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UNIT - II

OPERATIONS ON ONE RANDOM VARIABLE - EXPECTATION

EXPECTATION

The averaging process when applied to a random variable is called the expectation. It is denoted as $E[x]$ and it is read as expected value of x or mean value of x

$$E[x] = \bar{x}$$

EXPECTED VALUE OF RANDOM VARIABLE

If X is a continuous RV then, expected value of X is

$$E[x] = \bar{x} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

Where, $f_X(x)$ is probability density funcn

If X is discrete RV then, expected value of X is

$$E[x] = \bar{x} = \sum_{i=1}^{N} x_i\, P(x_i)$$

EXPECTATION OF A FUNCTION OF A RV

A random variable X can be derived by finding the expected value of a real funcn $g(x)$ of X. It can be shown that the expected value is given by,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \rightarrow g(x) \text{ is CRV}$$

$$E[g(x)] = \sum_{i=1}^{N} g(x_i) \cdot P(x_i) \rightarrow g(x) \text{ is discrete RV.}$$

$f_X(x)$ is replaced by the conditional density $f_X(x/B)$ where B is any event defined in the sample space, we have the conditional expected value of X. denoted as $E[x/B]$

$$E[x/B] = \int_{-\infty}^{\infty} x \cdot f_X(x/B)\, dx$$

Let $B = X \leq b$ then,

$$f_X(x/X \leq b) = \begin{cases} \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\, dx}, & x \leq b \\ 0, & x > b \end{cases}$$

$$E\left[\frac{X}{X \leq b}\right] = \frac{\int_{-\infty}^{\infty} x \cdot f_X(x)\, dx}{\int_{-\infty}^{b} f_X(x)\, dx}$$

* MOVEMENTS

The expected value of a funcn $g(x)$ of a RV 'X' is
The expected value of a funcn. There are 2 types in calculating movements.

1. Movements about the origin
2. Movements about the mean

MOVEMENT ABOUT THE ORIGIN.

The funcn $g(x) = x^n$, $n = 0, 1, 2, 3 \ldots$ denote the Movements about the origin of the RV 'X'. nth order movement by m_n then,

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

Zero order movement $m_0 = E[X^0] = E[1] = 1$

first order movement $m_1 = E[X^1] = E(X) = \bar{X}$

2nd order movement $m_2 = E[X^2]$

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MOVEMENT ABOUT THE MEAN

The movements about the mean is also called as the central movements. It is denoted by μ_n & is defined as the expected value of the funcn, $g(x) = (x-\bar{x})^n$,

$n = 0, 1, 2 \ldots$

$$\mu_n = E[(x-\bar{x})^n], \quad n = 0, 1, 2 \ldots$$

$$\mu_0 = E[(x-\bar{x})^0] = E[1] = 1$$

$$\mu_1 = E[(x-\bar{x})^1] = E(x] - E[\bar{x}]$$

$$= \bar{x} - \bar{x} = 0$$

$$\mu_2 = E[(x-\bar{x})^2] = E[x^2 - 2x\bar{x} + (\bar{x})^2]$$

$$= E[x^2] - 2\bar{x}E[x] + (\bar{x})^2$$

$$= E[x^2] - 2(\bar{x})^2 + (\bar{x})^2$$

$$= E[x^2] - (\bar{x})^2$$

$$= m_2 - m_1^2$$

VARIANCE

The 2nd central movement is called Variance.

(or)

The diff. b/w 2nd order movement & mean square.

It is denoted by σ_x^2.

$$\sigma_x^2 = \mu_2 = E[(x-\bar{x})^2] = m_2 - m_1^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_x(x)\, dx$$

(or)

$$\sigma_x^2 = E[x^2] - (\bar{x})^2 = m_2 - m_1^2$$

SKEW

The 3rd central movement $\mu_3 = E[(x-\bar{x})^3]$ is a measure of the symmetry of $f_x(x)$ about $\bar{x}$. It will be called skew of the density funcn.

$$\mu_3 = E[(x-\bar{x})^3] = \int_{-\infty}^{\infty} (x-\bar{x})^3 f_x(x)\, dx$$

COEFFICIENT OF SKEWNESS | SKEWNESS OF DENSITY FUNCTION

The normalised 3rd central movement μ_3 / σ_x^3 is know as the skewness of density funcn (or) Coeff. of skewness

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* **STANDARD DEVIATION**

The +ve sq. root of variance of a Random variable 'X' is known as standard deviation. It is denoted by σ_x

$$\sigma_x = \sqrt{var(x)}$$

$$= \sqrt{\sigma_x^2}$$

* **Variance & Their properties**

VARIANCE

$$\mu_2 = \sigma_x^2 = Var(x) = E\left[(x-\bar{x})^2\right] = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_x(x)dx$$

$$= E[x]^2 - E[\bar{x}]^2 = E[x^2] - (\bar{x})^2 = m_2 - m_1^2$$

PROPERTIES

1. The variance of the const. is '0'

$E[K^2] + E[\bar{K}^2] - 2\bar{K}E[K]$

$$\sigma_K^2 = E\left[(K-\bar{K})^2\right] = E\left[(K-K)^2\right] = 0$$

2. The variance of the random variable ax is equal to the a^2 times the variance of x

$$\sigma_{ax}^2 = a^2\sigma_x^2$$

$$\sigma_{ax}^2 = E\left[(ax - \bar{ax})^2\right] = E\left[(ax - a\bar{x})^2\right] = E\left[a^2(x-\bar{x})^2\right] = a^2E\left[(x-\bar{x})^2\right]$$

$$= a^2\sigma_x^2$$

3. The variance of the RV x+b is equal to the variance of x

$$\sigma_{x+b}^2 = \sigma_x^2$$

$$\sigma_{x+b}^2 = E\left[(x+b - \overline{x+b})^2\right] = E\left[(x+b-\bar{x}-b)^2\right]$$

$$= E\left[(x-\bar{x})^2\right] = \sigma_x^2$$

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4. The variance of the RV $ax+b$ is equal to a^2 times of the variance of X

$$\sigma_{ax+b}^2 = E\left[\left(ax+b - \overline{(ax+b)}\right)^2\right]$$

$$= E\left[\left(ax+b - a\bar{x} - \cancel{b}\right)^2\right]$$

$$= E\left[a^2 (x-\bar{x})^2\right]$$

$$= a^2 \sigma_x^2$$

$$\boxed{\sigma_{ax+b}^2 = a^2 \sigma_x^2}$$

5. If two R Variables X & Y, variance of $X \pm Y$ is equal to $\sigma_x^2 + \sigma_y^2 \pm 2\underbrace{\{E[XY] - E[X]E[Y]\}}_{\text{co-variance}}$

$$\sigma_{x\pm y}^2 = E\left[\left(x \pm y - \overline{x \pm y}\right)^2\right] = E\left[(x \pm y)^2 + (\overline{x \pm y})^2 - 2(x \pm y)(\overline{x \pm y})\right]$$

$$= E\left[x^2 + y^2 \pm 2xy + \bar{x}^2 + \bar{y}^2 \pm 2\bar{x}\bar{y} - 2(x \pm y)(\bar{x} \pm \bar{y})\right]$$

$$= E\left[x^2 + y^2 \pm 2xy + \bar{x}^2 + \bar{y}^2 \pm 2\bar{x}\bar{y} - 2x\bar{x} \mp 2x\bar{y} \mp 2y\bar{x} - 2y\bar{y}\right]$$

$$= E\left[x^2 + y^2 \pm 2xy + \bar{x}^2 + \bar{y}^2 \pm 2\bar{x}\bar{y} - \underbrace{2x\bar{x}}_{\bar{x}^2} \ldots\right]$$

$$= \underbrace{E[x^2] + E(y^2)}\ \pm 2E[xy] - \bar{x}^2 - \bar{y}^2 \mp 2\bar{x}\bar{y}$$

$$= \sigma_x^2 + \sigma_y^2 \pm 2\{E[xy] - E[x]E[y]\}$$

6. If 2 RV's X & Y are statistically independent then,

$$\sigma_{x\pm y}^2 = \sigma_x^2 + \sigma_y^2$$

$$\sigma_{x\pm y}^2 = \sigma_x^2 + \sigma_y^2 \pm 2\{E[xy] - E[x]E[y]\}$$

X & Y are statistically independent, then.

$$E[xy] = E[x]E[y] = \bar{x}\bar{y}$$

$$\therefore \sigma_{x\pm y}^2 = \sigma_x^2 + \sigma_y^2 \pm 2\{\bar{x}\bar{y} - \bar{x}\bar{y}\}$$

$$\sigma_{x\pm y}^2 = \sigma_x^2 + \sigma_y^2$$

* DISTRIBUTION FUNCTIONS

1. Binomial Distribution

Consider a random experiment that has only 2 possible outcomes. for ex, in the coin tossing experiment, success corresponds to the falling of the head & failure corresponds to the falling of the tail. Assume that these outcomes have probabilities p & q respectively such that $p+q=1$. If the exp. is repeated 'n' times independently with 2 possible outcomes, they are called "Bernoulli Trails".

