Remarks on Sharkovsky's Theorem

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Recent publication of a paper on Sharkovsky's Theorem in this MONTHLY **[8]** is a good occasion for making several historical comments on this beautiful theorem.

The original paper of Sharkovsky **[12]** was published in Russian and has been translated into English only recently **[13]**. As a result, some authors citing **[12]** may be not fully aware of the contents of this paper. Moreover, there was a subsequent paper by Sharkovsky **[14]**, that in some sense completed his theorem.

Consider the *Sharkovsky ordering* of the set of natural numbers:

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 3 \cdot 2 \prec 5 \cdot 2 \prec 7 \cdot 2 \prec 9 \cdot 2 \prec \cdots$$
$$\prec 3 \cdot 2^2 \prec 5 \cdot 2^2 \prec 7 \cdot 2^2 \prec 9 \cdot 2^2 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1 .$$

Let I be either the real line or an interval. If $f{:}I \to I$ is a continuous map, then a set $P = \{x_1, x_2, \dots, x_n\}$ such that $f(x_1) = x_2, f(x_2) = x_3, \dots, f(x_n) = x_1$, is called a *cycle* or a *periodic orbit*. The *period* of a cycle P is the number of its elements.

The three parts of the full Sharkovsky Theorem are:

Theorem 1. *Let f: $I \to I$ be a continuous map. If f has a cycle of period n and if n appears before k in the Sharkovsky ordering, then f has a cycle of period k.*

Theorem 2. *For every k there exists a continuous map f: $I \to I$ that has a cycle of period k, but has no cycles of period n for any n appearing before k in the Sharkovsky ordering.*

Theorem 3. *There exists a continuous map f: $I \to I$ that has a cycle of period 2^n for every n and has no cycles of any other periods.*

In most papers and books dealing with Sharkovsky's Theorem, this name is applied only to Theorem 1. However, the original statement of Sharkovsky's Theorem is stronger. It is equivalent to Theorem 1 plus the assertion that if n appears before k in the Sharkovsky ordering then there exists a continuous map f: $I \to I$ with a cycle of period k but with no cycle of period n. Moreover, the arguments given in **[12]** also prove Theorem 2. Theorem 3 is proved in **[14]**. Thus, "Sharkovsky's Theorem" properly refers to the union of all three theorems.

The first proofs of Theorem 1 were difficult to follow. I remember that when I learned of this theorem, I tried to read the proof in **[12]**. The idea was clear, but the details were messy. This was apparently also an impression of Štefan, who wrote another proof **[15]**. However, when I tried to read Štefan's proof, I also found that the idea was clear, but the details were messy. Therefore I decided to write my own proof. When I tried to read it several months later, I realized that I did no better: the idea was clear, but the details were messy. The standard proof is now easy to follow complete detail; it was discovered almost simultaneously by many mathematicians (see e.g. **[3]**, **[4]**, **[10]**, **[16]**).

The standard proof of Theorem 2 uses examples of maps having only cycles of odd period n and periods following n in the Sharkovsky ordering, and the "square root" construction. Various presentations of this proof (including **[8]**) are only small modifications of the proof in **[12]**. Many of them leave details to the reader (as for instance in [**7**, pp. 66–68]).

There was an interesting question connected with these proofs. Suppose f has a cycle P of period n and no cycle of any period preceding n in the Sharkovsky ordering. What does P look like? This question has been answered in **[2]**, **[6]**, and **[9]**. Problems of this type led to the development of *combinatorial dynamics*.

To prove Theorem 3, one has to give an example of what is called a map *of type* 2^∞. Several kinds of examples are known, but the most important are the ones that are smooth and unimodal ("unimodal" means "with one interior local extremum"). For these maps, as well as for one-parameter families containing them, one observes interesting geometric structure both in the parameter space and on the interval. This observation led to the development of the so called *Feigenbaum Theory* (see **[5**, pp. 199–238**]**). A very short proof of Theorems 2 and 3 together can be given by looking at the family of truncated tent maps (trapezoidal maps); see **[1]**. However, this proof is not constructive. The real career of Sharkovsky's Theorem began with the publication of the paper *Period three implies chaos* by Li and Yorke in this MONTHLY **[11]**, although the authors did not even know about Sharkovsky's Theorem when they wrote their paper.

REFERENCES

1. L. Alsedà, J. Llibre, and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, Adv. Ser. in Nonlinear Dynamics **5**, World Scientific, Singapore, 1993.
2. L. Alsedà, J. Llibre, and R. Serra, Minimal periodic orbits for continuous maps of the interval, *Trans. Amer. Math. Soc.*, **286** (1984). 595–627.
3. L. Block, J. Guckenheimer, M. Misiurewicz, and L.-S. Young, Periodic points and topological entropy of one dimensional maps, *Global Theory of Dynamical Systems*, Lecture Notes in Math., 819, Springer, Berlin, 1980, pp. 18–34.
4. U. Burkart, Interval mapping graphs and periodic points of continuous functions, *J. Combin. Theory Ser. B*, **32** (1982), 57–68.
5. P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Progr. in Phys. **1**, Birkhäuser, Boston, 1980.
6. W. A. Coppel, Šarkovskii-minimal orbits, *Math. Proc. Cambr. Phil. Soc.* **93** (1983), 397–408.
7. R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Second Edition, Addison-Wesley, Redwood City, California, 1989.
8. S. Elaydi, On a converse of Sharkovsky's Theorem, *Amer. Math. Monthly* **103** (1996), 386–392.
9. C.-W. Ho, On the structure of the minimum orbits of periodic points for maps of the real line, preprint, Southern Illinois Univ., Edwardsville, IL (1984).
10. C.-W. Ho and C. Morris, A graph theoretic proof of Sharkovsky's theorem on the periodic points of continuous functions, *Pacific J. Math.* **96** (1981), 361–370.
11. T.-Y. Li and J. Yorke, Period three implies chaos, *Amer. Math. Monthly* **82** (1975), 985–992.
12. A. N. Sharkovsky, Co-existence of the cycles of a continuous mapping of the line into itself, *Ukrain. Math. Zh.* **16** **(1)** (1964), 61–71 (Russian).
13. A. N. Sharkovsky, Coexistence of the cycles of a continuous mapping of the line into itself, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **5** (1995), 1263–1273.
14. A. N. Sharkovsky, On cycles and structure of a continuous map, *Ukrain. Math. Zh.* **17** **(3)** (1965), 104–111 (Russian).
15. P. Štefan, A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line, *Comm. Math. Phys.* **54** (1977), 237–248.
16. P. D. Straffin Jr., Periodic points of continuous functions, *Math. Mag.* **51** (1978), 99–105.

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