The Cross-Power density spectrum

For two real random processes $X(t)$ & $Y(t)$, define $x_T(t)$ & $y_T(t)$ as truncated ensemble members, that is

$$x_T(t) = \begin{cases} x(t) , & -T < t < T \\ 0 , & \text{elsewhere} \end{cases}$$

and
$$y_T(t) = \begin{cases} y(t) , & -T < t < T \\ 0 , & \text{elsewhere} \end{cases}$$

As a consequence, they will process F.Ts that denote by $X_T(\omega)$ & $Y_T(\omega)$, respectively.

$$x_T(t) \Leftrightarrow X_T(\omega)$$

$$y_T(t) \Leftrightarrow Y_T(\omega)$$

The cross power $P_{xy}(T)$ is the two processes with in the interval $(-T, T)$ by

$$P_{xy}(T) = \frac{1}{2T} \int_{-T}^{T} x_T(t)\, y_T(t)\, dt = \frac{1}{2T} \int_{-T}^{T} x(t)\, y(t)\, dt$$

using parseval's theorem

$$P_{xy}(T) = \frac{1}{2T} \int_{-T}^{T} x(t)\, y(t)\, dt = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \frac{X_T^*(\omega)\, Y_T(\omega)}{2T} \cdot d\omega$$

Now we obtain the average cross-power, by taking the expected value and letting $T \to \infty$

$$P_{XY} = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} E[x(t)\, y(t)]\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underset{T\to\infty}{Lt} \frac{E[X_T^*(\omega)\, Y_T(\omega)]}{2T}\, d\omega$$

$$P_{XY} = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t,t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underset{T\to\infty}{Lt} \frac{E[X_T^*(\omega)\, Y_T(\omega)]}{2T}\, d\omega$$

It is clear that integrand involving ω can be defined as a cross-power density spectrum, it is denoted by $S_{XY}(\omega)$

$$\therefore\ S_{XY}(\omega) = \underset{T\to\infty}{Lt} \frac{E[X_T^*(\omega) \cdot Y_T(\omega)]}{2T}$$

Thus
$$P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \cdot d\omega$$

Slly

$$S_{YX}(\omega) = \underset{T\to\infty}{Lt} \frac{E[Y_T^*(\omega)\, X_T(\omega)]}{2T}$$

$$P_{YX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega)\, d\omega = P_{XY}^*$$

Properties :-

1. $S_{xy}(\omega) = S^*_{yx}(\omega)$, Note $R_{xy}(\tau) = R_{yx}(-\tau)$

Proof :- $$S_{xy}(\omega) = \int_{-\alpha}^{\alpha} R_{xy}(\tau) \cdot e^{-j\omega\tau} \, d\tau$$

$$= \int_{-\alpha}^{\alpha} R_{yx}(-\tau) \cdot e^{-j\omega\tau} \, d\tau$$

$$= \int_{-\alpha}^{\alpha} R_{yx}(\tau) \cdot e^{j\omega\tau} \, d\tau$$

$$= S_{yx}(-\omega) = S^*_{yx}(\omega)$$

2. $Re[S_{xy}(\omega)]$ is an even function of ω and $Im[S_{xy}(\omega)]$ is an odd function of ω

Proof :- $$S_{xy}(\omega) = \int_{-\alpha}^{\alpha} R_{xy}(\tau) \cdot e^{-j\omega\tau} \, d\tau$$

$$= \int_{-\alpha}^{\alpha} R_{xy}(\tau) \left[\cos\omega\tau - j\sin\omega\tau\right] d\tau$$

$$= \int_{-\alpha}^{\alpha} R_{xy}(\tau) \cos\omega\tau \, d\tau - j\int_{-\alpha}^{\alpha} R_{xy}(\tau) \sin\omega\tau \, d\tau$$

$$= Re[S_{xy}(\omega)] - j\, Im[S_{xy}(\omega)]$$

where $Re[S_{xy}(\omega)] = \int_{-\alpha}^{\alpha} R_{xy}(\tau) \cos\omega\tau \, d\tau$ is an even function of ω

and $Im[S_{xy}(\omega)] = \int_{-\alpha}^{\alpha} R_{xy}(\tau) \sin\omega\tau \, d\tau$ is an odd function of ω