The probability of r heads & '$n-r$' tails in 'n' independent trails in a specific order (HTHHT-----) is given by the compound probability.

$$P(HTHHT---) = \underbrace{P(H)P(H)P(H)\cdots}_{r\ times} \underbrace{P(T)P(T)P(T)----}_{n-r\ times}$$

$$= p^r q^{n-r}$$

→ One way to obtain head exactly r times & tail exactly '$n-r$' times would be to have the sequence where for the 'r' times head & for remaining '$n-r$' times, the tail. The no. of such sequences out of 'n' independent trails is given by

$$\binom{n}{r} \text{ or } ({}^nC_r) = \frac{n!}{r!(n-r)!}$$

∴ The probability of 'r' success in 'n' trails in any order is given by,

$$P(x=r) = {}^nC_r\, p^r q^{n-r}, \quad r = 0, 1, 2, ----n$$

This eqn is called Binomial distribution of order n & parameter 'p'

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The Binomial density funcn is given by,

$$f_X(x) = \sum_{r=0}^{n} {}^nC_r p^r q^{n-r} \delta(x-r)$$

The Binomial distribution funcn is given by,

$$F_X(x) = \sum_{r=0}^{n} {}^nC_r p^r q^{n-r} u(x-r)$$

→ Let us calculate some of the statistical quantities of binomial distribution

1. Mean:

$$\text{Mean} = E[X] = \sum_{i=0}^{n} x_i P(x_i)$$

$$E[X=r] = \sum_{r=0}^{n} r P(X=r)$$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r p^r q^{n-r}$$

Since first term of summation is zero for $r=0$,

$$E[X=r] = \sum_{r=1}^{n} r \cdot \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

$$= \sum_{r=1}^{n} \frac{n!}{(r-1)!(n-r)!} p^r q^{n-r}$$

$$= \sum_{r=1}^{n} \frac{(n)(n-1)!}{(r-1)!(n-1-(r-1))!} p\, p^{r-1} q^{n-1-(r-1)}$$

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!(n-1-(r-1))!} p^{r-1} q^{n-1-(r-1)}$$

$$(p+q)^n = \sum_{r=0}^{n} {}^nC_r p^r q^{n-r}$$

$$= np(p+q)^{n-1}$$

$$m_1 = np = \mu$$

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2nd order moment

$$m_2 = E[r^2] = \sum_{r=0}^{n} r^2 \cdot p(r)$$

$$= \sum_{r=0}^{n} r^2 \cdot {}^nC_r p^r q^{n-r}$$

$$= \sum_{r=1}^{n} r^2 \, {}^nC_r \, p^r q^{n-r}$$

$$= \sum_{r=1}^{n} \left((r)(r-1) + r \right) {}^nC_r \, p^r q^{n-r}$$

$$= \sum_{r=1}^{n} (r)(r-1) \, . {}^nC_r \, p^r q^{n-r} + \underbrace{\sum_{r=1}^{n} r \, {}^nC_r \, p^r q^{n-r}}_{np}$$

$$= \sum_{r=2}^{n} (r)(r-1) \frac{n!}{r!\,(n-r)!} p^r q^{n-r} + np$$

$$= \sum_{r=2}^{n} \frac{\cancel{(r)}\,\cancel{(r-1)}\,(n)(n-1)(n-2)!}{\cancel{(r)}\,\cancel{(r-1)}\,(r-2)!\,((n-2)-(r-2))!} p^2 . p^{r-2} q^{(n-2)-(r-2)} + np$$

$$= (n)(n-1)\,p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!\,((n-2)-(r-2))!} p^{r-2} q^{(n-2)-(r-2)} + np$$

$$= (n^2-n)\,p^2 \underbrace{(p+q)^{n-2}}_{(1)^{n-2}} + np$$

$$\boxed{m_2 = n^2p^2 - np^2 + np}$$

Variance $(\sigma^2) = m_2 - m_1^2$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

$$\boxed{\sigma^2 = npq}$$

* Standard deviation

$\sigma = \sqrt{\text{Variance}}$

$= \sqrt{npq}$

Q. When a fair coin is tossed 200 times, Determine the mean variance & standard deviation

Given, $n = 200$ $p = 1/2$ $q = 1/2$

mean $= np = 200 \times \frac{1}{2} = 100$

Variance $= npq = 200 \times \frac{1}{2} \times \frac{1}{2} = 50$

Standard deviation $= \sqrt{npq} = \sqrt{50} = 7.071$

8 find the probability in tossing a fair coin 5 times, there will appear

a. 3 heads

b. 3 tails & 2 heads

c. at least one head

d. not more than one tail.

a. $r = 3$

$$P(3heads) = {}^5C_3 (1/2)^3 (1/2)^{5-3} \Rightarrow {}^5C_3 \left(\frac{1}{2}\right)^5$$

$$\Rightarrow \frac{5!}{2!\,3!} \times \frac{1}{2^5} = \frac{\cancel{120}}{\cancel{32} \times \cancel{6} \times 2}$$

$$= \frac{5}{16}$$

b. $P(3tails \; \& \; 2heads)$

$r = 2$ ($\because$ success is heads)

$${}^5C_2 \left(\frac{1}{2}\right)^2 (1/2)^3 \Rightarrow \frac{5!}{2!\,3!} \times \left(\frac{1}{2}\right)^5 = \frac{5}{16}$$

c. $P(\text{at least one head})$

$$P(r \geq 1) = \sum_{r=1}^{5} {}^5C_r \, p^r q^{n-r} = 1 - p(r<1)$$

$$= 1 - {}^5C_0 (1/2)^0 (1/2)^5$$

$$= 1 - \frac{1}{32} = \frac{31}{32}$$

$$= 0.96875$$

d. $P(1 \text{ tail } \& \; 0 \text{ tail})$

$$P(5heads \; \& \; 4heads) = {}^5C_4 \left(\frac{1}{2}\right)^5 + {}^5C_5 \left(\frac{1}{2}\right)^5$$

$$= \left(\frac{1}{2}\right)^5 (5 + 1)$$

$$= \frac{6}{32} = \frac{3}{16} = 0.1875.$$

Applications

The binomial distribution can be applied to many games of chance, many experiments having only 2 possible outcomes in any give trail, detection problems in RADAR & SONAR

* Poisson Distribution.

$$\lim_{n\to\infty} P(X=r) = \frac{e^{-\mu}\mu^r}{r!}, \quad r = 0,1,2,\ ---- \infty$$

Mean $= \mu = np$

Probability of success $\Rightarrow p = \frac{\mu}{n}$

Probability of failure $\Rightarrow q = 1-p = 1-\frac{\mu}{n}$

Proof :-

$$P(X=r) = {}^nC_r\, p^r q^{n-r}$$

$$= \frac{n!}{r!(n-r)!}\left(\frac{\mu}{n}\right)^r\left(1-\frac{\mu}{n}\right)^r$$

$$= \frac{n(n-1)(n-2)---(n-r+1)\cancel{(n-r)!}}{r!\cancel{(n-r)!}} \times \frac{\mu^2}{n^r}\,\frac{(1-\mu/n)^n}{(1-\mu/n)^r}$$

$$= \frac{\mu^r}{r!}\,\frac{n(n-1)(n-2)---(n-r+1)}{n^r}\,\frac{(1-\mu/n)^n}{(1-\mu/n)^r}$$

$$= \frac{\mu^r}{r!}\,\frac{\cancel{n^r}.1.(1-1/n)(1-2/n)--(1-\frac{(r+1)}{n})}{\cancel{n^r}}\,\frac{(1-\mu/n)^n}{(1-\mu/n)^r}$$

$$P(X=r) = \frac{\mu^r}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)---\left(1-\frac{(r+1)}{n}\right)\frac{(1-\mu/n)^n}{(1-\mu/n)^r}$$

