Advanced Structural Analysis

Stiffness Matrix

This lecture notes discusses the following :

① Definition of stiffness.

② Types of Displacements

③ Stiffness matrix formation.

(i) For a beam element using

(ii) For a plane frame

④ Problems on analysis of Beams and frames using stiffness matrix.

Stiffness Definition:

It is a force or moment required to produce a unit deflection or rotation.

Consider a structural element with a single degree of freedom.

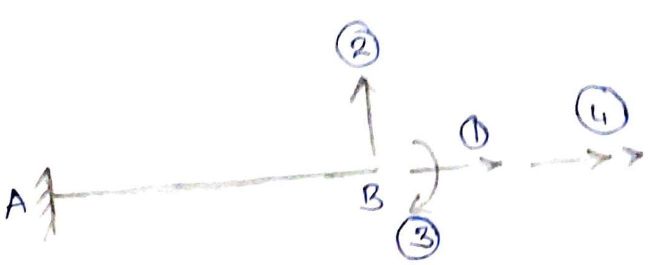
The spring AB shown in the figure, is fixed at end A and has a single degree of freedom at end B along coordinate 1.

The stiffness of the spring is defined as the force k_{11} required for a unit displacement at coordinate 1.

$$\boxed{\text{Stiffness} = \frac{P_1}{\Delta_1} = k_{11}}$$

Types of Displacement

Consider a structural element with multiple degrees of freedom.



The structural member AB of uniform cross-section, is fixed at end A. The end B will have the following 4 types of displacements.

① Axial displacement Δ_1 at coordinate 1.

If a axial force P_1 is applied at coordinate 1, displacement Δ_1 at coordinate 1 is given by the equation

$$\Delta_1 = \frac{P_1 L}{AE}$$

where, L = length of the member

A = Cross-sectional area of the member

E = modulus of elasticity.

By definition, the axial stiffness is the force required for unit displacement along coordinate 1.

if $\Delta_1 = 1$

$$1 = \frac{P_1 L}{AE}$$

$$P_1 = \frac{AE}{L}$$

$\therefore$ Axial stiffness, $k_{11} = \frac{AE}{L}$

In the case of rigid-jointed frames, the axial displacements are small as compared to transverse displacements. Hence, in the analysis of rigid jointed frames the members are considered to be inifinitely stiff w.r.t axial displacements.

Transverse Displacement :-

The force P_2 required at coordinate 2 for displacement Δ_2 at coordinate 2 without any displacement at coordinate 1, 3 and 4 is given by the equation.

$$P_2 = \frac{12EI\,\Delta_2}{L^3}$$

Hence by definition, the stiffness with respect to transverse displacement may be written as

$$\text{transverse stiffness, } K_{22} = \frac{12EI}{L^3}$$

The above equation is based on the assumption that end A, known as the far end is fixed.

If far-end A is hinged, the force P_2 required at coordinate 2 for a displacement Δ_2 at coordinate 2 without any displacement at coordinates 1, 3 and 4 is given by

$$P_2 = \frac{3EI\,\Delta_3}{L^3}$$

Hence, by definition, the stiffness with respect to transverse displacement may be written as

$$\text{transverse stiffness, } K_{22} = \frac{3EI}{L^3}$$

Bending or Flexural Displacement :-

The force P_3 required at coordinate 3 for displacement Δ_3 at coordinate 3 without any displacement at coordinates 1, 2 and 4 is given by the equation.

$$P_3 = \frac{4EI\Delta_3}{L}$$

By definition, the stiffness w.r.t flexural displacement may be written as

$$\text{flexural stiffness}, \ k_{33} = \frac{4EI}{L}$$

The above equation is based on the assumption that far-end A is fixed. If far-end A is hinged, the force P_2 required at coordinate 3 for a displacement Δ_3 at coordinate 3 without any displacement at coordinates 1, 2 and 4 is given by

$$P_3 = \frac{3EI\Delta_3}{L}$$

By definition, the stiffness with respect to flexural displacement may be written as

$$\text{flexural stiffness}, \ k_{33} = \frac{3EI}{L}$$

Torsional Displacement or Twist :

