$$\boxed{(SNR)_{IDPRa} = \frac{2}{N_0} V^2 T} \longrightarrow (I)$$

→ (I) ⇒ SNR ↑es with ↑ng bit duration 'T'
& that it depends on V^2T

→ V^2T is the normalized energy of bit signal.

→ ∴, a bit keyed by a narrow, high ampl. sgl
&
" " " " " " wide, low ampl sgl
are equally effective, provided V^2T is kept constant.

→ Note: -

→ That integrator filters the signal & the noise such that signal V/g ↑es linearily with time

→ While Std. deviation (rms value) of noise ↑ more slowly, as $\sqrt{T}$

→ Thus, the integrator enhances the signal relative to noise & this enhancement ↑es with time. ⇐ (I)

Probability of Error for Baseband Receiver (or Integrate-&-dump Rxr)

1. Since the function of a Rxr of a data txn is to distinguish bit 1 from bit '0' in presence of noise
2. A Most ✗. characteristics is probability that an error will be made in such a determination.
3. Noise interference leads to wrong decision at the Rxr.
4. Probability of error Pe is the good measure for performance of the detector.

Derivation :-

1. o/p of integrator is at $t = T$

$$y(T) = s_o(T) + n_o(T)$$

2. for the Pulse of ampl V, $s_o(T)$ is

$$s_o(T) = \frac{VT}{\tau}$$

3. for -ve pulse of ampl $-V$, $s_o(T)$ is

$$s_o(T) = -\frac{VT}{\tau}$$

4. $\therefore$

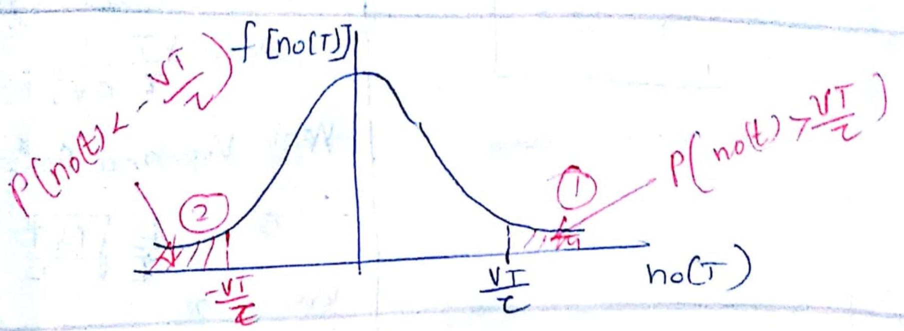
$$\boxed{\begin{aligned} y(T) &= \frac{VT}{\tau} + n_o(T) \quad \text{for } +V \rightarrow (1) \\ &\& \\ y(T) &= \frac{-VT}{\tau} + n_o(T) \quad \text{for } -V \rightarrow (2) \end{aligned}}$$

Pe of error in I & dump Rxr

Input $x(t)$	Value of $n_o(t)$ for error in o/p	o/p $y(T)$	Prob. of error
$-V$ (0)	Error introduced if $n_o(t) > \frac{VT}{2}$	+ve	'Pe' can be obtained by evaluating Prob. that $n_o(t) > \frac{VT}{2}$
$+V$ (1)	" if $n_o(t) < -\frac{VT}{2}$	−ve	$P\left[n_o(t) < -\frac{VT}{2}\right]$

1) These probabilities can be obtained from PDF of $n_o(t)$.

