Module 28:Cross Power density spectrum

Objective: To determine the relationship between two time series as a function of frequencyusing Cross spectral analysis .

Module Description:

- **Using Cross-Spectral Density** i.e. **cross**-correlation, the **power** shared by a given frequency for the two signals using its squared module, and the phase shift between the two signals at that frequency using its argument can be found

❖ Defining the Power Spectral Density of a random Process

- .Let

$$X_T(t) = X(t) \qquad -T < t < T$$
$$= 0 \qquad \text{otherwise}$$
$$= X(t)rect(\frac{t}{2T})$$

and $Y_T(t) = \begin{matrix} Y(t) for -T < t < T \\ 0 \ otherwise \end{matrix}$

where $rect(\frac{t}{2T})$ is the unity-amplitude rectangular pulse of width $2T$ cantered at origin. As $t \to \infty$, $X_T(t)$ will represent the random process $X(t)$. .Similarly $Y_T(t)$

- Define $F[X_T(t)] = X_T(\omega) = \int_{-T}^{T} X_T(t).e^{-j\omega t}\,.dt = \int_{-T}^{T} X(t).e^{-j\omega t}\,.dt$
- $F[Y_T(t)] = Y_T(\omega) = \int_{-T}^{T} Y_T(t).e^{-j\omega t}\,.dt = \int_{-T}^{T} Y(t).e^{-j\omega t}\,.dt$
- Consider the generalized Parseval's relation

$.\int_{-\infty}^{\infty} X(t).Y(t)\,.dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega).Y^*(\omega)\,.d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega).X^*(\omega)\,.d\omega$

- Therefore, the average power P_{XY} is
- $\frac{1}{2T}\int_{-T}^{T} X_T(t)Y_T(t).dt = \frac{1}{2T}\int_{-T}^{T} X(t).Y(t).dt$

- The average power is given by

$$\frac{1}{2T}E\left[\int_{-T}^{T} X(t).Y(t).dt\right] = \frac{1}{2T}E\left[\int_{-\infty}^{\infty} X^*{}_T(\omega)Y_T(\omega)\,.d\omega\right] = \left[\int_{-\infty}^{\infty} \frac{E[X^*{}_T(\omega)Y_T(\omega)]}{2T}d\omega\right]$$

- where $E\left[\int_{-\infty}^{\infty} \frac{X^*{}_T(\omega)Y_T(\omega)}{2T}d\omega\right]$ is the contribution to the average powerP_{XY} at frequency ω and represents the cross power spectral density of $X_T(t)$ and $Y_T(t)$. As $T \to \infty$, the left-hand side in the above expression represents the average power P_{XY} . Therefore, the cross PSD $S_{xy}(\omega)$ of the process X(t) and Y(t) is defined in the limiting sense by

$$S_{xy}(\omega) = \lim_{T\to\infty} \frac{E[X^*_T(\omega)Y_T(\omega)]}{2T}$$

<u>Relation Between cross Power-spectral Density and Cross Correlation function of the Random Processes</u>

- We have PSD $S_{xy}(\omega) = \lim_{T\to\infty} \frac{E[X^*_T(\omega)Y_T(\omega)]}{2T}$
- $X^*_T(\omega) = \int_{-T}^{T} X(t).e^{j\omega t}.dt$ and $Y_T(\omega) = \int_{-T}^{T} Y(t).e^{-j\omega t}.dt$
- $S_{xy}(\omega) = \lim\limits_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} X(t_1).e^{j\omega t_1}.dt_1.\int_{-T}^{T} Y(t_2).e^{-j\omega t_2}.dt_2\right]$
- $= \lim\limits_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T}\int_{-T}^{T} X(t_1)\,Y(t_2).e^{-j\omega(t_2-t_1)}.dt_1 dt_2.\right]$
- $= \lim\limits_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E[X(t_1)\,Y(t_2)].e^{-j\omega(t_2-t_1)}.dt_1 dt_2$
- $= \lim\limits_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xy}(t_1,t_2)\,e^{-j\omega(t_2-t_1)}.dt_1 dt_2$
- Consider the inverse Fourier Transform of cross PSD i.e. $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\,e^{j\omega\tau}.d\omega$
- $F^{-1}[S_{xy}(\omega)] =$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xy}(t_1,t_2)\,e^{-j\omega(t_2-t_1)}.dt_1 dt_2]\,e^{j\omega\tau}.d\omega$$

- $= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xy}(t_1, t_2) \frac{1}{2\pi} . \int_{-\infty}^{\infty} e^{-j\omega(t_2 - t_1)} . d\omega . dt_1 dt_2$

- Since, $F[\delta(t)] = 1, \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 . e^{j\omega t} \, d\omega = \delta(t)$
- On similar lines, $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\tau - t_2 + t_1)} \, d\omega = \delta(\tau - t_2 + t_1)$