Taking $\lim_{n\to\infty}$ on both sides, we know that,

$$\lim_{x\to\infty}(1+1/x)^x = e^1, \quad \lim_{n\to\infty}(1-\mu/n)^n = e^{-\mu}$$

$$\lim_{n\to\infty}(1-\mu/n)^r = 1$$

$$\lim_{n\to\infty} P(X=r) = \frac{\mu^r}{r!}.1.1.1.---.1.\frac{e^{-\mu}}{1}$$

$$= \frac{\mu^r . e^{-\mu}}{r!}, \quad r = 0,1,2 -$$

Poisson density function is given by

$$f_X(x) = \sum_{r=0}^{\infty} \frac{e^{-\mu} \cdot \mu^{r}}{r!} \delta(x-r)$$

Poisson distribution function is given by

$$F_X(x) = \sum_{x=0}^{\infty} \frac{e^{-\mu} \mu^{r}}{r!} u(x-x)$$

Statistical parameters.

1. Mean.

$$m_1 = E[x=r] = \sum_{r=0}^{\infty} r \cdot P(r)$$

$$= \sum_{r=0}^{\infty} r \cdot \frac{e^{-\mu} \cdot \mu^{r}}{r!}$$

$$= \sum_{r=1}^{\infty} \frac{r \cdot e^{-\mu} \cdot \mu^{r}}{(r)(r-1)!} = \sum_{r=1}^{\infty} \frac{e^{-\mu} \cdot \mu^{x-1} \cdot \mu}{(r-1)!}$$

$$= \mu \cdot \sum_{r=1}^{\infty} \frac{e^{-\mu} \cdot \mu^{x-1}}{(x-1)!}$$

$$= \mu e^{-\mu} \left(1 + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \cdots\right)$$

$$= \mu e^{-\mu} \cdot e^{x} = \mu.$$

$$\boxed{\text{mean} = \mu}$$

2nd order moment,

$$m_2 = E[r^2] = \sum_{r=0}^{\infty} r^2 \cdot p(r)$$

$$= \sum_{r=0}^{\infty} r^2 \cdot \frac{e^{-\mu} \mu^{r}}{r!}$$

$$= \sum_{r=0}^{\infty} [r(r-1)+r] \cdot \frac{e^{-\mu} \mu^{r}}{r!}$$

$$= \sum_{r=1}^{\infty} (r)(r-1) \frac{e^{-\mu} \mu^{r}}{r!} + \sum_{r=1}^{\infty} \frac{e^{-\mu} \mu^{r}}{r!} \cdot r$$

$$= \sum_{r=2}^{\infty} (r)(r-1) \frac{e^{-\mu} \cdot \mu^{r}}{r!} + \mu$$

$$\sum_{r=2}^{\infty} \frac{(r)(r-1) \cdot e^{-\mu} \cdot \mu^{r-2} \cdot \mu^{2}}{(r)(r-1)(r-2)!} + \mu$$

$$\mu^2 e^{-\mu} \sum_{r=0}^{\infty} \frac{\mu^{r-2}}{(r-2)!} + \mu \quad = \mu^2 \cdot \cancel{e^{-\mu}} \cancel{e^{\mu}} + \mu$$

$$\boxed{m_2 = \mu^2 + \mu}$$

$$\text{Variance}(\sigma^2) = m_2 - m_1^2$$

$$= \mu^2 + \mu - \mu^2$$

$$\boxed{\sigma^2 = \mu}$$

Standard deviation $= \sqrt{\text{variance}} = \sqrt{\mu}$ //

$$\boxed{\sigma = \sqrt{\mu}}$$

Applications

It is mostly applied to counting type problems

1. No. of telephone calls made during a period of time
2. The no. of defective elements in a given sample.
3. The no. of e^{-}'s emitted from a cathode in a given time interval.
4. The no. of items waiting in a queue.

Q. If a random variable has a poisson distribution such that $p(1) = P(2)$, find $p(4)$

$$\lim_{n\to\infty} P(r) = \frac{e^{-\mu} \cdot \mu^{r}}{r!}$$

$$\frac{\cancel{e^{-\mu}} \cdot \mu^{1}}{1!} = \frac{\cancel{e^{-\mu}} \cdot \mu^{2}}{2!}$$

$$\mu = 2! = 2$$

$$P(4) = \frac{e^{-2} \cdot 2^{4}}{4!} = \frac{e^{-2} \times \cancel{16}\, \cancel{4}\, 2}{\cancel{4} \times \cancel{6}\, 3} = \frac{2}{3} e^{-2}$$

$$= 0.09022$$

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Q. 20% of the bolts produced in a factory are found to be defective. Find the probability that in a sample of 10 bolts choosen at random, exactly 2 will be defective by using

a. Binomial distribution

b. Poisson's distribution

a. $n = 10$, $r = 2$, $P = 0.2\left(\because \frac{20}{100}\right)$ $q = 1 - P = 0.8$

$$P(x=r) = {}^{n}C_{r}\, p^{r} q^{n-r}$$

$(r=2)$

$$= {}^{10}C_{2} (0.2)^{2} (0.8)^{8}$$

$$= \frac{10!}{8!\,2!} \times (0.2)^{2} \times (0.8)^{8}$$

$$= 45 \times (0.2)^{2} \times (0.8)^{8} = 0.3019$$

b. $P(x=r) = \dfrac{e^{-\mu}\mu^{r}}{r!}$ where, $\mu = np = 10 \times 0.2 = 2$.

$$P(x=2) = \frac{e^{-2} \cdot 2^{2}}{2!} = e^{-2} \times 2 = 0.2706$$

Q. If the mean & variance of the binomial distribution are 6 & 1.5 respectively. find the $E[x - P(x \geq 3)]$

Given, $\text{mean}(\mu) = np = 6$

$$\sigma^{2} = \mu = 1.5 = npq.$$

$$\therefore (np)q = 1.5$$

$$6.q = 1.5 \Rightarrow q = \frac{\cancel{1.5}^{0.5}}{\cancel{6}_{2}} = 0.25$$

$$n = 6/p = \frac{6}{1-q} = \frac{6}{0.75} = \frac{\cancel{600}^{24}}{\cancel{75}_{3}} = 8$$

$\therefore n = 8$, $P = 0.75$, $q = 0.25$

$$P(x \geq 3) = 1 - P(x < 3)$$

$$= 1 - \{P(0) + P(1) + P(2)\}$$

$$= 1 - \{{}^{8}C_{0}(0.75)^{0}(0.25)^{8} + {}^{8}C_{1}(0.75)^{1}(0.25)^{7} + {}^{8}C_{2}(0.75)^{2}(0.25)^{6}\}$$

$= 1 - 0.0004$

$= 0.995$

$E(x - P(x \geq 3)] = E(x) - E[P(x \geq 3)]$

$= np - E(0.995)$

$= 6 - 0.995$

$= 5.005$

UNIFORM DISTRIBUTION FUNCTION.

def A continuous random variable X is said to have a uniform distribution over an interval (a, b), if it's density funcn $f_X(x)$ is a constant 'c' over the entire range of x i.e,

$$f_X(x) = \begin{cases} c, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

We know that the total probability is always 1. ie, the area under the density curve unity.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\int_a^b c \cdot dx = 1$$

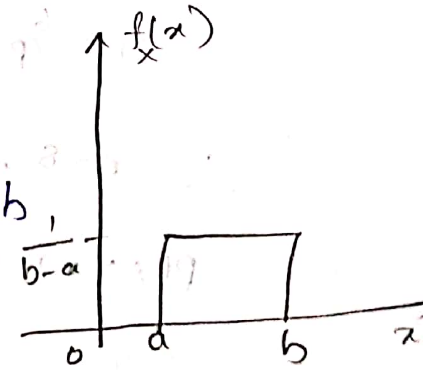
$$cx \Big|_a^b = 1$$

$$\Rightarrow c(b-a) = 1$$

$$\boxed{c = \frac{1}{b-a}}$$

∴ The uniform density funcn,

$$\therefore f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise} \end{cases}$$



