Computational Finance: T-Exercise 13 solution

// Group 3 // Alex Kukuk // Anton Gorev // Mykhaylo Cherner

a) 1. To represent logarithmic stock process $X(t) := f(S(t)) = log(S(t)$ as an Itô process, we need to apply Itô's formula for continuous processes:

$$df(S(t)) = f'(S(t))dS(t) + \frac{1}{2}f''(S(t))d[S,S](t)$$

First dirivative: $f'(S(t)) = \frac{1}{S(t)}$, second derivative: $f''(S(t)) = -\frac{1}{S^2(t)}$ and the quadratic covariation process of $dS(t)$ is given by: $d[S,S](t) = \sigma^2 S^2(t)dt$, consequently: $dX(t) := f(S(t)) = dlog(S(t)) =$

$$\begin{aligned} dX(t) &= \frac{1}{S(t)}\Big(\mu S(t)dt + \sigma S(t)dW(t)\Big) + \frac{1}{2}\Big(-\frac{1}{S^2(t)}\Big)\sigma^2 S^2(t)dt \\ dX(t) &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t) \end{aligned}$$

Furthermore it is possible to represent $dX(t)$ as $X(t) - X(0)$, substituting then $X(t)$ by $log(S(t))$ yields:

$$log(S(t)/S(0)) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t)$$
$$S(t) = S(0)exp\Big((\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t)\Big)$$

Since μ and σ do not depend on t it is possible to represent this process as follows:

$$S(t) = S(0)exp\Big((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\Big)$$

2. Quadratic variation process of $X(t)$ is:

$$d[X,X](t) = \sigma^2 dt$$

beacause quadratic variation of processes with finite variation is equal zero and quadratic variation of process with finite variation and a stochastic process is equal zero $\Big(dtdt = dtdW(t) = dW(t)dt = 0$ and $dW(t)dW(t) = dt\Big)$. Therefore the only relevant part is $\sigma dW(t)$ since $W(t)$ is a stochastic process.

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b) $V_t(\phi) = \phi_t^0 B_t + \phi_t^1 S_t$, substituting ϕ_t^1 by $\phi_t^1 = \frac{V_t(\phi)}{2S_t}$, as given in the exercise, yields: $\phi_t^0 = \frac{V_t(\phi)}{2B_t}$ and by self financing property of the portfolio we obtain:

$$\begin{aligned} dV_t(\phi) &= \phi_t^0 dB_t + \phi_t^1 dS_t \\ dV_t(\phi) &= \phi_t^0 r B_t dt + \phi_t^1 \Big(\mu S_t dt + \sigma S_t dW_t\Big) \end{aligned}$$

Substituting in the equation above obtained ϕ_t^0 and ϕ_t^1:

$$dV_t(\phi) = V_t(\phi)\frac{1}{2}\Bigg(\Big(\mu + r\Big)dt + \sigma dW_t\Bigg)$$

To obtain $V_t(\phi)$ via the stocahstic exponential we can represent $dV_t(\phi)$ as $V_t(\phi)dY(t)$ with $Y(t) = \frac{1}{2}\Bigg(\Big(\mu + r\Big)t + \sigma W_t\Bigg)$, appying formula (3.11):

$$V_t(\phi) = V_0(\phi)\mathcal{E}(Y)(t) = exp\Bigg(\frac{1}{2}\Bigg(\Big(\mu + r - \frac{1}{4}\sigma^2\Big)t + \sigma W_t\Bigg)\Bigg), \quad V_0(\phi) = 1$$

And therefore value process $V_t(\phi)$ is a geometric Brownian motion.