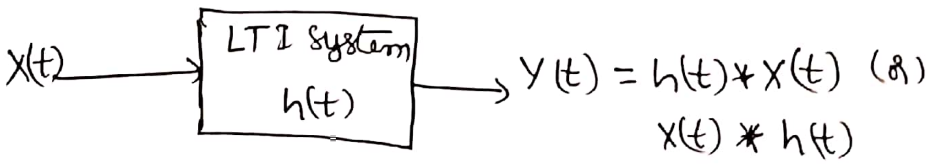
UNIT-VI

Linear Systems With Random Inputs

Response of Linear Systems for random Inputs :-

Consider a continuous LTI system with impulse response $h(t)$. Assume that the system is always causal and stable. When a continuous time random process $X(t)$ is applied to LTI system, the ouput response is also a continuous time random process $Y(t)$. If the random processes X and Y are discrete time signals then the linear system is called a discrete time system.

System Response :- Let a random process $X(t)$ be applied to a continuous LTI system whose impulse response is $h(t)$, then the output response $Y(t)$ is also a random process. It can be expressed by the convolution integral, $Y(t) = h(t) * X(t)$.



i.e, the output response is $Y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot X(t-\tau)\, d\tau$.

Mean and Mean-Squared Value of System Response ;-

Consider the random process $X(t)$ is Wide Sense Stationary (WSS) process.

Mean Value of system response $= E[Y(t)] = \bar{Y}$

$$\bar{Y} = E[Y(t)] = E[h(t) * X(t)]$$
$$= E\left[\int_{-\alpha}^{\alpha} h(\tau) \cdot X(t-\tau)\, d\tau\right]$$
$$= \int_{-\alpha}^{\alpha} h(\tau) \cdot E[X(t-\tau)] \cdot d\tau$$

But $E[X(t-\tau)] = \bar{X} =$ constant, since $X(t)$ is WSS.

$$\therefore \bar{Y} = E[Y(t)] = \bar{X} \cdot \int_{-\alpha}^{\alpha} h(\tau) \cdot d\tau$$

Also, if $H(\omega)$ is the Fourier transform of $h(t)$, then

$$H(\omega) = \int_{-\alpha}^{\alpha} h(t) \cdot e^{-j\omega t}\, dt.$$

at $\omega = 0$, $H(0) = \int_{-\alpha}^{\alpha} h(t)\, dt$ is called the zero-frequency response of the system. Substituting this we get

$$\boxed{\bar{Y} = E[Y(t)] = \bar{X}\, H(0) \text{ is constant.}}$$

Thus the mean value of the system response (or) output response $Y(t)$ of a WSS random process is equal to the product of the mean value of the input process and the zero-frequency response of the system.

<u>Mean Squared Value of system response</u> $= E[Y^2(t)]$

$$E[Y^2(t)] = E[(h(t) * X(t))^2]$$
$$= E[\{h(t) * X(t)\}\{h(t) * X(t)\}]$$

$$E[Y^2(t)] = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2) \cdot x(t-\tau_2)\, d\tau_2\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\tau_1) \cdot x(t-\tau_2) \cdot h(\tau_1) \cdot h(\tau_2) \cdot d\tau_1 \cdot d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[x(t-\tau_1) \cdot x(t-\tau_2)] \cdot h(\tau_1) \cdot h(\tau_2) \cdot d\tau_1 \cdot d\tau_2$$

Where τ_1 and τ_2 are shifts in time intervals.

If input $x(t)$ is a WSS random process, then

$$E[x(t-\tau_1)\, x(t-\tau_2)] = R_{xx}(\tau_1 - \tau_2)$$

$$\therefore \boxed{E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau_1-\tau_2) \cdot h(\tau_1) \cdot h(\tau_2) \cdot d\tau_1 \cdot d\tau_2}$$

This expression is independent of time t. And it represents the output power.

