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A TOPOLOGIST'S VIEW OF SHARKOVSKY'S THEOREM

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ABSTRACT. A theorem due to A. N. Sharkovsky gives a surprising answer to the following question: Let R be the real line and f: $R \to R$ a continuous function. If f has a periodic point of period k, must f have periodic points of other periods as well? Here necessary and sufficient conditions for a topological space X are discussed so that X can replace R in Sharkovsky's theorem. It is, e.g., shown that Sharkovsky's theorem holds for an ordered set in the order topology if and only if it is connected, and that Sharkovsky's sequence is still sharp for such spaces if they contain an arc.

1. The problem. Let f: $R \to R$ be a continuous function on the real line R. A point $x \in R$ is a *periodic point* of f of *period* $k > 0$ if $f^k(x) = x$ but $f^i(x) \neq x$ for $0 < i < k$. Hence x is a fixed point if $k = 1$. In recent years the following question has aroused interest: If f has a point of period k, must f also have points of other periods? The question has an intriguing answer which was found by the Russian mathematician A. N. Sharkovsky [13] in 1964.

THEOREM 1.1 (Sharkovsky's theorem). *Order the positive integers in a sequence as follows:*

(S) $3,5,7,\ldots,2\cdot 3,2\cdot 5,2\cdot 7,\ldots,2^2\cdot 3,2^2\cdot 5,2^2\cdot 7,\ldots,\ldots,2^3,2^2,2,1.$

(That is, first the odd integers $\geqslant 3$, then the powers of 2 times the odd integers,etc., and finally the powers of 2 backwards.) *If a continuous function* f: $R \to R$ *has a point of period* k, *then* f *has points of all periods which follow* k *in the sequence* (S).

We shall call the sequence (S) the *Sharkovsky sequence.* Sharkovsky's original proof of Theorem 1.1 is quite complicated, even in the somewhat improved English version given by P. Štefan [15] in 1977. But an efficient method of proof was found by P. D. Straffin [16] in 1978, who used directed graphs (so-called k-periodic digraphs, see §2) to represent information about the periodic points of f. Straffin

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could only prove a partial version of Theorem 1.1, but complete proofs based on Straffin's graphs were soon obtained independently by U. Burkhart [4] and by C.-W. Ho and C. Morris [7]. A similar graph-theoretic approach is due to L. Block, J. Guckenheimer, M. Misiurewicz and L. S. Young [2]. There also exist some proofs of partial versions of Theorem 1.1 which were obtained by Western mathematicians not yet aware of Sharkovsky's result (see, e.g., [5], [10] and [12]).

Sharkovsky's theorem is attractive to mathematicians because of its simple setting and surprising answer. And it also has applications, mainly in the area of population modelling (see, e.g., [10], [16]).

In spite of attempts to generalize Sharkovsky's theorem to special types of functions in higher-dimensional Euclidean spaces [8] it is clear that the theorem is basically a one-dimensional phenomenon. Extensions to continuous functions on the circle can be found in [2], [3] and [1], but the results get more complicated. All proofs of Sharkovsky's theorem depend strongly on the order relation of the real numbers, and a look at [15], [16], [2], [4] and [7] shows that not much else is used. We show here that the seemingly algebraic character of Sharkovsky's sequence is completely independent of the algebraic properties of the real numbers, and that Sharkovsky's theorem is true for ordered spaces under very general assumptions. To make this statement precise, we say that a (linearly) ordered set L with more than one point is a *linear continuum* [11], page 152 if

(i) L has the least upper bound property (or, equivalently, the greatest lower bound property),

(ii) L is order dense, i.e., if $x < y$, then there exists z so that $x < z < y$,

and give L the order topology [11], §2-3, page 84 ff. (Note that L is not a "continuum," which the topologist usually defines as a compact connected space, as L need not be compact.) The real line and intervals of the real line are examples of linear continua in the order topology. But there are many others, among them the long line and the unit square in the dictionary order. (See, e.g., [11], §2-3, page 152 ff. and [14], Counterexamples 45, 46, 48 for these and some other examples.) Background material and notation on linear continua can be found in [11]. We shall use that an ordered set in the order topology is a linear continuum if and only if it is connected

(see [11], Theorem 2.1, page 152 and Exercise 4, page 158; [14] Counterexample 39, pages 67-68), and also need a generalization of the intermediate value theorem which follows from [11], Theorem 2.3, page 154.