- $F^{-1}[S_{xy}(\omega)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xy}(t_1, t_2) \delta(\tau - t_2 + t_1) . dt_1 dt_2$

- $since\ \delta(\tau - t_2 + t_1) = 1\ at\ \ \tau - t_2 + t_1 = 0\ i.e.\ t_2 = \tau + t_1$

- $F^{-1}[S_{xy}(\omega)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xy}(t_1, \tau + t_1) \, dt_1$

- Let $t_1 = \tau \rightarrow dt_1 = d\tau$

- Hence, $F^{-1}[S_{xy}(\omega)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xy}(t, t + \tau) \, dt$
- The RHS of the above eq. is the time average of cross correlation function.
- Thus, Time average of cross-correlation function and the cross spectral density form a Fourier Transform Pair.
- If the processes X(t) and Y(t) are jointly WSS processes, the time average of $R_{xy}(t, t + \tau)$ will be $R_{xy}(\tau)$, since it is independent of time.
- Thus, for a two jointly WSS processes, cross-correlation and cross Spectral Density form a Fourier Transform Pair.
- $S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} \, d\tau$
- $R_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) \, e^{j\omega\tau} . d\omega$

Properties of cross Power Spectral Density:

- $S_{yx}(\omega) = S_{xy}(-\omega)$
- Real part of cross spectral density is an even function of ω and imaginary part is an odd function of ω
- $S_{xy}(\omega) = 0$ if X(t) and Y(t) are orthogonal
- $If\ X(t) and\ Y(t)$ are uncorrelated and of constant mean E(X) and E(Y) respectively, then , $S_{xy}(\omega) = 2\pi E(X)E(Y)\delta(\omega)$
- $\underline{Power\ Spectral\ density\ of\ sum\ of\ random\ processes}$

Considerthe random process $Z(t) = X(t) + Y(t)$ which is the sum of two jointly WSS random processes *X(t)* and *Y(t)*. We have,

$$R_{zz}(\tau) = E[Z(t).z(t+\tau)] = E[\{x(t) + y(t)\}\{x(t+\tau) + y(t+\tau)\}]$$

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + R_{xy}(\tau) + R_{yx}(\tau)$$

Taking the Fourier transform of both sides,

$$S_{zz}(\omega) = S_{xx}(\omega) + S_{yy}(\omega) + S_{xy}(\omega) + S_{yx}(\omega)$$

Since x(t) and y(t) are orthogonal, their cross spectral density is zero.

Hence,

$$S_{zz}(\omega) = S_{xx}(\omega) + S_{yy}(\omega)$$

(1) $S_{XY}(\omega) = S^*_{YX}(\omega)$

Note that $R_{XY}(\tau) = R_{YX}(-\tau)$

$$\begin{aligned}
\therefore S_{XY}(\omega) &= \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} R_{YX}(-\tau)e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} R_{YX}(\tau)e^{j\omega\tau}d\tau \\
&= S^*_{YX}(\omega)
\end{aligned}$$

(2) $\mathrm{Re}(S_{XY}(\omega))$ is an even function of ω and $\mathrm{Im}(S_{XY}(\omega))$ is an odd function of ω

We have

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)(\cos\omega\tau + j\sin\omega\tau)d\tau$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau)\cos\omega\tau d\tau + j\int_{-\infty}^{\infty} R_{XY}(\tau)\sin\omega\tau)d\tau$$

$$= \mathrm{Re}(S_{XY}(\omega)) + j\,\mathrm{Im}(S_{XY}(\omega))$$

where

$\mathrm{Re}(S_{XY}(\omega)) = \int_{-\infty}^{\infty} R_{XY}(\tau)\cos\omega\tau d\tau$ is an even function of ω and

$\mathrm{Im}(S_{XY}(\omega)) = \int_{-\infty}^{\infty} R_{XY}(\tau)\sin\omega\tau d\tau$ is an odd function of ω and

(3) $X(t)$ and $Y(t)$ are uncorrelated and have constant means, then

$$S_{XY}(\omega) = S_{YX}(\omega) = \mu_X \mu_Y \delta(\omega)$$

Observe that

$$\begin{aligned} R_{XY}(\tau) &= EX(t+\tau)Y(t) \\ &= EX(t+\tau)EY(t) \\ &= \mu_X \mu_Y \\ &= \mu_Y \mu_X \\ &= R_{XY}(\tau) \end{aligned}$$