The corresponding uniform distribution funcn is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx \quad (\text{or}) \quad F_X(x) = P(X \le x)$$

$$= \int_{a}^{x} \frac{1}{b-a}\,dx$$

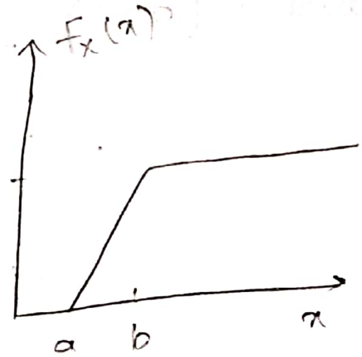
$$= \frac{1}{b-a} \Big|_{a}^{x}$$

If $x < a$, distribution is zero.

$$= \frac{1}{b-a}(x-a)$$

$$= \frac{x-a}{b-a}, \quad a \le x \le b$$

$$\therefore F_X(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1 & x > b \end{cases}$$



Statistical Parameters

1. Mean

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$= \int_{a}^{b} x \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{x^2}{2}\Big|_{a}^{b}\right)$$

$$= \frac{1}{b-a}\frac{(b^2-a^2)}{2}$$

$$= \frac{b+a}{2}$$

Mean $m_1 = E[X] = \dfrac{a+b}{2}$

2. Second order moment:

$$m_2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\,dx \Rightarrow \frac{1}{b-a} \times \frac{x^3}{3}\Big|_a^b$$

$$= \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right)$$

$$= \frac{1}{b-a}\left(\frac{(b-a)(a^2+b^2+ab)}{3}\right)$$

$$= \frac{a^2+b^2+ab}{3}$$

3. Variance (σ_x^2)

$$= m_2 - m_1^2$$

$$= \frac{a^2+b^2+ab}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{a^2+b^2+ab}{3} - \left(\frac{a^2+b^2+2ab}{4}\right)$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$$

APPLICATIONS

1. The random distribution of errors introduced in the round of process are uniformly distributed.
2. In digital communications, when a sample of the signal is rounded off to its nearest level (or) in a same, when a real no. is converted into an integer
3. The probability density funcn of errors are uniformly distributed.

Exponential Distribution Function

A continuous random variable X having the range $0 < x < \infty$ is said to have an exponential distribution if it has a probability density of the form,

$$f_X(x) = \begin{cases} a e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

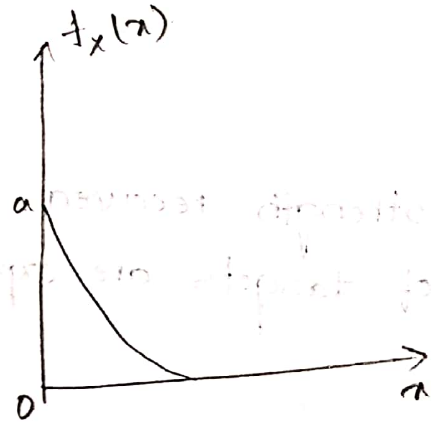
The corresponding distribution funcn is given by,

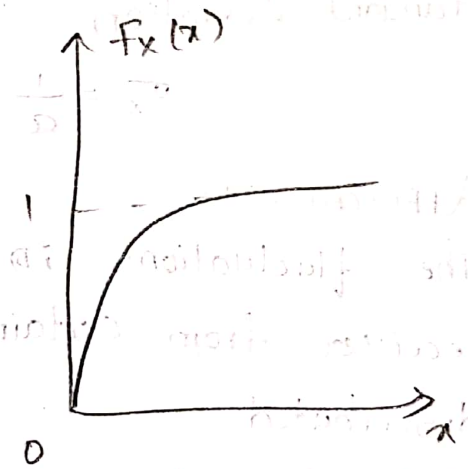
$$F_X(x) = \int \text{density}$$

$$F_X(x) = \int_{-\infty}^{\infty} f_X(x)\, dx$$

$$F_X(x) = \int_0^x a e^{-ax} dx = a \frac{e^{-ax}}{-a} \Big|_0^x$$

$$= \frac{e^{-ax} - e^{-0}}{-1} \Rightarrow 1 - e^{-ax}, \quad x \geq 0$$





Statistical Parameters

1. Mean

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_0^{\infty} x \cdot a e^{-ax} dx \Rightarrow a \int_0^{\infty} \underset{u}{x} \cdot \underset{v}{e^{-ax}} dx$$

$$\Rightarrow a \left\{ \frac{x e^{-ax}}{-a} \Big|_0^{\infty} - \frac{e^{-ax}}{a^2} \Big|_0^{\infty} \right\}$$

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$$a\left[\frac{0-0}{a} - \frac{(0-1)}{a^2}\right] = a\cdot\frac{1}{a^2} = \frac{1}{a}$$

2. Second order moment

$$m_2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_{0}^{\infty} x^2 . a . e^{-ax}\, dx$$

$$= a\int_{0}^{\infty} x^2 e^{-ax}\, dx = \frac{2}{a^2}$$

3. VARIANCE (σ_X^2)

$$\sigma_X^2 = m_2 - m_1^2$$

$$= \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$$

4. Standard deviation

$$\sigma_X = \frac{1}{a}$$

APPLICATIONS

1. The fluctuations in signal strength received by radar receiver from certain type of targets are exponentially distributed.
2. Rain drop sizes when a large no. of rain storm measurements are made.

4/1/20 RAYLEIGH DISTRIBUTION

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)\, e^{-(x-a)^2/b} & , x \geq a \\ 0 & , x < a \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$= \int_a^x \frac{2}{b}(x-a)\, e^{-(x-a)^2/b}\, dx$$

Let $\frac{(x-a)^2}{b} = t \Rightarrow \frac{2(x-a)\,dx}{b} = dt \Rightarrow dx = \frac{b}{2(x-a)} dt$

At $x = a \Rightarrow t_l = 0$

$x = x \Rightarrow t_u = (x-a)^2/b = t$

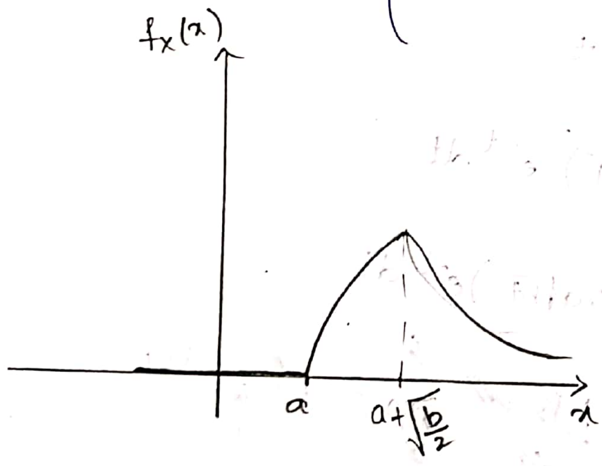
$$\therefore F_X(x) = \int_0^t \frac{2}{b}(x-a)\, e^{-t} \cdot \frac{dt\, b}{2(x-a)}$$

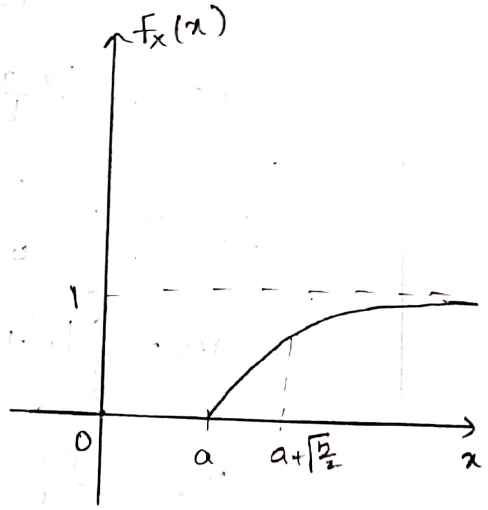
$$F_X(x) = \int_0^t e^{-t} dt \Rightarrow \frac{e^{-t}}{-1}\Big|_0^t = 1 - e^{-t}$$

$$= 1 - e^{-\left(\frac{(x-a)^2}{b}\right)} \quad x \geq a$$

$$0 \quad , \quad x < a$$

$$\therefore F_X(x) = \begin{cases} 1 - e^{-(x-a)^2/b} & x \geq a \\ 0 & , x < a \end{cases}$$





