Mathematics Magazine

ISSN: 0025-570X (Print) 1930-0980 (Online) Journal homepage: https://www.tandfonline.com/loi/umma20

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To cite this article: Philip D. Straffin (Jr.) (1978) Periodic Points of Continuous Functions, Mathematics Magazine, 51:2, 99-105, DOI: 10.1080/0025570X.1978.11976687

To link to this article: https://doi.org/10.1080/0025570X.1978.11976687

Published online: 13 Feb 2018.

Periodic Points of Continuous Functions

Analysis of the cyclic behavior of points under repeated application of a function yields insights into population patterns.

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Let $f\colon R \to R$ be a continuous function, and denote the nth iterate of f by f^n: $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for $n > 1$. A number x is said to be a point of period k for f if $f^k(x) = x$ and $f^i(x) \neq x$ for all $0 < i < k$. Suppose a continuous function f has points of period k; must it also have points of other periods $l \neq k$? The obvious answer would seem to be "no": why should there be any connection between points of period k and points of period l? Yet a little thought will show that there should be at least some results along these lines.

For instance, if a continuous function f has a point a of period $k > 1$, then it must also have a fixed point, that is, a point of period 1. To see this, suppose $f(a) > a$ and consider the sequence of points $a, f(a), f^2(a), f^3(a), \ldots, f^{k-1}(a), f^k(a) = a$. Then there must be a point $b = f^i(a)$ in the sequence such that $f(b) < b$, or else the sequence would constantly increase and could not return to a. But then the Intermediate Value Theorem (applied to $f(x) - x$) implies that there must be a number c between a and b such that $f(c) = c$. A symmetric argument works if $f(a) < a$.

Hence we have

$$\text{Period } k \Rightarrow \text{Period } 1 \text{ for all } k.$$

In 1975, Li and Yorke published (in [2]) a surprising theorem on this question: *If a continuous function $f\colon R \to R$ has a point of period* 3, *then it has points of all periods.* In other words,

$$\text{Period } 3 \Rightarrow \text{Period } l \text{ for all } l.$$

Clearly the periodic behavior of a function with points of all periods is extremely complex. In fact, considering the periodic behavior of physical and biological systems which can be modeled using such functions, Li and Yorke called such behavior "chaos", and titled their paper "Period Three Implies Chaos". Li and Yorke also produced a counterexample to show that Period 5 does not imply Period 3.

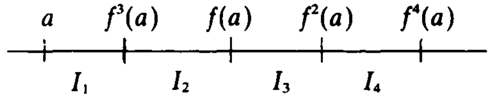
The purpose of this paper is to discuss general techniques for using a point of period k to deduce the existence of points of other periods. We will do this first by constructing an appropriate directed graph that will enable us to generalize Li and Yorke's result by showing, in section two below, that whenever a continuous function f has points of odd period k greater than one, then f has points of every period $l \geqq k - 1$. In other words,

$$\text{Period } k \Rightarrow \text{Period } l \text{ for all odd } k > 1 \text{ and all } l \geqq k - 1.$$