2) Prob. density of noise sample $n_o(T)$ is Gaussian



3) PDF of Gaussian distⁿ fn: is given by std: relation as

$$\boxed{f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(\mu - x)^2}{2\sigma^2}}}$$

μ: mean
σ: SD
σ^2: variance

4) Evaluate PDF of White Gaussian Noise

if $n_o(t) > \frac{VT}{2}$ error occurs if $x(t) = -V$

5) ∵ Gaussian ; $\mu = 0$

$$\therefore f[n_o(T)] = \frac{1}{2\pi\sigma^2}\, e^{-\frac{[n_o(T)]^2}{2\sigma^2}} \longrightarrow (3)$$

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(1) $$P_e = \int_{\frac{VT}{\tau}}^{\infty} f[n_o(T)]\, d\, n_o(T) \rightarrow (4)$$

$$= \int_{\frac{VT}{\tau}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{n_o^2(T)}{2\sigma^2}} d\, n_o(T)$$

$$P_e = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \int_{\frac{VT}{\tau}}^{\infty} e^{-\frac{n_o^2(T)}{2\sigma^2}} d n_o(T) \rightarrow (5)$$

Assume $x = \frac{n_o(T)}{\sigma\sqrt{2}}$ $\rightarrow$ (6)

then $dx = \frac{1}{\sigma\sqrt{2}} d[n_o(T)] \rightarrow (7)$ $\Rightarrow d[n_o(T)] = \sigma\sqrt{2}\, dx$

Limits

Upper limit

if $n_o(T) = \infty$

$$\boxed{x = \infty}$$

Lower limit

$n_o(T) = \frac{VT}{\tau}$

$x = \frac{VT}{\tau} \frac{1}{\sigma\sqrt{2}}$

Noise Variance $\sigma_0^2 = \frac{N_0}{2} \cdot \frac{T}{\tau^2}$

$\therefore \sigma = \frac{1}{\tau}\sqrt{\frac{N_0 T}{2}}$

Sub in x

$$x = \frac{V\cancel{T}^{\sqrt{T}}}{\cancel{\tau}} \cdot \frac{1}{\cancel{\sqrt{2}}} \cdot \frac{\cancel{\tau}\cancel{\sqrt{2}}}{\sqrt{N_0}\cancel{\sqrt{T}}}$$

$$\therefore \boxed{x = V\sqrt{\frac{T}{N_0}}}$$

(2) $$\therefore P_e = \frac{1}{\cancel{\sigma} \cdot \cancel{\sqrt{2}} \cdot \sqrt{\pi}} \int_{x = V\sqrt{\frac{T}{N_0}}}^{\infty} e^{-x^2} [\cancel{\sqrt{2}}\, \cancel{\sigma}]\, dx$$

(3) $$P_e = \frac{1}{2}\left(\frac{1}{\sqrt{\pi}} \times 2 \int_{x = V\sqrt{\frac{T}{N_0}}}^{\infty} e^{-x^2} dx\right) = \operatorname{erfc}(x)$$

$$Pe = \frac{1}{2}\left(\frac{2}{\sqrt{\pi}}\int_{x=V\sqrt{\frac{T}{N_0}}}^{\infty} e^{-x^2}\,dx\right)$$

$$= \frac{1}{2}\,\mathrm{erfc}(x) \qquad \text{where } x = V\sqrt{\frac{T}{N_0}}$$

$$\boxed{Pe = \frac{1}{2}\,\mathrm{erfc}\left(V\sqrt{\frac{T}{N_0}}\right)} \rightarrow (8)$$

$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{V^2 T}{N_0}\right)^{1/2}$$

$$\boxed{Pe(0) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{E_s}{N_0}\right)^{1/2}} \rightarrow (9)$$

$E_b = V^2 T$ (or) $E_s = V^2 T$

where $E_s = V^2 T$ is the signal energy of a bit.

- If sgl $-V$ were held instead of $+V$ during some bit interval, from the symmetry of situation that the $P_e(1)$ will be same as equ (9).