THEOREM 1.2 (Intermediate value theorem). *Let* $[a,b]$ *be a closed interval in the linear continuum* L *with the order topology. If* $f: [a,b] \to L$ *is a continuous function and* y *is a point of* L *lying between* f(a) *and* f(b), *then there exists a point* $x \in [a,b]$ *such that* $f(x) = y$.

Our first result (Theorem 2.1) extends Sharkovsky's theorem to all linear continua in the order topology. The proof in §2 is modelled on the proof of Sharkovsky's theorem in [16] and [7], and apart from an inspection of the arguments in [16] and [7] nothing is needed but a more careful proof of three lemmas (Lemmas 2.2, 2.3 and 2.4). In the next paragraph we investigate the necessity of the assumptions in Theorem 2.1. To do so, let us define that a topological space X is a *Sharkovsky space* if Theorem 1.1 (with R replaced by X) is true. The property of being a Sharkovsky space is a topological property, for if $h: X \to Y$ is a homeomorphism from a topological space X onto a topological space Y, then the continuous functions $f: X \to X$ and $g = h \circ f \circ h^{-1}: Y \to Y$ have periodic points of the same periods. Every Sharkovsky space is connected (Theorem 3.3), and every retract of a Sharkovsky space is again a Sharkovsky space (Theorem 3.4). These topological properties are used to show that linear continua are very general representatives of Sharkovsky spaces. We prove that an ordered set in the order topology is a Sharkovsky space if and only if it is a linear continuum (Corollary 3.4). The same may be true for all partially ordered topological spaces, but we can only prove it for dendrites (Theorem 3.7). In the final §4 we look at the "sharpness" of Sharkovsky's sequence. It is known that for continuous functions $f: R \to R$ Sharkovsky's result is the best possible, i.e., that no implications in the sequence (S) from right to left exist. (See, e.g., [15], page 245, [7], Proposition 3.8, [4], pages 66-67.) For linear continua we can only obtain this result if they contain an arc (Theorem 4.2). It is not known whether this assumption is necessary.

I wish to thank Paula Gray [6], who during her final undergraduate year at Carleton University collected much of the background material concerning

Sharkovsky's theorem for me.

2. Sharkovsky's theorem for linear continua. The purpose of this paragraph is to prove the following extension of Theorem 1.1.

THEOREM 2.1 (Sharkovsky's theorem for linear continua). *Any linear continuum is a Sharkovsky space.*

The proof of Theorem 1.1 in [16] and [7] uses two simple lemmas from analysis concerning continuous functions on the real line. They were first stated by T.-Y. Li and J. A. Yorke [10], Lemmas 0 and 1, and are cited in [16] as Lemmas 1 and 2. In the proof of these two lemmas standard properties of the reals are used, but we show, in Lemmas 2.2 and 2.3, that they still hold for linear continua.

LEMMA 2.2. *Let* L *be a linear continuum in the order topology, let* I *and* J *be closed intervals in* L *and let* $f: L \to L$ *be a continuous function. If* $J \subset f(I)$, *then there exists a closed interval* $Q \subset I$ *so that* $f(Q) = J$.

PROOF. We select points $p,q \in I$ with $p < q$ so that $J = [f(p),f(q)]$ or $J = [f(q),f(p)]$, and define $p \leqslant r < q$ by

$$r = \text{lub}\{x \in [p,q] \mid f(x) = f(p)\}.$$

We claim that $f(r) = f(p)$. For otherwise we can find (as L is Hausdorff) an open set V so that $f(r) \in V$ and $f(p) \notin V$, and by continuity of f an open neighbourhood U of r such that $f(U) \subset V$. As L is order dense, we can choose $p \leqslant r' < r$ with $[r',r] \subset U$, hence $f[r',r] \subset V$ implies $f(p) \notin f([r',r])$. But this contradicts the definition of r as a lub.

