Chapter 9

Random Processes through Linear Systems

In this chapter we study how random processes behave when they pass through linear time invariant systems. We will restrict ourselves to the class of W.S.S. random processes.

9.1 Review of Linear Systems

LTI System and Convolution

Recall that a linear time invariant system consists of two properties:

- Linearity: If $X_1(t) \to Y_1(t)$ and $X_2(t) \to Y_2(t)$, then
$$aX_1(t) + bX_2(t) \to aY_1(t) + bY_2(t).$$
- Time invariant: If $X(t) \to Y(t)$, then
$$X(t+\tau) \to Y(t+\tau).$$

All LTI systems satisfy the convolution property. That is, the output $Y(t)$ is a convolution of the input $X(t)$ with a filter (or kernel) $h(t)$:

$$Y(t) = h(t) * X(t), \tag{9.1}$$

where $*$ denotes the convolution operation, defined as

$$h(t) * X(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau.$$

For discrete time signals, convolution is defined as

$$h[n] * X[n] = \sum_{k=-\infty}^{\infty} h[k]X[n-k].$$

Continuous-time and Discrete-time Fourier Transforms

In this chapter we will mainly use continuous-time Fourier transform. For any function h satisfying (See Oppenheim and Willsky)

- it is absolute-integrable function, i.e.,

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty,$$

- h has finite number of discontinuity
- h has bounded variation

then the continuous Fourier transform exists and is defined as

$$H(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(t) e^{j\omega t} dt. \tag{9.2}$$

For notational simplicity, we write $H = \mathcal{F}(h)$. The inverse Fourier transform is $h = \mathcal{F}^{-1}(H)$.

Example. The "DC" value of the Fourier transform is

$$H(0) = \int_{-\infty}^{\infty} h(t) e^{j0t} dt = \int_{-\infty}^{\infty} h(t) dt.$$

Occasionally we will use the discrete-time Fourier transform. Discrete-time Fourier transform of a function h is defined as

$$H(e^{j\omega}) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} h[n] e^{j\omega n}. \tag{9.3}$$

9.2 Mean and Autocorrelation through LTI Systems

Mean Function through LTI System

Consider a W.S.S. random process $X(t)$. Since we assume that $X(t)$ is W.S.S., the mean function of $X(t)$ stays a constant, i.e., $\mu_X(t) = \mu_X$.

Proposition 1. *If $X(t)$ passes through an LTI system to yield $Y(t)$, then the mean function of $Y(t)$ is*

$$\mathbb{E}[Y(t)] = \mu_X \int_{-\infty}^{\infty} h(\tau) d\tau = \mu_X H(0). \tag{9.4}$$

Proof.

$$\begin{aligned}\mu_Y(t) = \mathbb{E}[Y(t)] &= \mathbb{E}\left[\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau\right] \\ &= \int_{-\infty}^{\infty} h(\tau)\mathbb{E}[X(t-\tau)]d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\mu_X d\tau\end{aligned}$$

where the last equality holds because $X(t)$ is W.S.S. so that $\mathbb{E}[X(t-\tau)] = \mu_X$. □

For discrete-time signals, the proposition becomes

$$\mathbb{E}\Big[Y[n]\Big] = \mu_X \sum_{k=-\infty}^{\infty} h[k] = \mu_X H(e^{j0}). \tag{9.5}$$

Autocorrelation Function through LTI System

Proposition 2. *If $X(t)$ passes through an LTI system to yield $Y(t)$, then the autocorrelation function of $Y(t)$ is*

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_X(\tau+s-r)dsdr. \tag{9.6}$$

Proof.

$$\begin{aligned}R_Y(\tau) &= \mathbb{E}[Y(t)Y(t+\tau)] \\ &= \mathbb{E}\left[\int_{-\infty}^{\infty} h(s)X(t-s)ds \int_{-\infty}^{\infty} h(r)X(t+\tau-r)dr\right] \\ &\overset{(a)}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\mathbb{E}\left[X(t-s)X(t+\tau-r)dsdr\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_X(\tau+s-r)dsdr,\end{aligned}$$

where in (a) we assume that integration and expectation are interchangeable. □

For discrete-time signals, the above proposition becomes

$$R_Y[k] = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h[m]h[n]R_X[k+m-n]. \tag{9.7}$$

