PROBABILITY OF ERROR OF THE MATCHED FILTER

(1) 1st find SNR
(2) 2nd find P_e

(1) P_e of matched filter, may be found by evaluating max SNR $\left[\frac{P_o^2(T)}{\sigma_o^2}\right]_{max}$ with $G_n(f) = \frac{N_o}{2}$

(2) $SNR_{max} = \rho^2_{max} = \left[\frac{P_o^2(T)}{\sigma_o^2}\right]_{max} = \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)} df \rightarrow (1)$

sub $G_n(f) = \frac{N_o}{2}$ in (1)

(3) & $H(f) = k \frac{P^*(f)}{G_n(f)} e^{-j2\pi f T} \rightarrow (2)$

(4) $\rho_{max}^2 = \frac{2}{N_o} \int_{-\infty}^{\infty} |P(f)|^2 df \rightarrow (3)$

&

(5) $H(f) = k \cdot \frac{2}{N_o} \cdot P^*(f)\, e^{-j2\pi f T} \rightarrow (4)$

(6) From Parseval's theorem,

$$\int_{-\infty}^{\infty} |P(f)|^2 df = \int_{-\infty}^{\infty} p^2(t)\,dt = \int_0^T p^2(t)\,dt \rightarrow (5)$$

b/as $p(t)$ persists for only a time T.

(7) with $p(t) = s_1(t) - s_2(t)$ & using equ (5), rewrite (3) as

$$\rho_{max}^2 = \frac{2}{N_o} \int_0^T p^2(t)\,dt$$

$$= \frac{2}{N_o} \int_0^T [s_1(t) - s_2(t)]^2 dt \rightarrow (6)$$

$$= \frac{2}{N_o} \int_0^T [s_1^2(t) + s_2^2(t) - 2 s_1(t)\, s_2(t)]\, dt$$

$$= \frac{2}{N_0}\left\{\int_0^T s_1^2(t)\,dt + \int_0^T s_2^2(t)\,dt - 2\int_0^T s_1(t)\,s_2(t)\,dt\right\}$$

→ $$D_{max}^2 = \frac{2}{N_0}\left(E_{s1} + E_{s2} - 2E_{s12}\right) \rightarrow (7)$$

→ Here E_{s1} & E_{s2} are energies resply in $s_1(t)$ & $s_2(t)$ while E_{s12} is the energy due to correlation b/w $s_1(t)$ & $s_2(t)$.

→ Let $s_1(t)$ have an energy E_{s1} if $s_2(t)$ is to have same energy, then optimum choice of $s_2(t)$ is

$$s_2(t) = -s_1(t) \rightarrow (8)$$

→ Choice is optimum in that it yields a max. o/p sgl $p_0^2(T)$ for a given sgl energy.

→ Letting $s_2(t) = -s_1(t)$, we find

$$E_{s1} = E_{s2} = -E_{s12} = E_s \rightarrow (9)$$

sub (9) in (7)

→ $$D_{max}^2 = \frac{2}{N_0}(4E_s)$$

$$\boxed{D_{max}^2 = 8.\frac{E_s}{N_0}} \rightarrow (10) = \left[\frac{p_0^2(T)}{\sigma_0^2}\right]_{max}$$

Pe

→ Pe for opt filter is

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left[\frac{s_{01}(T) - s_{02}(T)}{2\sqrt{2}\,\sigma_0}\right] \rightarrow (4) \text{ in OF}$$

→ rewrite using $p_0(T) = s_{01}(T) - s_{02}(T)$

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$\rightarrow$ $$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{s_{o1}(T) - s_{o2}(T)}{2\sqrt{2}\,\sigma_o}\right]$$

$$= \frac{1}{2}\,\text{erfc}\left[\frac{p_o(T)}{2\sqrt{2}\,\sigma_o}\right]$$

$\rightarrow$ $$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{p_o^2(T)}{8\,\sigma_o^2}\right]^{1/2} \longrightarrow (11)$$

