Computational Finance: T-Exercise 08 solution

// Group 3 // Alex Kukuk // Anton Gorev // Mykhaylo Cherner

For representation of d-dimensional functions as Itô processes it is needed to apply Itô's formula of the form:

$$df(X(t)) = \sum_{i=1}^{d} \partial_i f(X(t))dX(t) + \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij} f(X(t))d[X_i, X_j](t)$$

a) However, it might be, that the actual task is to represent $X(t) = tW(t)$ as an Itô process since it was not so obvious, here is also representation of $dX(t)$:

$$\partial_1 tW(t) = W(t) \quad \partial_2 tW(t) = t \quad \partial_{11} = \partial_{22} = 0$$

$$\begin{aligned} d(tW(t)) &= \partial_1 tW(t)dt + \partial_2 tW(t)dW(t) \\ d(tW(t)) &= W(t)dt + tdW(t) \end{aligned}$$

b) Representation of $Y(t) = \frac{W(t)}{1+t}$ as an Itô process:

$$\partial_1 Y(t) = (1+t)^{-1} \quad \partial_2 Y(t) = -\frac{W(t)}{(1+t)^2}$$

$$\partial_{11} Y(t) = 0 \quad \partial_{22} Y(t) = 2\frac{W(t)}{(1+t)^3}$$

$$d[t,t] = 0, \text{since t is deterministic}$$

$$dY(t) = (1+t)^{-1}dW(t) - \frac{W(t)}{(1+t)^2}dt$$

It is also possible to make use of the IBP (integretion by parts):

$$dY_t = W_t d(1+t)^{-1} + \frac{1}{1+t}dW_t + d[t, W]_t$$

It is possible to take a derivative of $(1+t)^{-1}$ in respect to dt and rearange the equation such that: $d(1+t)^{-1} = -(1+t)^{-2}dt$, consequently:

$$dY(t) = (1+t)^{-1}dW(t) - \frac{W(t)}{(1+t)^2}dt$$

Computational Finance: T-Exercise 09 solution

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a) $E(t) := \frac{1}{D(t)}$ is obviously a function of D(t) and therefore we can apply Itô's formula to represent process E as a Itô process:

$$
\begin{aligned}
dE(t) &= -\frac{1}{D^2(t)}dD(t) + \frac{1}{2}\frac{2}{D^3(t)}d[D,D](t) \\
dE(t) &= -\frac{1}{D^2(t)}\Big(D(t)\mu dt + D(t)\sigma dW(t)\Big) \\
&\quad + \frac{1}{D^3(t)}\Big(D(t)\mu dt + D(t)\sigma dW(t)\Big)^2 \\
dE(t) &= -\frac{1}{D(t)}\Big(\mu dt + \sigma dW(t)\Big) \\
&\quad + \frac{1}{D^3(t)}\Big((D(t)\mu dt)^2 + 2D(t)\mu dt D(t)\sigma dW(t) + (D(t)\sigma dW(t))^2\Big) \\
dE(t) &= -\frac{1}{D(t)}\Big(\mu dt + \sigma dW(t)\Big) + \frac{1}{D(t)}\sigma^2 dt \\
dE(t) &= \frac{1}{D(t)}\Big(\sigma^2 - \mu\Big)dt - \frac{1}{D(t)}\sigma dW(t) \\
dE(t) &= E(t)\Big(\sigma^2 - \mu\Big)dt - E(t)\sigma dW(t)
\end{aligned}
$$

b) To compute $D(t)$ via the stochastic exponential we can represent $dD(t)$ as $D(t)dY(t)$ where $Y(t) = \mu t + \sigma W(t)$ and applying formula (3.11) from the script, we get:

$$
\begin{aligned}
D(t) &= D(0)\mathcal{E}(Y)(t) = D(0)exp\Big((\mu t + \sigma W(t)) - \frac{1}{2}[Y,Y](t)\Big) \\
D(t) &= D(0)\mathcal{E}(Y)(t) = D(0)exp\Big((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\Big)
\end{aligned}
$$

To compute $E(t)$ via the stochastic exponential we represent $dE(t)$ as $E(t)dX(t)$ with $X(t) = (\sigma^2 - \mu)t - \sigma W(t)$, applying formula (3.11):

$$
\begin{aligned}
E(t) &= E(0)\mathcal{E}(X)(t) = E(0)exp\Big((\sigma^2 - \mu)t - \sigma W(t) - \frac{1}{2}[X,X](t)\Big) \\
E(t) &= E(0)\mathcal{E}(X)(t) = E(0)exp\Big((\frac{1}{2}\sigma^2 - \mu)t - \sigma W(t)\Big)
\end{aligned}
$$

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c) Pluging in $\mu = \frac{1}{2}\sigma^2$ in $D(t)$ and $E(t)$ gives us following equations:

$$D(t) = D(0)exp(\sigma W(t)) \quad \text{and} \quad E(t) = E(0)exp(-\sigma W(t))$$

Drift is not present in both equations anymore and evolution of $D(t)$ and $E(t)$ is determined by diffusion coefficient. Since the diffusion coefficient is in the equations with different signs, evolution of US-Dollar and Euro is always going in opposite direction. When US-Dollar depreciates then Euro appreciates and vice versa.
Expected Values of US-Dollar and Euro are the same and therefore no currency has an advantage over the other!