Proposition 3. *If $X(t)$ passes through an LTI system to yield $Y(t)$, then the power spectral density of $Y(t)$ is*

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega). \tag{9.8}$$

Proof. By definition, power spectral density $S_Y(\omega)$ is the Fourier transform of the autocorrelation function $R_Y(\omega)$. Therefore,

$$\begin{aligned}
S_Y(\omega) &= \int_{-\infty}^{\infty} R_Y(\tau) e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_X(\tau+s-r)dsdr\; e^{-j\omega\tau} d\tau
\end{aligned}$$

Letting $u = \tau + s - r$, we have

$$\begin{aligned}
S_Y(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_X(u)e^{-j\omega(u-s+r)}dsdrdu \\
&= \int_{-\infty}^{\infty} h(s)e^{j\omega s}ds \int_{-\infty}^{\infty} h(r)e^{-j\omega r}dr \int_{-\infty}^{\infty} R_X(u)e^{-j\omega u}du = \overline{H(\omega)}H(\omega)S_X(\omega),
\end{aligned}$$

where $\overline{H(\omega)}$ is the complex conjugate of $H(\omega)$. □

For discrete-time signals, the above proposition becomes

$$S_Y(e^{j\omega}) = |H(e^{j\omega})|^2 S_X(e^{j\omega}). \tag{9.9}$$

Example 1: A W.S.S. process $X(t)$ has a correlation function

$$R_X(\tau) = \text{sinc}(\pi\tau).$$

Suppose that $X(t)$ passes through an LTI system with input/output relationship

$$2\frac{d^2}{dt^2}Y(t) + 2\frac{d}{dt}Y(t) + 4Y(t) = 3\frac{d^2}{dt^2}X(t) - 3\frac{d}{dt}X(t) + 6X(t).$$

Find $R_Y(\tau)$.

Solution: The sinc function has a Fourier transform given by

$$\text{sinc}(Wt) \underset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{W}\text{rect}\left(\frac{\omega}{2W}\right).$$

Therefore, the auto-correlation function is

$$R_X(\tau) = \text{sinc}(\pi\tau) \quad \underset{\mathcal{F}}{\longleftrightarrow} \quad \frac{\pi}{\pi}\text{rect}\left(\frac{\omega}{2\pi}\right).$$

By taking the Fourier transform on both sides, we have

$$S_X(\omega) = \begin{cases} 1, & -\pi \le \omega \le \pi, \\ 0, & \text{elsewhere.} \end{cases}$$

The system response can be found from the differential equation as

$$\begin{aligned} H(\omega) &= \frac{3(j\omega)^2 - 3(j\omega) + 6}{2(j\omega)^2 + 2(j\omega) + 4} \\ &= \frac{3\left[(2-\omega^2) - j\omega\right]}{2\left[(2-\omega^2) + j\omega\right]}. \end{aligned}$$

Taking the magnitude square yields

$$\begin{aligned} |H(\omega)|^2 &= \frac{3\left[(2-\omega^2) - j\omega\right]}{2\left[(2-\omega^2) + j\omega\right]} \frac{3\left[(2-\omega^2) + j\omega\right]}{2\left[(2-\omega^2) - j\omega\right]} \\ &= \frac{9}{4}\frac{(2-\omega^2)^2 + \omega^2}{(2-\omega^2)^2 + \omega^2} = \frac{9}{4}. \end{aligned}$$

Therefore, the output power spectral density is

$$\begin{aligned} S_Y(\omega) &= |H(\omega)|^2 S_X(\omega) \\ &= \frac{9}{4} S_X(\omega). \end{aligned}$$

Taking the inverse Fourier transform, we have

$$R_Y(\tau) = \frac{9}{4}\text{sinc}(\pi\tau).$$

Example 2: A random process $X(t)$ has zero mean and $R_X(t,s) = \min(t,s)$. Consider a new process $Y(t) = e^t X(e^{-2t})$.

(a) Is $Y(t)$ W.S.S.?

(b) Suppose $Y(t)$ passes through an LTI system yo yield an output $Z(t)$ according to

$$\frac{d}{dt}Z(t) + 2Z(t) = \frac{d}{dt}Y(t) + Y(t).$$

Find $R_Z(\tau)$.