From the equation of torsion, the angle of twist Δ_4 due to the torque P_4 is given by the equation

$$\Delta_4 = \frac{P_4 L}{GK}$$

where G = shear modulus of elasticity

K = torsion constant

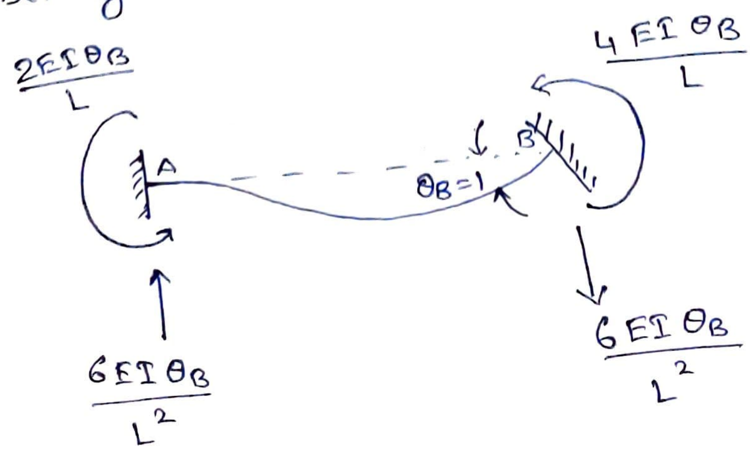
Torsional stiffness, which is defined as the torque required for a unit angle of twist, is obtained by putting $\Delta_4 = 1$. Hence torsional stiffness is given by

$$\text{torsional stiffness},\ K_{44} = \frac{GK}{L}$$

For Example :-

① If a unit rotation is provided at end B ($\theta_B = 1$) :

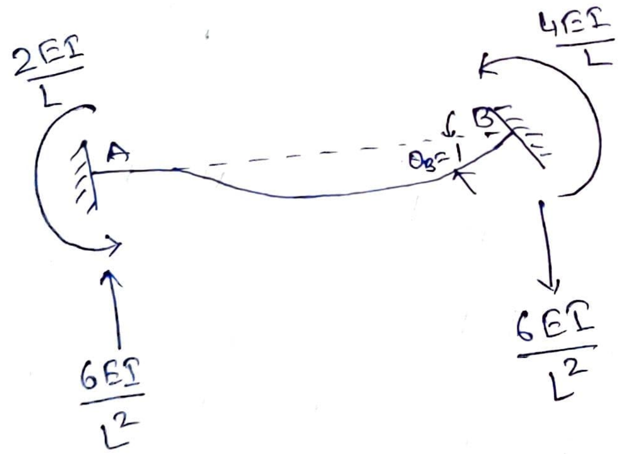
The member gets deformed to



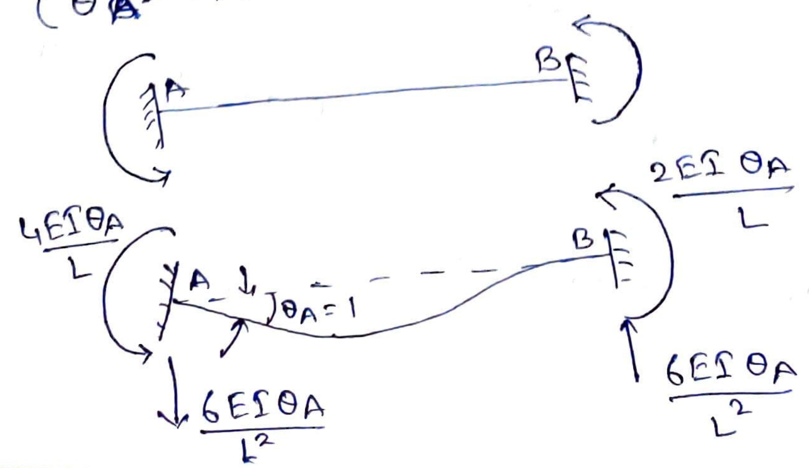
- At support A, the member is getting rotated in clockwise direction. In order to maintain the member in equilibrium an anticlockwise moment ($\frac{2EI\theta_B}{L}$) is provided.

