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# Computational Finance: T-Exercise 19 solution

Characteristic function $\mathcal{X}(u)$ of $X(t) = X_0 + \mu t + \sigma W(t) + \sum_{j=1}^{N(t)} Y_j$ can be obtained in the following way:

$$\begin{aligned}
\mathcal{X}(u) &= E\big[e^{iuX(t)}\big] \\
&= E\Big[exp\Big(iu\big(X_0 + \mu t + \sigma W(t) + \sum_{j=1}^{N(t)} Y_j\big)\Big)\Big] \\
&= e^{iu(X_0+\mu t)} E\big[e^{iu\sigma W(t)}\big] E\big[e^{\sum_{j=1}^{N(t)} Y_j}\big]
\end{aligned}$$

Since $X_0$ and $\mu t$ are not stochastic processes. And $E(WY) = E(W)E(Y)$ if W and Y are independent.

$$\mathcal{X}(u) = e^{iu(X_0+\mu t)} e^{(-u^2\sigma^2 t/2)} E\big[\sum_{n=0}^{\infty} \mathcal{I}_{\{N(t)=n\}} exp\big(\sum_{j=1}^{n} Y_j\big)\big]$$

Given properties a) and b) from the exercise.

$$\begin{aligned}
\mathcal{X}(u) &= e^{iu(X_0+\mu t)} e^{(-u^2\sigma^2 t/2)} E\big[\sum_{n=0}^{\infty} \mathcal{I}_{\{N(t)=n\}}\big] \prod_{j=1}^{n} E\big[e^{Y_j}\big] \\
&= e^{iu(X_0+\mu t)} e^{(-u^2\sigma^2 t/2)} \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \prod_{j=1}^{n} e^{ium - u^2 s^2/2}
\end{aligned}$$

As N(t) is Poisson random variable with Parameter $\lambda t$. And $Y_1, Y_2, ...$ are normally distributed with mean m and variance $s^2$.

$$\begin{aligned}
\mathcal{X}(u) &= e^{iu(X_0+\mu t)} e^{(-u^2\sigma^2 t/2)} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t e^{ium - u^2 s^2/2})^n}{n!} \\
&= e^{iu(X_0+\mu t)} e^{(-u^2\sigma^2 t/2)} exp\big[(e^{ium - s^2u^2/2} - 1)\lambda t\big] \\
&= exp\Big[iu(X_0 + \mu t) + \lambda t[e^{ium - u^2 s^2/2} - 1] - u^2\sigma^2 t/2\Big]
\end{aligned}$$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, I guess Hint b) was meant this way and the sum of the indicator function should start at $n = 0$ and not $n = 1$, therefore this property of the exponential function was used. Alternatively it is possible to take the charecteristic function of the Poisson distribution $M_X(t) = e^{\lambda(e^t - 1)}$, substitute t with $Y_1, Y_2, ...$ and $\lambda$ with $\lambda t$. Same result is calculated, other steps are the same.