Handbook of Magma Functions

Volume 4

Local Arithmetic Fields

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Editors

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MAGMA
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Handbook of Magma Functions

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Chapter 47

p-ADIC RINGS AND THEIR EXTENSIONS

47.1 Introduction

Magma supports finite extensions of the ring $\mathbf{Z}_p$ of p-adic integers or the field $\mathbf{Q}_p$ of p-adic numbers. Within this chapter, we mean these objects if we refer to local rings or fields. Section 47.2 provides more background information on the theory behind the p-adics.

Magma has two different models for working with these locals: fixed precision rings (`RngPadRes` and `RngPadResExt`, with element types `RngPadResElt` and `RngPadResExtElt`) and free precision rings (`RngPad` and `FldPad`, with element types `RngPadElt` and `FldPadElt`). The merits of each model are discussed in Section 47.3.1.

Magma also contains a type of local field where extensions can be made by any irreducible polynomial. For more information on these local fields, see Chapter 48.

47.2 Background

The p-adic field $\mathbf{Q}_p$ arises naturally as the completion of $\mathbf{Q}$ with respect to an absolute value function $|x|_p = p^{-v_p(x)}$, where $v_p(x)$ is the p-adic *valuation* of x (that is, a power of p such that $x = p^{v_p(x)}\frac{a}{b}$ but $p \not| ab$). The *ring of integers* of $\mathbf{Q}_p$, denoted $\mathbf{Z}_p$, is the set of all elements of non-negative valuation. The ring $\mathbf{Z}_p$ has a unique maximal ideal, generated by the prime p; the *residue class field* K is the quotient $\mathbf{Z}_p/p\mathbf{Z}_p$. Any element x of $\mathbf{Q}_p$ can be expressed as a power series in the prime p, so that $x = \sum_{i=v}^{\infty} a_i p^i$, where v is the valuation of x, a_v is non-zero, and each a_i is a lift of an element from the residue class field. In more general terms, $\mathbf{Q}_p$ is a local field, with its ring of integers $\mathbf{Z}_p$ being a local ring. A uniformizer π of $\mathbf{Q}_p$ is the prime p.

More generally, consider an irreducible polynomial over some local field L_1 (such as $\mathbf{Q}_p$). Then the extension given by adjoining a root α of this polynomial to L_1, $L_2 = L_1[\alpha]$, is also a local field. Let π_1 and π_2 be uniformizers of L_1 and L_2, respectively. Then $\pi_2^e = \pi_1 u$, where u is a unit of L_2. The number e is the *ramification degree* of L_2 over L_1, and divides the degree n of the extension. If $e = n$, we say L_2 is *totally ramified* over L_1; if $e = 1$, we say L_2 is *unramified* over L_1. The degree of the residue class field K_2 of L_2 over the residue class field K_1 of L_1 is $f = \frac{n}{e}$. Finite extensions in Magma must be either *unramified* or *totally ramified*; Magma also allows towers of extensions to be built.

It is well known that up to isomorphism there is only one degree n unramified extension of L, which can be obtained by adjoining a $p^n - 1$-th root of unity ζ to L. This extension is Galois, with Galois group isomorphic to the cyclic group of order n. The Galois group is generated by the Frobenius automorphism σ, which takes ζ to ζ^p. For some applications, it is necessary to have a fast Frobenius action, so Magma supports such a representation. However, the defining polynomial in this representation is particularly dense and can be expensive to construct, hence it is ill-suited for general applications. An unramified extension can only be defined by *inertial* polynomials, which are polynomials that are irreducible over the residue class field. Magma allows an unramified extension to be defined by any inertial polynomial.

All totally ramified extensions contain a root of an Eisenstein polynomial of L. An Eisenstein polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ satisfies $v_\pi(a_n) = 0$, $v_\pi(a_i) \geq 1$ for all $0 < i < n$, and $v_\pi(a_0) = 1$. Magma allows a totally ramified extension to be defined by any Eisenstein polynomial; note that this does not allow arbitrary representations of totally ramified extensions to be constructed.

47.3 Overview of the p-adics in Magma

47.3.1 p-adic Rings

Since p-adic rings are completions (like the real numbers), it is difficult to represent their elements in an exact form. As described in Section 47.2, any element x of a local ring L can be expressed as a power series in the uniformizing element of L. Most problems can be solved by computing in a finite quotient of L, which is equivalent to truncating the infinite expansion of x at some point. More precisely, we work in the quotient rings $L/\pi^k L$ for non-negative k, which is called the *precision* of the ring L.

Unfortunately, working with finite approximations to elements does have problems. Some operations on these approximations will yield results with reduced precision. For example, in a ring constructed with precision k the quotient of two elements with valuations $v_1 \geq v_2$ lies in the local ring, but can only be determined up to $k - v_2$ digits. Also, it is only possible in general to construct an approximation to the GCD of two polynomials f and g over a local ring.

Magma offers two models for computing in a local ring L. The first model (the *fixed precision model*) allows the user to work with the quotient rings defined above. These quotient rings are finite structures, and their elements can easily be represented exactly; hence, many exact algorithms can be applied to them. On the other hand, precision management is left to the user, since operations which may lose precision, such as division, will still return results to the full precision of the ring. In the second model (the *free precision model*), the user works more directly with L, instead of a finite quotient of L. Each element is still represented by a finite approximation, however, these approximations can be of varying precisions, thus freeing the user from precision management. However, these rings are inexact (for instance, there is no zero element of the ring, only approximations to it), and hence many algorithms in Magma which are designed for exact rings may cause significant precision loss. The situation is analogous to the support for real numbers in Magma.

47.3.2 p-adic Fields

A local field in Magma is the field of fractions of a local ring. Representing a local field on a machine causes some trouble, since there is no algebraic structure in which truncations of the elements could be interpreted (as in the case of the finite quotients for the local rings). Moreover, all structural information of a local field is already contained in its ring of integers, so that the main motivation for supporting local fields is to provide some basic arithmetic.

An element x of a local field L is stored internally as $x = \pi^v u + O(\pi^{v+k})$, where v is the (possibly negative) valuation of x, and u is a unit known to precision k in the ring of integers of L. It can happen that operations in a field yield elements with no known digits at all, since the unit parts may not overlap.

47.3.3 Free Precision Rings and Fields

Free precision rings and fields can be created with bounded or unbounded precision. If they are created with some bounded precision k, then no element in the structure can have precision greater than k. Thus, bounded free precision rings are "inexact" versions of the fixed precision rings. Unbounded precision structures can have elements with arbitrarily large, but not infinite, precision. In general, it is recommended that free precision structures are utilized unless speed is absolutely critical.

47.3.4 Precision of Extensions

In Magma, all measurements of precision are made with respect to a valuation function v_π which takes the uniformizer π of the local ring L to unity (throughout this chapter, the subscript will be dropped on v_π where there is no ambiguity). This can have surprising results. For instance, consider a local ring L_1 which extends L_2, such that the ramification degree of L_1 over L_2 is e. Let π_1 and π_2 be the uniformizing elements of L_1 and L_2, respectively. Then, $v_{\pi_1}(\pi_1) = 1$, and $v_{\pi_1}(\pi_2) = e$. Hence, an element of precision k in L_2 will have precision ek in when mapped into L_1. Thus, it is important to always remember in which ring precision is determined.

47.4 Creation of Local Rings and Fields

A local ring in Magma can be constructed in two ways: as either a p-adic ring, or as an extension of another local ring. Magma supports the construction of towers of extensions of local rings; the only restriction is that each extension must be either unramified or totally ramified. As discussed in Section 47.2, Magma requires that the defining polynomial is either inertial or Eisenstein.

Additionally, the user must specify whether to construct a fixed precision or free precision structure, and, if necessary, assign a precision to the structure. For local rings the precision is interpreted as an absolute precision, specifying to what precision the element is known, but for local fields it is interpreted as a relative precision, specifying to what precision the unit part of the element is known.

47.4.1 Creation Functions for the p-adics

```
pAdicRing(p, k)
```

```
pAdicField(p, k)
```

Given a prime integer p and non-negative single-precision integer k, construct the bounded free precision ring (field) of p-adic integers with maximum precision k.

```
pAdicRing(p)
```

```
pAdicField(p)
```

`Precision` RngIntElt *Default :* 20

Given a prime integer p, construct the unbounded free precision ring (field) of p-adic numbers. The optional parameter `Precision`, which must be a non-negative single precision integer, controls the default precision to which elements are created, e.g., when coercing precise elements such as integers or rationals into the ring.

```
pAdicQuotientRing(p, k)
```

Given a prime integer p and non-negative single precision integer k, construct the fixed precision quotient ring $\mathbf{Z}_p/p^k\mathbf{Z}_p$.

```
quo< L | x >
```

Given a local ring L, construct the quotient ring L/xL, where x is an element of L.

Example H47E1

The creation of p-adic rings using the above functions is illustrated below.

```
> R := pAdicRing(5);
> R;
5-adic ring
> R`DefaultPrecision;
20
> R!1;
1 + O(5^20)
> R := pAdicRing(5 : Precision := 20);
> R!1;
1 + O(5^20)
> Q := quo<R | 5^20>;
> Q;
Quotient of the 5-adic ring modulo the ideal generated by 5^20
> Q!1;
1
> Q eq pAdicQuotientRing(5, 20);
true
```

47.4.2 Creation of Unramified Extensions

```
UnramifiedExtension(L, n)
```

```
ext< L | n >
```

`Cyclotomic` BoolElt *Default :* `false`

`GNBType` RngIntElt *Default :* 0

Given a local ring or field L and a positive single precision integer n, construct the default unramified extension of L of degree n. If K is the residue class field of L, then the defining polynomial of the default degree n extension of K is lifted to be an inertial polynomial of L; this polynomial is used as the defining polynomial of the extension. If `Cyclotomic` is `true`, then the lift of the defining polynomial will be such that $p^n - 1$-st root of unity will be adjoined to L (this representation makes computation of the Frobenius automorphism particularly efficient). The angle bracket notation can be used to assign a name to the generator of the extension, e.g. `K<t> := UnramifiedExtension(L, n)`. If `Cyclotomic` is `false` but `GNBType` is $t > 0$ then a Gaussian normal basis of type t is used. This allows extremely fast multiple Frobenius computations but multiplication is slower than the usual or `Cyclotomic` representation if $t > 2$. Because of this, currently only 1 or 2 are legal values for t. Only certain extension degrees will have a Gaussian normal basis of type 1 or 2. To determine if this is true, the `HasGNB` functions described below may be used.

```
UnramifiedQuotientRing(K, k)
```

Given a finite field K and a non-negative single precision integer k, construct the fixed precision quotient ring which has residue class field K and precision k. The angle bracket notation can be used to assign a name to the generator of the extension, e.g. `L<t> := UnramifiedExtension(K, f)`.

```
UnramifiedExtension(L, f)
```

```
ext< L | f >
```

Given a local ring or field L and a polynomial f with coefficients coercible to L, construct the unramified extension of L defined by f. The polynomial f must be an inertial polynomial over L. The angle bracket notation can be used to assign a name to the generator of the extension, e.g. `K<t> := UnramifiedExtension(L, f)`. Free precision rings can only be extended by a polynomial if they are of bounded precision, in which case f must be specified to the maximum precision of the ring.

```
IsInertial(f)
```

Given a polynomial f with coefficients over a local ring or field L, return `true` if and only if f is an inertial polynomial. A polynomial is inertial over L if it is irreducible over the residue class field of L.

```
HasGNB(R, n, t)
```

Given a local ring or field, returns `true` iff the unramified extension of degree n can be generated by a Gaussian Normal Basis (GNB) of Type t. A GNB allows particularly fast multiple Frobenius and Norm computations. Multiplication will tend to be slower though, unless $t = 1$.

```
CyclotomicUnramifiedExtension(R, f)
```

```
CyclotomicUnramifiedExtension(R, f)
```

```
CyclotomicUnramifiedExtension(R, f)
```

```
CyclotomicUnramifiedExtension(R, f)
```

Given a local ring of field R, construct the unramified degree f extension by adjoining a $p^f - 1$-th root of unity to R. Functionally equivalent to calling `UnramifiedExtension(R, f:Cyclotomic := true)`.

Example H47E2

The creation of unramified extensions of local rings using the above functions is illustrated below.

```
> R1 := pAdicRing(2, 20);
> R2 := ext<R1 | 5>;
> R2;
Unramified extension defined by the polynomial x^5 + x^2 + 1
 over 2-adic ring mod 2^20
> DefiningPolynomial(R2);
x^5 + x^2 + 1
> R3 := ext<R1 | 5 : Cyclotomic>;
> R3;
Cyclotomic unramified extension of degree 5 over 2-adic ring mod 2^20
> DefiningPolynomial(R3);
x^5 + 426248*x^4 - 14172*x^3 - 147105*x^2 + 293314*x - 1
> R3.1^(2^5-1);
1
> P1<x> := PolynomialRing(R1);
> f1 := x^3 + 3*x + 1;
> IsInertial(f1);
true
> R4 := ext<R1 | f1>;
> R4;
Unramified extension defined by the polynomial x^3 + 3*x + 1
 over 2-adic ring mod 2^20
> P2<y> := PolynomialRing(R2);
> f2 := y^3 + 3*y + 1;
> IsInertial(f2);
true
> ext<R2 | f2>;
Unramified extension defined by the polynomial x^3 + 3*x + 1
```

```
 over Unramified extension defined by the polynomial x^5 + x^2 + 1
 over 2-adic ring mod 2^20
```

47.4.3 Creation of Totally Ramified Extensions

```
TotallyRamifiedExtension(L, f)
```

```
ext< L | f >
```

Given a local ring or field L and a polynomial f with coefficients coercible to L, construct the totally ramified extension of L defined by f. The polynomial f must be an Eisenstein polynomial, that is, the leading coefficient is a unit, the constant coefficient has valuation 1 and all other coefficients have valuation greater than or equal to 1. The angle bracket notation can be used to assign a name to the generator of the extension, e.g. `K<t> := TotallyRamifiedExtension(L, f)`. Free precision rings can only be extended by a polynomial if they are of bounded precision, in which case f must be specified to the maximum precision of the ring.

```
IsEisenstein(f)
```

Given a polynomial f with coefficients over a local ring or field L, return `true` if and only if f is an Eisenstein polynomial over L. An Eisenstein polynomial satisfies the following properties: the leading coefficient is a unit, the constant coefficient has valuation 1 and all other coefficients have valuation greater than or equal to 1.

Example H47E3

The creation of totally ramified extensions of local rings using the above functions is illustrated below.

```
> L1<a> := ext<pAdicRing(5, 20) | 4>;
> L1;
Unramified extension defined by the polynomial x^4 + 4*x^2 + 4*x + 2
 over 5-adic ring mod 5^20
> L2<b> := ext<L1 | x^4 + 125*x^2 + 5>;
> L2;
Totally ramified extension defined by the polynomial x^4 + 125*x^2 + 5
 over Unramified extension defined by the polynomial x^4 + 4*x^2 + 4*x + 2
 over 5-adic ring mod 5^20
> P<y> := PolynomialRing(L2);
> L3<c> := TotallyRamifiedExtension(L2, y^3 + b^4*a^5*y + b*a^2);
> L3;
Totally ramified extension defined by the polynomial x^3 + ((500*a^3 + 500*a^2 +
    250*a)*b^2 + 20*a^3 + 20*a^2 + 10*a)*x + a^2*b
 over Totally ramified extension defined by the polynomial x^4 + 125*x^2 + 5
 over Unramified extension defined by the polynomial x^4 + 4*x^2 + 4*x + 2
 over 5-adic ring mod 5^20
```

If the precision of the base ring is only 1, then it is not possible to construct a ramified extension, as there is not enough precision to allow the constant coefficient to be non-zero to that precision.

```
> R<x> := PolynomialRing(Integers());
> L<a> := UnramifiedExtension(pAdicRing(5, 1), 3);
>  TotallyRamifiedExtension(L, x^4 + 5);
>> TotallyRamifiedExtension(L, x^4 + 5);
                           ^

Runtime error in 'TotallyRamifiedExtension': Polynomial must be Eisenstein
> L<a> := UnramifiedExtension(pAdicRing(5, 2), x^5 + x^2 + 2);
> TotallyRamifiedExtension(L, x^4 + 5);
Totally ramified extension defined by the polynomial x^4 + 5 over Unramified
extension defined by the polynomial x^5 + x^2 + 2 over 5-adic ring mod 5^2
> ext<L | x^4 + 125*x^2 + 5>;
Totally ramified extension defined by the polynomial x^4 + 5 over Unramified
extension defined by the polynomial x^5 + x^2 + 2 over 5-adic ring mod 5^2
```

47.4.4 Creation of Unbounded Precision Extensions

Suppose we have an unbounded precision local ring or field L, and we wish to create a finite extension of it. If we need the default degree n unramified extension, then we can use the construction functions defined in Section 47.4.2 to construct this extension. However, suppose we wish to define the extension by some polynomial f. As there is no upper bound on the precision of elements of L, it is impossible for us to represent the polynomial f sufficiently precisely, and hence we cannot use the creation functions defined in previous sections for this task. To allow such extensions to be created, Magma allows extensions to be defined by a map $\phi : \mathbf{Z}_{\geq 0} \to R[x]$, where R is a ring whose elements are coercible to the quotient rings $L/\pi^k L$ for all $k \in \mathbf{Z}_{\geq 0}$. The map ϕ, given an input precision k, returns the defining polynomial of the extension to precision k. Internally, whenever Magma needs to represent an element of the extension to some precision, it will use ϕ to compute the defining polynomial up to this precision. Magma may call ϕ on any precision between zero and the precision of the most precise element created by the user.

```
ext< L | m >
```

Given a free precision local ring or field L and a map m with domain $\mathbf{Z}$ and codomain $R[x]$, where elements of R are coercible to the quotient rings $L/\pi^k L$ for all $k \in \mathbf{Z}_{\geq 0}$, construct an extension of L defined by m. Given a non-negative single precision integer k, the map m must return the defining polynomial of the extension to precision k, as a polynomial over R. The map m's behaviour for other input values is undefined. Internally, Magma will coerce the value returned by the map m to be a polynomial over $L/\pi^k L$. Examples of suitable codomains R include the integers, rationals, or L itself.

Example H47E4

The creation of extensions of local rings using maps is illustrated below. We show how it is possible to define an extension of a free precision ring using an "exact" polynomial.

```
> R := pAdicRing(2);
> Z := Integers();
> P<x> := PolynomialRing(Z);
> m := map<Z -> P | k :-> x^3 + x + 1>;
> R2 := ext<R | m>;
> R2;
Unramified extension defined by a map over 2-adic ring
> DefiningPolynomial(R2);
(1 + O(2^20))*$.1^3 + O(2^20)*$.1^2 + (1 + O(2^20))*$.1 + 1 + O(2^20)
> R2‘DefaultPrecision := 1000;
> DefiningPolynomial(R2);
(1 + O(2^1000))*$.1^3 + O(2^1000)*$.1^2 + (1 + O(2^1000))*$.1 + 1 + O(2^1000)
```

47.4.5 Creation of Related Rings

`IntegerRing(F)`

`Integers(F)`

`RingOfIntegers(F)`

Given a local field F, construct the ring of integers R of F. The ring R is the set of elements of F of non-negative valuation.

`RingOfIntegers(R)`

Given a ring R, this function simply returns it, it is provided to support generic functionality for finite extensions of rings and fields.

`FieldOfFractions(R)`

Given a local ring R, construct the field of fractions F of R. The relative precision of F is equal to the precision of R.

`SplittingField(f, R)`

Given a polynomial f over the integers and a p-adic ring R, compute an extension S over R such that f splits into linear factors over S. The algorithms uses the R4-methods as developed by Pauli ([Pau01]).

`AbsoluteTotallyRamifiedExtension(R)`

Given a tower of ramified extensions over some unramified ring S, compute a more efficient representation of R, ie. an extension of S that is totally ramified and defined by a single Eisenstein polynomial. The map returned allows to convert between the new and old representations.

47.4.6 Other Elementary Constructions

```
Composite(R, S)
```

For two p-adic fields that are normal over $\mathbf{Q}_p$, compute the compositum of R and S, ie. the smallest field containing both R and S.

47.4.7 Attributes of Local Rings and Fields

```
L`DefaultPrecision
```

Used to retrieve or set the default precision of the local ring or field L. This attribute is only relevant if L is an unbounded free precision ring, in which case this will change the precision with which elements are created by default. For bounded precision structures, the default precision of the ring is equal to the upper bound on precision; attempting to set this attribute will result in an error in this case.

47.5 Elementary Invariants

These functions return some simple information partially defining a local ring.

```
Prime(L)
```

Given a local ring or field L, return the prime p defining the p-adic ring or field underlying L. This is also the characteristic of the residue class field of L.

```
InertiaDegree(L)
```

Return the inertia degree of the local ring or field L over its coefficient ring.

```
InertiaDegree(K, L)
```

Return the inertia degree of the local ring or field K relative to its subring L.

```
AbsoluteInertiaDegree(L)
```

Return the inertia degree of the local ring or field L over the p-adic ring.

```
RamificationDegree(L)
RamificationIndex(L)
```

Return the ramification degree of the local ring or field L over its coefficient ring.

```
RamificationDegree(K, L)
RamificationIndex(K, L)
```

Return the ramification degree of the local ring or field K relative to its subring L.

```
AbsoluteRamificationDegree(L)
AbsoluteRamificationIndex(L)
```

Return the ramification degree of the local ring or field L over the p-adic ring.

`AbsoluteDegree(L)`

The degree of L over $\mathbf{Z}_p$.

`Degree(L)`

Return the degree of the local ring or field L over its coefficient ring.

`Degree(K, L)`

Return the degree of the local ring or field K relative to its subring L.

`DefiningPolynomial(L)`

Return the minimal polynomial of the generator of L over its coefficient ring. If L is p-adic, the polynomial $x - 1$ is returned. For free precision rings and fields, the coefficients of the defining polynomial are given to the default precision of L.

`DefiningPolynomial(K, L)`

Return the minimal polynomial of the generator of K over its coefficient ring L.

`DefiningMap(L)`

Given a free precision local ring or field L, return the map that was used to define the extension (see Section 47.4.4 for information on defining extension by maps). If a map was not used, then an error is raised.

`HasDefiningMap(L)`

Given a free precision local ring or field L, return `true` if L is defined by a map; if so, the defining map is also returned.

`PrimeRing(L)`

`PrimeField(L)`

`pAdicRing(L)`

`pAdicField(L)`

Given a local ring or field L, return the p-adic ring or field which is a subring of L.

`BaseRing(L)`

`CoefficientRing(L)`

`BaseField(L)`

`CoefficientField(L)`

`BaseRing(L)`

Given a local ring or field L, return the base ring of L.

`ResidueClassField(L)`

Given a local ring or field L, return the residue class field K of L, and a map from L to K.

`ResidueSystem(R)`

Given a p-adic ring or field R, compute a set of representatives of the residue class field of R as elements of R.

`UniformizingElement(L)`

Given a local ring or field L, return the uniformizing element of L.

`L . 1`

Given a local ring or field L, return an element α of L such that if K is L's base ring or field, then the powers of α give a basis of L as a vector space over K.

`Precision(L)`

Given a local ring or field L, return the precision with which L has been created. If L is a local ring this is the maximum absolute precision to which its elements can be created. If L is a local field this is the maximum relative precision to which its elements can be created. If L is an unbounded free precision ring or field, then infinity is returned.

`HasPRoot(R)`

Given a local ring R extending $\mathbf{Z}_p$ for some prime p, decide if R contains a primitive p-th root of unity.

`HasRootOfUnity(L, n)`

Given a local ring L and some positive integer n, decide if L contains a primitive nth root of unity.

`Discriminant(R)`

Compute the discriminant of the local ring R over its coefficient ring. Since R is defined by either an inertial polynomial or an Eisenstein one, this is equivalent to computing the discriminant of the defining polynomial.

`Discriminant(K, k)`

Given p-adic rings K/k, compute the discriminant of K as an extension of k.

`AdditiveGroup(R)`

The additive group of the p-adic quotient R as an abelian group and the isomorphism from this group back to R.

Example H47E5

We illustrate the functions in this section for rings. Similar constructions can be used for fields.

```
> Zp := pAdicRing(5, 20);
> I<a> := UnramifiedExtension(Zp, 3);
> R<x> := PolynomialRing(I);
> L<b> := ext<I | x^3 + 5*a*x^2 + 5>;
> Prime(L);
5
> InertiaDegree(L);
1
```

The inertia degree of L is returned as 1 because L has been defined as a totally ramified extension of I. However, the inertia degree of L over Z_p is 3, because I itself is an unramified extension of Z_p.

```
> InertiaDegree(L, Zp);
3
> Degree(L);
3
> Degree(L, Zp);
9
> DefiningPolynomial(L);
x^3 + 5*a*x^2 + 5
> P<y> := PolynomialRing(Zp);
> DefiningPolynomial(I);
y^3 + 3*y + 3
> BaseRing(L);
Unramified extension defined by the polynomial x^3 + 3*x + 3
 over 5-adic ring mod 5^20
> PrimeRing(L);
5-adic ring mod 5^20
> PrimeRing(I);
5-adic ring mod 5^20
> ResidueClassField(L);
Finite field of size 5^3
Mapping from: RngPad: L to GF(5^3)
> ResidueClassField(I);
Finite field of size 5^3
Mapping from: RngPad: I to GF(5^3)
```

Here, we see that the residue class fields of I and L are identical. This is due to the fact that L is a totally ramified extension of I.

```
> UniformizingElement(L);
b
> Precision(L);
60
> Precision(I);
20
```

```
> R<a> := ext<pAdicRing(2) | 2>;
> DefiningPolynomial(R);
(1 + O(2^20))*$.1^2 + (1 + O(2^20))*$.1 + 1 + O(2^20)
> Precision(R);
Infinity
```

`AbsoluteRootNumber(K)`

The root number of a p-adic extension $K/\mathbf{Q}_p$ can be computed as in §3.3 of [JR06], with the result returned (using their convention and not Deligne's) as an element of $\mathbf{Q}(\zeta_4)$.

`RootNumber(K)`

The root number of a relative p-adic extension can be computed by using `AbsoluteRootNumber` and an induction relation.

Example H47E6

We compute the root numbers of some p-adic extensions.

```
> K := ext<pAdicField(2,20)|Polynomial([10,0,1])>; K;
Totally ramified extension defined by the polynomial x^2 + 10
 over 2-adic field mod 2^20
> RootNumber(K);
-zeta_4
> K := pAdicField(3,40);
> A := AllExtensions(K,3);
> _<x> := PolynomialRing(Integers(K)); // printing
> [<RootNumber(FieldOfFractions(a)),DefiningPolynomial(a)> : a in A];
[
    <zeta_4, x^3 + 3*x + 3>,
    <-zeta_4, x^3 + 6*x + 3>,
    <1, x^3 + 3*x^2 + 3>,
    <1, x^3 + 6*x^2 + 3>,
    <1, x^3 + 6*x^2 + 12>,
    <1, x^3 + 6*x^2 + 21>,
    <zeta_4, x^3 + 3>,
    <zeta_4, x^3 + 12>,
    <zeta_4, x^3 + 21>,
    <1, x^3 + 2*x + 1>
]
```

We take an example where the relative and absolute root numbers differ.

```
> K := pAdicField(3,20);
> L := ext<K|Polynomial([3,0,1])>; // both ramified
> M := ext<L|Polynomial([L.1,0,1])>; // both ramified
> RootNumber(M); // relative for M/L
```

```
zeta_4
> AbsoluteRootNumber(M); // absolute for M/Q3
-zeta_4
> RootNumber(L); // intermediate field
zeta_4
> X := AbsoluteTotallyRamifiedExtension(Integers(M));
> RootNumber(FieldOfFractions(X));
-zeta_4
```

47.6 Operations on Structures

`AssignNames(~L, S)`

`AssignNames(~L, S)`

Assign a name to the generator of L. The sequence must have only one element, which must be a string. This element is assigned to be the name of the generator when L is considered as a linear associative algebra over its base ring.

`Characteristic(L)`

`Characteristic(L)`

`Characteristic(L)`

The characteristic of the local ring or field L.

`#L`

The number of elements in the local ring or field L. The cardinality is finite only if L is a quotient ring or a bounded free precision ring.

Iterating over the elements of a local ring is possible if it is bounded, but it will take time in proportion to the cardinality of L. It is recommended only in the case of “small” local rings (i.e., rings for which the precision is be very small).

`Name(L, k)`

Given a local ring or field L and an integer k, return the generator of L if k is 1; otherwise, raise an error.

`ChangePrecision(L, k)`

`ChangePrecision(~L, k)`

`ChangePrecision(L, k)`

`ChangePrecision(~L, k)`

`ChangePrecision(L, k)`

`ChangePrecision(~L, k)`

`ChangePrecision(L, k)`

`ChangePrecision(~L, k)`

`ChangePrecision(L, k)`

`ChangePrecision(~L, k)`

`ChangePrecision(L, k)`

`ChangePrecision(~L, k)`

Given a local ring or field L and a non-negative single precision integer k, change the maximum precision with which elements can be created to be k. Depending on how L and its subrings have been constructed, there may be an upper bound (possibly infinite for free structures) on the precision to which L can be changed. For instance, the precision to which a defining polynomial has been given places a bound on the precision of the extension — no defining polynomial can be expanded beyond the precision with which it was originally specified.

`L eq K`

Given local rings or fields L and K, return whether or not L and K are the same object.

`L ne K`

Given local rings or fields L and K, return whether or not L and K are different objects.

Example H47E7

```
> Zp := pAdicRing(5, 20);
> I<a> := UnramifiedExtension(Zp, 3);
> R<x> := PolynomialRing(I);
> L<b> := ext<I | x^3 + 5*a*x^2 + 5>;
> ChangePrecision(Zp, Infinity());
5-adic ring
> L;
Totally ramified extension defined by the polynomial x^3 + 5*a*x^2 + 5
 over Unramified extension defined by the polynomial x^3 + 3*x + 3
 over 5-adic ring mod 5^20
```

```
> ChangePrecision(~L, 50);
> L;
Totally ramified extension defined by the polynomial x^3 + 5*$.1*x^2 + 5
 over Unramified extension defined by the polynomial x^3 + 3*x + 3
 over 5-adic ring mod 5^17
> #L;
87581154020301066932733098955619758205O1\
6371367282235734816767743111942667866287\
592914886772632598876953125
> AssignNames(~L, ["t"]);
> L.1;
t
> b;
b
> L eq ChangePrecision(L, 10);
false
```

Note that b is an element of the original ring L with precision 60 which is why it retains its print name.

47.6.1 Ramification Predicates

`IsRamified(R)`

`IsUnramified(R)`

`IsTotallyRamified(R)`

Return whether the local ring or field extension R is ramified, unramified or totally ramified.

`IsTamelyRamified(R)`

`IsWildlyRamified(R)`

Return whether the local ring or field extension R is tamely ramified (the prime does not divide the ramification degee) or wildly ramified (the prime does divide the ramification degree).

47.7 Element Constructions and Conversions

Local ring elements are implemented using a balanced mod representation. This allows small negative elements x to be represented as x rather than $p^k - x$ where k is a p-adic precision.

47.7.1 Constructions

To simplify the creation of elements in a local ring, various coercions are provided. The most obvious is to regard the integer ring or rational field as embedded in the p-adic ring or field. But there is a range of coercions available, including that of elements from the residue class field.

To create an element of a local field not lying in the local ring, constructors are provided that create an element coercible into the ring, and which increase or decrease this element's valuation in the field.

```
Zero(L)
```

Given a local ring or field L, create the additive identity of L. Note that if L is an unbounded precision ring, this will only be the zero element to the default precision of the ring, and hence only an approximation to the additive identity of L. The valuation and hence the absolute precision of the zero of a field is infinite.

```
One(L)
```

Given a local ring or field L, create the multiplicative identity of L. Note that if L is an unbounded precision structure, this will only be the one element to the default precision of the ring, and hence only an approximation to the multiplicative identity of L.

```
Random(L)
```

Given a local ring or field L, return a random element of L, which must be a quotient ring or bounded precision ring. The element will have the default precision of the ring.

```
Representative(L)
```

Return an element of the local ring or field L.

```
elt< L | u >
L ! u
```

Coerce the object u into the local ring or field L. The resulting element will have as much precision as possible. The element u is allowed to be one of the following:

(i) An integer.

(ii) An element of $\mathbf{Z}/p^m\mathbf{Z}$; (where m is a precision).

(iii) An element of the residue class field of L.

(iv) An element of a local ring or field with something in common with L.

(v) A rational number. If L is a ring then u must not have valuation in the denominator.

(vi) An element of a valuation ring over the rationals with the same prime as L.

(vii) A sequence s. In this case, the sequence s is coerced to a sequence t over the base ring or field of L. This sequence is coerced to $\sum_{i=1}^{\#t} t[i]L.1^{i-1}$.

`elt< L | u, r >`

Create an element of the local ring or field L by coercing u into L and returning with it precision r.

`elt< L | v, u, r >`

Create an element of the local ring or field L by coercing u into L, multiplying by the v-th power of the uniformizing element and returning it with precision r.

`BigO(x)`

`O(x)`

For an element x of a local ring or field L of valuation v, create an element of valuation v and relative precision 0. For rings this is the zero element in the quotient $L/\pi^v L$.

`UniformizingElement(L)`

Given a local ring or field L, return the uniformizing element of L to the default precision of L.

Example H47E8

Here we illustrate the usage of element constructors for local fields and imprecise zeros.

```
> Zp := pAdicRing(5, 20);
> I<a> := UnramifiedExtension(Zp, 3);
> R<x> := PolynomialRing(I);
> L<b> := ext<I | x^3 + 5*a*x^2 + 5>;
> K<pi> := ext<BaseField(FieldOfFractions(L)) | x^2 + 5>;
> K;
Totally ramified extension defined by the polynomial x^2 + 5
 over Unramified extension defined by the polynomial x^3 + 3*x + 3
 over 5-adic field mod 5^20
> elt<K | 64>;
64 + O(pi^40)
> P := PrimeField(K);
> elt<K | 2, 3/4, 6>;
pi^2*7 + O(pi^6)
> 4*$1;
pi^2*3 + O(pi^6)
> K!3/5;
pi^-2*3 + O(pi^38)
```

```
> O(K!40^200);
O(pi^400)
> Precision($1);
0
> O(K!0);
O(pi^40)
> O(K!1);
O(1)
> R<x> := PolynomialRing(Integers());
> K<pi> := ext<pAdicField(5, 100) | x^2 + 5>;
> elt<K | 64>;
64 + O(pi^200)
> Precision($1);
200
> K!3/10;
-pi^-2*394430452610505902705864282641393114836603217554511502385139465332031 1 +
O(pi^198)
```

Example H47E9

There is a subtle interplay between the default precision of rings and fields and coercion of sequences, which we demonstrate here. We construct the 5-adic field P, and an unramified extension of R. We set the default precision of R to be higher than that of P.

```
> P := pAdicRing(5);
> R := ext<P | 2>;
> P‘DefaultPrecision;
20
> R‘DefaultPrecision := 40;
> x := Random(R);
> x;
-1579801843431963201369145587*R.1 - 6805757304589750390333394769 + O(5^40)
> s := [-6805757304589750390333394769, -1579801843431963201369145587];
> R!s;
-13585890629962*R.1 + 47482939261481 + O(5^20)
> P‘DefaultPrecision := 40;
> R!s;
-1579801843431963201369145587*R.1 - 6805757304589750390333394769 + O(5^40)
```

As can be seen from the above, it is the default precision of the base ring, not the ring itself, which determines the precision of elements created by sequences.

47.7.2 Element Decomposers

`ElementToSequence(x)`

`Eltseq(x)`

`Coefficients(x)`

Given an element x of a degree n extension K of L, these functions return a sequence s of elements of L such that $x = \sum_{i=1}^{n} s[i]K.1^{i-1}$.

`Coefficient(x, i)`

Equivalent to, but more efficient than, `Coefficients(x)[i]`.

Example H47E10

We want to perform a Galois descent for a polynomial, i.e. we interpret the product of the Galois conjugates of a polynomial in a subring.

```
> p := 3;
> L<a> := UnramifiedExtension(pAdicRing(p), 4 : Cyclotomic);
> R<x> := PolynomialRing(L);
> g := x^2 + (a+a^-2)*x + (a^-1+a^3+1);
> g;
(1 + O(3^20))*x^2 + (-151999392*a^3 - 428033534*a^2 + 1509587217*a - 64512399 +
    O(3^20))*x + 2*a^3 + 307453058*a^2 - 1732356354*a - 151999391 + O(3^20)
> a2 := a^(p^2);
> g2 := R ! [ &+[Eltseq(c)[i]*a2^(i-1) : i in [1..4]] : c in Eltseq(g) ];
```

Here `Eltseq` is being used to replace occurrences of a in the element by a_2.

```
> h := g * g2;
> h;
(1 + O(3^20))*x^4 + (774759822*a^3 - 1559939781*a^2 + 644136111*a + 143311364 +
    O(3^20))*x^3 + (73899384*a^3 - 333497478*a^2 + 1655979363*a + 158024680 +
    O(3^20))*x^2 + (989668189*a^3 - 661853000*a^2 - 1685887308*a + 122035547 +
    O(3^20))*x - 1630826887*a^3 - 328410694*a^2 - 175290219*a + 1601599448 +
    O(3^20)
```

The polynomial g_2 is the image of g under the automorphism induced by the square of the Frobenius automorphism. The product $h = gg_2$ has coefficients in the unramified extension of degree 2, which is the fixed field under the square of the Frobenius and is generated by a^{10}. Next, we determine a representation for the polynomial h with coefficients lying in this unramified extension of degree 2.

```
> K<b> := UnramifiedExtension(pAdicRing(p), 2 : Cyclotomic);
> S<y> := PolynomialRing(K);
> M := RMatrixSpace(PrimeRing(L), 2, 4) ! 0;
> V := RSpace(PrimeRing(L), 4);
> M[1] := V ! Eltseq(a^0);
> M[2] := V ! Eltseq(a^10);
> sol := [ Solution(M, V ! Eltseq(c)) : c in Eltseq(h) ];
> h2 := S ! [ K ! Eltseq(s) : s in sol ];
> h2;
(1 + O(3^20))*y^4 + (-1237301755*b + 505889265 + O(3^20))*y^3 + (-542504216*b +
    1258167831 + O(3^20))*y^2 + (-280210015*b - 1205051127 + O(3^20))*y +
    1213139407*b + 1602993886 + O(3^20)
```

47.8 Operations on Elements

47.8.1 Arithmetic

For local ring elements the usual operations are available. The quotient of two elements can be computed when the result lies in the ring (i.e. the valuation of the dividend is not smaller than that of the divisor). The precision of the result is reduced by the valuation of the divisor. The result of an operation with elements of reduced precision will have as much precision as possible. For addition and subtraction this is the minimum of the precisions of the two elements. For multiplication it is the minimum of $v_1 + k_2$ and $v_2 + k_1$, where v_i and k_i are the valuations and precisions of the two elements, respectively.

For local field elements the operations are performed with the maximum precision possible. For multiplication and division this is the minimum of the (relative) precisions of the two elements. For addition and subtraction of elements x and y with valuations v_x and v_y and precisions k_x and k_y the precision of $x \pm y$ is $\min(v_x + k_x, v_y + k_y) - v_p(x \pm y)$, which may even be 0.

`-x`

The negative of the element x.

`x + y`

The sum of the elements x and y.

`x - y`

The difference of the elements x and y.

`x * y`

The product of the elements x and y.

`x ^ k`

The k-th power of the element x. If k has valuation (when coerced) and x has precision less than that of its parent ring, the power x^k will have more precision than x.

`x div y`

The quotient of the elements x and y. For elements of a local ring, this results in an error if the valuation of x is smaller than that of y.

```
x div:= y
```

Mutation operation: replace the element x by its quotient upon division by y. For elements of a local ring, this results in an error if the valuation of x is smaller than that of y.

```
x / y
```

The quotient of the elements x and y. For elements of a local ring, the result will be returned in its field of fractions.

```
IsExactlyDivisible(x, y)
```

Return `true` if x can be exactly divided by y; in this case also return the quotient.

Example H47E11

Division of non-units in the ring will yield a result in the field of fractions. In that case the result has reduced precision. To ensure that the result is returned as a ring element, the operator `div` should be used.

```
> R := pAdicRing(2);
> pi := UniformizingElement(R);
> pi;
2 + O(2^20)
> 1 / pi;
2^-1 + O(2^18)
> Parent($1);
2-adic field
>  1 div pi;
>> 1 div pi;
     ^

Runtime error in 'div': Division is not exact
> IsExactlyDivisible(1, pi);
false
> IsExactlyDivisible(pi^2, pi);
true 2 + O(2^20)
```

47.8.2 Equality and Membership

It is possible to test whether two local ring or field elements are equal only in quotient rings. Equality testing in free precision rings is disabled, as there are several reasonable definitions of equality in an imprecise ring.

```
x eq y
```

Given local ring or field elements x and y, return `true` if and only if x and y are identical in value. This operator is only defined in fixed precision rings.

`x ne y`

Given local ring or field elements x and y, return `true` if and only if x and y are not identical in value. This operator is only defined in fixed precision rings.

`x in L`

Return `true` if and only if x lies in the local ring or field L.

`x notin L`

Return `true` if and only if x does not lie in the local ring or field L.

Example H47E12

We demonstrate how an unramified extension can be constructed from a given degree. The idea is to interpret the minimal polynomial of a primitive element in the residue class field as a polynomial over $\mathbf{Z}_p$. The quotient of $\mathbf{Z}_p[x]$ by the ideal generated by this polynomial is isomorphic to the unramified extension and $a := \overline{x}$ is a first approximation for the $p^f - 1$-th root of unity. This is improved by iterating $a \mapsto a^{p^f}$ until this remains fixed.

```
> p := 2;
> f := 5;
> Zp := pAdicRing(p, 25);
> R<x> := PolynomialRing(Zp);
> g := R ! MinimalPolynomial(GF(p,f).1);
> Q<r> := quo<R | g>;
> a := [ r, r^(p^f) ];
> while a[#a] ne a[#a-1] do
>     print a[#a];
>     Append(~a, a[#a]^(p^f));
> end while;
34*r^4 - 44*r^3 + 58*r^2 - 23*r + 36
12522914*r^4 + 12522004*r^3 - 12174790*r^2 - 8200343*r - 10407260
8242594*r^4 + 12409364*r^3 + 5143098*r^2 + 15781737*r - 3636572
5490082*r^4 + 8804884*r^3 - 11109830*r^2 + 11456361*r + 11698852
-15481438*r^4 - 5875180*r^3 + 5667386*r^2 + 7262057*r - 884060
> [ Minimum([ Valuation(c) : c in Eltseq(a[i] - a[i-1]) ]) : i in [2..#a-1] ];
[ 1, 6, 11, 16, 21 ]
```

The last statement demonstrates the convergence of the process. The polynomial defining the unramified extension could now easily be obtained as the minimal polynomial of the fixed element.

```
> U := ext<Zp | f>;
> MinimalPolynomial(U ! Eltseq(a[#a]));
x^5 - 13205240*x^4 - 3159900*x^3 - 13778593*x^2 + 9730498*x - 1
```

47.8.3 Properties

Local ring and field elements can be tested for certain properties. However, note that in free precision rings and fields, exact zero, one, and minus one elements do not exist, only approximations to such elements — in these cases, the predicates will always return `false`.

`IsZero(x)`

Given an element x of a local ring or field, return `true` if and only if x is the zero element of its parent ring or field.

`IsOne(x)`

Given an element x of a local ring or field, return `true` if and only if x is the one element of its parent ring or field.

`IsMinusOne(x)`

Given an element x of a local ring or field, return `true` if and only if x is the minus one element of its parent ring or field.

`IsUnit(x)`

Given an element x of a local ring or field, return `true` if and only if x is a unit, that is, x has valuation 0.

`IsIntegral(x)`

Given an element x of a local ring or field, return `true` if and only if x has non-negative valuation.

47.8.4 Precision and Valuation

Note that the precision of an element may be modified.

`Parent(x)`

Return the parent local ring or field of the element x.

`Precision(x)`

For a local ring element x, returns the precision to which it is known; for a local field element x, returns the precision to which its unit part is known.

`AbsolutePrecision(x)`

Returns the precision of a local ring or field element x.

`RelativePrecision(x)`

Returns the difference between the absolute precision and the valuation of the local ring or field element x.

`ChangePrecision(x, k)`

`ChangePrecision(~x, k)`

`ChangePrecision(x, k)`

`ChangePrecision(~x, k)`

Given a local ring or field element or polynomial over a local ring or field x and a non-negative single precision integer k, change the precision of x to k. If k is smaller than the precision of x, x is truncated; if k is larger, then an element y is returned such that $v(x - y) \geq k$. Note that the precision of an element cannot be changed to be greater than its parent. Any attempt to do so will result in the element gaining the precision of its parent.

`Expand(x)`

Change the precision of the local ring or field element x to be the maximum precision allowed by its parent. This results in an error if x is an element of an unbounded free precision ring.

`Valuation(x)`

Return the valuation of the local ring or field element x. This is always bounded by the absolute precision of the element.

Example H47E13

The uses and properties of relative and absolute precision for field elements are illustrated here, as well as the results of `Expand`.

```
> R<x> := PolynomialRing(Integers());
> K<d> := ext<ext<pAdicField(5, 100) | 2> |  x^2 + 5>;
> x := d + d^7;
> x;
-d*124 + O(d^200)
> AbsolutePrecision(x);
200
> RelativePrecision(x);
199
> RelativePrecision(x + O(d^10));
9
> AbsolutePrecision(x + O(d^10));
10
> RelativePrecision(ChangePrecision(x, 19));
19
> RelativePrecision(ChangePrecision(x, 10));
10
> AbsolutePrecision(ChangePrecision(x, 10));
11
> Expand(ChangePrecision(x, 10));
```

```
d*3001 + O(d^201)
> Valuation(x);
1
> ChangePrecision(x, 20) - (x + O(d^21));
O(d^21)
```

The last two lines show that changing the precision of an element is equivalent to adding imprecision to the element effectively cancelling off all terms beyond that of relative valuation 20.

47.8.5 Logarithms and Exponentials

`Log(x)`

The logarithm of the local ring or field element x, returned to the precision of x. Note that the power series of the logarithm function only converges if the valuation of $x-1$ is positive. For ring elements x, the answer lies in the ring (and not its field of fractions) only if the valuation of $x-1$ is greater than or equal to the ramification degree of the ring divided by the prime.

The rate of convergence of `Log` is dependent on the valuation of $x-1$. The greater the valuation the faster the convergence, as is illustrated in the example below.

`Exp(x)`

The exponential of the local ring or field element x, returned to the precision of x. Note that the power series of the exponential function only converges if the valuation of x is strictly larger than $e/(p-1)$, where e is the ramification degree of the parent ring of x.

Example H47E14

This example illustrates the relative timings of the `Log` function and the `Exp` function for rings and fields and for various valuations of $x-1$ or x as appropriate.

```
> K := ext<pAdicField(3, 20) | x^3 + x^2 + x + 2>;
> K<d> := ext<K | x^3 + 3*x^2 + 3*x + 3>;
> L<b> := IntegerRing(K);
> x := 1 + b;
> time Log(x);
Time: 0.070
> x := 1 + b^5;
> time Log(x);
874050819*b^2 + 1624571442*b - 914550768
Time: 0.010
> Valuation(Exp(Log(x)) - x);
60
> x := b^2;
> time Exp(x);
```

```
1418565628*b^2 + 1334033745*b + 905945461
Time: 0.170
> Valuation(Log(Exp(x)) - x);
60
> x := b^6;
> time Exp(x);
1700312013*b^2 - 72781965*b + 1129064707
Time: 0.020
> Valuation(Log(Exp(x)) - x);
60
```

47.8.6 Norm and Trace

`Norm(x)`

Given a local ring or field element x, return the norm of x over the base ring of its parent.

`Norm(x, R)`

Given a local ring or field element x, return the norm of x over the local ring or field R. The ring R must be a subring of the parent of x.

`Trace(x)`

Given a local ring or field element x, return the trace of x over the base ring of its parent. The trace is the product of all conjugates of x.

`Trace(x, R)`

Given a local ring or field element x, return the trace of x over the local ring or field R. The ring R must be a subring of the parent of x. The trace is the sum of all conjugates of x.

`MinimalPolynomial(x)`

Given a local ring or field element x, return the minimal polynomial of x over the base ring of its parent.

`MinimalPolynomial(x, R)`

Given a local ring or field element x, return the minimal polynomial of x over the local ring or field R. The ring R must be a subring of the parent of x.

`CharacteristicPolynomial(x)`

Given a local ring element x, return the characteristic polynomial of x over the base ring of its parent.

```
CharacteristicPolynomial(x, R)
```

Given a local ring or field element x, return the characteristic polynomial of x over the local ring or field R. The ring R must be a subring of the parent of x.

```
GaloisImage(x, i)
```

Given a local ring or field element x, return the image of x under the Frobenius automorphism composed with itself i times, that is, $a \mapsto a^{(p^i)}$ where a is a $p^f - 1$st root of unity of the parent of x.

Example H47E15

One important application of the p-adics is to counting points on elliptic curves over finite fields. This example demonstrates how a simple version of the Arithmetic-Geometric Mean (AGM) algorithm could be implemented in Magma.

```
> d := 50;
> FF := GF(2^d);
> E := EllipticCurve([FF | 1, 0, 0, 0, Random(FF)]);
> assert Degree(sub<BaseRing(E) | jInvariant(E)>) gt 2;
>
> n := (d + 1) div 2 + 3;
> R := ext<pAdicRing(2) | d>;
> R`DefaultPrecision := n;
>
> a6 := elt<R | jInvariant(E)^-1, 1>;
> lambda := 1 + 8 * a6;
>
> for k in [4..n] do
>     ChangePrecision(~lambda, k + 2);
>     lambda := (1 + lambda) * InverseSqrt(lambda) div 2;
> end for;
> lambda := 2 * lambda div (1 + lambda);
> Exp(Trace(Log(lambda)));
32196193 + O(2^26)
> Trace(E) mod 2^26;
32196193
```

```
EuclideanNorm(x)
```

Return the euclidean norm of the element x of a fixed precision local ring.

47.8.7 Teichmüller Lifts

Another important operation is the determination of the canonical *Teichmüller lift* of an element u in the residue field of a p-adic ring. By definition this is the unique root of unity of order prime to p which reduces to u modulo the maximal ideal.

```
TeichmuellerLift(u, R)
```

For a fixed precision quotient ring R of the p-adics or an unramified extension and a finite field element u, the function attempts to coerce u into the residue class field of R and then computes and returns its Teichmüller lift to R. This uses a fast iterative lifting method of Harley described in Chapter 12 of [C$^+$05]. For unramified extensions this works most efficiently with `Cyclotomic` or `Gaussian Normal` bases since it uses the Frobenius operation.

47.9 Linear Algebra

All linear algebra over local rings and fields in MAGMA is implemented in terms of the fixed precision model. Internally, linear algebra over free precision models is performed over an appropriate quotient ring. The advantage of this method is that exact algorithms can be used to solve linear systems; the disadvantage is that the answer may be returned to less precision than is theoretically possible.

47.10 Roots of Elements

Roots of local ring and field elements can be found to some precision.

```
SquareRoot(x)
```

```
Sqrt(x)
```

Given a local ring or field element x, return a square root of x. An error results if x is not a square. The result may have less precision than x.

```
IsSquare(x)
```

Return whether the local ring or field element x is the square of an element in its parent and if it is, the square root is returned. The result may have less precision than x.

```
InverseSquareRoot(x)
```

```
InverseSqrt(x)
```

Given a local ring or field element x, return an inverse square root of x. The element x must be a unit. An error results if x is not a square. The result may have less precision than x.

```
InverseSquareRoot(x, y)
```

```
InverseSqrt(x, y)
```

Given local ring or field elements x and y, return an inverse square root of x lifted from an initial approximation y. The element x must be a unit. An error results if x is not a square, or if y is not a valid initial approximation to an inverse square root of x. The result may have less precision than x.

```
Root(x, n)
```

Return an n-th root of x if one exists. An error results if x is not an n-th power. The result may have less precision than x.

```
IsPower(x, n)
```

Return whether x is an n-th power of some element belonging to its parent and if it is return an n-th root. The result may have less precision than x.

```
InverseRoot(x, n)
```

Given a local ring or field element x, return an inverse n-th root of x. The element x must be a unit. An error results if x is not an n-th power. The result may have less precision than x.

```
InverseRoot(x, y, n)
```

Given local ring or field elements x and y, return an inverse n-th root of x lifted from an initial approximation y. The element x must be a unit. An error results if x is not an n-th power, or if y is not a valid initial approximation to an inverse n-th root of x. The result may have less precision than x.

47.11 Polynomials

Various simple operations for polynomials as well as root finding and factorization functions have been implemented for polynomials over local rings and fields.

47.11.1 Operations for Polynomials

A number of functions for polynomials in general are applicable for polynomials over local rings and fields. Arithmetic functions, including `div` and `mod` can be used with such polynomials (though there may be some precision loss), as well as all the elementary functions to access coefficients and so forth. Derivatives can be taken and polynomials over local rings and fields can be evaluated at elements coercible into the coefficient ring. Along with GCD for these polynomials, the LCM of two polynomials can also be found.

Although the ring of polynomials over a local ring is not a principal ideal domain, it is useful to have a GCD function available. For example, for polynomials which are coprime over the local field, the ideal generated by the two polynomials contains some power of the uniformizing element of the local ring. This power determines whether an approximate factorization can be lifted to a proper factorization.

`GreatestCommonDivisor(f, g)`

`Gcd(f, g)`

`GCD(f, g)`

Determine the greatest common divisor of polynomials f and g where f and g are over a free precision local ring or field. The GCD returned is such that the cofactors of the polynomials will have coefficients in the ring (if the polynomial is not over a field). The process of computing the GCD of two polynomials may result in some inaccuracy. The GCD is computed by echelonizing the Sylvester matrix of f and g.

`f div g` `f mod g` `LeastCommonMultiple(f, g)`

`Coefficient(f, i)` `LeadingCoefficient(f)`

`Derivative(f)` `Evaluate(f, x)`

Example H47E16

This example illustrates the usage and results of the GCD functions. Note the precision loss in the answer.

```
> L := pAdicRing(5, 20);
> R<x> := PolynomialRing(L);
> elts := [Random(L) : i in [1..3]];
> f := (x - elts[1])^3 * (x - elts[2])^2 * (x - elts[3]);
> f;
x^6 + 31722977012336*x^5 - 34568128687249*x^4 +
    4655751767246*x^3 + 11683626356181*x^2 -
    29833674388290*x + 32360011367900
> GCD(f, Derivative(f));
(1 + O(5^18))*x^3 - (934087632277 + O(5^18))*x^2 -
    (89130566204 + O(5^18))*x + 1178670674955 + O(5^18)
> f mod $1;
O(5^18)*x^5 + O(5^18)*x^4 + O(5^18)*x^3 + O(5^18)*x^2 +
    O(5^18)*x + O(5^19)
> (x - elts[1])^2 * (x - elts[2]);
x^3 + 14324701430223*x^2 + 26613750293171*x +
    31696248799955
> ChangePrecision($1, 18);
(1 + O(5^18))*x^3 - (934087632277 + O(5^18))*x^2 -
    (89130566204 + O(5^18))*x + 1178670674955 + O(5^18)
```

`ShiftValuation(f, n)`

Shifts the valuation of each coefficient of f by n, ie. scales the polynomial by π^n.

47.11.2 Roots of Polynomials

47.11.2.1 Hensel Lifting of Roots

Roots of a polynomial f defined over a local ring or field can be found by first determining their valuation (using the Newton polygon), finding a first approximation over the finite field and finally improving this approximation until the Hensel condition $v(f(a)) > 2v(f'(a))$ is satisfied.

`NewtonPolygon(f)`

The Newton polygon of a polynomial $f = \sum_{i=0}^{n} c_i x^i$ (over a local ring or field) is the lower convex hull of the points $(i, v(c_i))$. The slopes of the Newton polygon determine the valuations of the roots of f in a splitting ring and the number of roots with that valuation. The faces of the Newton polygon can be determined using the function `Faces` which returns the faces expressed as the line $mx + ny = c$ which coincides with the face. The function `GradientVector` will return the m and n values from the line so that the valuation (gradient) can be calculated. The function `EndVertices` will return the end points of the face, the x coordinates of which will give the number of roots with valuation equal to the gradient of the face.

Newton polygons are discussed in greater detail in Chapter 54 and are illustrated below.

`ValuationsOfRoots(f)`

Return a sequence containing pairs which are valuations of roots of f and the number of roots of f which have that valuation.

Example H47E17

For a polynomial of the form $g := \prod(x - r_i)$ we demonstrate that the Newton polygon determines the valuations of the roots of g.

```
> Z3 := pAdicRing(3, 30);
> R<y> := PolynomialRing(Z3);
> pi := UniformizingElement(Z3);
> roots := [ pi^Random([0..3]) * Random(Z3) : i in [1..10] ];
> [ Valuation(r) : r in roots ];
[ 3, 1, 6, 3, 0, 3, 2, 3, 3, 2 ]
> g := &* [ y - r : r in roots ];
> N := NewtonPolygon(g);
> N;
Newton Polygon of y^10 + 44594997030169*y^9 - 85346683389318*y^8 +
    76213593390537*y^7 + 74689026811236*y^6 - 48671968754502*y^5 -
    58608670426020*y^4 - 63609139981179*y^3 + 77334553491246*y^2 +
    39962036019861*y - 94049035648173 over pAdicRing(3, 30)
> F := Faces(N);
> F;
[ <6, 1, 26>, <3, 1, 23>, <2, 1, 17>, <1, 1, 9>, <0, 1, 0> ]
> [GradientVector(F[i]) : i in [1 .. #F]];
```

```
[ <6, 1>, <3, 1>, <2, 1>, <1, 1>, <0, 1> ]
> [$1[i][1]/$1[i][2] : i in [1 ..#$1]];
[ 6, 3, 2, 1, 0 ]
> [EndVertices(F[i]) : i in [1 .. #F]];
[
    [ <0, 26>, <1, 20> ],
    [ <1, 20>, <6, 5> ],
    [ <6, 5>, <8, 1> ],
    [ <8, 1>, <9, 0> ],
    [ <9, 0>, <10, 0> ]
]
> [$1[i][2][1] - $1[i][1][1] : i in [1 .. #$1]];
[ 1, 5, 2, 1, 1 ]
```

So there is one root of valuation 6, five of valuation 3, two of valuation 2, one of valuation 1 and one root with valuation zero. This information could also have been gained using `ValuationsOfRoots`.

```
> ValuationsOfRoots(g);
[ <6, 1>, <3, 5>, <2, 2>, <1, 1>, <0, 1> ]
```

`HenselLift(f, x)`

`HenselLift(f, x, k)`

Return a root of the polynomial f over a local ring or field by lifting the approximate root x to a root with precision k (or the default precision of the structure if not specified). This results in an error, if the Hensel condition $v(f(x)) > 2v(f'(x))$ is not satisfied.

Example H47E18

This examples illustrates how Hensel lifting is used to compute square roots.

```
> Zx<x> := PolynomialRing(Integers());
> L1<a> := ext<pAdicRing(3, 20) | 2>;
> L2<b> := ext<L1 | x^2 + 3*x + 3>;
> R<y> := PolynomialRing(L2);
> c := (a+b)^42;
> r := L2 ! Sqrt(ResidueClassField(L2) ! c);
> r;
a
> rr := HenselLift(y^2-c, r);
> rr;
(-1513703643*a - 1674232545)*b - 1219509587*a + 760894776
> Valuation(rr^2 - c);
40
> ChangePrecision(rr, 1);
```

```
a + O(b)
```

For $p = 2$ the situation is a bit more difficult, since the derivative of $y^2 - c$ is not a unit at this point.

```
> Zx<x> := PolynomialRing(Integers());
> L1<a> := ext<pAdicRing(2, 20) | 2>;
> L2<b> := ext<L1 | x^2 + 2*x + 2>;
> R<y> := PolynomialRing(L2);
> c := (a+b)^42;
> r := L2 ! Sqrt(ResidueClassField(L2) ! c);
> r;
1
>  HenselLift(y^2-c, r);
>> HenselLift(y^2-c, r);
             ^
Runtime error in 'HenselLift': Hensel lift condition is not satisfied
```

We have to find a better approximation for the square root first.

```
> for d in GF(2,2) do
>     if Valuation((r + b*L2!d)^2 - c) gt 4 then
>         print L2!d;
>     end if;
> end for;
a + 1
> r +:= b * (a+1);
> HenselLift( y^2-c, r );
(-199021*a + 100463)*b + 204032*a - 31859 + O(b^38)
> ChangePrecision($1, 1);
1 + O(b)
> ChangePrecision($2, 2);
(a + 1)*b + 1 + O(b^2)
```

The square root is an element of reduced precision, since the proper root is only guaranteed to coincide with the approximation up to valuation 18.

47.11.2.2 Functions returning Roots

These functions determine the roots of a polynomial from the factorization of the polynomial.

`Roots(f)`

`Roots(f, R)`

`IsSquarefree` BoolElt *Default :* `false`

Return the roots of the polynomial f over a local ring or field R as a sequence of tuples of elements in R and multiplicities. If R is not specified it is taken to be the coefficient ring of f. If the polynomial is known to be squarefree, the root-finding algorithm may run considerably faster.

```
HasRoot(f)
```

Try to find a root of the polynomial f over a local ring or field. If a root is found, this function returns `true` and a root as a second value; otherwise it returns `false`.

Example H47E19

We generate the ramified extensions of $\mathbf{Z}_2$ of degree 2 by looping over some Eisenstein polynomials with small coefficients and checking whether a new polynomial has a root in one of the already known rings.

```
> Zx<x> := PolynomialRing(Integers());
> RamExt := [];
> for c0 in [2, -2, 6, -6] do
> for c1 in [0, 2, -2, 4, -4, 6, -6] do
>     g := x^2 + c1*x + c0;
>     new := true;
>     for L in RamExt do
>         if HasRoot(PolynomialRing(L)!g) then
>             new := false;
>             break L;
>         end if;
>     end for;
>     if new then
>         print "new field with polynomial", g;
>         Append(~RamExt, ext<pAdicRing(2, 20) | g>);
>     end if;
> end for;
> end for;
new field with polynomial x^2 + 2
new field with polynomial x^2 + 2*x + 2
new field with polynomial x^2 + 4*x + 2
new field with polynomial x^2 + 2*x - 2
new field with polynomial x^2 + 4*x - 2
new field with polynomial x^2 + 6
```

These are all such extensions, since the extensions of $\mathbf{Q}_2$ of degree 2 are in bijection to the 7 non-trivial classes of $\mathbf{Q}_2^*/(\mathbf{Q}_2^*)^2$ and one of these classes yields the unramified extension.

47.11.3 Factorization

It is possible to factorize (to some precision) polynomials over a local ring or field. Approximate factorizations can also be lifted to a factorization to greater precision.

```
HenselLift(f, s)
```

Given a sequence s of polynomials with coefficients that can be coerced into the coefficient ring of f such that $f \equiv \prod_{i=1}^{\#s} s[i]$ modulo π, $s[i]$ and $[j]$ are co–prime modulo π for all i and j, and $s[i]$ is monic for all i, find a more accurate factorization $t[1], t[2], \ldots t[\#s]$ such that $f \equiv \prod_{i=1}^{\#s} t[i]$ modulo π^k, where k is the minimum precision of the coefficients of f.

Example H47E20

If the reduction of a polynomial over the residue class field is not a power of an irreducible polynomial, the factorization into powers of different irreducibles can be lifted to a factorization over the local ring.

```
> Z2 := pAdicRing(2, 25);
> R<x> := PolynomialRing(Z2);
> f := &* [ x - i : i in [1..8] ];
> F2 := ResidueClassField(Z2);
> Factorization( PolynomialRing(F2)!f );
[
    <$.1, 4>,
    <$.1 + 1, 4>
]
> h1 := x^4;
> h2 := (x+1)^4;
> h := HenselLift(f, [h1, h2]);
> h[1], h[2], f - h[1]*h[2];
x^4 - 20*x^3 + 140*x^2 - 400*x + 384
x^4 - 16*x^3 + 86*x^2 - 176*x + 105
0
```

`IsIrreducible(f)`

Given a polynomial f with coefficients lying in a local ring or field L, return `true` if and only if f is irreducible over L. Currently, this only works over p-adic rings, unramified extensions of p-adic rings, totally ramified extensions of p-adic rings, and totally ramified extension of unramified extensions of p-adic rings.

`SquareFreeFactorization(f)`

Return a sequence of tuples of polynomials and multiplicities where the polynomials are not divisible by any square of a polynomial. The product of the polynomials to the corresponding multiplicities is the polynomial f (to some precision). Currently, this only works over p-adic rings, unramified extensions of p-adic rings, totally ramified extensions of p-adic rings, and totally ramified extension of unramified extensions of p-adic rings.

```
Factorization(f)
LocalFactorization(f)
```

Certificates	BoolElt	*Default :* false
IsSquarefree	BoolElt	*Default :* false
Ideals	BoolElt	*Default :* false
Extensions	BoolElt	*Default :* false

Return the factorization of the polynomial f over a local ring or field into irreducible factors as a sequence of pairs, the first entry giving the irreducible factor and the second its multiplicity.

Precision is important since for polynomials over rings of relatively small precision a correct factorization may not be possible and an error will result. A lower bound on the precision needed for the factorization to succeed is given by `SuggestedPrecision`; this precision may still be insufficient, however.

The precision the factorization is returned to is reduced for multiple factors.

The optional parameter `IsSquarefree` can be set to `true`, if the polynomial is known to be square-free. The `Certificates` parameter can be set to `true` to compute certificates for the irreducibility of the individual factors returned. This information can be used to compute the p-maximal order of the equation order defined by the factor. If the `Extensions` parameter is set to `true` then certificates will be returned which will include an extension given by each factor.

```
SuggestedPrecision(f)
```

For a polynomial f over a local ring or field, return a precision at which the factorization of f as given by `Factorization(f)` will be Hensel liftable to the correct factorization.

The precision returned is not guaranteed to be enough to obtain a factorization of the polynomial. It may be that a correct factorization cannot be found at that precision but may be possible with a little more precision.

Currently, this only works over p-adic rings, unramified extensions of p-adic rings, totally ramified extensions of p-adic rings, and totally ramified extension of unramified extensions of p-adic rings.

```
IsIsomorphic(f, g)
```

Given two irreducible polynomials over the same p-adic field, test if the extensions defined by them are isomorphic.

```
Distance(f, g)
```

Given two Eisenstein polynomials of the same degree and discriminant compute their distance, i.e. the minimal weighted valuation of the difference of the coefficients.

Example H47E21

The use of `SuggestedPrecision` along with `Factorization` is illustrated below for a few different cases.

```
> L<b> := ext<ext<pAdicRing(5, 20) | 2> |  y^2 + 5*y + 5>;
> R<x> := PolynomialRing(L);
> f := (x - 1)^3*(x - b)^2*(x - b^2 + b - 1);
> SuggestedPrecision(f);
80
> Factorization(f);
[
    <x - b + O(b^40), 2>,
    <x - 1 + O(b^40), 3>,
    <x + 6*b + 4 + O(b^40), 1>
]
1 + O(b^40)
> f := (x + 2)^3*(x + b)*(x + 3)^4;
> f;
x^8 + (b + 18 + O(b^40))*x^7 + (18*b + 138 + O(b^40))*x^6 + (138*b + 584 +
    O(b^40))*x^5 + (584*b + 1473 + O(b^40))*x^4 + (1473*b + 2214 + O(b^40))*x^3
    + (2214*b + 1836 + O(b^40))*x^2 + (1836*b + 648 + O(b^40))*x + 648*b +
    O(b^40)
> SuggestedPrecision(f);
80
> Precision(L);
80
> P<y> := PolynomialRing(Integers());
> R<b> := ext<ext<pAdicRing(3, 20) | 2> |  y^2 + 3*y + 3>;
> P<x> := PolynomialRing(R);
> f := x^12 + 100*x^11 + 4050*x^10 + 83700*x^9 + 888975*x^8 + 3645000*x^7 -
> 10570500*x^6 - 107163000*x^5 + 100875375*x^4 + 1131772500*x^3 -
> 329614375*x^2 + 1328602500*x + 332150625;
> SuggestedPrecision(f);
48
> Factorization(f);
[
    <x + 153560934 + O(b^40), 1>,
    <x - 1595360367 + O(b^40), 1>,
    <x + 1273329451*$.1 - 1037496066 + O(b^40), 1>,
    <x + -1273329451*$.1 - 97370567 + O(b^40), 1>,
    <x^4 + (-1598252738*$.1 - 309919655 + O(b^40))*x^3 + (-280421715*$.1 -
        102998055 + O(b^40))*x^2 + (1263867503*$.1 + 923047935 + O(b^40))*x +
        -1252549104*$.1 - 786365260 + O(b^40), 1>,
    <x^4 + (1598252738*$.1 - 600198580 + O(b^40))*x^3 + (280421715*$.1 +
        457845375 + O(b^40))*x^2 + (-1263867503*$.1 - 1604687071 + O(b^40))*x +
        1252549104*$.1 + 1718732948 + O(b^40), 1>
]
1 + O(b^40)
```

```
> R<b> := ext<ext<pAdicRing(3, 25) | 2> |  y^2 + 3*y + 3>;
> P<x> := PolynomialRing(R);
> f := x^12 + 100*x^11 + 4050*x^10 + 83700*x^9 + 888975*x^8 + 3645000*x^7 -
> 10570500*x^6 - 107163000*x^5 + 100875375*x^4 + 1131772500*x^3 -
> 329614375*x^2 + 1328602500*x + 332150625;
> Factorization(f);
[
    <x + 143905694229 + O(b^50), 1>,
    <x - 399882754935 + O(b^50), 1>,
    <x + 46601526664*$.1 + 229090274400 + O(b^50), 1>,
    <x + -46601526664*$.1 + 135887221072 + O(b^50), 1>,
    <x^4 + (242476655332*$.1 + 187976437999 + O(b^50))*x^3 + (372805509192*$.1 -
        38457626466 + O(b^50))*x^2 + (126788105939*$.1 + 60198382752 +
        O(b^50))*x + 347425890996*$.1 + 215394267602 + O(b^50), 1>,
    <x^4 + (-242476655332*$.1 - 296976872665 + O(b^50))*x^3 + (-372805509192*$.1
        + 63219964593 + O(b^50))*x^2 + (-126788105939*$.1 - 193377829126 +
        O(b^50))*x + -347425890996*$.1 + 367831095053 + O(b^50), 1>
]
1 + O(b^50)
```

Note that the polynomial itself must have coefficients with precision at least that given by `SuggestedPrecision` (and not just the coefficient ring) for `Factorization` to succeed. Sometimes this will not be possible if the coefficients of the polynomial are not known to sufficient precision.