- $Pe(0) \rightarrow$ 0 is Txed but ~~deci~~ wrongly decided as 1
  $Pe(1) \rightarrow$ 1 is " " " " '0'

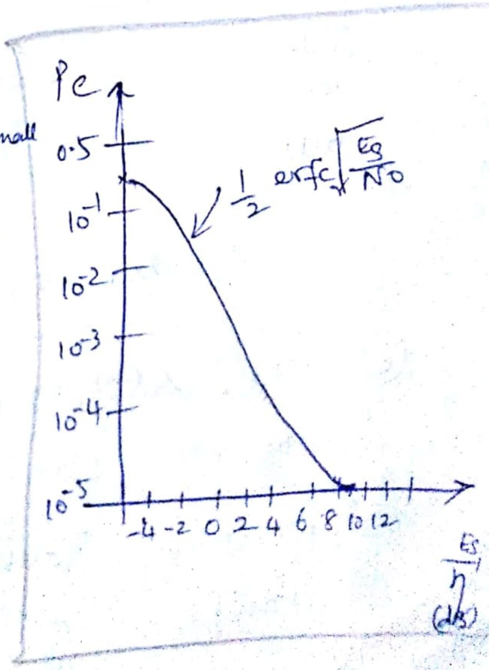
- $Pe(1) = P_e(0) = \frac{1}{2}$ considering equiprobable case with 1 & 0.

- Pe ↓es rapidly as $\frac{E_s}{N_0}$ ↑es

- $\boxed{(Pe)_{max} = \frac{1}{2}}$ when $\frac{E_b}{N_0}$ is small

- Thus, even if sgl is entirely lost in noise, so that any determination of Rxr is a guess

- Rxr can't be wrong more than half the time on average.



OPTIMUM FILTER

① Assume that fixed signal is a binary w/form.

② One binary digit (bit) is reped by a sgl wfm $s_1(t)$ which persist for time T

$s_1(t)$ = one bit, for T
$s_2(t)$ = another bit, for T

③ for eg, in case of txn at base band

$s_1(t) = +V$
$s_2(t) = -V$

④ for other modulation s/ms, different wfms are used.

for eg

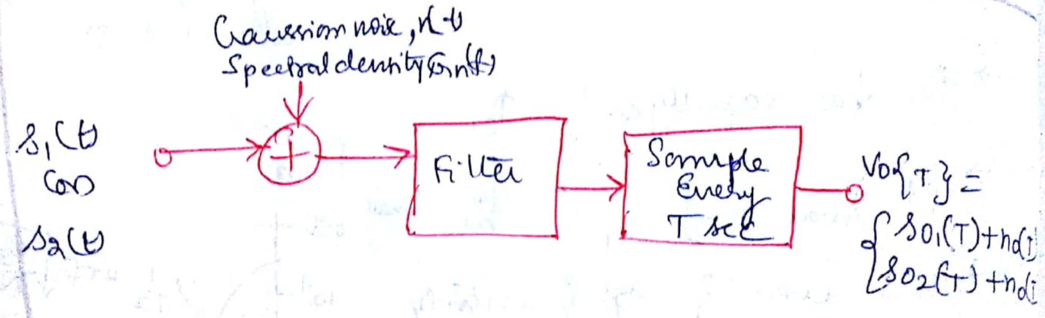
PSK signalling

$s_1(t) = A \cos w_0 t$
$s_2(t) = -A \cos w_0 t$

for FSK

$s_1(t) = A \cos(w_0 + \Omega)t$
$s_2(t) = A \cos(w_0 - \Omega)t$

⑤



A rxr for binary coded signaling

⑥ I/P, $s_1(t)$ (or) $s_2(t)$ is corrupted by addition of noise $n(t)$.

⑦ Noise is Gaussian & has a spectral density $G_n(f)$.

$G_n(f) = \frac{N_0}{2}$

① signal & noise are filtered & then sampled at end of each bit interval.

② o/p sample is either

$V_o(T) = s_{o1}(T) + n_o(T)$

(or)

$V_o(T) = s_{o2}(T) + n_o(T)$

③ Assume that immediately after each sample, every energy-storing element in the filter has been discharged.

④ In absence of noise, o/p sample would be

$V_o(T) = s_{o1}(T)$ (or) $s_{o2}(T)$

⑤ In presence of noise o/p sample would be:-

to minimize probability of error one should assume

⑥ $s_1(t)$ has been txed if $v_o(T)$ is closer to $s_{o1}(T)$ than to $s_{o2}(T)$.