Solution:

(a) In order to verify whether $Y(t)$ is W.S.S., we need to check the mean function and the auto-correlation function. The mean function is

$$\begin{aligned}\mathbb{E}[Y(t)] &= \mathbb{E}\left[e^t X(e^{-2t})\right] \\ &= e^t \mathbb{E}\left[X(e^{-2t})\right].\end{aligned}$$

Since $X(t)$ has zero mean, $\mathbb{E}[X(t)] = 0$ for all t. This implies that if $u = e^{-2t}$, then $\mathbb{E}[X(u)] = 0$ because u is just another time instant. Therefore, $\mathbb{E}[X(e^{-2t})] = 0$, and hence $\mathbb{E}[Y(t)] = 0$.

The auto-correlation can be found as

$$\begin{aligned}\mathbb{E}\left[Y(t+\tau)Y(t)\right] &= \mathbb{E}\left[e^{t+\tau}X(e^{-2(t+\tau)})e^t X(e^{-2t})\right] \\ &= e^{2t+\tau}\mathbb{E}\left[X(e^{-2(t+\tau)})X(e^{-2t}\right] \\ &= e^{2t+\tau}R_X(e^{-2(t+\tau)}, e^{-2t}) \\ &= e^{2t+\tau}\min(e^{-2(t+\tau)}, e^{-2t}) \\ &= e^{2t+\tau}\begin{cases} e^{-2(t+\tau)}, & \tau \geq 0 \\ e^{-2t}, & \tau < 0 \end{cases} \\ &= \begin{cases} e^{-\tau}, & \tau \geq 0 \\ e^{\tau}, & \tau < 0 \end{cases} \\ &= e^{-|\tau|}\end{aligned}$$

So, $R_Y(\tau) = e^{-|\tau|}$. Since $R_Y(\tau)$ is a function of τ, $Y(t)$ is W.S.S.

(b) The system response is given by

$$H(\omega) = \frac{1+j\omega}{2+j\omega}.$$

The magnitude is therefore

$$|H(\omega)|^2 = \frac{1+\omega^2}{4+\omega^2}.$$

Hence, the output auto-correlation function is

$$R_Y(\tau) = e^{-|\tau|} \longleftrightarrow S_Y(\omega) = \frac{2}{1+\omega^2},$$

and

$$\begin{aligned}S_Z(\omega) &= |H(\omega)|^2 S_Y(\omega) \\ &= \frac{1+\omega^2}{4+\omega^2}\frac{2}{1+\omega^2} = \frac{2}{4+\omega^2}.\end{aligned}$$

Therefore,

$$R_Z(\tau) = \frac{1}{2}e^{-2|\tau|}.$$

9.3 Cross-Correlation through LTI Systems

Before we discuss the cross-correlation function passing through an LTI system, we first define jointly W.S.S. for two arbitrary random processes $X(t)$ and $Y(t)$.

Jointly W.S.S. Processes

Definition 1. *Two random processes $X(t)$ and $Y(t)$ are jointly W.S.S. if*

1. *$X(t)$ is W.S.S. and $Y(t)$ is W.S.S.*
2. *$R_{X,Y}(t_1,t_2) = \mathbb{E}\left[X(t_1)Y(t_2)\right]$ is a function of $t_1 - t_2$.*

If $X(t)$ and $Y(t)$ are jointly W.S.S., then we write

$$R_{X,Y}(t_1,t_2) = R_{X,Y}(\tau) \stackrel{\text{def}}{=} \mathbb{E}\left[X(t+\tau)Y(\tau)\right].$$

The definition $R_{Y,X}(\tau)$ can be seen from the following Lemma.