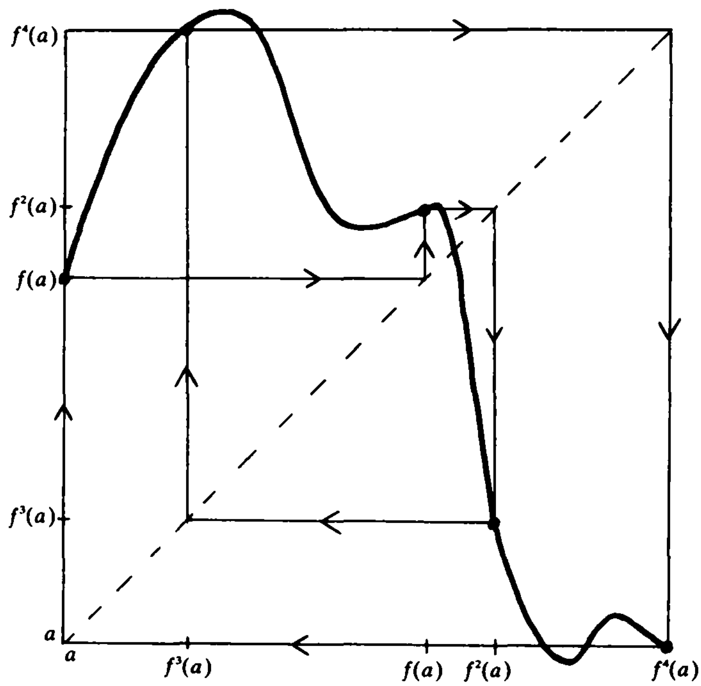
Unknown to Li and Yorke in 1975, and unknown to me when I gave the talk on which this article is based, the Russian mathematician A. N. Sharkovsky had already answered (in 1964) the general question of when period k implies period l. In section three we present Sharkovsky's theorem and show how the method of directed graphs gives a partial proof of his result. Finally, in section four we discuss the relevance of the material of the first three sections to recent work in population ecology.

1. Constructing periodic point digraphs

Consider a continuous function f which has a point x of period k. Among x and its iterates under f, let a be the smallest number. Then starting with a, the iterates of a are spread out along the real line to the right of a, and determine $k-1$ finite closed intervals. Call these intervals, from left to right, $I_1, I_2, \ldots, I_{k-1}$. For example, the function in FIGURE 1 has a point of period five, and gives rise to the intervals



Now consider $f(I_1)$. In our example, I_1 has endpoints a and $f^3(a)$, which get mapped to $f(a)$ and $f^4(a)$. Hence by the Intermediate Value Theorem, $f(I_1)$ must include all points between $f(a)$ and $f^4(a)$. In other words, $f(I_1) \supset I_3 \cup I_4$. Similarly, we get $f(I_2) \supset I_4$, $f(I_3) \supset I_2 \cup I_3$, and $f(I_4) \supset I_1$.



A continuous function with a point of period five.

FIGURE 1

One convenient way to organize this information is in a directed graph, usually called, for short, a digraph. Label the vertices of the digraph $I_1, I_2, \ldots, I_{k-1}$ and draw a directed arc from I_i to I_j if $f(I_i) \supset I_j$. (For our example, we get the digraph in FIGURE 2.) This construction starting with the periodic point and ending with the digraph is perfectly general: it works for any continuous function f with a point of period k, for any k. We will call a digraph arising from this construction a ***k*-periodic point digraph.** The interesting thing is that one can read from the digraph information about other periodic points which f must have.

THEOREM A. *If a k-periodic point digraph associated to f has a non-repetitive cycle of length l, then f must have a point of period l.*

Cycles, as we interpret them, are allowed to use a vertex or edge more than once; by a "non-repetitive cycle" we simply mean one that does not consist *entirely* of a cycle of smaller length traced several times. In the digraph of FIGURE 2, $I_1I_3I_2I_4I_1I_3I_2I_4I_1$ is a repetitive cycle of length eight (illegal), while $I_1I_3I_3I_3I_3I_3I_2I_4I_1$ is a non-repetitive cycle of length eight (legal).

The proof of Theorem A is modeled on Li and Yorke's proof. It uses two lemmas which are standard in analysis courses:

LEMMA 1. *Suppose I and J are closed intervals, f continuous, and $J \subset f(I)$. Then there is a closed interval $Q \subset I$ such that $f(Q) = J$.*

LEMMA 2. *Suppose I is a closed interval, f continuous, and $I \subset f(I)$. Then f has a fixed point in I.*

To prove the theorem, suppose we are given a non-repetitive sequence of closed intervals $I^0, I^1, \ldots, I^l = I^0$ (the superscript refers merely to the position in the sequence) such that $f(I^i) \supset I^{i+1}$. We wish to show that f has a point of period l. Consider the diagram:

$$\begin{array}{ccc} I^0 & \xrightarrow{f} & f(I^0) \\ & & \cup \\ & & I^1 \; . \end{array}$$

