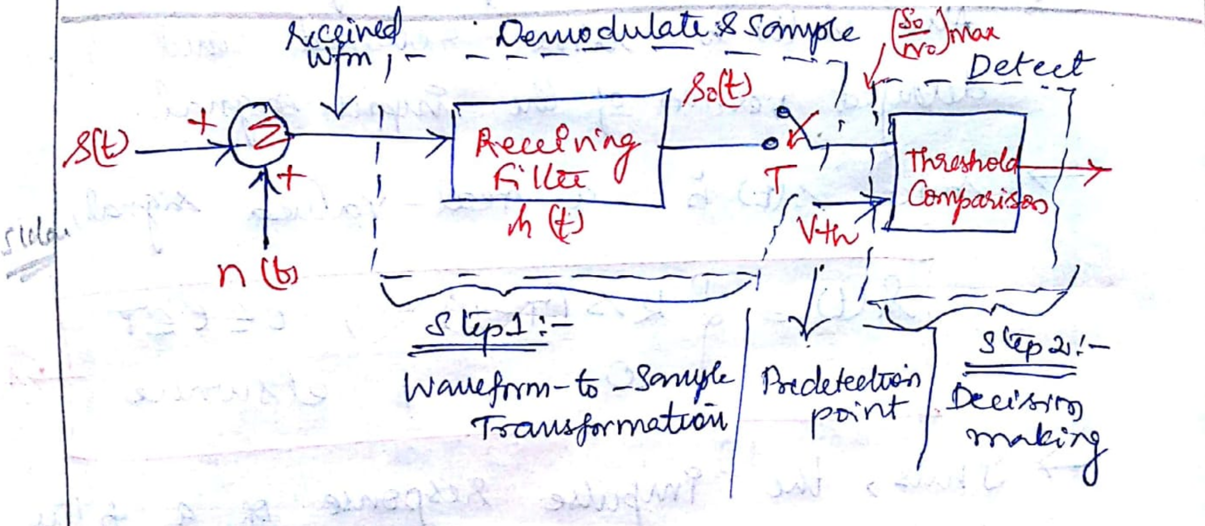
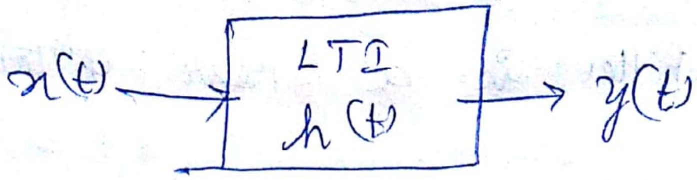
Matched Filter

1. Matched Filter is a linear filter.
2. Designed to provide maximum SNR at its o/p for a given transmitted symbol wfm.
3. It is an LTI filter
4. This filter is used for non-coherent detection of modulation techniques.

Matched Filter Receiver



5. Received signal and noise are passed through a filter having an impulse response $h(t)$.
6. Transfer function of the filter is $H(f)$.
7. Output of the filter is sampled for every 'T_b' secs.
8. Sampled value is compared with the threshold voltage to take a decision
9. whether the o/p is '1' (or) '0'. The errors will be minimum if $\left(\frac{S_o}{N_o}\right)_{max}$ at i/p of the comparator.



$$y(t) = x(t) * h(t) \longrightarrow (1)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \longrightarrow (2)$$

where $\boxed{h(t) = s(T-t)} \longrightarrow (3)$

→ Because impulse response of matched filter is matched to input signal.

→ Also $h(t)$ is time reversed and delayed version of the input signal.

→ Since $s(t)$ is a real-valued signal,

$$\boxed{h(t) = \begin{cases} k\,s(T-t), & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}} \longrightarrow (4)$$

→ Thus, the impulse response of a filter that produces the maximum output SNR is the mirror image of the message signal $s(t)$, delayed by the symbol time duration 'T'.