If we define $r < s \leqslant q$ by

$$s = \text{glb}\{x \in [r,q] \mid f(x) = f(q)\},$$

then $f(s) = f(q)$ can be proved analogously. We put $Q = [r,s] \subset I$, and show that $f(Q) = J$.

The intermediate value theorem 1.2 applied to $f|[r,s]$ shows that $J \subset f([r,s])$. But we also have $f([r,s]) \subset J$, for otherwise there exists $r < x < s$ with $f(x) \notin J$. If $f(x) < f(p) < f(q)$ or $f(q) < f(p) < f(x)$, then the intermediate value theorem 1.2 applied to $f|[x,s]$ asserts that $f(p) = f(x')$ for some $r < x < x' < s$, which contradicts the definition of r as a lub. If $f(x) < f(q) < f(p)$ or $f(p) < f(q) < f(x)$, then the

intermediate value theorem 1.2 applied to $f|[r,x]$ yields a contradiction to the definition of s as a glb. Therefore no such point can exist, and we see that $J = f(Q)$.

LEMMA 2.3. *Let* L *be a linear continuum in the order topology, let* I *be a closed interval in* L *and* $f: L \to L$ *a continuous function. If* $I \subset f(I)$, *then* f *has a fixed point in* I.

PROOF. We use Lemma 2.2 to choose a closed interval $Q \subset I$ with $f(Q) = I$, and we will show that f has a fixed point in Q.

Assume, by way of contradiction, that $f|Q$ is fixed point free. Then $Q \subset A \cup B$, where

$$A = \{x \in L | x < f(x)\},$$

$$B = \{x \in L | f(x) < x\}.$$

To see that A is open, pick for any $x \in A$ a point $x < z < f(x)$, and an open neighbourhood $U \subset (-\infty,z)$ of x with $f(U) \subset (z,\infty)$. Then $U \subset A$, so x is an interior point of A. Similarly we see that B is open. Hence $Q \cap A$ and $Q \cap B$ are open in Q, and $Q = (Q \cap A) \cup (Q \cap B)$. To see that $Q \cap A \neq \emptyset$, we write $I = [c,d]$. As $f(Q) = I$, there exists $x' \in Q$ with $f(x') = d$, and if $f|Q$ is fixed point free, then $x' \neq d$. As $Q \subset I$, we have $x' < f(x') = d$, so $x' \in Q \cap A$. Analogously we can find $x'' \in Q - \{c\}$ with $f(x'') = c$ and $x'' \in Q \cap B$. Hence $Q \cap A$ and $Q \cap B$ form a separation of the connected set Q, which is impossible. Therefore f must have a fixed point in Q.

We illustrate the behaviour of linear continua by proving one more lemma, as its standard proof for continuous functions $f: R \to R$ uses the difference function $f(x) - x$. (See, e.g., [16], page 99.) Lemma 2.4 extends the first part of Proposition 3.2 in [7] to linear continua.

LEMMA 2.4. *Let* L *be a linear continuum in the order topology and* $f: L \to L$ *a continuous function. If* f *has a periodic point, then* f *has a fixed point.*

PROOF. Let $f: L \to L$ have a point of period $k > 1$, and choose a in the sequence $a, f(a), f^2(a), \ldots, f^k(a) = a$ so that $a < f(a)$. As the orbit of a returns to a, there exists a point $b = f^i(a)$, for $i = 1,2,\ldots,k-1$, with $f(b) < b$. As in the proof of Lemma 2.3 it follows that the sets

$$A = \{x \in L | x < f(x)\},$$

$$B = \{x \in L | f(x) < x\}$$

are open. If f has no fixed point, then $L = A \cup B$, hence $a \in A$, $b \in B$, and $A \cap B = \emptyset$ imply that A and B form a separation of the connected space L. But this is impossible, so f must have a fixed point.