Lemma 1. *For any random processes $X(t)$ and $Y(t)$, the cross-correlation $R_{X,Y}(\tau)$ is related to $R_{Y,X}(\tau)$ as*

$$R_{X,Y}(\tau) = R_{Y,X}(-\tau). \tag{9.10}$$

Proof. Recall the definition of $R_{Y,X}(-\tau) = \mathbb{E}\left[Y(t-\tau)X(t)\right]$. It holds that

$$\begin{aligned} R_{Y,X}(-\tau) = \mathbb{E}\left[Y(t-\tau)X(t)\right] &= \mathbb{E}\left[X(t)Y(t-\tau)\right] \\ &= \mathbb{E}\left[X(t'+\tau)Y(t')\right] = R_{X,Y}(\tau), \end{aligned}$$

where we substituted $t' = t - \tau$. □

Example 3. Let $X(t)$ and $N(t)$ be two independent W.S.S. random processes with expectations $\mathbb{E}[X(t)] = \mu_x$ and $\mathbb{E}[N(t)] = 0$, respectively. Let $Y(t) = X(t) + N(t)$. We want to show that $X(t)$ and $Y(t)$ are jointly W.S.S., and we want to find $R_{X,Y}(\tau)$.

Solution.
Before we show the joint W.S.S. property of $X(t)$ and $Y(t)$, we first show that $Y(t)$ is W.S.S.:

$$\begin{aligned} \mathbb{E}[Y(t)] &= \mathbb{E}[X(t)+N(t)] = \mu_x \\ R_Y(t_1,t_2) &= \mathbb{E}\left[(X(t_1)+N(t_1))(X(t_2)+N(t_2))\right] \\ &= \mathbb{E}\left[(X(t_1)X(t_2)\right] + \mathbb{E}\left[(N(t_1)N(t_2)\right] \\ &= R_X(t_1-t_2) + R_N(t_1-t_2). \end{aligned}$$

Thus, $Y(t)$ is W.S.S.

To show that $X(t)$ and $Y(t)$ are jointly W.S.S., we need to check the cross-correlation function:

$$\begin{aligned}
R_{X,Y}(t_1,t_2) &= \mathbb{E}[X(t_1)Y(t_2)] \\
&= \mathbb{E}\left[X(t_1)(X(t_2)+N(t_2))\right] \\
&= \mathbb{E}\left[X(t_1)(X(t_2)\right] + \mathbb{E}\left[X(t_1)N(t_2))\right] \\
&= R_X(t_1,t_2) + \mathbb{E}[X(t_1)]\mathbb{E}[N(t_2)] = R_X(t_1,t_2).
\end{aligned}$$

Since $R_{X,Y}(t_1,t_2)$ is a function of t_1-t_2, and since $X(t)$ and $Y(t)$ are W.S.S., $X(t)$ and $Y(t)$ must be jointly W.S.S.

Finally, to find $R_{X,Y}(\tau)$, we substitute $\tau = t_1 - t_2$ and obtain $R_{X,Y}(\tau) = R_X(\tau)$.

We next define the cross power spectral density of two jointly W.S.S. processes as the Fourier transform of the cross-correlation function.

Definition 2. *The* ***cross power spectral density*** *of two jointly W.S.S. processes $X(t)$ and $Y(t)$ is defined as*

$$\begin{aligned}
S_{X,Y}(\omega) &= \mathcal{F}[R_{X,Y}(\tau)] \\
S_{Y,X}(\omega) &= \mathcal{F}[R_{Y,X}(\tau)]
\end{aligned}$$

The relationship between $S_{X,Y}$ and $S_{Y,X}$ can be seen from the following Lemma.

Proposition 4. *For two jointly W.S.S. random processes $X(t)$ and $Y(t)$, the cross power spectral density satisfies the property that*

$$S_{X,Y}(\omega) = \overline{S_{Y,X}(\omega)}, \tag{9.11}$$

where $\overline{(\cdot)}$ denotes the complex conjugate.

Proof. Since $S_{X,Y}(\omega) = \mathcal{F}[R_{X,Y}(\tau)]$ by definition, it follows that

$$\begin{aligned}
\mathcal{F}[R_{X,Y}(\tau)] = \int_{-\infty}^{\infty} R_{X,Y}(\tau)e^{-j\omega\tau}d\tau &= \int_{-\infty}^{\infty} R_{Y,X}(-\tau)e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} R_{X,Y}(\tau')e^{j\omega\tau'}d\tau' = \overline{S_{Y,X}(\omega)}.
\end{aligned}$$

□

Cross-correlation Function through LTI System

We now study the special case where $X(t)$ is the input to an LTI system, and $Y(t)$ is the output of the LTI system.