$\rightarrow$ combining equ (11) with (10) we get

$$(P_e)_{min} = \frac{1}{2}\,\text{erfc}\left\{\frac{1}{8}\left[\frac{p_o^2(T)}{\sigma_o^2}\right]_{max}\right\}^{1/2}$$

$$= \frac{1}{2}\,\text{erfc}\left\{\frac{1}{8}\cdot\frac{8E_s}{N_o}\right\}^{1/2}$$

$$\boxed{(P_e)_{min} = \frac{1}{2}\,\text{erfc}\left[\frac{E_s}{N_o}\right]^{1/2}} \longrightarrow (12)$$

$\rightarrow$ (12) $\Rightarrow$ P_e depends only on signal energy & not on sgl. waveshape.

$\rightarrow$ also (12) gives $(P_e)_{min}$ for the case of matched filter & when $s_1(t) = -s_2(t)$.

$\rightarrow$ In baseband rxr, for calculating its P_e we considered $s_1(t) = +V$ & $s_2(t) = -V$. & filter used was an integrator.

$\rightarrow$ $$\boxed{P_e(o) = \frac{1}{2}\,\text{erfc}\left(\frac{E_s}{N_o}\right)^{1/2}} \rightarrow (I) \qquad E_s = V^2T$$

$\rightarrow$ equ (12) & (I) are same.

$\rightarrow$ $\Rightarrow$ for a i/p sgl where $s_1(t) = +V$ & $s_2(t) = -V$, integrator is the matched filter.

→ when

$$s_1(t) = V, \quad 0 \le t \le T$$
$$s_2(t) = -V, \quad 0 \le t \le T$$

→ impulse response of the matched filter is

$$\boxed{h(t) = \frac{2R}{N_0}\left[s_1(T-t) - s_2(T-t)\right]} \rightarrow (13)$$

PROBABILITY ERROR of ASK, PSK & FSK

① It depends on signal nature i.e. ASK, (or) PSK

② Max SNR, $\rho_{max}^2 = \frac{2}{N_0}\int_0^T |p(t)|^2 dt$

consider if $0 \rightarrow s_1(t)$

$1 \rightarrow s_2(t)$

$p(t) = s_1(t) - s_2(t)$

within the bit duration 'T'.

③ $P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{1}{8} \cdot \rho_{max}^2\right]^{1/2}$

Pe of ASK

① signal nature

$1 \rightarrow$	$A\cos\omega t$
$0 \rightarrow$	0

i.e. like Unipolar format

② $\rho_{max}^2 = \frac{2}{N_0}\int_0^T |p(t)|^2 dt$

$= \frac{2}{N_0}\int_0^T |(A\cos\omega t - 0)|^2 dt$

$= \frac{2}{N_0}\int_0^T A^2\cos^2\omega t \, dt$

$= \frac{2A^2}{N_0}\int_0^T \cos^2\omega t \, dt$

$= \frac{2A^2}{N_0}\int_0^T \left(\frac{1+\cos 2\omega t}{2}\right) dt$

$= \frac{2A^2}{N_0}\left\{\int_0^T \frac{1}{2} \cdot dt + \int_0^T \frac{\cos 2\omega t}{2} dt\right\}$

$$= \frac{2A^2}{N_0}\left\{ \frac{T}{2} + \frac{1}{2}\left[\frac{\sin 2\omega t}{2\omega} \right]_0^T \right\}$$

$$d_{max}^2 = \frac{2A^2}{N_0}\left\{ \frac{T}{2} + \frac{1}{4\omega} \sin 2\omega T \right\}$$

Always $\omega >> \frac{1}{f}$ (or) (T)

where $\omega \to$ is carrier angular freq
$T = \frac{1}{f} \to$ symbol bit duration

$\therefore$ then term $\frac{\sin 2\omega T}{4\omega} \to \frac{1}{\text{large}}$ = very small