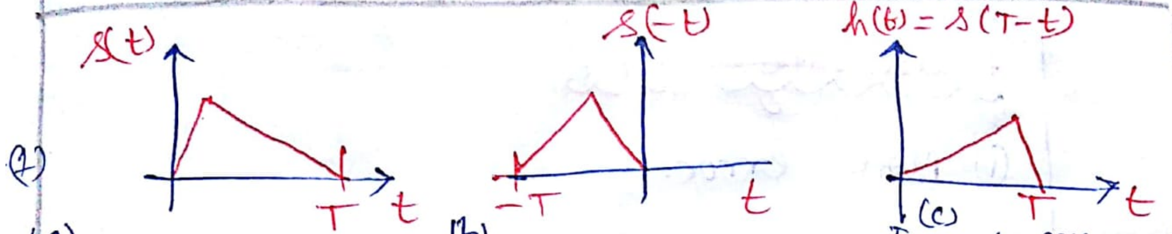
→ Delay of 'T' seconds make equ (4) causal.

→ i.e. delay of 'T' seconds makes $h(t)$ a function of the time in the interval $0 \le t \le T$.

→ Without delay of 'T' seconds, the response $s(-t)$ is unrealizable because it

→ describes a response as a function of -ve time.

(7)



(a) Signal waveform (b) Mirror image of s(t) (c) Impulse response of MF

→ equ (4) & fig(1) illustrate the matched filter's basic property :

→ Impulse response of the filter is a delayed version of mirror image (rotated on t=0 axis) of signal waveform

→ & the ~~[illegible]~~ mirror image delayed by 'T' seconds is $s(T-t)$.

$$y(t) = \int_{-\infty}^{\infty} [x(\tau)\, s(T-(t-\tau))]\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, s(T-t+\tau)\, d\tau$$

at $t=T$

$$y(T) = \int_0^T x(\tau)\, s(\cancel{T}-\cancel{T}+\tau)\, d\tau$$

$\because$ $s(t) = 0$ outside the interval $0 \le t \le T$.

$$\boxed{y(T) = \int_0^T x(\tau)\, s(\tau)\, d\tau} \rightarrow (5)$$

o/p of MF @ $t=T$.

Advantage of MF

① Analog multipliers/devices can be avoided

② No need of phase synchronization.

Disadvantage of MF

① More error.

Optimum Filter

A matched filter which operates with maximum SNR is called as an optimum Filter.

Output Signal to Noise Ratio of Matched Filter

$$y(t) = x(t) * h(t)$$

$$x(t) = s(t) + n(t) \longrightarrow (6)$$

$$y(t) = [s(t) + n(t)] * h(t)$$

$$s_o(t) + n_o(t) = s(t) * h(t) + n(t) * h(t)$$

$$s_o(t) = s(t) * h(t) \quad \& \longrightarrow (7)$$

$$n_o(t) = n(t) * h(t) \longrightarrow (8)$$

Signal Power

Fourier transform of equ (7)

$$\boxed{S_o(f) = S(f) \cdot H(f)} \longrightarrow (9)$$

where $H(f)$ is the T.F of linear filter.

Inverse F.T. of equ

$$\boxed{s_o(t) = \int_{-\infty}^{\infty} S_o(f)\, e^{j2\pi ft} \cdot df} \longrightarrow (10)$$

Sub. (9) in (10)

$$s_o(t) = \int_{-\infty}^{\infty} S(f) \cdot H(f)\, e^{j2\pi ft} \cdot df \longrightarrow (11)$$

→ Filter output is sampled at $t=T$ to give sample value $s_o(T)$.

→ Normalized signal power is given by $|s_o(T)|^2$

→ Using equation (10), with $t=T$, expression for Normalized signal Power is

$$\boxed{|s_o(T)|^2 = \left| \int_{-\infty}^{\infty} \underbrace{S(f)}_{f_2(t)} \cdot \underbrace{H(f)}_{f_1(t)}\, e^{j2\pi fT} \cdot df \right|^2} \longrightarrow (12)$$

(12) ⇒ Signal power at the i/p of comparator

Schwartz inequality:

$$\boxed{\left| \int_{-\infty}^{\infty} f_1(t)\, f_2(t)\, dt \right|^2 \leq \int_{-\infty}^{\infty} |f_1(t)|^2 dt \int_{-\infty}^{\infty} |f_2(t)|^2 dt}$$