& $s_2(t)$ " " " if " " " $s_{o2}(T)$ " " $s_{o1}(T)$.

⑦ Decision boundary is ∴ midway b/w $s_{o1}(T)$ & $s_{o2}(T)$.

⑧ For eg in baseband s/m of binary signalling,

where $s_{o1}(T) = \frac{VT}{\tau}$

$s_{o2}(T) = \frac{-VT}{\tau}$

∴ decision boundary is $V_o(T) = 0$.

①

⑨ ∴ Decision boundary is $\boxed{V_{th} = \frac{s_{o1}(T) + s_{o2}(T)}{2}}$

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① **Pe**

Suppose that $s_{o1}(T) > s_{o2}(T)$ & $s_2(t)$ was tred.

② If at sampling time, noise $n_o(T)$ is +ve & larger in magnitude than V-tg difference

i.e. $n_o(T) \geq V_{th} - s_{o2}(T)$

③ error occurs if

$$n_o(T) \geq \frac{s_{o1}(T) - s_{o2}(T)}{2} - s_{o2}(T)$$

$$\boxed{n_o(T) \geq \frac{s_{o1}(T) - s_{o2}(T)}{2}} \rightarrow ②$$

④ ie. an error [we decide that $s_1(t)$ is tred rather than $s_2(t)$] will result if

$$n_o(T) \geq \frac{s_{o1}(T) - s_{o2}(T)}{2}$$

⑤ Hence Pe is

$$\boxed{Pe = \int_{\frac{s_{o1}(T) - s_{o2}(T)}{2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n_o^2(T)}{2\sigma^2}} \, d\,n_o(T)} \rightarrow ③$$

⑥ Let $x = \frac{n_o(T)}{\sigma\sqrt{2}}$ $\therefore dx = \frac{d[n_o(T)]}{\sigma\sqrt{2}}$

$$\Rightarrow d[n_o(T)] = [\sigma\sqrt{2}]\, dx$$

⑦ **Limit**

if $n_o(T) = \infty$ then $x = \infty$

if $n_o(T) = \frac{s_{o1}(T) - s_{o2}(T)}{2}$ then $x = \frac{1}{\sigma\sqrt{2}} \cdot \frac{s_{o1}(T) - s_{o2}(T)}{2}$

① $P_e = \frac{1}{\sqrt{2}\sqrt{\pi}\,\cancel{\sigma}} \int_{x = \frac{s_{o1}(T) - s_{o2}(T)}{\sigma 2\sqrt{2}}}^{\infty} e^{-x^2} (\cancel{\sigma\sqrt{2}})\,dx$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{\pi}} \sqrt{2} \int_{x=}^{\infty} e^{-x^2}\,dx$

$= \frac{1}{2}\,\mathrm{erfc}(x)$

$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{s_{o1}(T) - s_{o2}(T)}{2\sqrt{2}\sigma}\right)} \rightarrow ④$$

② for case $s_{o1}(T) = \frac{VT}{\tau}$ & $s_{o2}(T) = -\frac{VT}{\tau}$

& $\sigma_1^2 = \overline{n_o^2(T)} = \frac{N_o}{2} \cdot \frac{T}{\tau^2}$

$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{V^2 T}{N_o}\right)}$$

③ Complementary error function is a monotonically ↓ing fn. of its argument.

④ Hence, P_e ↓es as the difference $s_{o1}(T) - s_{o2}(T)$ becomes larger & as rms noise v/g 'σ' becomes smaller.

⑦ Optimum filter, then is the filter which maximizes the ratio

⑧ $$\boxed{\gamma = \frac{s_{o1}(T) - s_{o2}(T)}{\sigma}} \rightarrow ⑤$$

⑨ Now calculate the T.F. $H(f)$ of this optimum filter.