The proof of Theorem 1.1 can now be completed as in [16] and [7]. Directed graphs (digraphs) can be used in the same manner as for $f: R \to R$. For, if a is a point of period k of the continuous function $f: L \to L$ on the linear continuum L in the order topology, then a and its iterates divide L into two rays and k - 1 closed intervals which can be labelled $I_1, I_2, \ldots, I_{k-1}$ "from left to right," i.e., so that $x_{j-1} < x_j$ for all $x_{j-1} \in I_{j-1}$ and $x_j \in I_j$, with $j = 2,3,\ldots,k$. If we direct the interval $I_i = [s_i, t_i]$ to the interval I_j if the endpoints of I_j lie between $f(s_i)$ and $f(t_i)$, then it follows from the intermediate value theorem 1.2 that all of I_j lies between $f(s_i)$ and $f(t_i)$. Hence the directed graph on the vertices labelled $I_1, I_2, \ldots, I_{k-1}$, called a k-*periodic digraph of* f, still reflects those properties of f which are used in the proofs of Sharkovsky's theorem 1.1 in [16] and [7]. We state the sequence of lemmas needed to complete the proof of Theorem 2.1, and refer to the places in [16] and [7] where their proofs can be found. A *non-repetitive cycle* of a k-periodic digraph is defined as in [16], page 101, as a cycle which does not consist entirely of a cycle of smaller length traced several times. The symbol L always denotes a linear continuum in the order topology, and $f: L \to L$ a continuous function.

LEMMA 2.5. *If a* k-*periodic digraph of* $f: L \to L$ *has a non-repetitive cycle of length* k, *then* f *has a periodic point of period* k.

PROOF. See [16], Theorem A, compare also [7], Theorem 2.1.

LEMMA 2.6. *If* $f: L \to L$ *has a point of odd period* $k \geqslant 3$, *then it has periodic points of all periods* $\geqslant k - 1$.

PROOF. See [16], Theorem B, compare also [7], Theorem 2.2.

LEMMA 2.7. *Let* $k \geqslant 2$. *Then every* k-*periodic digraph of* $f: L \to L$ *contains a cycle of length* k *which decomposes into two smaller non-repetitive cycles.*

PROOF. See [16], proof of Theorem B. This lemma replaces [7], Lemma 2.3, and we have stated it in the form in which it is used in the proof of the next lemma.

LEMMA 2.8. *If* $f: L \to L$ *has a point of odd period* $k \geqslant 3$, *then* f *has periodic points of all even periods.*

PROOF. See [7], Proposition 3.1.

LEMMA 2.9. *If* $f: L \to L$ *has a point of period* $k > 1$, *then it has a point of period* 2.

PROOF. See [7], second part of Proposition 3.2.

The proof of Theorem 2.1 can now be completed as in [7], proof of Theorem 3.3.

3. Necessary conditions for a Sharkovsky space. We have seen that an ordered set in the order topology is a Sharkovsky space if it has the lub property and is order dense. Now we will show that these two assumptions are necessary. Two simple examples illustrate the situation.

EXAMPLE 3.1. The subspace $X_1 = (-1,0) \cup (0,1)$ of R is order dense, but lacks the lub property. The continuous function $f: X_1 \to X_1$ defined by $f(x) = -x$ for all $x \in X_1$ has points of period two but no fixed point. Hence X_1 is not a Sharkovsky space.

EXAMPLE 3.2. The subspace $X_2 = [-2,-1] \cup [1,2]$ of R has the lub property, but is not order dense. Again the continuous function $f: X_2 \to X_2$ defined by $f(x) = -x$ for all $x \in X_2$ shows that X_2 is not a Sharkovsky space.

We now abstract the pattern of these two examples. It was pointed out in § 1 that an ordered set in the order topology is a linear continuum if and only if it is connected. The next result shows that the fact that the spaces X_1 and X_2 of Examples 3.1 and 3.2 are not connected is crucial.