<u>Autocorrelation function of system response</u> :—

The autocorrelation function of $Y(t)$ is

$$R_{yy}(t_1, t_2) = E[Y(t_1)\, Y(t_2)]$$

$$= E[\{h(t_1) * x(t_1)\}\{h(t_2) * x(t_2)\}]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1) \cdot x(t_1-\tau_1)\, d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2)\, x(t_2-\tau_2)\, d\tau_2\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1-\tau_1)\, x(t_2-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[x(t_1-\tau_1)\, x(t_2-\tau_2)] \cdot h(\tau_1)\, h(\tau_2) \cdot d\tau_1\, d\tau_2$$

We know that $E[X(t_1-\tau_1)X(t_2-\tau_2)] = R_{XX}(t_2-t_1+\tau_1-\tau_2)$

If input $X(t)$ is WSS-RP, Let the time difference $\tau = t_2 - t_1$ and $t = t_1$, then $E[X(t-\tau_1)X(t+\tau-\tau_2)] = R_{XX}(\tau+\tau_1-\tau_2)$

$$\therefore R_{YY}(t,t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, h(\tau_1)\, h(\tau_2) \cdot d\tau_1\, d\tau_2$$

$$\boxed{R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)}$$

Two facts results from above expressions

1. $Y(t)$ is WSS, if $X(t)$ is WSS because $R_{YY}(\tau)$ does not depend on 't' and $E[Y(t)]$ is a constant.
2. $R_{YY}(\tau)$ is the twofold convolution of the input autocorrelation function with the network's impulse response.

Cross-correlation functions between Input and output of the system :—

The cross-correlation function of $X(t)$ and $Y(t)$ is

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)]$$

$$= E\left[X(t)\int_{-\infty}^{\infty} h(\tau_1)\, X(t+\tau-\tau_1)\, d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} E[X(t)X(t+\tau-\tau_1)] \cdot h(\tau_1) \cdot d\tau_1$$

If $X(t)$ is WSS, then

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\tau_1) \cdot h(\tau_1)\, d\tau_1$$, which is the convolution $R_{XX}(\tau)$ with $h(\tau)$

$$\boxed{R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)}$$

A similar development shows that

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\tau_1)\, h(-\tau_1)\, d\tau_1$$

$$\boxed{R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)}$$

It is clear that ~~the~~ cross-correlation functions depend on τ and not on absolute time t. As a consequence of this fact x(t) and Y(t) are jointly WSS, if x(t) is WSS, because already Y(t) to be WSS.

Autocorrelation function and cross-correlation functions are seen to be related by

$$R_{YY}(\tau) = \int_{-\infty}^{\infty} R_{XY}(\tau+\tau_1)\, h(\tau_1)\, d\tau_1$$

(or) $\boxed{R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau) \quad \text{(or)} \quad h(-\tau) * R_{XY}(\tau)}$

slly $R_{YY}(\tau) = \int_{-\infty}^{\infty} R_{YX}(\tau-\tau_2) \cdot h(\tau_2)\, d\tau_2$

(or) $\boxed{R_{YY}(\tau) = R_{YX}(\tau) * h(\tau) \quad \& \quad h(\tau) * R_{YX}(\tau)}$

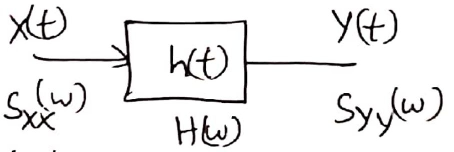
Spectral characteristics of System's Response :—

Consider that the random process X(t) is a WSS-RP with the autocorrelation function $R_{XX}(\tau)$ applied through an LTI system. The o/p response Y(t) is also a WSS-RP and the processes X(t) and Y(t) are jointly WSS. Now, we can obtain power spectral characteristics of the o/p process Y(t) by taking the Fourier transform of the correlation functions.

Power density spectrum of response :—

Consider that a random process $x(t)$ is applied on an LTI system having a transfer function $H(\omega)$. The o/p response is $y(t)$. If the power spectrum of the i/p process is $S_{xx}(\omega)$, then the power spectrum of the o/p response is given by

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$



Proof :— Let $R_{yy}(\tau)$ be the autocorrelation of the o/p response $y(t)$. Then the power spectrum of the response is the F.T of $R_{yy}(\tau)$.