⑩ For mathematical convenience we shall actually maximize γ^2 rather than γ.

Calculation of Optimum Filter T.F. H(f)

1. fundamental requirement of a binary encoded data rxr is
2. that it distinguishes vtgs $s_1(t) + n(t)$ & $s_2(t) + n(t)$.
3. Ability of the rxr to do so depends on how large a particular rxr can make 'ρ'!
4. * that here 'ρ' is proportional ~~to~~ not to $s_1(t)$ not to $s_2(t)$, but rather to the difference b/w them
5. for eg, in baseband s/m, we reped the sgls by vtg levels +V & −V.
6. Have adv. of requiring least average power to be tred.
7. Hence, while $s_1(t)$ (or $s_2(t)$) is the rxed signal, the signal which is to be compared with the noise.
8. i.e. sgl which is relevant in all our error probability calculations, is the difference signal.
9. Let $\boxed{p(t) = s_1(t) - s_2(t)} \rightarrow (1)$
10. Thus, for calculating the minimum error probability, we assume that input signal to optimum filter is $p(t)$
11. Corresponding output signal of filter is $\boxed{p_o(t) = s_{o1}(t) - s_{o2}(t)} \rightarrow (2)$

1) $p(t) \xleftrightarrow{FT} P(f)$

$p_o(t) \longleftrightarrow P_o(f)$

2) if $H(f)$ is the transfer function of the filter,

then

$$\boxed{P_o(f) = H(f)P(f)} \rightarrow (3)$$

3) $$P(f) \longrightarrow \boxed{\text{Opt filter } H(f)} \longrightarrow P_o(f)$$

4) at $t=T$. F.T

$$p_o(T) = \int_{-\infty}^{\infty} P_o(f)\, e^{j2\pi fT}\, df \rightarrow (4)$$

Sub (3) in (4)

$$\boxed{p_o(T) = \int_{-\infty}^{\infty} H(f)\, P(f)\, e^{j2\pi fT} \cdot df} \rightarrow (5)$$

5) Noise

~~at t=T~~

$$n(t) \longrightarrow \boxed{H(f)} \longrightarrow n_o(t)$$

i/p noise to opt. filter is $n(t)$
o/p noise from o/p filter is $n_o(t)$
which has a PSD $G_{no}(f)$ & related to PSD of
i/p noise $G_n(f)$ by

$$\boxed{G_{no}(f) = |H(f)|^2\, G_n(f)} \rightarrow (6)$$

6) Using Parseval's thm, normalized o/p noise power i.e. the noise variance is

$$\boxed{\sigma_o^2 = \int_{-\infty}^{\infty} G_{no}(f)\, df = \int_{-\infty}^{\infty} |H(f)|^2\, G_n(f)\, df} \rightarrow (7)$$

SNR

(1) $$\gamma^2 = \frac{p_0^2(T)}{\sigma_0^2} = \frac{\left| \int_{-\infty}^{\infty} H(f)\, P(f)\, e^{j2\pi fT}\, df \right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 G_n(f)\, df} \rightarrow (8)$$

(2) Sign has been included, in Numerator, to allow the use of Schwarz Inequality.

(3) It states that given arbitrary complex functions $X(f)$ and $Y(f)$ of a common variable f, then

$$\boxed{\left| \int_{-\infty}^{\infty} X(f)\, Y(f)\, df \right|^2 \le \int_{-\infty}^{\infty} |X(f)|^2 df \int_{-\infty}^{\infty} |Y(f)|^2 df} \rightarrow (9)$$

(4) equal sign applies when $\boxed{X(f) = k\, Y^*(f)} \rightarrow (10)$

where 'k' is an arbitrary constant
$Y^*(f)$ is the complex conjugate of $Y(f)$.