STATISTICAL PARAMETERS

1. Mean

$$E[x] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_a^{\infty} x \cdot \frac{2}{b}(x-a)\, e^{-(x-a)^2/b}\, dx$$

Let $(x-a)^2/b = t \Rightarrow \frac{2(x-a)dx}{b} = dt$

$\Rightarrow x = \sqrt{bt} + a$

$\therefore$ At $x = a, \; t_l = 0$

$x = \infty, \; t_u = \infty$

$$E[x] = \int_0^\infty (a+\sqrt{bt})\, e^{-t}\, dt$$

$$= \int_0^\infty a e^{-t} dt + \int_0^\infty \sqrt{bt}\, e^{-t} dt$$

2 $$= \frac{a \cdot e^{-t}}{-1}\Big|_0^\infty + \sqrt{b}\int_0^\infty t^{1/2} e^{-t}\, dt \quad \left(\because \Gamma n = \int_0^\infty t^{n-1} e^{-t} dt\right)$$

$$= a + \sqrt{b}\left\{\Gamma 3/2\right\}$$

$$= a + \sqrt{b}\left(\tfrac{1}{2}\Gamma \tfrac{1}{2}\right)$$

$$= a + \frac{\sqrt{\pi b}}{2} \quad = a + \sqrt{\frac{b\pi}{4}}$$

2. Second order moment (m_2)

$$m_2 = E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_a^\infty x^2 \cdot \frac{2}{b}(x-a)\, e^{-(x-a)^2/b}\, dx$$

Let, $\frac{(x-a)^2}{b} = t$

$$\therefore m_2 = \int_0^\infty (a+\sqrt{bt})^2 e^{-t}\, dt$$

$$m_2 = \int_0^\infty (a^2 + bt + 2a\sqrt{bt})e^{-t}\, dt$$

$$\cancel{m_2 = \frac{a^2 e^{-t}}{-1}\Big|_0^\infty + \frac{b\Gamma 2\, e^{-t}}{-1}\Big|_0^\infty + \frac{b\Gamma 3/2\, e^{-t}}{-1}}$$

$$m_2 = \frac{a^2 e^{-t}}{-1}\Big|_0^\infty + b\Gamma 2 + b\Gamma 3/2\,(2a)$$

$$= a^2 + b + \frac{\sqrt{\pi b}}{2}(2a)$$

$$= a^2 + b + a\sqrt{\pi b}$$

3. Variance (σ_x^2)

$$\sigma_x^2 = m_2 - m_1^2$$

$$= a^2 + b + a\sqrt{\pi b} - \left(a + \sqrt{\frac{b\pi}{4}}\right)^2$$

$$= a^2 + b + a\sqrt{\pi b} - a^2 - \frac{b\pi}{4} - 2a\sqrt{\frac{b\pi}{4}}$$

$$= b\left(1 - \frac{\pi}{4}\right)$$

$$\therefore \boxed{\sigma_x^2 = b\left(\frac{4-\pi}{4}\right)}$$

APPLICATIONS

1. It describes the envelope of White noise, When the noise is passed through a BPF
2. The relay density funcn has a relationship with the Gaussian density funcn.
3. Some types of signal fluctuations received by the receiver are modelled as relay's distribution

GAUSSIAN DISTRIBUTION (or) NORMAL DISTRIBUTION.

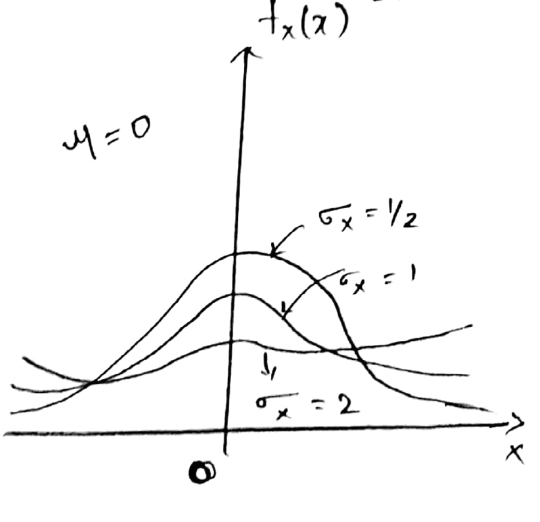
10/1/20 A Continuous R.V. X is said to have Normal / Gaussian distribution with parameters mean μ & variance (σ_x^2), It's density funcn is given by.

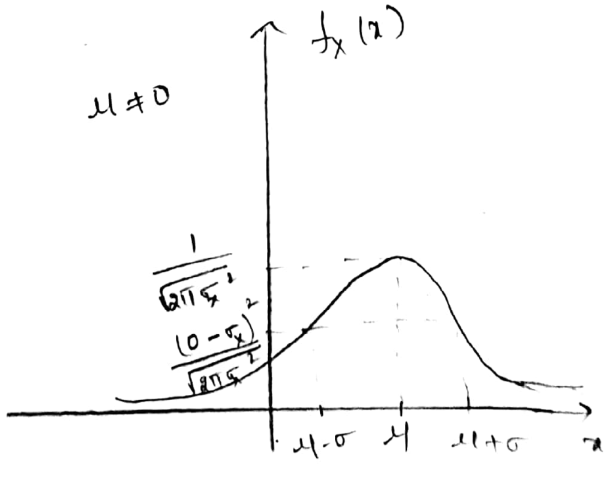
$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2} \quad \text{—— (1)}$$

and the corresponding Gaussian distribution funcn is given by $F_x(x)$

$$F_x(x) = \int_{-\infty}^{x} f_x(x)\, dx$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2} dx \quad \text{—— (2)}$$





→ If the Gaussian distribution funcn has $\mu = 0$ & $\sigma_X = 1$, then it is called a normalized Gaussian distribution funcn. It is denoted as $F(x)$ and is given by.

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \quad x \geq 0 \qquad (3)$$

$$F(-x) = 1 - F(x), \quad x < 0 \qquad (4)$$

Relationship between $F_X(x)$ & $F(x)$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_X^2}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_X}\right)^2} dx$$

Let $u = \frac{x-\mu}{\sigma_X} \Rightarrow du = \frac{dx}{\sigma_X} \Rightarrow dx = \sigma_X du$

$x = -\infty \Rightarrow u = -\infty$

$x = x \Rightarrow u = \frac{x-\mu}{\sigma_X}$

$$F_X(x) = \int_{-\infty}^{\frac{x-\mu}{\sigma_X}} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{u^2}{2}} \sigma_X du$$

$$= \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-u^2/2} du \qquad (5)$$

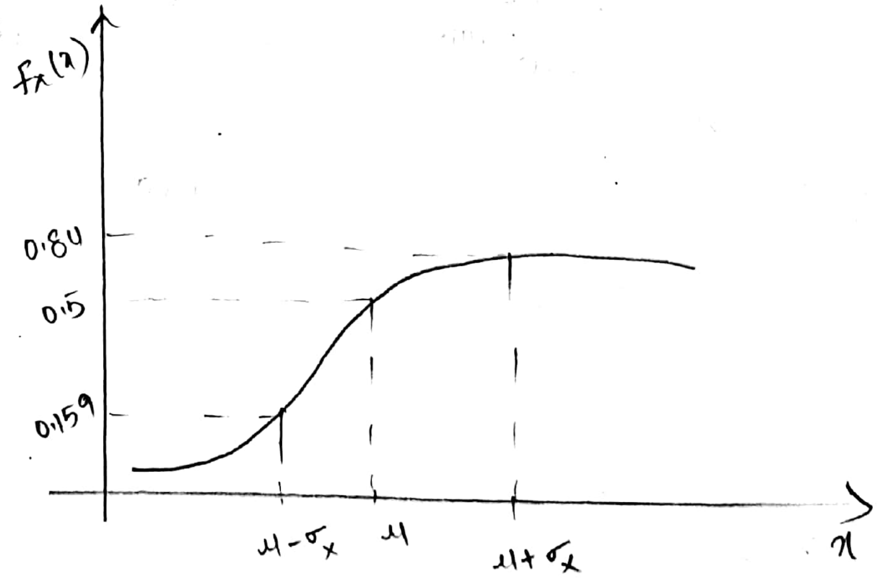
Compare eq (5) + eq (3)

$$\boxed{F_X(x) = F(u) = F\left(\frac{x-\mu}{\sigma_X}\right)} \qquad (6)$$

Peebles TB

↑

$F(0) = 0.5$



APPLICATIONS

1. The Gaussian probability density funcn is most important density funcn among all the density funcns in field of science and engineering.