Proposition 5. *If $X(t)$ passes through an LTI system to yield $Y(t)$, then the* ***cross-correlation*** *is*

$$R_{Y,X}(\tau) = h(\tau) * R_X(\tau). \tag{9.12}$$

Proof. Recall the definition of cross-correlation, we have

$$\begin{aligned} R_{Y,X}(\tau) &= \mathbb{E}[Y(t+\tau)X(t)] = \mathbb{E}\left[X(t)\int_{-\infty}^{\infty} X(t+\tau-r)h(r)dr\right] \\ &= \int_{-\infty}^{\infty} \mathbb{E}[X(t)X(t+\tau-r)]h(r)dr = \int_{-\infty}^{\infty} R_X(\tau-r)h(r)dr. \end{aligned}$$

□

Proposition 6. *If $X(t)$ passes through an LTI system to yield $Y(t)$, then the* ***cross power spectral density*** *is*

$$\begin{aligned} S_{Y,X}(\omega) &= H(\omega)S_X(\omega) \\ S_{X,Y}(\omega) &= \overline{H(\omega)}S_X(\omega) \end{aligned}$$

Proof. By taking the Fourier transform on $R_{Y,X}(\tau)$ we have that $S_{Y,X}(\omega) = H(\omega)S_X(\omega)$. Since $R_{X,Y}(\tau) = R_{Y,X}(-\tau)$, it holds that $S_{X,Y}(\omega) = \overline{H(\omega)}S_X(\omega)$. □

Example 4. Let $X(t)$ be a W.S.S. random process with

$$R_X(\tau) = e^{-\tau^2/2}, \quad H(\omega) = e^{-\omega^2/2}.$$

Find $S_{X,Y}(\omega)$, $R_{X,Y}(\tau)$, $S_Y(\omega)$ and $R_Y(\tau)$.

Solution. First, by Fourier transform table we know that $S_X(\omega) = \sqrt{2\pi}e^{-\omega^2/2}$. Since $H(\omega) = e^{-\omega^2/2}$, we have

$$S_{X,Y}(\omega) = H(\omega)S_X(\omega) = \sqrt{2\pi}e^{-\omega^2}.$$

The cross-correlation function is

$$R_{X,Y}(\omega) = \mathcal{F}^{-1}\left[\sqrt{2\pi}e^{-\omega^2}\right] = \frac{1}{\sqrt{2}}e^{-\frac{\tau^2}{4}}.$$

The power spectral density of $Y(t)$ is

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \sqrt{2\pi} e^{-\frac{3\omega^2}{2}}.$$

Therefore, the autocorrelation function of $Y(t)$ is

$$R_Y(\tau) = \mathcal{F}^{-1}\left[\sqrt{2\pi} e^{-\frac{3\omega^2}{2}}\right] = \frac{1}{\sqrt{3}} e^{-\tau^2/6}.$$

9.4 Optimal Linear Systems

The (discrete-time) optimal linear systems concern about the following problem: Let $X[n]$ be the input signal and $Y[n]$ be the observed output signal. The exact relationship between $X[n]$ and $Y[n]$ is unknown. However, statistical properties of $X[n]$ and $Y[n]$ are known, or at least can be estimated. These properties include $R_X[k]$, $R_Y[k]$ and $R_{X,Y}[k]$.

As suggested by its name, an optimal linear system imposes a linear model. Given $Y[n]$, we postulate that the best estimate of $X[n]$, denoted by $\widehat{X}[n]$, can be found using a linear combination of the $Y[n]$'s. Depending on how far we want to go in time (past and future), there are three types of estimates:

1. Finite Impulse Response (FIR)

$$\widehat{X}[n] = \sum_{k=n-K+1}^{n} h[n-k]Y[k]$$

2. Wiener Filter

$$\widehat{X}[n] = \sum_{k=-\infty}^{\infty} h[n-k]Y[k]$$

3. Causal Filter

$$\widehat{X}[n] = \sum_{k=0}^{\infty} h[n-k]Y[k]$$

In these three equations, the filter $h[n]$ is the subject of interest. The goal is to find the optimal $h[n]$ such that the error between the estimate $\widehat{X}[n]$ and the truth $X[n]$ is minimized, i.e.,

$$\underset{h[n]}{\text{minimize}} \quad \mathbb{E}\left[\left(X[n] - \widehat{X}[n]\right)^2\right].$$

The following sections will describe solutions to each of these models.