$$\therefore S_{yy}(\omega) = F\left[R_{yy}(\tau)\right]$$

$$= \int_{-\infty}^{\infty} R_{yy}(\tau) \cdot e^{-j\omega\tau}\, d\tau$$

we know that $R_{yy}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau+\tau_1-\tau_2) \cdot h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2$

then

$$S_{yy}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau+\tau_1-\tau_2) \cdot h(\tau_1)\, h(\tau_2) \cdot e^{-j\omega\tau} \cdot d\tau\, d\tau_1\, d\tau_2$$

Let $\tau+\tau_1-\tau_2 = t \Rightarrow d\tau = dt \qquad \tau = t-\tau_1+\tau_2$

$$\therefore S_{yy}(\omega) = \int_{-\infty}^{\infty} h(\tau_1) \cdot e^{j\omega\tau_1}\, d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2) \cdot e^{-j\omega\tau_2}\, d\tau_2 \cdot \int_{-\infty}^{\infty} R_{xx}(t)\, e^{-j\omega t}\, dt$$

$$= H(-\omega) \cdot H(\omega) \cdot S_{xx}(\omega) = H^*(\omega) \cdot H(\omega) \cdot S_{xx}(\omega)$$

$$\boxed{S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)}$$

Cross-power density spectrums of Input and output :—

The cross-power density Spectrum of I/p & O/p is

$S_{xy}(\omega) = S_{xx}(\omega) \cdot H(\omega)$ and

$S_{yx}(\omega) = S_{xx}(\omega) \cdot H(-\omega)$.

Proof :-

$$S_{xy}(\omega) = F[R_{xy}(\tau)]$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \cdot e^{-j\omega\tau} d\tau$$

we know that $R_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau - \tau_1) \cdot h(\tau_1)\, d\tau_1$

$$\therefore S_{xy}(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{xx}(\tau - \tau_1) \cdot h(\tau_1) \cdot e^{-j\omega\tau} \cdot d\tau \cdot d\tau_1$$

Let $\tau - \tau_1 = t \Rightarrow d\tau = dt \Rightarrow \tau = t + \tau_1$

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} h(\tau_1) \cdot e^{-j\omega\tau_1} \cdot d\tau_1 \cdot \int_{-\infty}^{\infty} R_{xx}(t) \cdot e^{-j\omega t} \cdot dt$$

$$= H(\omega) \cdot S_{xx}(\omega)$$

$$\boxed{S_{xy}(\omega) = S_{xx}(\omega) \cdot H(\omega)}$$

S|ly

$$\boxed{S_{yx}(\omega) = S_{xx}(\omega) \cdot H(-\omega)}$$

Ex 1:- A random process $X(t)$ is applied as input to a system whose impulse response is $h(t) = 3\,u(t)\cdot t^2\cdot \exp(-8t)$. If $E[X(t)] = 2$, what is the mean value of the system response $Y(t)$?

Sol:-

Mean value of the system response is $E[Y(t)] = \bar{Y}$

$$E[Y(t)] = E[h(t) * X(t)]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} h(\tau)\, E[X(t-\tau)]\cdot d\tau \qquad \because E[X(t-\tau)] = 2$$

$$= 2\int_{-\infty}^{\infty} 3\cdot u(\tau)\cdot \tau^2\cdot e^{-8\tau}\cdot d\tau$$

$$= 6\int_{0}^{\infty} \tau^2\cdot e^{-8\tau}\, d\tau$$

$$= 6\left[\tau^2\cdot \frac{e^{-8\tau}}{-8} - 2\tau\cdot \frac{e^{-8\tau}}{(-8)^2} + 2\cdot \frac{e^{-8\tau}}{(-8)^3}\right]_0^{\infty}$$

$$= 6\left[0 - 0 + 2\cdot \frac{(e^{-\infty} - e^{0})}{(-8)^3}\right]$$

$$= 6\left(\frac{1}{8^3}\right) = \frac{3}{256} = \text{constant}$$

Ex 2:- Let $X(t)$ be a zero-mean WSS process with $R_{XX}(\tau) = e^{-|\tau|}$. $X(t)$ is input to an LTI system with
$$|H(\omega)| = \begin{cases} \sqrt{1+(\omega)^2}, & |\omega| < 4\pi \\ 0, & \text{otherwise} \end{cases}$$
Let $Y(t)$ be the output.