(5) Apply (9) & (10) to eqn (8) by making identification

$$X(f) = H(f)\sqrt{G_n(f)} \rightarrow (11)$$

$$Y(f) = \frac{1}{\sqrt{G_n(f)}} P(f)\, e^{j2\pi fT} \rightarrow (12)$$

(6) $$\gamma^2 = \frac{p_0^2(T)}{\sigma_0^2} = \left| \int_{-\infty}^{\infty} X(f)\, Y(f)\, d(f) \right|^2 \le \int_{-\infty}^{\infty} |X(f)|^2 df \int_{-\infty}^{\infty} |Y(f)|^2$$

$$\gamma^2 = \frac{p_0^2(T)}{\sigma_0^2} = \frac{\left| \int_{-\infty}^{\infty} X(f)\, Y(f)\, d(f) \right|^2}{\int_{-\infty}^{\infty} |X(f)|^2 df} \le \int_{-\infty}^{\infty} |Y(f)|^2 df \rightarrow (13)$$

Compare (8) & (13)

$$\therefore \gamma^2 = \frac{p_0^2(T)}{\sigma_0^2} \le \int_{-\infty}^{\infty} |Y(f)|^2\, df$$

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① $\mu^2 = \frac{p_o^2(T)}{\sigma_o^2} \leq \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)} df.$ $\boxed{\because |e^{j2\pi f T}|^2 = 1}$

② μ^2 attains its max. value when equal sign condn is applied i.e. $X(f) = K\,\varphi^*(f)$

③ Optimum filter which yields such a max. ratio $\frac{p_o^2(T)}{\sigma_o^2}$ has a transfer function

Sub ⑪ & ⑫ in ⑩ we get

$$H(f)\sqrt{G_n(f)} = K \cdot \frac{1}{\sqrt{G_n(f)}} \cdot P^*(f)\, e^{-j2\pi f T}$$

$$\boxed{\therefore H(f) = K \cdot \frac{P^*(f)}{G_n(f)} e^{-j2\pi f T}} \rightarrow (14)$$

④ correspondingly, max ratio is

$$(SNR)_{max}\ \boxed{\mu^2_{max} = \left[\frac{p_o^2(T)}{\sigma_o^2}\right]_{max} = \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)} df} \rightarrow (15)$$

OPTIMUM FILTER REALIZATION USING MATCHED FILTER

① An optimum filter which yields a max. ratio $\frac{p_o^2(T)}{\sigma_o^2}$ is called a matched filter when the i/p noise is white.

② In this case $\boxed{G_n(f) = \frac{N_o}{2}} \rightarrow (16)$

③ sub (16) in (14)

$$\boxed{H(f) = K \cdot \frac{P^*(f)}{\frac{N_o}{2}} e^{-j2\pi f T}} \rightarrow (17)$$

(1) Impulsive Response of this filter.
i.e. Response of the filter to a unit strength impulse applied at $t=0$ is

$$H(f) = \frac{2k}{N_0} p^*(f) e^{-j2\pi f T}$$

$$h(t) = \text{Inv. F.T } [H(f)]$$

$$h(t) = \frac{2k}{N_0} \int_{-\infty}^{\infty} p^*(f) e^{-j2\pi f T} e^{j2\pi f t} . df$$

$$h(t) = \frac{2k}{N_0} \int_{-\infty}^{\infty} p^*(f) e^{j2\pi f(t-T)} . df \rightarrow (18)$$

(2) Physically realizable filter will have an impulse response which is real. (not complex)

(3) $\therefore h(t) = h^*(t) . \rightarrow (19)$

(4) Replacing RHS of eqn (18) by its complex conjugate,

(5) $$h(t) = \frac{2k}{N_0} \int_{-\infty}^{\infty} P(f) e^{j2\pi f(T-t)} df \rightarrow (20)$$

(6) $$h(t) = \frac{2k}{N_0} p(T-t) \rightarrow (21)$$

(7) $\because p(t) = s_1(t) - s_2(t) \rightarrow (22)$

$$\boxed{h(t) = \frac{2k}{N_0} [s_1(T-t) - s_2(T-t)]} \rightarrow (23)$$

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