(The arrow in this and subsequent diagrams represents an onto map.) Using Lemma 1, we can find a closed interval Q_1 to fill in the diagram:

$$\begin{array}{ccc} I^0 & \xrightarrow{f} & f(I^0) \\ \cup & & \cup \\ Q_1 & \xrightarrow{f} & I^1 \; . \end{array}$$

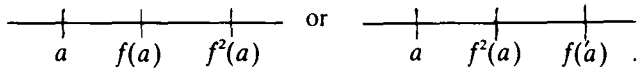
In fact, using Lemma 1 repeatedly, we can construct the following diagram:

$$\begin{array}{llllll}
I^0 \xrightarrow{f} & f(I^0) & & & & \\
\cup & \cup & & & & \\
Q_1 \xrightarrow{f} & I^1 \xrightarrow{f} & f(I^1) & & & \\
\cup & & \cup & & & \\
Q_2 \xrightarrow{\qquad f^2 \qquad} & & I^2 \xrightarrow{f} & f(I^2) & & \\
\cup & & & \cup & & \\
\vdots & & & \vdots & & \\
\cup & & & & & \\
Q_{l-1} \xrightarrow{\qquad\qquad f^{l-1} \qquad\qquad} & & & & I^{l-1} \xrightarrow{f} & f(I^{l-1}) \\
\cup & & & & & \cup \\
Q_l \xrightarrow{\qquad\qquad\qquad f^l \qquad\qquad\qquad} & & & & & I^l = I^0 \; .
\end{array}$$

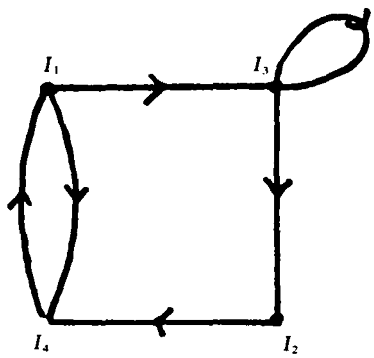
Now reading up the left hand column and across the bottom row, we see that $Q_l \subset I^0 = f^l(Q_l)$. Hence by Lemma 2, f^l has a fixed point x in Q_l. Since the sequence of intervals is non-repetitive, one can show that x is indeed a point of period l for f (rather than a point of period $< l$).

2. Analying periodic point digraphs

For $k = 3$, there are two possible orderings of points of period three:

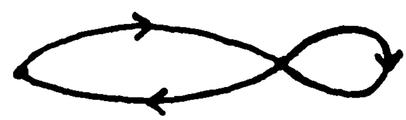


These orderings give rise to the same digraph, illustrated in FIGURE 3; we call this the Li-Yorke digraph. Since it clearly has non-repetitive cycles of all lengths, we recover the Li-Yorke theorem by applying Theorem A.



The periodic point digraph of the function in FIGURE 1.

FIGURE 2



The Li–Yorke digraph.