Example H47E22

In this example we demonstrate how factorizations of a rational polynomial over some local rings can give information on the Galois group.

```
> Zx<x> := PolynomialRing(Integers());
> g := x^5 - x + 1;
> Factorization(Discriminant(g));
[ <19, 1>, <151, 1> ]
> g2 := Factorization( PolynomialRing(pAdicRing(2, 10)) ! g );
> g2;
[
    <$.1^2 + 367*$.1 - 93, 1>,
    <$.1^3 - 367*$.1^2 - 386*$.1 + 11, 1>
]
> g3 := Factorization( PolynomialRing(pAdicRing(3, 10)) ! g );
> g3;
[
    <$.1^5 - $.1 + 1, 1>
]
> g7 := Factorization( PolynomialRing(pAdicRing(7, 10)) ! g );
> g7;
[
    <$.1^2 + 138071312*$.1 + 2963768, 1>,
    <$.1^3 - 138071312*$.1^2 + 132202817*$.1 - 84067650, 1>
]
```

This shows that the Galois group of g contains elements of orders 2, 3, 5 and 6, and therefore is isomorphic to S_5.

```
SplittingField(f)
```

`Std` BoolElt *Default : true*

Splitting field F of a squarefree polynomial f over a p-adic ring or field K, and the roots of f in F. The optional parameter `Std` (true by default) specifies whether F/K should be converted to a standard form — at most one ramified extension over one unramified extension of K.

Example H47E23

```
> K:=pAdicField(3,20);
> R<x>:=PolynomialRing(K);
> F:=SplittingField(x^9-3);
> Degree(F,K), RamificationDegree(F,K), InertiaDegree(F,K);
54 54 1
```

47.12 Automorphisms of Local Rings and Fields

The automorphisms of a local ring or field are determined by their images on the generators of the ring. All computations necessary to determine the automorphism group can be performed in the local ring.

```
Automorphisms(L)
```

Given a local ring or field L, returns the automorphisms of L over its p-adic sub-field $\mathbf{Q}_p$ as a sequence of maps of L into L.

```
Automorphisms(K, k)
```

Given a local ring or field K over k, returns the k-automorphisms of K as a sequence of maps of K into K.

```
AutomorphismGroup(L)
```

Return the automorphism group acting on L over its p-adic sub-field $\mathbf{Q}_p$ as a permutation group (representing the regular action). Also return the map from the permutation group to the group of automorphisms represented explicitly (i.e. like returned from the function above).

```
AutomorphismGroup(K, k)
```

Return the automorphism group acting on K over its p-adic k as a permutation group (representing the regular action). Also return the map from the permutation group to the group of automorphisms represented explicitly (i.e. like returned from the function above).

```
IsNormal(K)
```

Given a p-adic ring or field K, test if K is normal over it's prime field $\mathbf{Q}_p$, ie. if K admits exactly n automorphisms where n is the degree of K.

```
IsNormal(K, k)
```

Given a p-adic ring or field K, test if K is normal over the subfield k.

```
IsAbelian(K, k)
```

Given p-adic fields K/k, test if the automorphism group of K over k is abelian.

```
Continuations(m, L)
```

For an automorphism m of the p-adic ring L, compute all possible extensions of m to L.

```
IsIsomorphic(E, K)
```

For two p-adic rings or fields, test if they are isomorphic over $\mathbf{Q}_p$.

Example H47E24

We define an extension of $\mathbf{Z}_2$ with ramification degree 2 and inertia degree 2 and compute automorphisms.

```
> I<a> := ext<pAdicRing(2, 10) | 2>;
> R<x> := PolynomialRing(I);
> L<b> := ext<I | x^2 + 2*a*x + 2*a^2>;
> L;
Totally ramified extension defined by the polynomial
x^2 + (2*a)*x + -2*a - 2 over Unramified extension
defined by the polynomial x^2 + x + 1 over 2-adic ring
mod 2^10
> A := Automorphisms(L);
> [<A[i](a), A[i](b)> : i in [1 .. #A]];
[ <a, b + O(b^18)>, <a, -b + -2*a + O(b^18)>, <-a - 1,
a*b + O(b^18)>, <-a - 1, -a*b + 2*a + 2 + O(b^18)> ]
> AutomorphismGroup(L);
Permutation group acting on a set of cardinality 4
    Id($)
    (1, 2)(3, 4)
    (1, 3)(2, 4)
Mapping from: GrpPerm: $, Degree 4 to Power Structure
of Map given by a rule
```

```
GaloisGroup(f)
```

Galois group G of a squarefree polynomial f over a p-adic ring or field K. Returns G as a permutation group on the roots of f in its splitting field F, the roots themselves, and a map $G \to \mathrm{Aut}(F/K)$.

Example H47E25

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(K);
> G,r,act:=GaloisGroup(x^4-2);
> G;                                    // permutation group on roots in r
Permutation group G acting on a set of cardinality 4
(1, 4, 2, 3)
(1, 2)
> GroupName(G);
D4
> F<pi>:=Universe(r); F;                         // splitting field F of f
Totally ramified extension defined by the polynomial x^8 + 8*x^7 + 24*x^6 +
   32*x^5 + 18*x^4 + 8*x^3 + 12*x^2 + 8*x + 2
 over 2-adic field mod 2^20
> sigma:=act(G.1);                               // an automorphism of F/K
> sigma;
Mapping from: FldPad: F to FldPad: F given by a rule [no inverse]
> sigma(F.1);
(43690*pi^7 - 5*pi^6 + 43677*pi^5 - 15*pi^4 + 43687*pi^3 + 43684*pi - 3)*pi +
   O(pi^136)
```

47.13 Completions

Local Rings can be obtained by completing an order at a prime ideal (see Chapter 35 and `Completion` on page 3-906).

```
Completion(O, P)
```

```
Completion(K, P)
```

`Precision` RngIntElt *Default :* 20

The completion (as an unbounded precision local ring or field with default precision given by `Precision`) of the order or number field at the prime ideal P, and the embedding of the order or number field into the resulting local ring.

```
LocalRing(P, k)
```

The completion (as a local ring) of the order of the prime ideal P at P with precision k and the embedding of the order into the resulting local ring.

Example H47E26

Here we demonstrate the use of `Completion`.

```
> K := NumberField(x^6 - 5*x^5 + 31*x^4 - 85*x^3 + 207*x^2 - 155*x + 123);
> lp := Decomposition(K, 7);
> C, mC := Completion(K, lp[2][1]);
> C;
Totally ramified extension defined by a map over Unramified extension defined by
a map over 7-adic field
> mC;
Mapping from: FldNum: K to FldPad: C given by a rule
> mC(K.1);
(46564489*$.1 - 47959419)*C.1 - 116434149*$.1 - 61099304 + O(C.1^20)
> delta := (K.1 @ mC @@ mC) - K.1;
> delta;
833782149340252135048 8*K.1^5 - 69073506960056896464432*K.1^4 +
    189847416443444330877726*K.1^3 - 453361530291976951337876*K.1^2 +
    336979647814116799276099*K.1 - 267520869714197002579071
> // Check the accuracy of the mappings using the valuation of the difference
> Valuation(delta, lp[2][1]);
18
> C`DefaultPrecision := 30;
> mC(K.1);
(1090965976127*$.1 - 1208477074641)*C.1 - 589359803563*$.1 + 288063654676 +
O(C.1^30)
> delta := (K.1 @ mC @@ mC) - K.1;
> delta;
-619800242441603716728687734334907 83*K.1^5 +
    1189796803064803092593291088768968754*K.1^4 -
    3202353946933190588864309180653868957*K.1^3 +
    7537386928046164580731145031872017049*K.1^2 -
    5511297002936682579964210586013308810*K.1 +
    4438099444806431313582533435941098722
> Valuation(delta, lp[2][1]);
28
> C`DefaultPrecision := 10;
> mC(K.1);
(-7708*$.1 + 7759)*C.1 + 4747*$.1 - 5859 + O(C.1^10)
> delta := (K.1 @ mC @@ mC) - K.1;
> delta;
1908210240*K.1^5 - 7326424608*K.1^4 + 16701662320*K.1^3 - 35965440540*K.1^2 +
    41324075079*K.1 - 30476856505
> Valuation(delta, lp[2][1]);
8
```

47.14 Class Field Theory

The class field theory of local fields classifies abelian extensions of local field in a way similar to the way global class field theory deals with extensions of number fields and global function fields.

While the origins of local class field theory are, via completions and localisations, in the global case, today it is a theory in its own. Although local class field theory can be used to obtain global results, it has very powerful generalisations that the global case (currently) does not allow.

Local class fields are classified in terms of the norm group, ie. the multiplicative group of norms of elements, rather than some ideal or divisor class group as in the global case. Since the multiplicative group of a local field is far better understood than the ideal group of a global field, the theory is much more explicit and easier in the local case.

47.14.1 Unit Group

In contrast to the case of global fields, the multiplicative group of both p-adic rings and fields has a well understood structure which can be computed by algorithms developed and implemented by S. Pauli [Pau06]. It should be noted that all the unit group related functions operate on fixed-precision rings only.

```
PrincipalUnitGroupGenerators(R)
```

The principal units of a p-adic ring or field R are elements of the form $1 + \pi\mathbf{Z}_R$ where π is a uniformizing element of R and $\mathbf{Z}_R$ is the ring of integers. This function returns a sequence of generators for this group.

```
PrincipalUnitGroup(R)
```

The principal units of a p-adic ring or field R are elements of the form $1 + \pi\mathbf{Z}_R$ where π is a uniformizing element of R and $\mathbf{Z}_R$ is the ring of integers. This function returns an abstract abelian group isomorphic to the group of principal units and an explicit isomorphism, ie. a map between the abstract group and the p-adic ring or field.

```
UnitGroup(R)
```

Given a p-adic ring R of fixed precision, this function computes an abstract abelian group isomorphic to the unit group as well as an explicit map between the abstract group and R.

```
UnitGroup(F)
```

Given a p-adic field F of fixed precision, this function computes an abstract abelian group isomorphic to the multiplicative group of F as well as an explicit map between the abstract group and F.

```
UnitGroupGenerators(R)
```

Given a p-adic ring with fixed precision, this function computes generators for its unit group.

```
UnitGroupGenerators(F)
```

Given a p-adic field with fixed precision, this function computes generators for its multiplicative group.

```
pSelmerGroup(p,F)
```

Given a l-adic field F, return the p-Selmer group, i.e., the group F^*/F^{*p}, as an abstract group, as well as the map from F^* to the abstract group.

47.14.2 Norm Group

Given two p-adic field F/k the norm group of F in k, ie. the image of the norm map from F to k is the central object of local class field theory. Since the norm map will always operate on some multiplicative group, all functions in this section will take the map returned by `UnitGroup` as an argument as this then allows the convenient way of describing the norm group as a subgroup of some explicit finitely generated abelian group.

```
NormGroup(R, m)
```

Given a p-adic ring or field R extending S and a description of the unit group of S encoded by a map m from some abstract abelian group to S as computed by `UnitGroup`, compute the image of the norm map as a subgroup. The map returned is the embedding map returned form the subgroup constructor.

```
NormEquation(R, m, b)
```

Given a p-adic ring R defined over S, the unit group of S encoded by the map m as computed by `UnitGroup(S)` and some element $b \in S$, try to compute an element $a \in R$ such that the norm of a equals b. In case such an element exists, it is returned as a second value.

```
NormEquation(m1, m2, G)
```

Given two p-adic rings R and S and their unit groups U_R and U_S as parameterized by the maps $m_1 : U_R \to R$ and $U_S \to S$ as well as a subgroup $G < U_S$, compute the preimage of G under the norm map operating on the unit groups.

```
Norm(m1, m2, G)
```

Given two p-adic rings R and S and their unit groups U_R and U_S as parameterized by the maps $m_1 : U_R \to R$ and $U_S \to S$ as well as a subgroup $G < U_R$, compute the image of G under the norm map operating on the unit groups.

```
NormKernel(m1, m2)
```

Given two p-adic rings R and S and their unit groups U_R and U_S as parameterized by the maps $m_1 : U_R \to R$ and $U_S \to S$ compute the kernel of the norm map from U_R to U_S as a subgroup of U_R.

47.14.3 Class Fields

Class fields, that is abelian extensions are parameterized by their norm groups. Pauli, in [Pau06] gave explicit algorithms to solve the reverse problem of class field theory: given a suitable subgroup of some (abstractly given) multiplicatively group of some p-adic field, compute explicit defining equations for the class field.

`ClassField(m, G)`

Given a p-adic field S and its multiplicative group U_S specified by the map $m : U_S \to S$ and a suitable subgroup $G < U_S$, this function computes for each cyclic factor of U_S/G an explicit defining equation for the class field corresponding to this factor.

`NormGroupDiscriminant(m, G)`

Given a p-adic field S and its multiplicative group U_S specified by the map $m : U_S \to S$ and a suitable subgroup $G < U_S$, this function computes the valuation of the discriminant of the extension parameterized by G without computing explicit equations for it.

47.15 Extensions

It is a well known classical theorem that p-adic fields admit only finitely many different extensions of bounded degree (in contrast to number fields which have an infinite number of extensions of any degree). In his thesis, Pauli [Pau01a] developed explicit methods to enumerate those extensions.

`AllExtensions(R, n)`

`E`	RngIntElt	*Default :* 0
`F`	RngIntElt	*Default :* 0
`Galois`	BoolElt	*Default :* `false`
`vD`	RngIntElt	*Default :* -1
`j`	RngIntElt	*Default :* -1

Given a p-adic ring or field R and some positive integer n, compute all the extensions of R of degree n. At least one extension is given in every isomorphism class. The optional parameters can be used to impose restrictions on the fields returned. Note that j and vD must be unspecified (as -1) when $F \neq 1$ or when E and F are both not given.

The optional parameters can be used to limit the extensions in various ways:

`E`	specifies the ramification index. 0 implies no restriction.
`F`	specifies the inertia degree, 0 implies no restriction.
`vD`	specifies the valuation of the discriminant, -1 implies no restriction.
`j`	specifies the valuation of the discriminant via the formula $vD := n + j - 1$.
`Galois`	when set to `true`, limits the extensions to only list normal extensions.

```
NumberOfExtensions(R, n)
```

E	RngIntElt	*Default :* 0
F	RngIntElt	*Default :* 0
Galois	BoolElt	*Default :* false
vD	RngIntElt	*Default :* -1
j	RngIntElt	*Default :* -1

Given a p-adic ring or field R and some positive integer n, compute the number of extensions of R of degree n. Similarly to the above function, the optional parameters can be used to impose restrictions on the fields returned. Note that j and vD must be unspecified (as -1) when $F \neq 1$ or when E and F are both not given.

Note that the count will not be the same as `AllExtensions`, as the latter need only be up to isomorphism.

```
OreConditions(R, n, j)
```

Given a p-adic ring or field R and positive integers n and j, test if there exist totally ramified extensions of R of degree n with discriminant valuation $n + j - 1$.

Example H47E27

We follow Examples 9.1 and 9.2 from [PR01].
There are 54 (totally ramified) extensions of degree 9 and discriminant 3^{9+4-1} over $\mathbf{Q}_3$. There are six generating polynomials, each defining nine isomorphism classes. The possible (nontrivial) subfields of these have degree 3 and $j_0 = 1$, of which there are two defining polynomials each with three isomorphism classes. Each of these degree 3 fields then admits two extensions with $j_1 = 1$, which give six isomorphism classes. This gives a total of 27 degree 9 extensions that have a subfield of degree 3.

```
> R := pAdicRing(3,20);
> _<x> := PolynomialRing(R); // for printing
> NumberOfExtensions(R,9 : F:=1,j:=4);
54
> A9 := AllExtensions(R,9 : F:=1,j:=4);
> [DefiningPolynomial(a) : a in A9];
[ x^9 + 3*x^4 + 3, x^9 + 6*x^4 + 3,
  x^9 + 3*x^4 + 3*x^3 + 3, x^9 + 6*x^4 + 3*x^3 + 3,
  x^9 + 3*x^4 + 6*x^3 + 3, x^9 + 6*x^4 + 6*x^3 + 3 ]
> A3 := AllExtensions(R,3 : F:=1,j:=1);
> NumberOfExtensions(A3[1],3 : F:=1,j:=1);
6
> [DefiningPolynomial(a) : a in A3];
[ x^3 + 3*x + 3, x^3 + 6*x + 3 ]
> _<pi> := A3[1];
> _<y> := PolynomialRing(A3[1]);
> B3 := AllExtensions(A3[1],3 : F:=1,j:=1);
> [DefiningPolynomial(f) : f in B3];
[ y^3 + pi*y + pi, y^3 + 2*pi*y + pi ]
```

The other example concerns degree 10 extensions of $\mathbf{Q}_5$. Here there are 1818 total extensions, of which 1 is unramified and 2 have ramification degree 2, while 605 have ramification degree 5 and 1210 are totally ramified. With ramification degree 5, there are 145 defining polynomials over the unramified quadratic field, split into five j-groupings. As noted in [PR01], there is a further splitting in the $j = 4$ grouping. Similarly, there are 145 defining polynomials over either of the two tamely ramified extensions of degree 2 over $\mathbf{Q}_5$. The resulting fields are in fact isomorphic in pairs, but the **AllExtensions** function still lists both fields in each pair. So it returns 438 fields (1+2+145+290) rather than the stated 293 isomorphism classes (1+2+145+145).

```
> R := pAdicRing(5,20);
> NumberOfExtensions(R,10);
1818
> [NumberOfExtensions(R,10 : E:=e) : e in Divisors(10)];
[ 1, 2, 605, 1210 ]
> U := UnramifiedExtension(R,2);
> [#AllExtensions(U,5 : E:=5,j:=j0) : j0 in [1..5]];
[ 24, 24, 24, 48, 25 ]
> // compare the above/below to (#K)/N in Pauli-Roblot
> [#AllExtensions(R,10 : E:=10,j:=j0): j0 in [1..10]];
[ 8, 8, 8, 16, 0, 40, 40, 80, 40, 50 ] // twice P-R
```

47.16 Bibliography

[**C$^+$05**] H. Cohen et al. *Handbook of Elliptic and Hyperelliptic Curve Cryptography.* Discrete Mathematics and Its Applications. CRC Pr Llc, 2005.

[**JR06**] J. W. Jones and D. P. Roberts. A database of local fields. *J. Symb. Comput.*, 41(1):80–97, 2006.

[**Pau01a**] Sebastian Pauli. *Efficient Enumeration of Extensions of Local Fields with Bounded Discriminant.* PhD thesis, Concordia University, 2001.

[**Pau01b**] Sebastian Pauli. Factoring polynomials over local fields. *Journal of Symbolic Computation*, 32(5):533–547, 2001.

[**Pau06**] Sebastian Pauli. Constructing Class Fields over Local Fields. *J. Théorie des Nombres de Bordeaux*, 18(3):627–652, 2006.

[**PR01**] S. Pauli and X.-F. Roblot. On the computation of all extensions of a p-adic field of a given degree. *Math. Comp.*, 70(236):1641–1659, 2001.

48 GENERAL p-ADIC EXTENSIONS

Chapter 48

GENERAL p-ADIC EXTENSIONS

48.1 Introduction

The local fields described in this chapter are extensions of any p-adic local field in Magma by any irreducible polynomial over that field. The polynomial defining the extension is not required to be inertial or eisenstein, in contrast to the (older) extensions of p-adic fields. This allows ramified and inertial extensions to be made in one step rather than forcing such an extension to be split into two – being a ramified extension and an unramified extension.

Only fields are implemented in this way – no construction of a ring of integers is provided (although `IntegralBasis` gives a basis for it as a module over the base ring).

These local fields have type `RngLocA` with elements of type `RngLocAElt`, while the local fields described in the previous chapter 47 have type `FldPad`.

These fields (of type `RngLocA`) can be converted to the other representation (type `FldPad`) using `RamifiedRepresentation`, which returns an isomorphism between them. This can be used for calculations not supported for `RngLocA`.

48.2 Constructions

The main construction is to extend a local field (of either type) by adjoining the root of a univariate polynomial. Another construction is of the subfield generated by given elements.

```
LocalField(L, f)
```

Construct a local field F as an extension of the local field L by the polynomial f over L.

Example H48E1

We can use the 2 step local fields to help us find an irreducible polynomial over a p-adic field which can be used to extend that p-adic field in 1 step.

```
> P<x> := PolynomialRing(Integers());
> Zp := pAdicRing(7, 50);
> U := UnramifiedExtension(Zp, x^2 + 6*x + 3);
> R := TotallyRamifiedExtension(U, x^3 + 7*x^2 + 7*x + 7);
> L<a> := LocalField(pAdicField(7, 50), MinimalPolynomial(R.1 + U.1, Zp));
> L;
Extension of 7-adic field mod 7^50 by x^6 + (32 + O(7^50))*x^5 + (390 +
    O(7^50))*x^4 + (2284 + O(7^50))*x^3 + (6588 + O(7^50))*x^2 + (8744 +
    O(7^50))*x + 5452 + O(7^50)
```

We can also use any irreducible polynomial we can find to define a 1 step extension.

```
> LocalField(pAdicField(7, 50), x^6 - 49*x^2 + 686);
Extension of 7-adic field mod 7^50 by x^6 + O(7^50)*x^5 + O(7^50)*x^4 +
    O(7^50)*x^3 - (7^2 + O(7^52))*x^2 + O(7^50)*x + 2*7^3 + O(7^53)
```

sub< L | $a_1, ..., a_n$ >

sub< L | S >

Construct the local field F as a subfield of the local field L containing the elements a_i of L or the elements of the sequence S.

Example H48E2

```
> Qp := pAdicField(5, 20);
> P<x> := PolynomialRing(Qp);
> L<a> := LocalField(Qp, x^4 + 4*x^2 + 2);
> P<x> := PolynomialRing(L);
> LL<aa> := LocalField(L, x^4 + 4*L.1);
> r := (10236563738184*a^3 - 331496727861*a^2 + 10714284669258*a +
> 8590525712453*5)*aa^2
> -12574685904653*a^3 + 19786544763736*a^2 + 4956446023134*a + 37611818678747;
> S, m := sub< LL | r >;
> S;
Extension of Extension of 5-adic field mod 5^20 by x^4 + O(5^20)*x^3 + (4 +
    O(5^20))*x^2 + (4 + O(5^20))*x + 2 + O(5^20) by (O(5^20)*a^3 + O(5^20)*a^2 +
    O(5^20)*a + (1 + O(5^20)))*x^2 + (O(5^20)*a^3 + O(5^20)*a^2 + O(5^20)*a +
    O(5^20))*x + O(5^20)*a^3 + O(5^20)*a^2 + (4 + O(5^20))*a + O(5^20)
> m(S.1);
(O(5^20)*a^3 + O(5^20)*a^2 + O(5^20)*a + O(5^20))*aa^3 + (O(5^20)*a^3 +
    O(5^20)*a^2 + O(5^20)*a + (1 + O(5^20)))*aa^2 + (O(5^20)*a^3 + O(5^20)*a^2 +
    O(5^20)*a + O(5^20))*aa + O(5^20)*a^3 + O(5^20)*a^2 + O(5^20)*a + O(5^20)
```

48.3 Operations with Fields

```
BaseRing(L)
CoefficientRing(L)
```

Return the coefficient field of the local field L. This is the field which was extended to construct L.

```
DefiningPolynomial(L)
```

Return the polynomial used to define the local field L as an extension of its coefficient field.

```
Degree(L)
```

Return the degree of the local field L, that is, the degree of its defining polynomial.

```
Degree(L, R)
```

Return the degree of L as an extension of R where R is some coefficient ring of L.

```
InertiaDegree(L)
RamificationDegree(L)
RamificationIndex(L)
```

Return the degree of the inertial or totally ramified subfield of the local field L as an extension of the coefficient field of L.

```
Precision(L)
```

Return the precision of the local field L. This is the maximum number of digits which can occur in an element of L, the difference between the valuation of an element of L and the valuation of the term of highest valuation occurring in that element.

```
Prime(L)
```

Return the prime of the local field L. This is the same as the prime of the coefficient field of L.

Example H48E3

Continuing from the first example we have :

```
> CoefficientRing(L);
7-adic field mod 7^50
> DefiningPolynomial(L);
$.1^6 + (32 + O(7^50))*$.1^5 + (390 + O(7^50))*$.1^4 + (2284 + O(7^50))*$.1^3 +
    (6588 + O(7^50))*$.1^2 + (8744 + O(7^50))*$.1 + 5452 + O(7^50)
> Precision(L);
150
> Prime(L);
7
> Degree(L); RamificationDegree(L); InertiaDegree(L);
6
3
2
```

```
QuotientRepresentation(L)
```

Return the polynomial quotient ring which is isomorphic to the local field L and is used to represent L.

```
RamifiedRepresentation(L)
```

Return the local field isomorphic to the local field L constructed as an unramified then a ramified extension and the map from L into the isomorphic field.

Example H48E4

We create a 1 step local field and compute its representation as a 2 step local field.

```
> P<x> := PolynomialRing(Integers());
> L<a> := LocalField(pAdicField(7, 50), x^6 - 49*x^2 + 686);
> L;
Extension of 7-adic field mod 7^50 by x^6 + O(7^50)*x^5 + O(7^50)*x^4 +
    O(7^50)*x^3 - (7^2 + O(7^52))*x^2 + O(7^50)*x + 2*7^3 + O(7^53)
> QuotientRepresentation(L);
Univariate Quotient Polynomial Algebra in $.1 over 7-adic field mod 7^50
with modulus $.1^6 + O(7^50)*$.1^5 + O(7^50)*$.1^4 + O(7^50)*$.1^3 - (7^2 +
    O(7^52))*$.1^2 + O(7^50)*$.1 + 2*7^3 + O(7^53)
> RR, m := RamifiedRepresentation(L);
> RR;
Totally ramified extension defined by the polynomial x^2 +
    4170927323556945371133482017034370337886663*$.1^2 +
    586194602218356762336379252895075698537137*$.1 -
    1300941132246339981668875337559012140965597
 over Unramified extension defined by the polynomial x^3 + 6*x + 2
 over 7-adic field mod 7^50
```

```
> m(L.1);
RR.1 + O(RR.1^92)
> RR.1 @@ m;
O(7^48)*$.1^5 + O(7^48)*$.1^4 + O(7^49)*$.1^3 + O(7^49)*$.1^2 + $.1 + O(7^50)
> CoefficientRing(RR).1 @@ m;
O(7^34)*$.1^5 - (954564700580430506024960512238*7^-1 + O(7^35))*$.1^4 +
    O(7^36)*$.1^3 + (1031213687115590174398504554631*7^-1 + O(7^35))*$.1^2 +
    O(7^37)*$.1 + 1312848773660672951063501735 68*7 + O(7^36)
```

`AssignNames(~L, S)`

Assign the name in the sequence S to the generator of the extension defining the local field L.

`Name(L, i)`

Return the generator of the local field L which has assigned to it the name in the sequence S which was input to `AssignNames`. The only valid input for i is 1.

`Discriminant(L)`

Return the discriminant of the local field L.

`ResidueClassField(L)`

Return the residue class field of the maximal order of the local field L and the map between L and its residue class field.

`RelativeField(L, m)`

Return L as an extension of the domain of the map m which should be a map from a subfield of L (having the same coefficient ring as L) into L.

48.3.1 Predicates on Fields

`IsRamified(L)`

Return whether the local field L has a non trivial ramified subfield, that is, the ramification degree of L is greater than 1.

`IsTamelyRamified(L)`

`IsWildlyRamified(L)`

Return whether the local field L is tamely or wildly ramified.

`IsTotallyRamified(L)`

Return whether the local field L is a totally ramified extension, that is, L has a trivial inertial subfield.

`IsUnramified(L)`

Return whether the local field L is equal to its inertial subfield.

48.4 Maximal Order

`IntegralBasis(L)`

Return a basis for the maximal order of the local field L.

`IsIntegral(a)`

Return whether the local field element a lies in the maximal order of its parent L and a sequence giving the coordinates of a with respect to the integral basis of L if so.

Example H48E5

We construct a local field and compute an integral basis for it.

```
> P<x> := PolynomialRing(Integers());
> L := LocalField(pAdicField(7, 50), x^6 - 49*x^2 + 686);
> IntegralBasis(L);
[ 1 + O(7^50), (7^-1 + O(7^49))*$.1^2 + O(7^49)*$.1 + O(7^49), (7^-2 +
    O(7^48))*$.1^4 + O(7^48)*$.1^3 + O(7^48)*$.1^2 + O(7^50)*$.1 + O(7^50), $.1
    + O(7^50), (7^-1 + O(7^49))*$.1^3 + O(7^49)*$.1^2 + O(7^49)*$.1 + O(7^99),
    (7^-2 + O(7^48))*$.1^5 + O(7^48)*$.1^4 + O(7^48)*$.1^3 + O(7^50)*$.1^2 +
    O(7^50)*$.1 + O(7^50) ]
```

48.5 Homomorphisms

`hom< L → R | a >`

`hom< L → R | cfm, a >`

Return the homomorphism from the local field L into the ring R whose image of the generator of L is a and whose action on the coefficient field of L is given by cfm if given.

48.6 Automorphisms and Galois Theory

`FrobeniusAutomorphism(L)`

Return the automorphism of the unramified extension L which is the lift of the frobenius automorphism on the residue class field of L.

`AutomorphismGroup(L)`

Return the automorphism group of the local field L and a map from the group to the parent of automorphisms of L.

`DecompositionGroup(L)`

`InertiaGroup(L)`

`RamificationGroup(L, i)`

Return the subgroup of the automorphism group of the local field L whose elements are the automorphisms (represented as group elements) σ such that $v(\sigma(z) - z) \geq i+1$. The decomposition group is the -1th ramification group and the inertia group is the 0th ramification group.

`FixedField(L, G)`

Return the subfield of the local field L which is fixed by the automorphisms (represented as group elements) in the subgroup G of the automorphism group of L.

Example H48E6

The automorphism and inertia groups of a local field are computed and their fixed fields examined.

```
> P<x> := PolynomialRing(Integers());
> L := LocalField(pAdicField(7, 50), x^6 - 49*x^2 + 686);
> A, am := AutomorphismGroup(L);
> am(Random(A));
Mapping from: RngLocA: L to RngLocA: L
> $1(L.1);
-(2796746090469252651410760184885*7^-2 + O(7^34))*$.1^5 + O(7^35)*$.1^4 +
    (1035905251748988129458881464123*7^-1 + O(7^35))*$.1^3 + O(7^36)*$.1^2 -
    (1009443907710864908501983735501 + O(7^36))*$.1 + O(7^37)
> FixedField(L, A);
Extension of 7-adic field mod 7^50 by (1 + O(7^33))*x + O(7^33)
> InertiaGroup(L);
Permutation group acting on a set of cardinality 6
    Id($)
    (1, 2)(3, 5)(4, 6)
> FixedField(L, InertiaGroup(L));
Extension of 7-adic field mod 7^50 by (1 + O(7^37))*x^3 - (2*7^2 + O(7^37))*x^2
    + (7^4 + O(7^37))*x - 4*7^6 + O(7^37)
```

48.7 Elements Operations

```
L ! r
```

Return the element of the local field L described by r where r may be anything which is coercible into the quotient representation of L.

```
L . i
```

Return the generator of the local field L. The only valid input for i is 1.

```
InertialElement(L)
```

Return a generator for the inertial subfield of the local field L.

```
UniformizingElement(L)
```

Return an element of the local field L of valuation 1.

48.7.1 Arithmetic

```
a * b     a + b     a - b     - a     a ^ n     a / b
```

48.7.2 Predicates on Elements

```
a eq b
```

Return whether the local field elements a and b are considered equal.

```
IsOne(a)
IsMinusOne(a)
```

Return whether the local field element a is known to be 1 or -1 to the precision of the field.

```
IsWeaklyZero(a)
```

Return whether the local field element a is not known to be non zero.

```
IsZero(a)
```

Return whether the local field element a is known to be zero.

48.7.3 Other Operations on Elements

```
Valuation(a)
```

The valuation of the element a in a local field.

```
RelativePrecision(a)
```

The relative precision of the element a in a local field.

```
Eltseq(a)
```

Return the coefficients of powers of the generator of the parent of a in a.

```
RepresentationMatrix(a)
```

The representation matrix of the element a of a local field.

```
Norm(a)
Norm(a, F)
Trace(a)
Trace(a, F)
MinimalPolynomial(a)
MinimalPolynomial(a, F)
```

Return the norm, trace or minimal polynomial of a. If a coefficient field F of the parent L of a is given then the norm, trace or minimal polynomial will be that of a as an element of L represented as an extension of F.

Example H48E7

Continuing from the first example we have :

```
> UniformizingElement(L);
a^2 + (6 + O(7^50))*a + 3 + O(7^50)
> InertialElement(L);
a + O(7^50)
> Valuation(UniformizingElement(L));
1
> Valuation(InertialElement(L));
0
> Eltseq(UniformizingElement(L));
[ 3 + O(7^50), 6 + O(7^50), 1 + O(7^50) ]
```

48.8 Polynomial Factorization

Polynomials over local fields can be factored and their roots computed.

```
Factorization(f)
```

`Certificates` Bool<span style="font-variant:small-caps"></span>Elt *Default :* `false`

The factorization of the polynomial f over a local field defined by an arbitrary polynomial. The factorization is returned as a sequence of tuples of prime polynomials and exponents along with a scalar factor. If the parameter `Certificates` is `true` then certificates proving the primality of each prime are also returned.

```
SuggestedPrecision(f)
```

For a polynomial f over a general local field, return a precision at which the factorization of f as given by `Factorization(f)` will be Hensel liftable to the correct factorization.

The precision returned is not guaranteed to be enough to obtain a factorization of the polynomial. It may be that a correct factorization cannot be found at that precision but may be possible with a little more precision.

```
Roots(f)
Roots(f, R)
```

The roots of the polynomial f over the general local field R where R is taken to be the coefficient ring of f if it is not given.

Example H48E8

```
> Q3:=pAdicField(3,40);
> Q3X<x>:=PolynomialRing(Q3);
> L<a>:=LocalField(Q3,x^6-6*x^4+9*x^2-27);
> Factorization(Polynomial(L,x^6-6*x^4+9*x^2-27));
[
    <(O(3^38)*a^5 + O(3^38)*a^4 + O(3^39)*a^3 + O(3^39)*a^2 + O(3^40)*a + (1 +
        O(3^40)))*$.1 + (5*3^35 + O(3^38))*a^5 + O(3^38)*a^4 - (10*3^36 +
        O(3^39))*a^3 + O(3^39)*a^2 + (2701703435345984179 + O(3^40))*a +
        O(3^40), 1>,
    <(O(3^38)*a^5 + O(3^38)*a^4 + O(3^39)*a^3 + O(3^39)*a^2 + O(3^40)*a + (1 +
        O(3^40)))*$.1 + -(29642867960*3^-1 + O(3^23))*a^5 + O(3^24)*a^4 +
        (148214339800*3^-1 + O(3^24))*a^3 + O(3^25)*a^2 - (116512378274*3 +
        O(3^25))*a + O(3^26), 1>,
    <(O(3^38)*a^5 + O(3^38)*a^4 + O(3^39)*a^3 + O(3^39)*a^2 + O(3^40)*a + (1 +
        O(3^40)))*$.1 + -(29642867960*3^-1 + O(3^23))*a^5 + O(3^24)*a^4 +
        (148214339800*3^-1 + O(3^24))*a^3 + O(3^25)*a^2 - (349537134821 +
        O(3^25))*a + O(3^26), 1>,
    <(O(3^38)*a^5 + O(3^38)*a^4 + O(3^39)*a^3 + O(3^39)*a^2 + O(3^40)*a + (1 +
        O(3^40)))*$.1 + -(5*3^35 + O(3^38))*a^5 + O(3^38)*a^4 + (10*3^36 +
        O(3^39))*a^3 + O(3^39)*a^2 - (2701703435345984179 + O(3^40))*a +
        O(3^40), 1>,
    <(O(3^38)*a^5 + O(3^38)*a^4 + O(3^39)*a^3 + O(3^39)*a^2 + O(3^40)*a + (1 +
        O(3^40)))*$.1 + (29642867960*3^-1 + O(3^23))*a^5 + O(3^24)*a^4 -
        (148214339800*3^-1 + O(3^24))*a^3 + O(3^25)*a^2 + (116512378274*3 +
        O(3^25))*a + O(3^26), 1>,
    <(O(3^38)*a^5 + O(3^38)*a^4 + O(3^39)*a^3 + O(3^39)*a^2 + O(3^40)*a + (1 +
        O(3^40)))*$.1 + (29642867960*3^-1 + O(3^23))*a^5 + O(3^24)*a^4 -
        (148214339800*3^-1 + O(3^24))*a^3 + O(3^25)*a^2 + (349537134821 +
        O(3^25))*a + O(3^26), 1>
]
O(3^38)*a^5 + O(3^38)*a^4 + O(3^39)*a^3 + O(3^39)*a^2 + O(3^40)*a + 1 + O(3^40)
```

49 POWER, LAURENT AND PUISEUX SERIES

Chapter 49
POWER, LAURENT AND PUISEUX SERIES

49.1 Introduction

This chapter describes the operations on formal power, Laurent and Puiseux series available in Magma.

49.1.1 Kinds of Series

Internally Magma has only one kind of formal series, where general fractional exponents are allowed, but externally there are three kinds of series: power, Laurent and Puiseux. Power series must have a non-negative integral valuation and integral exponents, Laurent series must have an integral valuation and integral exponents (possibly negative), while Puiseux series may have a general rational valuation and general rational exponents. The main reason for the three kinds is simply to supply error checking for operations illegal to a specific kind, so the user will not move into the next kind of exponents inadvertently. Apart from these error checks, there is no difference at all between the different kinds of series, so one can simply use Puiseux series always if the restrictions on exponents are not desired, and this will cause no loss of efficiency or functionality at all even if all exponents remain integral or integral and non-negative.

In Magma, the set of all power series over a given base ring R is called a power series ring (denoted by $R[[x]]$) and the category names are `RngSerPow` for the ring and `RngSerPowElt` for the elements. The set of all Laurent series over a given base ring R is called a Laurent series ring (denoted by $R((x))$) and the category names are `RngSerLaur` for the ring and `RngSerLaurElt` for the elements. Finally, the set of all Puiseux series over a given base ring R is called a Puiseux series ring (denoted by $R\langle\langle x\rangle\rangle$) and the category names are `RngSerPuis` for the ring and `RngSerPuisElt` for the elements. For the rest of this chapter, the term "series ring" refers to any of the above rings and the corresponding virtual category name is `RngSer`; the term "series" refers to any of the above series kinds and the corresponding virtual category name is `RngSerElt`.

49.1.2 Puiseux Series

Puiseux series in a variable x are often mathematically defined to be Laurent series in another variable y say, where $y = x^{1/d}$, for a fixed positive integer d; this d is usually fixed for all the series under consideration. Magma is more general, in that although each series is internally represented in this way (i.e., its valuation, exponents and precision are thought to be divided by a single denominator associated with it), different Puiseux series may have different exponent denominators and may be freely mixed (for example, in addition, where the exponent denominator of the result will be derived from that of the arguments). Thus there is no restriction whatsoever to a fixed exponent denominator for a given Puiseux series ring.

49.1.3 Representation of Series

Formal series are stored in truncated form, unless only finitely many terms are non-zero. If a series has precision p, that means that its coefficients are unknown from exponent p onwards. This truncated form is similar to the floating point representation of real numbers except for one significant difference: because for series terms do not "carry" in arithmetic operations like they do in integer and real arithmetic, error propagation common in floating point methods does not occur. Consequently, given a result of any sequence of arithmetic operations with formal series, the coefficients of the known terms (the ones up to the given precision) will always be *exactly correct*, with no error (assuming the coefficient ring has exact arithmetic for its elements of course).

Elements

$$c_v x^v + c_{v+1} x^{v+1} + \ldots$$

(with $v \in \mathbf{Q}$ and $r_v \neq 0$) of series ring over a commutative ring R are stored in the form of approximations

$$c_v x^v + c_{v+1} x^{v+1} + \ldots + O(x^p)$$

to a certain *precision* $p \geq v$. The $O(x^p)$ notation is used to refer to terms of degree at least the precision p in x. For Laurent series v must be an integer and for power series v must be a non-negative integer. Note that for Puiseux series the above element is actually internally stored in the form

$$c_w x^{w/d} + c_{w+1} x^{(w+1)/d} + \ldots + O(x^{q/d})$$

where $v = w/d$ and $p = q/d$ and d (the exponent denominator) is minimal.

49.1.4 Precision

Associated with any series there are two types of precision: the *absolute precision* and the *relative precision*. For the element

$$c_v x^v + c_{v+1} x^{v+1} + \cdots + O(x^p)$$

(with $v \in \mathbf{Q}$ and $r_v \neq 0$), the absolute precision is p and the relative precision is $p - v$.

The absolute precision indicates the largest known term, and the relative precision indicates the size of the range over which terms are known. The two types of precision and the *valuation* of the series, which equals v in the above, are therefore always related via $p = v + r$.

49.1.5 Free and Fixed Precision

For each type of series ring, there are two sub-kinds: a *free precision* ring (the usual case), where elements of the ring have arbitrary precision, and a *fixed precision* ring, where all elements of the ring have a fixed precision. In the latter case, for power series rings the fixed precision is absolute, while for Laurent and Puiseux series ring the fixed precision is relative.

The free precision rings most closely resemble the mathematical objects $R[[x]]$ and $R((x))$; elements in these free rings and fields carry their own precision with them. Operations usually return results to a precision that is maximal given the input (and the nature of the operation). Operations which are given infinite precision series but which must return finite precision series (e.g., division) return series whose precision is the *default* precision for the series ring (this is different from the fixed precision of fixed precision rings); the default precision is 20 by default but may be set to another value at creation by a parameter (see below). Elements of free structures that have finite series expansion (i.e., polynomials) can be created and stored exactly, with infinite (absolute and relative) precision. Also note that the relative precision will be 0 for approximations to 0.

The structures with fixed precision, which we will sometimes refer to as *quotient* structures, behave differently. All elements in a *power* series ring of fixed precision have the same fixed *absolute* precision p, and the relative precision may be anything from 0 to p. This means that the ring with fixed precision p behaves like a quotient of a polynomial ring, $R[x]/x^p$. All elements in a *Laurent* or *Puiseux* series ring of fixed precision have the same fixed *relative* precision; the only exception to this rule is that 0 in the ring is stored as zero with infinite absolute precision. The absolute precision of elements in a free Laurent or Puiseux series ring can be anything.

49.1.6 Equality

Since a given series ring will contain series truncated at arbitrary precision, care has to be taken as to the meaning of equality of two elements. Two series are considered equal iff they are identical (the equality operator `eq` follows this convention). But we call two series A and B in the same ring *weakly equal* if and only if their coefficients are the same whenever both are defined, that is, if and only if for every $n \leq p$ the coefficients A_n and B_n are equal, where p is the minimum of the precisions of A and B. Thus, for example, $A = 3+x+O(x^2)$ and $B = 3+x+17x^2+O(x^4)$ are the same, but $C = 3+x+17x^2+x^3+O(x^4)$ is different from B. Note that A and C are equal under this definition, and hence weak equality is not transitive!

49.1.7 Polynomials over Series Rings

For a discussion of operations for polynomials over series rings see Chapter 54 and Section 49.7.

49.2 Creation Functions

49.2.1 Creation of Structures

```
PowerSeriesRing(R)
```

```
PowerSeriesRing(R, p)
```

`Global`	BoolElt	*Default :* `true`
`Precision`	RngIntElt	*Default :* 20

Given a commutative ring R, create the ring $R[[x]]$ of formal power series over R. If a second integer argument p is given, the resulting ring is a fixed precision series ring with fixed precision p; otherwise the resulting ring is a free precision series ring and the optional argument `Precision` may be used to set the default precision for elements of the power series ring (it will be 20 otherwise; see the section above on free and fixed precision). By default, a global series ring will be returned; if the parameter `Global` is set to `false`, a non-global series ring will be returned (to which a separate name for the indeterminate can be assigned). The angle bracket notation can be used to assign a name to the indeterminate: `S<x> := PowerSeriesRing(R)`.

```
LaurentSeriesRing(R)
LaurentSeriesRing(R, p)
```

`Global`	BoolElt	*Default :* `true`
`Precision`	RngIntElt	*Default :* 20

Given a commutative ring R, create the ring $R((x))$ of formal Laurent series over R. If a second integer argument p is given, the resulting ring is a fixed precision series ring with fixed precision p; otherwise the resulting ring is a free precision series ring and the optional argument `Precision` may be used to set the default precision for elements of the power series ring (it will be 20 otherwise; see the section above on free and fixed precision). By default, a global series ring will be returned; if the parameter `Global` is set to `false`, a non-global series ring will be returned (to which a separate name for the indeterminate can be assigned). The angle bracket notation can be used to assign a name to the indeterminate: `S<x> := LaurentSeriesRing(R)`.

```
PuiseuxSeriesRing(R)
PuiseuxSeriesRing(R, p)
```

`Global`	BoolElt	*Default :* `true`
`Precision`	RngIntElt	*Default :* 20

Given a commutative ring R, create the ring $R\langle\langle x\rangle\rangle$ of formal Puiseux series over R. If a second integer argument p is given, the resulting ring is a fixed precision series ring with fixed precision p; otherwise the resulting ring is a free precision series ring and the optional argument `Precision` may be used to set the default precision for elements of the power series ring (it will be 20 otherwise; see the section above on free and fixed precision). The optional argument `Precision` may be used to set the default precision for elements of the power series ring. By default, a global series ring will be returned; if the parameter `Global` is set to `false`, a non-global series ring will be returned (to which a separate name for the indeterminate can be assigned). The angle bracket notation can be used to assign a name to the indeterminate: `S<x> := PuiseuxSeriesRing(R)`.

Example H49E1

We demonstrate the difference between global and non-global rings. We first create the global power series ring over **Q** twice.

```
> Q := RationalField();
> P<x> := PowerSeriesRing(Q);
> PP := PowerSeriesRing(Q);
> P;
Power series ring in x over Rational Field
> PP;
Power series ring in x over Rational Field
> PP.1;
x
```

PP is identical to *P*. We now create non-global series rings (which are also different to the global series ring). Note that elements of all the rings are mathematically equal by automatic coercion.

```
> Pa<a> := PowerSeriesRing(Q: Global := false);
> Pb<b> := PowerSeriesRing(Q: Global := false);
> Pa;
Power series ring in a over Rational Field
> Pb;
Power series ring in b over Rational Field
> a;
a
> b;
b
> P;
Power series ring in x over Rational Field
> x;
x
> x eq a; // Automatic coercion
true
> x + a;
2*x
```

49.2.2 Special Options

`AssertAttribute(S,"DefaultPrecision", n)`

Procedure to change the default precision on a free series ring series *S*; the default precision will be set to *n*, which must be a non-negative integer.

`HasAttribute(S, "DefaultPrecision")`

Function that returns a Boolean indicating whether a default precision has been set on the free series ring *S* (which will always be `true`), as well as its (non-negative) integer value (which is 20 by default).

```
AssignNames(~S, ["x"])
```

Procedure to change the name of the 'indeterminate' transcendental element generating the series ring or field S; the name (used in printing elements of S) is changed to the string x. Note that no assignment to the identifier x is made (so x cannot be used for the specification of elements of S without further assignment).

```
Name(S, 1)
```

```
S . 1
```

Return the element of the series ring or field with a name attached to it, that is, return the 'indeterminate' transcendental element generating S over its coefficient ring.

49.2.3 Creation of Elements

The easiest way to create power and Laurent series in a given ring is to use the angle bracket construction to attach names to the indeterminate, and to use these names to express the series (see the examples). Below we list other options.

```
R . 1
```

```
UniformizingElement(R)
```

Return the generator (indeterminate) for the series ring R.

```
elt< R | v, [ a1, ..., ad], p >
```

Given a series ring R, integers v and p (where $p > 0$ or $p = \infty$), and a sequence $a = [a_1, \ldots, a_d]$ of elements of R, create the element in R with valuation v, known coefficients given by a and relative precision p. That is, this function returns the series $a_1x^v + \cdots + a_dx^{v+d-1} + O(x^{v+p})$, or, if $p = -1$, the exact series $a_1x^v + \cdots + a_dx^{v+d-1}$. If R is a power series ring, then v must be non-negative.

The integer v or the integer p or both may be omitted. If v is omitted, it will be set to zero by default; if p is omitted it will be taken to be $v + d$, where d is the length of the sequence a.

```
R ! s
```

Coerce s into the series ring R. Here s is allowed to be a sequence of elements from (or coercible into) the coefficient ring of R, or just an element from (or coercible into) R. A sequence $[a_1, \ldots, a_d]$ is converted into the series $a_1+a_2x^1+\cdots+a_dx^{d-1}+O(x^d)$.

`BigO(f)`

`O(f)`

Create the series $O(x^v)$ where x is the generator of the parent of f and v is the valuation of f. The most typical usage of this function is the expression $O(x^n)$ where x is the generator of a series ring, but a general series f is actually allowed.

`One(Q)` `Identity(Q)`

`Zero(Q)` `Representative(Q)`

49.3 Structure Operations

49.3.1 Related Structures

`Parent(R)` `Category(R)`

`BaseRing(R)`

`CoefficientRing(R)`

Return the coefficient ring of the series ring R.

`IntegerRing(R)`

`Integers(R)`

`RingOfIntegers(R)`

Return the power series ring which is the integer ring of the laurent series ring R.

`FieldOfFractions(R)`

Return the laurent series ring which is the field of fractions of the series ring R.

`ChangePrecision(R, r)`

`ChangePrecision(~R, r)`

Return a series ring identical to the series ring R but having precision r.

`ChangeRing(R, C)`

Return the series ring identical to the series ring R but having coefficient ring C.

`ResidueClassField(R)`

Return the residue class field of the series ring R (which will be the same as the coefficient ring of R) and the map from R into the residue class field.

49.3.2 Invariants

`Characteristic(R)`

`Precision(R)`

`GetPrecision(R)`

Return the precision of the fixed precision series ring R. If R is a fixed precision power series ring, then this is the fixed absolute precision for all elements of the ring. If R is a fixed precision Laurent series ring, then this is the maximum relative precision for all elements of the ring.

49.3.3 Ring Predicates and Booleans

`IsCommutative(Q)` `IsUnitary(Q)`

`IsFinite(Q)` `IsOrdered(Q)`

`IsField(Q)` `IsEuclideanDomain(Q)`

`IsPID(Q)` `IsUFD(Q)`

`IsDivisionRing(Q)` `IsEuclideanRing(Q)`

`IsPrincipalIdealRing(Q)` `IsDomain(Q)`

`R eq S` `R ne S`

49.4 Basic Element Operations

49.4.1 Parent and Category

`Parent(r)` `Category(r)`

49.4.2 Arithmetic Operators

`+ b` `- b`

`a + b` `a - b` `a * b` `a ^ k`

`a div b` `a / b`

49.4.3 Equality and Membership

`a eq b` `a ne b`

`a in R` `a notin R`

49.4.4 Predicates on Ring Elements

Note the definition of equality in the introduction to this Chapter. This not only affects the result of the application of `eq` and `ne`, but also that of `IsOne, IsZero` and `IsMinusOne`.

`IsZero(a)` `IsOne(a)` `IsMinusOne(a)`

`IsNilpotent(x)` `IsIdempotent(x)`

`IsUnit(a)` `IsZeroDivisor(x)` `IsRegular(x)`

`IsIrreducible(x)` `IsPrime(x)`

`IsWeaklyZero(f)`

Given a series f, return whether f is weakly zero, which is whether f is exactly zero or of the form $O(x^p)$ for some p.

`IsWeaklyEqual(f, g)`

Given series f and g, return whether f is weakly equal to g, which is whether $(f-g)$ is weakly zero (see `IsWeaklyZero`).

`IsIdentical(f, g)`

Given series f and g, return whether f is identical to g, which is whether f and g have exactly the same valuation, precision, and coefficients.

49.4.5 Precision

`AbsolutePrecision(f)`

Given a series f, this returns the absolute precision that is stored with f. If f is a series in x, the absolute precision of f is the exponent p such that x^p is the first term of f of which the coefficient is not known, that is, it is the least p such that $f \in O(x^p)$. If f is known exactly (in a free ring), the absolute precision is infinite and an error occurs. Note that the absolute precision may be a non-integral rational number if f is a Puiseux series.

```
RelativePrecision(f)
```

Given a series f, this returns the relative precision that is stored with f. The relative precision counts the number of coefficients of f that is known, starting at the first non-zero term. Hence the relative precision is the difference between the absolute precision and the valuation of f, and is therefore always non-negative; however, if f is exact, the relative precision is infinite and the value ∞ is returned. Note that the relative precision may be a non-integral rational number if f is a Puiseux series.

```
ChangePrecision(f, r)
ChangePrecision(~f, r)
```

The (non puiseux) series f with absolute precision r (which can be positive infinity).

49.4.6 Coefficients and Degree

```
Coefficients(f)
ElementToSequence(f)
Eltseq(f)
```

Let f be a series with coefficients in a ring R and with indeterminate x. This function returns the sequence Q of coefficients of f, the unscaled valuation v and the exponent denominator d of f (v is the true valuation of f multiplied by d). The i-th entry $Q[i]$ of Q equals the coefficient of

$$x^{\frac{v+i-1}{d}}$$

in f. Thus the first entry of Q is the 'first' (lowest order) non-zero coefficient of f, i.e., the coefficient of x^w where w is the true valuation of f.

```
Coefficient(f, i)
```

Given a series f with coefficients in a ring R, and a rational or integer i, return the coefficient of the i-th power of the indeterminate x of f as an element of R. If f is a Puiseux series i may be a (non-integral) rational; otherwise i must be an integer (and also must be non-negative if f is a power series). Also, i must be less than p, the precision of f.

```
LeadingCoefficient(f)
```

Given a series f with coefficients in a ring R, return the leading coefficient of f as an element of R, which is the first non-zero coefficient of f (i.e., the coefficient x^v in f, where x is the indeterminate of f and v is the valuation of f).

```
LeadingTerm(f)
```

Given a series f with coefficients in a ring R, return the leading term of f, which is the first non-zero term of f (i.e., the term of f whose monomial is x^v, where x is the indeterminate of f and v is the valuation of f).

`Truncate(f)`

Given a series f, return the exact series obtained by truncating f after the last known non-zero coefficient.

`ExponentDenominator(f)`

Given a series f, return the exponent denominator of f, i.e., the lowest common denominator of all the exponents of the non-zero terms of f (always an integer). For power series and Laurent series, this will always be 1 of course.

`Degree(f)`

Given a series f, return the degree of the truncation of f, that is, the exponent of the last known non-zero term. Note that this may be a non-integral rational number if f is a Puiseux series.

`Valuation(f)`

Given a series f, return the smallest integer v (possibly negative for Laurent series) such that the coefficient of x^v in f is not known to be zero. For the exact 0 element (in a free ring), the valuation is ∞. Note that the valuation may be a non-integral rational number if f is a Puiseux series.

`ExponentDenominator(f)`

The exponent denominator of the series f. This is the lowest common denominator of the exponents of the non-zero terms of f.

49.4.7 Evaluation and Derivative

`Derivative(f)`

Given a series $f \in R$, return the derivative of f with respect to its indeterminate, as an element of R. Note that the precision decreases by 1 (unless f has infinite precision).

`Derivative(f, n)`

Given a series $f \in R$ and an integer $n > 0$, return the n-th derivative of f with respect to its indeterminate, as an element of R. Note that the precision decreases by n (unless f has infinite precision).

`Integral(f)`

Given a series $f \in R$, return an anti-derivative F of f with respect to its indeterminate, which is an element of R which has derivative f. The coefficient of x^{-1} in f must be zero. Note that the precision of F will be exceeding that of f by 1 (unless f has infinite precision).

`Evaluate(f, s)`

Given an element f of a series ring over the coefficient ring R, and an element s of the ring S, return the value of $f(s)$ when the indeterminate x is evaluated at s. The result will be an element of the common overstructure over R and S.

`Laplace(f)`

The Laplace transform of the series f; if f has expansion $\sum_{i\geq 0} a_i x^i$, its Laplace transform has expansion $\sum_{i\geq 0}(i!a_i)x^i$. The valuation of f must be integral and non-negative.

49.4.8 Square Root

`SquareRoot(f)`

`Sqrt(f)`

Return the square root of the series f, f must have even valuation if it is a power or Laurent series.

49.4.9 Composition and Reversion

`Composition(f, g)`

Given elements f and g from the same series ring P, return their composition, defined by

$$f \circ g = \sum_{i<p} f_i(g^i),$$

where $f = \sum_{i<p} f_i x^i$.

`Reversion(f)`

`Reverse(f)`

Given a series f (in x, say), this returns the inverse of f under composition, that is, an element g of the same power series ring such that its composition with f equals x to the best possible precision. If f is a power or Laurent series, the valuation of f must be 1. If f is a Puiseux series, the valuation of f must be positive (but need not equal 1), and if the valuation of f is not 1, the leading coefficient of f must be 1.

`Convolution(f, g)`

Given elements f and g from the same series ring P, return their convolution $f * g$, defined by

$$f * g = \sum_{i<\min(p,q)} f_i g_i x^i,$$

where $f = \sum_{i<p} f_i x^i + O(x^p)$ and $g = \sum_{i<q} g_i x^i + O(x^q)$.

Example H49E2

We demonstrate the functions `Composition` and `Reversion`. First we check that `Arcsin` is the reversion of `Sin`.

```
> S<x> := PowerSeriesRing(RationalField());
> f := Sin(x);
> g := Arcsin(x);
> f;
x - 1/6*x^3 + 1/120*x^5 - 1/5040*x^7 + 1/362880*x^9 -
    1/39916800*x^11 + 1/6227020800*x^13 - 1/1307674368000*x^15 +
    1/355687428096000*x^17 - 1/121645100408832000*x^19 + O(x^21)
> g;
x + 1/6*x^3 + 3/40*x^5 + 5/112*x^7 + 35/1152*x^9 + 63/2816*x^11 +
    231/13312*x^13 + 143/10240*x^15 + 6435/557056*x^17 +
    12155/1245184*x^19 + O(x^21)
> Composition(f, g);
x + O(x^21)
> Composition(g, f);
x + O(x^21)
> Reversion(f) - g;
O(x^21)
> Reversion(g) - f;
O(x^21)
```

Next we compute the reversion of a series whose valuation is not 1.

```
> S<x> := PuiseuxSeriesRing(RationalField());
> f := x^3 - x^5 + 2*x^8;
> r := Reversion(f);
> f;
x^3 - x^5 + 2*x^8
> r;
x^(1/3) + 1/3*x + 4/9*x^(5/3) - 2/3*x^2 + 65/81*x^(7/3) -
    22/9*x^(8/3) + 5/3*x^3 - 208/27*x^(10/3) + 5005/729*x^(11/3)
    - 70/3*x^4 + 206264/6561*x^(13/3) - 50830/729*x^(14/3) +
    134*x^5 - 498674/2187*x^(16/3) + 31389020/59049*x^(17/3) +
    O(x^6)
> Composition(r, f);
x + O(x^18)
> Composition(f, r);
x + O(x^(20/3))
```

Finally we compute the reversion of a proper Puiseux series.

```
> f := x^(2/5) - x^(2/3) + x^(3/2) + O(x^2);
> r := Reversion(f);
> r;
x^(5/2) + 5/2*x^(19/6) + 145/24*x^(23/6) + 715/48*x^(9/2) +
    389795/10368*x^(31/6) - 5/2*x^(21/4) + O(x^(11/2))
> Composition(f, r);
```

```
x + O(x^4)
> Composition(r, f);
x + O(x^(11/5))
```

49.5 Transcendental Functions

In each of the functions below, the precision of the result will be approximately equal to the precision of the argument if that is finite; otherwise it will be approximately equal to the default precision of the parent of the argument. An error will result if the coefficient ring of the series is not a field. If the first argument has a non-zero constant term, an error will result unless the coefficient ring of the parent is a real or complex domain (so that the transcendental function can be evaluated in the constant term).

See also the chapter on real and complex fields for elliptic and modular functions which are also defined for formal series.

49.5.1 Exponential and Logarithmic Functions

`Exp(f)`

Given a series f defined over a field, return the exponential of f.

`Log(f)`

Given a series f defined over a field of characteristic zero, return the logarithm of f. The valuation of f must be zero.

Example H49E3

In this example we show how one can compute the Bernoulli number B_n for given n using generating functions. The function `BernoulliNumber` now actually uses this method (since V2.4). Using this method, the Bernoulli number B_{10000} has been computed, taking about 14 hours on a 250MHz Sun Ultrasparc (the computation depends on the new asymptotically fast methods for series division).

The exponential generating function for the Bernoulli numbers is:

$$E(x) = \frac{x}{e^x - 1}.$$

This means that the n-th Bernoulli number B_n is $n!$ times the coefficient of x^n in $E(x)$. The Bernoulli numbers $B_0, \ldots, B_n$ for any n can thus be calculated by computing the above power series and scaling the coefficients.

In this example we will compute B_{500}. We first set the final coefficient index n we desire to be 500. We then create the denominator $D = e^x - 1$ of the exponential generating function to precision $n+2$ (we need $n+2$ since we lose precision when we divide by the denominator and the valuation changes). For each series we create, we print the sum of it and $\mathrm{O}(x^{20})$, which will only print the terms up to x^{19}.

```
> n := 500;
> S<x> := LaurentSeriesRing(RationalField(), n + 2);
> time D := Exp(x) - 1;
Time: 0.040
> D + O(x^20);
x + 1/2*x^2 + 1/6*x^3 + 1/24*x^4 + 1/120*x^5 + 1/720*x^6 + 1/5040*x^7 +
    1/40320*x^8 + 1/362880*x^9 + 1/3628800*x^10 + 1/39916800*x^11 +
    1/479001600*x^12 + 1/6227020800*x^13 + 1/87178291200*x^14 +
    1/1307674368000*x^15 + 1/20922789888000*x^16 + 1/355687428096000*x^17 +
    1/6402373705728000*x^18 + 1/121645100408832000*x^19 + O(x^20)
```

We then form the quotient $E = x/D$ which gives the exponential generating function.

```
> time E := x / D;
Time: 5.330
> E + O(x^20);
1 - 1/2*x + 1/12*x^2 - 1/720*x^4 + 1/30240*x^6 - 1/1209600*x^8 +
    1/47900160*x^10 - 691/1307674368000*x^12 + 1/74724249600*x^14 -
    3617/10670622842880000*x^16 + 43867/5109094217170944000*x^18 + O(x^20)
```

We finally compute the Laplace transform of E (which multiplies the coefficient of x^i by $i!$) to yield the generating function of the Bernoulli numbers up to the x^n term. Thus the coefficient of x^n here is B_n.

```
> time B := Laplace(E);
Time: 0.289
> B + O(x^20);
1 - 1/2*x + 1/6*x^2 - 1/30*x^4 + 1/42*x^6 - 1/30*x^8 + 5/66*x^10 -
    691/2730*x^12 + 7/6*x^14 - 3617/510*x^16 + 43867/798*x^18 + O(x^20)
> Coefficient(B, n);
-165963806405685572298521230880771342066586643028066718923526509931553316412206\
96008401495608813577092146502532394280920785185799286021346378325274540909642O\
93250995316546673567548597903481761998372720984429108190814559782967498015988 9\
976244240633746601120703300698329029710482600069717866917229113749797632930033\
559794717838407415772796504419464932337498642714226081743688706971990010734262\
076881238322867559275748219588404488023034528296023051638858467185173202483888\
794342720837413737644410765563213220043477396887812891242952336301344808165757\
942109887803692579439427973561487863524556256869403384306433922049078300720480\
361757680714198044230522015775475287075315668886299978958150756677417180004362\
981454396613646612327019784141740499835461/8365830
> n;
500
```

49.5.2 Trigonometric Functions and their Inverses

`Sin(f)`

Given a series f defined over a field of characteristic zero, return the sine of f. The valuation of f must be zero.

`Cos(f)`

Given a series f defined over a field of characteristic zero, return the cosine of f. The valuation of f must be zero.

`Sincos(f)`

Given a series f defined over a field of characteristic zero, return both the sine and cosine of f. The valuation of f must be zero.

`Tan(f)`

Return the tangent of the series f defined over a field.

`Arcsin(f)`

Given a series f defined over a field of characteristic zero, return the inverse sine of f.

`Arccos(f)`

Given a series f defined over the real or complex field, return the inverse cosine of f.

`Arctan(f)`

Given a series f defined over a field of characteristic zero, return the inverse tangent of f.

49.5.3 Hyperbolic Functions and their Inverses

`Sinh(f)`

Given a series f defined over a field, return the hyperbolic sine of f.

`Cosh(f)`

Given a series f defined over a field, return the hyperbolic cosine of f.

`Tanh(f)`

Given a series f defined over a field, return the hyperbolic tangent of f.

`Argsinh(f)`

Given a series f defined over a field of characteristic zero, return the inverse hyperbolic sine of f.

`Argcosh(f)`

Given a series f defined over the real or complex field, return the inverse hyperbolic cosine of f.

`Argtanh(f)`

Given a series f defined over a field of characteristic zero, return the inverse hyperbolic tangent of f.

49.6 The Hypergeometric Series

For more information on the Hypergeometric Series, see [Hus87], page 176.

`HypergeometricSeries(a,b,c, z)`

Return the hypergeometric series $F(a,b,c;z)$ defined by

$$F(a,b,c;z) = \sum_{0\leq n} \frac{(a)_n(b)_n}{n!(c)_n z^n}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$.

49.7 Polynomials over Series Rings

Factorization is available for polynomials over series rings defined over finite fields. We recommend constructing polynomials from sequences rather than by addition of terms, especially over fields, to avoid some precision loss. For example, $x^4 + t$ will not have full precision in the constant coefficient as there will be a O term in the constant coefficient of x^4 which will reduce its precision as a field element. The equivalent polynomial `Polynomial([t, 0, 0, 0, 1])` will have more precision than that constructed as $x^4 + t$.

`HenselLift(f, L)`

Given a polynomial f over a series ring over a finite field or an extension of a series ring and a factorization L of f to precision 1 return a factorization of f known to the full precision of the coefficient ring of f.

`Factorization(f)`

`Certificates`	BoolElt	*Default :* `false`
`Ideals`	BoolElt	*Default :* `false`
`Extensions`	BoolElt	*Default :* `false`

The factorization of the polynomial f over a power series ring, laurent series field over a finite field or an extension of either into irreducibles.

If `Certificates` is set to `true`, a two-element certificate for each factor, proving its irreducibility, is returned.

If `Ideals` is set to `true`, two generators of some ideal for each factor are returned within the certificates.

If `Extensions` is set to `true`, an extension for each factor is returned within the certificates.

Example H49E4

We illustrate factorizations over series rings and the extensions which can be gained through them.