2. It gives accurate description of many practical random quantities. Especially in electronics & communication systems, the distribution of the noise signal, either internally generated or externally added exactly matches the Gaussian Probability density funcn.

3. The noise signal is an unwanted quantity which gets added to the information signal at each & every point in signal processing. It is possible to eliminate noise completely by knowing its behaviour using the Gaussian density funcn.

Q. Find the probability of the event $\{x \le 5.5\}$ for a Gaussian R.V having $\mu = 3$ & $\sigma_x = 2.5$. Given $F(1) = 0.8413$

$$F_x(x) = P(x \le x)$$

$$F_x(5.5) = P(x \le 5.5)$$

$$= F\left(\frac{x-\mu}{\sigma_x}\right)$$

$$= F\left(\frac{5.5-3}{2.5}\right) = F\left(\frac{2.5}{2.5}\right) = F(1) = 0.8413$$

Q. Assume that the height of clouds above the ground at some location is a Gaussian R.V. X with $\mu = 1830m$ & $\sigma_x = 460m$, find the probability that clouds will be higher than 2750 m (i.e $x > 2750m$). Given that, $F(2) = 0.9772$

$$F(x) = 1 - F(x)$$

$$F_x(x) = P(x \le x) = 1 - P(x > x)$$

$$\Rightarrow F(x > 2750) = 1 - P(x \le 2750)$$

$$= 1 - F_x(2750)$$

$$= 1 - F\left(\frac{2750-\mu}{\sigma_x}\right) = 1 - F(2)$$

$$= 0.0228$$

Q. Let x be a R.V. has a probability density func^n

$$f_X(x) = \begin{cases} \frac{\pi}{16}\cos\left(\frac{\pi x}{8}\right), & -4 \le x \le 4 \\ 0, & \text{elsewhere.} \end{cases}$$ find

(i) $E[X]$ (ii) $E[x^2]$ (iii) σ_X^2

S:

(i) $E[x] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$

$$= \int_{-4}^{4} x \cdot \frac{\pi}{16}\cos\left(\frac{\pi x}{8}\right) dx = \frac{\pi}{16}\int_{-4}^{4} \underset{u}{x} \underset{v}{\cos\left(\frac{\pi x}{8}\right)} dx$$

$$= \frac{\pi}{16}\left\{ \frac{x\sin\left(\frac{\pi x}{8}\right)}{\pi/8}\Bigg|_{-4}^{4} + \frac{\cos(\pi x/8)}{(\pi/8)^2}\Bigg|_{-4}^{4} \right\}$$

$$= \frac{\pi}{16}\left\{ \frac{4}{\pi/8} + \left(\frac{-4}{\pi/8}\right) + \frac{0}{(\pi/8)^2} + \frac{0}{(\pi/8)^2} \right\}$$

$$= \frac{\cancel{\pi}}{16}\left\{0\,\cancel{\frac{2\times 8^2}{\pi}}\right\} \Rightarrow \cancel{\frac{8}{\pi}}\quad 0 /\!/ .$$

(ii) $E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$

$$= \int_{-4}^{4} x^2 \cdot \frac{\pi}{16}\cos\left(\frac{\pi x}{8}\right) dx$$

$$= \frac{\pi}{16}\times 2 \int_{0}^{4} \underset{u}{x^2}\underset{v}{\cos\left(\frac{\pi}{8}x\right)} dx$$

$$= \frac{2\pi}{16}\left\{ \frac{x^2\sin\left(\frac{\pi}{8}x\right)}{\pi/8}\Bigg|_{0}^{4} - \int_{0}^{4} \frac{2x \sin\left(\left(\frac{\pi}{8}\right)x\right)}{\pi/8}\, dx \right\}$$

$$= \frac{\pi}{8}\left\{ \frac{16\times 8}{\pi} - \frac{16}{\pi}\left\{ \frac{-x\cos\left(\frac{\pi}{8}x\right)}{\pi/8}\Bigg|_{0}^{4} + \frac{\sin\left(\frac{\pi}{8}x\right)}{(\pi/8)^2}\Bigg|_{0}^{4} \right\}\right\}$$

$$= 16 - \left\{\frac{16}{\pi}\times\frac{8^2}{\pi^2}\right\}\frac{2\pi}{16}$$

$$= 16 - \frac{8^2\times 2}{\pi^2} - 16 - \frac{128}{\pi^2}$$

(iii) $\sigma_x^2 = m_2 - m_1^2$

$= 16 - \frac{128}{\pi^2} - (0)^2 = 16 - \frac{128}{\pi^2}$

Q A continuous RV X having uniform distribution in an interval $(0, 2\pi)$ find the ~~ex~~ $E[3X+2]$

$$f_X(x) = \frac{1}{\text{ULimit} - \text{LLimit}} = \frac{1}{2\pi - 0} = \frac{1}{2\pi}, \quad 0 \le x \le 2\pi$$

$$\therefore E[3X+2] = 3E[X] + 2$$

$$= 3\int_0^{2\pi} x \cdot \frac{1}{2\pi}\, dx + 2$$

$$= 3 \cdot \frac{x^2}{4\pi}\Big|_0^{2\pi} + 2$$

$$= \frac{3}{4\pi}(4\pi^2) + 2 = 3\pi + 2$$

17/1/20

CHARACTERISTIC FUNCTION

The characteristic funcn of a RV X is defined by,

$$\phi_X(\omega) = E[e^{j\omega x}] \quad \text{——— (1)} \qquad -\infty < \omega < \infty$$

where, $j = \sqrt{-1}$, It is the funcn of real variable

$$\phi_X(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x}\, dx$$

$$\phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x}\, dx \quad \text{——— (2)} \qquad \text{- F.T of } f_X(x)$$

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega) e^{-j\omega x}\, dx \quad \text{——— (3)} \qquad \text{- I.F.T of } \phi_X(\omega)$$

By formal differentiation of eq (2) n times w.r.t ω and setting $\omega = 0$ in the derivative, we may show that nth moment of X is given by m_n

$$m_n = (-j)^n \frac{d^n \phi_X(\omega)}{d\omega^n}\Big|_{\omega=0} \quad \text{——— (4)}$$

→ The major advantage of using $\phi_x(\omega)$ to find moments is that $\phi_x(\omega)$ is always exists.

→ It can be shown that the max. magnitude of characteristic funcn is unity & occurs at $\omega = 0$

i.e $|\phi_x(\omega)| \leq \phi_x(0) = 1$

Properties

1. $|\phi_x(\omega)| \leq 1$ i.e max magnitude of characteristic funcn is unity

2. $\phi_x(0) = 1$

3. $\phi_x(-\omega)$ and $\phi_x(\omega)$ are complex funcn's i.e $\overline{\phi_x(\omega)} = \phi_x(-\omega)$

Proof :

$$\phi_x(\omega) = E[e^{j\omega x}] = E[\cos\omega x + j\sin\omega x]$$

$$\overline{\phi_x(\omega)} = E[e^{j\omega x}]^* = E[e^{-j\omega x}] = E[\cos\omega x - j\sin\omega x]$$

$$= E[\cos(-\omega x) + j\sin(-\omega x)]$$

$$= \phi_x(-\omega)$$

4. $\phi_x(-\omega) = \phi_x(\omega)$, If x is even symmetry funcn.

