Computational Finance: T-Exercise 07 solution

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a) To represent some smooth function $f_i(x) := g(x)$ as an Itô process, we need to apply Itô's formula for continuous processes:

$$df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))d[X,X](t)$$

In the exercise X is some Brownian motion with drift $\mu \in \mathbb{R}$ and diffusion coefficient σ i.e. $X(t) = \mu + \sigma W(t)$ where W denotes Standard Brownian Motion. Applying Itô's formula as stated above to X, yields:

$$df(X(t)) = f'(X(t))\mu dt + f'(X(t))\sigma dW(t) + \frac{1}{2}f''(X(t))\sigma^2 dt$$
$$df(X(t)) = [f'(X(t))\mu + \frac{1}{2}f''(X(t))\sigma^2]dt + f'(X(t))\sigma dW(t)$$

1. Representation of $f_1(x) := x^3$ as an Itô process:

   $f_1'(x) = 3x^2 \qquad f_1''(x) = 6x$
   $d(f_1(X(t))) = (3X^2(t)\mu + 3X(t)\sigma^2)dt + 3X^2(t)\sigma dW(t)$

2. Representation of $f_2(x) := exp(x)$ as an Itô process:

   $f_2'(x) = exp(x) \qquad f_2''(x) = exp(x)$
   $d(f_2(X(t))) = (exp(X(t))\mu + \frac{1}{2}exp(X(t))\sigma^2)dt + exp(X(t))\sigma dW(t)$

3. Representation of $f_3(x) := 6x + 2$ as an Itô process:

   $f_3'(x) = 6 \qquad f_3''(x) = 0$
   $d(f_3(X(t))) = 6\mu dt + 6\sigma dW(t)$

b) Beacause quadratic variation of processes with finite variation is equal zero and quadratic variation of process with finite variation and a stochastic process is equal zero ($dtdt = dtdW(t) = dW(t)dt = 0$ and $dW(t)dW(t) = dt$). Therefore the only relevant part is $\sigma dW(t)$ since $W(t)$ is a stochastic process.

Quadratic variation processes of $[f_i(X)]_t$ for $i = 1, 2, 3$ are:

1. Quadratic variation process of $f_1(x) := x^3$

   $$d[f_1(X), f_1(X)]_t = [3\sigma X^2(t)]^2 dt = 9\sigma^2 X^4(t)t$$

2. Quadratic variation process of $f_2(x) := exp(x)$

   $$d[f_2(X), f_2(X)]_t = \Big(exp(X(t))\sigma\Big)^2 dt = exp\Big(2X(t)\Big)\sigma^2 t$$

3. Quadratic variation process of $f_3(x) := 6x + 2$

   $$d[f_3(X), f_3(X)]_t = (6\sigma)^2 t = 36\sigma^2 dt$$

Similarly as above: Covariance process of $[f_1(X), f_3(X)]_t$:

$$d[f_1(X), f_3(X)]_t = 3\sigma X^2(t) * 6\sigma dt = 18\sigma^2 X^2(t)dt$$