FIGURE 3

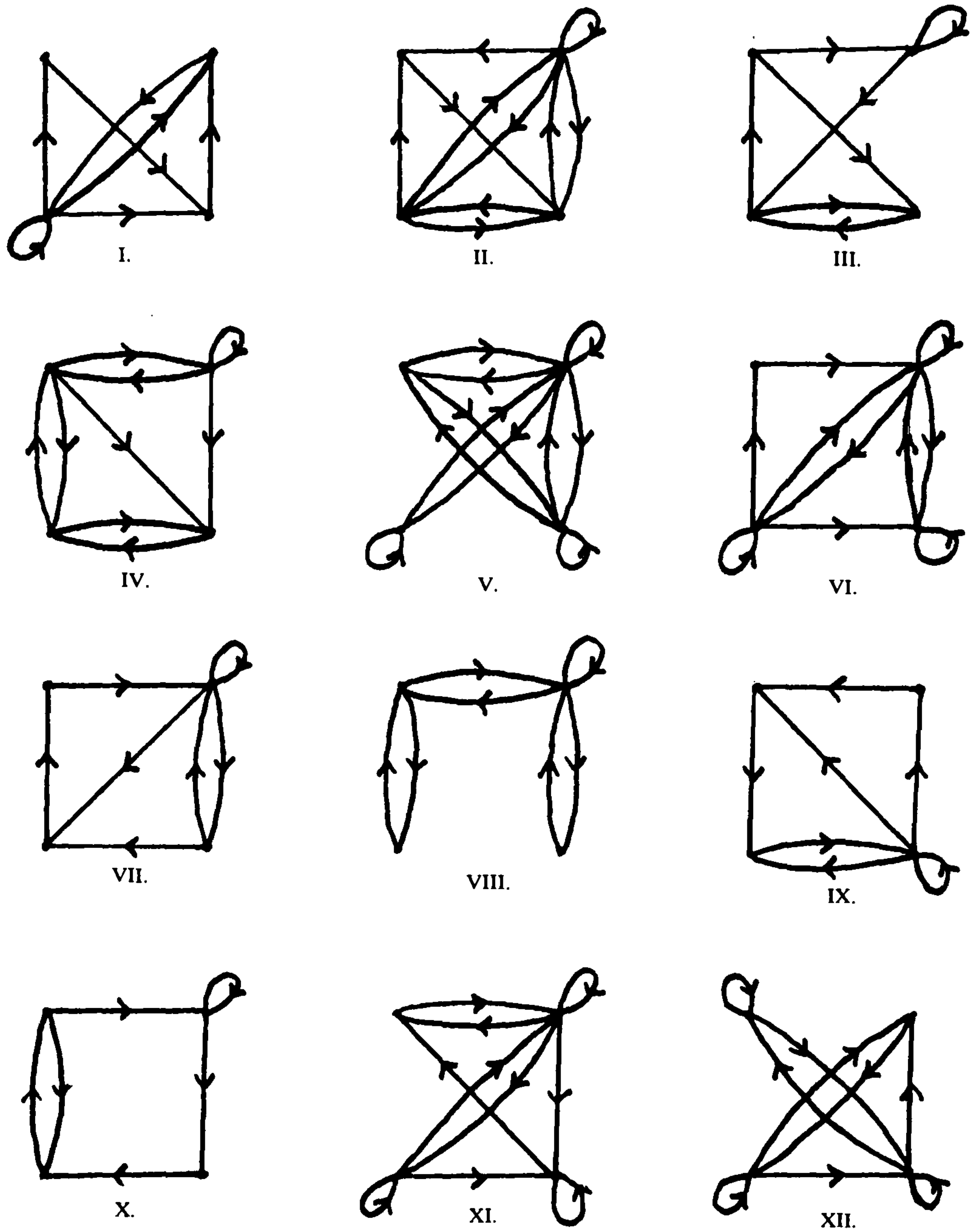
For $k = 5$, there are twelve possible digraphs, all shown in FIGURE 4. All except III and X contain the Li-Yorke digraph as a subgraph, and hence correspond to functions with points of all periods. You can check that III also has non-repetitive cycles of all lengths. X, our first example, has cycles of all lengths except three, and it corresponds to Li and Yorke's counterexample that Period 5 does not imply Period 3. However, we can conclude that

$$\text{Period } 5 \Rightarrow \text{Period } l \text{ for all } l \neq 3.$$

In general, the number of k-periodic point digraphs increases factorially with k, so that exhaustive analysis quickly becomes unmanageable. However, we can use an inductive technique to prove a general result:

THEOREM B. *If a continuous function $f\colon R \to R$ has a point of odd period $k > 1$, then it must have periodic points of all periods greater than or equal to $k - 1$.*