$\therefore$ neglect $\frac{1}{4\omega} \sin 2\omega T$

$$\therefore d_{max}^2 = \frac{\cancel{2}A^2}{N_0} \cdot \frac{T}{\cancel{2}} = \frac{A^2 T}{N_0}$$

$$\boxed{d_{max}^2 = \frac{A^2 T}{N_0}}$$

③ Pe

$$P_e = \frac{1}{2} \text{erfc}\left[\frac{1}{8} \cdot d_{max}^2\right]^{1/2}$$

$$P_e = \frac{1}{2} \text{erfc}\left[\frac{1}{8} \cdot \frac{A^2 T}{N_0}\right]$$

energy = Power x Time

$$\therefore E_s = \frac{A^2}{2} \times T$$

$$P_e = \frac{1}{2} \text{erfc}\left[\frac{1}{4} \cdot \underbrace{\frac{A^2}{2} \cdot T}_{E_s} \cdot \frac{1}{N_0}\right]$$

$$\boxed{P_e = \frac{1}{2} \text{erfc} \cdot \left[\frac{1}{4} \cdot \left(\frac{E_s}{N_0}\right)\right]}$$ for ASK

Pe ~~for~~ PSK

① Signal Nature

$$\boxed{\begin{array}{l} 1 \to A\cos\omega t \\ 0 \to -A\cos\omega t \end{array}}$$

180° PS

$$\therefore p(t) = A\cos\omega t - (-A\cos\omega t)$$

$$\boxed{p(t) = 2A\cos\omega t}$$

(2) $$d_{max}^2 = \frac{2}{N_0}\int_0^T |p(t)|^2 dt$$

$$= \frac{2}{N_0}\int_0^T |2A\cos\omega t|^2 dt$$

$$= \frac{2}{N_0}\int_0^T 4A^2\cos^2\omega t\, dt$$

$$= \frac{2}{N_0}\cdot 4A^2 \int_0^T \cos^2\omega t \cdot dt$$

$$= \frac{2}{N_0}\cdot 4A^2 \left[\frac{T}{2} + \cancel{\frac{\sin 2\omega T}{4\omega}}\right]_0^T$$

As $\omega \gg \frac{1}{f}$ (or T)

$$\therefore d_{max}^2 = \frac{\cancel{2}}{N_0}\cdot 4A^2 \cdot \frac{T}{\cancel{2}}$$

$$\boxed{d_{max}^2 = \frac{4A^2T}{N_0}}$$

(3) $$P_e = \frac{1}{2}\,\mathrm{erfc}\left[\frac{1}{8}\cdot d_{max}^2\right]^{1/2}$$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{1}{8}\cdot\frac{4A^2T}{N_0}\right]^{1/2}$$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\cancel{\frac{1}{4}}\cdot\cancel{4}\cdot\underbrace{\frac{A^2T}{2}}_{E_s}\cdot\frac{1}{N_0}\right]$$

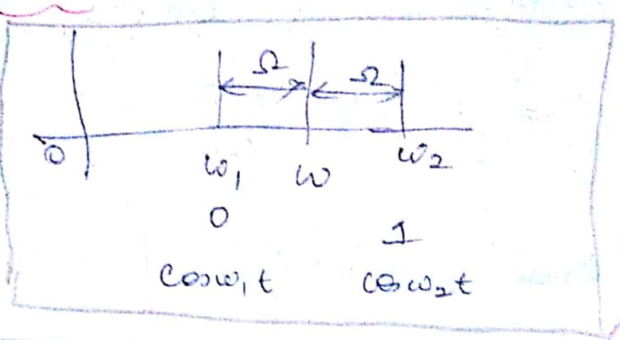
$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{E_s}{N_0}\right)}$$

(4) Compared with ASK's. Pe, PSK's Pe is better

i.e. than ASK.