THEOREM 3.3. *Every Sharkovsky space is connected.*

PROOF. We proceed indirectly. If the topological space X is not connected, then $X = A \cup B$, where A and B are non-empty closed subsets of X with $A \cap B = \emptyset$. We select points $a \in A$ and $b \in B$, and define $f: X \to X$ by

$$f(x) = \begin{cases} b \text{ for all } x \in A, \\ a \text{ for all } x \in B. \end{cases}$$

f is continuous by the Pasting Lemma [11], page 108, Theorem 7.3, and a and b are points of period two. But f has no fixed point, so X is not a Sharkovsky space.

COROLLARY 3.4. *Let* X *be an ordered set in the order topology. Then* X *is a Sharkovsky space if and only if it is a linear continuum.*

PROOF. This follows from Theorems 2.1 and 3.3.

The next example shows that the assumption in Theorem 2.1 that the order is linear cannot be weakened to a partial order.

EXAMPLE 3.5. Let T be the triod in the (x,y)-plane consisting of three line segments which join the points (0,-1), $(\frac{1}{2}\sqrt{3},\frac{1}{2})$, $(-\frac{1}{2}\sqrt{3},\frac{1}{2})$ to the origin. T inherits a topology as a subspace of the Euclidean plane. A partial order which is order dense and has the lub property is defined if we write $(x,y) \leqslant (x',y')$ whenever $y \leqslant y'$. Let $f: T \to T$ be the rotation through $2\pi/3$ with the origin as centre. Then all points of $T - \{(0,0)\}$ are points of period three and the origin is a fixed point, so T is not a Sharkovsky space.

In order to use the ideas of Example 3.5 to obtain a more general result, we first show that the property of being a Sharkovsky space is invariant under retraction.

THEOREM 3.6. *Let* X *be a retract of the topological space* Y. *If* Y *is a Sharkovsky space, then so is* X.

PROOF. Let $r: Y \to X$ be a retraction and $i: X \to Y$ the inclusion. If $f: X \to X$ is a continuous function, then the continuous function $g = i \circ f \circ r: Y \to Y$ has the same periodic and fixed points as f, so Theorem 3.6 follows.

REMARK. The converse of Theorem 3.6 is false, as a space Y which retracts to a Sharkovsky space X need not be a Sharkovsky space. To see this, let Y = T be the triod of Example 3.5 which retracts to the unit interval X = I.

We now look at partially ordered spaces in the context of Sharkovsky spaces. A *partially ordered topological space* X consists of a set with a partial order $\leqslant$ and a topology which has a subbasis for its closed sets consisting of the sets

$$L(a) = \{x \in X | x \leqslant a\},$$
$$M(a) = \{x \in X | a \leqslant x\}$$

for all $a \in X$ [17], page 148. This topology equals the order topology if the partial order is in fact a linear one. A subclass of partially ordered spaces are the *dendrites,* which are compact connected metric topological spaces in which every pair of points is separated by a third point. (See, e.g., [9], page 300 ff., or [19], page 88 ff. for their properties.) A characterization of dendrites as partially ordered topological spaces was given by L. E. Ward, Jr., [18], and the partial order is order dense [18], Theorem 1

and has the lub and glb property [17], page 148. The triod is an example of a dendrite, and a linearly ordered dendrite is an arc, i.e., a homeomorphic image of the unit interval $I = [0,1]$.

THEOREM 3.7. *Let* X *be a partially ordered space which is a dendrite. Then* X *is a Sharkovsky space if and only if it is a linear continuum.*

PROOF. In view of Theorems 2.1 and 3.3 it is sufficient to show that a dendrite which is a Sharkovsky space is linearly ordered. But this follows from Example 3.5 and Theorem 3.6, as well as the fact that any dendrite which is not an arc contains a triod to which it retracts [9], page 344.