Proof. We first show that for all $k > 1$, a k-periodic point digraph has a cycle of length k. To do this, recall the notation of section one. Set $I^0 = I_1$, which has a as one endpoint. Define I^n for $n \geqq 1$ to be the interval I_i which has $f^n(a)$ as one endpoint and is contained in $f(I^{n-1})$. This construction gives $I^k = I^0$, so that we obtain a cycle of length k. (For instance, in the example of section one, I^1 would be the interval which has $f(a)$ as one endpoint and is contained in $f(I^0) = f(I_1)$. This is I_3. You can check that the construction gives $I_1I_3I_3I_2I_4I_1$ as the cycle of length five.) Since there are only $k - 1$ vertices in a k-periodic point digraph, some vertex must be repeated in this k-cycle, and at this vertex, the k-cycle decomposes into two cycles of smaller length. (For instance, $I_1I_3I_3I_2I_4I_1$ decomposes into the 4-cycle $I_1I_3I_2I_4I_1$ and the loop at I_3.) Since the repeated interval has only two endpoints, it can appear only twice in the original k-cycle. Hence it can appear only once in each of the smaller cycles, and both of these smaller cycles must be non-repetitive. Finally, when k is odd, one of the smaller cycles must be of odd length.



All 5-periodic point digraphs.

FIGURE 4

The theorem can now be proved by induction on odd $k > 1$. We already have the result for $k = 3$ or 5. Suppose it is true for all odd numbers between 1 and k. If f has a point of period k, then we know its periodic point digraph has a cycle of length k, and this cycle decomposes into two smaller cycles, one of which is of odd length and non-repetitive. If this length is not 1, we are done by Theorem A and the induction hypothesis. If it is of length 1, (i.e., is a loop) then we can get non-repetitive cycles of all lengths greater than or equal to $k - 1$ by traveling the loop as often as we need, and then completing the complementary $k - 1$ cycle. Again, Theorem A immediately gives the desired result.

3. Sharkovsky's theorem

In 1964 A. N. Sharkovsky completely answered the question of when period k implies period l. His work [7] was published in the *Ukranian Journal of Mathematics* in Russian, and has not been translated. His result is remarkable, and deserves to be better known in the West than it has been.

THEOREM. *A continuous function $f: R \to R$ which has a point of period k, must also have a point of period l precisely when k precedes l in the following ordering*

$$3, 5, 7, 9, \ldots, 3\cdot 2, 5\cdot 2, \ldots, 3\cdot 2^2, 5\cdot 2^2, \ldots, \ldots, \cdot, \cdot, 2^3, 2^2, 2, 1$$

of all positive integers.

Thus our observation that period k implies period 1 for all k, just corresponds to 1 being at the end of Sharkovsky's ordering, and Li and Yorke's result just corresponds to 3 being at the beginning. Theorem B also clearly follows from Sharkovsky's Theorem. You can read off other interesting results, for instance that a continuous function $f: R \to R$ which has a point of period not equal to a power of 2, must have an infinite number of periodic points.

Sharkovsky's Theorem can be derived from three basic results:

(i) Period $k \Rightarrow$ Period 2, for all $k > 1$.
(ii) Any odd period $> 1 \Rightarrow$ all higher odd periods.
(iii) Any odd period $> 1 \Rightarrow$ all even periods.

Perhaps you can see, and Sharkovsky shows in his article, how to combine (i), (ii) and (iii) with careful consideration of f^{2^n} to obtain the general result. Of the three basic results, (i) is fairly easy to show directly. However, (ii) and (iii) are not obvious, and Sharkovsky's proof of them is long and very complicated. He constructs so many sequences of points that eight complex figures and most of the letters of the Greek alphabet are necessary to keep track of them. Sharkovsky's Theorem is an example of a common occurrence in mathematics — an elegant result whose first proof is extremely inelegant.

Theorem B comes close to providing a more elegant proof of Sharkovsky's Theorem for it embodies (ii) and most of (iii). To complete the proof of Sharkovsky's Theorem, we would only have to show that any odd period $k > 1$ implies all even periods between 2 and $k - 1$. Could the reader fill this gap using the method of k-periodic point digraphs?

Finally, we should note that it is possible to approach these questions using completely different techniques. For instance, John Guckenheimer [1] has recently succeeded in proving Sharkovsky's Theorem for a certain class of functions using the methods of symbolic dynamics.