$$\therefore S_{XY}(\omega) = S_{YX}(\omega) = \mu_X \mu_Y \delta(\omega)$$

(4) If $X(t)$ and $Y(t)$ are orthogonal, then

$$S_{XY}(\omega) = S_{YX}(\omega) = 0$$

If $X(t)$ and $Y(t)$ are orthogonal,

$$\begin{aligned} R_{XY}(\tau) &= EX(t+\tau)Y(t) \\ &= 0 \\ &= R_{XY}(\tau) \end{aligned}$$

$$\therefore S_{XY}(\omega) = S_{YX}(\omega) = 0$$

(5) The *cross power* P_{XY} between $X(t)$ and $Y(t)$ is defined by

$$P_{XY} = \lim_{T\to\infty} \frac{1}{2T} E \int_{-T}^{T} X(t)Y(t)dt$$

Applying Parseval's theorem, we get

$$P_{XY} = \lim_{T\to\infty} \frac{1}{2T} E \int_{-T}^{T} X(t)Y(t)dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} E \int_{-\infty}^{\infty} X_T(t)Y_T(t)dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} E \frac{1}{2\pi} \int_{-\infty}^{\infty} FTX_T^*(\omega)FTY_T(\omega)d\omega$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{EFTX_T^*(\omega)FTY_T(\omega)}{2T} d\omega$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)d\omega$$
$$\therefore\ P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)d\omega$$

Similarly,

$$P_{YX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega)d\omega$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}^*(\omega)d\omega$$
$$= P_{XY}^*$$

Illustrative Problems:

1.A random process is defined as $Y(t) = X(t).Cos(\omega_0 t + \theta)$, where $X(t)$ is a WSS process, ω_0 is a real constant and θ is a uniform random variable over$(0,2\pi)$ an is independent of X(t). Find the PSD of Y(t).

Soln.:$R_{yy}(\tau) = E[Y(t).Y(t+\tau)] = E[X(t).Cos(\omega_0 t + \theta).X(t+\tau).Cos(\omega_0(t+\tau)+\theta)]$

Since, θ and X(t) are independent of each other,

$$R_{yy}(\tau) = E[X(t).X(t+\tau)]E[Cos(\omega_0 t + \theta).Cos(\omega_0(t+\tau)+\theta)]$$

$$=\frac{1}{2} R_{xx}(\tau)Cos(\omega_0\tau)$$

PSD of Y(t) is Fourier Transform of $R_{yy}(\tau)$. i.e

$$.F\left\{\frac{1}{2} R_{xx}(\tau)Cos(\omega_0\tau)\right\} = \frac{\pi}{2}[S_{xx}(\omega+\omega_0) + S_{xx}(\omega+\omega_0)]$$

2.A random process is given by $Z(t) = A.X(t) + B.Y(t)$, where A and B are real constants and X(t) and Y(t) are jointly WSS processes.

(i) Find the Power spectrum of Z(t) (ii) Find the cross power spectrum $S_{XZ}(\omega)$

Soln.:

$$(i) R_{zz}(\tau) = E[Z(t).Z(t+\tau)] = E[\{AX(t) + BY(t)\}.[\{AX(t+\tau) + BY(t+\tau)\}.]]$$

$$= A^2 R_{xx}(\tau) + ABR_{xy}(\tau) + ABR_{yx}(\tau) + B^2 R_{yy}(\tau)$$

Power spectrum of Z(t) is $S_{zz}(\omega) = F[R_{zz}(\tau)]$

$$= A^2 S_{xx}(\omega) + B^2 S_{yy}(\omega) + ABS_{xy}(\omega) + ABS_{yx}(\omega)$$

(ii)$S_{XZ}(\omega) = F[R_{xz}(\tau)]$

$R_{xz}(\tau) = E[X(t).Z(t+\tau)] = E[X(t)\{A.X(t+\tau) + B.Y(t+\tau)\}]$

$= A.E[X(t)X(t+\tau)] + BE[X(t).Y(t+\tau)] = AR_{xx}(\tau) + B.R_{xy}(\tau)$

$$S_{XZ}(\omega) = AS_{XX}(\omega) + BS_{XY}(\omega)$$

3.A stationary random process X(t) has a spectral density $S_{xx}(\omega) = \frac{16}{\omega^2+16}$ and an independent stationary Y(t) has a spectral density $S_{yy}(\omega) = \frac{\omega^2}{\omega^2+16}$. Assuming X(t) and Y(t) are of zero mean, find the (i) PSD of U(t)=X(t)+Y(t) (ii) $S_{XY}(\omega)$ and $S_{XU}(\omega)$

Soln.:

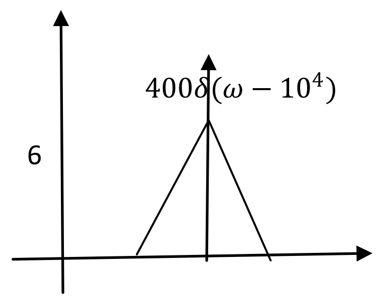
(i) PSD of U(t)= PSD of X(t) +PSD of Y(t) =1