Pe of FSK

① Signal Nature



$$\boxed{\begin{aligned} 1 &\rightarrow A\cos(\omega+\Omega)t \\ 0 &\rightarrow A\cos(\omega-\Omega)t \end{aligned}}$$

$\therefore p(t) = A\cos(\omega+\Omega)t - A\cos(\omega-\Omega)t$

② $\lambda^2_{max} = \frac{2}{N_0}\int_0^T |p(t)|^2 dt$

$= \frac{2}{N_0}\int_0^T \left| A\cos(\omega+\Omega)t - A\cos(\omega-\Omega)t \right|^2 dt$

$= \frac{2A^2}{N_0}\int_0^T \left| \cos(\omega+\Omega)t - \cos(\omega-\Omega)t \right|^2 dt$

$\lambda^2_{max} = \frac{2A^2}{N_0}\int_0^T \cos^2(\omega+\Omega)t\, dt + \frac{2A^2}{N_0}\int_0^T \cos^2(\omega-\Omega)t\, dt$

$- \frac{2A^2}{N_0}\int_0^T 2\cos(\omega+\Omega)\cos(\omega-\Omega)\, dt$

use $\int_0^T \cos at \cdot \cos bt \cdot dt = \frac{\sin(a-b)T}{2(a-b)} + \frac{\sin(a+b)T}{2(a+b)}$

$= \frac{2A^2}{N_0}\left[\frac{T}{2} + \frac{\sin 2(\omega+\Omega)T}{4(\omega+\Omega)} + \frac{T}{2} + \frac{\sin 2(\omega-\Omega)T}{4(\omega-\Omega)} - \frac{\sin 2\Omega T}{2\Omega} - \frac{\sin 2\omega T}{2\omega}\right]$

Assume $\omega \gg \Omega$, $\omega \gg \frac{1}{f}$ (or) T

$= \frac{2A^2}{N_0}\left[T - \frac{\sin 2\Omega T}{2\Omega}\right]$

$\lambda^2_{max} = \frac{2A^2 T}{N_0}\left[1 - \frac{\sin 2\Omega T}{2\Omega T}\right]$

to have max. value of λ^2_{max} $\left(\frac{\sin 2\Omega T}{2\Omega T}\right)$ should be minimum

$\frac{\sin 2\Omega T}{2\Omega T}$ min. value occurs at $2\Omega T =$

$\therefore \lambda_{max}^2 = \frac{2A^2T}{N_0}\left[1-(-0.212)\right] = \frac{2A^2T}{N_0}\left[1+0.2\right.$

$\lambda_{max}^2 = 2.424 \frac{A^2T}{N_0}$

$$\boxed{\lambda_{max}^2 = 2.42 \frac{A^2T}{N_0}}$$

(3) $P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{1}{8}\lambda_{max}^2\right]^{1/2}$

$= \frac{1}{2} \operatorname{erfc}\left[\frac{1}{8} \cdot 2.42 \cdot \frac{A^2T}{N_0}\right]^{1/2}$

$= \frac{1}{2} \operatorname{erfc}\left[0.3 \frac{A^2T}{N_0}\right]^{1/2}$

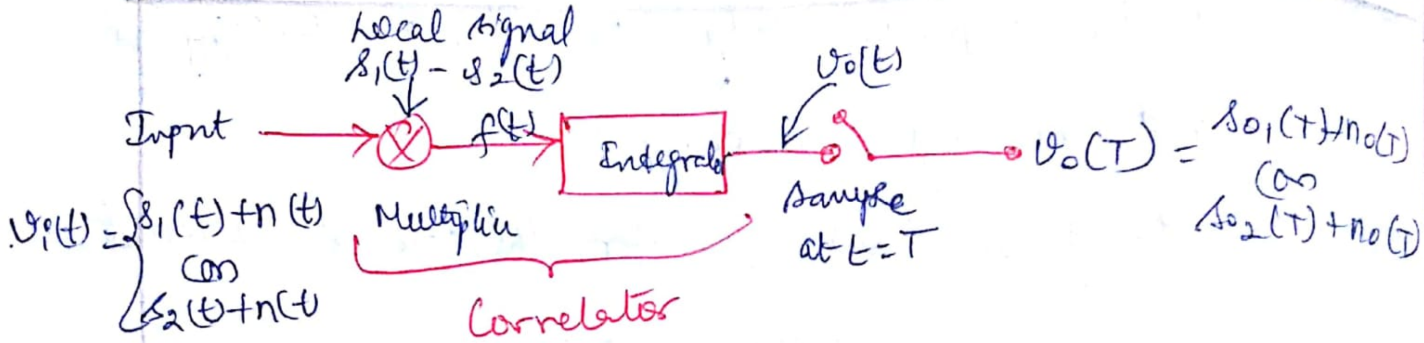
$$\boxed{P_e = \frac{1}{2} \operatorname{erfc}\left[0.6\left(\frac{E_s}{N_0}\right)\right]}$$

It is better than ASK but lesser than PS

COHERENT RECEPTION: CORRELATION

→ It is an alternative type of rxng s/m identical in performance with matched filter rxr.