```
> P<t> := PowerSeriesRing(GF(101));
> R<x> := PolynomialRing(P);
> Factorization(x^5 + t*x^4 - t^2*x^3 + (1 + t^20)*x^2 + t*x + t^6);
[
    <(1 + O(t^20))*x + t^5 + O(t^8), 1>,
    <(1 + O(t^20))*x + t + 100*t^4 + 100*t^5 + t^7 + O(t^8), 1>,
    <(1 + O(t^20))*x + 1 + 34*t^2 + 34*t^3 + 34*t^4 + 90*t^5 + 29*t^6 + 16*t^8 +
        32*t^9 + 6*t^10 + 66*t^11 + 41*t^12 + 93*t^13 + t^14 + 69*t^15 + 8*t^16
        + 61*t^17 + 86*t^18 + 19*t^19 + O(t^20), 1>,
    <(1 + O(t^20))*x^2 + (100 + 67*t^2 + 67*t^3 + 68*t^4 + 11*t^5 + 72*t^6 +
        100*t^7 + 85*t^8 + 69*t^9 + 94*t^10 + 31*t^11 + 59*t^12 + 16*t^13 +
        14*t^14 + 35*t^15 + 77*t^16 + 28*t^17 + 33*t^18 + 72*t^19 + O(t^20))*x +
        1 + 67*t^2 + 68*t^3 + 11*t^4 + 67*t^5 + 57*t^6 + 16*t^7 + 57*t^8 +
        21*t^9 + 68*t^10 + 68*t^11 + 61*t^12 + 98*t^13 + 4*t^14 + 21*t^15 +
        7*t^16 + 95*t^17 + 23*t^18 + 76*t^19 + O(t^20), 1>
]
1 + O(t^20)
> P<t> := PowerSeriesRing(GF(101), 50);
> R<x> := PolynomialRing(P);
> Factorization(x^5 + t*x^4 - t^2*x^3 + (1 + t^20)*x^2 + t*x + t^6 :
> Extensions);
[
    <x + t^5 + t^9 + 2*t^13 + t^16 + 5*t^17 + 6*t^20 + 14*t^21 + 26*t^24 +
        42*t^25 + 3*t^27 + 4*t^28 + 32*t^29 + 35*t^31 + 10*t^32 + 29*t^33 +
        46*t^35 + t^36 + 31*t^37 + O(t^38), 1>,
    <x + t + 100*t^4 + 100*t^5 + t^7 + 100*t^9 + t^10 + 4*t^11 + t^12 + 91*t^13
        + 86*t^14 + 98*t^15 + 15*t^16 + 7*t^17 + 83*t^18 + 10*t^19 + 23*t^20 +
        54*t^21 + 47*t^22 + 72*t^23 + 87*t^24 + 55*t^25 + 4*t^26 + 15*t^27 +
        90*t^28 + 82*t^29 + 97*t^30 + 15*t^31 + 23*t^32 + 86*t^33 + 57*t^34 +
        10*t^35 + 79*t^36 + 52*t^37 + O(t^38), 1>,
    <x + 1 + 34*t^2 + 34*t^3 + 34*t^4 + 90*t^5 + 29*t^6 + 16*t^8 + 32*t^9 +
        6*t^10 + 66*t^11 + 41*t^12 + 93*t^13 + t^14 + 69*t^15 + 8*t^16 + 61*t^17
        + 86*t^18 + 19*t^19 + 47*t^20 + 11*t^21 + 42*t^22 + 38*t^23 + 46*t^24 +
        90*t^25 + 14*t^26 + 7*t^27 + 89*t^28 + 85*t^29 + 70*t^30 + 24*t^31 +
        28*t^32 + 71*t^33 + 53*t^34 + 55*t^35 + 90*t^36 + 26*t^37 + 4*t^38 +
        56*t^39 + 44*t^40 + 8*t^41 + 25*t^42 + 94*t^43 + 14*t^44 + 92*t^45 +
        56*t^46 + 83*t^47 + 26*t^48 + 41*t^49 + O(t^50), 1>,
    <x^2 + (100 + 67*t^2 + 67*t^3 + 68*t^4 + 11*t^5 + 72*t^6 + 100*t^7 + 85*t^8
        + 69*t^9 + 94*t^10 + 31*t^11 + 59*t^12 + 16*t^13 + 14*t^14 + 35*t^15 +
        77*t^16 + 28*t^17 + 33*t^18 + 72*t^19 + 25*t^20 + 22*t^21 + 12*t^22 +
        92*t^23 + 43*t^24 + 15*t^25 + 83*t^26 + 76*t^27 + 19*t^28 + 3*t^29 +
        35*t^30 + 27*t^31 + 40*t^32 + 16*t^33 + 92*t^34 + 91*t^35 + 32*t^36 +
        93*t^37 + 84*t^38 + 98*t^39 + 85*t^40 + 54*t^41 + 25*t^42 + 88*t^43 +
        35*t^44 + 17*t^45 + t^46 + 39*t^47 + 89*t^48 + 67*t^49 + O(t^50))*x + 1
        + 67*t^2 + 68*t^3 + 11*t^4 + 67*t^5 + 57*t^6 + 16*t^7 + 57*t^8 + 21*t^9
        + 68*t^10 + 68*t^11 + 61*t^12 + 98*t^13 + 4*t^14 + 21*t^15 + 7*t^16 +
        95*t^17 + 23*t^18 + 76*t^19 + 62*t^20 + 59*t^21 + 66*t^22 + 35*t^23 +
        41*t^24 + 45*t^25 + 32*t^26 + 56*t^27 + 35*t^28 + 19*t^29 + 21*t^30 +
        59*t^31 + 50*t^32 + 72*t^33 + 58*t^34 + 75*t^35 + 59*t^36 + 76*t^37 +
        83*t^38 + 66*t^39 + 6*t^40 + 8*t^41 + 94*t^42 + 77*t^43 + 100*t^44 +
        30*t^45 + 72*t^46 + 26*t^47 + 54*t^48 + 21*t^49 + O(t^50), 1>
]
1 + O(t^50)
[
    rec<recformat<F: RngIntElt, Rho: RngUPolElt, E: RngIntElt, Pi: RngUPolElt,
    IdealGen1, IdealGen2: RngUPolElt, Extension> |
        F := 1,
        Rho := 1 + O($.1^50),
        E := 1,
        Pi := $.1 + O($.1^50),
        Extension := Power series ring in t over GF(101) with fixed absolute
            precision 50>,
    rec<recformat<F: RngIntElt, Rho: RngUPolElt, E: RngIntElt, Pi: RngUPolElt,
    IdealGen1, IdealGen2: RngUPolElt, Extension> |
        F := 1,
        Rho := 1 + O($.1^50),
        E := 1,
        Pi := $.1 + O($.1^50),
        Extension := Power series ring in t over GF(101) with fixed absolute
            precision 50>,
    rec<recformat<F: RngIntElt, Rho: RngUPolElt, E: RngIntElt, Pi: RngUPolElt,
    IdealGen1, IdealGen2: RngUPolElt, Extension> |
        F := 1,
        Rho := 1 + O($.1^50),
        E := 1,
        Pi := $.1 + O($.1^50),
        Extension := Power series ring in t over GF(101) with fixed absolute
            precision 50>,
    rec<recformat<F: RngIntElt, Rho: RngUPolElt, E: RngIntElt, Pi: RngUPolElt,
    IdealGen1, IdealGen2: RngUPolElt, Extension> |
        F := 2,
        Rho := (1 + O($.1^50))*$.1 + O($.1^50),
        E := 1,
        Pi := $.1 + O($.1^50),
        Extension := Extension of Power series ring in t over GF(101) with fixed
            absolute precision 50 by x^2 + (100 + O(t^50))*x + 1 + O(t^50)>
]
```

49.8 Extensions of Series Rings

Extensions of series rings are either unramified or totally ramified. Only series rings defined over finite fields can be extended. We recommend constructing polynomials from sequences rather than by addition of terms, especially over fields, to avoid some precision loss. For example, $x^4 + t$ will not have full precision in the constant coefficient as there will be a O term in the constant coefficient of x^4 which will reduce its precision as a field element. Extensions require full precision polynomials and some polynomials such as $x^4 + t$ may not have enough precision to be used to construct an extension whereas the equivalent `Polynomial([t, 0, 0, 0, 1])` will.

49.8.1 Constructions of Extensions

```
UnramifiedExtension(R, f)
```

Construct the unramified extension of R, a series ring or an extension thereof, defined by the inertial polynomial f, that is, adjoin a root of f to R.

```
TotallyRamifiedExtension(R, f)
```

`MaxPrecision` RngIntElt *Default :*

Construct a totally ramified extension of R, a series ring or an extension thereof, defined by the eisenstein polynomial f, that is, adjoin a root of f to R.

The parameter `MaxPrecision` defaults to the precision of R. It can be set to the maximum precision the coefficients of f are known to, which must not be less than the precision of R. This allows the precision of the result to be increased to the degree of f multiplied by this maximum precision.

The polynomial f may be given over a series ring or an extension of a series ring having a higher precision than R. This allows the precision of the result to be increased up to the precision the polynomial is known to (or `MaxPrecision` if set) without losing any of the polynomial known past the precision of R.

The precision of a ramified extension cannot be increased unless the defining polynomial is known to more precision than the coefficient ring, indicated either by providing the polynomial to greater precision or by setting `MaxPrecision`. This should be taken into account when constructing ramified extensions, especially if polynomials are to be factored over the extension.

```
ChangePrecision(E, r)
```

```
ChangePrecision(~E, r)
```

The extension of a series ring E with precision r. It is not possible to increase the precision of a ramified extension unless the parameter `MaxPrecision` was set on construction of the extension and r is less than or equal to this value multiplied by the ramification degree or the polynomial used in creating the extension was given to more precision than the coefficient ring of E.

```
FieldOfFractions(E)
```

The field of fractions of the extension of a series ring E.

Example H49E5

We show a simple creation of a two-step extension and change its precision.

```
> P<t> := PowerSeriesRing(GF(101), 50);
> PP<tt> := PowerSeriesRing(GF(101));
> R<x> := PolynomialRing(PP);
> U := UnramifiedExtension(P, x^2 + 2);
> T := TotallyRamifiedExtension(U, x^2 + tt*x + tt); T;
Extension of Extension of Power series ring in t over GF(101) with fixed
absolute precision 50 by x^2 + 2 + O(t^50) by x^2 + (t + O(t^50))*x + t +
    O(t^50)
> Precision($1);
100
> ChangePrecision($2, 200);
Extension of Extension of Power series ring in $.1 over GF(101) with fixed
absolute precision 100 by x^2 + 2 + O($.1^100) by x^2 + ($.1 + O($.1^100))*x +
    $.1 + O($.1^100)
> ChangePrecision($1, 1000);
Extension of Extension of Power series ring in $.1 over GF(101) with fixed
absolute precision 500 by x^2 + 2 + O($.1^500) by x^2 + ($.1 + O($.1^500))*x +
    $.1 + O($.1^500)
> ChangePrecision($1, 20);
Extension of Extension of Power series ring in $.1 over GF(101) with fixed
absolute precision 10 by x^2 + 2 + O($.1^10) by x^2 + ($.1 + O($.1^10))*x + $.1
    + O($.1^10)
```

Both U and T have a field of fractions.

```
> FieldOfFractions(U);
Extension of Laurent series field in $.1 over GF(101) with fixed relative
precision 50 by (1 + O($.1^50))*x^2 + O($.1^50)*x + 2 + O($.1^50)
> FieldOfFractions(T);
Extension of Extension of Laurent series field in $.1 over GF(101) with fixed
relative precision 50 by (1 + O($.1^50))*x^2 + O($.1^50)*x + 2 + O($.1^50) by (1
    + O($.1^50))*x^2 + ($.1 + O($.1^50))*x + $.1 + O($.1^50)
```

49.8.2 Operations on Extensions

`Precision(E)`

`GetPrecision(E)`

The maximum precision elements of the extension of a series ring E may have.

`CoefficientRing(E)`

`BaseRing(E)`

The ring the extension of a series ring E was defined over.

```
DefiningPolynomial(E)
```

The polynomial used to define the extension E.

```
InertiaDegree(E)
```

The degree of the extension E if E is an unramified extension, 1 otherwise.

```
RamificationIndex(E)
RamificationDegree(E)
```

The degree of the extension E if E is a totally ramified extension, 1 otherwise.

```
ResidueClassField(E)
```

The residue class field of the extension E, that is, $E/\pi E$.

```
UniformizingElement(E)
```

A uniformizing element for the extension E, that is, an element of E of valuation 1.

```
IntegerRing(E)
Integers(E)
RingOfIntegers(E)
```

The ring of integers of the extension E of a series ring if E is a field (extension of a laurent series ring).

```
E1 eq E2
```

Whether extensions $E1$ and $E2$ are considered to be equal.

```
E . i
```

The primitive element of the extension E, that is, the root of the defining polynomial of E adjoined to the coefficient ring of E to construct E.

```
AssignNames(~E, S)
```

Assign the string in the sequence S to be the name of the primitive element of E, that is, the root of the defining polynomial adjoined to the coefficient ring of E to construct E.

Example H49E6

A number of the operations above are applied to a two-step extension

```
> L<t> := LaurentSeriesRing(GF(53), 30);
> P<x> := PolynomialRing(L);
> U := UnramifiedExtension(L, x^3 + 3*x^2 + x + 4);
> P<y> := PolynomialRing(U);
> T := TotallyRamifiedExtension(U, Polynomial([t, 0, 0, 0, 1]));
> Precision(U);
30
> Precision(T);
120
> CoefficientRing(U);
Laurent series field in t over GF(53) with fixed relative precision 30
> CoefficientRing(T);
Extension of Laurent series field in t over GF(53) with fixed relative precision
30 by (1 + O(t^30))*x^3 + (3 + O(t^30))*x^2 + (1 + O(t^30))*x + 4 + O(t^30)
> DefiningPolynomial(U);
(1 + O(t^30))*x^3 + (3 + O(t^30))*x^2 + (1 + O(t^30))*x + 4 + O(t^30)
> DefiningPolynomial(T);
(1 + O(t^30))*y^4 + O(t^30)*y^3 + O(t^30)*y^2 + O(t^30)*y + t + O(t^30)
> InertiaDegree(U);
3
> InertiaDegree(T);
1
> RamificationDegree(U);
1
> RamificationDegree(T);
4
> ResidueClassField(U);
Finite field of size 53^3
Mapping from: RngSerExt: U to GF(53^3)
> ResidueClassField(T);
Finite field of size 53^3
Mapping from: RngSerExt: T to GF(53^3)
> UniformizingElement(U);
t + O(t^31)
> UniformizingElement(T);
(1 + O(t^30))*$.1 + O(t^30)
> Integers(T);
Extension of Extension of Power series ring in $.1 over GF(53) with fixed
absolute precision 30 by x^3 + (3 + O($.1^30))*x^2 + x + 4 + O($.1^30) by x^4 +
    $.1 + O($.1^30)
> U.1;
(1 + O(t^30))*$.1 + O(t^30)
> T.1;
```

```
(1 + O(t^30))*$.1 + O(t^30)
```

49.8.3 Elements of Extensions

`x * y` `x + y` `x - y` `- x` `x ^ n` `x div y` `x / y`

`x eq y` `IsZero(e)` `IsOne(e)` `IsMinusOne(e)` `IsUnit(e)`

`Valuation(e)`

The valuation of the element e of an extension of a series ring. This is the index of the largest power of π which divides e.

`RelativePrecision(e)`

The relative precision of the element e of an extension of a series ring. This is the number of digits of e (in π) which are known.

`AbsolutePrecision(e)`

The absolute precision of the element e of an extension of a series ring. This is the same as the sum of the relative precision of e and the valuation of e.

`Coefficients(e)`

`Eltseq(e)`

`ElementToSequence(e)`

Given an element e of an extension of a series ring, return the coefficients of e with respect to the powers of the uniformizing element of the extension.

Example H49E7

We show some simple arithmetic with some elements of some extensions.

```
> P<t> := PowerSeriesRing(GF(101), 50);
> R<x> := PolynomialRing(P);
> U<u> := UnramifiedExtension(P, x^2 + 2*x + 3);
> UF := FieldOfFractions(U);
> R<y> := PolynomialRing(U);
> T<tt> := TotallyRamifiedExtension(U, y^2 + t*y + t);
> TF<tf> := FieldOfFractions(T);
> UF<uf> := FieldOfFractions(U);
> u + t;
u + t + O(t^50)
> uf * t;
($.1 + O($.1^50))*uf + O($.1^51)
> tf eq tt;
false
> tf - tt;
```

```
O($.1^50)*tf + O($.1^50)
> IsZero($1);
false
> Valuation($2);
100
> Valuation(tt);
1
> Valuation(U!t);
1
> Valuation(T!t);
2
> RelativePrecision(u);
50
> AbsolutePrecision(u);
50
> AbsolutePrecision(uf);
50
> RelativePrecision(uf);
50
> RelativePrecision(u - uf);
0
> AbsolutePrecision(u - uf);
50
> u^7;
(13 + O(t^50))*u + 71 + O(t^50)
> Coefficients($1);
[
    71 + O(t^50),
    13 + O(t^50)
]
> tt^8;
(4*t^4 + 91*t^5 + 6*t^6 + 100*t^7 + O(t^50))*tt + t^4 + 95*t^5 + 5*t^6 + 100*t^7
    + O(t^50)
> Coefficients($1);
[ t^4 + 95*t^5 + 5*t^6 + 100*t^7 + O(t^50), 4*t^4 + 91*t^5 + 6*t^6 + 100*t^7 +
    O(t^50) ]
```

49.8.4 Optimized Representation

```
OptimizedRepresentation(E)
```

```
OptimisedRepresentation(E)
```

An optimized representation for the unramified extension E of a series ring. The defining polynomial of E must be coercible into the residue class field. The result is a series ring over the residue class field of E and a map from E to this series ring.

Example H49E8

We give a simple example of the use of `OptimizedRepresentation`.

```
> P<t> := PowerSeriesRing(GF(101), 50);
> R<x> := PolynomialRing(P);
> U<u> := UnramifiedExtension(P, x^2 + 2*x + 3);
> U;
Extension of Power series ring in t over GF(101) with fixed absolute precision
50 by x^2 + (2 + O(t^50))*x + 3 + O(t^50)
> OptimizedRepresentation(U);
Power series ring in $.1 over GF(101^2) with fixed absolute precision 50
Mapping from: RngSerExt: U to Power series ring in s over GF(101^2) with fixed
absolute precision 50 given by a rule
```

49.9 Bibliography

[**Hus87**] Dale Husemöller. *Elliptic Curves*, volume 111 of *Graduate Texts in Mathematics*. Springer, New York, 1987.

50 LAZY POWER SERIES RINGS

Chapter 50

LAZY POWER SERIES RINGS

50.1 Introduction

The lazy series rings RngLaz and their elements RngLazElt allow the creation of infinite precision series by providing series for which all coefficients can be calculated by some given formula. Only finitely many coefficients of such a series can be known at any one time but all infinitely many of the coefficients are knowable. Any coefficient of a lazy series can be generated and once a coefficient is computed it will be stored in the series for quick retrieval.

The simplest implementation of this idea is creating a series by providing a map. This map is a formula for computing coefficients of a series. Given the exponents of the variables of a term it will give you the coefficient of that term. Series with finitely many non–zero terms can be created in special ways and several usual arithmetic operations can be applied to lazy series. Such constructions yield lazy series with more complicated formulas for their coefficients.

Consider the series $\sum_{i=0}^{n} i * x^i$. A formula for the coefficients is given by $i \mapsto i$. This series can be created as a lazy series as follows:

```
> L<x> := LazyPowerSeriesRing(Integers(), 1);
> m := map<Integers() -> Integers() | i :-> i>;
> s := elt<L | m>;
> s;
Lazy power series
```

Currently s does not know what any of its coefficients are yet it is possible to calculate any coefficient of s.

```
> Coefficient(s, 0);
0
> Coefficient(s, 100);
100
> Coefficient(s, 100000000000000000000000);
100000000000000000000000
```

50.2 Creation of Lazy Series Rings

Both univariate and multivariate lazy series rings can be created.

`LazyPowerSeriesRing(C, n)`

The lazy power series ring with coefficient ring C and n variables. Any ring is valid input for C and n can be any positive integer.

`ChangeRing(L, C)`

Given a lazy series ring L defined over a ring R and some ring C, return the lazy series ring with coefficient ring C but the same number of variables as L. A map from L to the new lazy series ring is also returned which takes a series s in L to a series whose coefficients are those of s coerced into C.

Example H50E1

Here we illustrate the creation and printing of lazy power series rings.

```
> L := LazyPowerSeriesRing(Rationals(), 5);
> L;
Lazy power series ring in 5 variables over Rational Field
> ChangeRing(L, MaximalOrder(CyclotomicField(7)));
Lazy power series ring in 5 variables over Maximal Equation Order with defining
polynomial x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 over Z
```

50.3 Functions on Lazy Series Rings

Lazy series rings have variables, names for their variables and a coefficient ring.

`R . i`

The ith variable of the lazy power series ring R, where i is between 1 and the rank of R.

`AssignNames(~R, S)`

Given a lazy series ring R with n indeterminates and a sequence S of n strings, assign the elements of S to the names of the variables of R.

`BaseRing(R)`

`CoefficientRing(R)`

The coefficient ring of the lazy power series ring R. The coefficients of all the series in R will lie in this ring.

`Rank(R)`

The number of variables associated with the lazy power series ring R.

`R1 eq R2`

Return `true` if the lazy series rings $R1$ and $R2$ are the same ring, that is, they have the same coefficient ring and rank.

Example H50E2

The functions on lazy power series rings are illustrated here.

```
> L := LazyPowerSeriesRing(FiniteField(73), 7);
> L.4;
Lazy power series
> AssignNames(~L, ["a", "b", "c", "d", "fifth", "sixth", "seventh"]);
> L.4;
Lazy power series
```

The names for the variables of L are not used in default series printing. However they are used when printing an element to a given precision using `PrintToPrecision`.

```
> CoefficientRing(L);
Finite field of size 73
> Rank(L);
7
> L eq LazyPowerSeriesRing(CoefficientRing(L), Rank(L));
true
```

50.4 Elements

Lazy series may be created in a number of ways. Arithmetic for ring elements is available as well as a few predicates. Coefficients of the monomials in the series can be calculated.

50.4.1 Creation of Finite Lazy Series

`R ! c`

Return the series in the lazy series ring R with constant term c and every other coefficient 0 where c is any ring element coercible into the coefficient ring of R.

`R ! s`

Return the lazy series in the lazy series ring R whose coefficients are those of the lazy series s coerced into the coefficient ring of R.

`R ! S`

Return the series in the lazy series ring R whose coefficients are the elements of S where S is a sequence of elements each coercible into the coefficient ring of R. The coefficients are taken to be given in the order which `Coefficients` will return them and `PrintToPrecision` will print the terms of the series. Any coefficient not given is assumed to be zero so that all coefficients are calculable. The resulting series can only have finitely many non zero terms and all such non zero coefficients must be given in S.

```
LazySeries(R, f)
```

Create the lazy series in the lazy series ring R with finitely many non zero terms which are the terms of the polynomial f. The number of variables of the parent ring of f must be the same as the number of variables of R. The coefficients of f must be coercible into the coefficient ring of R.

It is also possible for f to be a rational function, p/q. The series created from f will be `LazySeries(R, p)*LazySeries(R, q)`$^{-1}$.

Example H50E3

Creation of series using the above functions is shown here.

```
> L := LazyPowerSeriesRing(AlgebraicClosure(), 3);
> LR := LazyPowerSeriesRing(Rationals(), 3);
> s := L!1;
> s;
Lazy power series
> LR!s;
Lazy power series
> P<x, y, z> := RationalFunctionField(Rationals(), 3);
> LazySeries(L, (x + y + 8*z)^7/(1 + 5*x*y*z + x^8)^3);
Lazy power series
```

50.4.1.1 Creation of Lazy Series from Maps

Creating a lazy series from a map simulates the existence of lazy series as series for which coefficients are not known on creation of the series but calculated by some rule given when creating the series.

This rule can be coded in a map where the input to the map is the exponents of the variables in a term and the output is the coefficient of the term described by the input. So for a lazy series ring with variables x, y and z, a general term of a series will look like $c_{ijk} * x_i^r * y_j^s * z_k^t$. The coefficient of this term, c_{ijk}, in a series defined by a map m is the result of $m(\langle i, j, k\rangle)$.

```
elt< R | m >
```

Creates a series in the lazy series ring R from the map m. The map m must take as input either an integer (when R is univariate only) or a tuple of integers (of length the number of variables of R) and return an element of the coefficient ring of R. This element will be taken as the coefficient of the term of the series whose variables have the exponents given in the input tuple, as described above.

Example H50E4

We first illustrate the univariate case.

```
> L<x> := LazyPowerSeriesRing(MaximalOrder(QuadraticField(5)), 1);
> Z := Integers();
> m := map<Z -> CoefficientRing(L) | t :-> 2*t>;
> s := elt<L | m>;
> PrintToPrecision(s, 10);
2*x + 4*x^2 + 6*x^3 + 8*x^4 + 10*x^5 + 12*x^6 + 14*x^7 + 16*x^8 + 18*x^9 +
    20*x^10
> Coefficient(s, 34);
68
> m(34);
68
> Coefficient(s, 2^30 + 10);
2147483668
> m(2^30 + 10);
2147483668
```

Example H50E5

And now for the multivariate case.

```
> L<x, y, z> := LazyPowerSeriesRing(AlgebraicClosure(), 3);
> Z := Integers();
> m := map<car<Z, Z, Z> -> CoefficientRing(L) | t :-> t[1]*t[2]*t[3]>;
> s := elt<L | m>;
> PrintToPrecision(s, 5);
x*y*z + 2*x^2*y*z + 2*x*y^2*z + 2*x*y*z^2 + 3*x^3*y*z + 4*x^2*y^2*z +
    4*x^2*y*z^2 + 3*x*y^3*z + 4*x*y^2*z^2 + 3*x*y*z^3
> Coefficient(s, [1, 1, 1]);
1
> m(<1, 1, 1>);
1
> Coefficient(s, [3, 1, 2]);
6
> m(<3, 1, 2>);
6
```

50.4.2 Arithmetic with Lazy Series

All the usual arithmetic operations are possible for lazy series.

`s + t`

The sum of the two lazy series s and t.

`-s`

The negation of the lazy series s.

`s - t`

The difference between lazy series s and t.

`s * t`

The product of the lazy series s and t.

`s + r`

`r + s`

The sum of the lazy series s and the element r of the coefficient ring of the parent ring of s.

`c * s`

`s * c`

The product of the lazy series s and the element c of the coefficient ring of the parent ring of s.

`s * n`

The product of the lazy series s and the monomial x^n, where x^n is $x_1^{n_1} \times \ldots \times x_r^{n_r}$ where r is the number of variables of the parent ring of s, n is the sequence $[n_1, \ldots, n_r]$ and $x = x_1, \ldots, x_r$ are the series variables of the parent ring of s.

`s ^ n`

Given a lazy series s and an integer n, return the nth power of s. It is allowed for n to be negative and inverses will be taken where possible.

Example H50E6

Here we demonstrate the above arithmetic operations.

```
> L<a, b, c, d> := LazyPowerSeriesRing(Rationals(), 4);
> (a + 4*b + (-c)*[8, 9, 2^30 + 10, 2] - d*b + 5)^-8;
Lazy power series
```

50.4.3 Finding Coefficients of Lazy Series

In theory, all the coefficients of most series can be calculated. In practice it is not possible to compute infinitely many. Some problems arise however when a series has been created using multiplication, inversion, evaluation or by taking square roots. For such series the jth coefficient for all $j < i$ must be known for the ith coefficient to be calculated. In such cases the exponents specified for each variable in the monomial whose coefficient is required must be small integers ($< 2^{30}$).

The default ordering of coefficients is by total degree of the corresponding monomials. This is the same order on multivariate polynomials by default. When drawn on paper or imagined in 3 or more dimensions this looks like a spiral if the coordinates representing the monomial exponents are joined. This is the same ordering which is used to compute the valuation of a series.

Once computed, coefficients are stored with the series so any subsequent call to these functions will be faster in non–trivial cases than the first call. It is possible to interrupt any of these functions and return to the prompt.

```
Coefficient(s, i)
```

Returns the coefficient in the univariate lazy series s of x^i where x is the series variable of the parent of s and i is a non negative integer.

```
Coefficient(s, T)
```

Returns the coefficient in the multivariate lazy series s of the monomial $x_1^{T_1} * \ldots * x_r^{T_r}$ where T is the sequence $[T_1, \ldots, T_r]$, $x_1, \ldots, x_r$ are the variables of the parent ring of s and r is the rank of the parent of s.

```
Coefficients(s, n)
```

```
Coefficients(s, l, n)
```

The coefficients of the lazy series s whose monomials have total degree at least l (0 if not given) and at most n where the monomials are ordered using the default "spiral" degree order. The bounds l and n must be non negative integers.

```
Valuation(s)
```

The valuation of the lazy series s. This is the exponent of the monomial with the first non–zero coefficient as returned by `Coefficients` above. The return value will be either `Infty`, an integer (univariate case) or a sequence (multivariate case).

```
PrintToPrecision(s, n)
```

Print the sum of all terms of the lazy series s whose degree is no more than n where n is a non negative integer. The series is printed using the "spiral" ordering.

```
PrintTermsOfDegree(s, l, n)
```

Print the sum of the terms of the lazy series s whose degree is at least l and at most n. The terms are printed using the spiral ordering. The bounds l and n must be non negative integers.

`LeadingCoefficient(s)`

The coefficient in the lazy series s whose monomial exponent is the valuation of s. That is, the first non–zero coefficient of s where the ordering which determines "first" is the "spiral" ordering used by `Coefficients` and `Valuation`.

`LeadingTerm(s)`

The term in the lazy series s whose monomial exponent is the valuation of s. That is, the first non–zero term of s where the ordering which determines "first" is the "spiral" ordering used by `Coefficients` and `Valuation`.

Example H50E7

In this example we look at some coefficients of an infinite series.

```
> L<a, b, c, d> := LazyPowerSeriesRing(Rationals(), 4);
> s := (1 + 2*a + 3*b + 4*d)^-5;
```

Find the coefficient of $a * b * c * d$.

```
> Coefficient(s, [1, 1, 1, 1]);
0
```

Find the coefficients of all monomials with total degree at most 6.

```
> time Coefficients(s, 6);
[ 1, -10, -15, 0, -20, 60, 180, 0, 240, 135, 0, 360, 0, 0, 240, -280, -1260, 0,
-1680, -1890, 0, -5040, 0, 0, -3360, -945, 0, -3780, 0, 0, -5040, 0, 0, 0,
-2240, 1120, 6720, 0, 8960, 15120, 0, 40320, 0, 0, 26880, 15120, 0, 60480, 0, 0,
80640, 0, 0, 0, 35840, 5670, 0, 30240, 0, 0, 60480, 0, 0, 0, 53760, 0, 0, 0, 0,
17920, -4032, -30240, 0, -40320, -90720, 0, -241920, 0, 0, -161280, -136080, 0,
-544320, 0, 0, -725760, 0, 0, 0, -322560, -102060, 0, -544320, 0, 0, -1088640,
0, 0, 0, -967680, 0, 0, 0, 0, -322560, -30618, 0, -204120, 0, 0, -544320, 0, 0,
0, -725760, 0, 0, 0, 0, -483840, 0, 0, 0, 0, 0, -129024, 13440, 120960, 0,
161280, 453600, 0, 1209600, 0, 0, 806400, 907200, 0, 3628800, 0, 0, 4838400, 0,
0, 0, 2150400, 1020600, 0, 5443200, 0, 0, 10886400, 0, 0, 0, 9676800, 0, 0, 0,
0, 3225600, 612360, 0, 4082400, 0, 0, 10886400, 0, 0, 0, 14515200, 0, 0, 0, 0,
9676800, 0, 0, 0, 0, 0, 2580480, 153090, 0, 1224720, 0, 0, 4082400, 0, 0, 0,
7257600, 0, 0, 0, 0, 7257600, 0, 0, 0, 0, 0, 3870720, 0, 0, 0, 0, 0, 0, 860160 ]
Time: 0.140
> #$1;
210
```

`PrintToPrecision` will display the monomials to which these coefficients correspond.

```
> time PrintToPrecision(s, 6);
1 - 10*a - 15*b - 20*d + 60*a^2 + 180*a*b + 240*a*d + 135*b^2 + 360*b*d +
    240*d^2 - 280*a^3 - 1260*a^2*b - 1680*a^2*d - 1890*a*b^2 - 5040*a*b*d -
    3360*a*d^2 - 945*b^3 - 3780*b^2*d - 5040*b*d^2 - 2240*d^3 + 1120*a^4 +
    6720*a^3*b + 8960*a^3*d + 15120*a^2*b^2 + 40320*a^2*b*d + 26880*a^2*d^2 +
    15120*a*b^3 + 60480*a*b^2*d + 80640*a*b*d^2 + 35840*a*d^3 + 5670*b^4 +
    30240*b^3*d + 60480*b^2*d^2 + 53760*b*d^3 + 17920*d^4 - 4032*a^5 -
    30240*a^4*b - 40320*a^4*d - 90720*a^3*b^2 - 241920*a^3*b*d - 161280*a^3*d^2
    - 136080*a^2*b^3 - 544320*a^2*b^2*d - 725760*a^2*b*d^2 - 322560*a^2*d^3 -
    102060*a*b^4 - 544320*a*b^3*d - 1088640*a*b^2*d^2 - 967680*a*b*d^3 -
    322560*a*d^4 - 30618*b^5 - 204120*b^4*d - 544320*b^3*d^2 - 725760*b^2*d^3 -
    483840*b*d^4 - 129024*d^5 + 13440*a^6 + 120960*a^5*b + 161280*a^5*d +
    453600*a^4*b^2 + 1209600*a^4*b*d + 806400*a^4*d^2 + 907200*a^3*b^3 +
    3628800*a^3*b^2*d + 4838400*a^3*b*d^2 + 2150400*a^3*d^3 + 1020600*a^2*b^4 +
    5443200*a^2*b^3*d + 10886400*a^2*b^2*d^2 + 9676800*a^2*b*d^3 +
    3225600*a^2*d^4 + 612360*a*b^5 + 4082400*a*b^4*d + 10886400*a*b^3*d^2 +
    14515200*a*b^2*d^3 + 9676800*a*b*d^4 + 2580480*a*d^5 + 153090*b^6 +
    1224720*b^5*d + 4082400*b^4*d^2 + 7257600*b^3*d^3 + 7257600*b^2*d^4 +
    3870720*b*d^5 + 860160*d^6
Time: 0.010
```

The valuation of s can be obtained as follows. The valuation of zero is a special case.

```
> Valuation(s);
[ 0, 0, 0, 0 ]
> Valuation(s*0);
Infinity
```

`CoefficientsNonSpiral(s, n)`

Returns the coefficients of the monomials in the lazy series s whose exponents are given by $[i_1, ..., i_r]$ where each $i_j \leq n_j$. The argument n may either be a non negative integer (univariate case) or a sequence of non negative integers of length r where r is the rank of the parent ring of s. The index of the $[i_1, ..., i_r]$-th coefficient in the return sequence is given by

$$\sum_{j=1}^{r} i_j * (\prod_{k=j+1}^{r} (n_k + 1)).$$

`Index(s, i, n)`

Return the index in the return value of `CoefficientsNonSpiral(s, n)` of the monomial in the lazy series s whose exponents are given by the (trivial in the univariate case) sequence i.

Example H50E8

We find the coefficients of the series used in the last example using the alternative algorithm.

```
> L<a, b, c, d> := LazyPowerSeriesRing(Rationals(), 4);
> s := (1 + 2*a + 3*b + 4*d)^-5;
> time CoefficientsNonSpiral(s, [3, 3, 3, 2]);
[ 1, -20, 240, 0, 0, 0, 0, 0, 0, 0, 0, 0, -15, 360, -5040, 0, 0, 0, 0, 0, 0, 0,
0, 0, 135, -3780, 60480, 0, 0, 0, 0, 0, 0, 0, 0, 0, -945, 30240, -544320, 0, 0,
0, 0, 0, 0, 0, 0, 0, -10, 240, -3360, 0, 0, 0, 0, 0, 0, 0, 0, 0, 180, -5040,
80640, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1890, 60480, -1088640, 0, 0, 0, 0, 0, 0, 0,
0, 0, 15120, -544320, 10886400, 0, 0, 0, 0, 0, 0, 0, 0, 0, 60, -1680, 26880, 0,
0, 0, 0, 0, 0, 0, 0, 0, -1260, 40320, -725760, 0, 0, 0, 0, 0, 0, 0, 0, 0, 15120,
-544320, 10886400, 0, 0, 0, 0, 0, 0, 0, 0, 0, -136080, 5443200, -119750400, 0,
0, 0, 0, 0, 0, 0, 0, 0, -280, 8960, -161280, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6720,
-241920, 4838400, 0, 0, 0, 0, 0, 0, 0, 0, 0, -90720, 3628800, -79833600, 0, 0,
0, 0, 0, 0, 0, 0, 0, 907200, -39916800, 958003200, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
Time: 0.370
> #$1;
192
```

It appears that the "spiral" algorithm may be faster since it computed more coefficients in less time. This may be because of the operations (inversion) involved in calculating s and that `CoefficientsNonSpiral` computed coefficients of larger degree monomials than `Coefficients`. In calculating these coefficients for larger degrees, coefficients which were not asked for may have been calculated. `Coefficients` will only need to calculate the coefficients which have been asked for — all intermediate calculations would have been asked for.

50.4.4 Predicates on Lazy Series

`s eq t`

Return `true` if the lazy series s and t are exactly the same series.

`IsZero(s)`

Return `true` if the lazy series s was created as the zero series.

`IsOne(s)`

Return `true` if the lazy series s was created as the one series.

`IsMinusOne(s)`

Return `true` if the lazy series s was created as the minus one series.

`IsUnit(s)`

Return `true` if the lazy series s is a unit.

`IsWeaklyZero(s, n)`

Return `true` if all the terms of the lazy series s with degree at most n are zero. Calling this function without the second argument returns `IsZero(s)`.

`IsWeaklyEqual(s, t, n)`

Return `true` if the terms of the lazy series s and t with degree at most n are the same. Calling this function without the third argument returns `s eq t`.

50.4.5 Other Functions on Lazy Series

`Derivative(s)`

`Derivative(s, v)`

`Derivative(s, v, n)`

Return the (nth) derivative of the lazy series s with respect to the vth variable of the parent ring of s. If v is not given, the parent ring of s must be univariate and the derivative with respect to the unique variable is returned. It is only allowed to give values of v from 1 to the rank of the parent of s and for n to be a positive integer.

`Integral(s)`

`Integral(s, v)`

Return the integral of the lazy series s with respect to the vth variable of the parent ring of s. If v is not given, the parent ring of s must be univariate and the integral with respect to the unique variable is returned. It is only allowed to give values of v from 1 to the rank of the parent of s.

`Evaluate(s, t)`

`Evaluate(s, T)`

Return the lazy series s evaluated at the lazy series t or the sequence T of lazy series. The series t or the series in T must have zero constant term so that every coefficient of the result can be finitely calculated.

Example H50E9

Some usage of `Evaluate` is shown below.

```
> R := LazyPowerSeriesRing(Rationals(), 2);
> AssignNames(~R, ["x","y"]);
> m := map<car<Integers(), Integers()> -> Rationals() | t :-> 1>;
> s := elt<R | m>;
> PrintToPrecision(s, 3);
1 + x + y + x^2 + x*y + y^2 + x^3 + x^2*y + x*y^2 + y^3
> R1 := LazyPowerSeriesRing(Rationals(), 1);
> AssignNames(~R1, ["z"]);
> m1 := map<car<Integers()> -> Rationals() | t :-> t[1]>;
> s1 := elt<R1 | m1>;
> PrintToPrecision(s1, 3);
z + 2*z^2 + 3*z^3
> e := Evaluate(s, [s1,s1]);
> PrintToPrecision(e, 10);
1 + 2*z + 7*z^2 + 22*z^3 + 67*z^4 + 200*z^5 + 588*z^6 + 1708*z^7 + 4913*z^8 +
    14018*z^9 + 39725*z^10
> Parent(e);
Lazy power series ring in 1 variable over Rational Field
> f := Evaluate(s1, s - 1);
> PrintToPrecision(f, 10);
x + y + 3*x^2 + 5*x*y + 3*y^2 + 8*x^3 + 18*x^2*y + 18*x*y^2 + 8*y^3 + 20*x^4 +
    56*x^3*y + 75*x^2*y^2 + 56*x*y^3 + 20*y^4 + 48*x^5 + 160*x^4*y + 264*x^3*y^2
    + 264*x^2*y^3 + 160*x*y^4 + 48*y^5 + 112*x^6 + 432*x^5*y + 840*x^4*y^2 +
    1032*x^3*y^3 + 840*x^2*y^4 + 432*x*y^5 + 112*y^6 + 256*x^7 + 1120*x^6*y +
    2496*x^5*y^2 + 3600*x^4*y^3 + 3600*x^3*y^4 + 2496*x^2*y^5 + 1120*x*y^6 +
    256*y^7 + 576*x^8 + 2816*x^7*y + 7056*x^6*y^2 + 11616*x^5*y^3 +
    13620*x^4*y^4 + 11616*x^3*y^5 + 7056*x^2*y^6 + 2816*x*y^7 + 576*y^8 +
    1280*x^9 + 6912*x^8*y + 19200*x^7*y^2 + 35392*x^6*y^3 + 47280*x^5*y^4 +
    47280*x^4*y^5 + 35392*x^3*y^6 + 19200*x^2*y^7 + 6912*x*y^8 + 1280*y^9 +
    2816*x^10 + 16640*x^9*y + 50688*x^8*y^2 + 103168*x^7*y^3 + 154000*x^6*y^4 +
    175344*x^5*y^5 + 154000*x^4*y^6 + 103168*x^3*y^7 + 50688*x^2*y^8 +
    16640*x*y^9 + 2816*y^10
> f := Evaluate(s1 + 1, s - 1);
> PrintToPrecision(f, 10);
1 + x + y + 3*x^2 + 5*x*y + 3*y^2 + 8*x^3 + 18*x^2*y + 18*x*y^2 + 8*y^3 + 20*x^4
    + 56*x^3*y + 75*x^2*y^2 + 56*x*y^3 + 20*y^4 + 48*x^5 + 160*x^4*y +
    264*x^3*y^2 + 264*x^2*y^3 + 160*x*y^4 + 48*y^5 + 112*x^6 + 432*x^5*y +
    840*x^4*y^2 + 1032*x^3*y^3 + 840*x^2*y^4 + 432*x*y^5 + 112*y^6 + 256*x^7 +
    1120*x^6*y + 2496*x^5*y^2 + 3600*x^4*y^3 + 3600*x^3*y^4 + 2496*x^2*y^5 +
    1120*x*y^6 + 256*y^7 + 576*x^8 + 2816*x^7*y + 7056*x^6*y^2 + 11616*x^5*y^3 +
    13620*x^4*y^4 + 11616*x^3*y^5 + 7056*x^2*y^6 + 2816*x*y^7 + 576*y^8 +
    1280*x^9 + 6912*x^8*y + 19200*x^7*y^2 + 35392*x^6*y^3 + 47280*x^5*y^4 +
    47280*x^4*y^5 + 35392*x^3*y^6 + 19200*x^2*y^7 + 6912*x*y^8 + 1280*y^9 +
    2816*x^10 + 16640*x^9*y + 50688*x^8*y^2 + 103168*x^7*y^3 + 154000*x^6*y^4 +
    175344*x^5*y^5 + 154000*x^4*y^6 + 103168*x^3*y^7 + 50688*x^2*y^8 +
    16640*x*y^9 + 2816*y^10
```

`SquareRoot(s)`

`Sqrt(s)`

The square root of the lazy series s.

```
IsSquare(s)
```

Return `true` if the lazy series s is a square and the square root if so.

```
PolynomialCoefficient(s, i)
```

Given a series s in a lazy series ring whose coefficient ring is a polynomial ring (either univariate or multivariate), consider the polynomial which would be formed if this series was written as a polynomial with series as the coefficients. For i a non negative integer and the coefficient ring of the series ring a univariate polynomial ring this function returns the series which is the ith coefficient of the polynomial resulting from the rewriting of the series.

When the coefficient ring of the parent of s is a multivariate polynomial ring i should be a sequence of non negative integers of length the rank of the coefficient ring. The series returned will be the series which is the coefficient of $x_1^{i_1} * \ldots * x_r^{i_r}$ in the rewritten series where r is the rank of the coefficient ring and x_j are the indeterminates of the coefficient ring.

51 ALGEBRAIC POWER SERIES RINGS

Chapter 51
ALGEBRAIC POWER SERIES RINGS

51.1 Introduction

Algebraic Power Series are a lazy representation of multivariate power series with fractional exponents, which are roots of univariate polynomials with coefficients in multivariate polynomial rings. The functionality allows the "lazy" computation of the power series expansion to any finite degree, this being well-determined by the defining algebraic equation.

The package was designed with the computation of formal resolutions of singularities of surfaces in mind but should provide a useful general tool for users. As well as allowing definition directly from a polynomial equation, the user can compose algebraic power series and recursively define series which are roots of polynomials whose coefficients are polynomial functions in other algebraic power series. There are also functions for basic arithmetic operations and tests for exact equality and the like.

The defined series must be expandable in fractional (positive) powers of the base variables. This is true for all roots of a *quasi-ordinary* polynomial possibly after a finite extension of the base field.

The package was designed and implemented by Tobias Beck at the RICAM institute in Linz, Austria. Some low-level adaptations for added efficiency and integration were carried out by the Magma group. The algorithms are described in [Bec07, Sec. 4].

51.2 Basics

In Magma, algebraic power series are represented in a hybrid lazy-exact way. Eventually every power series is given by a defining polynomial and a sufficiently large initial segment. Intermediate operations are represented in a lazy way. This makes it possible to compute both quickly and to high precision if necessary.

Note, however, that decision procedures may be very time intensive. In the sequel we have indicated in each function whether it is fast or whether it has to be used with care.

51.2.1 Data Structures

Algebraic power series are of type `RngPowAlgElt`.

There are two types of algebraic series: `Atomic` and `Substitution` which we sometimes refer to briefly as type A or type B. There is no difference in the functionality available but they are structurally different. The user can determine the type of a series, if (s)he desires, from the attribute `type` which is 0 for type A and 1 for type B.

`Atomic` series are the basic type that are given directly as the root of a univariate polynomial $p(z)$ with given initial expansion. $p(z)$ comes from a polynomial $f(z)$ with coefficients in a multivariate polynomial ring. Either $p = f$ or p is determined from f by evaluating the coefficients of f at a given array, *subs*, of algebraic power series. This allows the construction of algebraic power series as roots of polynomials over finitely-generated fields of already-constructed series. Most of the constructors return a series of this type.

`Substitution` series allow the composition of algebraic power series. The principal defining data is an algebraic power series s in n variables and an array of n algebraic series that are substituted into this. In fact, the substitution is not necessarily direct, but through n given elements in the dual lattice of the exponent lattice of s as explained in the constructor `EvaluationPowerSeries`.

Both types of series have an associated exponent lattice specified by two components: Γ, a sublattice of a standard integral lattice, and e, a positive integer. The expansion of the series will in general have fractional exponents and e is the LCM of the denominators of these (e may be 1). The finite expansions that are returned are always integral-exponent multivariate polynomials. The *actual* mathematical expansion is derived from this return value by dividing all exponents by e. With this scaling up of fractional exponents by e to get integral exponents, all exponent vectors for monomials occurring in the expansion (up to any degree) will lie in the lattice Γ. So the actual exponent lattice for the series is $(1/e)\Gamma$.

The user doesn't have to worry too much about the lattice Γ. It is automatically computed by most of the constructors for algebraic power series and the default of the standard integral lattice can always be used (assuming e is correct!). It's utility is that, in computing the exponent lattice for composite constructions on series, factors of e may be cancelled out from the resulting lattice. So an algebraic construction involving series with non-integral exponents may produce a result with only integral exponents. Such lattice computations are carried out automatically.

Important note: To speed up some of the basic internal polynomial operations, it is assumed that the domain of an algebraic power series is a multivariate polynomial ring with a *degree* ordering (`glex` or `grevlex`). Attempts to create series using polynomial rings with a non-degree ordering will result in a user error.

51.2.2 Verbose Output

A verbose flag `AlgSeries` exists which can take values `true`, `false`, 0 or 1. Setting to `true` (or 1) will output information on the progress of some of the potentially more time-consuming intrinsics.

51.3 Constructors

Using constructors one can construct power series starting from polynomial data or using other power series recursively.

`PolyToSeries(s)`

Given a multivariate polynomial s, returns the series representation of s.

```
AlgebraicPowerSeries(dp, ip, L, e)
```

subs SeqEnum *Default :* []

Define a power series root of a polynomial p using an initial expansion *ip* and its exponent lattice `1/e L`. The defining polynomial p is either *dp* when *subs* is empty or obtained by substituting the elements of *subs* into the variables of *dp*. The initial expansion has to be sufficiently long in order to uniquely identify a root, see [Bec07, Cond. 4.3]. In this initial finite expansion and in subsequent ones, the variables occurring actually represent e-th roots, *ie* $x_1 x_2^2$ is really $x_1^{1/e} x_2^{2/e}$. All exponents of monomials occurring in these expansions and the coefficients of p should lie in L (although this is not checked in many places), monomials of p giving true values rather than e-th roots.

There are simpler constructors where L is omitted (when it is assumed to be the standard integral lattice) or e is (when it is taken as 1).

As noted earlier, no strong checks are performed at construction time on the correctness of L or e or whether *ip* is indeed the initial expansion of a unique root in this "raw data" constructor. Incorrect initial data will only be revealed when failure occurs in further expansion of the series. The preferred methods of series creation are `ImplicitFunction`, `EvaluationPowerSeries` and `RationalPuiseux` because in these cases the sanity checks are more easily verified.

```
EvaluationPowerSeries(s, nu, v)
```

Given a series s, a sequence *nu* of vectors in the dual of its exponent lattice of s and a sequence v (of the same length) of power series in some other common domain (with compatible coefficient field). Returns the series obtained by substituting $\underline{x}^{\mu} \mapsto \prod_i \texttt{v[i]}^{\langle \texttt{nu[i]}, \mu \rangle}$. This requires that *nu* and v fulfill a certain condition on the orders to guarantee convergence of the resulting series, see [Bec07, Cond. 4.6].

```
ImplicitFunction(dp)
```

subs SeqEnum *Default :* []

The unique series with zero constant term defined by a polynomial $p \in k[x_1, .., x_n][z]$ or $k[[x_1, \dots, x_n]][z]$, fulfilling the conditions of the implicit function theorem, *i.e.*, $p(0, \dots, 0) = 0$ and $\partial p / \partial z(0, \dots, 0) \neq 0$. The polynomial p is equal to *dp* possibly substituted with the series in *subs* as in `AlgebraicPowerSeries`. *dp* should have coefficients in a multivariate polynomial ring.

51.3.1 Rational Puiseux Expansions

Let $p \in k[[x_1, \dots, x_n]][z]$ be a quasi-ordinary polynomial over a field k of characteristic zero. This means that p is non-zero, squarefree and monic (i.e., its leading coefficient in z is a unit in the power series ring) and if $d \in k[[x_1, \dots, x_n]]$ denotes its discriminant then $d = x_1^{e_1} \cdots x_n^{e_n} u(x_1, \dots, x_n)$ where u is a unit in the power series ring.

In this case the Theorem of Jung-Abhyankar states that p has deg(p) distinct Puiseux series roots, i.e., power series roots with fractionary exponents and coefficients in the algebraic closure of k.

These roots are computed by a generalization of the so called Newton-Puiseux algorithm. Also Duval's extension for computing rational parametrization has been implemented.

```
RationalPuiseux(p)
```

`Gamma`	Lattice	*Default :* `StandardLattice`
`subs`	SeqEnum	*Default :* []
`Duval`	BoolElt	*Default :* `false`
`OnlySingular`	BoolElt	*Default :* `false`
`ExtName`	MonStgElt	*Default :* "*gamma*"
`ExtCount`	RngIntElt	*Default :* 0

We first specify the behavior of this function in the case that no special value of *subs* has been given. This function assumes that p is a univariate polynomial over a multivariate polynomial ring $S = k[x_1, ..., x_r]$ and that p is quasi-ordinary. In this case it will compute a set of rational parametrizations of p. Note that for reasons of efficiency the user has to make sure that p is actually quasi-ordinary! (Otherwise, further processing of the output may result in runtime errors.)

The first return value will be the exponent lattice of the input polynomial in the usual format $< \Gamma_0, e_0 >$. If the parameter `Gamma` has been specified, then $\Gamma_0 =$ `Gamma` and $e_0 = 1$. In this case `Gamma` has to be an integral 'r'-dimensional lattice of full rank containing all the exponents of p. Otherwise Γ_0 will be set to the r-dimensional standard lattice and again $e_0 = 1$.

As a second value a complete list of rational parametrizations in the format $< \lambda, s, N, E >$ is returned. Here λ is a sequence of r field elements and s is a fractionary algebraic power series of type `RngPowAlgElt`. Let p_1 denote the image of p under the transformation $x^{\mu_i} \mapsto \lambda_i x^{\mu_i}$ where $(\mu_i)_i$ is the basis of the exponent lattice $e_0^{-1}\Gamma_0$ then s is a solution of p_1, i.e., we have $p_1(s) = 0$. Note that if neither `Gamma` nor `subs` have been supplied this just means that x_i is substituted by $\lambda_i x_i$. Finally N is the index of $e_0^{-1}\Gamma_0$ in the exponent lattice of s and E is the degree of the extension of the coefficient field needed for defining s.

The behavior described above corresponds to the Newton-Puiseux algorithm with Duval's trick. The field extensions that are used for expressing the series fulfill a certain minimality condition. If `Duval` is set to `false` then the function returns a complete set of representatives (up to conjugacy) of Puiseux series roots of the original polynomial p, in other words, the λ-vectors will always be vectors of ones.

If `OnlySingular` is set to `true` then only those parametrizations that correspond to singular branches are returned.

If the ground field has to be extended, the algebraic elements will be assigned the name `ExtName_`*i* where i starts from `ExtCount`. The last return value is the value of `ExtCount` plus the number of field extensions that have been introduced during the computation.

Finally, if the parameter `subs` is passed, then it has to be a sequence of r power series in a common domain and internally the variables in p will be substituted by the corresponding series. Again the resulting polynomial has to be quasi-ordinary. In this case Γ_0 and e_0 are determined by building the sum of the exponent lattices of all series in subs. The parameter Gamma then has no effect.

For further details on the algorithm and other references see [Bec07, Sec. 4.3]

Example H51E1

We illustrate the constructors by examples. For displaying results we already use the command Expand that will be explained later.

```
> Q := Rationals(); Qs<s> := FunctionField(Q);
> Qxy<x,y> := PolynomialRing(Q, 2, "glex");
> Qxyz<z> := PolynomialRing(Qxy);
> Qst<t> := PolynomialRing(Qs, 1, "glex");
> Qstu<u> := PolynomialRing(Qst);
```

One can consider polynomials as series.

```
> s0 := PolyToSeries(1 - 3*x + x^2*y + y^20);
> Expand(s0, 10);
true x^2*y - 3*x + 1
```

One can define series by the implicit function theorem at the origin.

```
> s1 := ImplicitFunction(z*(1 - x - y) - x - y);
> Expand(s1, 4);
true x^3 + 3*x^2*y + 3*x*y^2 + y^3 + x^2 + 2*x*y + y^2 + x + y
```

One can define a power series if an initial expansion is known. Note that the following power series has exponent lattice $\mathbf{Z}(\frac{1}{5}, -\frac{2}{5}) + \mathbf{Z}(\frac{2}{5}, \frac{1}{5})$ but its "expansions" are polynomials supported on $\mathbf{Z}(1, -2) + \mathbf{Z}(2, 1)$.

```
> defpol := (1+5*y+10*y^3+10*y^2+5*y^4+y^5)*z^5+(-20*y^3*x-
>   30*y^2*x-5*y^4*x-5*x-20*y*x)*z^4+(10*x^2+30*y^2*x^2+10*y^3*x^2+
>   30*x^2*y)*z^3+(-20*y*x^3-10*x^3-10*y^2*x^3)*z^2+
>   (5*y*x^4+5*x^4)*z-x^5-x^2*y;
> Gamma := Lattice(RMatrixSpace(Integers(), 2, 2) ! [1,-2, 2,1]);
> init := x^2*y;
> s2 := AlgebraicPowerSeries(defpol, init, Gamma, 5);
> Expand(s2, 20);
true
-x^2*y^16 + x^5*y^10 + x^2*y^11 - x^5*y^5 - x^2*y^6 + x^5 + x^2*y
```

We can "substitute" series into each other.

```
> X := AlgebraicPowerSeries(u^3-t+s*t^2, t, StandardLattice(1), 3);
> Y := PolyToSeries(t);
> duals := [RSpace(Integers(), 2) | [1, 3], [2, 1]];
> s3 := EvaluationPowerSeries(s2, duals, [X, Y]);
> Expand(s3, 13);
```

```
true (-1/9*s^2 - 1/3*s - 1)*t^10 + (-1/3*s + 1)*t^7 + t^4
```

We can compute all the Puiseux series roots of a quasi-ordinary polynomial up to conjugacy over $\mathbf{Q}$.

```
> qopol := z^6 + 3*x*y^2*z^4 + x*y*z^3 + 3*x^2*y^4*z^2 + x^3*y^6;
> _, prms := RationalPuiseux(qopol : Duval := false); prms;
[*
    <[ 1, 1 ], Algebraic power series -x*y, 3, 1>,
    <[ 1, 1 ], Algebraic power series gamma_0*x*y, 3, 2>,
    <[ 1, 1 ], Algebraic power series -x^2*y^5, 3, 1>,
    <[ 1, 1 ], Algebraic power series gamma_1*x^2*y^5, 3, 2>
*]
> Domain(prms[2][2]); ExponentLattice(prms[2][2]);
Polynomial ring of rank 2 over Number Field with defining
polynomial $.1^2 - $.1 + 1 over the Rational Field
Graded Lexicographical Order
Variables: x, y
<
    Lattice of rank 2 and degree 2
    Basis:
    ( 1  1)
    ( 1 -2),
    3
>
> Expand(prms[2][2], 15);
true x^3*y^9 + (gamma_0 - 1)*x^2*y^5 + gamma_0*x*y
```

We find that the sum over all field extensions $1 + 2 + 1 + 2 = 6$ is equal to the degree of the defining polynomial qopol. The third parametrization involves a field extension of $\mathbf{Q}$ by gamma_0 s.t. $\texttt{gamma\_0}^2 - \texttt{gamma\_0} + 1 = 0$ and an extension of the exponent lattice to $\mathbf{Z}(\frac{1}{3}, \frac{1}{3}) + \mathbf{Z}(\frac{1}{3}, -\frac{2}{3})$. It turns out that the field extension is not necessary if we are only interested in parametrizations.

```
> _, prms := RationalPuiseux(qopol : Duval := true); prms;
[*
    <[ -1, -1 ], Algebraic power series -x*y, 3, 1>,
    <[ -1, 1 ], Algebraic power series -x^2*y^5, 3, 1>
*]
```

No field extensions have been introduced, but this required the application of automorphisms $\mathbf{Q}[[x, y]] \to \mathbf{Q}[[x, y]]$ in advance (more precisely $x \mapsto -x, y \mapsto -y$ resp. $x \mapsto -x, y \mapsto y$). This time we can sum up the overall extension degrees (*i.e.*, for fields and lattices) $3 \cdot 1 + 3 \cdot 1 = 6$ to the degree of qopol.

51.4 Accessors and Expansion

The following functions provide an interface to conveniently extract information from a power series defined as above.

`Domain(s)`

Return the multivariate polynomial ring that is used for approximating the series s by its truncations.

`ExponentLattice(s)`

Return the exponent lattice $(1/e)\Gamma$ of the series as tuple (Γ, e) where Γ is an integral lattice and e is an integer.

`DefiningPolynomial(s)`

Return a defining polynomial of the series which is a squarefree univariate polynomial over the multivariate polynomial domain `Domain(s)`. In the case of series defined with substitutions, the computation may be expensive and can involve recursive resultant computations.

`Order(s)`

`TestZero` BoolElt *Default :* `false`

Given a series s, return the integral order (total degree of smallest non-zero term occurring) of its expansion as returned by `Expand`, *i.e.*, its fractionary order times the exponent denominator. If s is zero, this function will **not terminate**. Set `TestZero` to `true` to get a return value -1 in this case, but note that this involves the computationally complex call `IsZero`.

`Expand(s,ord)`

Given the power series β which is represented by s, let α be the result of substituting variables $x_i \mapsto x_i^e$ where e is taken from the output of `ExponentLattice(s)` (*i.e.*, α is β without exponent denominators). Returns `true` and the truncation of α modulo terms of order greater or equal `ord`. A return of `false` indicates that the representation is inconsistent (which should only happen when `RationalPuiseux` is called with non quasi-ordinary input or `AlgebraicPowerSeries` is used inconsistently).

Example H51E2

We can study the series `s3`.

```
> Domain(s3);
Polynomial ring of rank 1 over Univariate rational function
field over Rational Field
Graded Lexicographical Order
Variables: t
> ExponentLattice(s3);
<
    Standard Lattice of rank 1 and degree 1,
   3
>
> DefiningPolynomial(s3);
(s^45*t^45 - ... - 15*s^30*t^2 - s^30)*u^15 + ... +
(5*s^37*t^41 - ... + 120*s^31*t^19 - 30*s^30*t^18)*u^3 -
s^35*t^40 + ... - 5*s^31*t^21 + s^30*t^20
> Order(s3);
4
```

These commands reveal the following about s3: It is a power series in $\mathbf{Q}(s)[[t^{1/3}]]$, because it is approximated in $\mathbf{Q}(s)[t]$ and has exponent lattice $\frac{1}{3}\mathbf{Z}$. A defining polynomial in $\mathbf{Q}(s)[t][u]$ was also computed. (Recall that s3 has been defined recursively.) The order is $\frac{4}{3}$ which we know already from a previous expansion.

51.5 Arithmetic

There are functions to perform basic arithmetic operations (addition, subtraction etc.) on power series.

```
AlgComb(c,ss)
```

Given a polynomial *c* in *r* variables and a sequence *ss* of *r* power series (in a common domain with compatible coefficient field) return the series obtained by substituting the elements of *ss* for the variables of *c*. This allows the construction of completely arbitrary algebraic combinations.

```
s + t
s - t
s * t
```

Add, subtract or multiply two power series.

Example H51E3

One can easily substitute power series into polynomials.

```
> // construct the series s0^2+s1^2
> h0 := AlgComb(x^2 + y^2, [s0,s1]);
> Expand(h0, 3);
true 10*x^2 + 2*x*y + y^2 - 6*x + 1
```

This includes of course the ring operations.

```
> h1 := Add(s1, PolyToSeries(One(Qxy)));
> Expand(h1, 4);
true
x^3 + 3*x^2*y + 3*x*y^2 + y^3 + x^2 + 2*x*y + y^2 + x + y + 1
> h2 := Mult(h1, PolyToSeries(1 - x - y));
> Expand(h2, 4);
true 1
```

```
> h3 := Add(h2, PolyToSeries(-One(Qxy)));
> Expand(h3, 4);
true 0
```

51.6 Predicates

The following functions provide decision algorithms for algebraic power series. They may involve **recursive resultant computations**, hence, have a high complexity and should be used with care.

`IsZero(s)`

Decides if the series is zero.

`s eq t`

Decides if two series are equal.

`IsPolynomial(s)`

Decides whether the series is actually a polynomial (with integral exponents) in the multivariate polynomial domain as returned by `Domain(s)`. In the positive case also returns that polynomial. This function relies on `SimplifyRep`.

Example H51E4

The previous computations suggest that `h2` is 1, in particular it is polynomial (in contrast to `h1`). In this case `h3` would be zero.

```
> IsPolynomial(h1);
false
> IsPolynomial(h2);
true 1
> IsEqual(h2, PolyToSeries(One(Qxy)));
true
> IsZero(h3);
true
```

51.7 Modifiers

The following functions modify the representation of the power series or apply a simple automorphism.

```
ScaleGenerators(s,ls)
```

Let $\{\gamma_i\}_i$ be the basis (determined from the representation chosen by Magma) of the exponent lattice of the series s, and let $\sigma : \underline{x}^{\gamma_i} \mapsto ls[i]\underline{x}^{\gamma_i}$. Return the series $\sigma(s)$.

```
ChangeRing(s,R)
```

If R is a multivariate polynomial domain compatible with the approximation domain `Domain(s)`, return the same power series with new approximation domain R. This is sort of a "coercion between power series rings".

```
SimplifyRep(s)
```

`Factorizing` BoolElt *Default :* `true`

"Simplifies" the internal representation of a series. The result will be a series of atomic type without recursive (substitution) dependencies on other power series. The defining polynomial of the simplified series will be irreducible and therefore a minimal polynomial over `Domain(s)` (unless `Factorizing` is `false` when it will only be guaranteed to be squarefree). After the simplification, `DefiningPolynomial` returns this polynomial, which can be useful (*e.g.*, for `IsPolynomial`). However, experience shows that the resulting representation is in general neither simple nor more efficient for subsequent computations.

There is a **dangerous pitfall:**

Assume we have a series represented by a tree with nodes of type A and B. Assume further that the leaves have been constructed by `RationalPuiseux` with parameter `Gamma` set to some value. Then the intention was probably to work over the subring of a polynomial ring with restricted support. If now `SimplifyRep`, with `Factorizing` as `true`, is called, then a minimal polynomial over the whole polynomial ground ring is computed which is maybe not what one wants.

Example H51E5

We can modify `s2` by mapping generators (of Laurent polynomials) $x^{1/5}y^{-2/5} \mapsto 3x^{1/5}y^{-2/5}$ and $x^{2/5}y^{1/5} \mapsto 4x^{2/5}y^{1/5}$.

```
> Expand(ScaleGenerators(s2, [3,4]), 15);
true
64/81*x^2*y^11 - 64/3*x^5*y^5 - 16/9*x^2*y^6 + 48*x^5 + 4*x^2*y
```

One can naturally view `h1` as a series in $\mathbf{Q}(i)[[u,v]]$.

```
> Qi<i> := NumberField(R.1^2 + 1) where R is PolynomialRing(Q);
> Qiuv<u,v> := PolynomialRing(Qi, 2, "glex");
> h4 := ChangeRing(s1, Qiuv);
> Expand(h4, 4); Domain(h4);
true u^3 + 3*u^2*v + 3*u*v^2 + v^3 + u^2 + 2*u*v + v^2 + u + v
Polynomial ring of rank 2 over Qi
Graded Lexicographical Order
Variables: u, v
```

We have seen that the power series h3 is zero, but its representation does not show this immediately. We can "explicitize" its representation.

```
> SimplifyRep(h3 : Factorizing := true);
Algebraic power series
0
> DefiningPolynomial($1);
z
```

51.8 Bibliography

[**Bec07**] Tobias Beck. Formal Desingularization of Surfaces – The Jung Method Revisited –. Technical Report 2007-31, RICAM, December 2007.
URL:http://www.ricam.oeaw.ac.at/publications/reports/.

52 VALUATION RINGS

Chapter 52

VALUATION RINGS

52.1 Introduction

Magma currently supports basic operations in valuation rings obtained either from the rational field $\mathbf{Q}$ (and a finite prime p), or from a field of rational functions over a field (and an irreducible polynomial, or the infinite prime).

52.2 Creation Functions

52.2.1 Creation of Structures

```
ValuationRing(Q, p)
```

Given the rational field Q and a rational prime number p, create the valuation ring R corresponding to the discrete non-Archimedean valuation v_p, consisting of rational numbers r such that $v_p(r) \geq 0$, that is, $r = \frac{x}{y} \in \mathbf{Q}$ such that $p \not| y$.

```
ValuationRing(F, f)
```

Given the rational function field F as a field of fractions of the univariate polynomial ring $K[x]$ over a field K, as well as a monic irreducible polynomial $f \in K[x]$, create the valuation ring R corresponding to the discrete non-Archimedean valuation v_f. Thus R consists of rational functions $\frac{g}{h} \in F$ with $v_f(g/h) \geq 0$, that is, with $f \not| h$.

