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Computational Finance: T-Exercise 23 solution

a) We are looking for the CDF of X

$$\begin{aligned}&X_1, ..., X_m\\&Z \text{ is a r.v.}\\&X = X_z = \sum X_{i\{z=i\}}\end{aligned}$$

By Law of total probability: $P(x \leq t) = \sum_{i=1}^{m} P(x \leq t|z = i)P(z = i)$ and $P(z = i) = \omega_i$, therefore:

$$\begin{aligned}F_x(t) &= P(x \leq t) = \sum_{i=1}^{m} P(x \leq t|z = i)P(z = i)\\&= \sum_{i=1}^{m} \omega_i P(x \leq t)\\&= \sum_{i=1}^{m} \omega_i F_i(x)\end{aligned}$$

b) First note: $F_i(t) =: \sum_{i=1}^{m} f_i(t) = \sum_{i=1}^{m} \omega_i f_i(x)$, so then:

$$\begin{aligned}\mathbb{E}\Big[(X-\mu)^j\Big] &= \int_{-\infty}^{+\infty} (x-\mu)^j \sum_{i=1}^{m} \omega_i f_i(x) dx\\&= \sum_{i=1}^{m} \omega_i \int_{-\infty}^{+\infty} (x-\mu)^j f_i(x) dx\\&= \sum_{i=1}^{m} \omega_i \mathbb{E}\Big[(x_i-\mu)^j\Big]\end{aligned}$$

Now we expand the given equation by $(-\mu_i + \mu_i)$:

$$= \sum_{i=1}^{m} \omega_i \mathbb{E}\Big[(X_i - \mu_i + \mu_i - \mu)^j\Big]$$

By Binomial Formula: $(a+b)^j = \sum_{k=0}^{j} \binom{j}{k} a^{j-k} b^k$ we get:

$$= \sum_{i=1}^{m} \omega_i \mathbb{E}\Big[\sum_{k=0}^{j} \binom{j}{k} (\mu_i - \mu)^{j-k} (X_i - \mu_i)^k\Big]$$

Finally we take $\sum_{k=0}^{j} \binom{j}{k} (\mu_i - \mu)^{j-k}$ out of the expactation:

$$= \sum_{i=1}^{m} \sum_{k=0}^{j} \binom{j}{k} \omega_i (\mu_i - \mu)^{j-k} \mathbb{E}\Big[(X_i - \mu_i)^k\Big]$$