a) find $E[Y(t)]$, b) $E[Y^2(t)]$, c) $R_{YY}(\tau)$

Sol:- Note that $X(t)$ is WSS, $X(t)$ & $Y(t)$ are jointly WSS, and therefore $Y(t)$ is WSS.

a) $E[Y(t)] = \bar{X} \cdot H(0)$

$= 0 \cdot 1 \qquad \because \bar{X} = 0,\ H(\omega) = \sqrt{1+(\omega)^2}$

$= 0 \qquad\qquad H(0) = 1$

c) $R_{yy}(\tau) = F^{-1}[S_{yy}(\omega)]$

where $S_{yy}(\omega) = S_{xx}(\omega) \cdot |H(\omega)|^2$

$$S_{xx}(\omega) = F[R_{xx}(\tau)] = \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-|\tau|} \cdot e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{\tau} \cdot e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-\tau} \cdot e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{\tau(1-j\omega)} d\tau + \int_{0}^{\infty} e^{-\tau(1+j\omega)} d\tau$$

$$= \left[\frac{e^{\tau(1-j\omega)}}{1-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-\tau(1+j\omega)}}{-(1+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{1-j\omega} + \frac{1}{1+j\omega} = \frac{2}{1+\omega^2}$$

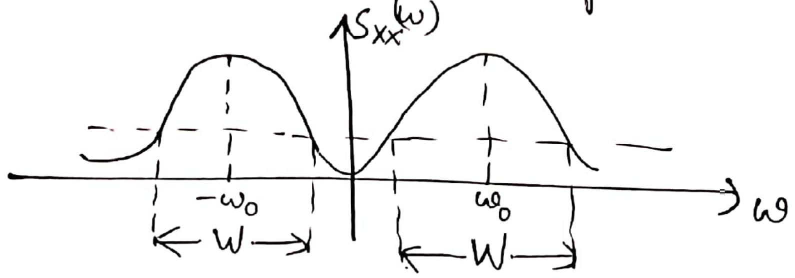
$$\therefore S_{yy}(\omega) = \frac{2}{1+\omega^2} \times 1+\omega^2 = 2, \quad |\omega| < 4\pi$$

$$R_{yy}(\tau) = F^{-1}[S_{yy}(\omega)] = \frac{1}{2\pi}\int_{-4\pi}^{4\pi} 2 \cdot e^{j\omega\tau} \cdot d\omega = 8\,\mathrm{sinc}(4\pi\tau)$$

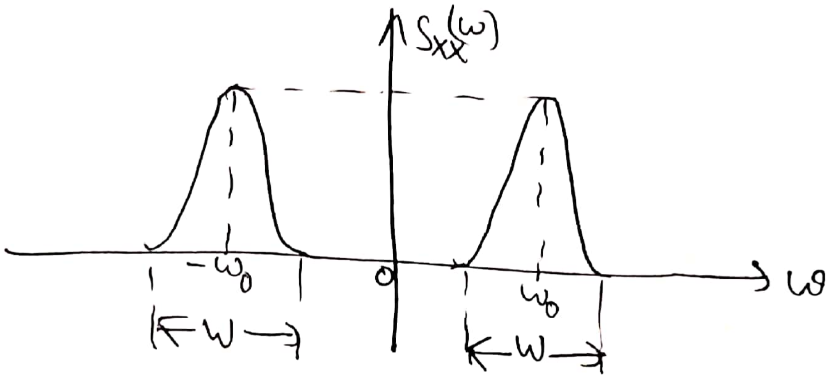
b) $E[Y^2(t)] = R_{yy}(0) = 8$

Band pass, Band-Limited and Narrowband processes and their Properties :-

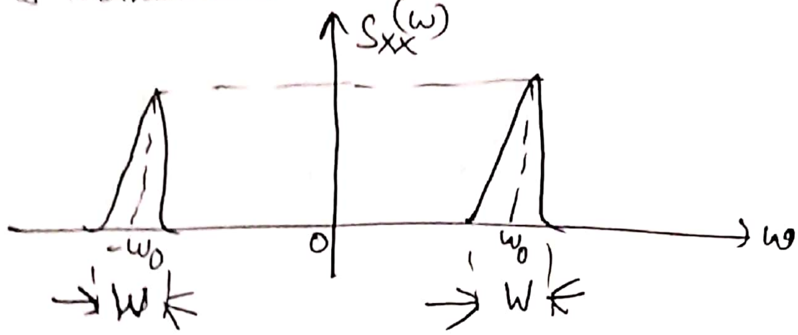
Band pass random processes :- A random process $X(t)$ is called a band pass process, if its power spectral density $S_{XX}(\omega)$ has significant components within a bandwidth 'W' that does not include $\omega=0$. But in practice, the spectrum may have a small amount of power spectrum at $\omega=0$. The spectrum components outside the band 'W' are very small and can be neglected.