3. $X(t)$ & $Y(t)$ are uncorrelated and have constant means, then $S_{xy}(\omega) = S_{yx}(\omega) = \mu_x \mu_y \delta(\omega) = \bar{X}\,\bar{Y}\,\delta(\omega)\,2\pi$

Proof :-

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) \cdot e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} E[X(t)Y(t+\tau)] \cdot e^{-j\omega\tau}\, d\tau.$$

$$= \int_{-\infty}^{\infty} \underbrace{E[X(t)] \cdot E[Y(t+\tau)]}_{S.I \to \text{uncorrelated}} \cdot e^{-j\omega\tau}\, d\tau.$$

$$= \int_{-\infty}^{\infty} \bar{X} \cdot \bar{Y} \cdot e^{-j\omega\tau}\, d\tau$$

$$= \bar{X} \cdot \bar{Y} \cdot \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega\tau}\, d\tau$$

$$= \bar{X} \cdot \bar{Y} \cdot \delta(\omega) \times 2\pi \qquad \because \delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{-j\omega\tau}\, d\tau$$

4. If $X(t)$ and $Y(t)$ are orthogonal, then

$$S_{xy}(\omega) = S_{yx}(\omega) = 0.$$

Proof :-

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) \cdot e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} E[X(t)Y(t+\tau)] \cdot e^{-j\omega\tau}\, d\tau.$$

$$= 0 \qquad \because E[X(t) \cdot Y(t+\tau)] = 0, \text{ orthogonal}$$

5. The cross power P_{XY} between $X(t)$ and $Y(t)$ is defined by $P_{XY} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X(t)Y(t)]\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\cdot d\omega$.

Proof :-

$$P_{XY} = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} E[X_T(t)\,Y_T(t)]\,dt$$

Applying Parseval's theorem, we get

$$P_{XY} = \lim_{T\to\infty} \frac{1}{2T}\cdot\frac{1}{2\pi} \int_{-\infty}^{\infty} E[X_T^*(\omega)\cdot Y_T(\omega)]\,d\omega$$

$\because x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_T^*(\omega)\cdot e^{-j\omega t}\,d\omega$

$$= \frac{1}{2\pi}\lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{E[X_T^*(\omega)\cdot Y_T(\omega)]}{2T}\cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[X_T^*(\omega)\,Y_T(\omega)]}{2T}\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\,d\omega$$

6. $A[R_{XY}(t, t+\tau)] \longleftrightarrow S_{XY}(\omega)$

$A[R_{YX}(t, t+\tau)] \longleftrightarrow S_{YX}(\omega)$

=

Relationship between Cross-power density spectrum and Cross-correlation function:

State:- The inverse Fourier transform of the ~~power~~ cross-power density spectrum is the time average of the cross-correlation function, that is

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} A[R_{XY}(t,t+\tau)] \cdot e^{-j\omega\tau} \cdot d\tau .$$

$$A[R_{XY}(t,t+\tau)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) \cdot e^{j\omega\tau}\, d\omega$$

Proof:- The cross-power density spectrum

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega) \cdot Y_T(\omega)]}{2T}$$

where $X_T^*(\omega) = \int_{-T}^{T} x(t_1) \cdot e^{j\omega t_1}\, dt_1$

$$Y_T(\omega) = \int_{-T}^{T} y(t_2) \cdot e^{-j\omega t_2} \cdot dt_2$$

$$\therefore S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} x(t_1) \cdot e^{j\omega t_1} dt_1 \int_{-T}^{T} y(t_2) \cdot e^{-j\omega t_2} dt_2\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E[x(t_1) \cdot y(t_2)] \cdot e^{-j\omega(t_2-t_1)}\, dt_1\, dt_2$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{XY}(t_1,t_2) \cdot e^{-j\omega(t_2-t_1)} \cdot dt_1 \cdot dt_2$$