- The direction of reaction at support A depends on the direction of the moment at point B. Similarly, the direction of reaction at point B depends upon the direction of the moment at support A respectively, so the member remains in equilibrium.

By the definition of stiffness, $\theta_B = 1$ then the forces at the supports turns as



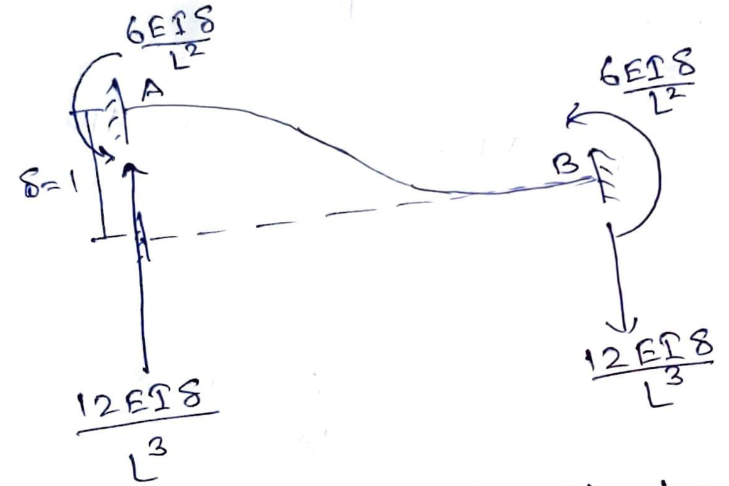
② Now, In a similar manner as above, a rotation is at point A ($\theta_A = 1$). then



③ A unit deflection at end A ($\delta_A = 1$): (upward direction)

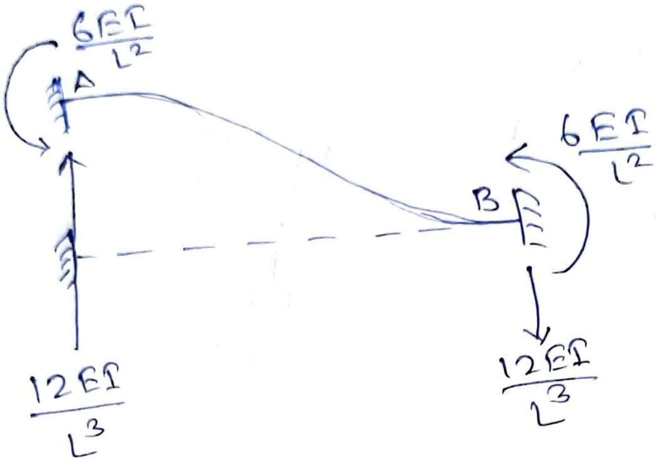
Now Considering a beam member of length 'L' having fixed end supports A and B.

~~The~~ At support A, the unit deflection is given the the forces at the ends ~~of the~~ supports will be as follows.



The ~~end~~ moment at support 'A' will be in anticlockwise direction with a magnitude of $\frac{6EI\delta}{L^2}$. The ~~end~~ moment at support 'B' will be in anticlockwise direction with a magnitude of $\frac{6EI\delta}{L^2}$. Now, the reaction at support 'A' will be in upward direction with a magnitude of $\frac{12EI\delta}{L^3}$. The reaction at support 'B' will be downward direction with a magnitude of $\frac{12EI\delta}{L^3}$

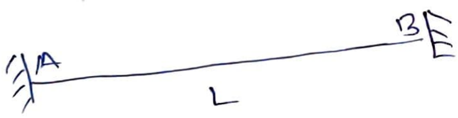
As per the definition of stiffness, ~~th~~ substituding the value of 'δ' as 1, then we get the stiffness forces as



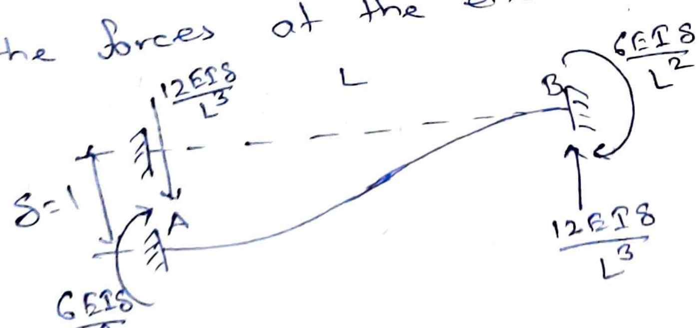
(downward direction)

④ Unit deflection at end A ($\delta_A = 1$)

A beam of length 'L' having fixed end support at A and B.