```
ValuationRing(F)
```

Given the rational function field F as a field of fractions of the univariate polynomial ring $K[x]$ over a field K, create the valuation ring R corresponding to v_∞, consisting of $\frac{g}{h} \in F$ such that $\deg(h) \geq \deg(g)$.

52.2.2 Creation of Elements

```
V ! r
```

Given a valuation ring V and an element of the field of fractions F of V (from which V was created), coerce the element r into V. This is only possible for elements $r \in F$ for which the valuation on V is non-negative, an error occurs if this is not the case.

52.3 Structure Operations

52.3.1 Related Structures

`Category(V)` `Parent(V)` `PrimeRing(V)` `Center(V)`

`FieldOfFractions(V)`

Return field of fractions of the valuation ring V, which is the rational field or the function field from which V was created.

52.3.2 Numerical Invariants

`Characteristic(V)`

52.4 Element Operations

52.4.1 Arithmetic Operations

`+ v` `- v`

`v + w` `v - w` `v * w` `v ^ k` `v / w`

`v +:= w` `v -:= w` `v *:= w`

`v div w`

The quotient q of the division with remainder $v = qw + r$ of the valuation ring elements v and w, where the remainder will have valuation less than that of w; if the valuation of v is greater than or equal than that of w, this simply returns the quotient v/w, if the valuation of w exceeds that of v it returns 0.

52.4.2 Equality and Membership

`v eq w` `v ne w`

`v in V` `v notin V`

52.4.3 Parent and Category

`Parent(v)` `Category(v)`

52.4.4 Predicates on Ring Elements

```
IsZero(n)    IsOne(n)    IsMinusOne(n)
IsNilpotent(n)    IsIdempotent(n)
IsUnit(n)    IsZeroDivisor(n)    IsRegular(n)
```

52.4.5 Other Element Functions

```
EuclideanNorm(v)
Valuation(v)
```

Given an element v of a valuation ring V, return the valuation (associated with V) of v.

```
Quotrem(v, w)
```

Given two elements v, w of a valuation ring V with associated valuation ϕ, return a quotient and remainder q and r in V such that $v = qw + r$ and $0 \leq \phi(r) < \phi(w)$. If $\phi(v) < \phi(w)$ this simply returns $q = 0$ and $r = v$, and if $\phi(v) \geq \phi(w)$ then it returns $q = v/w$ and $r = 0$.

```
GreatestCommonDivisor(v, w)
Gcd(v, w)
```

This function returns a greatest common divisor of two elements v, w in a valuation ring V. This will return u^m, where $m = \min(\phi(v), \phi(w))$ is the minimum of the valuations of v and w $m = \min(\phi(v), \phi(w))$ and u is the uniformizing element of V (with valuation $\phi(u) = 1$).

```
ExtendedGreatestCommonDivisor(v, w)
Xgcd(v, w)
XGCD(v, w)
```

This function returns a greatest common divisor $z \in V$ of two elements v, w in a valuation ring V as well as multipliers $x, y \in V$ such that $xv + yw = z$. The principal return value will be $z = u^m$, where $m = \min(\phi(v), \phi(w))$ is the minimum of the valuations of v and w $m = \min(\phi(v), \phi(w))$ and u is the uniformizing element of V (with valuation $\phi(u) = 1$).

53 GALOIS RINGS

Chapter 53
GALOIS RINGS

53.1 Introduction

Magma provides facilities for computing with Galois rings. The features are currently very basic, but advanced features will be available in the near future, including support for the creation of subrings and appropriate embeddings, allowing lattices of compatible embeddings, just as for finite fields.

A Galois ring R in Magma is considered as a finite algebraic extension of $\mathbf{Z}_{p^a}$ (where p is prime) by a monic polynomial $D \in \mathbf{Z}[x]$ which is irreducible modulo p. Thus R is presented as the polynomial quotient ring

$$\mathbf{Z}_{p^a}[x]/\langle D\rangle$$

and is usually written as $\mathbf{GR}(p^a, d)$, where d is the degree of D. The cardinality of R is easily seen to be p^{ad}.

R has a unique maximal ideal generated by p, and the quotient ring $R/\langle p\rangle$ is a finite field isomorphic to $\mathbf{Z}_p[x]/\langle D\rangle$, where D is here considered as a polynomial in $\mathbf{Z}_p[x]$ (the coefficients are reduced modulo p). This finite field is called the *residue field* of R. In the following, we will also call the integer residue ring $\mathbf{Z}_{p^a}$ the *base ring* of R, because this is the subring of R generated by 1 and we can think of R as an extension of $\mathbf{Z}_{p^a}$.

For a non-zero element x of R, the valuation of x is defined to be the largest power of p which divides the coefficients of x, where x is considered as a polynomial in $\mathbf{Z}_p[x]/\langle D\rangle$. x is a unit if and only if the valuation of x is zero.

Because of the valuation defined on them, Galois rings are Euclidean rings, so they may be used in Magma in any place where general Euclidean rings are valid. This includes many matrix and module functions, and the computation of Gröbner bases. Linear codes over Galois rings will be supported in the near future.

53.2 Creation Functions

53.2.1 Creation of Structures

```
GaloisRing(q, d)
```

```
GR(q, d)
```

Given integers q, $d \geq 1$, where $q = p^a$ for prime p and $a \geq 1$, create the default Galois ring $\mathbf{GR}(p^a, d)$. The defining polynomial used to construct the ring will be that used for $\mathbf{F}_{p^d}$, lifted to $\mathbf{Z}_{p^a}$. If p is very large, it is advised to use the next function instead, because Magma must first factor q completely.

The angle bracket notation can be used to assign names to the generator; e.g.: `R<w> := GaloisRing(2, 3)`.

```
GaloisRing(p, a, d)
GR(p, a, d)
```

Check BoolElt *Default :* `true`

Given a prime p and integers $a, d \geq 1$, create the default Galois ring $\mathbf{GR}(p^a, d)$. The defining polynomial used to construct the ring will be that used for $\mathbf{F}_{p^d}$, lifted to $\mathbf{Z}_{p^a}$.

By default p is checked to be a strong pseudoprime for 20 random bases b with $1 < b < p$; if the parameter `Check` is `false`, then no check is done on p at all (this is useful when p is very large and one does not wish to perform an expensive primality test on p).

```
GaloisRing(q, D)
GR(q, D)
```

Given an integer q, where $q = p^a$ for prime p and $a \geq 1$, and a monic polynomial D over $\mathbf{Z}$ such that D is irreducible mod p, create the Galois ring $R = \mathbf{GR}(p^a, D)$. The coefficients of D are reduced modulo p^a, and R is constructed to behave like the polynomial quotient ring $\mathbf{Z}_{p^a}[x]/\langle D\rangle$. If p is very large, it is advised to use the next function instead, because Magma must first factor q completely.

```
GaloisRing(p, a, D)
GR(p, a, D)
```

Check BoolElt *Default :* `true`

Given a prime p, an integer $a \geq 1$, and a monic polynomial D over $\mathbf{Z}$ such that D is irreducible mod p, create the Galois ring $R = \mathbf{GR}(p^a, D)$. The coefficients of D are reduced modulo p^a, and R is constructed to behave like the polynomial quotient ring $\mathbf{Z}_{p^a}[x]/\langle D\rangle$. The parameter `Check` is as above.

53.2.2 Names

```
AssignNames(~R, [f])
```

Procedure to change the name of the generating element in the Galois ring R to the contents of the string f. When R is created, the name will be `R.1`.

This procedure only changes the name used in printing the elements of R. It does *not* assign to an identifier called f the value of the generator in R; to do this, use an assignment statement, or use angle brackets when creating the ring.

Note that since this is a procedure that modifies R, it is necessary to have a reference `~R` to R in the call to this function.

```
Name(R, 1)
```

Given a Galois ring R, return the element which has the name attached to it, that is, return the element `R.1` of R.

53.2.3 Creation of Elements

```
R . 1
```

```
Generator(R)
```

The generator for R as an algebra over its base ring $\mathbf{Z}_{p^a}$. Thus, if R is viewed as $\mathbf{Z}_{p^a}[x]/\langle D\rangle$, then `R.1` corresponds to x in this presentation.

```
R ! a
```

Given a Galois ring R create the element specified by a; here a is allowed to be an element coercible into R, which means that a may be

(i) An element of R;

(ii) An integer, to be identified with a modulo the characteristic p^a of R;

(iii) An element of the base ring $\mathbf{Z}_{p^a}$ of R, to be identified with the corresponding element of R.

(iv) A sequence of elements of the base ring $\mathbf{Z}_{p^a}$ of R. In this case the element $a_0 + a_1 w + \cdots + a_{n-1}w^{n-1}$ is created, where $a = [a_0, \ldots a_{n-1}]$ and w is the generator `R.1` of R over $\mathbf{Z}_{p^a}$.

```
One(R)    Identity(R)
Zero(R)   Representative(R)
```

These generic functions create 1, 1, 0, and 0 respectively, in any Galois ring.

```
Random(R)
```

Create a pseudo-random element of Galois ring R.

53.2.4 Sequence Conversions

```
ElementToSequence(a)
```

```
Eltseq(a)
```

Given an element a of the Galois ring R, return the sequence of coefficients $[a_0, \ldots, a_{n-1}]$ in the base ring $\mathbf{Z}_{p^a}$ of R (where R is $\mathbf{Z}_{p^a}[x]/\langle D\rangle$), such that $a = a_0 + a_1 w + \cdots + a_{n-1}w^{d-1}$, with w the generator of R, and d the degree of D.

Example H53E1

We can define the Galois ring $\mathbf{GR}(2^3, 2)$ using the default function:

```
> R<w> := GaloisRing(2^3, 2);
> R;
GaloisRing(2, 3, 2)
```

We note that R has characteristic 8 and that $w^2 + w + 1 = 0$ in R.

```
> R!8;
```

```
0
> R!9;
1
> w;
w
> 4*w;
4*w
> 4*w + 4*w;
0
> w^2;
7*w + 7
> w^2 + w + 1;
0
```

We can list all the elements of R by simply looping over R:

```
> [x: x in R];
[ 0, 1, 2, 3, 4, 5, 6, 7, w, w + 1, w + 2, w + 3, w + 4, w + 5, w + 6,
    w + 7, 2*w, 2*w + 1, 2*w + 2, 2*w + 3, 2*w + 4, 2*w + 5, 2*w + 6, 2*w
    + 7, 3*w, 3*w + 1, 3*w + 2, 3*w + 3, 3*w + 4, 3*w + 5, 3*w + 6,
    3*w + 7, 4*w, 4*w + 1, 4*w + 2, 4*w + 3, 4*w + 4, 4*w + 5, 4*w +
    6, 4*w + 7, 5*w, 5*w + 1, 5*w + 2, 5*w + 3, 5*w + 4, 5*w + 5, 5*w
    + 6, 5*w + 7, 6*w, 6*w + 1, 6*w + 2, 6*w + 3, 6*w + 4, 6*w + 5,
    6*w + 6, 6*w + 7, 7*w, 7*w + 1, 7*w + 2, 7*w + 3, 7*w + 4, 7*w +
    5, 7*w + 6, 7*w + 7 ]
```

We see that the elements of R can be considered as polynomials of degree at most 1, with coefficients in the range $\{0 \dots 7\}$.
We can easily create elements of R also using the `!` operator, and use the `Eltseq` function to recover the corresponding sequence of coefficients.

```
> R ! [1, 2];
2*w + 1
> Eltseq(2*w + 1);
[ 1, 2 ]
> Eltseq(w);
[ 0, 1 ]
```

53.3 Structure Operations

53.3.1 Related Structures

`Category(R)` `Parent(R)` `Centre(R)`

`PrimeRing(R)` `PrimeField(R)`

`FieldOfFractions(R)`

53.3.2 Numerical Invariants

`Characteristic(R)`

The characteristic of R, which is p^a where R is considered as $\mathbf{Z}_{p^a}[x]/\langle D\rangle$.

`#R`

The cardinality of R, which is p^{ad} where R is considered as $\mathbf{Z}_{p^a}[x]/\langle D\rangle$ and d is the degree of D.

`Degree(R)`

The degree of R, which is the degree of D, where R is considered as $\mathbf{Z}_{p^a}[x]/\langle D\rangle$.

`ResidueField(R)`

The residue field of R, which is the finite field $\mathbf{Z}_p[x]/\langle D\rangle$, where R is considered as $\mathbf{Z}_{p^a}[x]/\langle D\rangle$.

53.3.3 Ring Predicates and Booleans

The following functions are described for rings in general in Section 17.4.3.

`IsCommutative(R)` `IsUnitary(R)`

`IsFinite(R)` `IsOrdered(R)`

`IsField(R)` `IsEuclideanDomain(R)`

`IsPID(R)` `IsUFD(R)`

`IsDivisionRing(R)` `IsEuclideanRing(R)`

`IsPrincipalIdealRing(R)` `IsDomain(R)`

`R eq G` `R ne G`

53.4 Element Operations

See also Section 17.5.

53.4.1 Arithmetic Operators

`+ a` `- a`

`a + b` `a - b` `a * b` `a ^ k`

`a +:= b` `a -:= b` `a *:= b`

53.4.2 Euclidean Operations

`a div b` `a mod b` `Quotrem(a, b)`

`GCD(a, b)` `LCM(a, b)` `XGCD(a, b)`

53.4.3 Equality and Membership

`a eq b` `a ne b`

`a in R` `a notin R`

53.4.4 Parent and Category

`Parent(a)` `Category(a)`

53.4.5 Predicates on Ring Elements

`IsZero(a)` `IsOne(a)` `IsMinusOne(a)`

`IsNilpotent(a)` `IsIdempotent(a)`

`IsUnit(a)` `IsZeroDivisor(a)`

Example H53E2

This simple example shows how one can compute Gröbner bases over Galois rings. We create an ideal in 3 variables over a Galois ring and compute its Gröbner basis.

```
> R<w> := GaloisRing(3, 3, 2);
> #R;
729
> P<x,y,z> := PolynomialRing(R, 3);
> I := ideal<P | [x^2 - w*y, 3*y^3 - 3*w*x*z, 9*z^5 - 9*w]>;
> GroebnerBasis(I);
[
    x^2 + 26*w*y,
    3*x*y^3 + (6*w + 6)*y*z,
    3*x*z + (15*w + 3)*y^3,
    9*x + (9*w + 9)*y^3*z^4,
    3*y^6 + (21*w + 15)*y*z^2,
    9*z^5 + 18*w
]
```

Notice that the leading coefficients include 3 and 9, which are not units in the ring. See Chapter 111 for much more information on such Gröbner bases.

54 NEWTON POLYGONS

Chapter 54
NEWTON POLYGONS

54.1 Introduction

This chapter introduces functions which allow the construction and simple study of Newton polygons. It allows data from a number of different contexts to be used to construct the polygons. It also covers different interpretations of Newton polygons, and translation among these interpretations. Recall that a Newton polygon is the intersection of finitely many rational half spaces in the rational plane. An advantage of this definition is that it emphasizes that Newton polygons are often thought of as being noncompact. However, they are not implemented here in this way. Instead polygons are interpreted as the convex hull of finitely many points of the plane (possibly including some points at $+\infty$ along the axes).

Any Newton polygon is contained in some virtual cartesian product of the rational field with itself (virtual since this plane is not a structure that is intended to be accessible to the user or which is characteristic of the polygon). The first and second coordinate functions of this plane are referred to as the x and y coordinates respectively. Points of the plane will always be written $\langle a, b\rangle$ while faces of polygons will be written $\langle a, b, c\rangle$. Geometrically, a face $\langle a, b, c\rangle$ is a one-dimensional boundary intersection of N with the line $ax + by = c$.

The *standard Newton polygon* of a polynomial $f = f(u, v)$ is, by definition, the convex hull of the points $\langle a, b\rangle$, so-called *Newton points*, ranging over monomials $u^a v^b$ having nonzero coefficient in f together with the points $+\infty$ on the two axes. The infinite points, however, are not listed among the vertices of the polygon; they are simply a convenient way of hiding all Newton points other than those on the 'lower lefthand' faces of the hull of those points. A similar definition applies to polynomials $f(y)$ whose coefficients lie in some field of fractional power series, or *Puiseux field*, $k\langle\langle x\rangle\rangle$. Newton Polygons can also be created for polynomials over local rings (and fields). The defining points are computed as $\langle i, v(a_i)\rangle$ where a_i is the coefficient of the i-th power of the generator and v denotes the valuation function on the ring. Infinite points arise only as the valuation of zero coefficients.

The main intended application of Newton polygons is to the Newton–Puiseux analysis of singular points of plane curves, or put another way, the factorization of polynomials defined over Puiseux fields.

The examples below should be thought of as running consecutively in a single Magma session.

```
> R<x,y> := PolynomialRing(Rationals(),2);
> f := x^5 + x^2*y + x^2*y^3 + y^4 + x^3*y^5 + y^2*x^7;
> N := NewtonPolygon(f);
> N;
Newton Polygon of x^7*y^2 + x^5 + x^3*y^5 + x^2*y^3 + x^2*y + y^4 over
```

```
Rational Field
> Faces(N);
[ <3, 2, 8>, <1, 3, 5> ]
> Vertices(N);
[ <0, 4>, <2, 1>, <5, 0> ]
```

This is the standard Newton polygon associated with the polynomial f. Only those vertices and faces of the convex hull of points which correspond to the monomials appearing in f which 'face' the origin are considered to be vertices and faces of the polygon. As already mentioned, one interpretation of this is to think of the points $+\infty$ on each axis as being included among the defining points of N, and then N is the convex hull of its defining points.

To consider N as the compact convex hull of only the monomials of f simply use alternative vertex and face functions.

```
> AllVertices(N);
[ <0, 4>, <2, 1>, <5, 0>, <7, 2>, <3, 5> ]
> AllFaces(N);
[ <3, 2, 8>, <1, 3, 5>, <-1, 1, -5>, <-3, -4, -29>, <1, -3, -12> ]
```

However, where there is a choice of interpretation, Magma always interprets N in the way it was defined. So for example, the functions which test for faces and vertices compare a given value with the sequence returned by the functions `Faces(N)` or `Vertices(N)`, which are fixed the first time they are calculated, rather than with any other collections of faces or vertices.

```
> IsFace(N,<3/4,3/6,2>);
true <3, 2, 8>
> IsFace(N,<-3,-4,-29>);
false
```

Notice that faces are reduced to normal integral form and that the correct form is returned as the second return value of the test function.

When a polygon is created using a polynomial, the restriction of the polynomial to faces is important characteristic data.

```
> FaceFunction(Faces(N)[1]);
x^2*y + y^4
```

The face function is simply the sum of those monomial terms of f (they keep their coefficients) whose corresponding Newton point lies on the given face.

54.2 Newton Polygons

All polygons are determined by a finite collection of points in the rational plane. For Magma, these points are the most basic attribute of any Newton polygon. They are always determined and recorded on creation of a polygon. Throughout this chapter, for a Newton polygon N, these points are denoted by P_N. As seen in the introduction, the main class of polygons is that comprising polygons in the first quadrant of the plane and including the points $+\infty$ on the two axes. But there are other useful types, especially when calculating factorizations of univariate polynomials over series rings. The data distinguishing the different flavours of Newton polygon is the collection of lines and points that are considered to be faces and vertices.

54.2.1 Creation of Newton Polygons

These are the functions available for constructing Newton polygons and retrieving the points which describe them.

```
NewtonPolygon(f)
```

`Faces` MonStgElt *Default : "Inner"*

The standard Newton polygon of a polynomial f in two variables. This is the hull of the Newton points of the polynomial together with the points $+\infty$ on each axis. The horizontal and vertical 'end' faces are not listed among the faces of the polygon; the points at infinity are not listed among the vertices of the polygon.

The parameter `Faces` can have the value `"Inner"`, `"Lower"` or `"All"`. This determines which faces are returned by the intrinsic `Faces`.

```
NewtonPolygon(f)
```

`SwapAxes` BoolElt *Default :* `false`

`Faces` MonStgElt *Default :*

The standard Newton polygon of a polynomial in one variable defined over a series ring or a local ring or field. A value of `true` for `SwapAxes` is only valid if the polynomial is over a series ring. If `SwapAxes` is set to `true` then the exponents of the series variable will be plotted on the horizontal axis and the exponents of the polynomial on the vertical axis.

For a polynomial over a series ring, the hull includes the points $+\infty$ on each axis. For a polynomial over a local ring, the infinite points are not included.

The parameter `Faces` can have the value `"All"`, `"Inner"` or `"Lower"`. This determines which faces are returned by the intrinsic `Faces`. The default for series rings is `"Inner"` and for local rings is `"Lower"`.

```
NewtonPolygon(f, p)
```

`Faces` MonStgElt *Default : "Inner"*

The newton polygon of f where p is a prime used for valuations of the coefficients of f. The polynomial f may be over the integers or rationals or a number field or algebraic function field or an order thereof. The prime p may be an integer or a prime ideal. The newton polygon will have points (i, v_i) where i is the exponent of a term of f and v_i is the valuation of the coefficient of the ith term. The points at $+\infty$ on each axis are included.

The parameter Faces can have the value "Inner", "Lower" or "All". This determines which faces are returned by the intrinsic Faces.

`NewtonPolygon(f, p)`

Faces MonStgElt *Default : "Inner"*

The newton polygon of the polynomial f where the place p of an algebraic function field is the prime used for determining the valuations of the coefficients of f. The points at $+\infty$ on each axis are included.

The parameter Faces can have the value "Inner", "Lower" or "All". This determines which faces are returned by the intrinsic Faces.

`NewtonPolygon(C)`

The standard Newton polygon of the defining polynomial of the curve C.

`NewtonPolygon(V)`

Faces MonStgElt *Default : "All"*

The Newton polygon that is the compact convex hull of the set or sequence V of points of the form $\langle a, b\rangle$ where a, b are integers or rational numbers.

The parameter Faces can have the value "All", "Lower" or "Inner". This determines which faces are returned by the intrinsic Faces.

`DefiningPoints(N)`

The points of the rational plane used in the initial creation of N. Applying this function to two polygons allows their defining points to be compared. No explicit function is provided for testing whether defining points of two polygons are equal.

Example H54E1

Some ways of creating Newton Polygons from polynomials are shown below.

```
> P<y> := PuiseuxSeriesRing(Rationals());
> R<x> := PolynomialRing(P);
> f := 3*x^4 + (5*y^3 + 4*y^(1/4))*x^3 + (7*y^2 + 1/2*y^(1/3))*x^2 + 6*x + y^(
> 4/5);
> N := NewtonPolygon(f);
> N;
Newton Polygon of 3*x^4 + (4*y^(1/4) + 5*y^3)*x^3 + (1/2*y^(1/3) + 7*y^2)*x^2 +
6*x + y^(4/5) over Puiseux series field in y over Rational Field
> P<x> := PolynomialRing(Integers());
> L := ext<ext<pAdicRing(5, 100) | 3> | x^2 + 5>;
> R<x> := PolynomialRing(L);
> f := 3*x^4 + 75*x^3 + 78*x^2 + 10*x + 750;
> NR := NewtonPolygon(f);
> NR;
Newton Polygon of 3*x^4 + 75*x^3 + 78*x^2 + 10*x + 750 over L
```

Newton Polygons can also be created by specifying the defining points that the polygon must enclose.

```
> N2 := NewtonPolygon({<2, 0>, <0, 3>, <4, 1>});
> N2;
Newton Polygon with defining points {(0, 3), (2, 0), (4, 1)}
> N6 := NewtonPolygon({<1, 4>, <1, 6>, <2, 4>, <3, 1>, <6, 1>, <5, 2>, <4, 5>,
> <4, 7>, <6, 6>, <7, 7>, <2, 7>, <5, 9>, <8, 4>, <8, 6>, <8, 8>, <7, 9>});
> N6;
Newton Polygon with defining points {(1, 4), (1, 6), (2, 4), (2, 7), (3, 1), (4,
5), (4, 7), (5, 2), (5, 9), (6, 1), (6, 6), (7, 7), (7, 9), (8, 4), (8, 6), (8,
8)}
```

These polygons will be referred to in later examples.

54.2.2 Vertices and Faces of Polygons

Both the vertices $\langle a, b\rangle$ and faces $\langle a, b, c\rangle$ (representing $ax + by = c$) of a given polygon N are computed as needed. As seen above, these will be a particular choice of possible faces and vertices determined by the data used to create the polygon. They can be recovered using the `Faces()` and `Vertices()` intrinsics. A different choice of faces and vertices, those faces and vertices of the compact convex hull of the defining points say, can be made using the other intrinsics below.

Recall that P_N denotes the set of points used in the definition of the Newton polygon N whether they arise as the powers of monomials appearing in a polynomial or have been given explicitly as a sequence of pairs.

`Faces(N)`

The sequence of faces $\langle a, b, c\rangle$ (representing $ax + by = c$) of N listed anticlockwise. How this is interpreted in terms of the points used to create N depends on the creation function used (see Section 54.2.1). The faces are listed anticlockwise starting with the face with its left endpoint being the lowest of the leftmost points.

`InnerFaces(N)`

Those faces of the compact convex hull of P_N starting at the lowest of the leftmost points which have strictly negative gradient.

`LowerFaces(N)`

Those faces of the compact convex hull of P_N which bound it below in the y direction.

`OuterFaces(N)`

The union of lower faces which aren't inner faces and the faces which bound the compact convex hull of P_N above in the y-direction (ignoring infinite points).

`AllFaces(N)`

The faces of the compact convex hull of P_N.

Example H54E2

Using some of the polygons defined before the different types of faces are illustrated.

```
> Faces(N);
[ <4, 5, 4> ]
> InnerFaces(N);
[ <4, 5, 4> ]
> OuterFaces(N);
[ <0, 1, 0>, <-1, -4, -4>, <-11, -60, -48> ]
> AllFaces(N);
[ <4, 5, 4>, <0, 1, 0>, <-1, -4, -4>, <-11, -60, -48> ]
> Faces(NR);
[ <4, 1, 6>, <2, 1, 4>, <0, 1, 0> ]
> InnerFaces(NR);
[ <4, 1, 6>, <2, 1, 4> ]
> LowerFaces(NR);
[ <4, 1, 6>, <2, 1, 4>, <0, 1, 0> ]
```

For the polynomial over the Puiseux Field it is no coincidence that `InnerFaces` and `Faces` return the same sequences. Similarly, for the polynomial over the local ring `Faces` is defined to be `LowerFaces`. For both, this is the category of faces that gives the most information for the purposes that the polygon is used. It can also be noted that combining `InnerFaces` and `OuterFaces` will give `AllFaces` with no repetitions (though repetitions will occur if the polygon has only one face and this face is an inner face).

`Vertices(N)`

The sequence of vertices of N. The vertices will be listed anticlockwise from the lowest of the leftmost points.

`InnerVertices(N)`

The sequence of vertices which arise as endpoints of inner faces.

`LowerVertices(N)`

The sequence of vertices which arise as endpoints of lower faces.

`OuterVertices(N)`

The sequence of vertices which arise as endpoints of outer faces.

`AllVertices(N)`

The sequence of vertices of the compact convex hull of P_N.

Example H54E3

This example illustrates the types of vertices that can be calculated. Note that the printing of these polygons, created from their defining points, changes as more information is calculated. This would occur in the same manner if faces were being calculated instead of vertices.

```
> InnerVertices(N2);
[ <0, 3>, <2, 0> ]
> N2;
Newton Polygon with vertices {(0, 3), (2, 0)} and defining points {(0, 3), (2,
0), (4, 1)}
> InnerVertices(N6);
[ <1, 4>, <3, 1> ]
> N6;
Newton Polygon with vertices {(1, 4), (3, 1)} and defining points {(1, 4), (1,
6), (2, 4), (2, 7), (3, 1), (4, 5), (4, 7), (5, 2), (5, 9), (6, 1), (6, 6), (7,
7), (7, 9), (8, 4), (8, 6), (8, 8)}
> Vertices(N2);
[ <0, 3>, <2, 0>, <4, 1> ]
> Vertices(N6);
[ <1, 4>, <3, 1>, <6, 1>, <8, 4>, <8, 8>, <7, 9>, <5, 9>, <2, 7>, <1, 6> ]
> AllVertices(N2);
[ <0, 3>, <2, 0>, <4, 1> ]
> N2;
Newton Polygon with vertices {(0, 3), (2, 0), (4, 1)} and defining points {(0,
3), (2, 0), (4, 1)}
> AllVertices(N6);
[ <1, 4>, <3, 1>, <6, 1>, <8, 4>, <8, 8>, <7, 9>, <5, 9>, <2, 7>, <1, 6> ]
> N6;
Newton Polygon with vertices {(1, 4), (3, 1), (6, 1), (8, 4), (8, 8), (7, 9),
(5, 9), (2, 7), (1, 6)} and defining points {(1, 4), (1, 6), (2, 4), (2, 7), (3,
1), (4, 5), (4, 7), (5, 2), (5, 9), (6, 1), (6, 6), (7, 7), (7, 9), (8, 4), (8,
6), (8, 8)}
```

Here `Vertices` has been defined to be `AllVertices`. All the known vertices of the polygon are printed when the polygon is printed. There is some overlap between the inner and outer vertices as is shown below. Every vertex is either an inner vertex or an outer vertex with some being both. Not all defining points are vertices.

```
> OuterVertices(N6);
[ <3, 1>, <6, 1>, <8, 4>, <8, 8>, <7, 9>, <5, 9>, <2, 7>, <1, 6>, <1, 4> ]
> OuterVertices(N2);
[ <2, 0>, <4, 1>, <0, 3> ]
```

`EndVertices(F)`

A sequence containing the two end vertices of the face $F = \langle a, b, c\rangle$.

`FacesContaining(N,p)`

Those faces of the polygon N returned by `Faces` on which the point $p = \langle a, b\rangle$ lies.

Example H54E4

Using some of the example polygons that have been created above, we illustrate the simple use of `EndVertices` and `FacesContaining`.

```
> AN := AllFaces(N);
> AN;
[ <4, 5, 4>, <0, 1, 0>, <-1, -4, -4>, <-11, -60, -48> ]
> A6 := AllFaces(N6);
> A6;
[ <3, 2, 11>, <0, 1, 1>, <-3, 2, -16>, <-1, 0, -8>, <-1, -1, -16>, <0, -1, -9>,
<2, -3, -17>, <1, -1, -5>, <1, 0, 1> ]
> AllVertices(N);
[ <0, 4/5>, <1, 0>, <4, 0>, <3, 1/4> ]
> AllVertices(N6);
[ <1, 4>, <3, 1>, <6, 1>, <8, 4>, <8, 8>, <7, 9>, <5, 9>, <2, 7>, <1, 6> ]
> EndVertices(AN[1]);
[ <0, 4/5>, <1, 0> ]
> EndVertices(AN[4]);
[ <0, 4/5>, <3, 1/4> ]
> EndVertices(A6[1]);
[ <1, 4>, <3, 1> ]
> EndVertices(A6[5]);
[ <7, 9>, <8, 8> ]
> EndVertices(A6[9]);
[ <1, 4>, <1, 6> ]
> FacesContaining(N, <1, 0>);
[ <4, 5, 4> ]
> FacesContaining(N6, <1, 0>);
[]
> FacesContaining(N6, <4, 1>);
[ <0, 1, 1> ]
> FacesContaining(N, <4, 1>);
[]
> FacesContaining(N6, <3, 1>);
[ <3, 2, 11>, <0, 1, 1> ]
```

`GradientVector(F)`

The a and b values of the line describing the face F of the form $a * x + b * y = c$ where a, b and c are integers.

`GradientVectors(N)`

A sequence containing the gradient vectors of the faces of the newton polygon N.

```
Weight(F)
```

The c value of the line describing the face F of the form $a * x + b * y = c$ where a, b and c are integers.

```
Slopes(N)
```

The slopes of the faces of the newton polygon N.

```
InnerSlopes(N)
LowerSlopes(N)
AllSlopes(N)
```

The slopes of the polygon N corresponding of `InnerFaces`, `LowerFaces` and `AllFaces` respectively.

Example H54E5

In this example `GradientVector` and `Weight` can be seen to be access functions on the components of a face of a polygon.

```
> A := AllFaces(N);
> A;
[ <4, 5, 4>, <0, 1, 0>, <-1, -4, -4>, <-11, -60, -48> ]
> f := A[3];
> GradientVector(f);
<-1, -4>
> Weight(f);
-4
```

The gradient of the face can now be easily computed as shown.

```
> a := GradientVector(f)[1];
> b := GradientVector(f)[2];
> -a/b;
-1/4
```

54.2.3 Tests for Points and Faces

Once more, recall that P_N denotes the finite set of points in the plane used to define the Newton polygon N. Whether or not a point is considered to lie in a polygon depends on what are considered to be its faces. MAGMA always uses the list of faces returned by `Faces(N)` when testing points. Of course, this is not always the case in applications. One must to perform other tests explicitly when there is doubt.

```
IsFace(N, F)
```

Return `true` if and only if the tuple $F = \langle a, b, c\rangle$ describes a line coinciding with a face of the polygon N as returned by `Faces`.

```
IsVertex(N, p)
```

Return true if and only if the point $p = \langle a, b\rangle$ of the rational plane (given as a tuple) is a vertex of the polygon N as returned by `Vertices`.

```
IsInterior(N,p)
```

Return true if and only if the point $p = \langle a, b\rangle$ given as a tuple lies strictly in the interior of the polygon N.

```
IsBoundary(N, p)
```

Return true if and only if the point $p = \langle a, b\rangle$ given as a tuple lies on the boundary of the polygon N, that is, the point is contained in a face of N.

```
IsPoint(N,p)
```

Return true if and only if the point $p = \langle a, b\rangle$ (given as a tuple) lies on the polygon N.

54.3 Polynomials Associated with Newton Polygons

The polynomial used to define a polygon can be recovered, but more usefully so can those restrictions of that polynomial to parts of the polygon, the so-called face functions in particular.

Note that most of these functions will return an error if N was not defined in terms of a polynomial.

```
HasPolynomial(N)
```

Return true if and only if the polygon N was defined as the Newton polygon of some polynomial.

```
Polynomial(N)
```

The polynomial used to define the polygon N.

```
ParentRing(N)
```

The parent ring of the polynomial of the polygon N.

```
IsNewtonPolygonOf(N, f)
```

Return whether the newton polygon N is defined by the polynomial f.

```
FaceFunction(F)
```

If the polygon N is defined by a polynomial in two variables f this returns those monomial terms of f whose corresponding Newton points lie on the face F. On the other hand, if N is determined by a univariate polynomial over a series ring, this returns the univariate polynomial supported on the face F.

```
IsDegenerate(F)
```

Return `true` if the face function along F is not squarefree.

```
IsDegenerate(N)
```

Return `true` if a face function on some face of N is degenerate.

54.4 Finding Valuations of Roots of Polynomials from Newton Polygons

Newton polygons can be used to find the valuations of roots of the polynomial from which the polygon was created at the prime used in the creation (given implicitly or explicitly). The following functions use Newton polygons to calculate the valuations of the roots of the polynomial paired with the number of roots with that valuation.

```
ValuationsOfRoots(f)
```

The valuations of the roots of f, where f is a polynomial over a local ring or a series ring.

```
ValuationsOfRoots(f, p)
```

The valuations of the roots of f with respect to p where p may be either a prime integer, a prime ideal of a number field or a place of a function field.

54.5 Using Newton Polygons to Find Roots of Polynomials over Series Rings

The operations described in this section are relevant for polynomials over series rings. There are two main algorithms involved.

```
SetVerbose("Newton", v)
```

Set the verbose printing to level v for `PuiseuxExpansion`, `ExpandToPrecision`, `DuvalPuiseuxExpansion`, `Roots` and `ImplicitFunction`. A level of 1 will mean that any partial solutions that could not be expanded to the precision requested will be printed before an error is returned except for `Roots`. The polynomials used in forming extensions will also be printed before the extension is computed. In `Roots`, the algorithm used to compute the expansions will be printed. When Walker's algorithm is being used the current value of the denominator will be printed. For `ImplicitFunction` a warning about a potentially bad value of d will be printed if the value of d given is not divisible by the exponent denominator of some coefficient of f. A level of 2 will print the last polynomials calculated during the newton polygon part of `PuiseuxExpansion` and `DuvalPuiseuxExpansion` and some evaluated polynomials during `ImplicitFunction`.

54.5.1 Operations not associated with Duval's Algorithm

```
PuiseuxExpansion(f, n)
```

`PreciseRoot`	BoolElt	*Default :* `false`
`TestSquarefree`	BoolElt	*Default :* `true`
`NoExtensions`	BoolElt	*Default :* `false`
`LowerFaces`	BoolElt	*Default :* `true`
`OneRoot`	BoolElt	*Default :* `false`
Verbose	`Newton`	*Maximum :* 2

This function implements the algorithm described in [Wal78].

Return a sequence of partial expansions of the roots of the polynomial f over a series ring as puiseux series. The roots are returned with relative precision at least n/d where d is the least common multiple of exponent denominator for the series expansion and the exponent denominators of the coefficients of the f. An input of $n = 0$ will return the expansions calculated by the newton polygon part of the algorithm and these will be to the precision of what is known. The coefficient ring of the series ring containing the coefficients of f must always be a field and unless extensions are not required it must be able to be extended.

If the coefficient ring of the series ring is a finite field whose characteristic is less than or equal to the degree of the polynomial then the denominators computed in the newton polygon part of the algorithm may not be bounded and the function will return an error. However, it is possible for some polynomials that the denominators will be bounded. This is stated by [Gri95], pg 269 – 272.

Care needs to be taken with polynomials whose coefficients have low precision. The algorithm must extract from f the squarefree part and in doing so lose even more precision. The result is that the algorithm may not have enough precision to calculate the expansions correctly. A solution is to set `TestSquarefree` to `false` if the polynomial is known to have no multiple roots. However this will not solve all precision problems and the answer is only as good as the precision allows it to be.

If `PreciseRoot` is set to `true` then the partial expansions to be returned are checked and if any are exact roots of f they are returned with full precision. If `NoExtensions` is set to `true` then expansions are found within the puiseux series ring only. By default, the coefficient ring of the series ring is extended to find all the partial expansions. If `LowerFaces` is set to `false` then expansions with negative valuations will not be found. If `OneRoot` is set to true then representatives of conjugate roots only will be found instead of each of the roots individually.

Note that it may useful to define the polynomial over an algebraically closed field (via `AlgebraicClosure`), so that all roots may be found.

```
ExpandToPrecision(f, c, n)
```

PreciseRoot	BoolElt	*Default :* false
TestSquarefree	BoolElt	*Default :* true
Verbose	Newton	*Maximum :* 2

Given a polynomial f over a Puiseux series ring and a partial root c of that polynomial (found by `PuiseuxExpansion` for example) continue to expand that root until it has relative precision n/d where d is the least common multiple of the exponent denominator of c and the exponent denominators of the coefficients of f. If c is given to greater precision (or length greater than n if its precision is infinite), the relative precision of c is reduced to n/d. An error results if c is not a partial expansion to precision n/d. If `PreciseRoot` is true then the partial expansion to be returned is checked and if it is an exact root of f it is returned with full precision. All input is checked for being an exact root regardless. If `TestSquarefree` is `false` the polynomial will not be made squarefree. This may avoid some loss of precision but may result in some unique partial roots not being recognized as unique partial roots and as such they cannot be expanded.

An error may result if c is a partial root of f but the exponent denominator of the full expansion is greater than that of c. Therefore for c to be expanded it must have the same denominator as the expansion it is part of. This will rarely be an issue for those partial expansions resulting from `PusieuxExpansion` which did not encounter problems with precision in the newton polygon part since the algorithm for this will find at least as much of the expansion as necessary to compute the exponent denominator.

```
ImplicitFunction(f, d, n)
```

Verbose	Newton	*Maximum :* 2

Return a root of the polynomial f over a series ring. The input d is the denominator (or a multiple of) the exponent denominator of the root. The root is given to absolute precision n/d. The evaluation of f at zero (polynomial) evaluated at zero (series) must be zero but that of its derivative must be nonzero.

Example H54E6

This example illustrates the joint use of `PuiseuxExpansion` and `ExpandToPrecision` which can be used together to gain and improve partial roots of a polynomial. The use of `ExpandToPrecision` following `PuiseuxExpansion` avoids the recalculation of information already known.

```
> P<x> := PuiseuxSeriesRing(Rationals());
> R<y> := PolynomialRing(P);
> f := y^3 + 2*x^-1*y^2 + 1*x^-2*y + 2*x;
> c := PuiseuxExpansion(f, 0);
> A<a> := Parent(c[1]);
> N<n> := CoefficientRing(A);
> Q<q> := PolynomialRing(A);
> c;
```

```
[
    -2*a^3 + O(a^4),
    -a^-1 + n*a + O(a^2),
    -a^-1 - n*a + O(a^2)
]
> [ExpandToPrecision(f, c[i], 10) : i in [1 .. #c]];
[
    -2*a^3 - 8*a^7 - 56*a^11 + O(a^13),
    -a^-1 + n*a + a^3 + 5/4*n*a^5 + 4*a^7 + O(a^9),
    -a^-1 - n*a + a^3 - 5/4*n*a^5 + 4*a^7 + O(a^9)
]
```

The same results could have been gained using `PuiseuxExpansion` with the required precision in the first place.

```
> c := PuiseuxExpansion(f, 10);
> A<a> := Parent(c[1]);
> N<n> := CoefficientRing(A);
> c;
[
    -2*a^3 - 8*a^7 - 56*a^11 + O(a^13),
    -a^-1 + n*a + a^3 + 5/4*n*a^5 + 4*a^7 + O(a^9),
    -a^-1 - n*a + a^3 - 5/4*n*a^5 + 4*a^7 + O(a^9)
]
```

However, asking for more precision requires time so that if it is not necessary the extra calculation can be avoided and if more precision happens to be required then it can be gained without recalculation. `ExpandToPrecision` is also called on only one root so that if only one expansion is required using `PusieuxExpansion` and then `ExpandToPrecision` will not calculate any unnecessary information.

```
> time c := PuiseuxExpansion(f, 100);
Time: 2.810
> time c := PuiseuxExpansion(f, 10);
Time: 0.060
> A<a> := Parent(c[1]);
> N<n> := CoefficientRing(A);
> time ExpandToPrecision(f, c[1], 100);
-2*a^3 - 8*a^7 - 56*a^11 - 480*a^15 - 4576*a^19 - 46592*a^23 -
    496128*a^27 - 5457408*a^31 - 61529600*a^35 - 707266560*a^39 -
    8257566720*a^43 - 97654702080*a^47 - 1167349284864*a^51 -
    14082308833280*a^55 - 171221451538432*a^59 -
    2096081963188224*a^63 - 25814314231136256*a^67 -
    319605795242639360*a^71 - 3975750610806374400*a^75 -
    49666299938073477120*a^79 - 622818862289639178240*a^83 -
    7837247078959687925760*a^87 - 98931046460491133091840*a^91 -
    1252424949872174982758400*a^95 - 15897106567806080658702336*a^99
    + O(a^103)
```

```
Time: 0.410
```

`IsPartialRoot(f, c)`

Return `true` if the series c can be expanded to at least one root of the polynomial f.

`IsUniquePartialRoot(f, c)`

`TestSquarefree` BoolElt *Default :* `true`

Return `true` if the series c can be expanded to exactly one distinct root of the polynomial f. By default f will have multiple factors removed to allow partial expansions of multiple roots to be recognized as being unique. If `TestSquarefree` is set to `false` then f will be taken as given which may avoid errors due to lost precision but may not pick partial expansions of multiple roots as being unique and as such is best used when f is squarefree or the expansion is known to be of a single root.

Example H54E7

The above 2 functions can be used to reduce the occurrence of errors from `ExpandToPrecision` by checking that the input can be expanded. Errors resulting from a lack of precision which means that the expansion cannot be calculated to the requested precision are the only errors that cannot be removed. Only unique partial roots can be expanded. If a partial root is not unique then calling `PuiseuxExpansion` will provide several further partial expansions of the partial root that will themselves be unique and so can be used to calculate several expansions of the original.

```
> P<x> := PuiseuxSeriesRing(Rationals());
> R<y> := PolynomialRing(P);
> f := (y^2 - x^3)^2 - y*x^6;
> IsPartialRoot(f, x^(3/2));
true
>  ExpandToPrecision(f, x^(3/2), 10);
>> ExpandToPrecision(f, x^(3/2), 10);
                         ^

Runtime error in 'ExpandToPrecision': Element is not a unique partial
root of the polynomial
> IsUniquePartialRoot(f, x^(3/2));
false
> c := PuiseuxExpansion(f, 0);
> A<a> := Parent(c[1]);
> N<n> := CoefficientRing(A);
> Q<q> := PolynomialRing(A);
> c;
[
    a^(3/2) + 1/2*a^(9/4) + O(a^(5/2)),
    a^(3/2) - 1/2*a^(9/4) + O(a^(5/2)),
    -a^(3/2) + 1/2*n*a^(9/4) + O(a^(5/2)),
    -a^(3/2) - 1/2*n*a^(9/4) + O(a^(5/2))
]
> IsUniquePartialRoot(f, x^(3/2) + 1/2*x^(9/4));
true
> ExpandToPrecision(f, x^(3/2) + 1/2*x^(9/4), 10);
x^(3/2) + 1/2*x^(9/4) - 1/64*x^(15/4) + O(x^4)
>  ExpandToPrecision(f, x^(3/2) + x^2, 30);
>> ExpandToPrecision(f, x^(3/2) + x^2, 30);
                    ^

Runtime error in 'ExpandToPrecision': Element is not a partial root of
the polynomial
> IsPartialRoot(f, x^(3/2) + x^2);
false
```

So if `IsPartialRoot` returns `false` then no expansion can be made. If `IsUniquePartialRoot` returns `false` (but `IsPartialRoot` returns `true`) then several expansions can be made after calling `PuiseuxExpansion`.

`PuiseuxExponents(p)`

Given a series expansion return the sequence of exponents $[a/b]$ of the non zero terms of the series p up to and including the first one where b is the global denominator for the series.

`PuiseuxExponentsCommon(p, q)`

Given two series return the sequence of exponents $[a/b]$ of the non zero initial terms of the series p and q which are equal up to but not including the first unequal terms.

Example H54E8

This example illustrates how `PuiseuxExponents` and `PuiseuxExponentsCommon` can be used on output from `PusieuxExpansion`. (Similar can be done with related functions and general series).

```
> P<x> := PuiseuxSeriesRing(FiniteField(5, 3));
> R<y> := PolynomialRing(P);
> f := (1+x)*y^4 - x^(-1/3)*y^2 + y + x^(1/2);
> time c := PuiseuxExpansion(f, 5);
Time: 0.030
> c;
[
    4*x^(1/2) + x^(2/3) + 3*x^(5/6) + x^(7/6) + O(x^(4/3)),
    x^(1/3) + x^(1/2) + 4*x^(2/3) + 2*x^(5/6) + O(x^(7/6)),
    4*x^(-1/6) + 2*x^(1/3) + O(x^(2/3)),
    x^(-1/6) + 2*x^(1/3) + O(x^(2/3))
]
> PuiseuxExponents(c[1]);
[ 1/2, 2/3, 5/6 ]
> PuiseuxExponents(c[3]);
```

```
[ -1/6 ]
> P<x> := PuiseuxSeriesRing(FiniteField(5, 3));
> R<y> := PolynomialRing(P);
> f := ((y^2 - x^3)^2 - y*x^6)^2 - y*x^15;
> c := PuiseuxExpansion(f, 0);
> A<a> := Parent(c[1]);
> N<n> := CoefficientRing(A);
> Q<q> := PolynomialRing(A);
> c;
[
    4*a^(3/2) + 4*a^(9/4) + 4*a^3 + O(a^(13/4)),
    4*a^(3/2) + 4*a^(9/4) + a^3 + O(a^(13/4)),
    4*a^(3/2) + a^(9/4) + 4*a^3 + O(a^(13/4)),
    4*a^(3/2) + a^(9/4) + a^3 + O(a^(13/4)),
    a^(3/2) + 3*a^(9/4) + 4*a^3 + O(a^(13/4)),
    a^(3/2) + 3*a^(9/4) + a^3 + O(a^(13/4)),
    a^(3/2) + 2*a^(9/4) + 4*a^3 + O(a^(13/4)),
    a^(3/2) + 2*a^(9/4) + a^3 + O(a^(13/4))
]
> PuiseuxExponentsCommon(c[1], c[1]);
[ 3/2, 9/4, 3 ]
> PuiseuxExponentsCommon(c[1], c[2]);
[ 3/2, 9/4 ]
> PuiseuxExponentsCommon(c[1], c[3]);
[ 3/2 ]
> PuiseuxExponentsCommon(c[1], c[8]);
[]
```

54.5.2 Operations associated with Duval's algorithm

The following functions have a similar use to those given above but implement a different algorithm, namely that of [Duv89] which is faster and can handle larger degree polynomials. However, it can only be used with polynomials which are essentially over a laurent series ring and the coefficient ring of that laurent series ring has either characteristic zero or characteristic greater than the degrees of the squarefree factors of the polynomial.

```
DuvalPuiseuxExpansion(f, n)
```

`Version`	MonStgElt	*Default :* "*Rational*"
`TestSquarefree`	BoolElt	*Default :* `true`
`NoExtensions`	BoolElt	*Default :* `false`
`LowerFaces`	BoolElt	*Default :* `true`
`OneRoot`	BoolElt	*Default :* `false`
Verbose	`Newton`	*Maximum :* 2

A sequence of parametrizations of puiseux expansions of roots of f, as puiseux series, where f is a polynomial over a series ring. The expansions will have at least n non zero terms (unless the expansion is finite and has less than n non zero terms), with more than n occurring only if n is less than the number of terms returned by the newton polygon part of the algorithm. The coefficients of f must have exponent denominator 1.

This algorithm is faster than that given by Walker and implemented in `PuiseuxExpansion`, since it doesn't calculate non zero terms explicitly and doesn't make all necessary extensions during the algorithm leaving some to be made when the series is computed from the parametrizations.

If f has coefficients with finite precision then the expansions can only be computed to as many non zero terms as can be known for that expansion. After this limit has been reached, an error results since the next non zero term is not known. If f has roots with finite puiseux expansions then if n is greater than the number of non zero terms in the expansion the expansion is returned with infinite precision.

If `Version` is set to `"Classical"` then the (slower) classical branch of the algorithm will be run which makes all extensions necessary for the computation of the expansions. It is still faster than `PusieuxExpansion` since it does not iterate through and calculate zero terms but will encounter the same problems that `PuiseuxExpansion` does with field extensions over the rationals. The classical version will return as many parametrizations as there are expansions and some of these parametrizations will give the same set of expansions.

If `NoExtensions` is set to `true` then only the expansions which lie in the puiseux series ring corresponding to the coefficient ring of f are calculated. Otherwise, all the expansions of roots of f are calculated regardless of where they lie. `LowerFaces` and `OneRoot` work as for `PuiseuxExpansion`.

This algorithm works with the squarefree part of f only. If any coefficient of f has low precision then this step may make it impossible for any information about the expansions to be gained due to a loss of further precision. A way around this is to set `TestSquarefree` to `false` if the polynomial is known to be squarefree. This may result in some information being returned but such information is only as good as the precision it was allowed.

```
ParametrizationToPuiseux(T)
```

The series that satisfy the parametrization `T`. These are found by evaluating `T[2]` at t where `T[1]` $= \lambda t^e$.

```
PuiseuxToParametrization(S)
```

A parametrization of the series S. It is the simplest one which takes the denominator out of S and makes it the exponent of the first entry in the parametrization.

Example H54E9

This example illustrates the use of `DuvalPuiseuxExpansion` and `ParametrizationToPuiseux` to gain the information given by `PuiseuxExpansion` and also compares the performance of the two algorithms. It also highlights some of the anomalies that may be encountered due to precision concerns.

```
> P<x> := PuiseuxSeriesRing(Rationals());
> R<y> := PolynomialRing(P);
> f := (y^2 - x^3)^2 - y*x^6;
> time D := DuvalPuiseuxExpansion(f, 0);
Time: 0.000
> D;
[
    <16*x^4, 64*x^6 + 256*x^9 + O(x^10)>
]
> time P := ParametrizationToPuiseux(D[1]);
Time: 0.060
> A<a> := Parent(P[1]);
> N<n> := CoefficientRing(A);
> P;
[
    a^(3/2) + 1/2*a^(9/4) + O(a^(5/2)),
    a^(3/2) - 1/2*a^(9/4) + O(a^(5/2)),
    -a^(3/2) + 1/2*n*a^(9/4) + O(a^(5/2)),
    -a^(3/2) - 1/2*n*a^(9/4) + O(a^(5/2))
]
> time c := PuiseuxExpansion(f, 0);
Time: 0.050
```

Here it can be seen that the newton polygon part of the algorithm is substantially faster using Duval's method, though the converting of the parametrization to a series is not as fast. Asking for more terms shows this more substantially.

```
> time D := DuvalPuiseuxExpansion(f, 10);
Time: 0.020
> D;
[
    <16*x^4, 64*x^6 + 256*x^9 - 512*x^15 + 2048*x^18 - 4608*x^21 + 56320*x^27 -
        294912*x^30 + 792064*x^33 - 12082176*x^39 + 68157440*x^42 + O(x^43)>
]
> time P := ParametrizationToPuiseux(D[1]);
Time: 0.129
> time c := PuiseuxExpansion(f, 10);
Time: 0.100
> A<a> := Parent(c[1]);
> N<n> := CoefficientRing(A);
> c;
[
    a^(3/2) + 1/2*a^(9/4) - 1/64*a^(15/4) + O(a^4),
    a^(3/2) - 1/2*a^(9/4) + 1/64*a^(15/4) + O(a^4),
    -a^(3/2) + 1/2*n*a^(9/4) + 1/64*n*a^(15/4) + O(a^4),
    -a^(3/2) - 1/2*n*a^(9/4) - 1/64*n*a^(15/4) + O(a^4)
]
> A<a> := Parent(P[1]);
> N<n> := CoefficientRing(A);
> P;
[
    a^(3/2) + 1/2*a^(9/4) - 1/64*a^(15/4) + 1/128*a^(9/2) - 9/4096*a^(21/4) +
        55/131072*a^(27/4) - 9/32768*a^(15/2) + 1547/16777216*a^(33/4) -
        11799/536870912*a^(39/4) + 65/4194304*a^(21/2) + O(a^(43/4)),
    a^(3/2) - 1/2*a^(9/4) + 1/64*a^(15/4) + 1/128*a^(9/2) + 9/4096*a^(21/4) -
        55/131072*a^(27/4) - 9/32768*a^(15/2) - 1547/16777216*a^(33/4) +
        11799/536870912*a^(39/4) + 65/4194304*a^(21/2) + O(a^(43/4)),
    -a^(3/2) + 1/2*n*a^(9/4) + 1/64*n*a^(15/4) - 1/128*a^(9/2) -
        9/4096*n*a^(21/4) - 55/131072*n*a^(27/4) + 9/32768*a^(15/2) +
        1547/16777216*n*a^(33/4) + 11799/536870912*n*a^(39/4) -
        65/4194304*a^(21/2) + O(a^(43/4)),
    -a^(3/2) - 1/2*n*a^(9/4) - 1/64*n*a^(15/4) - 1/128*a^(9/2) +
        9/4096*n*a^(21/4) + 55/131072*n*a^(27/4) + 9/32768*a^(15/2) -
        1547/16777216*n*a^(33/4) - 11799/536870912*n*a^(39/4) -
        65/4194304*a^(21/2) + O(a^(43/4))
]
```

It can be seen that the computation of the information is a lot faster using Duval's method. It is only the cosmetic of converting this information into series that could make this algorithm seem slow. But also note that there is much greater information given by Duval's algorithm. The equivalent information is given below.

```
> time D := DuvalPuiseuxExpansion(f, 3);
Time: 0.009
> time P := ParametrizationToPuiseux(D[1]);
Time: 0.049
> A<a> := Parent(P[1]);
> N<n> := CoefficientRing(A);
> P;
[
    a^(3/2) + 1/2*a^(9/4) - 1/64*a^(15/4) + O(a^4),
    a^(3/2) - 1/2*a^(9/4) + 1/64*a^(15/4) + O(a^4),
    -a^(3/2) + 1/2*n*a^(9/4) + 1/64*n*a^(15/4) + O(a^4),
    -a^(3/2) - 1/2*n*a^(9/4) - 1/64*n*a^(15/4) + O(a^4)
]
```

One thing that may be taken for granted from `PuiseuxExpansion` is that all the expansions lie in the same puiseux series ring. However, for `DuvalPuiseuxExpansion` this may not be the case. It will always be true that each of the parametrizations will lie in the same puiseux series ring but series resulting from different parametrizations may not. This occurs since some extensions are left to the stage of calculating the series from the parametrization to be made and for different parametrizations these extensions may be different.

```
> f := (-x^3 + x^4) - 2*x^2*y - x*y^2 + 2*x*y^4 + y^5;
> time D := DuvalPuiseuxExpansion(f, 0);
Time: 0.010
```

```
> D;
[
    <x^2, -x^2 - x^3 + O(x^4)>,
    <x^3, x + O(x^2)>
]
> time P := ParametrizationToPuiseux(D[1]);
Time: 0.000
> P;
[
    -x - x^(3/2) + O(x^2),
    -x + x^(3/2) + O(x^2)
]
> Parent(P[1]);
Puiseux series field in x over Rational Field
> time P := ParametrizationToPuiseux(D[2]);
Time: 0.030
> A<a> := Parent(P[1]);
> N<n> := CoefficientRing(A);
> P;
[
    a^(1/3) + O(a^(2/3)),
    n*a^(1/3) + O(a^(2/3)),
    (-n - 1)*a^(1/3) + O(a^(2/3))
]
> Parent(P[1]);
Puiseux series field in a over N
> N;
Number Field with defining polynomial $.1^2 + $.1 + 1 over the Rational Field
```

`DuvalPuiseuxExpansion` reacts differently to `PuiseuxExpansion` when given input which has finite expansions either due to finite precision or exact roots. These differences are shown below and are due to the fact that `DuvalPuiseuxExpansion` always looks for the next non zero term in an expansion whereas `PuiseuxExpansion` will calculate zero terms.

```
> f := y - x^3 - x^7 - x^76 + O(x^200);
> D := DuvalPuiseuxExpansion(f, 0);
> D;
[
    <x, x^3 + O(x^4)>
]
> D := DuvalPuiseuxExpansion(f, 3);
> D;
[
    <x, x^3 + x^7 + x^76 + O(x^77)>
]
>  D := DuvalPuiseuxExpansion(f, 4);
>> D := DuvalPuiseuxExpansion(f, 4);
                             ^

Runtime error in 'DuvalPuiseuxExpansion': Insufficient precision to calculate to
```

```
requested precision
> c := PuiseuxExpansion(f, 197);
> c;
[
    x^3 + x^7 + x^76 + O(x^200)
]
>  c := PuiseuxExpansion(f, 200);
>> c := PuiseuxExpansion(f, 200);
                        ^
Runtime error in 'PuiseuxExpansion': Insufficient precision to calculate to
requested precision
> f := y - x^3 - x^7 - x^76;
> D := DuvalPuiseuxExpansion(f, 0);
> D;
[
    <x, x^3 + O(x^4)>
]
> D := DuvalPuiseuxExpansion(f, 3);
> D;
[
    <x, x^3 + x^7 + x^76 + O(x^77)>
]
> D := DuvalPuiseuxExpansion(f, 4);
> D;
[
    <x, x^3 + x^7 + x^76>
]
> c := PuiseuxExpansion(f, 10);
> c;
[
    x^3 + x^7 + O(x^13)
]
> c := PuiseuxExpansion(f, 100);
> c;
[
    x^3 + x^7 + x^76 + O(x^103)
]
> c := PuiseuxExpansion(f, 200);
> c;
[
    x^3 + x^7 + x^76 + O(x^203)
]
> c := PuiseuxExpansion(f, 200 : PreciseRoot := true);
> c;
[
    x^3 + x^7 + x^76
]
```

The two methods can be combined. Given a series there is no way that Duval's REGULAR algorithm can be used to further the precision of an expansion. But `ExpandToPrecision` can be used to gain the extra precision. Using the REGULAR algorithm would be preferable since it is faster but this is not possible for the type of input that is available. This is explored in the case of our first example.

```
> f := (y^2 - x^3)^2 - y*x^6;
> time D := DuvalPuiseuxExpansion(f, 0);
Time: 0.009
> time P := ParametrizationToPuiseux(D[1]);
Time: 0.050
> A<a> := Parent(P[1]);
> N<n> := CoefficientRing(A);
> P;
[
    a^(3/2) + 1/2*a^(9/4) + O(a^(5/2)),
    a^(3/2) - 1/2*a^(9/4) + O(a^(5/2)),
    -a^(3/2) + 1/2*n*a^(9/4) + O(a^(5/2)),
    -a^(3/2) - 1/2*n*a^(9/4) + O(a^(5/2))
]
> time ExpandToPrecision(f, P[1], 20);
a^(3/2) + 1/2*a^(9/4) - 1/64*a^(15/4) + 1/128*a^(9/2) - 9/4096*a^(21/4) +
    O(a^(13/2))
Time: 0.070
> time D := DuvalPuiseuxExpansion(f, 5);
Time: 0.010
> time P := ParametrizationToPuiseux(D[1]);
Time: 0.059
```

It can be seen that using `ExpandToPrecision` is slower even than rerunning Duval's algorithm from the beginning. Even more so when it is remembered that running Duval's algorithm with the extra precision will give parametrizations of all the expansions of the roots of f and not just one expansion. This seems to still be the case when larger examples are considered so that if more terms of an expansion are required it is probably best to start from the beginning asking for these extra terms.

```
> time p1 := ExpandToPrecision(f, P[1], 50);
> A<a> := Parent(p1);
> N<n> := CoefficientRing(A);
> p1;
a^(3/2) + 1/2*a^(9/4) - 1/64*a^(15/4) + 1/128*a^(9/2) - 9/4096*a^(21/4) +
    55/131072*a^(27/4) - 9/32768*a^(15/2) + 1547/16777216*a^(33/4) -
    11799/536870912*a^(39/4) + 65/4194304*a^(21/2) - 189805/34359738368*a^(45/4)
    + 1584999/1099511627776*a^(51/4) - 2261/2147483648*a^(27/2) + O(a^14)
Time: 0.439
> time D := DuvalPuiseuxExpansion(f, 13);
Time: 0.009
> time P := ParametrizationToPuiseux(D[1]);
```

```
Time: 0.200
```

54.5.3 Roots of Polynomials

This section describes two similar functions that can be used for finding roots of polynomials over series rings in a similar way to finding roots of polynomials over any other ring for which roots can be computed in Magma.

`Roots(f)`

`Roots(f, n)`

Verbose `Newton` *Maximum :* 2

Find the roots of the polynomial f which lie in the coefficient ring of f. The first form of this function can be used on any polynomial over any ring for which Magma can compute roots. Since precision is an issue in series rings and some roots may have infinite expansions, the second version of this function which is specific to series rings allows a lower bound for the precision to which these roots will be known to be specified. The first computes the roots to at least the default precision of the ring if that ring has infinite precision otherwise it computes them to at least the precision of the ring. The first version will be enough when all the coefficients of the polynomial have infinite precision. The second version may be required when a precision other than that assumed by the first is sought which may be due to the impossibility of computing the roots to such a high precision as the default. The precision is relative to the least common multiple of the exponent denominators of the coefficients of f and the exponent denominator of the root. Roots which are known to be different but are identical to the precision specified will be returned as two distinct roots.

Duval's algorithm as implemented in `DuvalPuiseuxExpansion` will usually be used. Walker's algorithm as implemented in `PuiseuxExpansion` will be used if the polynomial has coefficients involving fractional powers or the characteristic of the coefficient ring of the series ring is less than the degree of a squarefree factor.

If Walker's algorithm is used and the characteristic of the field is less than the degree of the polynomial then the computation may not finish (see remarks under `PuiseuxExpansion`) and control will return to the user when interrupted.

If verbose printing of partial output that doesn't have enough precision is required the functions `PuiseuxExpansion` and `DuvalPuiseuxExpansion` should be used with the appropriate precision. `Roots` also requires that there is enough precision in the roots so that the multiplicities can be calculated correctly and the parts of the roots that are returned are distinct. Therefore an error will be given if there is not enough precision to calculate the part of the root that results from the newton polygon part of the algorithm and the root is not known to be a single root since the multiplicity may not be able to be calculated correctly. Information at such a low precision can be gained correctly by using the `PuiseuxExpansion` functions and determining the multiplicities manually.

`HasRoot(f)`

Return `true` if the polynomial f has a root in its coefficient ring and that root can be found to the fixed or default precision of the ring as applicable. A root is also returned in this case. If f is irreducible over its coefficient ring then return `false`.

Example H54E10

Below are some examples of the use of the `Roots` function.

```
> SetVerbose("Newton", 1);
> P<x> := PuiseuxSeriesRing(Rationals());
> R<y> := PolynomialRing(P);
> f := y^3 + 2*x^-1*y^2 + 1*x^-2*y + 2*x;
> Roots(f);
DUVAL :
[
    <-2*x^3 - 8*x^7 - 56*x^11 - 480*x^15 - 4576*x^19 + O(x^23), 1>
]
> f := f^2;
> Roots(f);
DUVAL :
[
   <-2*x^3 - 8*x^7 - 56*x^11 - 480*x^15 - 4576*x^19 + O(x^23), 2>
]
> f := y^3 + 2*x^-1*y^2 + 1*x^-2*y + 2*x;
> f +:= O(x^20)*(y^3 + y^2 + y + 1);
> f;
(1 + O(x^20))*y^3 + (2*x^-1 + O(x^20))*y^2 + (x^-2 + O(x^20))*y + 2*x + O(x^20)
>  Roots(f);
DUVAL :
>> Roots(f);
        ^

Runtime error in 'Roots': Roots not calculable to default precision
> Roots(f, 10);
DUVAL :
[
   <-2*x^3 - 8*x^7 - 56*x^11 + O(x^13), 1>
]
> f := f^2;
> f;
(1 + O(x^20))*y^6 + (4*x^-1 + O(x^19))*y^5 + (6*x^-2 + O(x^18))*y^4 + (4*x^-3 +
   4*x + O(x^18))*y^3 + (x^-4 + 8 + O(x^18))*y^2 + (4*x^-1 + O(x^18))*y + 4*x^2
   + O(x^21)
> Roots(f, 10);
DUVAL :
[
   <-2*x^3 - 8*x^7 - 56*x^11 + O(x^13), 2>
]
```

```
> f := (y - x^(1/4))*(y - x^(1/3));
> Roots(f);
WALKER :
[
    <x^(1/3) + O(x^2), 1>,
    <x^(1/4) + O(x^(23/12)), 1>
]
```

54.6 Bibliography

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55 SERIES RINGS OVER p-ADIC RINGS

Chapter 55

SERIES RINGS OVER p-ADIC RINGS

55.1 Introduction

Thanks to Xavier Caruso and David Lubicz, Magma now contains an implementation of linear algebra over $\mathbf{Z}_p[[u]]$ and related rings [CL12]. This is a first step in various calculations with p-adic Hodge theory, to be able to compute with modules over suitable approximations in these rings.

The package deliberately builds a wrapper level through user-defined types in Magma. While one could alternatively just do everything in terms of power (and Laurent) series rings over the p-adics directly, this extra layer is hoped to make it easier to use the functionality in a coherent way. Some of the common functions applicable to power series, matrices, and modules have been copied over, but not all of them. The underlying attributes can be accessed directly in the cases where there is no equivalent on the types described here. Most notably, vectors and matrices that are user-defined types do not seem yet to be accesible via $v[i]$ or $M[i, j]$.

The package has seen limited testing, with not every possibility of precision bounds (in both the p-adic and power series aspects) being considered in all cases, nor has much work been done for base rings that are extensions of $\mathbf{Z}_p$.

55.1.1 Background

Let R be a discrete valuation ring with residue field of characteristic p, and let ν be real. Recall that $f(u) = \sum_i a_i u^i$ where $a_i \in \mathrm{Frac}(R)$ converges for $|u| < 1/p^\nu$ if and only if $\inf_i v_R(i) + i\nu$ is bounded.

When ν is rational, this infinimum is attained. It is called the Gauss valuation, denoted by $v_\nu(f)$, and the Weierstrass degree $d_\nu(f)$ is the minimal i that realises this (by convention, the Weierstrass degree of the zero element is $-\infty$). An element is *distinguished* if its Gauss valuation is 0. We have that $v_\nu(fg) = v_\nu(f) + v_\nu(g)$ and $v_\nu(f + g) \geq \min\bigl(v_\nu(f), v_\nu(g)\bigr)$, while $d_\nu(fg) = d_\nu(f)d_\nu(g)$.

As it is not particularly clear how to represent elements of a convergent series ring in finite form, so we instead work with the "slope" ring of elements of nonnegative Gauss valuation for which $v_\nu(f) \geq 0$, namely

$$S^\nu(R) = \left\{\sum_{i=0}^{\infty} a_i u^i | v_R(a_i) + i \cdot \nu \geq 0, a_i \in \mathrm{Frac}(R)\right\}.$$

For instance, when $\nu = 0$, this ring is just the power series ring over R.

There are two related rings of interest. The first extends the coefficients to allow them to have negative valuation in $\mathrm{Frac}(R)$, namely $S_p^\nu(R) = S^\nu[1/\pi]$ where π uniformises R.

The second is a type of Laurent extension, namely writing $\nu = a/b$ in lowest terms, we have $S_u^\nu(R) = S^\nu[\pi^a/u^b]$.

For instance, the polynomial u^2/p is in $S^{1/2}(\mathbf{Q}_p)$ and $S^\nu(\mathbf{Q}_p)$ for all ν, and inverting it gives us $p/u^2 \in S_u^{1/2}(\mathbf{Q}_p)$. Note that as a power series u^2/p clearly has infinite radius of convergence, even though it is in not in the slope ring $S^0(\mathbf{Q}_p)$.

Another example, $p+u$ has Gauss valuation $\min(1,\nu)$, and thus the Weierstrass degree is 0 for $\nu \geq 1$ but degree 1 for $\nu < 1$.

Both of these "completion" rings are Euclidean, which allows us to construct a module theory over them. While internally MAGMA has generic code for handling modules over Euclidean rings, with these user-defined types the component parts are all taken from boilerplate package-level implementations.

55.1.2 Basic Operations

Here are the basic creations and operations for S^ν rings and elements. All of these have a direct analogue for S_p^ν and S_u^ν rings.

55.1.2.1 Creation Intrinsics

```
SnuRing (F, nu)
SpRing (F, nu)
SuRing (F, nu)
```

    Precision   RngIntElt   Default : ∞

Given a p-adic field (or ring) F and a slope ν (which can be rational or integral), create the slope ring (or the S_p^ν or S_u^ν ring). The precision refers to that of the power series ring.