Let $t_1 = t$, $t_2 - t_1 = \tau$, then $t_2 = t + \tau$

$$S_{xy}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} \int_{t-T}^{t+T} R_{xy}(t, t+\tau) \cdot e^{-j\omega\tau} \cdot dt\, d\tau$$

applying $T \to \infty$ to 2^{nd} integral and interchange

$$= \int_{-\infty}^{\infty} \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} R_{xy}(t, t+\tau) \cdot dt \cdot e^{-j\omega\tau} \cdot d\tau$$

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} A\left[R_{xy}(t, t+\tau)\right] \cdot e^{-j\omega\tau} d\tau$$

$$\therefore A\left[R_{xy}(t, t+\tau)\right] \xleftrightarrow{F.T} S_{xy}(\omega)$$

Slly

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} A\left[R_{yx}(t, t+\tau)\right] \cdot e^{-j\omega\tau} d\tau.$$

$$\therefore A\left[R_{yx}(t, t+\tau)\right] \xleftrightarrow{F.T} S_{yx}(\omega)$$

If $x(t)$ & $y(t)$ are J.W.S.S, then

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) \cdot e^{-j\omega\tau} \cdot d\tau \ \&$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau) \cdot e^{-j\omega\tau} \cdot d\tau$$

Ex: The cross-correlation function of two processes $X(t)$ & $Y(t)$ by $R_{XY}(t, t+\tau) = \frac{AB}{2}\{\sin(\omega_0\tau) + \cos[\omega_0(2t+\tau)]\}$ where A, B and ω_0 are constants. Find the cross power density spectrum.

Sol:

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} A[R_{XY}(t, t+\tau)] \cdot e^{-j\omega\tau} \, d\tau$$

where

$$A[R_{XY}(t, t+\tau)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{AB}{2}\{\sin(\omega_0\tau) + \cos[\omega_0(2t+\tau)]\}\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left\{\left[\frac{AB}{2}\sin(\omega_0\tau) \cdot t\right]_{-T}^{T} + \left[\frac{AB}{2}\frac{\sin\omega_0(2t+\tau)}{2\omega_0}\right]_{-T}^{T}\right\}$$

$$= \lim_{T\to\infty} \left\{\frac{1}{2T} \cdot \frac{AB}{2}\sin(\omega_0\tau) \cdot 2T + \frac{1}{2T} \cdot \frac{AB}{2} \cdot \frac{\sin\omega_0(2T+\tau) - \sin\omega_0(-2T+\tau)}{2\omega_0}\right\}$$

$$= \frac{AB}{2}\sin(\omega_0\tau) + 0$$

$$= \frac{AB}{2}\sin\omega_0\tau$$

$$\therefore S_{XY}(\omega) = \int_{-\infty}^{\infty} \frac{AB}{2}\sin\omega_0\tau \cdot e^{-j\omega\tau}\, d\tau$$

$$= -\frac{j\pi}{2}AB\{\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\}$$

Ex :- Find the cross-power spectral density, if

i) $R_{xy}(\tau) = \frac{A^2}{2}\sin(\omega_0\tau)$ ii) $R_{xy}(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)$

Sol :-

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\cdot e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{A^2}{2}\sin\omega_0\tau \cdot e^{-j\omega\tau}\, d\tau$$

$$= \frac{A^2}{2}\int_{-\infty}^{\infty}\left\{\frac{e^{j\omega_0\tau} - e^{-j\omega_0\tau}}{2j}\right\} e^{-j\omega\tau}\, d\tau$$

$$= \frac{A^2}{4j}\left\{\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)\tau}\, d\tau - e^{-j(\omega+\omega_0)\tau}\, d\tau\right\}$$

$$= \frac{A^2}{4j}\left\{2\pi\cdot\delta(\omega-\omega_0) - 2\pi\,\delta(\omega+\omega_0)\right\}$$

$$= \frac{-jA^2\pi}{2}\left\{\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right\}$$

$$S_{xy}(\omega) = \frac{jA^2\pi}{2}\left\{\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right\}$$

ii)

slly $S_{xy}(\omega) = F\left[\frac{A^2}{2}\cos(\omega_0\tau)\right]$

$$= \frac{A^2\pi}{2}\left\{\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right\}$$