(ii)$S_{XY}(\omega) = F[R_{xy}(\tau)] = F[E\{X(t).Y(t+\tau)\}] = F[E\{X(t)\}.E\{Y(t+\tau)\}] = 0$

$$S_{XU}(\omega) = F[R_{xU}(\tau)] = F[E\{X(t).U(t+\tau)\}] = F[E\{X(t).\{X(t+\tau) + Y(t+\tau)\}] =$$

$F[E\{X(t)\}.X(t+\tau)\} + E\{X(t)Y(t+\tau)\}] = F[R_{xx}(\tau)] + F[R_{xy}(\tau)] = S_{xx}(\omega) = \frac{16}{\omega^2+16}$

4.The PSD of a real process X(t) for positive frequencies is shown below:



The spectrum extends from $\omega = 9000\ rad\ to\ 11000\ rad$, centered at 10000rad. Find the Mean and MS value of X(t).

Soln.:

MS value is the area under the PSD curve. Hence,

$E[X^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega).d\omega = \frac{2}{2\pi}\int_0^{\infty} S_{xx}(\omega).d\omega, as\ PSD\ is\ an\ even\ function\ of\ frequency$

$$\frac{1}{\pi}\left[400 + 2x\frac{1}{2}x1000x6\right] = \frac{6400}{\pi}$$

Since, the power spectrum has no dc component, the dc component of the corresponding signal will have zero dc value. Hence, E[X(t)]=0.

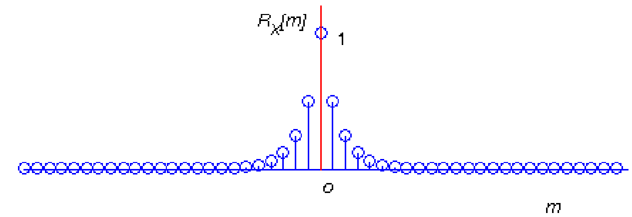
5. $R_X[m] = 2^{-|m|} \qquad m = 0, \pm1, \pm2, \pm3....$ Then

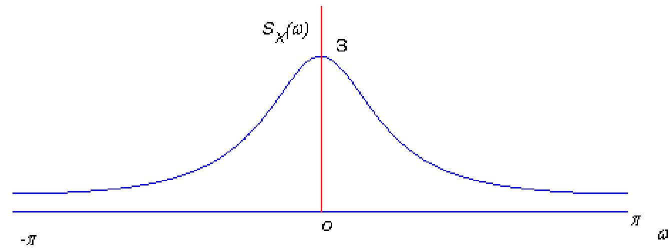
$$S_X(\omega) = \sum_{m=-\infty}^{\infty} R_X[m]e^{-j\omega m}$$

$$= 1 + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{1}{2}\right)^{|m|} e^{-j\omega m}$$

$$= \frac{3}{5 - 4\cos\omega}$$

The plot of the autocorrelation sequence and the power spectral density is shown in Fig. below.





Exercise Problems:

1.X(t) is WSS process with a PSD of $S_X(f)$. Find the PSD of Y(t)=X(2t-1).

2.The PSD of a real stationary random process X(t) is given by $S_X(f) = \begin{matrix} \frac{1}{W}\ for\ |f| \leq W \\ 0\ for\ |f| > W \end{matrix}$.

Then, find $E\left[\pi X(t)X(t - \frac{1}{4W})\right]$

3.Two random processes are given as $X(t) = Z_1(t) + 3Z_2(t-\tau)\ and\ \ Y(t) = 3Z_1(t-\tau) + Z_2(t+\tau)$

Where $Z_1(t)\ and\ Z_2(t)$ are independent white noise processes of zero mean and variance of 0.5. Find the autocorrelation of X(t), Y(t) and their cross correlation.

4. A real band limited random process X(t) has two sided PSD given by

$$S_x(f) = \begin{Bmatrix} \frac{10^{-6}(3000 - |f|\}watts}{Hz} for\ |f| \leq 3KHz \\ 0\ other\ wise \end{Bmatrix}$$

Where f is measured in Hz. The signal X(t) modulates a carrier cos16000πt and the resultant signal is passed through an ideal BPF of unity gain with centre frequency of 8KHz and bandwidth of 2KHz. Find the output power.

5.$X(t) = A.Cos(\omega_o t + \theta) and\ Y(t) = Z(t).Cos(\omega_o t + \theta)$ are two random processes, where A and ω_o are real positive constants. θ is a random variable and independent of Z(t), which is a random process with a constant mean $\bar{Z}$. Find the Cross spectral density of X(t) and Y(t).

&&&&&&&&