```
SnuRing (F)
SpRing (F)
SuRing (F)
SnuRing (p, e)
SnuRing (p)
SpRing (p, e)
SpRing (p)
SuRing (p, e)
SuRing (p)
```

    nu   FldRatElt   Default : 0
    Precision   RngIntElt   Default : ∞

The first three are the same as above, but with ν as a parameter. The others create the slope ring over $\mathbf{Q}_p$, with e being the precision of the p-adic ring in this case.

```
SnuRing (S, nu)
```
```
SpRing (S, nu)
```
```
SuRing (S, nu)
```

Given a power-series ring over a p-adic ring or field and a rational number (or integer), create the slope ring. For the S^ν_u ring, one gives a Laurent series ring.

```
SnuRing (S)
```
```
SpRing (S)
```
```
SnuRing (S)
```
```
SuRing (S)
```

Given an S^ν_p-ring, create the associated S^ν-ring, and vice-versa. Similarly with an S^ν_u-ring.

55.1.2.2 Access Functions

```
R eq S
```
```
R ne S
```

Two slope rings are equal if their underlying p-adic power series rings are the same (including precisions), and the slopes ν are the same.

```
Slope (S)
```

The slope of the S^ν ring.

```
Precision (S)
```

The precision of the underlying power-series ring.

```
CoefficientRing (S)
```

The underlying p-adic ring of the S^ν ring.

55.1.3 Element Operations

The full range of addition, subtraction, multiplication, division (where applicable) and powering are available, similarly with equality.

```
Parent (f)
```

Return the S^ν ring to which the element belongs.

```
IsWeaklyZero (f)
```

This is the proper way to test equality with elements of S^ν. It checks whether every (known) coefficient is zero to within the p-adic precision.

55.1.3.1 Valuation and Degree

`GaussValuation (f)`

`WeierstrassDegree (f)`

The Gauss valuation and Weierstrass degree respectively return these values.

`IsDistinguished (f)`

The `IsDistinguished` intrinsic returns whether the Gauss valuation is zero (this intrinsic is only for S^ν elements).

`LeadingTerm (f)`

Gives the first non(weakly)zero term in the S_p^ν-element, as a power series (or Laurent series). Useful in particular when examining vectors or matrices of such.

`WeierstrassTerm (f)`

Gives the term corresponding to the Weierstrass degree of the S_p^ν-element, as a power series (or Laurent series). Useful in particular when examining vectors or matrices of such.

`O (x)`

Gives the big-Oh value of the argument.

Example H55E1

Here are some basic operations with a ring of slope 1.

```
> S<u> := SnuRing (pAdicField (5, 15), 1); // slope 1 over 5-adics
> assert Slope(S) eq 1;
> f := 5*u + u^2 + 2*u^3 + u^5 + O(u^10);
> g := u^4/25 + u^6 + u^8 + O(u^10);
> GaussValuation (f); // both have Gauss valuation 2
2
> GaussValuation (g); // both have Gauss valuation 2
2
> q := g/f; q; // g has degree 4, so this division works
(5^-3 + O(5^12))*u^3 + (-5^-4 + O(5^11))*u^4 +
  (616*5^-5 + O(5^10))*u^5 + (-606*5^-6 + O(5^9))*u^6 +
  (9946*5^-7 + O(5^8))*u^7 + (-3761*5^-8 + O(5^7))*u^8 + O(u^9)
> LeadingTerm (q); // returned as a power series ring element
(5^-3 + O(5^12))*$.1^3
```

All the same functionality copies over to the S_p^ν rings, the main difference is that one can invert p directly there. There is automatic coercion from S^ν to S_p^ν rings in some cases, even when the slopes are not equal. The criterion is whether the result in the S_p^ν ring is valid.

```
> Sp<uu> := SpRing (pAdicField (5, 15), 1/2); // slope 1/2 over 5-adics
> assert Slope(Sp) eq 1/2;
> ff := 1/5 + uu + 5*uu^2 + O(uu^10);
> GaussValuation (ff);
-1
> gg := uu + 5^2*uu^2 + uu^3 - uu^6 + uu^9 + O(uu^10);
> h := S ! (ff * S!gg); // coerced back to the Snu-ring
> assert IsDistinguished (h);
> LeadingTerm (ff) * LeadingTerm (gg) eq LeadingTerm (h);
true
```

Similarly with S_u^ν rings, where Laurent series are now also allowed.

```
> Su<z> := SuRing (pAdicField (7, 25), 2/3); // slope 2/3 over 7-adics
> F := 7/z + 3*7 + z^2/7 + 7^2*z^3 + O(z^10);
> GaussValuation (F);
1/3
> WeierstrassTerm (F);
(7 + O(7^26))*$.1^-1
> SR<t> := SnuRing (Su);
> t*z/7; // automatic coercion into Su
(7^-1 + O(7^24))*z^2 + O(z^20)
```

55.1.4 Euclidean Algorithm

Given $A, B \in S^\nu$ with $v_\nu(A) \geq v_\nu(B)$ (so in particular B is nonzero) the quotient-remainder algorithm returns Q, R such that $A = BQ + R$ with R a polynomial of degree less that $d_\nu(B)$. In general the precision loss in this operation becomes more acute with larger ν. This precision loss percolates throughout all computations. Note that the result in general depends on ν.

The Weierstrass preparation theorem takes a distinguished element $f \in S^\nu$ and writes $f = UP$ where U is invertible in $S^\nu(R)$ and P is a polynomial of degree $d_\nu(f)$. The units of S^ν are precisely those elements with Gauss valuation and Weierstrass degree both 0.

The ring S^ν is not itself Euclidean, as we can only properly quotient elements as A/B when $v_\nu(B) \geq v_\nu(A)$, and this property need not be preserved in the Euclidean steps. For instance with $\nu = 1$, starting with $A = pu^2 + p^4$ and $B = p^2u$ we get $Q = u/p$ and $R = p^4$, but then $v_\nu(R) = 4 > 3 = v_\nu(B)$, so that we cannot divide B by R.

However, the ability to scale by arbitrary powers of the uniformiser implies that the ring S_p^ν is Euclidean with respect to the Weierstrass degree, and similarly the ring S_u^ν is a discrete valuation ring (hence Euclidean) for the valuation v_ν.

The extended gcd in S^ν or S_p^ν takes A, B with $v(A) \geq v(B)$ and returns (G, H, w, x, y, z) has $Aw+Bx = G$ where $v(B) = v(G)$ and $Ay+Bz = H$ with $v(H) > v(A)$ with $wz-xy = 1$ (consecutive Euclidean steps). The gcd is G in the case where H is the zero polynomial (this will always be true over S_p^ν).

Since S_u^ν is a discrete valuation ring, the quotient-remainder $A = qB + r$ always has $r = 0$ when $v(B) < v(A)$, and else $q = 0$. The *canonical* element of valuation v in S_u^ν is $u^y p^z$ where $0 \leq y < \text{denom}(\nu)$ is as small as possible and $y\nu + z = v$. The gcd of two elements A and B can always be taken to be a canonical element of valuation equal to the smaller of the valuations of A and B.

55.1.4.1 Intrinsics

```
WeierstrassPreparation (f)
```

Given a distinguished element f of S^ν (one of Gauss valuation 0), write $f = UP$ where U is invertible in S^ν and P is a polynomial (whose degree equals the Weierstrass degree of f).

```
Quotrem (A, B)
```

Given S^ν elements A and B with $v(A) \geq v(B)$, determine Q, R with $A = BQ + R$ where B is a polynomial whose degree is less than the Weierstrass degree of B.

```
Quotrem (A, B)
```

Given A and (nonzero) B which are S^ν_p elements, determine Q, R with $A = BQ + R$ where B is a polynomial whose degree is less than the Weierstrass degree of B.

```
Quotrem (A, B)
```

Given A and (nonzero) B which are S^ν_u elements, determine Q, R with $A = BQ + R$. Here when $v(A) \geq v(B)$ we have $Q = A/B$ and $R = 0$, and when $v(A) < v(B)$ we have $Q = 0$ and $R = A$.

```
ExtendedGcd (A, B)
```

Given S^ν elements A and B with $v(A) \geq v(B)$, try to compute the gcd. This returns a 6-tuple (G, H, w, x, y, z) with $Aw + Bx = G$, $Cw + Dy = H$, and $wz - xy = 1$. When $H = 0$ then G is the gcd. Alternatively, $v(H)$ will be some element with $v(H) > v(A)$, which precludes the Euclidean algorithm from progressing.

```
ExtendedGcd (A, B)
```

Given S^ν_p elements A and B compute the extended gcd. This returns a 6-tuple (G, H, w, x, y, z) with $Aw + Bx = G$, $Cw + Dy = H = 0$, and $wz - xy = 1$, where G is the gcd.

```
ExtendedGcd (A, B)
```

Given S^ν_u elements A and B compute the extended gcd. Since S^ν_u is a DVR, the gcd can be taken to be the `CanonicalElement` of the smaller valuation of A and B.

```
CanonicalElement (S, v)
```

The *canonical* element of valuation v in S^ν_u is $u^y p^z$ where $0 \leq y < \mathrm{denom}(\nu)$ is as small as possible and $y\nu + z = v$.

Example H55E2

Here are some basic examples of Euclidean operations.

```
> S<u> := SnuRing (pAdicField (5, 15), 1); // slope 1 over 5-adics
> f := 5*u + u^2 + 2*u^3 + u^5 + O(u^10);
> g := u^4/25 + u^6 + u^8 + O(u^10);
> q := g/f;
> assert IsDistinguished (q);
> U, P := WeierstrassPreparation (q); // loses a lot of precision
> U;
1 + O(5^15) + (-5^-1 + O(5^14))*u + (616*5^-2 + O(5^13))*u^2 +
 (-606*5^-3 + O(5^12))*u^3 + (9946*5^-4 + O(5^11))*u^4 +
 (-3761*5^-5 + O(5^10))*u^5 + O(u^6)
> P;
(5^-3 + O(5^12))*u^3 + O(u^6)
> q, r := Quotrem (g, f); assert r eq 0; assert q eq g/f;
> q; // same as above, though is O(u^10) not O(u^9)
(5^-3 + O(5^12))*u^3 + (-5^-4 + O(5^11))*u^4 +
  (616*5^-5 + O(5^10))*u^5 + (-606*5^-6 + O(5^9))*u^6 +
  (9946*5^-7 + O(5^8))*u^7 + (-3761*5^-8 + O(5^7))*u^8 +
  (-172699*5^-9 + O(5^6))*u^9 + O(u^10)
> q, r := Quotrem (f, g); // this is valid, as the valuations are equal
> q, r; assert f eq g*q + r;
(5^2 + O(5^17))*u + (-5^4 + O(5^17))*u^3 + (24*5^4 + O(5^17))*u^5 +
  (-23*5^6 + O(5^17))*u^7 + (551*5^6 + O(5^17))*u^9 + O(u^10)
(5 + O(5^16))*u + (1 + O(5^15))*u^2 + (2 + O(5^15))*u^3 + O(u^10)
> G, H, A, B := ExtendedGcd (f, g); assert H eq 0;
> G;
(-5 + O(5^16))*u + (-1 + O(5^15))*u^2 + (-2 + O(5^15))*u^3 + O(u^10)
> assert A*f + B*g eq G;

> f1 := 5*u^2 + 5^4;      // example from the text
> g1 := 5^2*u;            // where ExtendedGcd fails
> G, H := ExtendedGcd (f1, g1); assert H ne 0;
> H;
-5^4 + O(5^24) + O(u^20)
> GaussValuation (H); GaussValuation (g1);
4 3
```

The above will also work in S_p^ν. As noted above, the corresponding commands for S_u^ν rings are particularly simplified.

```
> Su<z> := SuRing (S);
> F := Su!f; assert GaussValuation(F) eq 2;
> G := Su!g; assert GaussValuation(G) eq 2;
> Q, R := Quotrem (F, G); assert R eq 0;
> g, _, A, B, C, D := ExtendedGcd (F, G);
> g; assert g eq CanonicalElement (Su, 2);
5^2 + O(5^17) + O(z^20)
```

```
> assert A*F + B*G eq g;
> assert C*F + D*G eq 0;
> assert A*D - B*C eq 1;
```

55.2 Matrices and Modules

Since S_p^ν and S_u^ν are Euclidean, we can work with canonical forms and modules over these rings. We consider S_p^ν first.

55.2.1 Matrices

The row-echelon form of a matrix over S_p^ν has pivot entries which are polynomials of the form

$$u^d + \sum_{j=0}^{d-1} b_j u^j \qquad \text{where} \qquad v(b_j) + \nu j > \nu d$$

for j in the sum, so that the Weierstrass degree is d and the Gauss valuation is $d\nu$.

The Hermite form additionally assures that the other entries in the columns with pivots are polynomials of smaller degree, via taking Quotrem. This can be made unique via suitable choices, and so provides a way of testing equality of modules.

The Smith form is computed by creating a matrix of pivots corresponding to a Hermite on the matrix and then a Hermite on the transpose. One also demands recursive divisibility of the resulting diagonal terms.

Over S_u^ν the picture is the same, except that pivot entries of the echelon form are all `CanonicalElements`, with similar changes thereon. E.g., the Hermite form will have elements of smaller valuation above the pivots.

```
SpMatrixSpace (S, r, c)
```

```
SuMatrixSpace (S, r, c)
```

Given an S_p^ν of S_u^ν ring and a (nonnegative) number of rows and columns, construct the matrix space.

```
IdentityMatrix (S, n)
```

```
IdentityMatrix (S)
```

```
ZeroMatrix (S)
```

```
ZeroMatrix (S, n)
```

```
ZeroMatrix (S, r, c)
```

```
IdentityMatrix (S, n)
```

```
IdentityMatrix (S)
```

```
ZeroMatrix (S)
```

```
ZeroMatrix (S, n)
ZeroMatrix (S, r, c)
```

Different ways of obtaining the zero matrix or identity matrix (if applicable) of a matrix ring.

```
SpMatrix (A)
SpMatrix (r, c, A)
SpMatrix (v)
SpMatrix (A)
SuMatrix (r, c, A)
SuMatrix (v)
```

Various ways of obtaining an `SpMat` or `SuMat`. The first takes a sequence of sequences of `SpElements` or `SuElements`, the second takes a row and column count and a sequence of `SpElements` or `SuElements`, and the third takes a sequence of `SpVecs` or `SuVecs`.

```
GaussValuations (M)
```

Returns a sequence of sequences of the Gauss valuations of the elements of a matrix. Since these can be `ExtendedReals`, the valuations themselves are put in an array rather than a matrix.

```
WeierstrassDegrees (M)
```

Returns a sequence of sequences of the Weierstrass degrees of the elements of a matrix. Since these can be `ExtendedReals`, the valuations themselves are put in an array rather than a matrix.

```
LeadingTerms (M)
```

Returns a matrix (defined over the underlying power or Laurent series ring) which has the leading term of each element.

```
WeierstrassTerms (M)
```

Returns a matrix (defined over the underlying power or Laurent series ring) which has the Weierstrass term of each element.

```
IsWeaklyZero (M)
```

Returns whether every element in the matrix is nearly zero.

```
EchelonForm (M)
```

`Transform` BoolElt *Default :* `false`

Returns the (row) echelon form E, and (if desired) the transform T such that $E = TM$.

`HermiteForm (M)`

`Transform` BoolElt *Default :* `false`

Returns the Hermite form H, and (if desired) the transform T such that $H = TM$.

`SmithForm (M)`

Returns the Smith form S, and (if desired) the transforms P, Q such that $S = PMQ$.

`Kernel (M)`

`Image (M)`

Return the kernel or image of an S_p^ν or S_u^ν matrix as a module space. The `Image` also has a `Transform` parameter that gives the inclusion.

Example H55E3

Here are some basic examples of operations with matrices over S_p^ν.

```
> S<u> := SpRing (pAdicField (3, 20), 1/2); // slope 1/2
> M := SpMatrix (2, 3, [ S | u^2/3,u/3^2,0, u^3,3^2*u,3*u ]);
> E, T := EchelonForm (M : Transform); assert E eq T*M;
> LeadingTerms (E);
[(3^-1 + O(3^19))*$.1^2  (3^-2 + O(3^18))*$.1  O($.1^20)]
[O($.1^20)               (3^2 + O(3^18))*$.1   (3 + O(3^21))*$.1]
> WeierstrassTerms (E); // not the same
[(3^-1 + O(3^19))*$.1^2  (3^-2 + O(3^18))*$.1    O($.1^20)]
[O($.1^20)              -(3^-1 + O(3^19))*$.1^2  (3 + O(3^21))*$.1]

> S<u> := SpRing (pAdicField (5, 10), 3/2); // slope 3/2
> M := SpMatrix (3,5,[S|5^3,u,u^2/5,0,0, 0,0,25,u^3/25,3*5, 0,0,0,0,5^2]);
> H, T := HermiteForm (M : Transform); assert H eq T*M;
> WeierstrassTerms(H);
[5^3 + O(5^13)  $.1        O($.1^20)     -(5^-5 + O(5^5))*$.1^5  O($.1^20)]
[O($.1^20)      O($.1^20)  5^2 + O(5^9)  (5^-2 + O(5^8))*$.1^3   O($.1^20)]
[O($.1^20)      O($.1^20)  O($.1^20)     O($.1^20)               5^2 + O(5^12)]
> X, P, Q := SmithForm (M : Transform); assert P*M*Q eq X;
> WeierstrassTerms(X); // note that 5^3 divides 5^2 in this ring!
[5^3 + O(5^13)  O($.1^20)     O($.1^20)      O($.1^20)  O($.1^20)]
[O($.1^20)      5^2 + O(5^9)  O($.1^20)      O($.1^20)  O($.1^20)]
[O($.1^20)      O($.1^20)     5^2 + O(5^12)  O($.1^20)  O($.1^20)]
```

Example H55E4

Here are similar examples over S_u^ν.

```
> S<u> := SuRing (pAdicField (5, 15), 2/3); // slope 2/3
> M := SuMatrix (2, 3, [ S | 5^2/u,0,u^2, u,5*u^2,5^3/u^2 ]);
> E, T := EchelonForm (M : Transform); assert E eq T*M;
> LeadingTerms (E);
```

```
[$.1        (5 + O(5^16))*$.1^2   (5^3 + O(5^18))*$.1^-2]
[O($.1^20)  5^3 + O(5^15)         (5^5 + O(5^20))*$.1^-4]
> WeierstrassTerms (E); // not the same, in lower right
[$.1        (5 + O(5^16))*$.1^2  (5^3 + O(5^18))*$.1^-2]
[O($.1^20)  5^3 + O(5^15)        -$.1^2]

> A := [S | 25/u,u,u^2/5,0,0, 0,0,25,125/u^2,50*u, 0,0,0,0,5*u^2];
> M := SuMatrix (3, 5, A);
> H, T := HermiteForm (M : Transform); assert H eq T*M;
> W := WeierstrassTerms(H); W;
[$.1^2 (5^-2 + O(5^13))*$.1^4 (5^-3 + O(5^12))*$.1^5 O($.1^20) O($.1^20)]
[O($.1^20) O($.1^20) 5^2 + O(5^17) (5^3 + O(5^18))*$.1^-2 O($.1^20)]
[O($.1^20) O($.1^20) O($.1^20) O($.1^20) (5 + O(5^16))*$.1^2]
> assert CanonicalElement (S, 4/3) eq S!W[1,1];
> assert CanonicalElement (S, 2) eq S!W[2,3];
> assert CanonicalElement (S, 7/3) eq S!W[3,5];
> X, P, Q := SmithForm (M : Transform); assert X eq P*M*Q;
> WeierstrassTerms(X);
[(5^-1 + O(5^14))*$.1^2 O($.1^20) O($.1^20) O($.1^20) O($.1^20)]
[O($.1^20) (5 + O(5^16))*$.1 O($.1^20) O($.1^20) O($.1^20)]
[O($.1^20) O($.1^20) (5 + O(5^16))*$.1^2 O($.1^20) O($.1^20)]
```

55.2.2 Modules

These are implemented as “vectors” over S_p^ν or S_u^ν.

`SpSpace (R, n)`

`SuSpace (R, n)`

Given an S_p^ν or S_u^ν ring and a nonnegative integer, construct the module of that degree.

`SpSpace (M)`

`SuSpace (M)`

Given an S_p^ν or S_u^ν matrix, construct the associated module space. This always computes the Hermite form to use as the basis.

`SpSpace (v)`

`SuSpace (v)`

Given a sequence of `SpVecs` or `SuVecs`, construct the module space generated by them. Again this always computes the Hermite form for the basis.

`Ambient (S)`

Given a module space, return the (full) ambient space of dimension equal to the degree.

`Parent (v)`

The Parent of an `SpVec` or `SuVec` is always its `Ambient` space.

`ZeroVector (S)`

Return the zero vector of a module space.

`SpVector (e)`

Given a sequence of S_p^ν or S_u^ν elements, construct the vector of them.

`Basis (S)`

Given a module space, return a basis as a sequence of vectors.

`BasisMatrix (S)`

Given a module space, return a basis as a matrix.

`Dimension (S)`

`Degree (S)`

The dimension and degree of a module space.

`LeadingTerms (v)`

The leading term of each element in an `SpVec` or `SuVec`.

`WeierstrassTerms (v)`

The Weierstrass term of each element in an `SpVec` or `SuVec`.

`GaussValuations (v)`

Returns a sequence of sequences of the Gauss valuations of the elements of a matrix. Since these can be `ExtendedReals`, the valuations themselves are put in an array rather than a vector.

`WeierstrassDegrees (v)`

Returns a sequence of sequences of the Weierstrass degrees of the elements of a vector. Since these can be `ExtendedReals`, the valuations themselves are put in an array rather than a vector.

`IsWeaklyZero (v)`

Return whether an `SpVec` or `SuVec` is weakly zero.

`M1 + M2`

`DirectSum (M1, M2)`

`M1 meet M2`

The sum and intersection of two module spaces. The use of `DirectSum` requires the summands to be disjoint.

`S * M`

The transformation of a module space by a matrix. The basis of the reuslting space will be given in Hermite form.

`IsConsistent (M, v)`

`IsConsistent (M, e)`

`IsConsistent (M, W)`

`v in M`

Determines whether $xM = v$ is solvable, and it so returns such an x. The second version takes an a sequence of vectors, and the third takes a matrix (and also includes the kernel as a third return value).

`IsSubspace (A, B)`

Determines whether A is a subspace of B, and if so gives an inclusion map. Equality can be determined via symmetric inclusion, or alternatively via the Hermite form (which should be unique) of the basis matrix.

Example H55E5

Here are some basic examples of usage for modules.

```
> S<u> := SpRing (pAdicField (5, 25), 1/2); // slope 1/2
> A := SpSpace (S, 3);
> A.1;
[ 1 + O(5^25) + O(u^20), O(u^20), O(u^20) ]
> M := SpMatrix (2, 3, [ S | 5*u,5,u^2, 1/5,u^3,u ]);
> B := SpSpace (M); assert Dimension(B) eq 2;
> LeadingTerms(BasisMatrix(B));
[5^-1 + O(5^24) O($.1^20)    $.1]
[O($.1^20)      -5 + O(5^26) (24 + O(5^25))*$.1^2]
> B eq SpSpace (VerticalJoin (M, M)); // also gives inclusion
true
> A meet B eq B;
true
> Image(M) eq B; // by definition
true
> IsConsistent (M, A.1);
false
> IsConsistent (M, B.1);
```

```
true
> SpSpace([B.1]) + SpSpace([B.2]) eq B;
true
> SpSpace([B.1]) + SpSpace([u * B.2]) eq B; // u does not invert
false
> SpSpace([1/5 * B.1]) + SpSpace([5 * B.2]) eq B;
true
> sub<B|[B.1+B.2]> + sub<B|[B.2-B.1]> eq sub<B|[B.1,B.2]>;
true
```

Example H55E6

Here is an example over S_p^ν with a nontrivial kernel.

```
> S<u> := SpRing (pAdicField (7, 30), 1); // slope 1
> M := SpMatrix(3,5,[S|1,0,u,u^2,u,  u^2/7,0,1,0,0, 0,0,1,0,u^4/7^2]);
> ROWS := Rows(M);
> V := u * ROWS[1] + u^2/7 * ROWS[3]; // create a kernel
> MAT := SpMatrix(ROWS cat [Universe(ROWS)|V]); // 4 x 5
> Dimension (SpSpace (MAT)); // 3
3
> K := Kernel (MAT); K;
SpSpace of degree 4, dimension 1 over Sp-slope ring of slope 1 given
 by Power series ring in $.1 over pAdicField(7, 30)
Basis given by:
[ (-1 + O(7^30))*u + O(u^21), O(u^21),
  (-7^-1 + O(7^29))*u^2 + O(u^21), 1 + O(7^30) + O(u^21) ]
> KK := Kernel(BasisMatrix(K)); KK;
SpSpace of degree 1, dimension 0 over Sp-slope ring of slope 1 given
 by Power series ring in $.1 over pAdicField(7, 30)
> assert ZeroVector(Ambient(KK)) in KK;
```

Example H55E7

Here is an example over S_u^ν.

```
> S<u> := SuRing (pAdicField (7, 20), 5/4); // slope 5/4
> A := SuSpace (S, 3);
> B := SuSpace ([A.1, A.3]); // dimension 2
> SuSpace([B.1]) + SuSpace([7^5/u^4 * B.2]) eq B; // slope 5/4
true
> Basis(sub<A|[A.1+A.3]>)[1] in B;
true
> M := SuMatrix (2, 3, [S | 7^2/u,7,5*7*u, 7,u^3,u^2 ]);
> V := SuSpace(M);
> I := V meet B; assert Dimension(I) eq 1;
> v := Basis(I)[1]; assert v in I;
> WeierstrassTerms(v);
```

```
(7 + O(7^21) O($.1^20) $.1^2)
> WeierstrassDegrees(v);
[ 0, -Infinity, 2 ]
> I + B eq B; assert v in B;
true
> J := VerticalJoin (M, M);
> SetSeed(1); // ensure example works
> JJ := (SuMatrixSpace(S,4,4)!RandomSLnZ(4,8,16)) * J;
> assert Dimension(Kernel(JJ)) eq 2;
> assert Dimension(Image(JJ)) eq 2;
```

Note that the final commands, with the dimension of the kernel/image of the given space, can sometimes fail for various reasons involving precision loss.

55.3 Bibliography

[**CL12**] X. Caurso and D. Lubicz. Linear algebra over $\mathbf{Z}_p[[u]]$ and related rings. 2012.

56 LOCAL GALOIS REPRESENTATIONS

Chapter 56
LOCAL GALOIS REPRESENTATIONS

56.1 Overview

This package provides functionality for working with Galois representations

$$\mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_m(\mathbf{C})$$

over p-adic fields K (type `FldPad`). The representations we consider are precisely those on the 'Galois side' of the local Langlands correspondence, or, technically, the *Frobenius-semisimple Weil-Deligne representations* over K. We refer the reader to Tate's article [Tat79, §4] for their basic properties.

In arithmetic geometry, such representations arise from l-adic étale cohomology of algebraic varieties over K for $l \neq p$. Magma includes Galois representations that come from finite Galois groups of p-adic extensions, local components of Dirichlet characters and Artin representations, local Galois representations coming from a Tate module of an elliptic curve or a modular form, as well as various constructions to produce new representations from existing ones, such as direct sums, tensor products, induction, restriction and semisimplification.

56.1.1 Notation and Printing

Suppose K is a finite extension of $\mathbf{Q}_p$, with ring of integers O, maximal ideal m and residue field $k = O/m \cong \mathbf{F}_q$. The absolute Galois group of K fits into an exact sequence

$$\begin{array}{ccccc} 1 \longrightarrow I_K \longrightarrow & \mathrm{Gal}(\bar{K}/K) & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow 1, \\ & \sigma & \mapsto & \sigma \bmod m & \end{array}$$

where I_K is the *inertia group* of K. The group $\mathrm{Gal}(\bar{k}/k)$ is topologically generated by the automorphism $x \mapsto x^q$, and any of its lifts to $\mathrm{Gal}(\bar{K}/K)$ is called an *arithmetic Frobenius element*, denoted Frob_K.

The *Weil group W_K of K* is a subgroup of $\mathrm{Gal}(\bar{K}/K)$ generated by the inertia group I_K and any Frobenius element. It fits in the same exact sequence as above, except that the profinite group $\mathrm{Gal}(\bar{k}/k) \cong \hat{\mathbf{Z}}$ is replaced by a copy of $\mathbf{Z}$ with discrete topology. A *Weil representation* is a continuous representation

$$\rho : W_K \longrightarrow \mathrm{GL}_m(\mathbf{C}).$$

By continuity, the image of inertia $\rho(I_K)$ is finite, and ρ is said to be *unramified* if it is trivial. We will always assume that ρ is *Frobenius-semisimple*, that is $\rho(\mathrm{Frob}_K)$ is a semisimple endomorphism of $\mathbf{C}^m$. Every such representation ρ is a direct sum of irreducible representations of the form

$$\psi \otimes R,$$

where $\psi : I_K \mapsto 1, \mathrm{Frob}_K \mapsto \alpha \in \mathbf{C}^\times$ is an unramified 1-dimensional character (uniquely determined by $\alpha \in \mathbf{C}^\times$) and R is a representation of $\mathrm{Gal}(F/K)$ for some finite Galois extension F/K.

Weil representations occur naturally as local components of Artin representations, and as representations associated to (H^1 of) elliptic curves and abelian varieties over K with potentially good reduction. (By the Néron-Ogg-Shafarevich criterion, potentially good reduction is *equivalent* to $\rho(I_K)$ being finite.) To deal with arbitrary reduction behaviour one considers, more generally, *Weil-Deligne* representations. These come from Galois representations

$$\rho : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_m(\mathbf{C})$$

that can be described as follows.

As before, ρ is a direct sum of indecomposable representations, and an indecomposable one is now of the form (cf. [Tat79, 4.1.5])

$$\rho = \psi \otimes \mathtt{SP}(n) \otimes R,$$

with ψ and R as above, and $\mathtt{SP}(n)$ is a 'special' representation ([Tat79, 4.1.4]),

$$\mathtt{SP}(n) = \mathbf{C}e_0 + \mathbf{C}e_1 + \ldots + \mathbf{C}e_{n-1}.$$

The e_i are eigenvectors for Frob_K with eigenvalues q^{-i}, and the action of inertia is nilpotent, and represented by a matrix N (see [Tat79, 4.1.2]) that takes $e_i \mapsto e_{i+1}$ and $e_{n-1} \mapsto 0$.

This is precisely how Galois representations are stored in Magma,

$$\rho = \bigoplus_i \psi_i \otimes \mathtt{SP}(n_i) \otimes R_i,$$

except that it is convenient to

(1) assume that all R_i factor through the same Galois group $G = \mathrm{Gal}(F/K)$, and

(2) allow ψ_i to be arbitrary unramified representations, not necessarily 1-dimensional. Such a ψ is a sum of unramified characters $\chi_j : \mathrm{Frob}_K \mapsto \alpha_j$, and ψ is uniquely determined by its *Euler factor*

$$\Psi(T) = \det(1 - \mathrm{Frob}_K^{-1} T|\psi) = \prod_j (1 - \alpha_j^{-1} T) \in \mathbf{C}[T].$$

In other words, every component of ρ can be represented (though not quite uniquely) by a triple $\langle \Psi, n, c \rangle$, where $\Psi \in \mathbf{C}[T]$, $n \in \mathbf{Z}$ and c is a character of a representation R of a finite group $\mathrm{Gal}(F/K)$. Throughout the chapter we will refer to R as the 'finite part' of an indecomposable representation $\psi \otimes \mathtt{SP}(n) \otimes R$, though it depends on the choice of such a presentation.

Galois representations have Magma type `GalRep`. The base field K, field F, the Galois group F/K, and the list of components of ρ can be obtained with `BaseField`, `Field`, `Group` and `Factorization`, respectively.

Example H56E1

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(Rationals());
```

First, construct a typical unramified representation specified by its Euler factor ψ.

```
> psi:=UnramifiedRepresentation(K,1+x^2); psi;
2-dim unramified Galois representation Unr(1+x^2) over Q2[20]
```

Next, construct a typical finite image representation, by picking a character of a finite Galois extension F/K, in this case an S_3-extension.

```
> S<z>:=PolynomialRing(K);
> F:=ext<K|z^3-2>;
> c:=GaloisRepresentations(F,K)[3]; c;
2-dim Galois representation (2,0,-1) with G=S3, I=C3, conductor 2^2 over Q2[20]
```

Here is a general Galois representation, the tensor product of all 3 terms, and another one — direct sum of the same three terms.

```
> psi*SP(K,2)*c;
8-dim Galois representation Unr(1+x^2)*SP(2)*(2,0,-1) with G=S3, I=C3,
   conductor 2^8 over Q2[20]
> psi+SP(K,2)+c;
6-dim Galois representation Unr(1+x^2) + (2,0,-1) + SP(2) with G=S3, I=C3,
   conductor 2^3 over Q2[20]
```

56.1.2 Conventions

There are various choices of signs in local class field theory, for which there appears to be no consensus in the literature. Our conventions follow Tate [Tat79], and are as follows. (We refer the reader to Tate's article [Tat79] for the definitions and properties of Weil-Deligne representations, and [Tat79, Del79] for their ϵ-factors.)

For a p-adic field K with ring of integers O, uniformizer π and residue field $\mathbf{F}_q$,

- A Frobenius element Frob_K is an arithmetic Frobenius, i.e. acts as $x \mapsto x^q$ on the residue field (**not** $x \mapsto x^{1/q}$).
- The local reciprocity map $\theta : K^\times \to \mathrm{Gal}(\bar{K}/K)^{ab}$ takes π to Frob_K^{-1} (**not** Frob_K).
- The local epsilon-factors $\epsilon(\chi) = \epsilon(\chi, \psi, dx)$ rely implicitly on the choice of a measure dx on O and an additive character $\psi : K \to \mathbf{C}$. Our choices are that dx is normalized, $\int_O dx = 1$, and

$$\psi(x) = \exp\bigl(2\pi i \ \mathrm{Tr}_{K/\mathbf{Q}_p}(x)\bigr),$$

viewing $\mathrm{Tr}_{K/\mathbf{Q}_p}(x) \in \mathbf{Q}_p$ as any rational number $a/p^n \in \mathbf{Q}$ in the same class mod $\mathbf{Z}_p$. Finally, for 1-dimensional ramified χ, the formula for $\epsilon(\chi)$ is as in [Tat79, 3.2.6.2],

$$\epsilon(\chi) = \int_{c^{-1}O^\times} \chi^{-1}(\theta(x))\psi(x),$$

with

$$v_K(c) = v_K(\text{Conductor}(\chi)) + v_p(\text{Discriminant}(O, \mathbf{Z}_p)).$$

- Euler factors of global Artin representations A (and of Dirichlet characters) are *the same* (**not** complex conjugate) as the local ones:

  `EulerFactor(A,p) = EulerFactor(GaloisRepresentation(A,p)).`

As the Artin L-function $L(A, s)$ is defined using arithmetic Frobenius and Galois representations using geometric Frobenius, this means that the Galois representation `GaloisRepresentation(A,p)` comes from the Galois action on the *dual* vector space of A.

56.1.3 Implementation Notes

Galois representations attached to Artin representations are computed using the machinery of [DD13]. Galois representations coming from elliptic curves rely partly on the theory of reconstructing representations from their Euler factors [DD15] (see §56.7.2), and Rachel Newton's tame local reciprocity formula [New12].

56.2 Creating Galois Representations

`ZeroRepresentation(K)`

Galois representation 0 over a p-adic field K. It is 0-dimensional, and $0 + A = A$, $0 \otimes A = 0$ for every Galois representation A.

Example H56E2

```
> K:=pAdicField(3,20);
> zero:=ZeroRepresentation(K);
> zero;
Galois representation 0 with G=C1, I=C1 over Q3[20]
> zero + CyclotomicCharacter(K) eq CyclotomicCharacter(K);
true
> zero*CyclotomicCharacter(K) eq zero;
true
```

`PrincipalCharacter(K)`

Principal character 1 of the absolute Galois group of K, as a Galois representation. It is a 1-dimensional unramified representation, same as `UnramifiedCharacter(K,1)`. Thus $1 \otimes A = 1$ for every Galois representation A.

Example H56E3

Take $K = \mathbf{Q}_3$, and F/K the unramified extension of degree 4, so that $G = \mathrm{Gal}(F/K) \cong C_4$. The 4 irreducible representations of G can be viewed as Galois representations, and the first one of these is the principal character (for any group).

```
> K:=pAdicField(3,20);
> one:=PrincipalCharacter(K);
> one;
1-dim trivial Galois representation 1 over Q3[20]
> F:=ext<K|4>;
> A1,A2,A3,A4:=Explode(GaloisRepresentations(F,K));
> A1 eq one;
true
```

```
CyclotomicCharacter(K)
```

Cyclotomic character over K. It is an unramified character (trivial on inertia) and takes the value q, the size of the residue field of K, on any Frobenius element.

Example H56E4

```
> K:=pAdicField(3,20);
> chi:=CyclotomicCharacter(K);
> chi,EulerFactor(chi);
1-dim unramified Galois representation Unr(1/3) over Q3[20]
-1/3*x + 1
> chi^3,EulerFactor(chi^3);
1-dim unramified Galois representation Unr(1/27) over Q3[20]
-1/27*x + 1
```

```
UnramifiedCharacter(K,c)
```

Galois representation over K given by an unramified character that sends the arithmetic Frobenius element $\mathrm{Frob}_K \mapsto c^{-1}$ (and, so, the geometric Frobenius element $\mathrm{Frob}_K^{-1} \mapsto c$.) The parameter c must be a non-zero complex number.

Example H56E5

```
> K:=pAdicField(3,20);
> assert UnramifiedCharacter(K,1) eq PrincipalCharacter(K);
> assert UnramifiedCharacter(K,1/3) eq CyclotomicCharacter(K);
> C<i>:=ComplexField();
> UnramifiedCharacter(K,2+i);
1-dim unramified Galois representation Unr(2+i) over Q3[20]
```

`UnramifiedRepresentation(K,CharPoly)`

Unique unramified Galois representation ρ over K with Euler factor $\det(1 - \mathrm{Frob}_K^{-1} \,|\rho) = \mathrm{CharPoly}$.

Example H56E6

```
> K:=pAdicField(37,20);
> R<x>:=PolynomialRing(Rationals());
> rho:=UnramifiedRepresentation(K,(1-37*x)*(1-3*x));
> rho;
2-dim unramified Galois representation Unr(1-40*x+111*x^2) over Q37[20]
> rho eq CyclotomicCharacter(K)^(-1)+UnramifiedCharacter(K,3);
true
```

`UnramifiedRepresentation(K,dim,dimcomputed,CharPoly)`

Unramified Galois representation over K of dimension *dim*, with Euler factor *CharPoly* computed up to and inclusive degree *dimcomputed*.

Example H56E7

Consider the hyperelliptic curve $C : y^2 = x^5 + x + 1$ over the p-adic field $\mathbf{Q}_{10007}$.

```
> _<x>:=PolynomialRing(Rationals());
> p:=10007;
> K:=pAdicField(p,20);
> _<X>:=PolynomialRing(K);
> C:=HyperellipticCurve(X^5+X+1); C;
Hyperelliptic Curve defined by y^2 = x^5 + O(10007^20)*x^4 + O(10007^20)*x^3 +
   O(10007^20)*x^2 + x + 1 + O(10007^20) over pAdicField(10007, 20)
```

The Galois representation A associated to $H^1(C)$ is unramified, of dimension 4, and could be defined by

```
UnramifiedRepresentation(K,1-ap*x+bp*x^2-p*ap+p^2);
```

if we find a_p and b_p by counting points of C over $\mathbf{F}_p$ and $\mathbf{F}_{p^2}$. The coefficient a_p can be computed very quickly:

```
> k:=ResidueClassField(Integers(K));
> _<X>:=PolynomialRing(k);
> Ck:=HyperellipticCurve(X^5+X+1);
> ap:=p+1-#Ck; ap;
-21
```

However, b_p would take a long time. If we are only interested in working with A up to degree 1 (e.g. to compute L-series of $C/\mathbf{Q}$ with $< 10^8$ terms), there is no reason to compute it. Instead, we can define an unramified Galois representation of degree 4, which is known to be computed only up to degree 1:

```
> A:=UnramifiedRepresentation(K,4,1,1-ap*x);
> A;
4-dim unramified Galois representation Unr(1+21*x+O(x^2)) over Q10007[20]
```

One can still take direct sums, and tensor products of such representations with (possibly ramified) Galois representations, and the Euler factors will still be correct up to degree 1:

```
> A*A;
16-dim unramified Galois representation Unr(1-441*x+O(x^2)) over Q10007[20]
> EulerFactor(A*A+1/CyclotomicCharacter(K));
-10448*x + 1
```

`SP(K,n)`

The n-dimensional indecomposable Galois representation SP(n) over a p-adic field K; see §56.1.1 for its description.

Example H56E8

```
> K:=pAdicField(3,20);
> SP(K,1) eq PrincipalCharacter(K);
true
> rho:=SP(K,2); rho;
2-dim Galois representation SP(2) over Q3[20]
> Degree(rho);
2
> Semisimplification(rho);
2-dim unramified Galois representation Unr(1-4/3*x+1/3*x^2) over Q3[20]
> $1 eq PrincipalCharacter(K)+CyclotomicCharacter(K);
true
> InertiaInvariants(rho);
1-dim unramified Galois representation Unr(1/3) over Q3[20]
> EulerFactor(rho);
-1/3*x + 1
```

`SP(K,f,n)`

Unramified twist $\psi\otimes\mathtt{SP}(n)$ over a p-adic field K, with ψ specified by its Euler factor f.

Example H56E9

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(Rationals());
> SP(K,1-x^2,2);
4-dim Galois representation Unr(1-x^2)*SP(2) over Q2[20]
> $1*$1;     // Tensor product with itself
16-dim Galois representation Unr(1-1/2*x^2+1/16*x^4) + Unr(1-2*x^2+x^4)*SP(3)
   over Q2[20]
```

56.2.1 Representations from Finite Extensions

```
GaloisRepresentations(F,K)
```

For a p-adic extension F/K, compute all irreducible Galois representations that factor through the (Galois closure of) F/K.

Example H56E10

We take F to be a degree 16 dihedral extension of $K = \mathbf{Q}_2$, and compute the irreducible characters of $\mathrm{Gal}(F/K)$, viewed as Galois representations over K.

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(K);
> F:=ext<K|x^8+2>;
> list:=GaloisRepresentations(F,K);
> list;
[
1-dim trivial Galois representation 1 over Q2[20],
1-dim Galois representation (1,1,-1,-1,1,1,1) with G=D8, I=D8, conductor 2^2
   over Q2[20],
1-dim Galois representation (1,1,-1,1,1,-1,-1) with G=D8, I=D8, conductor 2^3
   over Q2[20],
1-dim Galois representation (1,1,1,-1,1,-1,-1) with G=D8, I=D8, conductor 2^3
   over Q2[20],
2-dim Galois representation (2,2,0,0,-2,0,0) with G=D8, I=D8, conductor 2^8
   over Q2[20],
2-dim Galois representation (2,-2,0,0,0,-zeta(8)_8^3+zeta(8)_8,
   zeta(8)_8^3-zeta(8)_8) with G=D8, I=D8, conductor 2^10 over Q2[20],
2-dim Galois representation (2,-2,0,0,0,zeta(8)_8^3-zeta(8)_8,
   -zeta(8)_8^3+zeta(8)_8) with G=D8, I=D8, conductor 2^10 over Q2[20]
]
```

The first 5 characters are not faithful, and we can descend them to smaller quotients of $\mathrm{Gal}(F/K)$.

```
> min:=[Minimize(rho): rho in list | not IsFaithful(Character(rho))];
> min;
[
1-dim trivial Galois representation 1 over Q2[20],
1-dim Galois representation (1,-1) with G=C2, I=C2, conductor 2^2 over Q2[20],
1-dim Galois representation (1,-1) with G=C2, I=C2, conductor 2^3 over Q2[20],
1-dim Galois representation (1,-1) with G=C2, I=C2, conductor 2^3 over Q2[20],
2-dim Galois representation (2,-2,0,0,0) with G=D4, I=D4, conductor 2^8
   over Q2[20]
]
```

`GaloisRepresentations(f)`

For a polynomial f over a p-adic field K and splitting field F, returns irreducible representations of $\mathrm{Gal}(F/K)$.

Example H56E11

We construct 4 one-dimensional characters of $\mathbf{Q}_2(\zeta_8)/\mathbf{Q}_2$.

```
> K:=pAdicField(2,20);
> R:=PolynomialRing(K);
> GaloisRepresentations(R!CyclotomicPolynomial(8));
[
1-dim trivial Galois representation 1 over Q2[20],
1-dim Galois representation (1,-1,1,-1) with G=C2^2, I=C2^2, conductor 2^3
   over Q2[20],
1-dim Galois representation (1,1,-1,-1) with G=C2^2, I=C2^2, conductor 2^2
   over Q2[20],
1-dim Galois representation (1,-1,-1,1) with G=C2^2, I=C2^2, conductor 2^3
   over Q2[20]
]
```

`PermutationCharacter(F,K)`

For a p-adic extension F/K, compute $\mathbf{C}[\mathrm{Gal}(\bar{K}/K)/\mathrm{Gal}(\bar{K}/F)]$ as a Galois representation over K of degree $[F:K]$.

Example H56E12

Take $K = \mathbf{Q}_2$ and $F = \mathbf{Q}_2(\sqrt[3]{2})$. Then `PermutationCharacter(F,K)` is a 3-dimensional representation which is the trivial representation plus a 2-dimensional irreducible one.

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(K);
> F:=ext<K|x^3-2>;
> PermutationCharacter(F,K);
3-dim Galois representation (3,1,0) with G=S3, I=C3, conductor 2^2 over Q2[20]
> $1 - PrincipalCharacter(K);
2-dim Galois representation (2,0,-1) with G=S3, I=C3, conductor 2^2 over Q2[20]
```

```
A !! chi
```

Change a Galois representation by a finite representation with character χ, which must be a character of `Group(A)`, or a list of values that determine such a character.

Example H56E13

Take $K = \mathbf{Q}_2$ and $F = \mathbf{Q}_2(\sqrt[3]{2})$, so that $G = \mathrm{Gal}(F/K) \cong S_3$. Using `!!` we can start with any Galois representation whose finite part comes from this Galois group, and replace it by any other character of G.

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(K);
> F:=ext<K|x^3-2>;
> rho:=PermutationCharacter(F,K);
> rho!![1,1,1];
1-dim trivial Galois representation 1 over Q2[20]
> rho!![6,0,0];
6-dim Galois representation (6,0,0) with G=S3, I=C3, conductor 2^4 over Q2[20]
> rho!![0,0,0];   // but not [-1,0,0] - may not be virtual
Galois representation 0 with G=S3, I=C3 over Q2[20]
```

56.2.2 Local Representations of Global Objects

```
GaloisRepresentation(chi,p)
```

`Precision` RngIntElt *Default :* 40

Local Galois representation at p of a Dirichlet character χ.

Example H56E14

Local components of a Dirichet character χ of order 6 at $p = 2, 3, 7$.

```
> G<chi>:=FullDirichletGroup(7);
> GaloisRepresentation(chi,2);
1-dim unramified Galois representation (1,-zeta(3)_3-1,zeta(3)_3)
   with G=C3, I=C1 over Q2[40]
> GaloisRepresentation(chi,3);
1-dim unramified Galois representation (1,-1,-zeta(3)_3-1,zeta(3)_3,-zeta(3)_3,
   zeta(3)_3+1) with G=C6, I=C1 over Q3[40]
> GaloisRepresentation(chi,7);
1-dim Galois representation (1,-1,-zeta(3)_3-1,zeta(3)_3,-zeta(3)_3,zeta(3)_3+1)
   with G=C6, I=C6, conductor 7^1 over Q7[40]
```

By our convention, the character χ, the associated Artin representation, and the Galois representations associated to them all have the same Euler factors.

```
> loc1:=EulerFactor(chi,2);
> loc2:=EulerFactor(ArtinRepresentation(chi),2);
> loc3:=EulerFactor(GaloisRepresentation(chi,2));
> loc4:=EulerFactor(GaloisRepresentation(ArtinRepresentation(chi),2));
> [PolynomialRing(ComplexField(5))| loc1,loc2,loc3,loc4];
[
(0.50000 - 0.86603*$.1)*$.1 + 1.0000,
(0.50000 - 0.86603*$.1)*$.1 + 1.0000,
(0.50000 - 0.86603*$.1)*$.1 + 1.0000,
(0.50000 - 0.86603*$.1)*$.1 + 1.0000
]
```

`GaloisRepresentation(A,p)`

`Precision` RngIntElt *Default :* 40

`Minimize` BoolElt *Default :* `false`

Local Galois representation at p of an Artin representation A. This is the representation of a decomposition group at p of Gal($\bar{\mathbf{Q}}/\mathbf{Q}$) on the *dual* vector space of A. (The reason for the dual is that, by our convention, the global and the local Euler factors agree; see §56.1.2 and Example H56E14.) If `Minimize` is `true`, choose the field through which it factors to be as small as possible (automatic for faithful representations).

Example H56E15

Local components of an Artin representation. We take the Trinks' polynomial $x^7 - 7x - 3$ with Galois group PSL$(2,7)$ over $\mathbf{Q}$, one of its 7-dimensional representations A of conductor $3^8 7^8$, and compute its local components over $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{Q}_5$ and $\mathbf{Q}_7$.

```
> R<x>:=PolynomialRing(Rationals());
> K:=NumberField(x^7-7*x-3);
> GroupName(GaloisGroup(K));
PSL(2,7)
> A:=ArtinRepresentations(K)[5];
> GaloisRepresentation(A,2);
7-dim unramified Galois representation (7,0,0,0,0,0,0) with G=C7, I=C1
   over Q2[40]
> GaloisRepresentation(A,3);
7-dim Galois representation (7,-1,1) with G=S3, I=S3, conductor 3^8 over Q3[40]
> GaloisRepresentation(A,5);
7-dim unramified Galois representation (7,0,0,0,0,0,0) with G=C7, I=C1
   over Q5[40]
> GaloisRepresentation(A,7);
7-dim Galois representation (7,1,1,0,0) with G=C7:C3, I=C7:C3, conductor 7^8
   over Q7[40]
> Conductor(A) eq 3^8*7^8;
true
```

```
GaloisRepresentation(E)
```

`Minimize` BoolElt *Default :* `true`

Local Galois representation of (the first l-adic étale cohomology group of) an elliptic curve over a p-adic field. If `Minimize` is `true` (default), choose the field through which it factors to be as small as possible.

Example H56E16

Take an elliptic curve $E/\mathbf{Q}_5$, with additive (potentially good) reduction of type II.

```
> K:=pAdicField(5,20);
> E:=EllipticCurve([K|0,5]);
> E;
Elliptic Curve defined by y^2 = x^3 + O(5^20)*x + (5 + O(5^21))
   over pAdicField(5, 20)
> loc:=LocalInformation(E); loc;
<5 + O(5^21), 2, 2, 1, II, true>
```

Its Galois representation is an unramified twist of a representation with finite image that factors through the dihedral extension $\mathbf{Q}_5(\zeta_6, \sqrt[6]{5})$ of $\mathbf{Q}_5$.

```
> A:=GaloisRepresentation(E); A;
2-dim Galois representation Unr(sqrt(5)*i)*(2,-2,0,0,-1,1) with G=D6, I=C6,
   conductor 5^2 over Q5[20]
> Field(A);
Totally ramified extension defined by the polynomial x^6 - 5
 over Unramified extension defined by the polynomial x^2 + 4*x + 2
 over 5-adic field mod 5^20
```

```
GaloisRepresentation(E,p)
```

`Precision` RngIntElt *Default :* 40

`Minimize` BoolElt *Default :* `true`

Local Galois representation of (the first l-adic étale cohomology group of) an elliptic curve over $\mathbf{Q}$ at p. If `Minimize` is `true` (default), choose the field through which it factors to be as small as possible.

Example H56E17

We take the elliptic curve 20a1 over $\mathbf{Q}$ and compute its local Galois representation at a prime $p = 3$ of good reduction, $p = 5$ of (non-split) multiplicative reduction and $p = 2$ of additive reduction.

```
> E:=EllipticCurve("20a1");
> GaloisRepresentation(E,3);
2-dim unramified Galois representation Unr(1+2*x+3*x^2) over Q3[40]
> GaloisRepresentation(E,5);
2-dim Galois representation Unr(-5)*SP(2) over Q5[40]
```

```
> GaloisRepresentation(E,2);
2-dim Galois representation Unr(sqrt(2)*i)*(2,0,-1) with G=S3, I=C3, conductor
   2^2 over Q2[40]
> EulerFactor($3),EulerFactor($2),EulerFactor($1);
3*x^2 + 2*x + 1
x + 1
1
```

`GaloisRepresentation(E,P)`

`Precision`	RngIntElt	*Default :* 40
`Minimize`	BoolElt	*Default :* `true`

Local Galois representation of (the first l-adic étale cohomology group of) an elliptic curve E over a number field F at a given prime ideal P. If `Minimize` is `true` (default), choose the field through which it factors to be as small as possible.

Example H56E18

```
> K:=CyclotomicField(5);
> E:=BaseChange(EllipticCurve("75a1"),K);
> P:=Ideal(Decomposition(K,5)[1,1]);
> GaloisRepresentation(E,P);
2-dim Galois representation Unr(sqrt(5)*i)*(2,0,-1) with G=S3, I=C3, conductor
   pi^2 over ext<Q5[10]|x^4-15*x^3-40*x^2-90*x-45>
```

`GaloisRepresentation(C)`

`Degree`	RngIntElt	*Default :* ∞
`Minimize`	BoolElt	*Default :* `false`

Galois representation associated to (H^1 of) a hyperelliptic curve C over a p-adic field. `Degree` specifies that Euler factors of unramified pieces should only be computed up to that degree. (See Example H56E7.) Setting `Minimize:=true` forces the representation to be minimized. (See `Minimize`.)

Example H56E19

We take a curve C over $K = \mathbf{Q}_{23}$ of conductor 23^2 and compute its Galois representation.

```
> K:=pAdicField(23,20);
> R<x>:=PolynomialRing(K);
> C:=HyperellipticCurve(-x,x^3+x^2+1);    // genus 2, conductor 23^2
> A:=GaloisRepresentation(C); A;
4-dim Galois representation Unr(1-46*x+529*x^2)*SP(2) over Q23[20]
```

If F/K is a finite extension, then the base change of A to F is the same as the Galois representation of C/F:

```
> F:=ext<K|2>;
> BaseChange(A,F);
4-dim Galois representation Unr(1-1058*x+279841*x^2)*SP(2) over ext<Q23[20]|2>
> GaloisRepresentation(BaseChange(C,F));
4-dim Galois representation Unr(1-1058*x+279841*x^2)*SP(2) over ext<Q23[20]|2>
```

`GaloisRepresentation(C,p)`

`Degree` RngIntElt *Default :* ∞

`Minimize` BoolElt *Default :* `false`

Galois representation associated to (H^1 of) a hyperelliptic curve $C/\mathbf{Q}$ at p. `Degree` specifies that Euler factors of unramified pieces should only be computed up to that degree. (See Example H56E7.) Setting `Minimize:=true` forces the representation to be minimized. (See `Minimize`.)

Example H56E20

```
> R<x>:=PolynomialRing(Rationals());
> C:=HyperellipticCurve((x^2+5)*(x+1)*(x+2)*(x+3));
> GaloisRepresentation(C,5);   // bad reduction
4-dim Galois representation Unr(1+2*x+5*x^2) + Unr(5)*SP(2) over Q5[20]
> GaloisRepresentation(C,11);  // good reduction
4-dim unramified Galois representation Unr(1-2*x+6*x^2-22*x^3+121*x^4)
   over Q11[5]
> GaloisRepresentation(C,997: Degree:=1);  // don't count pts over GF(997^2)
4-dim unramified Galois representation Unr(1+26*x+O(x^2)) over Q997[5]
```

`GaloisRepresentation(C,P)`

`Degree` RngIntElt *Default :* ∞

`Minimize` BoolElt *Default :* `false`

Galois representation associated to (H^1 of) a hyperelliptic curve C over a number field at a prime ideal P. `Degree` specifies that Euler factors of unramified pieces should only be computed up to that degree. (See Example H56E7.) Setting `Minimize:=true` forces the representation to be minimized. (See `Minimize`.)

Example H56E21

We take a curve of genus 4 over $\mathbf{Q}(\zeta_{11})$ and compute its Galois representation at a unique prime P above 11.

```
> K<zeta>:=CyclotomicField(11);
> R<x>:=PolynomialRing(K);
> C:=HyperellipticCurve(x^9+x^2+(zeta-1));
> P:=Ideal(Decomposition(K,11)[1,1]);
> GaloisRepresentation(C,P);
8-dim Galois representation Unr(1-44*x^3+1331*x^6) + Unr(11)*SP(2)
   over ext<Q11[2]|x^10+22*x^9+55*x^8+44*x^7-33*x^6-22*x^5-22*x^4-33*x^3+44*x^2+
   55*x+11>
```

`GaloisRepresentation(f,p)`

Precision RngIntElt *Default :* 40

Local Galois representation at p of a modular form f. Currently only implemented when p^2 does not divide the level.

Example H56E22

We take a rational modular form of weight 4 and level 5, and compute its Galois representations at $p = 3$ (unramified principal series) and $p = 5$ (Steinberg).

```
> f:=Newforms("5k4")[1,1];
> GaloisRepresentation(f,3);
2-dim unramified Galois representation Unr(1-2*x+27*x^2) over Q3[40]
> GaloisRepresentation(f,5);
2-dim Galois representation Unr(-25)*SP(2) over Q5[40]
```

56.3 Basic Invariants

```
BaseField(A)
```

The base field K of the Galois representation $A : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_m(\mathbf{C})$.

```
Degree(A)
```

```
Dimension(A)
```

Degree (=dimension) m of a Galois representation $A : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_m(\mathbf{C})$

Example H56E23

```
> K:=pAdicField(3,20);
> R<x>:=PolynomialRing(K);
> F:=ext<K|x^3-3>;
> list:=GaloisRepresentations(F,K);
> forall{A: A in list | BaseField(A) eq K};
true
> [Degree(A): A in list];
[ 1, 1, 2 ]
```

```
Group(A)
```

```
GaloisGroup(A)
```

Finite Galois group $\mathrm{Gal}(F/K)$ that computes the finite part of a Galois representation, where F is `Field(A)` and K is `BaseField(A)`.

Example H56E24

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(K);
> list1:=[PrincipalCharacter(K),CyclotomicCharacter(K),SP(K,3)];
> [Degree(A): A in list1];
[ 1, 1, 3 ]
> [GroupName(Group(A)): A in list1];
[ C1, C1, C1 ]
> list2:=GaloisRepresentations(x^4-2);
> [Degree(A): A in list2];
[ 1, 1, 1, 1, 2 ]
> [GroupName(Group(A)): A in list2];
[ D4, D4, D4, D4, D4 ]
> list1[1] eq list2[1];
true
```

```
FrobeniusElement(A)
```

An arithmetic Frobenius element of `Group(A)` for a Galois representation A.

Example H56E25

Take $K = \mathbf{Q}_2$ and $F = \mathbf{Q}_2(\zeta_5)$, a degree 4 unramified extension of K. A Frobenius element $\mathrm{Frob} \in \mathrm{Gal}(F/K)$ is characterized by the property that $\mathrm{Frob}(x) \equiv x^q \mod m_F$. In this example, $q = 2$ (size of the residue field of K).

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(K);
> A:=GaloisRepresentations(x^4+x^3+x^2+x+1)[4]; A;
1-dim unramified Galois representation (1,-1,-zeta(4)_4,zeta(4)_4)
   with G=C4, I=C1 over Q2[20]
> frob:=FrobeniusElement(A); frob;
(1, 2, 4, 3)
> F<u>:=Field(A);
> Valuation(Automorphism(A,frob)(u)-u^2) gt 0;
true
```

`Character(A)`

Character of the finite part of a Galois representation A. For this to be well-defined, A must have only one component (and not several components with different characters).

Example H56E26

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(K);
> A1:=PrincipalCharacter(K);
> Character(A1);
( 1 )
> A2:=CyclotomicCharacter(K);
> Character(A2);
( 1 )
> A3:=PermutationCharacter(ext<K|3>,K);
> Character(A3);
( 3, 0, 0 )
> A1+A2+A3;
5-dim unramified Galois representation Unr(1-3/2*x+1/2*x^2) + (3,0,0)
   with G=C3, I=C1 over Q2[20]
```

`Field(A)`

Given a Galois representation A, return the p-adic field F such that the finite part of A factors through $\mathrm{Gal}(F/K)$, $K =$ `BaseField(A)`.

```
DefiningPolynomial(A)
```

For a Galois representation A over K, returns a polynomial over K whose splitting field is F =`Field(A)`. The Galois group $\mathrm{Gal}(F/K)$ is represented as a permutation group `Group(A)` on the roots of this polynomial.

Example H56E27

In this example $K = \mathbf{Q}_5$ and F/K is a dihedral extension of degree 12, represented as a splitting field of a degree 6 polynomial.

```
> K:=pAdicField(5,20);
> E:=EllipticCurve([K|0,5]);
> A:=GaloisRepresentation(E); A;
2-dim Galois representation Unr(sqrt(5)*i)*(2,-2,0,0,-1,1) with G=D6, I=C6,
   conductor 5^2 over Q5[20]
> Field(A);
Totally ramified extension defined by the polynomial x^6 - 5
 over Unramified extension defined by the polynomial x^2 + 4*x + 2
 over 5-adic field mod 5^20
> DefiningPolynomial(A);
x^6 + O(5^20)*x^5 + O(5^20)*x^4 + O(5^20)*x^3 + O(5^20)*x^2 + O(5^20)*x - 5 +
   O(5^20)
```

```
Automorphism(A,g)
```

The automorphism of `Field(A)/BaseField(A)` given by g.

Example H56E28

In this example $K = \mathbf{Q}_5$ and F/K is a dihedral extension of degree 12, represented as a splitting field of the polynomial $x^6 - 5$.

```
> R<x>:=PolynomialRing(Rationals());
> K:=NumberField(x^6-5);
> a:=ArtinRepresentations(K)[6];
> A:=GaloisRepresentation(a,5); A;
2-dim Galois representation (2,-2,0,0,-1,1) with G=D6, I=C6, conductor 5^2
   over Q5[40]
> F:=Field(A); F;
Totally ramified extension defined by the polynomial x^6 - 5
 over Unramified extension defined by the polynomial x^2 + 4*x + 2
 over 5-adic field mod 5^40
> DefiningPolynomial(A);
x^6 - 5
```

The 12 elements $\sigma \in \mathrm{Gal}(F/K) \cong D_6$ act on $\pi = \sqrt[6]{5}$ by multiplying it by 6th roots of unity, with $\sigma(\pi) = \pi$ for 2 of them, and $v(\sigma(\pi) - \pi) = 1$ for the other 10.

```
> pi:=UniformizingElement(F);
> autF:=[* Automorphism(A,g): g in Group(A) *];
> [Valuation(sigma(pi)-pi): sigma in autF];
[ 241, 1, 1, 1, 1, 1, 240, 1, 1, 1, 1, 1 ]
```

```
EulerFactor(A)
```

R RNG *Default :*

Euler factor (=local polynomial) of a Galois representation A over a p-adic field K. It is defined by

$$P(T) = \det(1 - \mathrm{Frob}_K^{-1} T | A^{I_K}),$$

and has degree `Dimension(A)` if and only if A is unramified. The coefficient ring of P (rational/complex/cyclotomic field) may be specified with the optional parameter R.

Example H56E29

```
> G<chi>:=DirichletGroup(5);
> A:=GaloisRepresentation(chi,2);
> EulerFactor(A);
x + 1
> A:=GaloisRepresentation(chi,5);
> EulerFactor(A);
1
```

```
IsZero(A)
```

Return `true` if A is the Galois representation 0.

```
IsOne(A)
```

Return `true` if A is the trivial 1-dimensional Galois representation.

```
Factorization(A)
```

Returns the list of tuples $\langle\chi_i,n_i,\rho_i\rangle$, where A is the direct sum over i of twists by $\mathbf{SP}(n_i)$ by unramified representations with Euler factor χ_i, and a finite Weil representation given by a character ρ_i of `Group(A)`.

Example H56E30

```
> R<x>:=PolynomialRing(ComplexField());    // prettier print for complex polys
> K:=pAdicField(2,20);
> S:=SP(K,2);
> S; Factorization(S);
2-dim Galois representation SP(2) over Q2[20]
[*
<-x + 1, 2, ( 1 )>
*]
> A:=Semisimplification(S);
> A; Factorization(A);
2-dim unramified Galois representation Unr(1-3/2*x+1/2*x^2) over Q2[20]
[*
<1/2*x^2 - 3/2*x + 1, 1, ( 1 )>
*]
> [Factorization(I)[1]: I in Decomposition(A)];
[ <-x + 1, 1, ( 1 )>, <-1/2*x + 1, 1, ( 1 )> ]
```

56.3.1 Ramification

`InertiaGroup(A)`

For a Galois representation A over a p-adic field K this is the image of inertia $I_K \subset \mathrm{Gal}(\bar{K}/K)$ under the semisimplification of A. Equivalently, if

$$A = \psi \otimes \mathtt{SP}(n) \otimes R,$$

as in §56.1.1, with ψ unramified and R a representation of a finite Galois group $\mathrm{Gal}(F/K)$, this is the image of the inertia subgroup of $\mathrm{Gal}(F/K)$ under R. If F is chosen to be minimal possible (so that R is faithful), then `InertiaGroup(A)` simply *is* the inertia subgroup of $\mathrm{Gal}(F/K)$.

Example H56E31

Take $K = \mathbf{Q}_3$ and $F = K(\zeta_6, \sqrt[6]{3})$, a D_6-extension of K. For each of the 6 irreducible representations of $\mathrm{Gal}(F/K)$ we compute their inertia (=ramification) groups:

```
> K:=pAdicField(3,20);
> R<x>:=PolynomialRing(K);
> list:=GaloisRepresentations(x^6-3);
> [GroupName(InertiaGroup(A)): A in list];
[ C1, C2, C1, C2, S3, S3 ]
```

`InertiaGroup(A,n)`

The nth (lower) ramification subgroup of `InertiaGroup(A)`.

Example H56E32

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(K);
> list:=GaloisRepresentations(x^8-2);
> a:=list[#list]; a;
2-dim Galois representation (2,-2,0,0,0,zeta(8)_8^3+zeta(8)_8,
   -zeta(8)_8^3-zeta(8)_8) with G=SD16, I=SD16, conductor 2^10 over Q2[20]
> [GroupName(InertiaGroup(a,n)): n in [1..17]];
[ SD16, SD16, C8, C8, C4, C4, C4, C4, C2, C2, C2, C2, C2, C2, C2, C2, C1 ]
```

`IsUnramified(A)`

Return `true` if a Galois representation is unramified.

Example H56E33

```
> K:=pAdicField(2,20);
> IsUnramified(CyclotomicCharacter(K));
true
> IsUnramified(SP(K,2));
false
> IsUnramified(Semisimplification(SP(K,2)));
true
```

`IsRamified(A)`

Return `true` if a Galois representation is ramified.

Example H56E34

```
> K:=pAdicField(2,20);
> IsRamified(CyclotomicCharacter(K));
false
> IsRamified(SP(K,2));
true
> IsRamified(Semisimplification(SP(K,2)));
false
```

`IsTamelyRamified(A)`

Return `true` if a Galois representation A over a p-adic field K is tamely ramified. Equivalently, `InertiaGroup(A)` has order prime to p (and is then automatically cyclic).

