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Advanced Statistics II – Summer Term 2017

Problem Set 3

1. (Home Assignment 2)
   The random sample $\{X_i, i : 1 \to n\}$ is drawn from an exponential distribution with pdf
$$f(x;\theta) = \frac{1}{\theta}\exp\{-\frac{x}{\theta}\} \cdot I_{(0,\infty)}(x)\ , \quad \theta > 0\,.$$
   The following statistics are considered as estimators for θ:
$$T_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad T'_n = \left[\frac{1}{2n}\sum_{i=1}^{n} X_i^2\right]^{1/2} .$$
   (a) Write a function for each of the estimators.
   (b) Write a function that computes bias, variance and MSE for an given estimator. (Hint: This function should return all of the three quantities!)
   (c) **Use your function from part b)** to discuss consistency in terms of MSE for each estimator. Therefore conduct a simulation study for the following parameters: $MC = 100$, $n = 1, \cdots, 100$, $\theta = 0.1,\ 1,\ 5,\ 25$. Plot the MSE for each θ and estimator as a function of n. Arrange all graphs in one single figure. Save this figure as a pdf. Name all axes appropriately, add a meaningful legend and indicate in each graph to which parameter combination it refers. Which of the estimators is consistent? How does the MSE depend on n, θ?
   (d) Compare now the variances of both estimators for each θ. Plot the individual variances against n. Arrange all graphs in one single figure. Save this figure as a pdf. Name all axes appropriately, add a meaningful legend and indicate in each graph to which parameter combination it refers. Which estimator has the lower variance for small ($n < 10$) and/or for large $n > 75$?

2. Consider the following estimators $\hat{\mu}_X^{(j)}$ for the mean of a normally distributed population with $X \sim N(\mu, \sigma^2)$ and a random sample $\{X_i, i : 1 \to n\}$, where $3 < n < \infty$
$$\hat{\mu}_X^{(1)} = \frac{1}{n-1}\sum_{i=1}^{n} X_i; \quad \hat{\mu}_X^{(2)} = \frac{1}{3}\sum_{i=1}^{3} X_i; \quad \hat{\mu}_X^{(3)} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$
   (a) Write a function for each of the estimators (see PS 2).
   (b) Write a function that computes the CRLB for an (unbiased estimator) for μ.
   (c) Conduct a simulation study for the following parameters: $MC = 1000$, $n = 3, \cdots, 100$, $\mu = 0,\ 1,\ 5$, $\sigma = 1,\ 2,\ 5$. Plot the simulated variance for each estimator as a function of n as well as for each possible parameter combination. Add the CRLB to your plot. Arrange all graphs on one single figure. Save this figure as a pdf. Name all axes appropriately, add a meaningful legend and indicate in each graph to which parameter combination it refers. Repeat this procedure for every estimator. How does the variance depend on n, μ, σ? Is the Cramer-Rao variance inequality valid? Which of the three estimators (if any) reaches the CLRB?

Home Assignment 3 (2 points)

Consider the random sample $\{X_i, i : 1 \to n\}$ from density

$$f(x;\theta) = \frac{1}{\theta}\exp\{-\frac{x}{\theta}\} \cdot I_{(0,\infty)}(x)\ , \quad \theta > 0\,.$$

a) Write a function for the Cramér-Rao Lower Bound for the variance of unbiased estimators for θ, $1/\theta$ and $p(X > c)$.

b) Write a function for each of the estimators derived in class.

   Hint: Use $\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_{(c,\infty)}(x_i)$ as an estimator for $p(X > c)$.

c) Conduct a simulation study for the following parameters: $MC = 1000$; $n = 1, \cdots, 100$; $\theta = 0.1,\ 1,\ 2,\ 5$; $c = 1,\ 10$. Plot the simulated variance for each estimator as a function of n as well as for every θ (and c, for the last estimator). Add the CRLB to your plot. Arrange all graphs on one single figure. Save this figure as a pdf. Name all axes appropriately, add a meaningful legend and indicate in each graph to which parameter combination it refers. Use the estimators derived in class. For every estimator you have to create a pdf file.

d) Is the Cramer-Rao variance inequality valid? Which of the estimators reaches the CRLB? Does this depend on θ, c?

You are only allowed to use functions (and packages) that are part of basic R. Please use the provided script file for the first home assignment and fill out the header! Hand in your script file (with the necessary comments) until 30.5.17 8pm via mail.