5. $\phi_{ax+b}(\omega) = e^{jb\omega}\phi_x(a\omega)$

Proof :

$$\phi_{ax+b}(\omega) = E[e^{j\omega(ax+b)}] = E[e^{j\omega ax + j\omega b}]$$

$$= E[e^{j\omega ax} \cdot e^{j\omega b}]$$

$$= e^{j\omega b} E[e^{j\omega ax}]$$

$$= e^{j\omega b}\phi_x(a\omega)$$

6) If X & Y are 2 independent RV then,

$$\phi_{X+Y}(\omega) = \phi_X(\omega) \cdot \phi_Y(\omega)$$

Proof:

$$\phi_{X+Y}(\omega) = E\left[e^{j\omega(X+Y)}\right] = E\left[e^{j\omega X} \cdot e^{j\omega Y}\right]$$

$$= E\left[e^{j\omega X}\right] \cdot E\left[e^{j\omega Y}\right]$$

$$= \phi_X(\omega) \cdot \phi_Y(\omega).$$

MOMENT GENERATING FUNCTION

The moment Generating funcn of a RV 'X' about the origin is defined by,

$$M_X(v) = E\left[e^{vx}\right]$$

$$= \int_{-\infty}^{\infty} e^{vx} f_X(x)\, dx \qquad \text{CRV}$$

$$= \sum_{i=1}^{\infty} e^{vx_i} p(x_i) \qquad \text{D.R.V}$$

$$e^{vx} = 1 + \frac{vx}{1!} + \frac{v^2x^2}{2!} + \frac{v^3x^3}{3!} + \cdots + \frac{v^nx^n}{n!} + \cdots$$

$$E\left[e^{vx}\right] = 1 + \frac{vm_1}{1!} + \frac{v^2m_2}{2!} + \frac{v^3m_3}{3!} + \cdots + \frac{v^nm_n}{n!} + \cdots$$

$$m_r = E\left[x^r\right] = \int_{-\infty}^{\infty} x^r f_X(x)\, dx$$

r th order moment of x about origin is the coeffient of $\frac{v^r}{r!}$ in the expansion of $M_X(v)$ in the series of powers of 'v'

In order to find m_r, we have to differentiate the moment generating funcn r times w.r.t 'v' & setting v=0

$$m_r = \left.\frac{d^r M_X(v)}{dv^r}\right|_{v=0}$$

→ The main advantage of the moment generating funcn derives from it's ability to give the moments

→ the main disadvantage of MGF as opposed to the charac-teristic funcn is that it may not exist for all RV & all values of v

PROPERTIES

1. $M_{ax}(v) = M_x(av)$ where, a is const

Proof :

$$M_{ax}(v) = E[e^{vax}]$$
$$= E[e^{av \cdot x}]$$
$$= M_x(av)$$

2. Let $Y = ax+b$ where, x is a RV with moment Generating funcn $M_x(v)$ then, $M_y(v) = e^{bv} \cdot M_x(av)$

Proof :

$$M_y(v) = E[e^{vay}] = E[e^{v(a(x)+b)}]$$
$$= E[e^{vax} e^{vb}]$$
$$= e^{vb} \cdot E[e^{av \cdot x}] = e^{bv} \cdot M_x(av)$$

3. Let $Y = \frac{x+a}{b}$ where, x is a RV with moment Generating funcn $M_x(v)$ then $M_y(v) = e^{a/b} M_x(v/b)$.

Proof :

$$M_y(v) = E\left[e^{v\left(\frac{x+a}{b}\right)}\right] = E\left[e^{\frac{vx}{b}} \cdot e^{a} \cdot e^{-b}\right]$$
$$= E\left[e^{\frac{vx}{b}} \cdot e^{(a/b)}\right]$$
$$= e^{a/b} M_x(v/b)$$

4. If x & y are 2 independent RV's then, $M_{x+y}(v) = M_x(v) \cdot M_y(v)$

Proof :- $M_{x+y}(v) = E[e^{(x+y)v}] = M_x(v) \cdot M_y(v)$

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Q. find the mean value of the RV whose probability density funcn given by $f_X(x) = \frac{3}{5} 10^{-5} x(100-x)$ for, $0 \le x \le 100$

Mean value $E[x] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$$= \int_0^{100} x \frac{3}{5} \times 10^{-5} x(100-x)\, dx$$

$$= \frac{3}{5} 10^{-5} \int_0^{100} (100x - x^2)x\, dx$$

$$= \frac{3}{5} 10^{-5} \left\{ \frac{100x^3}{3}\Big|_0^{100} - \frac{x^4}{4}\Big|_0^{100} \right\} = \cancel{\frac{3}{5} 10^{-5} \left\{ 50x^2\Big|_0^{100} - \frac{x^3}{3}\Big|_0^{100} \right\}}$$

$$= \frac{3}{5} 10^{-5} \left\{ \frac{10^8}{3} - \frac{10^8}{4} \right\} = \frac{3}{5} 10^{-5} \left\{ \cancel{50 \times 10^4 - \frac{10^6}{3}} \right\}$$

$$= \frac{3}{5} \times 10^3 \left\{ \frac{1}{\frac{12}{4}} \right\} \qquad \cancel{= 3 - 2 = 1}$$

$$= 25 \times 2 = 50$$

20/1/20

Q. Given the following table.

x	-3	-2	-1	0	1	2	3
$P(x)$	0.05	0.1	0.3	0	0.3	0.15	0.1

find i) $E[x]$ (ii) $E[2x \pm 3]$ (iii) $E[4x+5]$ (iv) $V[x]$ (v) $V[2x \pm 1]$

(i) $E[x^2] = \sum_{x=-\infty}^{\infty} x^2 p(x)$

$$= \sum_{x=-3}^{3} x^2 p(x) = (-3)^2 p(-3) + 2^2 p(-2) + (-1)^2 p(-1) + 0P(0) + 1P(1) + 2^2 p(2)$$
$$+ 3^2 p(3)$$

$$= 9(0.05) + 4(0.1) + (0.3)(1) + (0)(0) + 1(0.3)$$
$$+ 4(0.15) + 9(0.1)$$

$$= -0.15 - 0.2 - 0.3 + 0 + 0.3 + 0.3 + 0.3$$

$$= 0.25$$

(ii) $E[2x \pm 3] = 2E[x] \pm 3$

$= 2(0.25) \pm 3 = 0.5 \pm 3$

$= 3.5$ (or) -2.5

(iii) $E[4x \pm 5] = 4E[x] + 5$

$= 4(0.25) + 5$

$= 1 + 5 = 6$

(iv) $V[x] = \sigma_x^2 = E[(x-\bar{x})^2] = E[x^2] - (E[x])^2$

$E[x^2] = \sum_{x=-\infty}^{\infty} x^2 p(x) = \sum_{x=-3}^{3} x^2 p(x)$

$= (-3)^2 p(-3) + (-2)^2 p(-2) + (-1)^2 p(-1) + 0p(0) + 1^2 p(1) + 2^2 p(2) + 3^2 p(3)$

$= 9(0.05) + 4(0.1) + 1(0.3) + 0 + 1(0.3) + 4(0.15) + 9(0.1)$

$= 0.45 + 0.4 + 0.3 + 0.3 + 0.6 + 0.9 \Rightarrow E[x^2] = 2.95$

$V[x] = E[x^2] - (E(x))^2 = 2.95 - (0.25)^2 = 2.95 - 0.0625$

$V[x] = 2.8875$

(v) $V(2x \pm 3) = 2^2 V[x]$

$= 4(2.8875)$ $\quad (\because \sigma^2_{ax+b} = a^2 \sigma_x^2)$

$= 11.55$

Q. S.T the distribution funcn for which the characteristic funcn $e^{-|\omega|}$ has the density funcn $f_x(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x \leq \infty$

Given characteristic funcn $\phi_x(\omega) = e^{-|\omega|}$

$$f_x(x) \xleftrightarrow{F.T} \phi_x(\omega)$$

$$f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(\omega) e^{-j\omega x} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\omega|} e^{-j\omega x} d\omega$$

$$= \frac{1}{2\pi} \left\{ \int_{-\infty}^{0} e^{\omega} e^{-j\omega x} d\omega + \int_{0}^{\infty} e^{-\omega} e^{-j\omega x} d\omega \right\}$$

$$= \frac{1}{2\pi} \left\{ \int_{-\infty}^{0} e^{\omega(1-jx)} d\omega + \int_{0}^{\infty} e^{-\omega(1+jx)} d\omega \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{e^{\omega(1-jx)}}{1-jx} \Big]_{-\infty}^{0} + \frac{e^{-\omega(1+jx)}}{-(1+jx)} \Big]_{0}^{\infty} \right\}$$

$$= \frac{1}{2\pi}\left\{\frac{e^{0}-e^{-\infty}}{1-jx} - \frac{(e^{-\infty}-e^{0})}{1+jx}\right\} \quad \left[\phi_x(\omega) = E[e^{j\omega x}] = \int e^{j\omega x} f_x(x)\,dx\right]$$

$$= \frac{1}{2\pi}\left\{\frac{1}{1-jx} + \frac{1}{1+jx}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1+jx+1-jx}{1+x^2}\right\} = \frac{1}{\pi(1+x^2)}$$

Q. The RV 'x' has the characteristic funcn is given by,

$\phi_x(\omega) = \begin{cases} 1-|\omega| & |\omega| \le 1 \\ 0 & \omega > 1 \end{cases}$. find the PDF ($f_x(x)$) of the RV 'x'.