4. Population ecology

Recent concern about the pure mathematical question of the nature of periodic points of continuous functions was generated by ecologists, who in turn had been stimulated by earlier work of a meteorologist. References can be found in [2], [3], [4], and [5]. The ecological problem is to describe the behavior over time of, say, an insect population with discrete generations, which might behave according to the equation

$$x_{t+1} = rx_t(1 - x_t).$$

Here x_t is the size of the tth generation, r is the "intrinsic rate of increase", and $(1 - x_t)$ is a damping term due to environmental limitations. In this model, periodic points of the function $f(x) = rx(1 - x)$ would correspond to cyclical behavior of the insect population.

What happens for this function is that as r increases, more and more periodic points begin to appear. Points of period 2 appear as soon as $r > 3.00$, period 4 comes in at $r \approx 3.45$, then periods 8, 16, etc. All periods of the form 2^n are in by $r \approx 3.57$, when points of other periods begin to appear. The first points of odd period appear when $r \approx 3.68$, and period 3 finally comes in at $r \approx 3.83$. As you can see, the first appearance of points of different periods appears to run exactly backwards through

Sharkovsky's ordering, as r increases between 3.00 and 3.83. If this is true, and no one has yet produced a formal proof that it is, we would have one family of counterexamples, much simpler than the counterexamples Sharkovsky originally constructed in [7], to show that the Sharkovsky result is strict: no other implications are possible. Ecologically, an infinite number of periodic points by $r = 3.57$, and points of all periods by $r = 3.83$, means that for r this large, periodic behavior becomes chaotic. An excellent survey of these kinds of results from an ecological point of view has recently appeared in [6].

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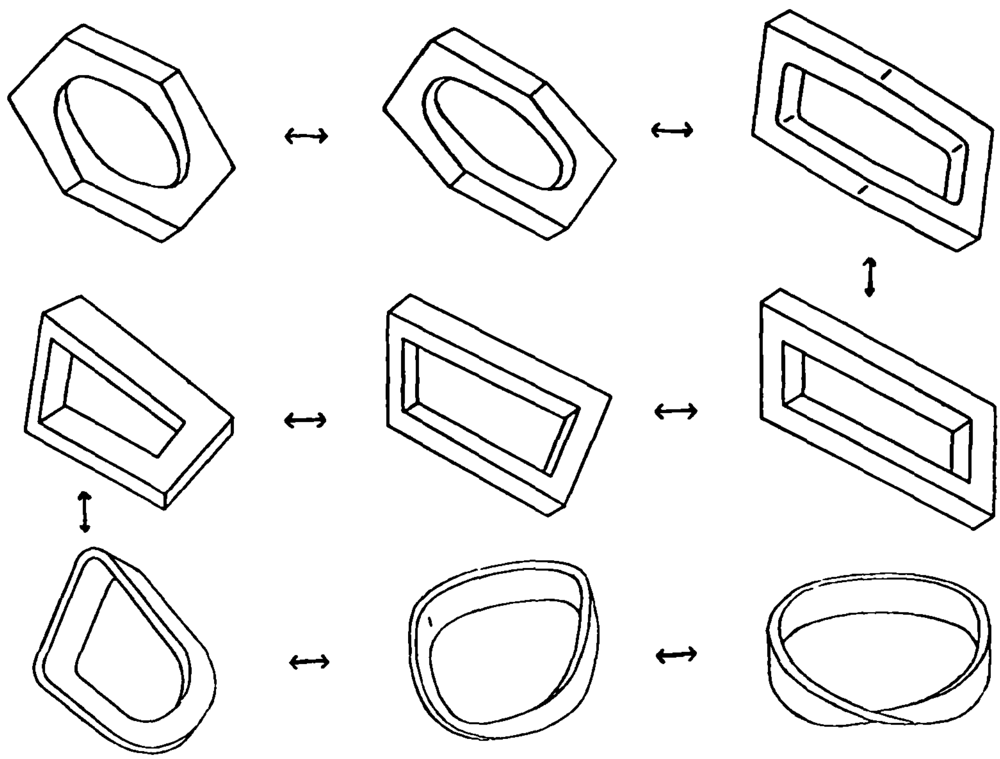

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