Only when $\boxed{f_1(t) = k f_2^*(t)}$

where 'k' is an arbitrary constant

'$*$' indicates complex conjugate

$$\left| \int_{-\infty}^{\infty} S(f) \cdot H(f)\, e^{j2\pi f T} df \right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 df \times \int_{-\infty}^{\infty} |S(f)|^2 \left(e^{j2\pi fT}\right)^2 df \longrightarrow (13)$$

Only when $\boxed{H(f) = k\, S^*(f)\, e^{-j2\pi fT}} \longrightarrow (14)$

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Noise Power

$$\text{Noise } (PSD)_{o/p} = \frac{N_0}{2} \, |H(f)|^2 \quad \text{watts/Hz}$$

$$\text{o/p Noise Power} = \int_{-\infty}^{\infty} (\text{Noise PSD}) \cdot df$$

$$N_0(T)^2 = \int_{-\infty}^{\infty} \frac{N_0}{2} \, |H(f)|^2 \cdot df$$

$$\boxed{N_0(T)^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} \rightarrow (5)$$

$(SNR)_{o/p}$

$$(SNR)_{o/p} = \frac{|s_0(T)|^2}{|N_0(T)|^2}$$

$$\leq \frac{\cancel{\left(\int_{-\infty}^{\infty} |H(f)|^2 df\right)} \int_{-\infty}^{\infty} |S(f)|^2 df}{\frac{N_0}{2} \times \cancel{\int_{-\infty}^{\infty} |H(f)|^2 df}}$$

$$\leq \frac{\int_{-\infty}^{\infty} |S(f)|^2 df}{N_0/2}$$

$$\boxed{(SNR)_{o/p} \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |S(f)|^2 df} \rightarrow (16)$$

$$\because |e^{j2\pi fT}| = 1$$

→ Need to determine TF $H(f)$ of the filter such that $(SNR)_{o/p}$ at $t=T$ given by equ (16) is maximum.

→ (or) Need to determine $H(f)$ that optimizes the $(SNR)_{o/p}$.

→ equality holds good if & only if

$$\boxed{H(f) = k S^*(f)}$$

$S^*(f)$ is the complex conjugate of $S(f)$.

→ equ (17) ⇒ $(SNR)_{o/p}$ depends only on signal energy of input signal $s(t)$ given by

Parseval's theorem: $\int_{-\infty}^{\infty} |S(f)|^2 df$

and

PSD of white noise $\frac{N_0}{2}$.

→ $$\boxed{\text{Maximum } (SNR)_{o/p} = \frac{2}{N_0}\int_{-\infty}^{\infty} |S(f)|^2 df} \rightarrow (17)$$

$$\boxed{\max (SNR)_{o/p} = \frac{2E}{N_0}} \rightarrow (18)$$

→ where the energy E of input signal $s(t)$ is

$$\boxed{E = \int_{-\infty}^{\infty} |S(f)|^2 df} \rightarrow (19)$$

→ Thus max $(SNR)_{o/p}$ depends on the input signal energy & PSD of noise.

→ not on the particular shape of waveform that is used.

$s(t) \xleftrightarrow{F} S(f)$
$|S(f)|^2 \leftrightarrow$ ESD Tells
$\int$ ESD → Energy | $\int$ PSD → Power

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Optimum Filter Transfer function $H_{opt}(f)$

→ The equality in eqn (18) holds only if the optimum filter T.F. $H_{opt}(f)$ is used,

→ such that

$$\boxed{H(f) = H_{opt}(f) = k\, S^*(f)\, e^{-j2\pi fT}} \rightarrow (20)$$

(or)