`IsWildlyRamified(A)`

Return `true` if a Galois representation A over a p-adic field K is wildly ramified, i.e. not tamely ramified. Equivalently, `InertiaGroup(A)` has non-trivial p-Sylow.

Example H56E35

Galois representations attached to elliptic curves are always tamely ramified when $p \geq 5$, but may be wildly ramified when $p = 2$ or 3.

```
> E:=EllipticCurve("75a1");
> A5:=GaloisRepresentation(E,5); A5;
2-dim Galois representation Unr(sqrt(5)*i)*(2,0,-1) with G=S3, I=C3, conductor
   5^2 over Q5[40]
> IsWildlyRamified(A5);
false
> E:=EllipticCurve("256a1");
> A2:=GaloisRepresentation(E,2); A2;
2-dim Galois representation Unr(sqrt(2))*(2,-2,0,0,0) with G=D4, I=C4,
   conductor 2^8 over Q2[40]
> IsWildlyRamified(A2);
true
```

`InertiaInvariants(A)`

Inertia invariants of a Galois representation A. This is an unramified Galois representation.

Example H56E36

```
> K:=pAdicField(5,20);
> E:=BaseChange(EllipticCurve("15a1"),K);
> A:=GaloisRepresentation(E); A;
2-dim Galois representation Unr(5)*SP(2) over Q5[20]
> I:=InertiaInvariants(A); I;
1-dim trivial Galois representation 1 over Q5[20]
> Dimension(A),Dimension(I);
2 1
```

`ConductorExponent(A)`

Conductor exponent of a Galois representation.

`Conductor(A)`

Conductor of a Galois representation.

Example H56E37

```
> K:=pAdicField(2,40);
> E:=BaseChange(EllipticCurve("256a1"),K);
> A:=GaloisRepresentation(E); A;
2-dim Galois representation Unr(sqrt(2))*(2,-2,0,0,0) with G=D4, I=C4,
   conductor 2^8 over Q2[40]
> ConductorExponent(A);
8
> Conductor(A);
2^8 + O(2^48)
> Conductor(E);    // same, by definition
2^8 + O(2^48)
```

```
EpsilonFactor(A)
```

Epsilon-factor $\epsilon(A)$ of a Galois representation over a p-adic field. Currently only implemented in a few basic cases, and returns 0 otherwise. See also Example H56E52 in §56.7.1.

```
RootNumber(A)
```

Root number $\epsilon(A)/|\epsilon(A)|$ of a Galois representation. Currently only implemented in a few basic cases, and returns 0 otherwise. See also Example H56E52 in §56.7.1.

Example H56E38

```
> E:=EllipticCurve("98a1");
> A:=GaloisRepresentation(E,7); A;
2-dim Galois representation Unr(7)*SP(2)*(1,-1) with G=C2, I=C2, conductor 7^2
   over Q7[40]
> RootNumber(A);
-1
> RootNumber(E,7);  // same
-1
```

56.3.2 Semisimplicity and Irreducibles

`IsIrreducible(A)`

Return `true` if a Galois representation A is irreducible.

Example H56E39

We take the polynomial $x^8 - 6$ with Galois group $C_8 : C_2^2$ over $\mathbf{Q}$, its irreducible 4-dimensional Artin representation A, and compute whether its local components over $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{Q}_5$ and $\mathbf{Q}_7$ are irreducible.

```
> R<x>:=PolynomialRing(Rationals());
> K:=NumberField(x^8-6);
> GroupName(GaloisGroup(K));
C8:C2^2
> assert exists(A){A: A in ArtinRepresentations(K) | Degree(A) eq 4};
> A;
Artin representation C8:C2^2: (4,-4,0,0,0,0,0,0,0,0,0) of ext<Q|x^8-6>
> [IsIrreducible(GaloisRepresentation(A,p)): p in PrimesUpTo(10)];
[ true, false, false, false ]
```

`IsIndecomposable(A)`

Return `true` if a Galois representation A is indecomposable (for semisimple representations, i.e. Weil representations, same as irreducible).

`IsSemisimple(A)`

Return `true` if a Galois representation A is semisimple, i.e. a Weil representation.

`Semisimplification(A)`

Semisimplification of a Galois representation A.

`Decomposition(A)`

Decompose A into indecomposable (for semisimple representations same as irreducible) consituents and return them as a sequence, possibly with repetitions.

Example H56E40

```
> K:=pAdicField(2,20);
> S:=SP(K,2);
> IsIndecomposable(S);
true
> IsIrreducible(S);
false
> IsSemisimple(S);
false
> Decomposition(S);
[ 2-dim Galois representation SP(2) over Q2[20] ]
> Decomposition(Semisimplification(S));
[
1-dim trivial Galois representation 1 over Q2[20],
1-dim unramified Galois representation Unr(1/2) over Q2[20]
]
```

56.4 Arithmetic

`A1 + A2`

Direct sum of two Galois representations, both defined over the same p-adic field K.

Example H56E41

```
> K:=pAdicField(2,20);
> SP(K,1)+SP(K,2)+SP(K,3);
6-dim Galois representation 1 + SP(2) + SP(3) over Q2[20]
```

`A1 - A2`

Assuming $A2$ is a Galois subrepresentation of $A1$, compute $A1 - A2$.

Example H56E42

We take $K = \mathbf{Q}_2$, F its unique degree 3 unramified extension, and compute the regular representation $\mathrm{Gal}(F/K)$ minus the trivial representation, as a Galois representation over K.

```
> K:=pAdicField(2,20);
> F:=ext<K|3>;
> PermutationCharacter(F,K)-PrincipalCharacter(K);
2-dim unramified Galois representation (2,-1,-1) with G=C3, I=C1 over Q2[20]
```

`A1 * A2`

Tensor product of two Galois representations, both defined over the same p-adic field K.

Example H56E43

We take $K = \mathbf{Q}_2$ and compute SP(K,3)$\otimes$SP(K,3); this is basically the Clebsch-Gordan decomposition.

```
> K:=pAdicField(2,20);
> SP(K,3)*SP(K,3);
9-dim Galois representation Unr(1/4) + Unr(1/2)*SP(3) + SP(5) over Q2[20]
```

`A1 / A2`

Tensor A_1 with A_2^{-1}, for 1-dimensional A_2. (A_1 may also be the integer 0 or the integer 1.)

`A ^ n`

Tensor power of a Galois representation. The power n should be a non-negative integer for a general representation, but may be negative for 1-dimensional representations, and an arbitrary complex number for (powers of) the cyclotomic character.

Example H56E44

```
> K:=pAdicField(2,20);
> CyclotomicCharacter(K)^(1/2);
1-dim unramified Galois representation Unr(1/2*sqrt(2)) over Q2[20]
> SP(K,2)^2 / CyclotomicCharacter(K);
4-dim Galois representation 1 + Unr(2)*SP(3) over Q2[20]
```

`A1 eq A2`

Return `true` if the two Galois representations are equal.

Example H56E45

```
> K:=pAdicField(2,20);
> w:=CyclotomicCharacter(K);
> A:=Semisimplification(SP(K,2));
> A eq w^0+w;
true
```

`Determinant(A)`

Determinant of a Galois representation (a 1-dimensional Galois representation).

Example H56E46

```
> K:=pAdicField(5,20);
> E:=EllipticCurve([K|0,5]);
> A:=GaloisRepresentation(E); A;
2-dim Galois representation Unr(sqrt(5)*i)*(2,-2,0,0,-1,1) with G=D6, I=C6,
   conductor 5^2 over Q5[20]
> Determinant(A) eq CyclotomicCharacter(K)^(-1);
true
```

`TateTwist(A,n)`

Tate twist $A(n)$ of a Galois representation. So $A(n) = A \otimes w^{\otimes n}$ where w is the cyclotomic character.

Example H56E47

```
> K:=pAdicField(5,20);
> R<x>:=PolynomialRing(K);
> A:=GaloisRepresentations(x^2-2)[2];
> A;
1-dim unramified Galois representation (1,-1) with G=C2, I=C1 over Q5[20]
> TateTwist(A,1);
1-dim unramified Galois representation Unr(1/5)*(1,-1) with G=C2, I=C1
   over Q5[20]
> [EulerFactor(TateTwist(A,n)): n in [-2..2]];
[ 25*x + 1, 5*x + 1, x + 1, 1/5*x + 1, 1/25*x + 1 ]
```

56.5 Changing Precision

`Precision(A)`

Get p-adic precision of the base field of a Galois representation.

`ChangePrecision(~A,Prec)`

Destructively change p-adic precision of the base field of a Galois representation.

`ChangePrecision(A,Prec)`

Change p-adic precision of the base field of a Galois representation.

Example H56E48

```
> K:=pAdicField(5,20);
> R<x>:=PolynomialRing(K);
> A:=GaloisRepresentations(x^4-5)[3];
> A;
1-dim Galois representation (1,1,-1,-1) with G=C4, I=C4, conductor 5^1
   over Q5[20]
> Precision(A);
20
> ChangePrecision(~A,40);
> Precision(A);
40
> ChangePrecision(A,20);
1-dim Galois representation (1,1,-1,-1) with G=C4, I=C4, conductor 5^1
   over Q5[20]
```

56.6 Changing Fields

`Minimize(A)`

`To` GalRep *Default :*

Replace `Group(A)` by its smallest possible quotient through which all components of A factor. If `To` is specified, instead replace `Group(A)` by `Group(To)`, assuming the A factors through it.

Example H56E49

We take an extension F of $K = \mathbf{Q}_3$ with Galois group $F_5 = C_5 : C_4$ of order 20. It has five irreducible representations. Four of them are 1-dimensional, and so they actually factor through a smaller Galois group (C_1, C_2 or C_4). Minimize descends them to these Galois groups, although they are of course still the same as representations of the absolute Galois group.

```
> K:=pAdicField(3,20);
> R<x>:=PolynomialRing(K);
> list:=GaloisRepresentations(x^5-3);
> list;
[
1-dim trivial Galois representation 1 over Q3[20],
1-dim unramified Galois representation (1,1,-1,-1,1) with G=F5, I=C5
   over Q3[20],
1-dim unramified Galois representation (1,-1,-zeta(4)_4,zeta(4)_4,1)
   with G=F5, I=C5 over Q3[20],
1-dim unramified Galois representation (1,-1,zeta(4)_4,-zeta(4)_4,1)
   with G=F5, I=C5 over Q3[20],
4-dim Galois representation (4,0,0,0,-1) with G=F5, I=C5, conductor 3^4
   over Q3[20]
]
> [Minimize(A): A in list];
[
1-dim trivial Galois representation 1 over Q3[20],
1-dim unramified Galois representation (1,-1) with G=C2, I=C1 over Q3[20],
1-dim unramified Galois representation (1,-1,-zeta(4)_4,zeta(4)_4)
   with G=C4, I=C1 over Q3[20],
1-dim unramified Galois representation (1,-1,zeta(4)_4,-zeta(4)_4)
   with G=C4, I=C1 over Q3[20],
4-dim Galois representation (4,0,0,0,-1) with G=F5, I=C5, conductor 3^4
   over Q3[20]
]
> forall{A: A in list | A eq Minimize(A)};
true
```

Finally, we illustrate how the parameter To may be used to descend a Galois representation to a specific Galois group, in this case the Galois group $\mathrm{Gal}(F/K) \cong C_4$ of the degree 4 unramified extension of K.

```
> F:=ext<K|4>;                      // Take F = degree 4 unr. ext. of K, and
> B:=PermutationCharacter(F,K);    // any B with BaseField(B)=K, Field(B)=F
> list[2];
1-dim unramified Galois representation (1,1,-1,-1,1) with G=F5, I=C5 over Q3[20]
> Minimize(list[2]: To:=B);
1-dim unramified Galois representation (1,1,-1,-1) with G=C4, I=C1 over Q3[20]
> Minimize(list[2]);
1-dim unramified Galois representation (1,-1) with G=C2, I=C1 over Q3[20]
```

```
Restriction(A,L)
BaseChange(A,L)
```

Base change (restriction) of a Galois representation A over K over a finite extension L/K.

Example H56E50

We take a 2-dimensional irreducible representation of $\mathrm{Gal}(\mathbf{Q}_2(\zeta_3, \sqrt[3]{2})) \cong S_3$ and check that its base change to $\mathbf{Q}_2(\zeta_3)$ is reducible.

```
> K:=pAdicField(2,20);
> R<x>:=PolynomialRing(K);
> A:=GaloisRepresentations(x^3-2)[3]; A;
2-dim Galois representation (2,0,-1) with G=S3, I=C3, conductor 2^2 over Q2[20]
> L:=ext<K|2>;
> R:=Restriction(A,L); R;
2-dim Galois representation (2,-1,-1) with G=C3, I=C3, conductor 2^2
   over ext<Q2[20]|2>
> IsIrreducible(A),IsIrreducible(R);
true false
```

```
Induction(A,K0)
```

Induction of a Galois representation A over K to a subfield $K_0 \subset K$.

Example H56E51

```
> K0:=pAdicField(2,20);     // K0=Q2
> K:=ext<K0|2>;             // K=Q2(zeta_3)
> R<x>:=PolynomialRing(K);
> A:=GaloisRepresentations(x^3-102)[3];
> A;                        // 1-dim character over K
1-dim Galois representation (1,-zeta(3)_3-1,zeta(3)_3) with G=C3, I=C3,
   conductor 2^1 over ext<Q2[20]|2>
> Induction(A,K0);          // Induced to K0
2-dim Galois representation (2,0,-1) with G=S3, I=C3, conductor 2^2 over Q2[20]
```

56.7 Advanced Examples

56.7.1 Example: Local and Global Epsilon Factors for Dirichlet Characters

Example H56E52

We illustrate the relation between local and global epsilon-factors for Dirichlet characters (or Artin representations). Pick any Dirichlet character χ; in this example we take the unique odd one of conductor $5 \cdot 19$, order 6 and $\mathrm{Im}\,\chi(2) > 0$.

```
> D:=FullDirichletGroup(5*19);
> L:=[chi: chi in Elements(D) | (Order(chi) eq 6) and
>   (Conductor(chi) eq 5*19) and IsOdd(chi) and (Imaginary(chi(2)) ge 0)];
> assert #L eq 1;
> chi:=L[1];
```

We compute its local Galois representations at all the bad places: 5, 19 and ∞. (See §133.9.1 for Hodge structures at ∞.)

```
> G5:=GaloisRepresentation(chi,5); G5;
1-dim Galois representation (1,-1,-zeta(3)_3-1,zeta(3)_3,zeta(3)_3+1,-zeta(3)_3)
   with G=C6, I=C2, conductor 5^1 over Q5[40]
> G19:=GaloisRepresentation(chi,19); G19;
1-dim Galois representation (1,-1,zeta(3)_3,-zeta(3)_3-1,zeta(3)_3+1,-zeta(3)_3)
   with G=C6, I=C6, conductor 19^1 over Q19[40]
> Ginfty:=HodgeStructure(ArtinRepresentation(chi)); Ginfty;
Hodge structure of weight 0 given by <0,0,1>
```

Now compute the corresponding local root numbers - the way the conventions are set up for Artin representations, at ∞ it is the root number of the Hodge structure that enters the functional equation.

```
> localrootno:=[ComplexField()| RootNumber(G5),RootNumber(G19),
>               RootNumber(Ginfty)];
> globalrootno:=&*localrootno;
```

The global root number is the sign of the L-series of χ, and we check that it agrees with the one determined numerically from the functional equation:

```
> L:=LSeries(chi);
> ok:=CheckFunctionalEquation(L);
> Sign(L);
0.910747215816471738723996800097 + 0.412964294924567358770699943664*I
> globalrootno;
0.910747215816471738723996800108 + 0.412964294924567358770699943647*I
```

56.7.2 Example: Reconstructing a Galois Representation from its Euler Factors

Example H56E53

Every semisimple Galois representation A over a p-adic field K can be uniquely recovered from its Euler factors over the extensions of K ([DD15, Thm. 1]). We illustrate this with a Galois representation attached to an elliptic curve

$$E/K : y^2 = x^3 - 26x, \qquad K = \mathbf{Q}_{13}.$$

```
> K:=pAdicField(13,20);
> E:=EllipticCurve([K|-26,0]);
> A:=GaloisRepresentation(E);
> Degree(A),IsSemisimple(A);
2 true
```

Without looking at A, let us reconstruct it from its Euler factors over extensions of K. First we determine the inertia group `InertiaGroup(A)` by looking for a field over which A is unramified (i.e. E has good reduction). As the residue characteristic is > 3, one of the fields $\mathbf{Q}_{13}(\sqrt[d]{13})$ will do, for $d = 1, 2, 3, 4$ or 6.

```
> R<x>:=PolynomialRing(K);
> [EulerFactor(BaseChange(A,ext<K|x^d-13>)): d in [1,2,3,4,6]];
[ 1, 1, 1, 13*x^2 + 4*x + 1, 1 ]
```

We see that the representation becomes unramified over $L = \mathbf{Q}_{13}(\sqrt[4]{13})$, a cyclic extension of degree 4, but not over its subfields. So the inertia group must be C_4,

$$\mathrm{InertiaGroup}(A) \;\cong\; I_{L/K} \;=\; \mathrm{Gal}(L/K) \;\cong\; C_4.$$

Over L the representation $U = \mathrm{Res}_L\, A$ is unramified, and there it is determined by its Euler factor. It is a sum of two unramified characters, $\mathrm{Frob}_L^{-1} \mapsto -2 \pm 3i$.

```
> L:=ext<K|x^4-13>;
> f1:=EulerFactor(BaseChange(A,L));
> U:=UnramifiedRepresentation(L,f1); U;
2-dim unramified Galois representation Unr(1+4*x+13*x^2) over
   ext<Q13[20]|x^4-13>
> Decomposition(U);
[
1-dim unramified Galois representation Unr(-2+3*i) over ext<Q13[20]|x^4-13>,
1-dim unramified Galois representation Unr(-2-3*i) over ext<Q13[20]|x^4-13>
]
```

We are ready to reconstruct A. Its restriction to inertia is a faithful 2-dimensional representation of C_4, of determinant 1 (as it comes from an elliptic curve), so it must be $\sigma \oplus \sigma^{-1}$ where σ is one of the faithful 1-dimensional characters of C_4.

```
> GroupName(InertiaGroup(A)),IsUnramified(Determinant(A));  // just checking
C4 true
```

```
> list:=GaloisRepresentations(x^4-13);
> sigma:=[g: g in list | Order(Character(g)) eq 4][1];
> sigma;
1-dim Galois representation (1,-1,zeta(4)_4,-zeta(4)_4) with G=C4, I=C4,
   conductor 13^1 over Q13[20]
```

Since $\mathrm{Frob}_K = \mathrm{Frob}_L$ commutes with inertia in the Galois group

$$\mathrm{Gal}(K^{nr}(\sqrt[4]{13})/K) \cong \mathrm{Gal}(K^{nr}/K) \times \mathrm{Gal}(L/K) \cong \hat{\mathbf{Z}} \times C_4$$

through which A factors, $A(\mathrm{Frob}_K)$ and $A(\sigma)$ are simultaneously diagonalizable in $\mathrm{GL}_2(\mathbf{C})$, so A must be one of the following two representations A_1 and A_2:

```
> Qi<i>:=CyclotomicField(4);
> A1 := sigma * UnramifiedCharacter(K,-2-3*i) +
>       sigma^(-1)*UnramifiedCharacter(K,-2+3*i);
> A2 := sigma * UnramifiedCharacter(K,-2+3*i) +
>       sigma^(-1)*UnramifiedCharacter(K,-2-3*i);
```

Finally, to determine which one it is, we pick *another* extension of K where A becomes unramified and compare the Euler factors. Then we see that A must be A_2.

```
> L2:=ext<K|x^4-26>;
> EulerFactor(BaseChange(A,L2));
13*x^2 - 6*x + 1
> EulerFactor(BaseChange(A1,L2));
13*x^2 + 6*x + 1
> EulerFactor(BaseChange(A2,L2));
13*x^2 - 6*x + 1
> A eq A2;
true
```

56.8 Bibliography

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PART VIII
MODULES

57 INTRODUCTION TO MODULES

Chapter 57
INTRODUCTION TO MODULES

57.1 Overview

This section of the Handbook describes the Magma facilities for linear algebra and module theory. Since this topic is absolutely fundamental for much of algebra, it is important that the reader understand how linear algebra is presented in Magma. The structures covered under this heading include:

(i) Vector spaces;

(ii) Inner product spaces;

(iii) Modules defined over any ring or algebra

(iv) $R[G]$-modules, where R is a ring and G is a group;

(v) Linear transformations and R-module homomorphisms.

Although vector spaces are, of course, subsumed under general modules, we present a separate treatment of them, firstly because of their importance and secondly because their theory is somewhat cleaner than that of a general module. Magma users who are unfamiliar with the language of module theory will find a self-contained treatment of the vector space machinery in Chapter 28.

In the Magma universe, rectangular matrices are regarded as forming a module (actually a bimodule). We shall regard a rectangular matrix as the concrete realization of a linear transformation or R-module homomorphism. Thus, an $m \times n$ matrix over a ring R is considered to be an element of the module $\mathrm{Hom}_R(M, N)$. Reflecting the dual nature of matrices, the $\mathrm{Hom}_R(M, N)$ operations include the standard module-theoretic operations as well as operations that interpret an element of $\mathrm{Hom}_R(M, N)$ as a homomorphism.

57.2 General Modules

A module M is always regarded as a submodule or quotient module of the free module $S^{(n)}$, for some ring or algebra S. The types of module that are definable in the system fall into three classes:

(a) **Abstract Modules**: Given a ring R, a set M and a mapping $\phi : R \times M \to M$, the pair (M, ϕ) will be referred to as an *abstract* R-module. Because of the very general nature of this construction, only the basic arithmetic operations may be applied to modules of this type.

(b) **Modules with Scalar Action**: Given a general ring R, an R-module with *scalar action* is a submodule or quotient module of the free R-module $R^{(n)}$, where the action is that of ring multiplication in R.

(c) **Modules with Matrix Action**: Let R be a PIR and suppose S is a R-algebra. Thus there exists a ring homomorphism $\phi : R \to S$, and so S is a (left) R-module with the R-action defined by $r * s = \phi(r) * s$. Indeed, any S-module M is a (left) S-module with action defined by $r * m = \phi(r) * m$. Furthermore, if $\phi(R)$ lies in the centre of S, then S acts on M as a ring of R-module endomorphisms. Consequently, M is an S-module. We take M to be the free R-module $R^{(n)}$, and so the action of S on M is given by the action of a subring of $M_n(R)$ on M. Thus, given an R-algebra S, an S-module of the form $M = R^{(n)}$ may be specified by giving M together with a homomorphism of S into $M_n(R)$.

57.3 The Presentation of Submodules

Let N be a submodule of the module $M = R^{(m)}$. For simplicity, assume that N is free, and has dimension n where $n < m$. There are two ways in which N may be viewed:

(a) As a submodule embedded in M. Thus, though N is a module of dimension n, its elements are regarded as elements of M.

(b) As the module $R^{(n)}$ represented on a reduced basis, together with a morphism ϕ defining the inclusion of N into M.

The presentation (a) is the usual way submodules are regarded in elementary linear algebra. However, this presentation is inconvenient for more advanced applications. For example, many of the major functions available for studying an $R[G]$-module N expect that N is given relative to a reduced basis. We shall refer to a submodule presentation (a) as the *embedded presentation*, and (b) as the *reduced presentation*.

To provide the user with the maximum flexibility, Magma supports both forms of submodule presentation for the important classes of modules. Usually, a module calculation commences with the definition of one or two modules from which further modules are created by the operations of forming submodules, quotient modules and extensions. Let us call these latter modules *descendants* of the original module. Magma provides parallel creation functions which allow the user to choose the form of submodule presentation. Once that choice has been made, all descendants of the initial module(s) will follow the same presentation convention.

The module creation functions that select the embedded form are usually of the form *Qualifier*`Space`, while those that adopt the standard form are usually of the form *Qualifier*`Module`. Thus in the case of vector spaces the function `KSpace(K, n)` constructs the n-dimensional vector space over the field K, where submodules are to be presented in embedded form. On the other hand, `RModule(K, n)` constructs the n-dimensional vector space over the field K, where submodules are to be presented in reduced form.

58 FREE MODULES

Chapter 58
FREE MODULES

58.1 Introduction

58.1.1 Free Modules

This chapter describes the facilities provided for modules in the following representations:

Tuple Modules: Modules whose elements are n-tuples over a fixed ring R, i.e., modules $R^{(n)}$;

Matrix Modules: Modules whose elements are homomorphisms of modules, i.e., $Hom_R(M, N)$. The elements of these modules are $m \times n$ matrices over the ring R;

The ring R acts on the right of the module element by scalar multiplication. If R is not an Euclidean Domain then, currently, only arithmetic with vectors is supported. In particular, the ability to work with submodules and quotient modules is restricted to situations where R is either a field or Euclidean Domain.

In the first part of the chapter we describe the operations that apply to modules generally, while in the second half we describe the creation of modules $Hom_R(M, N)$ together with the operations that are specific to them. Insofar as elementary module-theoretic operations are concerned, there is no real difference between tuple modules and matrix modules except for the input and display of elements. Many special operations provided for matrices are described in the chapter on matrices.

The reader is referred to the chapter on vector spaces for descriptions of the extensive functionality provided for modules over fields.

58.1.2 Module Categories

The family of all finitely generated modules over a given ring R forms a category, while the set of all finitely generated modules forms a family of categories indexed by the ring R. In this family of categories, objects are modules and the morphisms are module homomorphisms. The category name for modules is `ModRng`. We distinguish the following subcategories of `ModRng`:

`ModTupFld` - the category of modules of n-tuples over a field;

`ModMatFld` - the category of modules of $m \times n$ matrices over a field;

`ModTupEd` - the category of modules of n-tuples over an euclidean domain.

`ModTupRng` - the category of modules of n-tuples over a ring;

`ModMatRng` - the category of modules of $m \times n$ matrices over a ring;

58.1.3 Presentation of Submodules

Let N be a free submodule of the R-module M. We have two alternative ways of presenting N. Firstly, we can present it on a set of generators that are elements of M; we call such a presentation an *embedded presentation.* Alternatively, given that N has rank r, we can present it as the module $S^{(r)}$, with appropriate action induced from the action of R on M. We call this presentation of N a *reduced presentation.*

The user can control the method of submodule presentation at the time of creation of an initial module through selection of the appropriate creation function. Thus, the function `RModule` will create a module with the convention that it and all its submodules and quotient modules will have their submodules presented in reduced form. The use of `RSpace`, on the other hand, signifies that submodules are to be presented in embedded form.

58.1.4 Notation

Throughout this chapter, R will denote a ring (possibly a field) while K will denote a field. The letters M and N will denote modules, while U and V will denote vector spaces.

58.2 Definition of a Module

58.2.1 Construction of Modules of n-tuples

```
RSpace(R, n)
```

```
RModule(R, n)
```

Given a ring R and a non-negative integer n, create the free right R-module $R^{(n)}$, consisting of all n-tuples over R. The module is created with the standard basis, $e_1, \ldots, e_n$, where e_i $(i = 1, \ldots, n)$ is the vector containing a 1 in the i-th position and zeros elsewhere.

The function `RModule` creates a module in reduced mode while `RSpace` creates a module in embedded mode.

```
RSpace(R, n, F)
```

Given a ring R, a non-negative integer n and a square $n \times n$ symmetric matrix F, create the free right R-module $R^{(n)}$ (in embedded form), with inner product matrix F. This is the same as `RSpace(R, n)`, except that the functions `Norm` and `InnerProduct` (see below) will be with respect to the inner product matrix F.

Example H58E1

We construct the module consisting of 6-tuples over the integers.

```
> Z := IntegerRing();
> M := RModule(Z, 6);
> M;
RModule M of dimension 6 with base ring Integer Ring
```

58.2.2 Construction of Modules of $m \times n$ Matrices

`RMatrixSpace(R, m, n)`

The module comprising all $m \times n$ matrices over the ring R.

58.2.3 Construction of a Module with Specified Basis

`RModuleWithBasis(Q)`

`RSpaceWithBasis(Q)`

`RSpaceWithBasis(a)`

Given a sequence Q (or matrix a) of k independent vectors each lying in a module M, construct the submodule of M of dimension k whose basis is Q (or the rows of a). The basis is echelonized internally but all functions which depend on the basis of the space (e.g. `Coordinates`) will use the given basis.

`RMatrixSpaceWithBasis(Q)`

The module of $m \times n$ matrices whose basis is given by the linearly independent matrices of the sequence Q.

58.3 Accessing Module Information

`M . i`

Given an R-module M and a positive integer i, return the i-th generator of M. The integer i must lie in the range $[1, r]$, where r is the number of generators for M.

`CoefficientRing(M)`

`BaseRing(M)`

`CoefficientRing(M)`

`BaseRing(M)`

`CoefficientField(M)`

`BaseField(M)`

Given an R-module M which is defined as a submodule of $S^{(n)}$, return the ring S.

`Generators(M)`

The generators for the R-module M, returned as a set.

`OverDimension(M)`

Given an R-module M which is an embedded submodule of the module $S^{(n)}$, return n.

```
OverDimension(u)
```

Given an element u of an embedded submodule of the module $S^{(n)}$, return n.

```
Moduli(M)
```

The column moduli of the module M over a euclidean domain.

```
Parent(u)
```

Given an element u belonging to the R-module M, return M.

```
Generic(M)
```

Given an R-module M which is a submodule of the module $R^{(n)}$, return the module $R^{(n)}$ as an R-module.

58.4 Standard Constructions

Given one or more existing modules, various standard constructions are available to construct new modules.

58.4.1 Changing the Coefficient Ring

```
ChangeRing(M, S)
```

Given a module M with base ring R, together with a ring S, construct the module N with base ring S obtained by coercing the components of elements of M into N, together with the homomorphism from M to N.

```
ChangeRing(M, S, f)
```

Given a module M with base ring R, together with a ring S, and a homomorphism $f : R \to S$, construct the module N with base ring S obtained by mapping the components of elements of M into N by f, together with the homomorphism from M to N.

```
ChangeUniverse(~x, R)
```

Change the coefficient ring of x to be R.

58.4.2 Direct Sums

```
DirectSum(M, N)
```

Given R-modules M and N, construct the direct sum D of M and N as an R-module. The embedding maps from M into D and from N into D respectively, and the projection maps from D onto M and from D onto N respectively are also returned.

```
DirectSum(Q)
```

Given a sequence Q of R-modules, construct the direct sum D of these modules. The embedding maps from each of the elements of Q into D and the projection maps from D onto each of the elements of Q are also returned.

58.5 Construction of Elements

elt< M | a_1, ..., a_n >

Given a module M with base module $S^{(n)}$, and elements $a_1, \ldots, a_n$ belonging to S, construct the element $m = (a_1, \ldots, a_n)$ of M. Note that if m is not an element of M, an error will result.

`M ! Q`

Given the module M with base module $S^{(n)}$, and elements $a_1, \ldots, a_n$ belonging to S, construct the element $m = (a_1, \ldots, a_n)$ of M. Note that if m is not an element of M, an error will result.

`CharacteristicVector(M, S)`

Given a submodule M of the module $R^{(n)}$ together with a set S of integers lying in the interval $[1, n]$, return the characteristic number of S as a vector of R.

`Zero(M)`

`M ! 0`

The zero element for the R-module M.

`Random(M)`

Given a module M defined over a finite ring or field, return a random vector.

Example H58E2

We create the module of 4-tuples over the polynomial ring $\mathbf{Z}[x]$ and define various elements.

```
> P<x> := PolynomialRing(IntegerRing());
> M := RModule(P, 4);
> a := elt< M | 1+x, -x, 2+x, 0 >;
> a;
(x + 1    -x x + 2     0)
> b := M ! [ 1+x+x^2, 0, 1-x^7, 2*x ];
> b;
(x^2 + x + 1           0    -x^7 + 1         2*x)
> zero := M ! 0;
> zero;
(0 0 0 0)
```

58.5.1 Deconstruction of Elements

```
ElementToSequence(u)
```

```
Eltseq(u)
```

Given an element u belonging to the R-module M, return u in the form of a sequence Q of elements of R. Thus, if u is an element of $R^{(n)}$, then $Q[i] = u[i], 1 \leq i \leq n$, while if u is an element of $R^{(m \times n)}$, then $Q[(i-1)n+j] = u[i,j], 1 \leq i \leq m, 1 \leq j \leq n$.

58.5.2 Operations on Module Elements

58.5.2.1 Arithmetic

```
u + v
```

Sum of the elements u and v, where u and v lie in the same R-module M.

```
-u
```

Additive inverse of the element u.

```
u - v
```

Difference of the elements u and v, where u and v lie in the same R-module M.

```
x * u
```

Given an element x belonging to a ring R, and an element u belonging to the left R-module M, return the (left) scalar product $x * u$ as an element of M.

```
u * x
```

Given an element x belonging to a ring R, and an element u belonging to the right R-module M, return the (right) scalar product $u * x$ as an element of M.

```
u / x
```

Given a non-zero element x belonging to a field K, and an element u belonging to the right K-module M, return the scalar product $u * (1/x)$ as an element of M.

58.5.2.2 Indexing

```
u[i]
```

Given an element u belonging to a submodule M of the R-module $R^{(n)}$ and a positive integer i, $1 \leq i \leq n$, return the i-th component of u (as an element of the ring R).

```
u[i] := x
```

Given an element u belonging to a submodule M of the R-module $T = R^{(n)}$, a positive integer i, $1 \leq i \leq n$, and an element x of the ring R, redefine the i-th component of u to be x. The parent of u is changed to T (since the modified element u need not lie in M).

58.5.2.3 Normalization

```
Normalize(u)
```
```
Normalise(u)
```

The element u must belong to an R-module, where R is either a field, the ring of integers or a univariate polynomial ring over a field. Assume that the vector u is non-zero. If R is a field then `Normalize` returns $\frac{1}{a} * u$, where a is the first non-zero component of u. If R is the ring of integers, `Normalize` returns $\epsilon * u$, where ϵ is $+1$ if the first non-zero component of u is positive, and -1 otherwise. If R is the polynomial ring $K[x]$, K a field, then `Normalize` returns $\frac{1}{a} * u$, where a is the leading coefficient of the first non-zero (polynomial) component of u. If u is the zero vector, it is returned as the value of this function.

```
Rotate(u, k)
```

Given a vector u, return the vector obtained from u by rotating by k coordinate positions.

```
Rotate(~u, k)
```

Given a vector u, destructively rotate u by k coordinate positions.

Example H58E3

We illustrate the use of the arithmetic operators for module elements by applying them to elements of the module of 4-tuples over the polynomial ring $\mathbf{Z}[x]$.

```
> P<x> := PolynomialRing(IntegerRing());
> M := RModule(P, 4);
> a :=  M ! [ 1+x, -x, 2+x, 0 ];
> b := M ! [ 1+x+x^2, 0, 1-x^7, 2*x ];
> a + b;
(x^2 + 2*x + 2    -x    -x^7 + x + 3    2*x)
> -a;
(-x - 1      x -x - 2      0)
> a - b;
(       -x^2          -x x^7 + x + 1        -2*x)
> (1-x + x^2)*a;
(x^3 + 1    -x^3 + x^2 - x    x^3 + x^2 - x + 2    0)
> a*(1-x);
(    -x^2 + 1      x^2 - x -x^2 - x + 2            0)
> a[3];
x + 2
> a[3] := x - 2;
> a;
(x + 1    -x x - 2     0)
> ElementToSequence(a - b);
[
    -x^2,
    -x,
    x^7 + x - 3,
    -2*x
]
> Support(a);
{ 1, 2, 3 }
```

58.5.3 Properties of Vectors

`IsZero(u)`

Returns `true` if the element u of the R-module M is the zero element.

`Depth(v)`

The index of the first non-zero entry of the vector v (0 if none such).

`Support(u)`

A set of integers giving the positions of the non-zero components of the vector u.

`Weight(u)`

The number of non-zero components of the vector u.

58.5.4 Inner Products

`(u, v)`

`InnerProduct(u, v)`

Return the inner product of the vectors u and v with respect to the inner product defined on the space. If an inner product matrix F is given when the space is created, then this is defined to be $u \cdot F \cdot v^{tr}$. Otherwise, this is simply $u \cdot v^{tr}$.

`Norm(u)`

Return the norm product of the vector u with respect to the inner product defined on the space. If an inner product matrix F is given when the space is created, then this is defined to be $u \cdot F \cdot u^{tr}$. Otherwise, this is simply $u \cdot u^{tr}$.

58.6 Bases

The application of the functions in this section is restricted either to vector spaces or to torsion-free modules over a Euclidean Domain.

For a full description of the basis functions for a module defined over a field, the reader is referred to the chapter on vector spaces.

```
Basis(M)
```

The current basis for the free R-module M, R an ED, returned as a sequence of module elements.

```
Rank(M)
```

The rank of the free R-module M.

```
Coordinates(M, u)
```

Given a vector u belonging to the rank r free R-module M, R an Euclidean Domain, with basis $u_1, \ldots, u_r$, return a sequence $[a_1, \ldots, a_r]$ giving the coordinates of u relative to the M-basis: $u = a_1 * u_1 + \cdots + a_r * u_r$.

58.7 Submodules

58.7.1 Construction of Submodules

Submodules may be defined for any type of module. However, functions that depend upon membership testing are only implemented for modules over Euclidean Domains (EDs). The conventions defining the presentations of submodules are as follows:

If M has been created using the function `RSpace`, then every submodule of M is given in terms of a generating set consisting of elements of M, i.e. by means of an embedded generating set.

If M has been created using the function `RModule`, then every submodule of M is given in terms of a reduced basis.

```
sub< M | L >
```

Given an R-module M, construct the submodule N generated by the elements of M specified by the list L. Each term L_i of the list L must be an expression defining an object of one of the following types:

(a) A sequence of n elements of R defining an element of M;

(b) A set or sequence whose terms are elements of M;

(c) A submodule of M;

(d) A set or sequence whose terms are submodules of M.

The generators stored for N consist of the elements specified by terms L_i together with the stored generators for submodules specified by terms of L_i. Repetitions of an element and occurrences of the zero element are removed (unless N is trivial).

The constructor returns the submodule N and the inclusion homomorphism $f : N \to M$.

Example H58E4

We construct a submodule of the 4-dimensional vector space over the field of rational function $F[x]$, where F is $\mathbf{F}_5$.

```
> P := PolynomialRing(GF(5));
> R<x> := FieldOfFractions(P);
> M := RSpace(R, 4);
> N := sub< M | [1, x, 1-x, 0], [1+2*x-x^2, 2*x, 0, 1-x^4 ] >;
> N;
Vector space of degree 4, dimension 2 over Field of Fractions in x over
Univariate Polynomial Algebra over GF(5)
Generators:
(1    x    4*x + 1    0)
(4*x^2 + 2*x + 1    2*x    0    4*x^4 + 1)
Echelonized basis:
(1  0  3/(x + 4)  (x^3 + x^2 + x + 1) / (x + 4))
(0  1  (4*x^2 + 2*x + 1) / (x^2 + 4*x)  (4*x^3 + 4*x^2 + 4*x + 4) / (x^2 + 4*x))
```

58.7.2 Operations on Submodules

58.7.3 Membership and Equality

The following operations are only available for submodules of $R^{(n)}$, $\mathrm{Hom}_R(M,N)$ and $R[G]$, where R is a Euclidean Domain. If the modules involved are $R[G]$-modules, the operators refer to the underlying R-module.

`u in M`

Returns `true` if the element u lies in the R-module M, where u and M belong to the same R-module.

`u notin M`

Returns `true` if the element u does not lie in the R-module M, where u and M belong to the same R-module.

`N subset M`

Returns `true` if the R-module N is contained in the R-module M, where M and N belong to a common R-module.

`N notsubset M`

Returns `true` if the R-module N is not contained in the R-module M, where M and N belong to a common R-module.

`M eq N`

Returns `true` if the R-modules N and M are equal, where N and M belong to a common R-module.

```
M ne N
```

Returns `true` if the R-modules N and M are not equal, where N and M belong to a common R-module.

58.7.4 Operations on Submodules

The following operations are only available for submodules of $R^{(n)}$, $\mathrm{Hom}_R(M,N)$ and $R[G]$, where R is a Euclidean Domain. If the modules involved are $R[G]$-modules, the operators refer to the underlying R-module.

```
M + N
```

Sum of the submodules M and N, where M and N belong to a a common R-module.

```
M meet N
```

Intersection of the submodules M and N, where M and N belong to a common R-module.

58.8 Quotient Modules

58.8.1 Construction of Quotient Modules

```
quo< M | L >
```

Given an R-module M, construct the quotient module $P = M/N$, where N is the submodule generated by the elements of M specified by the list L. Each term L_i of the list L must be an expression defining an object of one of the following types:

(a) A sequence of n elements of R defining an element of M;

(b) A set or sequence whose terms are elements of M;

(c) A submodule of M;

(d) A set or sequence whose terms are submodules of M.

The generators constructed for N consist of the elements specified by terms L_i together with the stored generators for submodules specified by terms of L_i.

The constructor returns the quotient module P and the natural homomorphism $f : M \to P$.

58.9 Homomorphisms

Throughout this part of the chapter, when discussing the set of all R-homomorphisms from the R-module M into the R-module N, it will be assumed that R is a **commutative** ring. We further assume that M and N are free R-modules and that bases for these modules are present. The module $\mathrm{Hom}_R(M, N)$ will be identified with the module of $m \times n$ matrices over R. Thus, an element of $\mathrm{Hom}_R(M, N)$ is represented as a matrix relative to the bases of the generic modules corresponding to M and N. For this reason, we will refer to these modules as *matrix modules*.

We remind the reader that submodules of $\mathrm{Hom}_R(M, N)$ are always presented in embedded form. If the user wishes to have submodules presented in reduced form then he/she should use the natural isomorphism between $R^{(m\times n)}$ and $R^{(mn)}$.

It should be noted that essentially operation defined for tuple modules, their elements and submodules applies to matrix modules. Thus, all of the operations discussed earlier in this chapter apply to matrix modules.

The modules M and N may themselves be matrix modules. In this case, the resulting matrix module has either a right or left action and an element belonging to it transforms a (homomorphism) element of M into a (homomorphism) element of N.

The function `Reduce` may be used to construct for a matrix module H the matrix module H' equivalent to H whose elements are with respect to the actual bases of the domain and codomain of elements of H (not the generic bases of the domain and codomain).

58.9.1 $\mathrm{Hom}_R(M, N)$ for R-modules

```
Hom(M, N)
```

If M is the tuple module $R^{(m)}$ and N is the tuple module $R^{(n)}$, create the module $\mathrm{Hom}_R(M, N)$ as the (R, R)-bimodule $R^{(m\times n)}$, represented as the set of all $m \times n$ matrices over R. The module is created with the standard basis, $\{E_{ij} \mid i = 1\ldots, m, j = 1\ldots, n\}$, where E_{ij} is the matrix having a 1 in the (i, j)-th position and zeros elsewhere.

```
RMatrixSpace(R, m, n)
```

Given a ring R and positive integers m and n, construct $H = \mathrm{Hom}(M, N)$, where $M = R^{(m)}$ and $N = R^{(n)}$, as the free (R, R)-bimodule $R^{(m\times n)}$, consisting of all $m \times n$ matrices over R. The module is created with the standard basis, $\{E_{ij} \mid i = 1\ldots, m, j = 1\ldots, m\}$. Note that the modules M and N are created by this function and may be accessed as `Domain(H)` and `Codomain(H)`, respectively.

Example H58E5

We construct the vector spaces V and W of dimensions 3 and 4, respectively, over the field of two elements and then define M to be the module of homomorphisms from V into W.

```
> F2 := GaloisField(2);
> V  := VectorSpace(F2, 3);
> W  := VectorSpace(F2, 4);
> M  := Hom(V, W);
> M;
Full KMatrixSpace of 3 by 4 matrices over GF(2)
```

58.9.2 $\mathrm{Hom}_R(M, N)$ for Matrix Modules

`Hom(M, N, "right")`

Suppose M is a matrix module over the coefficient ring R whose elements are a by b matrices and have domain D and codomain C. Suppose also that N is a matrix module over the coefficient ring R whose elements are a by c matrices and have domain D and codomain C'. Then the homomorphism module $H = \mathrm{Hom}(M, N)$ with right multiplication action exists and consists of all b by c matrices over R which multiply an element of M on the right to yield an element of N. This function constructs H explicitly. The domain of elements of H is then M and the codomain of elements of H is N and the elements are b by c matrices over R which multiply an element of M on the right to yield an element of N. Note that if M and N are proper submodules of their respective generic modules, then H may be a proper submodule of its generic module, and the correct basis of H will be explicitly constructed.

`Hom(M, N, "left")`

Suppose M is a matrix module over the coefficient ring R whose elements are a by c matrices and have domain D and codomain C. Suppose also that N is a matrix module over the coefficient ring R whose elements are b by c and have domain D' and codomain C. Then the homomorphism module $H = \mathrm{Hom}(M, N)$ with left multiplication action exists and consists of all b by a matrices over R which multiply an element of M on the left to yield an element of N. This function constructs H explicitly. The domain of elements of H is then M and the codomain of elements of H is N and the elements are b by a matrices over R which multiply an element of M on the right to yield an element of N. Note that if M and N are proper submodules of their respective generic modules, then H may be a proper submodule of its generic module, and the correct basis of H will be explicitly constructed.

Example H58E6

We construct two homomorphism modules H_1 and H_2 over $\mathbf{Q}$ and then the homomorphism module $H = \mathrm{Hom}(H_1, H_2)$ with right matrix action.

```
> Q := RationalField();
> H1 := sub<RMatrixSpace(Q, 2, 3) | [1,2,3, 4,5,6], [0,0,1, 1,3,3]>;
> H2 := sub<RMatrixSpace(Q, 2, 4) | [6,5,7,1, 15,14,16,4], [0,0,0,0, 1,2,3,4]>;
> H := Hom(H1, H2, "right");
> H: Maximal;
KMatrixSpace of 3 by 4 matrices and dimension 1 over Rational Field
```

```
Echelonized basis:

[   1    2    3    4]
[-1/2   -1 -3/2   -2]
[   0    0    0    0]

> H1.1 * H.1;
[  0   0   0   0]
[3/2   3 9/2   6]
> H1.1 * H.1 in H2;
true
> Image(H.1): Maximal;
KMatrixSpace of 2 by 4 matrices and dimension 1 over Rational Field
Echelonized basis:

[0 0 0 0]
[1 2 3 4]

> Kernel(H.1): Maximal;
KMatrixSpace of 2 by 3 matrices and dimension 1 over Rational Field
Echelonized basis:

[ 1  2  6]
[ 7 14 15]

> H1 := sub<RMatrixSpace(Q,2,3) | [1,2,3, 4,5,6]>;
> H2 := sub<RMatrixSpace(Q,3,3) | [1,2,3, 5,7,9, 4,5,6]>;
> H := Hom(H1, H2, "left");
> H: Maximal;
KMatrixSpace of 3 by 2 matrices and dimension 1 over Rational Field
Echelonized basis:

[1 0]
[1 1]
[0 1]
> Image(H.1);
KMatrixSpace of 3 by 3 matrices and dimension 1 over Rational Field
> Kernel(H.1);
KMatrixSpace of 2 by 3 matrices and dimension 0 over Rational Field
```

58.9.3 Modules $\mathrm{Hom}_R(M, N)$ with Given Basis

`RMatrixSpaceWithBasis(Q)`

Given a sequence Q of k independent matrices each lying in a matrix space $H = \mathrm{Hom}(M, N)$, where $M = R^{(m)}$ and $N = R^{(n)}$, construct the subspace of H of dimension k whose basis is Q. The basis is echelonized internally but all functions which depend on the basis of the matrix space (e.g. `Coordinates`) will use the given basis Q.

`KMatrixSpaceWithBasis(Q)`

Given a sequence Q of k independent matrices each lying in a matrix space $H = \mathrm{Hom}(M, N)$, where $M = K^{(m)}$ and $N = K^{(n)}$, with K a field, construct the subspace of H of dimension k whose basis is Q. The basis is echelonized internally but all functions which depend on the basis of the matrix space (e.g. `Coordinates`) will use the given basis Q.

58.9.4 The Endomorphsim Ring

`EndomorphismAlgebra(M)`

If M is the free R-module $R^{(m)}$, create the matrix algebra $\mathrm{Mat}_m(R)$. The algebra is created with the standard basis, $\{E_{ij} \mid i = 1 \ldots, m, j = 1 \ldots, m\}$, where E_{ij} is the matrix having a 1 in the (i, j)-th position and zeros elsewhere.

Example H58E7

We construct the endomorphism ring of the 4-dimensional vector space over the rational field.

```
> Q  := RationalField();
> R4 := RModule(Q, 4);
> M  := EndomorphismAlgebra(R4);
> M;
Full Matrix Algebra of degree 4 over Rational Field
```

58.9.5 The Reduced Form of a Matrix Module

```
Reduce(H)
```

Suppose H is a matrix module whose elements have domain A and codomain B. Suppose first that A and B are tuple modules (R-spaces, R-modules, or RG-modules) and that A has degree a and dimension d while B has degree b and dimension e. (For the reduced cases (R-modules or RG-modules), a equals d and b equals e). The elements of H have the natural representation with respect to the standard embedded basis of the generic modules of A and B. Thus H has degree a by b. So one can multiply a 1 by a vector of A directly by an element h of H (in the natural matrix way) to get a 1 by b vector of B. Now suppose A and B are in the embedded form (R-space) and h is in H. Then h is an a by b matrix but there is a corresponding d by e matrix h' which gives the same transformation of h from A to B but is *with respect to the bases of* A *and* B. We call h' the *reduced* form of h. Also, there is the *reduced module* H' corresponding to H. This function constructs the reduced module H' corresponding to H, together with the epimorphism f from H onto H'. Note that if A and B are in reduced form, then H' is the same as H.

Suppose secondly that A and B are matrix modules themselves. Suppose $A = \mathrm{Hom}(D_1, C_1)$, $B = \mathrm{Hom}(D_1, C_2)$, and $H = \mathrm{Hom}(A, B)$ with the right multiplication action. Suppose also that A has degree r by s and dimension d while B has degree r by t, and dimension e. Then H would have degree s by t so an element h of H would be s by t and would multiply a r by s element of A on the right to yield an r by t element of B. Then the reduced matrix h' corresponding to a matrix h of H would be a d by e matrix corresponding to the bases of A and B. This function similarly constructs the reduced module H' corresponding to H, together with the epimorphism f from H onto H'. Note also that in this case the domain and codomains of H' are the generic R-spaces (tuple modules) corresponding to A (of dimension d) and B (of dimension e). Similarly, for the left multiplication action there is the corresponding reduced module constructed in the obvious way.

Note also that the kernel of the epimorphism f is the submodule of H which consists of all matrices which transform all elements of A to the zero element of B.

Example H58E8

We demonstrate the function `Reduce` for a homomorphism module from one vector space to another.

```
> V1 := sub<VectorSpace(GF(3), 3) | [1,0,1], [0,1,2]>;
> V2 := sub<VectorSpace(GF(3), 4) | [1,1,0,2], [0,0,1,2]>;
> H := Hom(V1, V2);
> H;
KMatrixSpace of 3 by 4 matrices and dimension 8 over GF(3)
> R, f := Reduce(H);
> R;
Full KMatrixSpace of 2 by 2 matrices over GF(3)
> H.1;
```

```
[1 0 0 0]
[0 1 0 2]
[0 1 0 2]
> f(H.1);
[1 0]
[0 0]
> V1.1;
(1 0 1)
> V1.1 * H.1;
(1 1 0 2)
> Coordinates(V2, V1.1 * H.1);
[ 1, 0 ]
> Coordinates(V2, V1.2 * H.1);
[ 0, 0 ]
> Kernel(f): Maximal;
KMatrixSpace of 3 by 4 matrices and dimension 4 over GF(3)
Echelonized basis:

[1 0 0 0]
[2 0 0 0]
[2 0 0 0]

[0 1 0 0]
[0 2 0 0]
[0 2 0 0]

[0 0 1 0]
[0 0 2 0]
[0 0 2 0]

[0 0 0 1]
[0 0 0 2]
[0 0 0 2]
> R.1@@f;
[1 0 0 0]
[0 1 0 2]
[0 1 0 2]
```

Example H58E9

We demonstrate the function `Reduce` for a homomorphism module from one homomorphism module to another. Note that the reduced module has the same dimension as the original module but larger degrees!

```
> V1 := VectorSpace(GF(3), 2);
> V2 := VectorSpace(GF(3), 3);
> V3 := VectorSpace(GF(3), 4);
> H1 := Hom(V1, V2);
> H2 := Hom(V1, V3);
> H := Hom(H1, H2, "right");
> H1;
Full KMatrixSpace of 2 by 3 matrices over GF(3)
> H2;
Full KMatrixSpace of 2 by 4 matrices over GF(3)
> H;
Full KMatrixSpace of 3 by 4 matrices over GF(3)
> R,f := Reduce(H);
> R;
KMatrixSpace of 6 by 8 matrices and dimension 12 over GF(3)
> X := H.1;
> X;
[1 0 0 0]
[0 0 0 0]
[0 0 0 0]
> f(X);
[1 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
> Domain(X);
Full KMatrixSpace of 2 by 3 matrices over GF(3)
> Domain(f(X));
Full Vector space of degree 6 over GF(3)
> Image(X): Maximal;
KMatrixSpace of 2 by 4 matrices and dimension 2 over GF(3)
Echelonized basis:

[1 0 0 0]
[0 0 0 0]

[0 0 0 0]
[1 0 0 0]

> Image(f(X));
Vector space of degree 8, dimension 2 over GF(3)
Echelonized basis:
(1 0 0 0 0 0 0 0)
(0 0 0 0 1 0 0 0)
> Kernel(X): Maximal;
KMatrixSpace of 2 by 3 matrices and dimension 4 over GF(3)
Echelonized basis:

[0 1 0]
[0 0 0]
```

```
[0 0 1]
[0 0 0]

[0 0 0]
[0 1 0]

[0 0 0]
[0 0 1]

> Kernel(f(X)): Maximal;
Vector space of degree 6, dimension 4 over GF(3)
Echelonized basis:
(0 1 0 0 0 0)
(0 0 1 0 0 0)
(0 0 0 0 1 0)
(0 0 0 0 0 1)
```

58.9.6 Construction of a Matrix

`M ! Q`

Given the matrix bimodule M over the ring R, and the sequence $Q = [a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn}]$ whose terms are elements of the ring R, construct the $m \times n$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

as an element of M. In the context of the `sub` or `quo` constructors the coercion clause `M !` may be omitted.

Example H58E10

We create the 4×4 Hilbert matrix $h4$ as an element of the endomorphism ring of the 4-dimensional vector space over the rational field.

```
> Q  := RationalField();
> R4 := RModule(Q, 4);
> M  := EndomorphismAlgebra(R4);
> h4 := M ! [ 1/i :  i in [1 .. 16 ] ];
> h4;
[   1  1/2  1/3  1/4]
[ 1/5  1/6  1/7  1/8]
[ 1/9 1/10 1/11 1/12]
[1/13 1/14 1/15 1/16]
```

58.9.7 Element Operations

All operations that apply to elements of tuple modules also apply to elements of matrix modules. Here, we confine our discussion to those operations which are special to matrix modules.

Throughout this section, M is a submodule of $R^{(m)}$, N is a submodule of $R^{(n)}$ and a is a homomorphism belonging to $\mathrm{Hom}_R(M, N)$, where R is a Euclidean Domain.

See also the chapter on general matrices for many other functions applicable to matrices (e.g., `EchelonForm`).

```
u * a
```

```
a(u)
```

Given an element u belonging to the module M, return the image of u under the homomorphism a as an element of the module N.

```
a * b
```

Given a homomorphism a belonging to a submodule of $\mathrm{Hom}(M, N)$, and a homomorphism b belonging to a submodule of $\mathrm{Hom}(N, P)$, return the composition of the homomorphisms a and b as an element of $\mathrm{Hom}(M, P)$. Note that if $\mathrm{Hom}(M, P)$ does not already exist, it will be created.

```
a ^ -1
```

Given a homomorphism a belonging to a submodule of $\mathrm{Hom}(M, N)$ with M and N having the same dimension, return the inverse of a as an element of $\mathrm{Hom}(N, M)$.

```
Codomain(S)
```

Given a submodule S of the module $\mathrm{Hom}(M, N)$, return the module N.

```
Codomain(a)
```

The codomain N of the homomorphism a belonging to $\mathrm{Hom}(M, N)$.

```
Cokernel(a)
```

The cokernel for the homomorphism a belonging to the module $\mathrm{Hom}(M, N)$.

```
Domain(S)
```

The domain M of the submodule S belonging to the module $\mathrm{Hom}(M, N)$.

```
Domain(a)
```

The domain M of the homomorphism a belonging to the module $\mathrm{Hom}(M, N)$.

```
Image(a)
```

The image of the homomorphism a belonging to the module $H = \mathrm{Hom}(M, N)$, returned as a submodule of N. Note that if the domain and codomain of a are matrix modules themselves, the image will be with respect to the appropriate action (right or left).

`Kernel(a)`

`NullSpace(a)`

The kernel of the homomorphism a belonging to the module $\mathrm{Hom}(M, N)$, returned as a submodule of M. Note that if the domain and codomain of a are matrix modules themselves, the kernel will be with respect to the appropriate action (right or left).

`Morphism(M, N)`

Assuming the R-module M was created as a submodule of the module N, return the matrix defining the inclusion homomorphism $\phi : M \to N$ as an element of $\mathrm{Hom}_R(M, N)$. Thus ϕ gives the correspondence between elements of M (represented with respect to the standard basis of M) and elements for N.

`Rank(a)`

The dimension of the image of the homomorphism a, i.e. the rank of a.

`IsBijective(a)`

Returns `true` if the homomorphism a belonging to the module $\mathrm{Hom}(M, N)$ is a bijective mapping.

`IsInjective(a)`

Returns `true` if the homomorphism a belonging to the module $\mathrm{Hom}(M, N)$ is an injective mapping.

`IsSurjective(a)`

Returns `true` if the homomorphism a belonging to the module $\mathrm{Hom}_R(M, N)$ is a surjective mapping.

Example H58E11

We illustrate some of these operations in the context of the module $\mathrm{Hom}_R(M, N)$, where M and N are, respectively, the 4-dimensional and 3-dimensional vector spaces over $GF(8)$.

```
> K<w> := GaloisField(8);
> V3 := VectorSpace(K, 3);
> V4 := VectorSpace(K, 4);
> M := Hom(V4, V3);
> A := M ! [1, w, w^5, 0,  w^3, w^4, w, 1,  w^6, w^3, 1, w^4 ];
> A;
[  1   w w^5]
[  0 w^3 w^4]
[  w   1 w^6]
[w^3   1 w^4]
> Rank(A);
3
> Image(A);
```

```
Full Vector space of degree 3 over GF(2^3)
> Kernel(A);
Vector space of degree 4, dimension 1 over GF(2^3)
Echelonized basis:
(  1 w^5   1   1)
> Cokernel(A);
Vector space of degree 3, dimension 0 over GF(2^3)
```

59 MODULES OVER DEDEKIND DOMAINS

Chapter 59
MODULES OVER DEDEKIND DOMAINS

59.1 Introduction

Since the structure theory for modules over arbitrary orders (which are in general not Dedekind domains) is very unsatisfactory, modules over orders in Magma are always modules over some maximal order of a number field or function field, they form a magma of type `ModDed`.

Let k be a number field or function field and $\mathcal{O}_k$ its ring of integers. Since $\mathcal{O}_k$ is a Dedekind domain, every finitely generated torsion free module M over $\mathcal{O}_k$ has a representation as a direct sum

$$M = \sum_{i=1}^{m} \mathcal{A}_i \alpha_i = \{\sum_{i=1}^{m} a_i \alpha_i \mid a_i \in \mathcal{A}_i\}$$

with (fractional) ideals $\mathcal{A}_i$ and elements $\alpha_i \in kM \cong k^r$.

A (not necessarily direct) sum $\sum_{i=1}^{m} \mathcal{A}_i \alpha_i$ will be represented as a pseudo–matrix $(\mathcal{A}|A)$ where $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)^t$ is a column vector of ideals and $A = (\alpha_1, \ldots, \alpha_m)^t \in k^{m\times r}$ is a matrix. The ideals $\mathcal{A}_i$ are called coefficient ideals.

This pseudo–matrix is called a pseudo–basis iff the sum is direct. A pseudo–matrix $(\mathcal{A}|A)$ is in Hermite normal form iff there are $s \le m$, $1 \le i_1 < i_2 < \cdots < i_s$ such that $A_{j,l} = 0$ $(1 \le l < i_j)$, $A_{j,i_j} = 1$ and $A_{j,l}$ is reduced modulo $\mathcal{A}_j {\mathcal{A}_l}^{-1}$. For $j > s$ we have $A_{j,l} = 0$.

This normal form is unique if a suitable reduction is used.

As a consequence of this normalisation, usually $\alpha_i \notin M$. To be precise: $\alpha_i \in M$ iff $1 \in \mathcal{A}_i$.

All modules are in Hermite normal form, i.e. every module is represented by a pseudo–basis in Hermite normal form.

General (non torsion free) modules are represented as quotients of a torsion free module M and a submodule S. Elements of $Q := M/S$ are represented as elements of M, arithmetic in Q is reduced to arithmetic in M followed by a reduction modulo the pseudo–basis of S.

59.2 Creation of Modules

Modules over Dedekind domains can be created from orders of number fields and function fields and combinations of ideals and vector space elements. Submodules and quotient modules by submodules can also be created.

`Module(O, n)`

Create the free module O^n where O is a Dedekind domain.

`Module(O)`

Create the relative order O as a module over its coefficient ring. Also returns the map from the resulting module into O.

`Module(I)`

Create the ideal I of a relative order O as a module over the coefficient ring of the order. Also returns the map from the module into O.

`Module(S)`

Create a module from the sequence of tuples of ideals of a Dedekind domain and `ModElts` with entries in the Dedekind domain or its field of fractions. The elements of the resulting module will be the sum of products of an element of an ideal and the corresponding `ModElt`. Also returns the map from the vector space into the module.

`Module(S)`

Create the module which is equal to the direct sum of the ideals in the sequence.

`Module(S)`

Create the module which is freely generated by the elements of the sequence S. The elements of the sequence must be `ModElts` with entries in a Dedekind domain or field of fractions of a Dedekind domain. Also returns the map from the vector space into the module.

Example H59E1

The creation of some simple modules is shown.

```
> x := ext<Integers()|>.1;
> M := MaximalOrder(x^2 + 5);
> Module(M, 5);
Module over Maximal Equation Order with defining polynomial x^2 + 5 over Z
 generated by: (in echelon form)
Principal Ideal of M
Generator:
    M.1 * ( M.1 0 0 0 0 )
Principal Ideal of M
Generator:
    M.1 * ( 0 M.1 0 0 0 )
Principal Ideal of M
```

```
Generator:
    M.1 * ( 0 0 M.1 0 0 )
Principal Ideal of M
Generator:
    M.1 * ( 0 0 0 M.1 0 )
Principal Ideal of M
Generator:
    M.1 * ( 0 0 0 0 M.1 )
> I := 1/5*M;
> Module([I, I^3, I^8]);
Module over Maximal Equation Order with defining polynomial x^2 + 5 over Z
 generated by: (in echelon form)
Fractional Principal Ideal of M
Generator:
    1/5*M.1 * ( M.1 0 0 )
Fractional Principal Ideal of M
Generator:
    1/125*M.1 * ( 0 M.1 0 )
Fractional Principal Ideal of M
Generator:
    1/390625*M.1 * ( 0 0 M.1 )
> V := RModule(M, 3);
> Module([<I, V![0, 1, 0]>, <I^4, V![2, 3, 5]>]);
Module over Maximal Equation Order with defining polynomial x^2 + 5 over Z
Fractional Principal Ideal of M
Generator:
    1/5*M.1 car Fractional Principal Ideal of M
Generator:
    1/125*M.1
```

The same can be done using orders of function fields.

```
> P<x> := PolynomialRing(Rationals());
> P<y> := PolynomialRing(P);
> F<c> := FunctionField(x^2 - y);
> M := MaximalOrderFinite(F);
> Module(M, 5);
Module over Maximal Equation Order of F over Univariate Polynomial Ring in x over
Rational Field
 generated by: (in echelon form)
Ideal of M
Generator:
1 * ( 1 0 0 0 0 )
Ideal of M
Generator:
1 * ( 0 1 0 0 0 )
Ideal of M
Generator:
1 * ( 0 0 1 0 0 )
```

```
Ideal of M
Generator:
1 * ( 0 0 0 1 0 )
Ideal of M
Generator:
1 * ( 0 0 0 0 1 )
> I := 1/5*M;
> Module([I, I^3, I^8]);
Module over Maximal Equation Order of F over Univariate Polynomial Ring in x over
Rational Field
 generated by: (in echelon form)
Ideal of M
Generator:
1/5 * ( 1 0 0 )
Ideal of M
Generator:
1/125 * ( 0 1 0 )
Ideal of M
Generator:
1/390625 * ( 0 0 1 )
> V := RModule(M, 3);
> Module([<I, V![0, 1, 0]>, <I^4, V![2, 3, 5]>]);
Integral Module over Maximal Equation Order of F over Univariate Polynomial Ring
in x over Rational
Field
Ideal of M
Generator:
1/5 car Ideal of M
Generator:
1/125
```

---

sub< M | m >

sub< M | m1, .., mn >

Construct the submodule of the module M generated by the elements in the sequence or list of elements m. Also returns the inclusion map of the submodule into M.

quo< M | S >

quo< M | m >

quo< M | m1, .., mn >

Construct the quotient of the module M by the submodule S or the submodule generated by the elements of the sequence or list of elements m. Also returns the inclusion map of the quotient module into M.

Example H59E2

Use of the sub and quo constructors is illustrated below. Let M and V be as above when they were referring to number fields.

```
> Mod := Module([V|[0,1,0], [4,4,0]]);
> S1 := sub<Mod | >;
> S1;
Integral Module over Maximal Equation Order with defining polynomial x^2 + 5
over Z
(0)
> Q1 := quo<Mod | Mod>;
> Q1;
Quotient of Module over Maximal Equation Order with defining polynomial x^2 + 5
over Z
Principal Ideal of M
Generator:
    4/1*M.1 car Principal Ideal of M
Generator:
    M.1
by Integral Module over Maximal Equation Order with defining polynomial
x^2 + 5 over Z
Principal Ideal of M
Generator:
    4/1*M.1 car Principal Ideal of M
Generator:
    M.1
> S2 := sub<Mod | Mod.2>;
> S2;
Integral Module over Maximal Equation Order with defining polynomial x^2 + 5
over Z
Principal Ideal of M
Generator:
    M.1
> Q2 := quo<Mod | Mod.2>;
> Q2;
Quotient of Module over Maximal Equation Order with defining polynomial x^2 + 5
over Z
Principal Ideal of M
Generator:
    4/1*M.1 car Principal Ideal of M
Generator:
    M.1
by Integral Module over Maximal Equation Order with defining polynomial
x^2 + 5 over Z
Principal Ideal of M
Generator:
    M.1
> S3 := sub<Mod | 4*Mod.1, Mod.2>;
> S3;
Integral Module over Maximal Equation Order with defining polynomial x^2 + 5
over Z
Principal Ideal of M
Generator:
    4/1*M.1 car Principal Ideal of M
Generator:
    M.1
> Q3 := quo<Mod | >;
> Q3;
Integral Module over Maximal Equation Order with defining polynomial x^2 + 5
over Z
Principal Ideal of M
Generator:
    4/1*M.1 car Principal Ideal of M
Generator:
    M.1
> Q4 := quo<Mod | S1>;
> Q4;
Quotient of Module over Maximal Equation Order with defining polynomial x^2 + 5
over Z
Principal Ideal of M
Generator:
    4/1*M.1 car Principal Ideal of M
Generator:
    M.1
by Integral Module over Maximal Equation Order with defining polynomial
x^2 + 5 over Z
(0)
```

59.3 Elementary Functions

Various simple properties of a module can be retrieved using the following functions.

`BaseRing(M)`

`CoefficientRing(M)`

The Dedekind domain which M is a module over.

`Degree(M)`

The dimension of the vector space the module M embeds into.

`Ngens(M)`

`NumberOfGenerators(M)`

The minimum number of vectors and ideals which generate the module M.

`M . i`

The vector of the *ith* vector and ideal pair generating the module M.

`Determinant(M)`

The determinant of the module M.

`Dimension(M)`

The dimension of the vector space spanned by the module M over its coefficient ring. This is the same as the number of generators of a pseudo basis of M.

