{17}$ and
$\frac{2}{15} \mapsto \frac{4}{15} \mapsto \frac{8}{15} \mapsto \frac{14}{15} \mapsto \frac{2}{15}$.

It follows that $\lambda_2 = \frac{4}{5}$, $\lambda_3 = \frac{6}{7}$ and that $\lambda_4 = \frac{14}{17}$. Lemma 2.3.4 below implies that the values of λ_n are between $\frac{4}{5}$ and $\frac{6}{7}$ for all $n \geq 2$.

The above lemma applies, in particular, to $\lambda = \lambda_n$. Consequently, the point λ_n is periodic under T_{λ_n}, with primitive period n. So $n \in \mathbf{Per}(T_{\lambda_n})$, hence $\mathbf{S}(n) \subseteq \mathbf{Per}(T_{\lambda_n})$. We shall prove in Theorem 2.3.5 below the converse of this inclusion.

Lemma 2.3.2. *Let $\lambda > 0$ and $k \in \mathbf{Per}(T_\lambda)$. If $\lambda \leq \lambda_k$ then λ is a periodic point under T_λ with primitive period equal to k.*

Proof. Let B be a periodic T_λ-orbit with primitive period k. It is sufficient to show that $\lambda \in B$. Assume the contrary. Then T cannot assume a value greater than or equal to λ in any point of B, for otherwise T_λ would have there the value λ. Consequently, T and T_λ agree on B. It follows that B is a periodic T-orbit with primitive period k. By the definition of λ_k, this implies that $\max\{B\} \geq \lambda_k \geq \lambda$. On the other hand, $B = T_\lambda[B] \subseteq [0;\lambda]$ and $\lambda \notin B$, so $\max\{B\} < \lambda$ (B is finite). Contradiction. □

Corollary 2.3.3. *If $k \in \mathbf{Per}(T_{\lambda_n})$ and $k \neq n$ then $\lambda_k < \lambda_n$.*

Proof. If $\lambda_n \leq \lambda_k$ then Lemma 2.3.2 would imply that λ_n is periodic under T_{λ_n} with primitive period k. But by Lemma 2.3.1 the point λ_n has primitive period n under T_{λ_n}. This would imply that $k = n$. □

Corollary 2.3.4. $\forall k, n \in \mathbb{N} : n \prec k \Rightarrow \lambda_k < \lambda_n$.

Proof. If $k, n \in \mathbb{N}$ and $n \prec k$ then by Lemma 2.3.1 and Šarkovskij's Theorem, $k \in \mathbf{S}(n) \subseteq \mathbf{Per}(T_{\lambda_n})$. Now apply Corollary 2.3.3. □

Theorem 2.3.5. $\forall n \in \mathbb{N} : \mathbf{Per}(T_{\lambda_n}) = \mathbf{S}(n)$.

Proof. We have seen already that the inclusion $\mathbf{S}(n) \subseteq \mathbf{Per}(T_{\lambda_n})$ holds. Assume that this inclusion is proper: there exists $k \in \mathbf{Per}(T_{\lambda_n}) \setminus \mathbf{S}(n)$. Since $n \in \mathbf{S}(n)$, it follows that $k \neq n$, so by Corollary 2.3.3 above we may conclude that $\lambda_k < \lambda_n$. On the other hand, the assumption that $k \notin \mathbf{S}(n)$ implies that $k \prec n$, hence $\lambda_n < \lambda_k$ by Lemma 2.3.4 (with k and n are interchanged). This contradiction completes the proof. □

In the proof of the above theorem we have nowhere assumed that n is not a power of 2, so the result also holds if n is a finite power of 2. We conclude this section by showing that there is a value λ_∞ such that $\mathbf{Per}(T_{\lambda_\infty}) = \mathbf{S}(2^\infty)$. It will turn out that the following number does the job:

$$\lambda_\infty := \sup\{\, \lambda_{2^n} \,\vdots\, n \in \mathbb{N} \,\}\,.$$

Lemma 2.3.6. *For all $n, m \in \mathbb{Z}^+$ and every odd integer $q \geq 3$ we have*

$$\lambda_{2^n} < \lambda_\infty < \lambda_{q2^m}\,.$$

Proof. In the Šarkovskij ordering we have $q \cdot 2^m \prec 3 \cdot 2^{m+1} \prec 2^{n+1} \prec 2^n$. So Lemma 2.3.4 above implies $\lambda_{2^n} < \lambda_{2^{n+1}} < \lambda_{3\cdot 2^{m+1}} < \lambda_{q2^m}$. Using the definition of λ_∞ we get the desired inequalities. □

Theorem 2.3.7. $\mathbf{Per}(T_{\lambda_\infty}) = \mathbf{S}(2^\infty)$.

Proof. Consider arbitrary $n \in \mathbb{Z}^+$. Then by definition $\lambda_\infty \geq \lambda_{2^n}$, hence Lemma 2.3.1 (with n replaced by 2^n) implies that $2^n \in \mathbf{Per}(T_{\lambda_\infty})$. As this holds for every $n \in \mathbb{Z}^+$, this means that $\mathbf{S}(2^\infty) \subseteq \mathbf{Per}(T_{\lambda_\infty})$.