→



→ Input is a binary data wfm $s_1(t)$ or $s_2(t)$ corrupted by noise $n(t)$.

→ Bit length is T.

→ Rxed sgl $(v_i(t))$ + noise is Xed by locally gened wfm $s_1(t) - s_2(t)$.

→ o/p of Xer is passed thro' an ∫r whose o/p is sampled at $t = T$.

→ Immediately after each sampling, at

beginning of each new bit interval, all energy-storing elements in integrator are discharged.

→ This type of rxr is called Correlator

→ ∵ we are correlating the rxed sgl & noise with wfm $s_1(t) - s_2(t)$

→ $$\boxed{s_o(T) = \frac{1}{\tau}\int_0^T s_i(t)[s_1(t) - s_2(t)]dt} \rightarrow (1)$$

$$\boxed{n_o(T) = \frac{1}{\tau}\int_0^T n(t)[s_1(t) - s_2(t)]dt} \rightarrow (2)$$

→ where $s_i(t)$ is either $s_1(t)$ (or) $s_2(t)$ &

where 'τ' is constant of integrator

i.e. for o/p is $\frac{1}{\tau}$ times [?] of its i/p.

→ Now compare these o/ps with matched filter o/ps.

→ if $h(t)$ is impulsive response of matched filter, then o/p of matched filter $v_o(t)$ can be found using convolution integral

→ $$v_o(t) = \int_{-\infty}^{\infty} v_i(\lambda)\, h(t-\lambda)\, d\lambda = \int_0^T v_i(\lambda)\, h(t-\lambda) d\lambda \rightarrow (3)$$

∵ bit interval is from 0 to T.

→ $h(t)$ for matched filter is

$$h(t) = \frac{2k}{N_0}[s_1(T-t) - s_2(T-t)] \rightarrow (4)$$

→ ∴ $$h(t-\lambda) = \frac{2k}{N_0}[s_1(T-t+\lambda) - s_2(T-t+\lambda)] \rightarrow (5)$$

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Sub (5) in (3)

→ $v_0(t) = \frac{2k}{N_0}\int_0^T v_i(\lambda)\left[s_1(T-t+\lambda) - s_2(T-t+\lambda)\right]d\lambda$ → (6)

→ ∵ $v_i(\lambda) = s_i(\lambda) + n(\lambda)$
&
$v_0(t) = s_0(t) + n_0(t)$

→ at $t=T$ yields

→ $\boxed{s_0(T) = \frac{2k}{N_0}\int_0^T s_i(\lambda)\left[s_1(\lambda) - s_2(\lambda)\right]d\lambda}$ → (7)

where $s_i(\lambda)$ is $= s_1(\lambda)$ (or) $s_2(\lambda)$

→ |||ly $\boxed{n_0(T) = \frac{2k}{N_0}\int_0^T n(\lambda)\left[s_1(\lambda) - s_2(\lambda)\right]d\lambda}$ → (8)

→ Thus $s_0(T)$ & $n_0(T)$ as calcd from eqs (1) & (2) for correlation rxr &

→ as calcd from eqs (7) & (8) for matched filter rxr are identical

→ Hence the performance of the 2 rxrs are identical.

→ Simply MF & correlator are not 2 distinct, independent techniques which happens to yield same result.

→ But they are 2 techniques of synthesizing the optimum filter $h(t)$.

→ For an AWGN chl, when txed sgls are equally likely, optimum rxr which minimizes average Pe is a correlation rxr.

→ Appln: It is used in PSK