At support A, a unit deflection is given in the downword direction, the forces at the ends supports will be



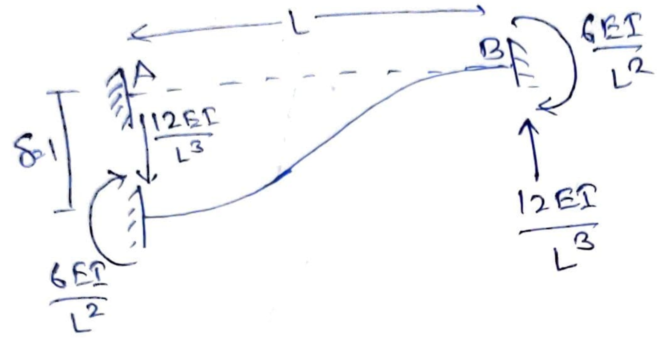
The ~~and~~ moment at support 'A' will be in clockwise direction with a magnitude of $\frac{6EI\delta}{L^2}$. The moment at support 'B' will be in clockwise direction with a magnitude of $\frac{6EI\delta}{L^2}$. The reaction at support A will be in downward direction with a magnitude of

$\frac{12EI\delta}{L^3}$. The reaction at support 'B' will be in upward direction

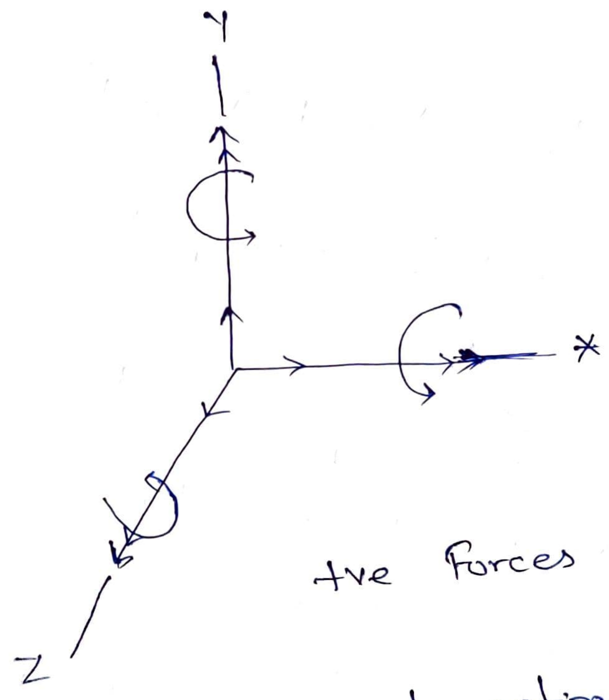
with a magnitude of $\frac{12EI\delta}{L^3}$.

As the definition of stiffness, substituting $\delta = 1$, then we get

the stiffness values as.



Sign Conventions



+ve Forces and Moments

All the anticlockwise direction moments acting along the direction are positive, and all the forces acting along the direction are positive.

Forces are represented by single arrow. ($\longrightarrow$)

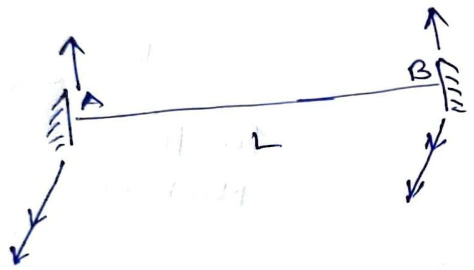
Moments are represented by double arrow ($\twoheadrightarrow$)

STIFFNESS MATRIX FORMATION:

(i) For a beam element:

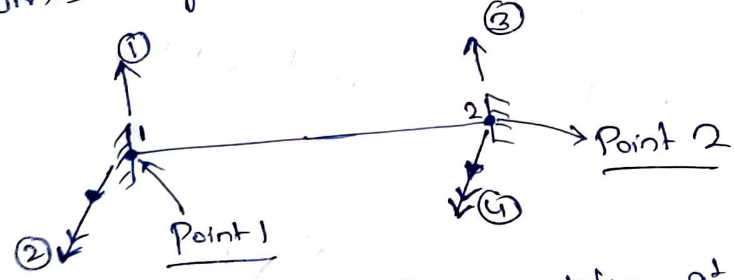
Consider a beam element of length L and fixed at both the ends.

Step 1: Draw the forces and moments



Force in the Y-direction and moment in the Z-direction.

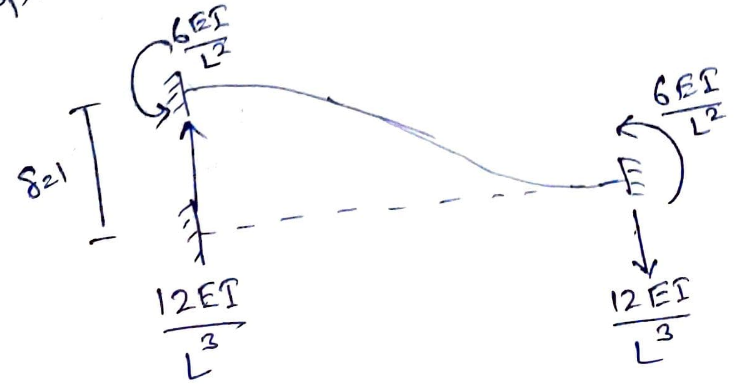
Step 2: Numbering the forces, Moments and Points.



Write down the stiffness forces acting at the points.

① Displacement due to force 1:

unit displacement of point 1 due to force 1:



Stiffness forces

$K_{11} = \frac{12EI}{L^3}$

K_{11} → Due to force 1; ↙ Force/Reaction Number 1

$K_{21} = \frac{6EI}{L^2}$

K_{21} → Due to force 1; ↙ Force/Reaction Number 2

$K_{31} = -\frac{12EI}{L^3}$

K_{31} → Due to force 1; ↙ Force/Reaction Number 3

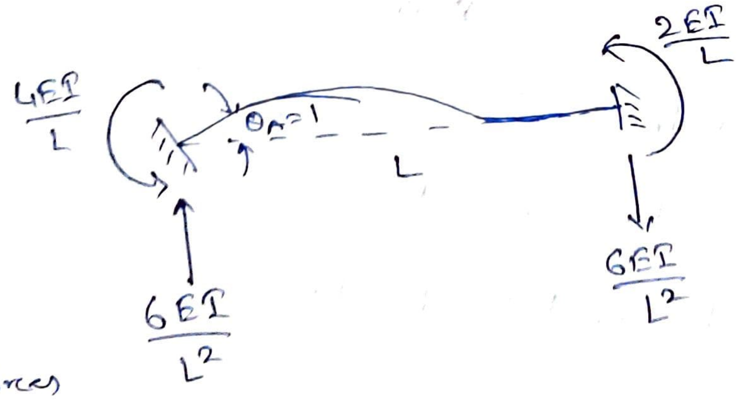
$K_{41} = \frac{6EI}{L^2}$

K_{41} → Due to force 1; ↙ Force/Reaction Number 4

Displacement due to force 2:-

Unit displacement at point 1 due to force 2:

Force 2 is moment, hence a unit rotation is applied at point 1.