→ $h(t) = \text{Inver. F.T}\{k S^*(f) e^{-j2\pi fT}\} \rightarrow (21)$

→ $$h_{opt}(t) = \int_{-\infty}^{\infty} H_{opt}(f) e^{j2\pi ft}\, df$$

$$= \int_{-\infty}^{\infty} k\, S^*(f)\, e^{-j2\pi fT} e^{j2\pi ft}\, df$$

$$= k \int_{-\infty}^{\infty} S^*(f)\, e^{-j2\pi f(T-t)}\, df$$

$$h_{opt}(t) = k \int_{-\infty}^{\infty} S(-f)\, e^{-j2\pi f(T-t)}\, df$$

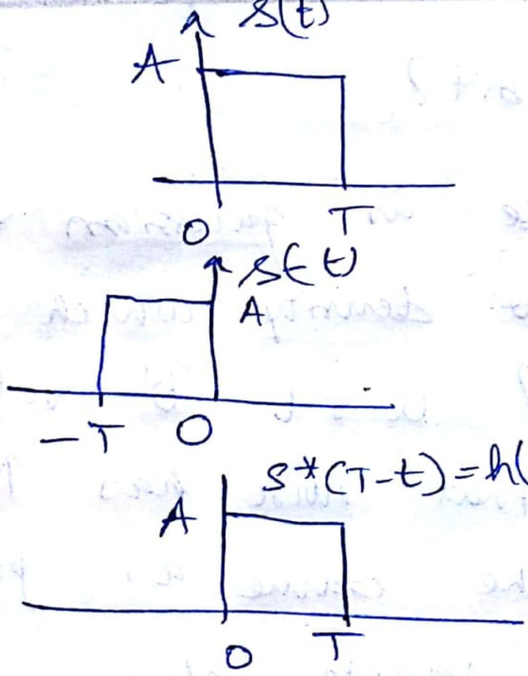
$$h_{op}(t) = k\left[\int_{-\infty}^{\infty} S(f)\, e^{j2\pi f(T-t)}\, df\right]^* \quad \because S^*(f) = S(-f)$$

$$\boxed{\therefore h_{op}(t) = S^*(T-t)} \rightarrow (22) \quad \text{when } k=1$$

→ if $s(t)$ is a real-valued sgl, we can write

$$\boxed{h(t) = \begin{cases} k\, s(T-t), & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}}$$

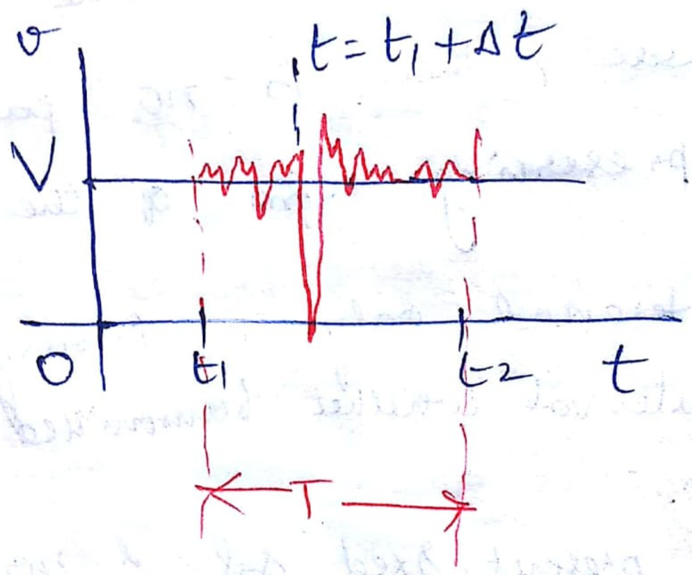
→ To determine impulse response of filter the following procedure is used.



→ Impulse response of the filter is same as the input signal (or)

→ Impulse response is matched to input signal

→ So, the filter is called "Matched Filter".



