// Group 3 // Anton Gorev // Mykhaylo Cherner

Computational Finance: T-Exercise 14 solution

Given model can be represented as: $V(t)(\phi) = \phi_0(t)S_0(t) + \phi_1(t)S_1(t) + \phi_2(t)S_2(t)$ and after discounting: $\hat{V}(\phi) = \phi_0 + \phi_1 S_1(t) + \phi_2 S_2(t)$ since $S_0(t) = 1$,
by self-financing property and substituting $dS_1(t)$ and $dS_2(t)$ in the equation we get:

$$d\hat{V}(\phi) = \phi_1 S_1(t)\big(3dt + dW_1(t) - dW_2(t)\big) + \phi_2 S_2(t)\big(1dt - dW_1(t) + dW_2(t)\big)$$

To eliminate risk and create and arbitrage we need to show, that it is possible to start of with a portfolio value of zero at time zero and be able to have a portfolio of bigger than zero at sime time $t > 0$. Therefore I suggest to set ϕ in such a way, that the amount invested in the second stock is half of the amount of the difference of the portfolio value and amount of money invested in the bond (S_0). This is intuitive if we set value of the portfolio to zero, we will be able to invest in the second stock only the amount that we borrow from shortin the bond:

$$\phi_2 = \frac{V(\phi) - \phi_0}{2S_2(t)}$$

By substituting ϕ_2 into the first equation we get ϕ:

$$\phi_1 = \frac{V(\phi) - \phi_0}{2S_1(t)}$$

finally substituting everything in $d\hat{V}(\phi)$ we get:

$$\begin{aligned} d\hat{V}(\phi) &= \frac{V(\phi) - \phi_0}{2}\big(3dt + dW_1(t) - dW_2(t)\big) + \frac{V(\phi) - \phi_0}{2}\big(1dt - dW_1(t) + dW_2(t)\big) \\ &= (V(\phi) - \phi_0)2dt \end{aligned}$$

Initial portfolio value must be EQUAL zero. When we to set the initial portfolio value to zero and $\phi_0 < 0$, then it is obvious that the amount of money, that we borrow on the market by shorting the bond, will grow positively and since in this setting bond is set to one and is independent of t, portfolio value will increase by the difference of $-\phi_0 2dt + \phi_0$. Therefore there is some "free money" in the market, namely arbitrage.

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Computational Finance: T-Exercise 15 solution

Price of an option at time t is given by: $V(t) = B(t)E_Q(f(S(T))/B(T)|\mathcal{F}_t)$
at time t_0 price of $V(t_0)$ is:

$$V(t_0) = S(t_0)\Big[\Phi(A) - e^{-r(T-t_0)}\Phi(B)\Big]$$

$$\begin{aligned} A &= \left(\frac{log\left(\frac{S(t_0)}{S(t_0)}\right) + r(T-t_0) + \frac{\sigma^2}{2}(T-t_0)}{\sigma\sqrt{T-t_0}}\right) \\ &= \left(\frac{(r+\frac{\sigma^2}{2})\sqrt{T-t_0}}{\sigma}\right) \\ B &= \left(\frac{log\left(\frac{S(t_0)}{S(t_0)}\right) - r(T-t_0) + \frac{\sigma^2}{2}(T-t_0)}{\sigma\sqrt{T-t_0}}\right) \\ &= \left(\frac{(r-\frac{\sigma^2}{2})\sqrt{T-t_0}}{\sigma}\right) \end{aligned}$$

(i) at time $t < t_0$ we get:

$$E\big[V(t_0)|\mathcal{F}_t\big] = E\big[S(t_0)|\mathcal{F}_t\big]\Big[\Phi(A) - e^{-r(T-t_0)}\Phi(B)\Big]$$

since $S(t_0) = S(t)exp\Big[(r-\frac{\sigma^2}{2})(t_0-t) + \sigma\sqrt{t_0-t}x\Big]$ we compute:

$$\begin{aligned} E\big[S(t_0)|\mathcal{F}_t\big] &= \int_{-\infty}^{+\infty} S(t)exp\Big[(r-\frac{\sigma^2}{2})(t_0-t) + \sigma\sqrt{t_0-t}x\Big]\frac{1}{\sqrt{2\pi}}exp\big[-\frac{x^2}{2}\big]dx \\ &= S(t)exp\Big[(r-\frac{\sigma^2}{2})(t_0-t)\Big]e^{\sigma^2(t_0-t)/2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}exp\big[-\frac{1}{2}(x-\sigma\sqrt{t_0-t})^2\big] \\ &= S(t)exp\Big[(r(t_0-t)\Big] \end{aligned}$$

Finaly we can write down Black-Scholes price of a forward start call

$$V(t) = e^{r(t_0-t)}V(t_0) = v(t, S(t)) = S(t)e^{r(t_0-t)}\Big[\Phi(A) - e^{-r(T-t_0)}\Phi(B)\Big]$$

(ii) at time $t \geq t_0$: this is the regular BS-Formula with $K := S(T_0)$ and d1,d2 as specified in the lecture (3.23), namely:

$$V(t) = v(t, S(t)) = S(t)\Phi(d1) - Ke^{-r(T-t)}\Phi(d2)$$

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Computational Finance: T-Exercise 15 solution

To find out the replicating strategy $\phi = (\phi_0, \phi_1)$ of a forward call, we use a formula from the lecture notes: $\phi_1(t) = \partial_2 v(t, S(t))$ and $\phi_0(t) = \frac{v(t,S(t))-\phi_1(t)}{B(t)}$.
We have already calculated $v(t, S(t))$ for a forward call option and therefore we can just plug in $v(t, S(t))$ in the equations for ϕ_0 and ϕ_1 above and compute ϕ_0 and ϕ_1 for a forward call option:

For $t < t_0$:

$$\phi_1(t) = \frac{v(t, S(t))}{S(t)}$$

$$\phi_0(t) = 0$$

For $t \geq t_0$:

$$\phi_1(t) = \Phi(d_1)$$

$$\phi_0(t) = -S(t_0)exp(-r(T-t)\Phi(d1))$$