Band-Limited random process :- A random process $X(t)$ is said to be band-limited, if its power spectrum components are zero outside the frequency band of width 'W' that does not include $\omega=0$. The power density spectrum of the band-limited band pass is



Narrowband random processes :— A band limited random Process is said to be a narrowband process, if the bandwidth 'W' is very small compared to the band centre frequency, i.e $W \ll \omega_0$ where W = bandwidth and ω_0 is the frequency at which the power spectrum is maximum.



Representation of a narrowband process :— For any arbitrary WSS random process N(t), the quadrature form of narrowband Process can be represented as $N(t) = X(t)\cos\omega_0 t - Y(t)\sin\omega_0 t$.
Where X(t) and Y(t) are called the in-phase & quadrature phase components of N(t).

They can be expressed as

$$X(t) = A(t) \cdot \cos[\theta(t)]$$

$$Y(t) = A(t) \cdot \sin[\theta(t)]$$

and the relationship between the processes A(t) & $\theta(t)$ are given by

$$A(t) = \sqrt{X^2(t) + Y^2(t)} \quad \&$$

$$\theta(t) = \tan^{-1}\left(\frac{Y(t)}{X(t)}\right)$$

Properties of Band-limited Random processes :—

Let $N(t)$ be any band-limited WSS-RP with zero mean value and a power spectral density, $S_{NN}(\omega)$. If the RP is represented by
$N(t) = X(t)\cos\omega_0 t - Y(t)\sin\omega_0 t$.

1. If $N(t)$ is WSS, then $X(t)$ & $Y(t)$ are Jointly WSS.
2. If $N(t)$ has zero mean. i.e $E[N(t)]=0$, then $E[X(t)]=E[Y(t)]=0$.
3. The mean-square values of the processes are equal

   i.e $E[N^2(t)] = E[X^2(t)] = E[Y^2(t)]$.
4. Both processes $X(t)$ & $Y(t)$ have the same autocorrelation functions.

   i.e $R_{XX}(\tau) = R_{YY}(\tau)$
5. The cross-correlation functions of $X(t)$ & $Y(t)$ are given by

   $R_{YX}(\tau) = -R_{XY}(\tau)$. If the processes are orthogonal, then

   $R_{XY}(\tau) = R_{YX}(\tau) = 0$
6. Both $X(t)$ & $Y(t)$ have the same power spectral densities.

$$S_{YY}(\omega) = S_{XX}(\omega) = \begin{cases} S_N(\omega-\omega_0) + S_N(\omega+\omega_0) & , |\omega| \le \omega_0 \\ 0 & , \text{otherwise} \end{cases}$$

7. The cross-power spectrums are $S_{XY}(\omega) = -S_{YX}(\omega)$
8. If $N(t)$ is a Gaussian-RP, then $X(t)$ & $Y(t)$ are jointly Gaussian
9. The relationship between autocorrelation & power spectrum $S_{NN}(\omega)$ is

$$R_{XX}(\tau) = \frac{1}{\pi}\int_0^{\infty} S_{NN}(\omega)\cdot\cos[(\omega-\omega_0)\tau]\, d\omega = R_{YY}(\tau)$$

$$R_{XY}(\tau) = \frac{1}{\pi}\int_0^{\infty} S_{NN}(\omega)\sin[(\omega-\omega_0)\tau]\cdot d\omega = -R_{YX}(\tau)$$

10. If $N(t)$ is zero-mean Gaussian and its Pdf, $S_{NN}(\omega)$ is symmetric about $\pm\omega_0$, then $X(t)$ & $Y(t)$ are S.I.