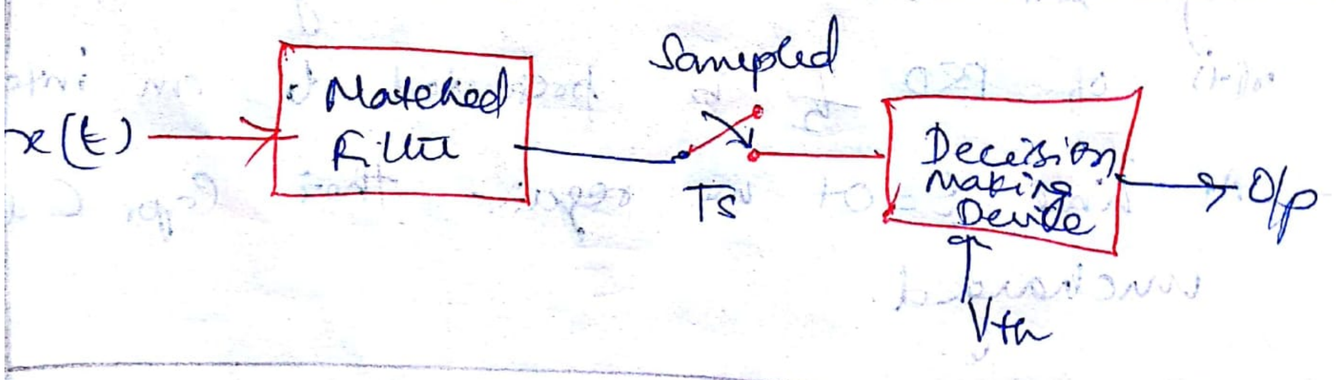
Noise may cause an error in determination of a fixed Vtg level

Peak Signal to RMS Noise Output Voltage Ratio

Base-band Receiver

→ coherent reception is not possible in base-band reception since it doesn't undergo modulation.

→ Base-band Receiver is similar to Non-coherent receiver.

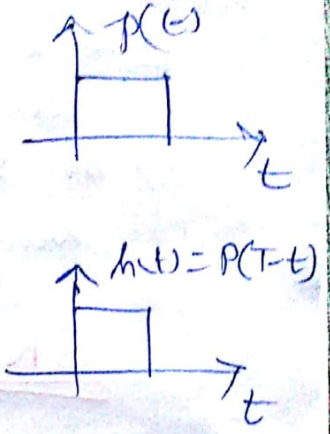


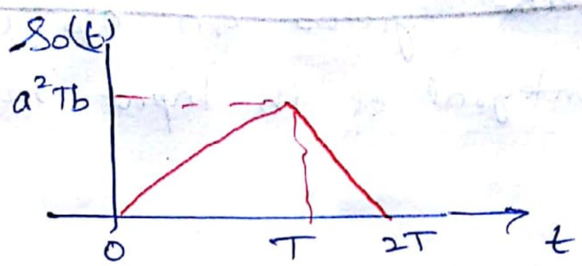
→ if $x(t) = \sum_k a_k \, p(t - kT)$

then

→ $h(t) = p(T - t)$

→ o/p of matched filter is convolution of 2 pulses





→ i/p to comparator is either $a^2 T_b$ (or) 0 if Unipolar NRZ signalling is used.

→ max value at the o/p of filter is equal to energy of the signal

→ time at which max value occurs is equal to duration of the i/p sgl.

→ Duration to be determined always w.r.t. zero. ~~time~~

→ Matched filter can be physically realised as a LPF

→ LPF can be realised using RC components but o/p is unstable.

→ ∴ we use active LPF using Op-amp.

→ Matched filter can be realized as integrator using op-amp.

→ Since matched filter is realised using Integrator & Sampler,
→ so it also called as Integrator & Dump receiver.

SNR

→ Integrator yields an o/p which is the integral of its input multiplied by $\frac{1}{RC}$.

→ using $\tau = RC$, we have.

→ o/p of the integrator

$y(T) =$ o/p at $t = T$

$$y(T) = \frac{1}{\tau}\int_0^T [s(t) + n(t)]dt$$

$$s_o(T) + n_o(T) = \frac{1}{\tau}\int_0^T s(t)dt + \frac{1}{\tau}\int_0 n(t)dt$$

→ since it is baseband receiver

$$s(t) = 1 = +V$$
$$= 0 = 0V$$

→ when $s(t) = V(1)$

sampled voltage due to sgl is

$$s_o(T) = \frac{1}{\tau}\int_0^T V dt$$

$$\boxed{s_o(T) = \frac{VT}{\tau}}$$

$$|So(T)|^2 = PSD = \boxed{\frac{V^2T^2}{\tau^2} = |so(T)|^2}$$

→ Sample vtg due to noise is

$$no(T) = \frac{1}{\tau}\int_0^T n(t)\,dt$$

→ This noise-sampling vtg $no(T)$ is a gaussian random variable in constrast with $n(t)$, which is gaussian RP.