`Contents(M)`

UseBasis BoolElt *Default :* false

The contents of the module M, ie. the gcd of the ideals obtained by multiplying the coefficient ideals by the ideal generated by the coefficients in the corresponding generators. The parameter UseBasis decides whether a pseudo basis or pseudo generators are used.

`Simplify(M)`

UseBasis BoolElt *Default :* false

Computes a module of contents 1 by scaling each coefficient ideal by the inverse of the contents of the module M. The parameter UseBasis determines if the operations are performed on the pseudo generators or the pseudo basis of M.

`EmbeddingSpace(M)`

The canonical vector space containing the module M, ie.. M tensored with the field of fractions of the coefficient ring.

Example H59E3

The use of some elementary functions on a module is shown below.

```
> P<x> := PolynomialRing(Rationals());
> P<y> := PolynomialRing(P);
> F<c> := FunctionField(x^2 - y);
> M := MaximalOrderFinite(F);
> Vs := RModule(M, 2);
> s := [Vs | [1, 3], [2, 3]];
> Mods := Module(s);
> CoefficientRing(Mods);
Maximal Equation Order of F over Univariate Polynomial Ring in x over Rational
Field
> Mods.1;
(1 0)
> Determinant(Mods);
Ideal of M
Generator:
```

```
-3
> Vs := RSpace(M, 2);
> s := [Vs | [1, 3], [2, 3]];
> Mods := Module(s);
> sMods := sub<Mods | Mods!Vs![1, 3]>;
> qMods := quo<Mods | sMods>;
> Degree(Mods);
2
> Ngens(Mods);
2
> Ngens(sMods);
1
> Degree(sMods);
2
> Degree(qMods);
2
> Ngens(qMods);
2
> Determinant(Mods);
Ideal of M
Basis:
[1]
>  Determinant(sMods);
>>   Determinant(sMods);
               ^
Runtime error in ‘Determinant’: Module must be square
> Determinant(qMods);
Ideal of M
Basis:
[1]
```

59.4 Predicates on Modules

Modules and potential elements can be tested against each other for a few properties.

`M eq N`

Return `true` if M and N are equal as modules.

`x in M`

Return `true` if x can be coerced into the module M.

`M subset N`

Return `true` if M is a submodule of N. An embedding map of M in N can be returned by `IsSubmodule(M, N)`.

59.5 Arithmetic with Modules

Some arithmetic operations can be carried out involving modules and their elements and compatible ideals to gain more modules.

```
I * M
```

```
M * I
```

The module generated by the products of the ideals of the module M with I.

```
M1 + M2
```

The union of the modules $M1$ and $M2$.

```
DirectSum(M1, M2)
```

```
DirectSum(S)
```

The direct sum D of the modules $M1$ and $M2$ or the modules in the sequence S together with the embedding maps $M1 \to D, M2 \to D, \ldots$ and the projection maps $D \to M1, D \to M2, \ldots$ returned in sequences only when a sequence S is input.

```
u * I
```

```
I * u
```

The module containing elements which are products of the Dedekind module element u and an element lying in the ideal I.

Example H59E4

Some module predicates and arithmetic are shown below.

```
> P<x> := PolynomialRing(Integers());
> K :=  NumberField([x^5 + 3, x^2 + 2]);
> M := MaximalOrder(K);
> Vs := RModule(M, 2);
> s := [Vs | [1, 3], [2, 3]];
> Mods := Module(s);
> sMods := sub<Mods | Mods!Vs![1, 3]>;
> Mods eq sMods;
false
> [1, 0] in Mods;
true
> [1, 0] in sMods;
false
> sMods subset Mods;
true
> Vs := RSpace(M, 2);
> s := [Vs | [Random(M, 3), 3], [2, Random(M, 2)]];
> Mods := Module(s);
> sMods := sub<Mods | Mods!s[1]>;
> (7*M + 11*K.1*M)*sMods;
Module over Maximal Equation Order with defining polynomial x^5 + [3, 0] over
```

```
its ground order
 generated by:
Ideal of M
Two element generators:
    7/1*$.1*M.1
    11/1*$.1*M.2 * ( M.1 + (-$.1 - 2/1*$.2)*M.2 + $.2*M.3 + (2/1*$.1 +
2/1*$.2)*M.5 3/1*$.1*M.1 )
 in echelon form:
Ideal of M
Two element generators:
    21/1*$.1*M.1
    33/1*$.1*M.2 * ( 1/3*$.1*M.1 + (-1/3*$.1 - 2/3*$.2)*M.2 + 1/3*$.2*M.3 +
(2/3*$.1 + 2/3*$.2)*M.5 M.1 )
> Mods + sMods;
Module over Maximal Equation Order with defining polynomial x^5 + [3, 0] over
its ground order
 generated by: (in echelon form)
Ideal of M
Two element generators:
    148919257164/1*$.1*M.1
    (148919257048/1*$.1 + 26/1*$.2)*M.1 + (32/1*$.1 + 148919257015/1*$.2)*M.2 +
    (196/1*$.1 + 148919257117/1*$.2)*M.3 + (148919257163/1*$.1 + 76/1*$.2)*M.4 +
    (148919257077/1*$.1 + 148919257116/1*$.2)*M.5 * ( M.1 0 )
Ideal of M
Two element generators:
    3/1*$.1*M.1
    ($.1 - $.2)*M.1 + -M.3 + M.4 + ($.1 + $.2)*M.5 * ( (-19866460521/1*$.1 -
33/1*$.2)*M.1 + (1/3*$.1 + 1/3*$.2)*M.4 + (1/3*$.1 + 1/3*$.2)*M.5 M.1 )
> 4*sMods;
Module over Maximal Equation Order with defining polynomial x^5 + [3, 0] over
its ground order
 generated by:
Principal Ideal of M
Generator:
    4/1*$.1*M.1 * ( (2/1*$.1 - 2/1*$.2)*M.1 + ($.1 + 2/1*$.2)*M.2 + (2/1*$.1 +
2/1*$.2)*M.3 + (2/1*$.1 - $.2)*M.5 3/1*$.1*M.1 )
 in echelon form:
Principal Ideal of M
Generator:
    12/1*$.1*M.1 * ( (2/3*$.1 - 2/3*$.2)*M.1 + (1/3*$.1 + 2/3*$.2)*M.2 +
(2/3*$.1 + 2/3*$.2)*M.3 + (2/3*$.1 - 1/3*$.2)*M.5 M.1 )
```

59.6 Basis of a Module

The basis of a module is given by vectors. However, more complete information can be supplied which includes the ideals.

```
Basis(M)
```

A sequence of vectors which correspond to a pseudo basis of the module M.

```
PseudoBasis(M)
```

A sequence of tuples containing ideals and vectors which generate the module M. The vectors are guaranteed to be linearly independent.

```
PseudoGenerators(M)
```

A sequence of tuples containing ideals and vectors which generate the module M. This will return the data used to define the module, so that in contrast to `PseudoBasis` the vectors will in general not be independent.

59.7 Other Functions on Modules

Intersections of modules can taken. Several other functions are also available.

```
M1 meet M2
```

Return the intersection of the modules $M1$ and $M2$.

```
Dual(M)
```

The module dual to M.

```
ElementaryDivisors(M, N)
```

The elementary divisors (ideals) of the torsion part of the quotient R-module M/N: For $N \subseteq M$ we get

$$T(M/N) \cong \oplus_{i=1}^{n} R/\mathcal{A}_i$$

The $\mathcal{A}_i$ are unique if we require $R \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n$. The $\mathcal{A}_i$ are called the elementary divisors (or elementary ideals) of M/N. This corresponds to the Smith normal form for integral matrices.

```
SteinitzClass(M)
```

The Steinitz class of the module M.

```
SteinitzForm(M)
```

The Steinitz (almost–free) form of the module M.

Example H59E5

Some bases and other functions are demonstrated below.

```
> P<x> := PolynomialRing(Rationals());
> P<y> := PolynomialRing(P);
> F<c> := FunctionField(y^3 - x^3*y^2 + y - x^7);
> M := MaximalOrderFinite(F);
> Vs := RSpace(M, 2);
> s := [Vs | [1, Random(M, 3)], [Random(M, 3), 3]];
> Mods := Module(s);
> qMods := quo<Mods | Mods!s[2]>;
> Basis(Mods);
[
    ([ 1, 0, 0 ] [ 2*x^2 - 3*x + 1/2, -2/3*x^2 - x + 1/3, 2/3*x^2 - 2*x + 1/2
        ]),
    ([ -x^2 + x + 2/3, -1/3*x^2 - 1, -3/2*x^2 - 2/3*x + 1 ] [ 3, 0, 0 ])
]
> Basis(qMods);
[
    ([ 1, 0, 0 ] [ 2*x^2 - 3*x + 1/2, -2/3*x^2 - x + 1/3, 2/3*x^2 - 2*x + 1/2
        ]),
    ([ -x^2 + x + 2/3, -1/3*x^2 - 1, -3/2*x^2 - 2/3*x + 1 ] [ 3, 0, 0 ])
]
> PseudoBasis(Mods) eq PseudoBasis(qMods);
> Vs := RModule(M, 2);
> s := [Vs | [Random(M, 3), Random(M, 3)], [2, 3]];
> Mods := Module(s);
> sMods := sub<Mods | Mods!s[1]>;
> Mods meet sMods;
Module over Maximal Equation Order of F over Univariate Polynomial Ring in x
over Rational Field
Ideal of M
Generator:
(x + 1)*c^2 + (-3/2*x^2 - 3/2*x)*c + 1/2*x^2 + 2/3*x + 1/2
> ElementaryDivisors(Mods, sMods);
[ Ideal of M
Basis:
[1 0 0]
[0 1 0]
[0 0 1], Ideal of M
Generator:
0 ]
> Dual(Mods);
Module over Maximal Equation Order of F over Univariate Polynomial Ring in x
over Rational Field
Fractional ideal of M
Generator:
(-3*x^4 + 21*x^3 + 20*x^2 + 80*x - 387)/(x^17 - 24*x^16 + 192*x^15 - 510*x^14 -
    94/3*x^13 + 304/3*x^12 + 902/3*x^11 - 3109/3*x^10 - 389/9*x^9 + 664/9*x^8 +
    1094/3*x^7 - 1540/3*x^6 - 101/9*x^5 + 128/3*x^4 + 2000/27*x^3 - 4361/9*x^2 -
    427*x + 2056)*c^2 + (-3*x^9 + 48*x^8 - 189*x^7 - 24*x^6 + 3*x^5 - 48*x^4 +
    195*x^3 - 3*x^2 - x + 24)/(x^17 - 24*x^16 + 192*x^15 - 510*x^14 - 94/3*x^13
    + 304/3*x^12 + 902/3*x^11 - 3109/3*x^10 - 389/9*x^9 + 664/9*x^8 + 1094/3*x^7
    - 1540/3*x^6 - 101/9*x^5 + 128/3*x^4 + 2000/27*x^3 - 4361/9*x^2 - 427*x +
    2056)*c + (3*x^12 - 48*x^11 + 192*x^10 + 3*x^9 - 20*x^8 - 56*x^7 + 192*x^6 +
    3*x^5 - 5*x^4 - 8*x^3 + 272/3*x^2 + 128*x - 771)/(x^17 - 24*x^16 + 192*x^15
    - 510*x^14 - 94/3*x^13 + 304/3*x^12 + 902/3*x^11 - 3109/3*x^10 - 389/9*x^9 +
    664/9*x^8 + 1094/3*x^7 - 1540/3*x^6 - 101/9*x^5 + 128/3*x^4 + 2000/27*x^3 -
    4361/9*x^2 - 427*x + 2056) car Ideal of M
Generator:
1/3
> Dual(sMods);
Module over Maximal Equation Order of F over Univariate Polynomial Ring in x
over Rational Field
Fractional ideal of M
Generator:
(3/2*x^6 + 3*x^5 - 3/4*x^4 - 4*x^3 - 25/12*x^2 - 5/6*x - 1/2)/(x^17 + 3*x^16 +
    21/4*x^15 + 27/4*x^14 + 1/24*x^13 - 247/24*x^12 - 173/24*x^11 + 61/24*x^10 +
    305/72*x^9 + 17/72*x^8 - 4*x^7 - 53/12*x^6 - 1/8*x^5 + 23/6*x^4 +
    377/108*x^3 + 23/18*x^2 + 1/3*x + 1/8)*c^2 + (-5/2*x^9 - 5*x^8 - 1/4*x^7 +
    9/2*x^6 + 13/4*x^5 + 5/4*x^4 - 3/4*x^3 - 7/4*x^2 - 3/4*x)/(x^17 + 3*x^16 +
    21/4*x^15 + 27/4*x^14 + 1/24*x^13 - 247/24*x^12 - 173/24*x^11 + 61/24*x^10 +
    305/72*x^9 + 17/72*x^8 - 4*x^7 - 53/12*x^6 - 1/8*x^5 + 23/6*x^4 +
    377/108*x^3 + 23/18*x^2 + 1/3*x + 1/8)*c + (x^12 + 2*x^11 - 1/2*x^10 -
    7/2*x^9 - 8/3*x^8 - 5/12*x^7 + 11/4*x^6 + 19/4*x^5 - 1/4*x^4 - 25/6*x^3 -
    67/36*x^2 - 1/3*x - 1/4)/(x^17 + 3*x^16 + 21/4*x^15 + 27/4*x^14 + 1/24*x^13
    - 247/24*x^12 - 173/24*x^11 + 61/24*x^10 + 305/72*x^9 + 17/72*x^8 - 4*x^7 -
    53/12*x^6 - 1/8*x^5 + 23/6*x^4 + 377/108*x^3 + 23/18*x^2 + 1/3*x + 1/8)
> SteinitzClass(Mods) eq SteinitzClass(sMods);
false
> SteinitzForm(Mods);
Module over Maximal Equation Order of F over Univariate Polynomial Ring in x
over Rational Field
Ideal of M
Generator:
3 car Ideal of M
Generator:
1
> SteinitzForm(sMods);
Module over Maximal Equation Order of F over Univariate Polynomial Ring in x
over Rational Field
Ideal of M
Generator:
1
```

59.8 Homomorphisms between Modules

It is possible to create a homomorphism between two modules, take the image and kernel of such and verify that these are submodules of the codomain and domain respectively. The Hom–module can also be created as a module of a Dedekind domain.

```
hom< M -> N | T >
```

`ModuleBasis` BoolElt *Default :* `true`

Return a homomorphism from the module M into the module N as specified by T from which the images of the generators can be inferred. T may be a map between the vector spaces of same degree as M and N, a matrix over the field of fractions or a sequence of vectors. If `ModuleBasis` is `true` then the matrix will be taken to be a transformation between the modules and as such will be expected to have size `Dimension(M)*Dimension(N)` otherwise it will be interpreted as a transformation between the corresponding vector spaces and will be expected to have size `Degree(M)*Degree(N)`.

```
Hom(M, N)
```

The module of homomorphisms between the module M and the module N and the map from the hom–module to the collection of maps from M to N, (such that given an element of the hom–module a homomorphism from M to N is returned). The module is over the same Dedekind domain as M and N.

```
IsSubmodule(M, N)
```

Return `true` if M is a submodule of N and the map embedding M into N.

```
Morphism(M, N)
```

The map giving the morphism from the module M to the module N. Either M is a submodule of N, in which case the embedding of M into N is returned, or N is a quotient module of M, in which case the natural epimorphism from M onto N is returned.

Example H59E6

This example demonstrates the use of homomorphisms between modules over Dedekind domains. Let `M` and `V` be as above referring to function fields.

```
> S := [V|[0,1,0], [4,4,0]];
> Mod := Module(S);
> W := KModule(FieldOfFractions(M), 4);
> S := [W|[3, 2, 1, 0]];
> N := Module(S);
>  h := hom<Mod -> N | >;
>> h := hom<Mod -> N | >;
          ^
Runtime error in map< ... >: No images given
>  h := hom<Mod -> N | V.1, V.2, V.3>;
```

```
>> h := hom<Mod -> N | V.1, V.2, V.3>;
           ^
Runtime error in map< ... >: An image for each generator is required
> h := hom<Mod -> N | W![3, 2, 1, 0], W![3*(M!F.1 + 1), 2*(M!F.1 + 1),
> M!F.1 + 1, 0] >;
> h(Mod!(4*V.1));
( 4 )
> h(Mod!V![0, 1, 0]);
( x^2 + 1 )
> I := Image(h);
> I;
Module over Maximal Equation Order of F over Univariate Polynomial Ring in x
over Rational Field
Ideal of M
Generator:
1
> K := Kernel(h);
> K;
Integral Module over Maximal Equation Order of F over Univariate Polynomial Ring
in x over Rational Field
Ideal of M
Generator:
1
> IsSubmodule(K, Mod);
true Mapping from: ModDed: K to ModDed: Mod
> H, m := Hom(Mod, N);
> H; m;
Module over Maximal Equation Order of F over Univariate Polynomial Ring in x over
Rational Field
 generated by: (in echelon form)
Ideal of M
Generator:
1/4 * ( 1 0 )
Ideal of M
Generator:
1 * ( 0 1 )
Mapping from: ModDed: H to Power Structure of Map given by a rule [no inverse]
> m(H![5, 20]);
Mapping from: ModDed: Mod to ModDed: N
using
[5]
[20]
```

59.9 Elements of Modules

59.9.1 Creation of Elements

`M ! v`

Coerce v into an element of M. v can be a sequence of length dimension of M, a module element or vector or an element of another module over a Dedekind domain which is compatible with M.

Example H59E7

Let `Mod` and its submodules and quotient modules be as in the sub and quotient module example above.

```
> m := 4*Mod.1;
> m;
(4/1*M.1 0)
> Q1!m;
( 4/1*M.1 0 )
> Q2!m;
( 4/1*M.1 0 )
> m := Mod!m;
> Q3!m;
( 4/1*M.1 0 )
> Q4!m;
( 4/1*M.1 0 )
>  S1!m;
>> S1!m;
     ^
Runtime error in ‘!’: Illegal coercion
LHS: ModDed
RHS: ModDedElt
> S1!Mod!V!0;
(  )
> S2!Mod!Mod.2;
( M.1 )
> S3!Mod!(4*Mod.1);
( 4/1*M.1 0 )
```

59.9.2 Arithmetic with Elements

Basic arithmetic can be performed with elements of a module over a Dedekind domain.

```
x + y
```

The sum of the module elements.

```
x - y
```

The difference of the module elements.

```
u * c
```

```
c * u
```

The product of the module element u and the ring element c.

```
u / c
```

The product of u and $1/c$ if it lies in the parent module of u.

```
I * u
```

```
u * I
```

The module containing elements which are products of u and an element lying in I.

59.9.3 Other Functions on Elements

Elements of modules over a Dedekind domain can be tested for equality and represented as a sequence.

```
x eq y
```

Return `true` if x and y are the same element of a module.

```
IsZero(a)
```

Returns whether the module element a is zero.

```
ElementToSequence(a)
```

```
Eltseq(a)
```

The module element a expressed as a sequence.

59.10 Pseudo Matrices

A pseudo matrix, ie. an object of type `PMat`, is a sequence of ideals together with a matrix. Pseudo matrices arise naturally in the (computational) theory of finitely generated torsion free modules over Dedekind domains, they are a natural extension of ordinary matrices which should be thought of as pseudo matrices where all the ideals are generated by 1. Pseudo matrices are generally used to represent the module that is generated by the rows of the matrix scaled by the elements of the corresponding ideal. Thus, if the matrix is regular, a linear combination of the rows lies in the module if and only if the coefficient of the ith row is a member of the ith ideal. The ideals are therefore called coefficient ideals.

59.10.1 Construction of a Pseudo Matrix

`PseudoMatrix(I, m)`

Construct the pseudo matrix with coefficient ideals the elements of the sequence I and matrix m.

`PseudoMatrix(m)`

Construct the pseudo matrix with trivial coefficient ideals and the matrix m.

`PseudoMatrix(M)`

`Generators` BoolElt *Default :* `false`

Construct the pseudo matrix described by the pseudo basis of the module M. If `Generators` is `true` the pseudo matrix described by the pseudo generators of M is returned.

59.10.2 Elementary Functions

`CoefficientIdeals(P)`

Return the coefficient ideals of the pseudo matrix P.

`Matrix(P)`

Return the matrix of the pseudo matrix P.

`Order(pm)`

Return the order the pseudo matrix pm is over.

`Dimension(pm)`

The dimension of the pseudo matrix pm. This is the numer of columns of the matrix.

`Length(pm)`

The length or dimension of the pseudo matrix pm. This is the number of coefficient ideals of pm.

59.10.3 Basis of a Pseudo Matrix

```
Basis(P)
```

Return a list of sequences of ring elements corresponding to the entries of each row of the matrix of the pseudo matrix P.

59.10.4 Predicates

```
p1 eq p2
```

Return whether the pseudo matrices $p1$ and $p2$ are equal, that is, whether they have the same matrix and the same sequence of coefficient ideals.

59.10.5 Operations with Pseudo Matrices

```
Transpose(P)
```

Return the pseudo matrix whose coefficient ideals are the same as those of P but whose matrix is the transpose of the matrix of P. This function requires the matrix to be square.

```
HermiteForm(X)
```

Return the Hermite normal form H of the pseudo matrix X together with a regular transformation matrix such that the module generated by X is the same as the one generated by H and such that (for the matrix parts) $H = TX$ holds.

```
VerticalJoin(X, Y)
```

Return the pseudo matrix whose matrix is the vertical join of the matrices of the pseudo matrices X and Y with coefficient ideals the concatenation of those of X and Y.

```
X meet Y
```

Return the intersection of the pseudo matrices X and Y.

```
Module(X)
```

Return the module $\sum C_i * m_i$ where C_i are the coefficient ideals of the pseudo matrix X and m_i are the rows of the matrix of X.

```
I * X
X * I
```

The pseudo matrix whose coefficient ideals are those of the pseudo matrix X multiplied by I and whose matrix is the matrix of X.

60 CHAIN COMPLEXES

Chapter 60

CHAIN COMPLEXES

60.1 Complexes of Modules

Complexes of modules are a fundamental object in homological algebra. Magma supports the type ModCpx representing a complex of modules. Conceptually, a complex is an infinite sequence of modules, indexed by integers, with maps between successive modules such that the composition of any two maps is zero. Complexes are often written

$$\ldots \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} M_{n-2} \xrightarrow{f_{n-2}} \ldots$$

where the map f_n has domain M_n and codomain M_{n-1}. The indices on the modules and maps decrease to the right. In practice, Magma requires all but a finite number of the modules and maps to be zero.

The homomorphism from M_n to M_{n-1} in the complex is the n^{th} boundary map of the complex. The homology of the complex in degree n is the quotient of the kernel of the n^{th} boundary map by the cokernel of the boundary map of degree $n-1$. Magma computes the homology only if both boundary maps are defined. A complex is said to be *exact* if the image of each map is equal to the kernel of the next.

Currently, there are two types of modules over which complexes are supported:

(a) Modules over a basic algebra A (see Chapter 90);

(b) Modules over a multivariate polynomial ring over a field (see Chapter 115 and in particular the function FreeResolution).

Most of the functions in this chapter work for either type of module but exceptions are noted.

60.1.1 Creation

```
Complex(L, d)
```

Given a list L of maps between successive A-modules create the corresponding complex. The last term of the complex has degree d. This function returns an error if the maps don't actually form a complex.

```
Complex(f, d)
```

Given a map f between A-modules M and N, form the complex consisting of the two term complex whose only map is f. The term N is in degree d.

```
ZeroComplex(A, m, n)
```

Given a basic algebra A and integers m and n such that $m > n$, create a complex of modules over the basic algebra A, starting with a term of degree m and ending with a term of degree n, where all the modules and maps are zero.

```
Dual(C)
```

```
Dual(C, n)
```

The dual of the complex C over a basic algebra A as a complex over the opposite algebra of A. If an integer n is supplied then the last term of the dual complex is in degree n. Otherwise, the last term of the dual complex is in degree 0.

60.1.2 Subcomplexes and Quotient Complexes

Let C be a complex of A-modules, $M_m, \ldots, M_n$. A subcomplex S of C is a complex whose terms are submodules of the terms of C and whose maps are the restrictions of the maps of C to the terms of S.

```
sub< C | Q >
```

```
sub< C | L >
```

Given a complex C and a sequence Q of submodules of the terms of C, returns the smallest subcomplex whose terms contain the modules in Q. The input can also be a list L of sequences of elements of the successive terms of C. In this case the submodules generated by the elements is computed. The function also returns the chain map giving the inclusion of the subcomplex in C.

```
RandomSubcomplex(C, Q)
```

Given a chain complex C over a basic algebra in degrees a to $a-t+1$ and a sequence $Q = [q_1, \ldots, q_t]$ of natural numbers, the function creates the minimal chain complex whose term in degree $a-i+1$ is a submodule generated by q_i random elements of the term in degree $a-i+1$ of C. The function also returns the chain map that is the inclusion of the subcomplex into C.

```
quo< C | D >
```

```
quo< C | S >
```

```
quo< C | L >
```

Given a complex C and a subcomplex D of C, returns the quotient complex C/D together with the natural quotient map. If given as a sequence S of submodules of the terms of C or a list L of sequences of elements of the terms of C, then the submodule generated by S or L is created and the quotient computed. The function also returns the chain map of C on the quotient.

60.1.3 Access Functions

```
Degrees(C)
```

Returns the first and last degrees of the defined terms of the complex C.

```
Algebra(C)
```

Given a complex C over a basic algebra A, this function returns the algebra A.

`BoundaryMap(C, n)`

Returns the boundary map of the complex C from the term of degree n to the term of degree $n - 1$.

`BoundaryMaps(C)`

The list of boundary maps of the complex C.

`DimensionsOfHomology(C)`

Returns the list of the dimensions of the homology groups of the complex C.

`DimensionOfHomology(C, n)`

Returns the dimension of the homology group of the complex C in degree n.

`DimensionsOfTerms(C)`

Returns the list of the dimensions of the terms of the complex C.

`Term(C, n)`

The module in the complex C in degree n.

`Terms(C)`

The sequence of terms of the complex C.

60.1.4 Elementary Operations

`DirectSum(C, D)`

Returns the direct sum of the complex C and the complex D.

`Homology(C)`

`HomologyOfChainComplex(C)`

The sequence of homology groups of the complex C as a sequence of modules.

`Homology(C, n)`

Returns the homology group in degree n of the complex C, as an A-module.

`Prune(C)`

Returns the complex that consists of the terms of C with the right end term (term of lowest degree) removed.

`Prune(C,n)`

Returns the complex that consists of the terms of C with n terms removed from the right end.

```
Preprune(C)
```

Returns the complex that consists of the terms of C with the left end term (term of highest degree) removed.

```
Preprune(C,n)
```

Returns the complex that consists of the terms of C with n terms removed from the left end.

```
Shift(C, n)
```

Given the complex C in degrees r to s, returns the complex in shifted degrees $r+n$ to $s+n$. The integer n may be either positive or negative. The maps in the complex are all multiplied by the scalar $(-1)^n$.

```
ShiftToDegreeZero(C)
```

Given the complex C, returns the shift of C so that the last term, the term of lowest degree, is in degree 0.

```
Splice(C, D)
```

Given two complexes C and D over the same basic algebra, such that the end term of C coincides with the initial term of D, form the complex that corresponds to the concatenation of C and D (as sequences of maps) This function checks that the resulting sequence of maps forms a complex. The degrees of complex D remain unchanged while the degrees of the terms in complex C are changed to fit.

```
Splice(C, D, f)
```

The splice of the complex C with the complex D along the map f from the last term of C to the first term of D. The degree of the last term of the splice is the same as the degree of the last term of the complex D.

60.1.5 Extensions

The following are elementary operations related to extending complexes

```
LeftExactExtension(C)
```

Given a complex C of modules over a basic algebra, returns the complex of length one greater that is obtained by adjoining the inclusion map from the kernel of the boundary map in highest degree to the term of highest degree in C.

```
RightExactExtension(C)
```

Given a complex C of modules over a basic algebra, returns the complex of length one greater that is obtained by adjoining the quotient map to the cokernel of the boundary map of lowest degree from the term of lowest degree in C.

```
ExactExtension(C)
```

Returns the left and right exact extensions of the complex C.

```
LeftZeroExtension(C)
LeftZeroExtension(C, n)
```

Given a complex C of modules over a basic algebra, returns the complex of length one greater that is obtained by adjoining the zero map from the zero module to the term of highest degree in the complex. If a natural number n is included in the input then the operation is performed n times.

```
RightZeroExtension(C)
RightZeroExtension(C, n)
```

Given a complex C of modules over a basic algebra, returns the complex of length one greater that is obtained by adjoining the zero map to the zero module from the term of lowest degree in the complex. If a natural number n is included in the input then the operation is performed n times.

```
ZeroExtension(C)
```

Returns the left and right zero extensions of the complex C.

```
EqualizeDegrees(C, D)
EqualizeDegrees(Q)
```

Given complexes C and D over the same algebra, the function returns the complexes obtained by taking zero extensions of C and D, if necessary, so that both complexes have the same degrees. The input can also be given as a sequence Q of complexes, in which case the function returns the sequence of complexes obtained by taking zero extensions of the elements of Q, if necessary, until all of the elements of the sequence have the same degrees.

```
EqualizeDegrees(C, D, n)
```

The two complexes, C and D, with zero extension sufficient that the first and the shift by degree n of the second have the same degrees.

60.1.6 Predicates

The following functions return a Boolean value.

```
IsExact(C)
```

Returns `true` if and only if the complex C is an exact sequence, i.e. the image of each map in C is equal to the kernel of the succeeding map (with the obvious exceptions of the first and last maps of the complex). If the complex has only two terms then this is true trivially.

`IsShortExactSequence(C)`

Returns `true` if the complex C consists of a short exact sequence together with other terms that are zero. The function returns also the degrees of the complex of nonzero terms.

`IsExact(C, n)`

Returns `true` if and only if the complex C is an exact sequence in degree n, i.e. the image of the map in C into the term of degree n is equal to the kernel of the succeeding map.

`IsZeroComplex(C)`

Returns `true` if and only if the complex C is composed entirely of zero modules and maps.

`IsZeroMap(C, n)`

Returns `true` if and only if the boundary map in degree n of the complex C is the zero map.

`IsZeroTerm(C, n)`

Returns `true` if and only if the term in degree n of the complex C is the zero object.

Example H60E1

We construct the quiver algebra of a quiver with three nodes and three arrows going from node 1 to node 2, from 2 to 1 and 2 to 3. The relation is that $(ab)^3a = 0$ where a is the first arrow and b is the second. Then we construct projective and injective resolutions of the first simple module.

```
> ff := GF(8);
> FA<e1,e2,e3,a,b,c> := FreeAlgebra(ff,6);
> rrr := [a*b*a*b*a*b*a];
> D := BasicAlgebra(FA,rrr,3,[<1,2>,<2,1>,<2,3>]);
> D;
Basic algebra of dimension 23 over GF(2^3)
Number of projective modules: 3
Number of generators: 6
> DimensionsOfProjectiveModules(D);
[ 10, 12, 1 ]
> DimensionsOfInjectiveModules(D);
[ 8, 7, 8 ]
```

Here is the opposite algebra:

```
> OD := OppositeAlgebra(D);
reverse trees
> OD;
Basic algebra of dimension 23 over GF(2^3)
Number of projective modules: 3
Number of generators: 6
```

```
> s1 := SimpleModule(D,1);
> P,mu := ProjectiveResolution(s1,7);
> P;
Basic algebra complex with terms of degree 7 down to 0
Dimensions of terms: 12 12 12 12 12 12 12 10
> Q,nu := InjectiveResolution(s1,7);
> Q;
Basic algebra complex with terms of degree 0 down to -7
Dimensions of terms: 8 7 0 0 0 0 0 0
```

Note that the projective and injective resolutions are complexes with the appropriate augmentation and coaugmentation maps. First we form the two term complex whose boundary map is the composition of the augmentation and coaugmentation maps.

```
> theta := MapToMatrix(hom<Term(P,0)-> Term(Q,0)|mu*nu>);
> E := Complex(theta,0);
```

Then we splice all of this together.

```
> R := Splice(P,E);
> S := Splice(R,Q);
> S;
Basic algebra complex with terms of degree 8 down to -7
Dimensions of terms: 12 12 12 12 12 12 12 10 8 7 0 0 0 0 0 0
```

60.2 Chain Maps

A chain map from complex C to complex D is a sequence of homomorphisms between the terms of C and the terms of D such that the maps commute with the boundary homomorphisms on the two complexes. The chain map has degree zero if the map on C_n has it's image in D_n. Otherwise the degree of the chain map expresses the extent to which the degrees of the terms are raised or lowered by the chain map.

Chain maps do not have to be defined in every degree for which either its domain or codomain is defined. On the other hand it must be defined wherever possible, i.e. for any degree i for which the term of the domain is defined and for which the term of the codomain is defined in degree $i + n$ where n is the degree of the chain map.

The freedom of definition can cause problems with constructions such as the kernel or cokernel. That is, the kernel of a chain complex may not be a complex. For this reason we allow the Kernel and Cokernel functions to work only for proper chain maps where "proper" is defined as follows. Suppose C is a chain complex in degrees a to b and D is a complex in degrees u to v. A chain map in degree n from C to D is proper provided $u \geq a + n$ and $v \geq b + n$.

60.2.1 Creation

```
ChainMap(Q, C, D, n)
```

Returns the chain map given by the sequence of maps in Q. Each element in Q is a map from a term of the complex C to a corresponding term of the complex D. The integer n is the degree of the chain map. This means that for any i the term in degree i of C is mapped to the term of degree $n+i$ of D. There should be a map defined for any degree i in the range of degrees of C such that there is a corresponding term in degree $n+i$ for the complex D.

```
ZeroChainMap(C, D)
```

Given chain complexes C and D, construct the chain map from C to D all of whose terms are zero.

60.2.2 Access Functions

```
Degree(f)
```

Given a chain map f, returns the degree of f. Note that $f : C \to D$ has degree n if it takes the term in degree i of the complex C to the term in degree $i+n$ of the complex D.

```
ModuleMap(f, n)
```

Given a chain map $f : C \to D$ and an integer n, this function returns the map from the term in degree n of the complex C to the term in degree $n + \texttt{Degree}(f)$ of the complex D.

```
Kernel(f)
```

Returns the kernel complex of f and the inclusion of the kernel in the domain of f. The function is only defined for proper chain maps. If the chain map is not defined for a certain term of the domain then that term is equal to the term of the kernel complex. That is, if f is not defined on a term then the functions acts as if f were the zero map on that term.

```
Cokernel(f)
```

Returns the cokernel complex of f and the projection of the cokernel onto the codomain of f. The function is only defined for proper chain maps. If the chain map is not defined for a certain term of the codomain then that term is equal to the term of the cokernel complex. That is, if f is not defined on a term then the function acts as if f were the zero map on that term.

```
Image(f)
```

Returns the image complex of f and the inclusion of the image in the codomain of f and the projection of the domain of f on to the image. The function is only defined for proper chain maps.

60.2.3 Elementary Operations

```
f + g
```

The sum of two chain maps with the same domain and codomain.

```
a * g
```

The product of the chain map g by the scalar a in the base ring of the algebra.

```
f * g
```

The composition of the two chain maps f and g.

60.2.4 Predicates

The following functions return a Boolean value.

```
IsSurjective(f)
```

Returns `true` if the chain map f is a surjection in every degree, `false` otherwise.

```
IsInjective(f)
```

Returns `true` if the chain map f is an injection in every degree, `false` otherwise.

```
IsZero(f)
```

Returns `true` if the chain map f is zero in every degree, `false` otherwise.

```
IsIsomorphism(f)
```

Returns `true` if the chain map f is an isomorphism of chain complexes, `false` otherwise.

```
IsShortExactSequence(f, g)
```

Returns `true` if the sequence of chain complexes, $0 \to Domain(f) \to Domain(g) \to Codomain(g) \to 0$, where the internal maps are f and g, is exact.

```
IsChainMap(L, C, D, n)
```

Returns `true` if the list of maps L from the terms of complex C to the terms of the complex D is a chain map of degree n, i. e. it has the right length and the diagram commutes.

```
IsChainMap(f)
```

Returns `true` if the supposed chain map f really is a chain map, i. e. the diagrams commute.

```
IsProperChainMap(f)
```

Returns `true` if the chain map f is a proper chain map, a necessary condition for taking kernel and cokernel.

```
HasDefinedModuleMap(C,n)
```

Returns `true` if the module map in degree n of the complex C is defined.

Example H60E2

We from the basic algebra of the direct product of a cyclic group of order 3 with symmetric group on three letters over the field with three elements.

```
> FA<e1,e2,a,b> := FreeAlgebra(GF(3),4);
> MM:= [e1 +e2 - FA!1, a*b*a, b*a*b];
> BS3 := BasicAlgebra(FA, MM, 2, [<1,2>,<2,1>]);
> gg := CyclicGroup(3);
> BC3 := BasicAlgebra(gg,GF(3));
> A := TensorProduct(BS3,BC3);
> A;
Basic algebra of dimension 18 over GF(3)
Number of projective modules: 2
Number of generators: 6
```

Now we want the projective resolution of the second simple module as a complex. This we will manipulate and take a somewhat random complex.

```
> PR := ProjectiveResolution(SimpleModule(A,2),12);
> PR;
Basic algebra complex with terms of degree 12 down to 0
Dimensions of terms: 117 108 99 90 81 72 63 54 45 36 27 18 9
> PR := Prune(PR);
> PR := Prune(PR);
> PR := Prune(PR);
> PR;
Basic algebra complex with terms of degree 12 down to 3
Dimensions of terms: 117 108 99 90 81 72 63 54 45 36
> PR := Prune(PR);
> PR := Prune(PR);
> PR;
Basic algebra complex with terms of degree 12 down to 5
Dimensions of terms: 117 108 99 90 81 72 63 54
> PR := ZeroExtension(PR);
> PR;
Basic algebra complex with terms of degree 13 down to 4
Dimensions of terms: 0 117 108 99 90 81 72 63 54 0
> PR := Shift(PR,-4);
> PR;
Basic algebra complex with terms of degree 9 down to 0
Dimensions of terms: 0 117 108 99 90 81 72 63 54 0
> S := [* *];
> for i := 1 to 10 do
>    S[i] := [Random(Term(PR,10-i)),Random(Term(PR,10-i))];
> end for;
> C,mu := Subcomplex(PR,S);
> C;
Basic algebra complex with terms of degree 9 down to 0
Dimensions of terms: 0 36 67 64 62 63 58 54 51 0
```

```
> Homology(C);
[
    AModule of dimension 4 over GF(3),
    AModule of dimension 6 over GF(3),
    AModule of dimension 5 over GF(3),
    AModule of dimension 3 over GF(3),
    AModule of dimension 2 over GF(3),
    AModule of dimension 1 over GF(3),
    AModule of dimension 4 over GF(3),
    AModule of dimension 26 over GF(3)
]
[
    Mapping from: AModule of dimension 4 over GF(3) to AModule of dimension 4
    over GF(3),
    Mapping from: AModule of dimension 38 over GF(3) to AModule of dimension 6
    over GF(3),
    Mapping from: AModule of dimension 34 over GF(3) to AModule of dimension 5
    over GF(3),
    Mapping from: AModule of dimension 33 over GF(3) to AModule of dimension 3
    over GF(3),
    Mapping from: AModule of dimension 31 over GF(3) to AModule of dimension 2
    over GF(3),
    Mapping from: AModule of dimension 33 over GF(3) to AModule of dimension 1
    over GF(3),
    Mapping from: AModule of dimension 29 over GF(3) to AModule of dimension 4
    over GF(3),
    Mapping from: AModule of dimension 51 over GF(3) to AModule of dimension 26
    over GF(3)
]
>  D := Cokernel(mu);
>  D;
Basic algebra complex with terms of degree 9 down to 0
Dimensions of terms: 0 81 41 35 28 18 14 9 3 0
>  Homology(D);
[
    AModule of dimension 64 over GF(3)
    AModule of dimension 5 over GF(3),
    AModule of dimension 3 over GF(3),
    AModule of dimension 2 over GF(3),
    AModule of dimension 1 over GF(3),
    AModule of dimension 4 over GF(3),
    AModule of dimension 3 over GF(3),
    AModule of dimension 3 over GF(3)
]
[
    Mapping from: AModule of dimension 64 over GF(3) to AModule of dimension 64
    over GF(3),
    Mapping from: AModule of dimension 22 over GF(3) to AModule of dimension 5
    over GF(3),
    Mapping from: AModule of dimension 22 over GF(3) to AModule of dimension 3
    over GF(3),
    Mapping from: AModule of dimension 15 over GF(3) to AModule of dimension 2
    over GF(3),
    Mapping from: AModule of dimension 14 over GF(3) to AModule of dimension 1
    over GF(3),
    Mapping from: AModule of dimension 8 over GF(3) to AModule of dimension 4
    over GF(3),
    Mapping from: AModule of dimension 9 over GF(3) to AModule of dimension 3
    over GF(3),
    Mapping from: AModule of dimension 3 over GF(3) to AModule of dimension 3
    over GF(3)
]
```

60.2.5 Maps on Homology

`InducedMapOnHomology(f, n)`

The homomorphism induced on homology by the chain map f in degree n.

`ConnectingHomomorphism(f, g, n)`

The connecting homomorphism in degree n of the short exact sequence of chain complexes given by the chain maps f and g.

`LongExactSequenceOnHomology(f, g)`

The long exact sequence on homology for the exact sequence of complexes given by the chain maps f and g as a chain complex with the homology group in degree i for the Cokernel of the complex C appearing in degree $3i$.

Example H60E3

We create a basic algebra with two simple modules and four nonidempotent generators. The relations are given in the sequence rrr.

```
> ff := GF(3);
> p := Characteristic(ff);
> FA<e1, e2, y, x, a, b> := FreeAlgebra(ff,6);
> rrr := [y^p,x^p,x*y+y*x,x*a*b-a*b*x,y*a*b-a*b*y,(b*a)^2,(b*a)^2];
> A := BasicAlgebra(FA,rrr,2,[<1,1>,<1,1>,<1,2>,<2,1>]);
> A;
Basic algebra of dimension 81 over GF(3)
Number of projective modules: 2
Number of generators: 6
> DimensionsOfProjectiveModules(A);
```

```
[ 45, 36 ]
```

Now we generate the simple modules and their projective covers. We want a module PP that is the direct sum of two copies of each of the projective indecomposable module.

```
> S1 := SimpleModule(A,1);
> PP := ProjectiveModule(A,[2,2]);
```

Now we are going to create a complex all of whose terms are isomorphic to PP. To make it interesting we will insist that the first homomorphism have its image in the second radical layer of PP.

```
> J1 := JacobsonRadical(PP);
> theta1 := Morphism(J1,PP);
> J2 := JacobsonRadical(J1);
> theta2 := Morphism(J2,J1);
> theta := theta2*theta1;
> HomPJ := AHom(PP, J2);
> HomPJ;
KMatrixSpace of 162 by 150 matrices and dimension 300 over GF(3)
> gamma := Random(HomPJ)*theta;
> LL := [* gamma *];
```

So we now have the first boundary map of the complex. The other boundary maps are generated randomly.

```
> for i := 1 to 15 do
>     K, phi := Kernel(gamma);
>     HomPK := AHom(PP,K);
>     gamma := Random(HomPK)*MapToMatrix(phi);
>     LL := [* gamma *] cat LL;
> end for;
```

Now we make the list into a chain complex.

```
> C := Complex(LL,0);
> C;
Basic algebra complex with terms of degree 16 down to 0
Dimensions of terms: 162 162 162 162 162 162 162 162 162 162 162 162 162 162
162 162 162
> DimensionsOfHomology(C);
[ 30, 23, 34, 32, 35, 32, 35, 42, 30, 34, 32, 33, 29, 42, 21 ]
Now we will get a random subcomplex.
> a,b := Degrees(C);
> S, mu := RandomSubcomplex(C,[2: i in [1 .. a-b+1]]);
> S;
Basic algebra complex with terms of degree 16 down to 0
Dimensions of terms: 162 155 162 162 162 154 142 162 162 148 154 162 162 152
155 152 162
> DimensionsOfHomology(S);
```

```
[ 27, 27, 34, 32, 35, 33, 48, 42, 28, 44, 38, 33, 29, 48, 24 ]
```

Now take the quotient.

```
> Q,nu := quo<C|S>;
> Q;
Basic algebra complex with terms of degree 16 down to 0
Dimensions of terms: 0 7 0 0 0 8 20 0 0 14 8 0 0 10 7 10 0
> DimensionsOfHomology(Q);
[ 7, 0, 0, 0, 6, 18, 0, 0, 12, 6, 0, 0, 8, 5, 10 ]
```

Now we check to see if this is a short exact sequence of chain maps.

```
> IsShortExactSequence(mu,nu);
true
> lll := LongExactSequenceOnHomology(mu,nu);
Basic algebra complex with terms of degree 47 down to 3
Dimensions of terms: 27 30 7 27 23 0 34 34 0 32 32 0 35 35 6 33 32 18 48 35 0
42 42 0 28 30 12 44 34 6 38 32 0 33 33 0 29 29 8 48 42 5 24 21 10
> IsExact(lll);
true
```

61 MULTILINEAR ALGEBRA

Chapter 61
MULTILINEAR ALGEBRA

61.1 Introduction

This chapter discusses Magma operations with multilinear algebra. Wherever possible we follow the conventions in use in physics [Wey50, Chapter V], differential geometry [Lee13, Chapter 10], and algebra [Lan12]. Necessary categorical formalism is drawn largely from [Wil13]. The package covers:

(1) Tensors and multilinear functions and their associated groups and algebras.

(2) Spaces of tensors.

(3) Categories of tensors and tensor spaces.

(4) Exceptional tensors of octonions and Jordan algebras

61.1.1 Overview

Throughout this chapter the symbol $\rightarrowtail$ denotes a multilinear map $[\cdot] : U_v \times \cdots \times U_1 \rightarrowtail U_0$ of K-modules $\{U_v, \ldots, U_0\}$ known as the *frame* of a multilinear map or tensor. The module U_0 is reserved for the codomain which in turn makes reverse indexing the simplest notation. Every tensor in Magma is treated as an element of a tensor space which is, by default, is a universal tensor space:

$$\hom_K(U_v, \ldots, \hom_K(U_1, U_0) \cdots).$$

In Magma, the tensor space determines the associated multilinear function of a given tensor T. Evaluation of T mimics the application of a map $U_v \times \cdots \times U_1 \to U_0$, for instance, $< \mathbf{u_v}, \ldots, \mathbf{u_1} > @\mathbf{T}$. Special attention is given to bilinear maps $* : U_2 \times U_1 \rightarrowtail U_0$ including the ability to use infix notation $\mathbf{u_2} * \mathbf{u_1}$. Tensor spaces have type `TenSpc` and behave like modules in that they have subspaces and quotient spaces. Tensors have type `TenSpcElt` and behave in a similar way to Magma matrices.

A library of commonly used exceptional tensors is provided. These include octonion algebras and exceptional Jordan algebras.

Tensor categories, type `TenCat`, provide Magma with the information needed to interpret the contents of a tensor space. For example, one tensor category may treat a $(d \times d)$-matrix F over a field K as a linear map $K^d \to K^d$, while another assigns the same matrix to a bilinear form $K^d \times K^d \rightarrowtail K$. Functors are provided to change tensor categories and to define standard categories.

61.2 Tensors

A tensor has the following information associated with it:-

* A commutative ring K of coefficients.
* A valence v indicating the number of variables to include in its associated multilinear map.
* A list $[U_v, \ldots, U_0]$ of K-modules called the *frame*.
* A function $U_v \times \cdots \times U_1 \to U_0$ that is K-linear in each U_i. Tensors have type `TenSpcElt` and are formally elements of a tensor space (type `TenSpc`). By default a tensor's parent space is a universal tensor space:

  $$\hom_K(U_v, \ldots, \hom_K(U_1, U_0) \cdots) \cong \hom_K(U_v \otimes_K \cdots \otimes_K U_1, U_0).$$

  The left hand module is used primarily as it avoids the need to work with the equivalence classes of a tensor product. Operations such as linear combinations of tensors take place within a tensor space. A tensor space has attributes such as coefficients, valence, and frame.

When necessary, the user may further direct the operations on tensors to appropriate tensor categories (type `TenCat`). For instance covariant and contravariant variables can be specified and symmetry conditions can be imposed. If no tensor category is prescribed then a default tensor category is selected based on the method of creation.

61.2.1 Creating Tensors

61.2.1.1 Black-box Tensors

A user can specify a tensor by a black-box function that evaluates the required multilinear map.

```
Tensor(S, F)
```

```
Tensor(S, F, Cat)
```

Returns the tensor having the frame S and the tensor category Cat. The last entry is assumed to be the codomain of the multilinear map. The user-defined function F should take as input a tuple of elements of the domain and return an element of the codomain. If no tensor category is provided, then Albert's homotopism category is used.

```
Tensor(D, C, F)
```

```
Tensor(D, C, F, Cat)
```

Returns the tensor with tensor category Cat and frame specified by D and C, where D is a sequence of spaces in the domain and C the codomain. The user-defined function F should take as input a tuple of elements of D and return an element of C. If no tensor category is provided, then Albert's homotopism category is used.

Example H61E1

Tensors make it easy to create algebras that do not fit into traditional categories, such as algebras with triple products.

```
> K := GF(541);
> U := KMatrixSpace(K,2,3);
> my_prod := func< x | x[1]*Transpose(x[2])*x[3] >;
> T := Tensor([U,U,U,U], my_prod );
> T;
Tensor of valence 3, U3 x U2 x U1 >-> U0
U3 : Full KMatrixSpace of 2 by 3 matrices over GF(541)
U2 : Full KMatrixSpace of 2 by 3 matrices over GF(541)
U1 : Full KMatrixSpace of 2 by 3 matrices over GF(541)
U0 : Full KMatrixSpace of 2 by 3 matrices over GF(541)
> A := U![1,0,0,0,0,0];
> <A,A,A>@T;  // A is a generalized idempotent
[  1   0   0]
[  0   0   0]
```

We can experiment to see if this triple product is left associative.

```
> X := [Random(U) : i  in [1..5]];
> <<X[1],X[2],X[3]>@T,X[4],X[5]>@T eq
>    <X[1],<X[4],X[3],X[2]>@T,X[5]>@T;
true
```

To confirm this we can create a new tensor for the left triple-associators and check that its image is 0.

```
> my_left_asct := func<X|<<X[1],X[2],X[3]>@T,X[4],X[5]>@T
>     - <X[1],<X[4],X[3],X[2]>@T,X[5]>@T >;
> LT := Tensor([U: i in [0..5]], my_left_asct);
> I := Image(LT);
> Dimension(I);
0
```

61.2.1.2 Tensors with Structure Constant Sequences

Most computations with tensors T will be carried out using structure constants $T_{j_v \cdots j_0} \in K$. Here T is framed by free K-modules $[U_v, \ldots, U_0]$ with each U_i having an ordered bases $\mathcal{B}_i = [e_{i1}, \ldots, e_{id_i}]$. The interpretation of structure constants is that the associated multilinear function $[x_v, \ldots, x_1]$ from $U_v \times \cdots \times U_1$ into U_0 is determined on bases $\mathcal{B}_i$ as follows:

$$[e_{vj_v}, \ldots, e_{1j_1}] = \sum_{k=1}^{d_0} T_{j_v \cdots j_0} e_{0k}.$$

Structure constants are input and stored as sequences S in K according to the following assignment. Set $f : \mathbf{Z}^{v+1} \to \mathbf{Z}$ to be:

$$f(j_v, \ldots, j_0) = 1 + \sum_{s=0}^{v} (j_s - 1) \prod_{t=0}^{s-1} d_t.$$

So $S[f(j_v, \ldots, j_0)] = T_{j_v \cdots j_0}$ specifies the structure constants as a sequence.

Notes.

* MAGMA does not currently support the notion of a sparse sequence of structure constants. A user can provide this functionality by specifying the multilinear function associated with a tensor by means of a user program rather than by structure constants.

* Some routines in MAGMA require structure constant sequences. If they are not provided, MAGMA may compute and store a structure constant representation with the tensor.

* MAGMA does not separate structure constant indices that are contravariant. Instead contravariant variables are signaled by tensor categories. So Ricci-styled tensors $T_{a_p \cdots a_1}^{b_q \cdots b_1}$ should be input as $T_{a_{p+q} \cdots a_{1+q} b_q \cdots b_1}$ and the tensor category changed to mark $\{q..1\}$ as contravariant. Intrinsics are provided to facilitate this approach.

```
Tensor(R, D, S)
```

```
Tensor(D, S)
```

```
Tensor(R, D, S, Cat)
```

```
Tensor(D, S, Cat)
```

Let R be a commutative unital ring and D a sequence $[d_v, \ldots, d_0]$ giving the dimensions of the R-modules $U_v, \ldots, U_0$. The intrinsic returns the tensor with frame $\{U_v, \ldots, U_1\}$, codomain U_0 and multilinear map defined by the structure constant sequence S. If R is not specified then the parent ring of the first element of S is used. The default tensor category Cat is the homotopism category.

```
StructureConstants(T)
```

```
Eltseq(T)
```

Returns the sequence of structure constants of the given tensor T.

61.2.2 Bilinear Tensors

In the special case of bilinear maps $U_2 \times U_1 \rightarrowtail U_0$, the structure constant sequences may be presented as lists $[M_1, \ldots, M_a]$ of matrices. This can be considered as a left (resp. right) representation $U_2 \to \hom_K(U_1, U_0)$, (resp.$U_1 \to \hom_K(U_2, U_0)$). Alternatively, bilinear maps can be treated as *systems of bilinear forms* $[M_1, \ldots, M_a]$ where the matrices are the

Gram matrices of bilinear forms $\phi_i : U_2 \times U_1 \rightarrowtail K$. Here the associated bilinear map $U_2 \times U_1 \rightarrowtail U_0$ is specified by

$$(u_2, u_1) \mapsto (\phi_1(u_2, u_1), \ldots, \phi_a(u_2, u_1)).$$

`Tensor(M, s, t)`

`Tensor(M, s, t, C)`

Given a sequence of matrices M, the bilinear tensor specified by M is returned. The interpretation of the matrices as structure constant sequences is determined by the coordinates s and t which must be integers from the set $\{2, 1, 0\}$. Optionally a tensor category C can be specified.

`AsMatrices(T, s, t)`

`SystemOfForms(T)`

For a tensor T with frame $[K^{d_v}, \ldots, K^{d_0}]$, a list $[M_1, \ldots, M_d]$, $d = (d_v \cdots d_0)/d_s d_t$, of $(d_s \times d_t)$-matrices in K representing the tensor as an element of $\hom_K(K^{d_s} \otimes_K K^{d_t}, K^d)$ is returned. For the intrinsic `SystemOfForms`, T must have valence 2 and the implied values are $s = 2$ and $t = 1$.

Example H61E2

```
> T := Tensor(GF(3), [2 ,2 ,2], [1 ,0 ,0 ,1 ,0 ,1 , -1 ,0]);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 2 over GF(3)
U1 : Full Vector space of degree 2 over GF(3)
U0 : Full Vector space of degree 2 over GF(3)
> StructureConstants(T);
[ 1, 0, 0, 1, 0, 1, 2, 0 ]
```

Systems of forms can be useful in providing some visually useful information such as symmetry. The choice of coordinates influences the resulting list of matrices.

```
> AsMatrices(T,1,0);
[
    [1 0]
    [0 1],
    [0 1]
    [2 0]
]
> SystemOfForms (T);
[
    [1 0]
    [0 2],
    [0 1]
    [1 0]
]
> IsSymmetric (T);
true
```

Example H61E3

Systems of forms will always produce tensors of valance 2. In particular two tensors with a common structure constant sequence can be treated differently by changing the dimensions that frame the data. Thus a tensor is more than a list of coefficients.

```
> F := [ RandomMatrix(GF(2),2,3) : i in [1..4]];
> T := Tensor(F,2,1);
> T;
Tensor U2 x U1 >-> U0
U2 : Full Vector space of degree 2 over GF(2)
U1 : Full Vector space of degree 3 over GF(2)
U0 : Full Vector space of degree 4 over GF(2)
> T2 := Tensor( [2,2,2,2], [1..16] );
> F2 := AsMatrices(T2,3,2);
> T3 := Tensor(F2, 2, 1);
> Eltseq(T2) eq Eltseq(T3);
true
> T2 eq T3;
false
> Valence(T2);
3
> Valence(T3);
2
```

61.2.2.1 Tensors from Algebraic Objects

A natural source of tensors is an algebraic object with a distributive property. Each tensor is assigned a category relevant to its origin.

`Tensor(A)`

Given an algebra A with an element product that satisfies an appropriate distributive law, return the bilinear tensor given by the product in A.

`CommutatorTensor(A)`

Given an algebra A return the tensor corresponding to the bilinear commutator map $[a, b] = ab - ba$ on the algebra A.

`AssociatorTensor(A)`

Given an algebra A return the tensor corresponding to the trilinear associator map $[a, b, c] = (ab)c - a(bc)$ on the algebra A.

Example H61E4

```
> A := MatrixAlgebra( Rationals () , 5 );
> AC := CommutatorTensor( A );
> IsAlternating( AC ); // [X, X] = 0?
true
```

Do three random octonions satisfy the associate law? Hardly ever.

```
> O := OctonionAlgebra ( GF (541) , -1 , -1 , -1);
> T := AssociatorTensor ( O );
> < Random ( O ) , Random ( O ) , Random ( O ) > @ T eq 0;
false
```

But the left alternative law, $(aa)b = a(ab)$ is always satisfied, as octonions are alternative algebras.

```
> a := Random ( O );
> b := Random ( O );
> <a ,a ,b > @ T eq 0;
true
> IsAlternating ( T );
true
```

```
pCentralTensor(G, s, t)
```

```
pCentralTensor(G, p, s, t)
```

Returns the bilinear map of commutation from the associated graded Lie algebra of the lower exponent-p central series η of G. The bilinear map pairs η_s/η_{s+1} with η_t/η_{t+1} into η_{s+t}/η_{s+t+1}. If $s = t$ the tensor category is set to force $U_2 = U_1$; otherwise it is the general homotopism category.

Example H61E5

Groups have a single binary operation. So even when groups are built from rings it can be difficult to recover the ring from the group operations. Tensors supply one approach for that task.

```
> P := ClassicalSylow(SL(3,125), 5);
> Q := PCGroup(P);Ψ// The group Q wont have explicit knowledge of GF(125).
> Q;
GrpPC : Q of order 1953125 = 5^9
PC-Relations:
    Q.4^Q.1 = Q.4 * Q.7^4,
    Q.4^Q.2 = Q.4 * Q.8^4,
    Q.4^Q.3 = Q.4 * Q.9^4,
    Q.5^Q.1 = Q.5 * Q.8^4,
    Q.5^Q.2 = Q.5 * Q.9^4,
    Q.5^Q.3 = Q.5 * Q.7^3 * Q.8^3,
    Q.6^Q.1 = Q.6 * Q.9^4,
    Q.6^Q.2 = Q.6 * Q.7^3 * Q.8^3,
    Q.6^Q.3 = Q.6 * Q.8^3 * Q.9^3
> T := pCentralTensor(Q,1,1);
> F := Centroid(T);  // Recover GF(125)
> Dimension(F);
3
> IsSimple(F);
true
> IsCommutative(F);
true
```

Polarisation(f)

Polarization(f)

Returns the polarization of the homogeneous multivariate polynomial f as a tensor and as a multivariate polynomial. Note that the intrinsic does not normalize by $1/d!$, where d is the degree of f.

Example H61E6

The polarization intrinsic is applied to the polynomial $f = x^2 * y$. Because f is homogeneous of degree 3 with 2 variables, it is expected that the polarization will have 6 variables and that the corresponding multilinear form will be $K^2 \times K^2 \times K^2 \rightarrowtail K$. The polarization of f is given by $P(x_1, x_2, y_1, y_2, z_1, z_2) = 2(x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1)$.

```
> K<x,y> := PolynomialRing(Rationals(), 2);
> T,p := Polarization(x^2*y);
> p;
2*$.1*$.3*$.6 + 2*$.1*$.4*$.5 + 2*$.2*$.3*$.5
> T;
Tensor of valence 3, U3 x U2 x U1 >-> U0
U3 : Full Vector space of degree 2 over Rational Field
U2 : Full Vector space of degree 2 over Rational Field
U1 : Full Vector space of degree 2 over Rational Field
U0 : Full Vector space of degree 1 over Rational Field
> <[1,0],[1,0],[1,0]> @ T;
(0)
> <[1,0],[1,0],[0,1]> @ T;
(2)
```

61.2.2.2 New Tensors from Old

A number of new tensors may be constructed from a given tensor.

```
AlternatingTensor(T)
```

Returns the alternating tensor induced by the given tensor T. If the tensor is already alternating, then the given tensor is returned.

```
AntisymmetricTensor(T)
```

Returns the antisymmetric tensor induced by the given tensor T. If the tensor is already antisymmetric, then the given tensor is returned.

```
SymmetricTensor(T)
```

Returns the symmetric tensor induced by the given tensor T. If the tensor is already symmetric, then the given tensor is returned.

```
Shuffle(T, g)
```

Given a tensor T in $\hom(U_v, \dots, \hom(U_1, U_0) \cdots)$, this intrinsic generates the representation of T in $\hom(U_{v^g}, \dots, \hom(U_{1^g}, U_{0^g}) \dots)$, where g is a permutation of $\{0..v\}$. The permutation g may be given as an element of a permutation group or as a sequence. Both the image and pre-image of 0 under g will be replaced by their K-dual space. For cotensors, g must permute $\{1..v\}$.

61.2.3 Operations with Tensors

Magma supports two perspectives for operations with tensors. Firstly, tensors determine multilinear maps and so behave as functions. Secondly, tensors are elements of a tensor space and so behave as elements of a module.

61.2.3.1 Elementary Operations

Treating the tensor space as a K-module, we have the standard operations.

```
S + T
k * T
```

Given tensors S and T belonging to the same tensor space V, and an element k belonging to the coefficient ring K, this intrinsic returns the sum or scalar multiple of S and T as elements of v. The corresponding multilinear maps are the sum or scalar multiple of the multilinear maps.

```
TensorOnVectorSpaces(T)
```

Returns the tensor on vector spaces and an isotopism from the given tensor. This is a forgetful functor that forces all domain and codomain terms to be vector spaces.

```
AssociatedForm(T)
```

For a tensor T with multilinear maps $U_v \times \cdots \times U_1 \rightarrowtail U_0$, this intrinsic creates the associated multilinear form $U_v \times \cdots \times U_1 \times U_0^* \rightarrowtail K$. The valence is increased by 1.

```
Compress(T)
```

Returns the compression of the tensor T. This removes all 1-dimensional spaces in the domain.

Example H61E7

The forgetful functor is illustrated using the tensor corresponding to multiplication of elements in the Lie Algebra $D4$ defined over $\mathbf{F}_5$. Applying the forgetful functor to this tensor has the effect of replacing the algebra structures by their underlying vector spaces. However, it is still possible to use this new tensor on vector spaces to evaluate products of Lie algebra elements.

```
> L := LieAlgebra("D4",GF(5));
> T := Tensor(L);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Lie Algebra of dimension 28 with base ring GF(5)
U1 : Lie Algebra of dimension 28 with base ring GF(5)
U0 : Lie Algebra of dimension 28 with base ring GF(5)
> < L.2, L.11 > @ T;
(1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0)
>
> S := TensorOnVectorSpaces(T);
> S;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 28 over GF(5)
U1 : Full Vector space of degree 28 over GF(5)
U0 : Full Vector space of degree 28 over GF(5)
> V := Domain(S)[1];
> < L.2, L.11 > @ S eq < V.2, V.11 > @ S;
true
```

Now we compute the forms associated to S to get a trilinear map. We shuffle it by the permutation $(0, 3)$ and compress it. The result is just the shuffle of the original bilinear map S by $(0, 2, 1)$.

```
> AF := AssociatedForm(S);
> AF;
Tensor of valence 3, U3 x U2 x U1 >-> U0
U3 : Full Vector space of degree 28 over GF(5)
U2 : Full Vector space of degree 28 over GF(5)
U1 : Full Vector space of degree 28 over GF(5)
U0 : Full Vector space of degree 1 over GF(5)
> Eltseq(AF) eq Eltseq(S);
true
> < L.2, L.11, L.1 > @ AF;
```

```
(1)
> < L.2, L.11, L.2 > @ AF;
(0)
>
> U := Shuffle(AF,[3,1,2,0]);
> U;
Tensor of valence 3, U3 x U2 x U1 >-> U0
U3 : Full Vector space of degree 1 over GF(5)
U2 : Full Vector space of degree 28 over GF(5)
U1 : Full Vector space of degree 28 over GF(5)
U0 : Full Vector space of degree 28 over GF(5)
>
> Cmp := Compress(U);
> Shf := Shuffle(S,[2,0,1]);
> Cmp eq Shf;
true
```

61.2.3.2 General Properties

`Parent(T)`

Returns the tensor space that contains tensor T. The default space is the universal tensor space.

`Domain(T)`

Returns the domain of the tensor T as a list of modules.

`Codomain(T)`

Returns the codomain of the tensor T.

`Valence(T)`

Returns the valence of the tensor T.

`Frame(T)`

Returns the modules in the frame of tensor T in the form of a list containing the domain modules and the codomain.

`TensorCategory(T)`

Returns the underlying tensor category of tensor T.

`ChangeTensorCategory(T, C)`

`ChangeTensorCategory( T, C)`

Returns the tensor T with the given category.

`IsCovariant(T)`

`IsContravariant(T)`

Decides if the underlying category of tensor T is covariant or contravariant.

Example H61E8

```
> T := RandomTensor(GF(3),[3,4,5,6]);
> T;
Tensor of valence 3, U3 x U2 x U1 >-> U0
U3 : Full Vector space of degree 3 over GF(3)
U2 : Full Vector space of degree 4 over GF(3)
U1 : Full Vector space of degree 5 over GF(3)
U0 : Full Vector space of degree 6 over GF(3)
> Valence(T);
3
>
> BaseRing(T);
Finite field of size 3
>
> Frame(T);
[*
    Full Vector space of degree 3 over GF(3),
    Full Vector space of degree 4 over GF(3),
    Full Vector space of degree 5 over GF(3),
    Full Vector space of degree 6 over GF(3)
*]
>
> Domain(T);
[*
    Full Vector space of degree 3 over GF(3),
    Full Vector space of degree 4 over GF(3),
    Full Vector space of degree 5 over GF(3)
*]
>
> Codomain(T);
Full Vector space of degree 6 over GF(3)
>
> Parent(T); // Universal tensor space
Tensor space of dimension 360 over GF(3) with valence 3
U3 : Full Vector space of degree 3 over GF(3)
U2 : Full Vector space of degree 4 over GF(3)
U1 : Full Vector space of degree 5 over GF(3)
U0 : Full Vector space of degree 6 over GF(3)
>
> TensorCategory(T);
Tensor category of Valence 3 (->,->,->,->) ({ 1 },{ 2 }, { 0 },{ 3 })
>
> Cat := TensorCategory([-1,-1,1,1],{{i} : i in [0..3]});
> ChangeTensorCategory(~T, Cat);
> TensorCategory(T);
```

```
Tensor category of Valence 3 (<-,<-,->,->) ({ 1 },{ 2 }, { 0 },{ 3 })
```

`Image(T)`

Returns the (categorical) image of the tensor T along with a map to the vector space. Thus, if the type of the codomain of T is not `ModTupRng`, the returned map is an isomorphism from the codomain of T to the free R-module R^{d_0}.

`BaseRing(T)`

`BaseField(T)`

Returns the base ring or field of the tensor.

`NondegenerateTensor(T)`

Returns the nondegenerate multilinear map associated to T along with a homotopism from the given tensor to the returned nondegenerate tensor.

`IsNondegenerate(T)`

Decides whether T is a nondegenerate multilinear map.

`FullyNondegenerateTensor(T)`

Returns the fully nondegenerate multilinear map associated to tensor T along with a cohomotopism from the given tensor to the returned tensor.

`IsFullyNondegenerate(T)`

Return `true` if the tensor T is a fully nondegenerate multilinear map.

`IsAlternating(T)`

Return `true` if the tensor T is an alternating tensor.

`IsAntisymmetric(T)`

Return `true` if the tensor T is an antisymmetric tensor.

`IsSymmetric(T)`

Return `true` if the tensor T is a symmetric tensor.