Assume that this inclusion is proper and let $k \in \mathbf{Per}(T_{\lambda_\infty}) \setminus \mathbf{S}(2^\infty)$. Then there are $m \in \mathbb{Z}^+$ and $q \in \mathbb{N}$, q odd and $q \neq 1$, such that $k = q \cdot 2^m$, hence Lemma 2.3.6 implies that $\lambda_\infty < \lambda_k$. So by Lemma 2.3.2 with $\lambda = \lambda_\infty$, k is the primitive period of the point λ_∞ under T_{λ_∞}. This implies that k is the unique element of $\mathbf{Per}(T_{\lambda_\infty}) \setminus \mathbf{S}(2^\infty)$. Therefore,

$$\mathbf{Per}(T_{\lambda_\infty}) = \mathbf{S}(2^\infty) \cup (*)\,,$$

where $(*)$ is a singleton subset of $\mathbb{N}$. But there is no possible choice for $(*)$ such that $\mathbf{S}(2^\infty) \cup (*)$ is a Šarkovskij tail, unless $(*)$ is the empty set. This contradiction shows that $\mathbf{Per}(T_{\lambda_\infty}) = \mathbf{S}(2^\infty)$. □

2.4 The double of a mapping

There is yet another method to obtain a mapping that has $\mathbf{S}(2^\infty)$ as its set of primitive periods. It is based on a construction that assigns to any continuous mapping

$f: [0;1] \to [0;1]$ a mapping $\tau(f): [0;1] \to [0;1]$ such that the restriction of the mapping $\tau(f)^2$ to the interval $[0; \frac{1}{3}]$ is conjugate to f. Therefore, $\tau(f)$ is sometimes called the *double* of f.

In the remainder of this section, f is an arbitrary continuous mapping of the unit interval $[0;1]$ into itself. For every such a mapping f we define the continuous mapping $\tau(f) : [0;1] \to [0;1]$ by

$$\tau(f) := \begin{cases} \frac{1}{3}\big(f(3x) + 2\big) & \text{for } 0 \le x \le \frac{1}{3}\,, \\ \big(f(1) + 2\big)(-x + \frac{2}{3}) & \text{for } \frac{1}{3} \le x \le \frac{2}{3}\,, \\ x - \frac{2}{3} & \text{for } \frac{2}{3} \le x \le 1\,. \end{cases}$$

Lemma 2.4.1.
(1) *The mapping $\tau(f)$ has a unique invariant point and all other periodic points have even periods. In addition all non-invariant periodic orbits have a point in the interval $I := [0; \frac{1}{3}]$.*
(2) *The mapping $\varphi: x \mapsto \frac{1}{3}x : [0;1] \to I$ is a conjugation from the dynamical system $([0;1], f)$ to the system $(I, \tau(f)^2|_I)$.*

Proof. (1) Clearly, $\tau(f)$ maps the two intervals $I := [0; \frac{1}{3}]$ and $J := [\frac{2}{3}; 1]$ into each other: $\tau(f)[I] \subseteq J$ and $\tau(f)[J] = I$. This implies that a periodic point in one of these interval must have an even primitive period and that its orbit has a point in I. Moreover, the mapping $\tau(f)$ has an invariant point in the open interval $(\frac{1}{3}; \frac{2}{3})$ and it is easily seen to be the unique invariant point under $\tau(f)$. On this interval the derivative of $\tau(f)$ exists and is in absolute value greater than or equal to 2. Hence by Proposition 2.1.4, for any point x in this interval the distance of $\tau(f)^n(x)$ to the invariant point increases with n, unless x is equal to this invariant point or $\tau(f)^n(x)$ leaves this interval, i.e., ends up in I or J. In the latter case, it will never come back into $(\frac{1}{3}; \frac{2}{3})$ under any iterate of $\tau(f)$, so in that case the point x is not periodic. In particular, the orbit of any non-invariant periodic point is included in $I \cup J$, hence has a point in I and has even period.
(2) The proof consists of a straightforward computation, which we leave to the reader. However, the following geometric argument may be enlightening: on I, the mapping $\tau(f)$ followed by a translation over $-\frac{2}{3}$ is conjugate to f on $[0;1]$, because it is just the mapping $\varphi \circ f \circ \varphi^{\leftarrow}$. However, the translation over $-\frac{2}{3}$ can be seen as another application of $\tau(f)$, because that is just the action of the mapping $\tau(f)$ on the interval $[\frac{2}{3}; 1]$. □

Lemma 2.4.2. *For every $n \in \mathbb{N}$, the mapping f has a periodic orbit with primitive period n iff the mapping $\tau(f)$ has a periodic orbit with primitive period $2n$.*

Proof. Let x be a periodic point under f with period n (not necessarily the primitive period). It follows from Lemma 2.4.1 (2) that the point $\frac{1}{3}x$ is periodic under $\tau(f)^2$ with period n, hence it is periodic under $\tau(f)$ with period $2n$. Conversely, by Lemma 2.4.1 (1) a periodic orbit under $\tau(f)$ has even period, say $2n$ with $n \in \mathbb{N}$, and it has a point in the