→ Variance of $no(T)$

$$(No(T))^2 = \text{PSD of o/p}$$
$$= \text{PSD of gaussian}$$
$$= \text{Variance} \times \frac{T}{\tau^2}$$

$$|No(T)|^2 = \sigma_o^2 = \overline{no^2(T)} = \boxed{\frac{No}{2}\cdot\frac{T}{\tau^2} = |No(T)|^2}$$

→ o/p of Sor, before sampling switch is

$$vo(t) = so(t) + no(t)$$

→ $so(t)$ is a ramp, in each bit interval of duration T.

→ at end of interval ramp attains vtg. $so(T)$ which is $+\frac{VT}{\tau}$ (or) $-\frac{VT}{\tau}$, depending on whether the bit is 1 (or) a '0'.

→ At the end of each interval the switch sw_1 closes momentarily to discharge capr. so that $s_o(t)$ drops to 0!

→ noise $n_o(t)$, also starts each interval with $n_o(0) = 0$ &

→ has random value $n_o(T)$ at end of each interval.

→ Sampling switch sw_2 closes briefly just before closing of sw_1 &

→ ~~r~~ is

$$\boxed{v_o(T) = s_o(T) + n_o(t)}$$

→ $(SNR)_{I \& D\ Rav}$

$$SNR = \frac{[s_o(T)]^2}{|N_o(T)|^2} = \frac{V^2 T^{\cancel{2}}}{\cancel{\tau^2}} \times \frac{2\cancel{\tau^2}}{N_o \cancel{T}}$$

$$SNR = \frac{2V^2T}{N_o}$$

$$\boxed{(SNR)_{\text{Integrator \& Dump Rav}} = \frac{2}{N_o} \cdot V^2 T}$$

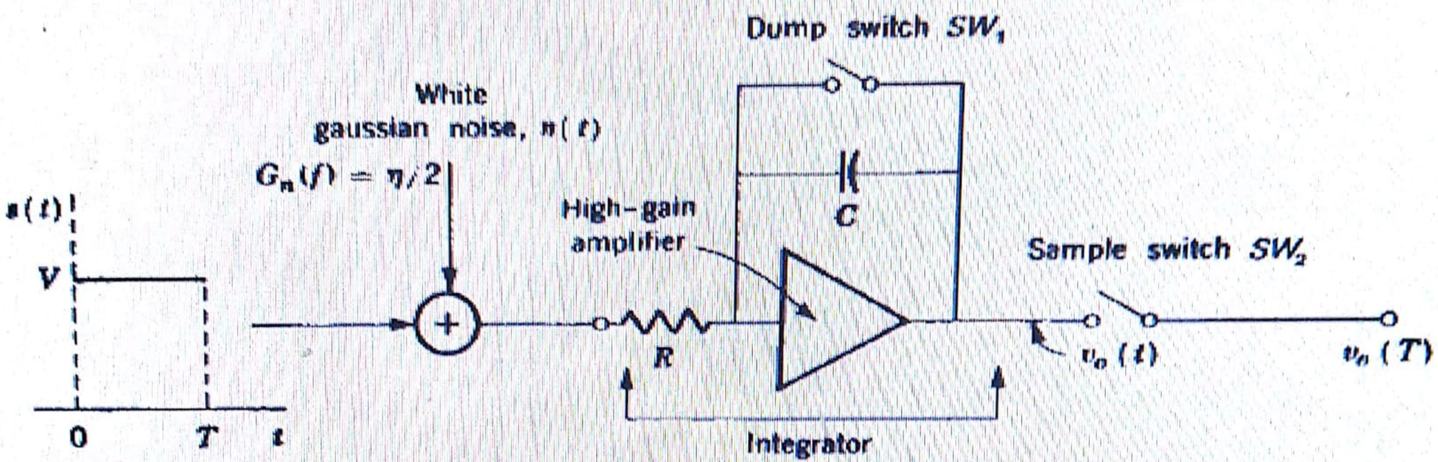


Figure 11.1-2 A receiver for a binary coded signal.