Given CF $\phi_x(\omega) = \begin{cases} 1-|\omega| & |\omega| \le 1 \to -1 \le \omega \le 1 \\ 0 & \omega > 1 \end{cases}$

$$f_x(x) \xleftrightarrow{F.T} \phi_x(\omega)$$

$$f_x(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_x(\omega)\, e^{-j\omega x}\, d\omega$$

$$f_x(x) = \frac{1}{2\pi}\int_{-1}^{1} (1-|\omega|)\, e^{-j\omega x}\, d\omega$$

$$f_x(x) = \frac{1}{2\pi}\left\{\int_{-1}^{0}(1+\omega)\, e^{-j\omega x}\, d\omega + \int_{0}^{1}(1-\omega)\, e^{-j\omega x}\, d\omega\right\}$$

$$f_x(x) = \frac{1}{2\pi}\left\{\int_{-1}^{0} e^{-j\omega x}\, d\omega + \int_{-1}^{0} \omega e^{-j\omega x}\, d\omega + \int_{0}^{1} e^{-j\omega x}\, d\omega - \int_{0}^{1} \omega e^{-j\omega x}\, d\omega\right\}$$

$$f_x(x) = \frac{1}{2\pi}\left\{\left.\frac{e^{-j\omega x}}{-jx}\right|_{-1}^{0} + \left.\frac{\omega e^{-j\omega x}}{-jx}\right|_{-1}^{0} - \left.\frac{e^{-j\omega x}}{(-jx)^2}\right|_{-1}^{0} + \left.\frac{e^{-j\omega x}}{-jx}\right|_{0}^{1} - \left.\frac{\omega e^{-j\omega x}}{-jx}\right|_{0}^{1} + \left.\frac{e^{-j\omega x}}{(-jx)^2}\right|_{0}^{1}\right\}$$

$$f_x(x) = \frac{1}{2\pi}\left\{\cancel{\frac{1}{-jx}} + \cancel{\frac{e^{jx}}{jx}} + \left(\cancel{\frac{-e^{jx}}{jx}}\right) + \frac{1}{x^2} - \frac{e^{jx}}{x^2} - \cancel{\frac{e^{-jx}}{jx}} + \cancel{\frac{1}{jx}} + \cancel{\frac{e^{-jx}}{jx}} - \frac{e^{-jx}}{x^2} + \frac{1}{x^2}\right.$$

$$f_x(x) = \frac{1}{2\pi}\left\{\frac{1}{x^2} + \frac{1}{x^2} + \left(-\frac{e^{jx}}{x^2}\right) + \left(-\frac{e^{-jx}}{x^2}\right)\right\}$$

$$f_x(x) = \frac{1}{2\pi x^2}\left\{2 + \left(-e^{jx} - e^{-jx}\right)\right\}$$

$$f_X(x) = \frac{1}{2\pi x^2}\left\{2-\left(e^{jx}+e^{-jx}\right)\right\}$$

$$f_X(x) = \frac{1}{2\pi x^2}\left\{2-2\left(\frac{e^{jx}+e^{-jx}}{2}\right)\right\}$$

$$f_X(x) = \frac{1}{2\pi x^2}\left\{2-2\cos x\right\}$$

$$f_X(x) = \frac{1}{2\pi x^2}\, 2\left\{1-\cos x\right\}$$

$$f_X(x) = \frac{1-\cos x}{\pi x^2}$$

$$\therefore \text{PDF } (f_X(x)) = \frac{1-\cos x}{\pi x^2}$$

Q. find the characteristic funcn & moment generating funcn of a uniformly distributed RV in the range [0,1] & hence find m_1.

MGF, $M_X(v) = E[e^{vx}] = \int_{-\infty}^{\infty} e^{vx} f_X(x)\,dx = \int_0^1 e^{vx}\cdot 1\,dx = \frac{e^v - e^0}{v} = \frac{e^v-1}{v}$

Cf, $\phi_X(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx = \int_0^1 e^{j\omega x}\cdot 1\,dx$

$$= \left.\frac{e^{j\omega x}}{j\omega}\right|_0^1$$

$$= \frac{e^{j\omega}-1}{j\omega} = \phi_X(\omega)$$

$(0,1) \longrightarrow (a,b)$

U.D.F $\quad f_X(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1$

nth order moment,

MGF $\quad m_n = \left.\frac{d^n}{dv^n} M_X(v)\right|_{v=0} \Rightarrow m_1 = \left.\frac{d}{dv}\left(\frac{e^v-1}{v}\right)\right|_{v=0} \Rightarrow \left.\frac{ve^v - e^v + 1}{v^2}\right|_{v=0}$

$\Downarrow$ L-Hospital

$$\left.\frac{ve^v}{2v}\right|_{v=0} \Rightarrow \frac{e^0}{2} \Rightarrow 1/2$$

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CF $\quad m_n = (-j)^n \frac{d^n}{d\omega^n} \phi_x(\omega) \Big|_{\omega=0}$

$$m_1 = -j \frac{d}{d\omega} \left(\frac{e^{j\omega}-1}{j\omega} \right) \Big|_{\omega=0}$$

$$= -j \left\{ \frac{j e^{j\omega}(j\omega) - j(e^{j\omega}-1)}{(j\omega)^2} \right\} \Big|_{\omega=0}$$

$\Downarrow$ L-hospital

$$= -j \left\{ \frac{-(j\omega e^{j\omega} + e^{j\omega}) - j(j e^{j\omega})}{-2\omega} \right\} \Big|_{\omega=0}$$

$\Downarrow$ L-hospital

$$= -j \left\{ \frac{-je^{j\omega} - j(j\omega e^{j\omega} + e^{j\omega}) + j e^{j\omega}}{-2} \right\} \Big|_{\omega=0} \Rightarrow -j\left(\frac{-j - \cancel{j} + \cancel{j}}{-2} \right) = \frac{1}{2}$$

Q. The characteristic funcn of a RV 'x' is given by,

$\phi_x(\omega) = \frac{1}{(1-j2\omega)^{N/2}}$. find mean & 2nd order moment of x.

CF $\quad m_n = (-j)^n \frac{d^n}{d\omega^n} \phi_x(\omega) \Big|_{\omega=0}$

$$m_1 = (-j) \frac{d}{d\omega} (\phi_x(\omega)) \Big|_{\omega=0} = (-j) \frac{d}{d\omega} \left(\frac{1}{1-j2\omega} \right)^{N/2} \Big|_{\omega=0}$$

$$= -j \left\{ \frac{N}{2} \left(\frac{-(-2j)}{((1-2j\omega)^2)^{\frac{N}{2}-1}} \right) \right\} \Big|_{\omega=0}$$

$$= -j \left\{ \frac{N}{2} (2j) \right\}$$

mean $= m_1 = N$

$$m_2 = (-j)^2 \frac{d^2}{d\omega^2} \phi_x(\omega) \Big|_{\omega=0}$$

$$= -\frac{d^2}{d\omega^2} \left(\frac{1}{(1-2j\omega)^{N/2}} \right) \Big|_{\omega=0}$$

$$= -\frac{d^2}{d\omega^2} (1-2j\omega)^{-N/2} \Big|_{\omega=0}$$

$$\frac{N}{2}(2j)\ \frac{d}{d\omega}(1-2j\omega)^{-\frac{N}{2}-1}\Big|_{\omega=0}$$

$$-jN\left(\frac{-N}{2}-1\right)(-2j)(1-2j\omega)^{-\frac{N}{2}-2}\Big|_{\omega=0}$$

$$\Rightarrow -jN\left(\frac{-N}{2}-1\right)(-2j) = 2N\left(\frac{N}{2}+1\right)$$

$$= N(N+2)$$