Example H61E9

```
> J := Matrix(GF(9),[[0,1,-1],[1,-1,-1],[-1,-1,1]]);
> M := DiagonalJoin(J,ZeroMatrix(GF(9),3,3));
> M;
[    0     1     2     0     0     0]
[    1     2     2     0     0     0]
[    2     2     1     0     0     0]
[    0     0     0     0     0     0]
[    0     0     0     0     0     0]
[    0     0     0     0     0     0]
>
> T := Tensor([M,-M],2,1);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 6 over GF(3^2)
U1 : Full Vector space of degree 6 over GF(3^2)
U0 : Full Vector space of degree 2 over GF(3^2)
>
> Image(T);
Vector space of degree 2, dimension 1 over GF(3^2)
Generators:
(    1     2)
Echelonized basis:
(    1     2)
Mapping from: Full Vector space of degree 2 over GF(3^2) to
Full Vector space of degree 2 over GF(3^2) given by a rule
>
> Radical(T);
<
    Vector space of degree 6, dimension 3 over GF(3^2)
    Echelonized basis:
    (    0     0     0     1     0     0)
    (    0     0     0     0     1     0)
    (    0     0     0     0     0     1),
    Vector space of degree 6, dimension 3 over GF(3^2)
    Echelonized basis:
    (    0     0     0     1     0     0)
    (    0     0     0     0     1     0)
    (    0     0     0     0     0     1)
>
```

The tensor we built above is degenerate because it has a nontrivial radical. We will construct the nondegenerate tensor as well as the fully nondegenerate tensor.

```
> ND := NondegenerateTensor(T);
> ND;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 3 over GF(3^2)
```

```
U1 : Full Vector space of degree 3 over GF(3^2)
U0 : Full Vector space of degree 2 over GF(3^2)
>
> FN := FullyNondegenerateTensor(T);
> FN;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 3 over GF(3^2)
U1 : Full Vector space of degree 3 over GF(3^2)
U0 : Vector space of degree 2, dimension 1 over GF(3^2)
Generators:
(    1    2)
Echelonized basis:
(    1    2)
```

61.2.3.3 As Multilinear Maps

Regarding tensors as multilinear maps makes it possible to define the operations of composition and evaluation for tensors.

`x @ T`

Evaluates the tensor T at $x \in U_v \times \cdots \times U_1$.

`T * f`

The tensor formed by composing the tensor T with the map f is constructed.

`S eq T`

Returns `true` if the tensors S and T have the same multilinear map.

61.2.3.4 Operations with Bilinear Maps

Tensors of valence 2, also known as bilinear tensors, or as bilinear maps, are commonly described as distributive products. Examples include the product of two elements of an algebra, the action of a ring on a module, or an inner product. Magma supports this interpretation by permitting an infix $x * y$ notation for the evaluation of bilinear tensors. This is achieved by creating special types `BmpU[Elt]`, `BmpV[Elt]`, and `BmpW[Elt]` for the terms of the frame of a bilinear tensor.

`x * y`

If x and y are associated to the bilinear map B, these operations return `<x,y> @ B`.

`x * B`

`B * y`

Given a bilinear tensor B with frame $[U_2, U_1, U_0]$, $x * B$ returns the action on the right as a linear map $L : U_1 \to U_0$ given by $vL = x * v$, where x is an element of U_2. If x is a subspace of U_2, then this returns a sequence of maps, one for each element in the basis for x. The left-hand action is given by $B * y$. If B is valence 1, then the image of either x or y is returned.

```
Parent(x)
```

The parent space of the bilinear map element x is returned.

```
Parent(X)
```

The original bilinear map from which the space X originated is returned.

```
LeftDomain(B)
```

The left domain U_2 of the bilinear map B with frame $[U_2, U_1, U_0]$ is returned. This operation is used when enabling the use of infix notation.

```
RightDomain(B)
```

The right domain U_1 of the bilinear map B with frame $[U_2, U_1, U_0]$ is returned This operation is used when enabling the use of infix notation.

```
IsCoercible(S,x)
```

Returns `true` if x can be coerced the left domain (right domain) S of a bilinear map. If successful it returns the coerced element.

```
u1 eq u2
```

Returns `true` if the elements $u1$ and $u2$ from the left domain of a bilinear map are equal.

```
v1 eq v2
```

Returns `true` if the elements $v1$ and $v2$ from the right domain of a bilinear map are equal.

```
U1 eq U2
```

Returns `true` if the subspaces $U1$ and $U2$ of the left domain of a bilinear map are equal.

```
V1 eq V2
```

Returns `true` if the subspaces $V1$ and $V2$ of the right domain of a bilinear map are equal.

Example H61E10

If $B : U_2 \times U_1 \rightarrowtail U_0$ is a tensor, then for $x \in U_2$, $x * B : U_1 \to U_0$ is a valence 1 tensor. Similarly, for $y \in U_1$, $B * y : U_2 \to U_0$. Furthermore, $x * B * y$ can be used in place of $< x, y > @B$.

```
> B := RandomTensor(GF(5),[4,3,5]);
> B;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 4 over GF(5)
U1 : Full Vector space of degree 3 over GF(5)
U0 : Full Vector space of degree 5 over GF(5)
> [3,2,0,1]*B*[1,1,2];
(2 1 1 3 4)
> [1,0,0,0]*B;
Tensor of valence 1, U1 >-> U0
U1 : Full Vector space of degree 3 over GF(5)
U0 : Full Vector space of degree 5 over GF(5)
> B*[0,2,0];
Tensor of valence 1, U1 >-> U0
U1 : Full Vector space of degree 4 over GF(5)
U0 : Full Vector space of degree 5 over GF(5)
```

Example H61E11

Suppose G is a p-group and $B : U_2 \times U_1 \rightarrowtail U_0$ is the tensor given by commutation where $U_2 = U_1 = G/\Phi(G)$ and U_0 is the next factor of the exponent-p central series of G.

```
> G := SmallGroup(512,10^6);
> B := pCentralTensor(G,2,1,1);
> B;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 5 over GF(2)
U1 : Full Vector space of degree 5 over GF(2)
U0 : Full Vector space of degree 4 over GF(2)
> U := LeftDomain(B);   //U2
> V := RightDomain(B);  //U1
> U![0,1,0,1,0] * V![1,0,0,0,0];
(1 0 0 1)
> U!(G.2*G.4) * V!G.1;
(1 0 0 1)
```

This notation can be used to evaluate subspaces.

```
> H := sub< G | G.2,G.4 >;
> U!H * V!G.1;
Vector space of degree 4, dimension 2 over GF(2)
Generators:
(0 0 0 1)
(1 0 0 0)
Echelonized basis:
```

```
(1 0 0 0)
(0 0 0 1)
> U!H * V!G;
Vector space of degree 4, dimension 3 over GF(2)
Generators:
(1 0 0 0)
(0 0 1 0)
(0 0 0 1)
(1 0 0 0)
(1 0 0 0)
Echelonized basis:
(1 0 0 0)
(0 0 1 0)
(0 0 0 1)
```

61.2.3.5 Manipulating Tensor Data

The data associated with a tensor can be accessed in different ways. In the case of tensors given by structure constants this access is the multidimensional analog of choosing a row or column of a matrix. Other operations are a generalization of the transpose of a matrix. Magma does these operations with a degree of efficiency, that is, it may not physically move the values in a structure constant sequence but instead permute the lookup coordinates of the values.

`Slice(T, grid)`

`InducedTensor(T, grid)`

Returns the slice of the structure constants running through the given grid. For a tensor framed by free modules $[U_v, \dots, U_0]$ with $d_i = \dim U_i$, a grid is a sequence $[G_v, \dots, G_0]$ of subsets $G_i \subseteq \{1..d_i\}$. The slice is the list of entries in the structure constants of the tensor indexed by $G_v \times \cdots \times G_0$. `Slice` returns the structure constants whereas `InducedTensor` produces a tensor with these structure constants.

Example H61E12

```
> U := VectorSpace(Rationals(),4);
> V := VectorSpace(Rationals(),3);
> W := VectorSpace(Rationals(),2);
> TS := TensorSpace([U,V,W]);
> T := TS![ i : i in [1..24] ];
> Slice(T,[{1..4},{1..3},{1..2}]);  // structure constants
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17,
18, 19, 20, 21, 22, 23, 24 ]
```

The data `Slice` extracts from the structure constants of tensor T can be used as the structure constants for a new tensor.

```
> Slice(T,[{1..4},{2},{1}]);
```

```
[ 3, 9, 15, 21 ]
>
> W1 := VectorSpace(Rationals(),1);
> pi := hom< W -> W1 | <W.1,W1.1>, <W.2,W1!0> >; // project
> Eltseq( (T*V.2)*pi );
[ 3, 9, 15, 21 ]
```

However, in this example, the tensors are not the same. If the induced tensor is compressed, then they become equal.

```
> T_ind := InducedTensor(T,[{1..4},{2},{1}]);
> T_ind;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 4 over Rational Field
U1 : Full Vector space of degree 1 over Rational Field
U0 : Full Vector space of degree 1 over Rational Field
> S := (T*V.2)*pi;
> S;
Tensor of valence 1, U1 >-> U0
U1 : Full Vector space of degree 4 over Rational Field
U0 : Full Vector space of degree 1 over Rational Field
>
> Compress(T_ind) eq S;
true
```

`Foliation(T, i)`

If T is a tensor contained in $\hom(U_v \otimes \cdots \otimes U_1, U_0)$, and i is the index of some term of the frame of T, this intrinsic returns the matrix representing the linear map $U_i \to \hom(\bigotimes_{j\neq i} U_i, U_0)$ using the bases of each U_j. If $i = 0$, then the returned matrix is given by the representation $U_0^* \to \hom(\bigotimes U_i, K)$.

`AsTensorSpace(T, i)`

If T is a tensor contained in $\hom(U_v \otimes \cdots \otimes U_1, U_0)$, and i is the index of some term of the frame of T, this intrinsic returns the associated tensor space of T at $i > 0$ together with a matrix given by the foliation of T at i. The returned tensor space is framed by $U_v \times \cdots \times U_{i+1} \times U_{i-1} \times \cdots \times U_1 \rightarrowtail U_0$ and is generated by the tensors T_u for each u in the basis of U_i.

`AsCotensorSpace(T)`

Given a tensor T, this intrinsic returns the associated cotensor space of T together with a matrix given by the foliation of T at 0. The returned cotensor space has frame $U_v \times \cdots \times U_1 \rightarrowtail K$ and is generated by the tensors Tf for each f in the basis of U_0^*.

`AsTensor(S)`

Given a tensor space S, this intrinsic returns a tensor belonging to S. If S is contravariant, then the returned tensor has the frame $U_v \times \cdots \times U_1 \rightarrowtail T$. If S is covariant, then the returned tensor has the frame $T \times U_v \times \cdots \times U_1 \rightarrowtail U_0$. Note that `AsTensor` is "inverse" to `AsCotensorSpace` and `AsTensorSpace` when $i = v$.

Example H61E13

```
> T := RandomTensor( GF(7), [2,3,4] );
> Foliation(T,0);
[3 4 3 3 6 4]
[0 6 4 3 3 4]
[0 0 6 2 3 3]
[4 6 4 0 0 3]
>
> Slice(T,[{1,2},{1..3},{3}]); // row 3
[ 0, 0, 6, 2, 3, 3 ]
>
> Slice(T,[{2},{1},{1..4}]); // col 4
[ 3, 3, 2, 0 ]
```

The cotensor space associated to a tensor is the subcotensor space spanned by multilinear forms obtained from T. For a fixed basis of U_0, we get a basis for the cotensor space by projecting onto various basis vectors of U_0.

```
> CT := AsCotensorSpace(T);
> CT;
Cotensor space of dimension 4 over GF(7) with valence 1
U2 : Full Vector space of degree 2 over GF(7)
U1 : Full Vector space of degree 3 over GF(7)
>
> S := Random(CT);
> MS := KMatrixSpace(GF(7),2,3);
> SystemOfForms(S) subset sub<MS|SystemOfForms(T)>;
true
```

The same applies for the associated tensor space to a tensor, where a basis is obtained by projecting onto the basis vectors of U_i.

```
> TS := AsTensorSpace(T,1);
> TS;
Tensor space of dimension 3 over GF(7) with valence 1
U1 : Full Vector space of degree 2 over GF(7)
U0 : Full Vector space of degree 4 over GF(7)
>
> S := Random(TS);
> MS := KMatrixSpace(GF(7),2,4);
> AsMatrices(S,1,0) subset sub<MS|AsMatrices(T,2,0)>;
true
```

61.2.4 Invariants of Tensors

To access the projections or the objects acting on a specific factor, the following should be used.

```
Induce(X, i)
```

Returns the induced sub-object associated to the ith factor of the associated tensor and a projection from the given object to the returned sub-object.

61.2.4.1 Standard Invariants

We integrate the invariant theory associated to bilinear and multilinear maps into the realm of tensors.

```
Radical(T, s)
```

Returns the sth (categorical) radical as a subspace of U_s along with an isomorphism from U_s to K^{d_s}. If U_s is already a vector space, then the returned map is the identity.

```
Radical(T)
```

Returns the tuple of all the s-radicals for each $s \in \{1, ..., v\}$.

```
Coradical(T)
```

Returns the (categorical) coradical of T and a vector space surjection from the codomain to the coradical.

61.2.4.2 Invariants for Bilinear Tensors

The following are used only for tensors of valence 2.

```
AdjointAlgebra(B)
```

Returns the adjoint $*$-algebra of the given bilinear map.

Example H61E14

```
> V := VectorSpace( GF(5), 10 );
> E := ExteriorCotensorSpace( V, 2 );
> E;
Cotensor space of dimension 45 over GF(5) with valence 1
U2 : Full Vector space of degree 10 over GF(5)
U1 : Full Vector space of degree 10 over GF(5)
>
> T := Random(E);
> S := Random(E);
> CT := SubtensorSpace(E,[T,S]);
> T2 := AsTensor(CT);
> T2;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 10 over GF(5)
```

```
U1 : Full Vector space of degree 10 over GF(5)
U0 : Full Vector space of degree 2 over GF(5)
>
> A := AdjointAlgebra(T2);
> RecognizeStarAlgebra(A);
true
> Star(A);
Mapping from: AlgMat: A to AlgMat: A given by a rule [no
inverse]
```

```
LeftNucleus(B)
```

Returns the left nucleus of the bilinear map B as a subalgebra of $\mathrm{End_K(U_2)} \times \mathrm{End_K(U_0)}$.

```
MidNucleus(B)
```

Returns the mid nucleus of the bilinear map B as a subalgebra of $\mathrm{End_K(U_2)} \times \mathrm{End_K(U_1)}$.

```
RightNucleus(B)
```

Returns the right nucleus of the bilinear map B as a subalgebra of $\mathrm{End_K(U_1)} \times \mathrm{End_K(U_0)}$.

61.2.4.3 Invariants of General Multilinear Maps

The following functions can be used for general multilinear maps.

```
Centroid(T)
```

Returns the centroid of the tensor as a subalgebra of $\prod_{i=0}^{v} \mathrm{End_K(U_i)}$.

Example H61E15

```
> A := MatrixAlgebra(GF(3),25);
> f := RandomIrreduciblePolynomial(GF(3),25);
> S := sub< A | CompanionMatrix(f) >; // GF(3^25) inside A
> T := Tensor(S);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Matrix Algebra of degree 25 with 1 generator over GF(3)
U1 : Matrix Algebra of degree 25 with 1 generator over GF(3)
U0 : Matrix Algebra of degree 25 with 1 generator over GF(3)
> C := Centroid(T);
> Dimension(C);
25
> Ngens(C);
1
```

```
DerivationAlgebra(T)
```

Given a tensor T, returns the derivation Lie algebra of the tensor as a Lie subalgebra of $\prod_{i=0}^{v} \mathrm{End}_{\mathrm{K}}(\mathrm{U_i})$.

```
Nucleus(T, s, t)
```

Returns the st-nucleus ($s \neq t$) of the tensor as a subalgebra of $\mathrm{End}_{\mathrm{K}}(\mathrm{U_i}) \times \mathrm{End}_{\mathrm{K}}(\mathrm{U_j})$, where $i = \max(s, t)$ and $j = \min(s, t)$.

Example H61E16

```
> H := QuaternionAlgebra(Rationals(),-1,-1);
> T := Tensor(H);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Quaternion Algebra with base ring Rational Field,
defined by i^2 = -1, j^2 = -1
U1 : Quaternion Algebra with base ring Rational Field,
defined by i^2 = -1, j^2 = -1
U0 : Quaternion Algebra with base ring Rational Field,
defined by i^2 = -1, j^2 = -1
> D := DerivationAlgebra(T);
> SemisimpleType(D);
A1
```

Now we verify that the mid nucleus of T is the quaternion algebra.

```
> ChangeTensorCategory(~T,HomotopismCategory(2));
> N := Nucleus(T,2,1);
> Dimension(N);
4
> N.1^2 eq N!-1;
true
> N.2^2 eq N!-1;
true
> N.1*N.2 eq -N.2*N.1;
true
```

If the centroid of a tensor is a commutative local ring, we can rewrite a tensor over its centroid. We employ the algorithms developed by Brooksbank and Wilson [BW15] to efficiently determine if a matrix algebra is cyclic.

```
TensorOverCentroid(T)
```

If the given tensor T is framed by K-vector spaces, then the returned tensor is framed by E-vector spaces where E is the residue field of the centroid. The returned homotopism is an isotopism of the K-tensors.

Example H61E17

This will "forget" the field structure of our group, so that the tensor given by the commutator will be over $\mathbf{F}_2$. We will use the centroid to rewrite our tensor as an alternating form over $\mathbf{F}_{1024}$.

```
> G := ClassicalSylow( GL(3,1024), 2 );
> P := PCPresentation( UnipotentMatrixGroup(G) );
> T := pCentralTensor(P,1,1);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 20 over GF(2)
U1 : Full Vector space of degree 20 over GF(2)
U0 : Full Vector space of degree 10 over GF(2)
> TC := TensorOverCentroid(T);
> TC;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 2 over GF(2^10)
U1 : Full Vector space of degree 2 over GF(2^10)
U0 : Full Vector space of degree 1 over GF(2^10)
> IsAlternating(TC);
true
```

We include some well-known polynomial invariants for bilinear maps.

`Discriminant(B)`

Returns the discriminant of the bilinear map.

`Pfaffian(B)`

Returns the Pfaffian of the antisymmetric bilinear map.

Example H61E18

```
> J := Matrix(GF(7),[[0,1],[-1,0]]);
> J;
[0 1]
[6 0]
> M := [ InsertBlock(ZeroMatrix(GF(7),4,4),J,i,i)
>      : i in [1..3] ];
> M;
[
    [0 1 0 0]
    [6 0 0 0]
    [0 0 0 0]
    [0 0 0 0],
    [0 0 0 0]
    [0 0 1 0]
    [0 6 0 0]
    [0 0 0 0],
    [0 0 0 0]
    [0 0 0 0]
    [0 0 0 1]
    [0 0 6 0]
]
> T := Tensor(M,2,1);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 4 over GF(7)
U1 : Full Vector space of degree 4 over GF(7)
U0 : Full Vector space of degree 3 over GF(7)
> Discriminant(T);
$.1^2*$.3^2
> Pfaffian(T);
$.1*$.3
```

61.3 Exporting Tensors

Tensors can be used to define algebraic structures such as groups and algebras.

```
HeisenbergAlgebra(B)
```

Returns the Heisenberg algebra A induced by the bilinear tensor B. If $B = \circ : U \times V \rightarrowtail W$ is a bilinear map of K-vector spaces, and U, V, and W are isomorphic, then A is the algebra over U with the given product. If U and V are isomorphic but not with W, then A is the algebra over $U \oplus W$ with the given product. If U is not isomorphic to V, then it creates a new bilinear map $\bullet : (U \oplus V) \times (U \oplus V) \rightarrowtail W$, where

$$(u, v) \bullet (u', v') = u \circ v'.$$

```
HeisenbergLieAlgebra(B)
```

Returns the Heisenberg Lie algebra with Lie bracket given by the alternating bilinear tensor induced by B.

```
HeisenbergGroup(B)
```

Returns the class 2, exponent p, Heisenberg p-group with commutator given by the alternating bilinear tensor induced by B. This uses the pQuotient functions to convert a finitely presented group into a polycyclic group.

Example H61E19

```
> T := RandomTensor(GF(3),[10,10,4]);
> V := Domain(T)[1];
> V.1*T*V.2;
(0 0 1 0)
> A := HeisenbergAlgebra(T);
> A;
Algebra of dimension 14 with base ring GF(3)
> A.1*A.2;
(0 0 0 0 0 0 0 0 0 0 0 0 1 0)
```

Now we create the Lie algebra. The tensor T is not alternating, so it will induce an alternating tensor form which to create the Lie algebra.

```
> L := HeisenbergLieAlgebra(T);
> L.1*L.2;
(0 0 0 0 0 0 0 0 0 0 2 0 1 0)
> T2 := AlternatingTensor(T);
> V.1*T2*V.2;
(2 0 1 0)
```

Now we create the Heisenberg 3-group G from T. Because this algorithm uses the p-quotient algorithms, the commutator on G will not be identical to the induced alternating tensor T_2. Instead, it will be pseudo-isometric to the T_2.

```
> G := HeisenbergGroup(T);
> (G.2,G.1);  // Defining word for 1st gen in Frattini
G.11
```

61.4 Tensor Spaces

In Magma a tensor space is a parent type for tensors. It behaves as a module but also maintains an interpretation of its elements as a multilinear map. Each tensor space further maintains a tensor category which is assigned to its tensors.

61.4.1 Constructions of Tensor and Cotensor Spaces

61.4.1.1 Universal Tensor Spaces

Construction of universal tensor spaces is modeled after construction of free modules and matrix spaces. For efficiency reasons, the actual representation may vary based on the parameters, e.g. it may be a space of structure constants, black-box functions, or systems of forms. So access to the tensors in these tensor space should be made through the provided functions.

```
KTensorSpace(K, S)
```

```
KTensorSpace(K, S, C)
```

For a field K and sequence $S = [d_v, \cdots, d_0]$, returns the universal tensor space $\hom_K(K^{d_v}, \dots, \hom_K(K^{d_1}, K^{d_0})\cdots) \cong K^{d_v\cdots d_0}$ with covariant tensor category given by C. The default is Albert's homotopism category.

```
RTensorSpace(R, S)
```

```
RTensorSpace(R, S, C)
```

For a commutative and unital ring R and sequence $S = [d_v, \cdots, d_0]$, returns the universal tensor space $\hom_R(R^{d_v}, \dots, \hom_R(R^{d_1}, R^{d_0})\cdots) \cong R^{d_v\cdots d_0}$ with covariant tensor category given by C. The default is Albert's homotopism category.

```
TensorSpace(S)
```

```
TensorSpace(S, C)
```

Given a sequence $S = [U_v, \dots, U_0]$ of K-modules returns a universal tensor space equivalent to $\hom_K(U_v \otimes_K \cdots \otimes_K U_1, U_0)$ with covariant tensor category given by C. The default is Albert's homotopism category.

```
TensorSpace(V, p, q)
```

```
TensorSpace(K, d, p, q)
```

Returns the signatured (p, q)-tensor space over the vector space $V = K^d$. The first p indices are covariant and the last q indices are contravariant. This is functionally equivalent to creating a universal tensor space from the sequence $[V, \dots,_p V, V^*, \dots,_q V^*]$ and the tensor category with arrows $[1, \dots,_p 1, -1, \dots,_q -1]$ and duplicates $\{\{p+q, \dots, 1+q\}, \{q, \dots, 1\}, \{0\}\}$.

Example H61E20

```
> TS := KTensorSpace(Rationals(),[ i : i in [3..7] ]);
> TS;
Tensor space of dimension 2520 over Rational Field with
valence 4
U4 : Full Vector space of degree 3 over Rational Field
U3 : Full Vector space of degree 4 over Rational Field
U2 : Full Vector space of degree 5 over Rational Field
U1 : Full Vector space of degree 6 over Rational Field
```

```
U0 : Full Vector space of degree 7 over Rational Field
>
> TS.1;
Tensor of valence 4, U4 x U3 x U2 x U1 >-> U0
U4 : Full Vector space of degree 3 over Rational Field
U3 : Full Vector space of degree 4 over Rational Field
U2 : Full Vector space of degree 5 over Rational Field
U1 : Full Vector space of degree 6 over Rational Field
U0 : Full Vector space of degree 7 over Rational Field
```

Give a tensor space a frame.

```
> R := pAdicRing(3,6);
> Fr := [RSpace(R,5), sub<RSpace(R,3)|[0,1,0],[0,0,1]>,
>   RSpace(R,2)];
> TS := TensorSpace(Fr);
> TS;
Tensor space of dimension 20 over 3-adic ring mod 3^6 with
valence 2
U2 : Full RSpace of degree 5 over pAdicRing(3, 6)
U1 : RSpace of degree 3, dimension 2 over pAdicRing(3, 6)
Generators:
(0 1 0)
(0 0 1)
Echelonized basis:
(0 1 0)
(0 0 1)
U0 : Full RSpace of degree 2 over pAdicRing(3, 6)
>
> TS.10;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full RSpace of degree 5 over pAdicRing(3, 6)
U1 : RSpace of degree 3, dimension 2 over pAdicRing(3, 6)
Generators:
(0 1 0)
(0 0 1)
Echelonized basis:
(0 1 0)
(0 0 1)
U0 : Full RSpace of degree 2 over pAdicRing(3, 6)
```

With signatured tensor spaces, the tensor category is not immediately obvious at print out. Instead, one can glean categorical information using `TensorCategory` on a tensor space.

```
> TS := TensorSpace( VectorSpace(GF(3),3), 2, 4 );
> TS;
Tensor space of dimension 729 over GF(3) with valence 6
U6 : Full Vector space of degree 3 over GF(3)
U5 : Full Vector space of degree 3 over GF(3)
U4 : Full Vector space of degree 3 over GF(3)
```

```
U3 : Full Vector space of degree 3 over GF(3)
U2 : Full Vector space of degree 3 over GF(3)
U1 : Full Vector space of degree 3 over GF(3)
U0 : Full Vector space of degree 1 over GF(3)
>
> TensorCategory(TS);
Tensor category of Valence 6 (<-,<-,->,->,->,->,==)
({ 0 },{ 5 .. 6 },{ 1 .. 4 })
```

We see that the indices 5 and 6 are contravariant and both are linked together as well. Furthermore the covariant indices are 1–4, which are also linked together.

61.4.1.2 Universal Cotensor Spaces

We only consider cotensor spaces over fields.

```
KCotensorSpace(K, S)
KCotensorSpace(K, S, C)
```

For a field K and sequence $S = [d_v, \ldots, d_1]$ returns the universal cotensor space $\hom_K(K^{d_v} \otimes \cdots \otimes K^{d_1}, K) \cong K^{d_v \cdots d_1}$ with the given contravariant tensor category C. The default is Albert's homotopism category.

```
CotensorSpace(S)
CotensorSpace(S, C)
```

Given a sequence $S = [U_v, \ldots, U_1]$ of K-vector spaces returns the universal tensor space equivalent to $\hom_K(U_v \otimes_K \cdots \otimes_K U_1, K)$ with contravariant tensor category given by C. The default is Albert's homotopism category.

Example H61E21

```
> CT := KCotensorSpace(GF(2),[ i : i in [5..7] ]);
> CT;
Cotensor space of dimension 210 over GF(2) with valence 2
U3 : Full Vector space of degree 5 over GF(2)
U2 : Full Vector space of degree 6 over GF(2)
U1 : Full Vector space of degree 7 over GF(2)
>
> CT.1;
Cotensor of valence 2, U3 x U2 x U1 >-> K
U3 : Full Vector space of degree 5 over GF(2)
U2 : Full Vector space of degree 6 over GF(2)
U1 : Full Vector space of degree 7 over GF(2)
>
>
> Cat := CotensorCategory([1,0,-1],{{1},{2},{3}});
> Fr := [ VectorSpace(GF(8),4) : i in [1..3] ];
> CT := CotensorSpace( Fr, Cat );
> CT;
Cotensor space of dimension 64 over GF(2^3) with valence 2
U3 : Full Vector space of degree 4 over GF(2^3)
U2 : Full Vector space of degree 4 over GF(2^3)
U1 : Full Vector space of degree 4 over GF(2^3)
>
> TensorCategory(CT);
Cotensor category of valence 2 (->,==,<-) ({ 1 },{ 2 },{ 3 })
```

61.4.1.3 Some Standard Constructions

We include some subspaces generated by well-known tensors.

```
AlternatingSpace(T)
```

Returns the sub(co-)tensor space generated by all the alternating (co-)tensors contained in the given (co-)tensor space.

```
AntisymmetricSpace(T)
```

Returns the sub(co-)tensor space generated by all the antisymmetric (co-)tensors contained in the given (co-)tensor space.

```
SymmetricSpace(T)
```

Returns the sub(co-)tensor space generated by all the symmetric (co-)tensors contained in the given (co-)tensor space.

```
ExteriorCotensorSpace(V, n)
```

Returns the cotensor space given by the nth exterior power of the vector space V.

```
SymmetricCotensorSpace(V, n)
```

Returns the cotensor space given by the nth symmetric power of the vector space V.

Example H61E22

```
> TS := KTensorSpace(GF(3), [3,3,3,2]);
> TS;
Tensor space of dimension 54 over GF(3) with valence 3
U3 : Full Vector space of degree 3 over GF(3)
U2 : Full Vector space of degree 3 over GF(3)
U1 : Full Vector space of degree 3 over GF(3)
U0 : Full Vector space of degree 2 over GF(3)
>
> SS := SymmetricSpace(TS);
> AsMatrices(Random(SS),3,2);
[
    [1 1 1]
    [1 0 0]
    [1 0 0],
    [2 2 2]
    [2 0 1]
    [2 1 0],
    [1 0 0]
    [0 1 2]
    [0 2 2],
    [2 0 1]
    [0 1 1]
    [1 1 2],
    [1 0 0]
    [0 2 2]
    [0 2 1],
    [2 1 0]
    [1 1 2]
    [0 2 0]
]
```

For $V = \mathbf{F}_{25}^6$, we construct the fourth exterior power of V, $\Lambda^4(V)$ as a sub cotensor space of dimension $\binom{6}{4=15}$.

```
> V := VectorSpace(GF(25),6);
> E := ExteriorCotensorSpace(V,4);
> E;
Cotensor space of dimension 15 over GF(5^2) with valence 3
U4 : Full Vector space of degree 6 over GF(5^2)
U3 : Full Vector space of degree 6 over GF(5^2)
U2 : Full Vector space of degree 6 over GF(5^2)
U1 : Full Vector space of degree 6 over GF(5^2)
> T := Random(E);
> IsAlternating(T);
true
```

61.4.1.4 Subspaces as Closures

`DerivationClosure(TS, O)`

Returns the derivation closure of the given tensor space TS, with frame $U_2 \times U_1 \rightarrowtail U_0$, with operators $O \subseteq \mathrm{End}(\mathrm{U}_2) \times \mathrm{End}(\mathrm{U}_1) \times \mathrm{End}(\mathrm{U}_0)$. Currently, this only works for tensor spaces of valence 2. This is the subspace whose tensors' derivation algebra contains O.

`DerivationClosure(TS, T)`

Returns the derivation closure of the given tensor space TS, with frame $U_2 \times U_1 \rightarrowtail U_0$, with operators $O \subseteq \mathrm{End}(\mathrm{U}_2) \times \mathrm{End}(\mathrm{U}_1) \times \mathrm{End}(\mathrm{U}_0)$. Currently, this only works for tensor spaces of valence 2. This is the subspace whose tensors' derivation algebra contains the derivation algebra of T.

`NucleusClosure(TS, O, s, t)`

Returns the nucleus closure of the tensor space TS, with frame $U_2 \times U_1 \rightarrowtail U_0$, with operators $O \subseteq \mathrm{End}(\mathrm{U}_\mathrm{s}) \times \mathrm{End}(\mathrm{U}_\mathrm{t})$. Currently, this only works for tensor spaces of valence 2. This returns the subspace whose tensors' st-nuclues contains O.

`NucleusClosure(TS, T, s, t)`

Returns the nucleus closure of the tensor space TS, with frame $U_2 \times U_1 \rightarrowtail U_0$, with operators $O \subseteq \mathrm{End}(\mathrm{U}_\mathrm{s}) \times \mathrm{End}(\mathrm{U}_\mathrm{t})$. Currently, this only works for tensor spaces of valence 2. This returns the subspace whose tensors' st-nuclues contains the st-nucleus of T.

Example H61E23

The derivation closure of the tensor in the universal tensor space is 1-dimensional.

```
> Fr := [ KMatrixSpace(GF(3),2,3),
>   KMatrixSpace(GF(3),3,2),KMatrixSpace(GF(3),2,2) ];
> F := func< x | x[1]*x[2] >;
> T := Tensor(Fr,F);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full KMatrixSpace of 2 by 3 matrices over GF(3)
U1 : Full KMatrixSpace of 3 by 2 matrices over GF(3)
U0 : Full KMatrixSpace of 2 by 2 matrices over GF(3)
>
> TS := Parent(T);
> TS;
Tensor space of dimension 144 over GF(3) with valence 2
U2 : Full Vector space of degree 6 over GF(3)
U1 : Full Vector space of degree 6 over GF(3)
U0 : Full Vector space of degree 4 over GF(3)
>
> D := DerivationClosure(TS,T);
> D;
Tensor space of dimension 1 over GF(3) with valence 2
U2 : Full Vector space of degree 6 over GF(3)
U1 : Full Vector space of degree 6 over GF(3)
U0 : Full Vector space of degree 4 over GF(3)
```

Example H61E24

```
> H := ClassicalSylow( GL(3,125), 5 ); // Heisenberg group
> T := pCentralTensor(H,5,1,1);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 6 over GF(5)
U1 : Full Vector space of degree 6 over GF(5)
U0 : Full Vector space of degree 3 over GF(5)
```

The centroid of T will be 3-dimensional and is isomorphic to $\mathbf{F}_{125}$.

```
> C := Centroid(T);
> C;
Matrix Algebra of degree 15 with 3 generators over GF(5)
>
> S := TensorOverCentroid(T);
> S;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 2 over GF(5^3)
U1 : Full Vector space of degree 2 over GF(5^3)
U0 : Full Vector space of degree 1 over GF(5^3)
>
> TS := Parent(S);
> N := NucleusClosure(TS,S,2,1);
> N;
Tensor space of dimension 1 over GF(5^3) with valence 2
U2 : Full Vector space of degree 2 over GF(5^3)
U1 : Full Vector space of degree 2 over GF(5^3)
U0 : Full Vector space of degree 1 over GF(5^3)
```

Compare this closure with the closure of our original tensor over $\mathbf{F}_5$.

```
> NT := NucleusClosure(Parent(T),T,2,1);
> NT;
Tensor space of dimension 36 over GF(5) with valence 2
U2 : Full Vector space of degree 6 over GF(5)
U1 : Full Vector space of degree 6 over GF(5)
U0 : Full Vector space of degree 3 over GF(5)
```

61.4.2 Operations on Tensor Spaces

61.4.2.1 Membership and Comparison with Tensor Spaces

We define some intuitive functions for tensor spaces, similar to those found for modules.

`T in TS`

Decides if T is contained in the tensor space TS.

`IsCoercible(TS, T)`

`TS ! T`

Decides if the tensor T can be coerced into the tensor space TS. If so, the tensor is returned as an element of TS.

`IsCoercible(TS, S)`

`TS ! S`

Decides if the sequence S can be coerced into the tensor space TS as a tensor. If so, the corresponding tensor is returned.

`S eq T`

Decides if the tensor spaces S and T are equal.

`S subset T`

Decides if S is a subset of the tensor space T.

Example H61E25

```
> TS := KTensorSpace( GF(2), [2,3,2] );
> TS;
Tensor space of dimension 12 over GF(2) with valence 2
U2 : Full Vector space of degree 2 over GF(2)
U1 : Full Vector space of degree 3 over GF(2)
U0 : Full Vector space of degree 2 over GF(2)
>
> S := [ Random(GF(2)) : i in [1..12] ];
> T := TS!S;
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 2 over GF(2)
U1 : Full Vector space of degree 3 over GF(2)
U0 : Full Vector space of degree 2 over GF(2)
>
> T eq Tensor(GF(2),[2,3,2],S);
true
```

We demonstrate how to coerce into the symmetric cube of $V = \mathbf{Q}^{10}$ and construct a subcotensor space.

```
> V := VectorSpace(Rationals(),10);
> SS := SymmetricCotensorSpace(V,3);
> SS;
Cotensor space of dimension 220 over Rational Field with valence 2
U3 : Full Vector space of degree 10 over Rational Field
U2 : Full Vector space of degree 10 over Rational Field
U1 : Full Vector space of degree 10 over Rational Field
>
> CT := SubtensorSpace(SS,SS![1..1000]);
> CT;
Cotensor space of dimension 1 over Rational Field with valence 2
U3 : Full Vector space of degree 10 over Rational Field
U2 : Full Vector space of degree 10 over Rational Field
U1 : Full Vector space of degree 10 over Rational Field
>
> CT subset SS;
true
> CT.1 in SS;
true
> SS.2 in CT;
false
```

61.4.2.2 Tensor Spaces as Modules

We view a tensor space as a K-module, so we have notions of generators, dimension (if it is free), and cardinality.

`Generators(T)`

Returns a sequence of generators for the tensor space.

`T . i`

Returns the ith generator of the tensor space T.

`NumberOfGenerators(T)`

`Ngens(T)`

Returns the number of generators of the tensor space T.

`Dimension(T)`

Returns the dimension of the tensor space T as a free K-module.

`#T`

Returns the size of the tensor space, provided it is finite.

`Random(T)`

Provided the base ring has a random algorithm in Magma, it returns a random element of the tensor space T.

```
RandomTensor(R, S)
```

```
RandomTensor(R, S, C)
```

Provided R has a random algorithm in Magma, it returns a random tensor from the tensor space $\hom_R(R^{d_v}, \dots, \hom_R(R^{d_1}, R^{d_0}) \cdots)$ with category C. The default category is the homotopism category.

Example H61E26

```
> TS := KTensorSpace( GF(9), [2,2,2,2] );
> TS;
Tensor space of dimension 16 over GF(3^2) with valence 3
U3 : Full Vector space of degree 2 over GF(3^2)
U2 : Full Vector space of degree 2 over GF(3^2)
U1 : Full Vector space of degree 2 over GF(3^2)
U0 : Full Vector space of degree 2 over GF(3^2)
>
> Ngens(TS);
16
> #TS eq 9^Ngens(TS);
true
>
> Eltseq(TS.2);
[ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

We can use `RandomTensor` to get a random tensor or cotensor from scratch.

```
> T := RandomTensor(GF(3),[2,2,2]);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 2 over GF(3)
U1 : Full Vector space of degree 2 over GF(3)
U0 : Full Vector space of degree 2 over GF(3)
>
> Cat := CotensorCategory([1,1,1],{{1,2,3}});
> T := RandomTensor(GF(3),[2,2,2],Cat);
> T;
Cotensor of valence 2, U3 x U2 x U1 >-> K
U3 : Full Vector space of degree 2 over GF(3)
U2 : Full Vector space of degree 2 over GF(3)
U1 : Full Vector space of degree 2 over GF(3)
```

61.4.2.3 Properties of Tensor Spaces

We define some functions to access basic properties of tensor spaces.

```
Valence(T)
```

Returns the valence of the tensor space.

```
Frame(T)
```

Returns the list of modules in the frame of the tensor space.

```
BaseRing(T)
BaseField(T)
```

Returns the base ring (or field) of the tensor space.

```
TensorCategory(T)
```

Returns the underlying tensor category of the tensor space.

```
IsCovariant(T)
IsContravariant(T)
```

Decides if the underlying tensor category is covariant or contravariant.

```
ChangeTensorCategory(T, C)
ChangeTensorCategory( T, C)
```

Returns the tensor category with the given tensor category.

```
IsAlternating(T)
```

Decides if every tensor in the tensor space is an alternating tensor.

```
IsAntisymmetric(T)
```

Decides if every tensor in the tensor space is an antisymmetric tensor.

```
IsSymmetric(T)
```

Decides if every tensor in the tensor space is a symmetric tensor.

```
UniversalTensorSpace(T)
UniversalCotensorSpace(T)
Generic(T)
```

Returns the universal (co-)tensor space with the same frame and category as T.

Example H61E27

```
> TS := KTensorSpace( GF(23), [3,4,5,6] );
> TS;
Tensor space of dimension 360 over GF(23) with valence 3
U3 : Full Vector space of degree 3 over GF(23)
U2 : Full Vector space of degree 4 over GF(23)
U1 : Full Vector space of degree 5 over GF(23)
U0 : Full Vector space of degree 6 over GF(23)
>
> Valence(TS);
3
> Frame(TS);
[*
    Full Vector space of degree 3 over GF(23),
    Full Vector space of degree 4 over GF(23),
    Full Vector space of degree 5 over GF(23),
    Full Vector space of degree 6 over GF(23)
*]
> TensorCategory(TS);
Tensor category of Valence 3 (->,->,->,->)
({ 1 },{ 2 },{ 0 },{ 3 })
>
> Cat := TensorCategory([1,1,-1,-1],{{0},{1},{2},{3}});
> ChangeTensorCategory(~TS,Cat);
> TensorCategory(TS);
Tensor category of Valence 3 (->,->,<-,<-)
({ 1 },{ 2 },{ 0 },{ 3 })
```

We construct the universal cotensor space of the symmetric cube, $S^3(V)$, of a vector space V.

```
> V := VectorSpace( GF(3), 5 );
> S := SymmetricCotensorSpace(V,3);
> S;
Cotensor space of dimension 30 over GF(3) with valence 2
U3 : Full Vector space of degree 5 over GF(3)
U2 : Full Vector space of degree 5 over GF(3)
U1 : Full Vector space of degree 5 over GF(3)
> UniversalCotensorSpace(S);
Cotensor space of dimension 125 over GF(3) with valence 2
U3 : Full Vector space of degree 5 over GF(3)
U2 : Full Vector space of degree 5 over GF(3)
U1 : Full Vector space of degree 5 over GF(3)
>
> IsSymmetric(S);
true
```

61.5 Tensor Categories

Magma allows tensors and tensor spaces to change categories. Unless a user specifies otherwise, all tensors are assigned a category that is natural to the method by which it was created. For example a tensor created from an algebra will be assigned an algebra category, whereas a tensor created by structure constants will be assigned the Albert homotopism category. Tensor categories influence the behavior of commands such as kernels and images as well as the algebraic invariants such as derivation algebras of a tensor.

Our conventions follow [Wil13]. In particular given a tensor T framed by $[U_v, \dots, U_0]$ then a tensor category for T will specify a function $A : \{0 \dots v\} \to \{-1, 0, 1\}$ along with a partition $\mathcal{P}$ of $\{0, \dots, v\}$ such that the following rules apply to the tensors and morphisms in the category.

(i) for each tensor T framed by $[U_v, \dots, U_0]$, if $X \in \mathcal{P}$, then

$$\forall i, j \in X, \quad U_i = U_j.$$

(ii) Given a second tensor S framed by $[V_v, \dots, V_0]$, a morphism $f : T \to S$ (Magma type `Hmtp`) will be a list $[f_v, \dots, f_0]$ of homomorphisms as follows:

(Covariant) if $A(i) = 1$ then $f_i : U_i \to V_i$;

(Constant) if $A(i) = 0$ then $U_i = V_i$ and $f_i = 1_{U_i}$; or else

(Contravariant) $A(i) = -1$ and $f_i : U_i \leftarrow V_i$. So if $A(0) = 1$ then

$$\left\langle \sum_{i \in A^{-1}(1)} u_i f_i + \sum_{j \notin A^{-1}(-1)} v_j \right\rangle_S = \left\langle \sum_{i \in A^{-1}(1)} u_i + \sum_{j \notin A^{-1}(-1)} v_j f_j \right\rangle_T f_0;$$

if $A(0) = 0$ then

$$\left\langle \sum_{i \in A^{-1}(1)} u_i f_i + \sum_{j \notin A^{-1}(-1)} v_j \right\rangle_S = \left\langle \sum_{i \in A^{-1}(1)} u_i + \sum_{j \notin A^{-1}(-1)} v_j f_j \right\rangle_T;$$

else $A(0) = -1$ and

$$\left\langle \sum_{i \in A^{-1}(1)} u_i f_i + \sum_{j \notin A^{-1}(-1)} v_j \right\rangle_S f_0 = \left\langle \sum_{i \in A^{-1}(1)} u_i + \sum_{j \notin A^{-1}(-1)} v_j f_j \right\rangle_T.$$

Magma manages internally the differences between vectors and covectors and more generally tensors and cotensors. Both types are issued the Magma type `TenSpcElt`. For operations sensitive to the difference, Magma stores a value of co/contra-variance of the tensor as a property of the tensor category. This the third general property stored in Magma's tensor category type `TenCat`.

We use the phrase tensor category exclusively for categories that describe tensors and tensor spaces. In other words, the data structure of a tensor category is a function $A : \{0, \dots, v\} \to \{-1, 0, 1\}$ and a partition P of $\{0, \dots, v\}$. It is useful to distinguish from tensors and cotensors at the categorical level, so a tensor category is either covariant or contravariant as well (in the latter case, referred to as a cotensor category).

61.5.1 Creating Tensor Categories

`TensorCategory(A, P)`

Sets up a covariant tensor space category with specified direction of arrows A, and a partition P indicating variables to be treated as equivalent. The fiber $A^{-1}(1)$ denotes the covariant variables, $A^{-1}(0)$ identifies the constant variables, and $A^{-1}(-1)$ marks the contra-variant variables.

`CotensorCategory(A, P)`

Sets up a contra-variant tensor space category with specified direction of arrows A, and a partition P indicating variables to be treated as equivalent. The fiber $A^{-1}(1)$ denotes the covariant variables, $A^{-1}(0)$ identifies the constant variables, and $A^{-1}(-1)$ marks the contra-variant variables.

`HomotopismCategory(v : `*parameters*`)`

`Contravariant` BoolElt *Default :* `false`

Returns Albert's homotopism category – all modules categories are covariant and no duplicates considered. Set the optional parameter `Contravariant` to `true` to make it a cotensor category.

`CohomotopismCategory(v)`

Returns the cohomotopism category – all domain modules categories are covariant, the codomain is contravariant, and no duplicates considered.

`AdjointCategory(v, s, t)`

`LinearCategory(v, s, t)`

Returns the tensor category where all modules are constant except in position s and t. Both s and t are in $\{0, \ldots, v\}$. Position s is covariant, position t is contravariant.

Example H61E28

```
> Cat := TensorCategory([1,-1,0],{{0},{1},{2}});
> Cat;
Tensor category of Valence 2 (->,<-,==) ({ 1 },{ 2 },{ 0 })
>
> TS := KTensorSpace(GF(5),[5,3,4],Cat);
> TS;
Tensor space of dimension 60 over GF(5) with valence 2
U2 : Full Vector space of degree 5 over GF(5)
U1 : Full Vector space of degree 3 over GF(5)
U0 : Full Vector space of degree 4 over GF(5)
> TensorCategory(TS);
Tensor category of Valence 2 (->,<-,==) ({ 1 },{ 2 },{ 0 })
>
```

```
> IsContravariant(TS);
false
```

All the tensor constructors that allow a `TenCat` input can be used to make cotensors.

```
> Cat := HomotopismCategory(2 : Contravariant := true);
> Cat;
Cotensor category of valence 2 (->,->) ({ 1 },{ 2 })
> T := Tensor(GF(5),[2,2],[1..4],Cat);
> T;
Cotensor of valence 1, U2 x U1 >-> K
U2 : Full Vector space of degree 2 over GF(5)
U1 : Full Vector space of degree 2 over GF(5)
```

61.5.2 Operations on Tensor Categories

We have basic operations for tensor categories.

`C1 eq C2`

Decides if the tensor categories are the same.

`Valence(C)`

Returns the valence of the tensor category.

`Arrows(C)`

Returns the sequence of arrows of the tensor category. A -1 signifies an a contravariant index, a 0 signifies a constant index, and a 1 signifies a covariant index.

`RepeatPartition(C)`

Returns the repeat partition for the tensor category.

`IsCovariant(C)`

`IsContravariant(C)`

Decides if the tensor category is covariant or contravariant.

Example H61E29

```
> C1 := TensorCategory([1,1,-1,1],{{0,3},{1},{2}});
> C1;
Tensor category of Valence 3 (->,->,<-,->) ({ 1 },{ 2 },
{ 0, 3 })
>
> A := map< {0..3} -> {-1,0,1} | x :-> 1 >;
> C2 := TensorCategory(A,{{0..3}});
> C2;
Tensor category of Valence 3 (->,->,->,->) ({ 0 .. 3 })
>
> C1 eq C2;
false
> RepeatPartition(C2);
    { 0 .. 3 }
> Valence(C2);
3
> Arrows(C2);
[ 1, 1, 1, 1 ]
```

61.5.3 Categorical Operations

61.5.4 Categorical Operations on Tensors

We include functions defined for the category of tensors. Most functions are currently defined only for Albert's homotopism category.

`Subtensor(T, S)`

Returns the smallest submap of T containing S.

`Subtensor(T, D, C)`

Returns the smallest submap of T containing D in the domain and C in the codomain.

`IsSubtensor(T, S)`

Decides whether S is a subtensor of T.

`LocalIdeal(T, S, I)`

Returns the local ideal of T at I constaining S.

`LocalIdeal(T, D, C, I)`

Returns the local ideal of T at I constaining D in the domain and C in the codomain.

`LocalIdeal(T, S, I)`

Returns the local ideal of T at I constaining S as a submap.

`IsLocalIdeal(T, S, I)`

Decides if S is a local ideal of T at I.

`Ideal(T, S)`

Returns the ideal of T containing S.

`Ideal(T, D, C)`

Returns the ideal of T containing D in the domain and C in the codomain.

`Ideal(T, S)`

Returns the ideal of T containing S as a submap.

`IsIdeal(T, S)`

Decides if S is an ideal of T.

`LocalQuotient(T, S, I : `*parameters*`)`

`Check` BoolElt *Default :* `true`

Returns the local quotient of T by S at I. If you know S is a local ideal of T at I, set `Check` to `false` to skip the verification. A homotopism is also returned, mapping from T to T/S.

`Quotient(T, S : `*parameters*`)`

`Check` BoolElt *Default :* `true`

`T / S`

Returns the quotient of T by S. If you know S is an ideal of T, set `Check` to `false` to skip the verification. A homotopism is also returned, mapping from T to T/S.

Example H61E30

```
> T := RandomTensor(GF(5),[4,4,2]);
> T := RandomTensor(GF(5),[4,4,2]);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 4 over GF(5)
U1 : Full Vector space of degree 4 over GF(5)
U0 : Full Vector space of degree 2 over GF(5)
>
> F := Frame(T);
> L := [* F[1]![1,1,1,0], F[2]![0,0,0,1], F[3]![0,0] *];
> S := Subtensor(T,L);
> S;
Tensor of valence 2, U2 x U1 >-> U0
```

```
U2 : Vector space of degree 4, dimension 1 over GF(5)
Generators:
(1 1 1 0)
Echelonized basis:
(1 1 1 0)
U1 : Vector space of degree 4, dimension 1 over GF(5)
Generators:
(0 0 0 1)
Echelonized basis:
(0 0 0 1)
U0 : Vector space of degree 2, dimension 1 over GF(5)
Generators:
(0 2)
Echelonized basis:
(0 1)
>
> IsSubtensor(T,S);
true
```

Now we construct the ideal of T containing S.

```
> I := Ideal(T,S);
> I;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Vector space of degree 4, dimension 1 over GF(5)
Generators:
(1 1 1 0)
Echelonized basis:
(1 1 1 0)
U1 : Vector space of degree 4, dimension 1 over GF(5)
Generators:
(0 0 0 1)
Echelonized basis:
(0 0 0 1)
U0 : Full Vector space of degree 2 over GF(5)
Generators:
(1 0)
(0 1)
>
> IsIdeal(T,I);
true
```

Finally, we construct the quotient of T by I.

```
> T/I;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 3 over GF(5)
U1 : Full Vector space of degree 3 over GF(5)
U0 : Full Vector space of degree 0 over GF(5)
Maps from U2 x U1 >-> U0 to V2 x V1 >-> V0.
```

```
U2 -> V2: Mapping from: Full Vector space of degree 4 over
GF(5) to Full Vector space of degree 3 over GF(5)
U1 -> V1: Mapping from: Full Vector space of degree 4 over
GF(5) to Full Vector space of degree 3 over GF(5)
U0 -> V0: Mapping from: Full Vector space of degree 2 over
GF(5) to Full Vector space of degree 0 over GF(5)
```

61.5.5 Categorical Operations on Tensor Spaces

We have categorical notions for tensor spaces as well.

`SubtensorSpace(T, L)`

Returns the subtensor space of T generated by the tensors in the sequence L.

`SubtensorSpace(T, t)`

Returns the subtensor space of T generated by the tensor t.

`IsSubtensorSpace(T, S)`

Decides if the tensor space S is a subtensor space of T.

`Quotient(T, S)`

`T / S`

Returns the quotient tensor space of T by S.

Example H61E31

```
> T := KTensorSpace(GF(2),[4,4,2]);
> T;
Tensor space of dimension 32 over GF(2) with valence 2
U2 : Full Vector space of degree 4 over GF(2)
U1 : Full Vector space of degree 4 over GF(2)
U0 : Full Vector space of degree 2 over GF(2)
>
> L := [ T.i : i in [1..Ngens(T)] | IsEven(i) ];
> S := SubtensorSpace(T, L);
> S;
Tensor space of dimension 16 over GF(2) with valence 2
U2 : Full Vector space of degree 4 over GF(2)
U1 : Full Vector space of degree 4 over GF(2)
U0 : Full Vector space of degree 2 over GF(2)
> SystemOfForms(Random(S));
[
    [0 0 0 0]
    [0 0 0 0]
    [0 0 0 0]
    [0 0 0 0],
    [1 0 0 0]
    [0 1 0 1]
    [1 0 0 0]
    [0 0 1 0]
]
```

Now we compute the quotient tensor space $Q = T/S$.

```
> Q := T/S;
> Q;
Tensor space of dimension 16 over GF(2) with valence 2
U2 : Full Vector space of degree 4 over GF(2)
U1 : Full Vector space of degree 4 over GF(2)
U0 : Full Vector space of degree 2 over GF(2)
> SystemOfForms(Random(Q));
[
    [0 0 0 0]
    [1 1 1 0]
    [1 0 1 1]
    [1 1 0 1],
    [0 0 0 0]
    [0 0 0 0]
    [0 0 0 0]
    [0 0 0 0]
]
> SystemOfForms(Q![1 : i in [1..32]]);
[
    [1 1 1 1]
    [1 1 1 1]
    [1 1 1 1]
    [1 1 1 1],
    [0 0 0 0]
    [0 0 0 0]
    [0 0 0 0]
    [0 0 0 0]
]
```

61.5.6 Homotopisms

Magma provides functions for homotopisms. Homotopisms are also equipped with a tensor category.

61.5.6.1 Constructions of Homotopisms

```
Homotopism(T, S, M)
```

```
Homotopism(T, S, M, C)
```

Returns the homotopism from T to S given by the list of maps M and the category C. The default tensor category is the same as tensor categories for T and S.

Example H61E32

```
> TS := KTensorSpace(GF(4),[2,3,4]);
> T := Random(TS);
> S := Random(TS);
> M := [* Random(KMatrixSpace(GF(4),i,i)) : i in [2..4] *];
> H := Homotopism(T,S,M);
> H;
Maps from U2 x U1 >-> U0 to V2 x V1 >-> V0.
U2 -> V2:
[    1     0]
[    0     0]
U1 -> V1:
[    0   $.1 $.1^2]
[    1     1   $.1]
[  $.1   $.1 $.1^2]
U0 -> V0:
[    1     1   $.1 $.1^2]
[    0     1 $.1^2   $.1]
[    0     0   $.1 $.1^2]
[  $.1 $.1^2   $.1     0]
>
>
> M[2] := map< Frame(TS)[2] -> Frame(TS)[2] | x :-> x >;
> H2 := Homotopism(T,S,M);
> H2;
Maps from U2 x U1 >-> U0 to V2 x V1 >-> V0.
U2 -> V2:
[    1     0]
[    0     0]
U1 -> V1: Mapping from: Full Vector space of degree 3
over GF(2^2) to Full Vector space of degree 3 over GF(2^2)
given by a rule [no inverse]
U0 -> V0:
[    1     1   $.1 $.1^2]
[    0     1 $.1^2   $.1]
[    0     0   $.1 $.1^2]
[  $.1 $.1^2   $.1     0]
```

61.5.6.2 Basic Operations with Homotopisms

We provide some operations for homotopisms.

`H1 * H2`

Returns the composition of the homotopisms H_1 and H_2.

`Domain(H)`

Returns the domain tensor of H.

`Codomain(H)`

Returns the codomain tensor of H.

`Maps(H)`

Returns the of maps for the various modules in the domain and codomain tensors.

`TensorCategory(H)`

Returns the tensor category of H.

`ChangeTensorCategory(H, C)`

`ChangeTensorCategory( H, C)`

Changes the tensor category of H to the given category.

`Kernel(H)`

Returns the kernel of H as an ideal of its domain tensor.

`Image(H)`

Returns the image of H as a submap of the codomain tensor.

Example H61E33

```
> T := RandomTensor(GF(7),[5,4,3]);
> F := Frame(T);
>
> I := [* hom< F[j] -> F[j] | [< F[j].i,F[j].i > :
>   i in [1..Dimension(F[j])]] > : j in [1..3] *];
> H := Homotopism(T,T,I);
>
> Image(H);
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 5 over GF(7)
U1 : Full Vector space of degree 4 over GF(7)
U0 : Full Vector space of degree 3 over GF(7)
> Kernel(H);
Tensor of valence 2, U2 x U1 >-> U0
U2 : Vector space of degree 5, dimension 0 over GF(7)
U1 : Vector space of degree 4, dimension 0 over GF(7)
```

```
U0 : Vector space of degree 3, dimension 0 over GF(7)
```

If the tensor is over vector spaces, then matrices can be used to create a homotopism.

```
> M := [* RandomMatrix(GF(7),i,i) : i in [5,4,3] *];
> G := Homotopism(T,T,M);
> G;
Maps from U2 x U1 >-> U0 to V2 x V1 >-> V0.
U2 -> V2:
[5 3 6 0 4]
[6 0 0 1 1]
[6 4 3 1 5]
[3 4 6 1 4]
[2 4 1 3 2]
U1 -> V1:
[5 5 0 3]
[3 5 1 3]
[3 6 1 5]
[2 3 5 4]
U0 -> V0:
[6 4 1]
[2 6 5]
[1 2 3]
```

Homotopisms can be composed so long as Magma can compose each of the individual maps.

```
> G*G;
Maps from U2 x U1 >-> U0 to V2 x V1 >-> V0.
U2 -> V2:
[3 6 3 0 5]
[0 5 1 4 2]
[1 5 0 2 1]
[2 4 4 2 2]
[4 2 0 0 5]
U1 -> V1:
[4 3 6 0]
[4 6 0 6]
[4 3 4 3]
[0 4 0 0]
U0 -> V0:
[3 1 1]
[1 5 5]
[6 1 6]
```

We can change the underlying category for the homotopism G to get a different morphism.

```
> Cat := TensorCategory([1,-1,1],{{0},{1},{2}});
> G := Homotopism(T,T,M,Cat);
> G;
Maps from U2 x U1 >-> U0 to V2 x V1 >-> V0.
U2 -> V2:
```

```
[5 3 6 0 4]
[6 0 0 1 1]
[6 4 3 1 5]
[3 4 6 1 4]
[2 4 1 3 2]
U1 <- V1:
[5 5 0 3]
[3 5 1 3]
[3 6 1 5]
[2 3 5 4]
U0 -> V0:
[6 4 1]
[2 6 5]
[1 2 3]
>
> Image(G);
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 5 over GF(7)
U1 : Vector space of degree 4, dimension 0 over GF(7)
U0 : Vector space of degree 3, dimension 2 over GF(7)
Echelonized basis:
(1 0 4)
(0 1 3)
>
> Kernel(G);
Tensor of valence 2, U2 x U1 >-> U0
U2 : Vector space of degree 5, dimension 0 over GF(7)
U1 : Full Vector space of degree 4 over GF(7)
U0 : Vector space of degree 3, dimension 1 over GF(7)
Echelonized basis:
(1 4 0)
```

61.6 Some Extended Examples

We include some examples to demonstrate some of the uses of this package.

61.6.1 Distinguishing Groups

Example H61E34

In 2004, S. E. Payne asked if two elation groups were isomorphic but suspected they were not [Pay]. The first group, G_f, is the elation group of the generalized quadrangle $H(3, q^2)$, the Hermitian geometry. This group is defined as a Heisenberg group whose bilinear map is the usual dot product.

```
> p := 3;
> e := 4;
> q := p^e; // q = 3^e >= 27
> F := [KSpace(GF(q),2), KSpace(GF(q),2), KSpace(GF(q),1)];
>
> DotProd := function(x)
>   return KSpace(GF(q),1)!(x[1]*Matrix(2,1,x[2]));
> end function;
>
> DoubleForm := function(T)
>   F := SystemOfForms(T)[1];
>   K := BaseRing(F);
>   n := Nrows(F);
>   m := Ncols(F);
>   MS := KMatrixSpace(K,n,m);
>   Z := MS!0;
>   M1 := HorizontalJoin(Z,-Transpose(F));
>   M2 := HorizontalJoin(F,Z);
>   D := VerticalJoin( M1, M2 );
>   return Tensor( D, 2, 1 );
> end function;
>
> f := DoubleForm( Tensor( F, DotProd ) );
> f;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 4 over GF(3^4)
U1 : Full Vector space of degree 4 over GF(3^4)
U0 : Full Vector space of degree 1 over GF(3^4)
>
> IsAlternating(f);
true
> Gf := HeisenbergGroup(f);
```

Now we define Payne's second group, $G_{\bar{f}}$, which is the elation group of the Roman quadrangle with parameters (q^2, q). In this example, $\bar{f}$ is a biadditive map, but is bilinear over the prime field $\mathbf{F}_3$. Therefore, we construct a vector space isomorphism from $\mathbf{F}_3^e$ to $\mathbf{F}_{3^e}$ and the bilinear commutator map, induced by $\bar{f}$. Hence, $G_{\bar{f}}$ is the Heisenberg group of this bilinear commutator map.

```
> n := PrimitiveElement(GF(q)); // nonsquare
> MS := KMatrixSpace(GF(q),2,2);
> A := MS![-1,0,0,n];
> B := MS![0,1,1,0];
> C := MS![0,0,0,n^-1];
> F1 := Frame(f);
> F2 := [KSpace(GF(p),4*e), KSpace(GF(p),4*e), >   KSpace(GF(p),e)];
>
> // take 1/3^r root
> Root := function(v,r)
>   k := Eltseq(v)[1];
>   K := Parent(k);
>   if k eq K!0 then return k; end if;
>   R<x> := PolynomialRing(K);
>   f := Factorization(x^(3^r)-k)[1][1];
>   return K!(x-f);
> end function;
>
> // biadditive map defining elation grp
> RomanGQ := function(x)
>   u := Matrix(1,2,x[1]);
>   v := Matrix(2,1,x[2]);
>   M := [A,B,C];
>   f := &+[Root(u*M[i]*v,i-1) : i in [1..3]];
>   return KSpace(GF(q),1)![f];
> end function;
>
> // vector space isomorphisms
> phi := map< F2[1] -> F1[1] |
>   x :-> F1[1]![ GF(q)![ s : s in Eltseq(x)[i+1..e+i] ] :
>     i in [0,e,2*e,3*e] ] >;
> gamma := map< F1[3] -> F2[3] |
>   x :-> F2[3]!&cat[ Eltseq(s) : s in Eltseq(x) ] >;
>
> // bilinear commutator from RomanGQ
> RomanGQComm := function(x)
>   x1 := Eltseq(x[1]@phi)[1..2];
>   x2 := Eltseq(x[1]@phi)[3..4];
>   y1 := Eltseq(x[2]@phi)[1..2];
>   y2 := Eltseq(x[2]@phi)[3..4];
>   comm := RomanGQ( <x2,y1> ) - RomanGQ( <y2,x1> );
>   return comm @ gamma;
> end function;
>
> f_bar := Tensor( F2, RomanGQComm );
> f_bar;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 16 over GF(3)
U1 : Full Vector space of degree 16 over GF(3)
U0 : Full Vector space of degree 4 over GF(3)
>
```

```
> IsAlternating(f_bar);
true
> Gfb := HeisenbergGroup(f_bar);
```

The groups G_f and $G_{\bar{f}}$ have order 3^{20} and are class 2, exponent 3, and minimally generated by 16 elements. In other words, the groups G_f and $G_{\bar{f}}$ are central extensions of $\mathbf{Z}_3^{16}$ by $\mathbf{Z}_3^4$ and have exponent 3. Using standard heuristics, these groups are indistinguishable. However, the invariants associated to their exponent-p central tensor are vastly different, and thus, they determine that these groups are nonisomorphic. We show that the centroids of the tensors are not isomorphic.

```
> Tf := pCentralTensor(Gf,1,1);
> Tf;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 16 over GF(3)
U1 : Full Vector space of degree 16 over GF(3)
U0 : Full Vector space of degree 4 over GF(3)
>
> Tfb := pCentralTensor(Gfb,1,1);
> Tfb;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 16 over GF(3)
U1 : Full Vector space of degree 16 over GF(3)
U0 : Full Vector space of degree 4 over GF(3)
>
> Cf := Centroid(Tf);
> Cfb := Centroid(Tfb);
> Dimension(Cf) eq Dimension(Cfb);
false
```

61.6.2 Simplifying Automorphism Group Computations

Example H61E35

We construct a class 2, exponent p, p-group G which is a quotient of a maximal unipotent subgroup of $\mathrm{GL}(3, 317^4)$.

```
> p := 317;
> e := 4;
> H := ClassicalSylow( GL(3,p^e), p );
> U := UnipotentMatrixGroup(H);
> P := PCPresentation(U);
> Z := Center(P);
>
> N := sub< P | >;
> while #N lt p^2 do
>   N := sub< P | Random(Z), N >;
> end while;
>
```

```
> G := P/N;
> G;
GrpPC : G of order 10246902931634286779441449 = 317^10
PC-Relations:
    G.5^G.1 = G.5 * G.9^62 * G.10^133,
    G.5^G.2 = G.5 * G.9^312 * G.10^295,
    G.5^G.3 = G.5 * G.9^316,
    G.5^G.4 = G.5 * G.10^316,
    G.6^G.1 = G.6 * G.9^312 * G.10^295,
    G.6^G.2 = G.6 * G.9^316,
    G.6^G.3 = G.6 * G.10^316,
    G.6^G.4 = G.6 * G.9^138 * G.10^163,
    G.7^G.1 = G.7 * G.9^316,
    G.7^G.2 = G.7 * G.10^316,
    G.7^G.3 = G.7 * G.9^138 * G.10^163,
    G.7^G.4 = G.7 * G.9^188 * G.10^50,
    G.8^G.1 = G.8 * G.10^316,
    G.8^G.2 = G.8 * G.9^138 * G.10^163,
    G.8^G.3 = G.8 * G.9^188 * G.10^50,
    G.8^G.4 = G.8 * G.9^125 * G.10^151
```

We construct the exponent-p central tensor of G and compute its adjoint $*$-algebra A.

```
> T := pCentralTensor(G,1,1);
> T;
Tensor of valence 2, U2 x U1 >-> U0
U2 : Full Vector space of degree 8 over GF(317)
U1 : Full Vector space of degree 8 over GF(317)
U0 : Full Vector space of degree 2 over GF(317)
>
> A := AdjointAlgebra(T);
> Dimension(A);
16
> star := Star(A);
```

If $V = G/\Phi(G)$ is the Frattini quotient of G, then our goal is to get the cotensor space $V \wedge_A V$. Note that $\dim V \wedge V = 28$, so standard methods will compute a stabilizer of $\mathrm{GL}(8, 317)$ inside $V \wedge V$. We will decrease the size of the ambient space resulting in an easier stabilizer computation.

```
> V := Domain(T)[1];
> E := ExteriorCotensorSpace(V,2);
> E;
Cotensor space of dimension 28 over GF(317) with valence 1
U2 : Full Vector space of degree 8 over GF(317)
U1 : Full Vector space of degree 8 over GF(317)
```

Now we create a sub cotensor space S generated by all $(e_iX) \wedge e_j - e_i \wedge (e_jX)$ for $X \in A$, and then quotient $V \wedge V$ by S. The result is a 4 dimensional space.

```
> L := [];
> for E_gen in Generators(E) do
>   F := SystemOfForms(E_gen)[1];
>   for X in Basis(A) do
>     L cat:= [E!Eltseq(X*F - F*Transpose(X@star))];
>   end for;
> end for;
>
> S := SubtensorSpace(E,L);
> S;
Cotensor space of dimension 24 over GF(317) with valence 1
U2 : Full Vector space of degree 8 over GF(317)
U1 : Full Vector space of degree 8 over GF(317)
>
> Q := E/S;
> Q;
Cotensor space of dimension 4 over GF(317) with valence 1
U2 : Full Vector space of degree 8 over GF(317)
U1 : Full Vector space of degree 8 over GF(317)
```

61.7 Bibliography

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