QUESTION 3.8. *Is any partially ordered topological space a Sharkovsky space if and only if it is a linear continuum?*

4. The sharpness of Sharkovsky's sequence. For continuous functions on the real line or unit interval Sharkovsky's sequence is *sharp* in the sense that no implications from right to left are possible. This is proved by constructing, for every integer k, a continuous function which has a point of period k but no point of a period which precedes k in Sharkovsky's sequence. The construction of such functions was indicated by Š. Stefan [15], page 245 and is described in more detail by C.-W. Ho and C. Morris [7], Proposition 3.8. An outline of a proof is also given by U. Burkhart [1], pages 66-67. All proofs are similar, as they define f on a finite set of points (which correspond to the orbit of a point with period k) and then extend it by using the fact that any two intervals [a,b] and [c,d] of R are homeomorphic.

This method no longer works for all linear continua in the order topology. For let L be the unit square $I \times I$ in the dictionary order topology, let $I_1 = [(0,0),(0,1)]$ and $I_2 = L$. Then I_1 is homeomorphic to I, but I_2 is not path-connected (see [11], page 156, Example 6, or [14], page 73, Counterexample 48). Hence I_1 and I_2 are not homeomorphic, and there exists no continuous function $I_1 \to I_2$ which extends the function given by $(0,0) \to (0,0)$ and $(0,1) \to (1,1)$. There seems to be no easy way to adjust the usual proof that Sharkovsky's sequence is sharp for continuous functions $f: R \to R$ to continuous functions on arbitrary linear continua, but we shall obtain sharpness in a special case as a consequence of the next theorem.

THEOREM 4.1. *Let* X *be a topological space which retracts to an arc. Then*

there exists, for any integer k, *a continuous function* $f: X \to X$ *which has a point of period* k *but no point of a period which precedes* k *in Sharkovsky's sequence.*

PROOF. Let $h: I \to A$ be a homeomorphism from the unit interval I onto the arc A, let $r: X \to A$ be a retraction and $i: A \to X$ the inclusion. We choose $g: I \to I$ as a continuous function which has a point of period k but no point of any period which precedes k in Sharkovsky's sequence. The continuous function $g' = h \circ g \circ h^{-1}: A \to A$ has periodic points of the same period as g, and so has $f = i \circ g' \circ r: X \to X$.

THEOREM 4.2 (Sharpness of Sharkovsky's sequence). *Let* L *be a linear continuum in the order topology which contains an arc. Then* L *is a Sharkovsky space for which Sharkovsky's sequence is sharp.*

PROOF. Theorem 4.2 follows from Theorems 2.1 and 4.1 if we can show that L retracts to any arc A contained in it. To see this, we use that A is a closed interval in L. This is an easy consequence of the well-known fact that a homeomorphism from an ordered space onto another is either order preserving or order reversing, but here is a proof:

Let $h: I \to A$ be a homeomorphism from the unit interval I onto the arc A in L. If $h(0) < h(1)$, then the intermediate value Theorem 1.2 shows that $[h(0),h(1)] \subset h([0,1])$. But we also have $h([0,1]) \subset [h(0),h(1)]$, for otherwise there exists a point $0 < x < 1$ with $y = h(x) \notin [h(0),h(1)]$. If $y < h(0)$, then the intermediate value theorem 1.2 applied to $h|[x,1]$ shows that $h(x') = h(0)$ for some $x < x' < 1$, which cannot happen. Similarly we exclude the case $h(1) < y$, so in fact $h([0,1]) = [h(0),h(1)]$. If $h(1) < h(0)$, it follows analogously that $h([0,1]) = [h(1),h(0)]$.

As A is a closed interval, we can write $A = [a,b]$, and define a function $r: L \to A$ by

$$r(x) = \begin{cases} x & \text{for all } x \in A, \\ a & \text{for all } x \leqslant a \\ b & \text{for all } b \leqslant x. \end{cases}$$

r is continuous by the Pasting Lemma [11], page 108, Theorem 7.3, and is clearly a retraction. Hence Theorem 4.2 holds.

All examples mentioned in §1 contain an arc, so Sharkovsky's sequence is sharp for these spaces. Nevertheless we still have

QUESTION 4.3. *Can the assumption that* L *contains an arc be omitted from Theorem* 4.2?

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