Stiffness forces

$K_{12} = \frac{6EI}{L^2}$

K_{12} → Due to force 2; ↙ Force/Reaction Number 1

$$K_{22} = \frac{4EI}{L}$$

K_{22} → Force/Reaction Number 2 → Due to Force 2

$$K_{32} = \frac{-6EI}{L^2}$$

K_{32} → Force/Reaction Number 3 → Due to Force 2

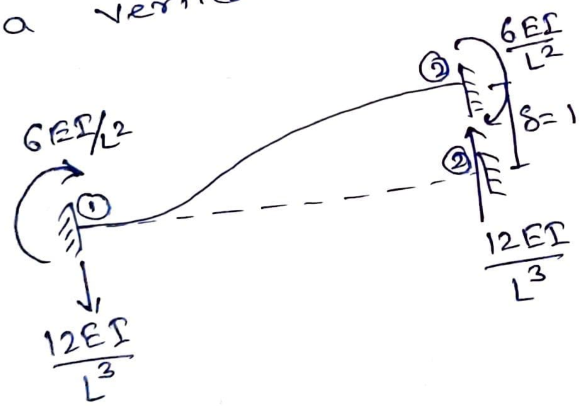
$$K_{42} = \frac{2EI}{L}$$

K_{42} → Force/Reaction number 4 → Due to Force 2

Displacement Due to Force 3:

Unit displacement at point 2 due to force 3:-

Force 3 is a vertical reaction acting a point 2.



Stiffness Forces

$$K_{13} = -\frac{12EI}{L^3}$$

K_{13} → Force/Reaction Number 1 → Due to Force 3

$$K_{23} = -\frac{6EI}{L^2}$$

K_{23} → Force/Reaction Number 2 → Due to Force 3

$$K_{33} = \frac{12EI}{L^3}$$

K_{33} → Force/Reaction Number 3 → Due to Force 3

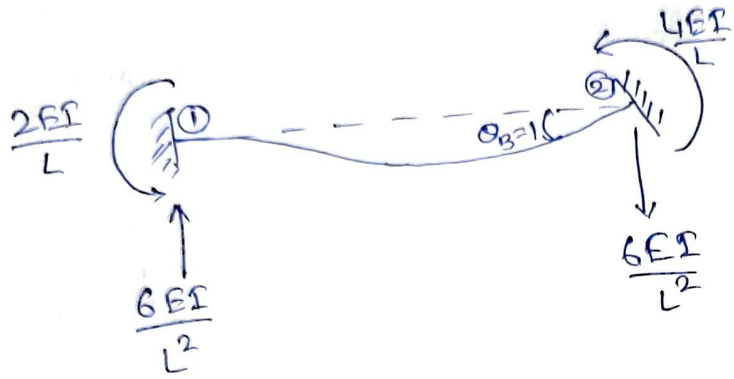
$$K_{43} = -\frac{6EI}{L^2}$$

K_{43} → Force/Reaction Number 4 → Due to Force 3

Displacement due to force 4:

Unit displacement at point 2 due to force 4:

Force 4 is moment hence a unit rotation is applied at point 2.



Stiffness forces

$K_{14} = \frac{6EI}{L^2}$

K_{14} → Due to force 4
↙ Force/Reaction Number 1

$K_{24} = \frac{2EI}{L}$

K_{24} → Due to force 4.
↙ Force/Reaction Number 2

$K_{34} = \frac{-6EI}{L^2}$

K_{34} → Due to force 4
↙ Force/Reaction number 3

$K_{44} = \frac{4EI}{L}$

K_{44} → Due to force 4.
↙ Force/reaction number 4

Finally putting all the stiffness forces in the matrix form.

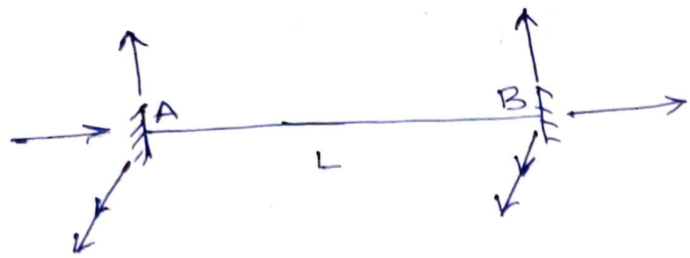
$$[K] = \begin{array}{c|cccc|c} & 1 & 2 & 3 & 4 & \\ \hline & K_{11} & K_{12} & K_{13} & K_{14} & 1 \\ & K_{21} & K_{22} & K_{23} & K_{24} & 2 \\ & K_{31} & K_{32} & K_{33} & K_{34} & 3 \\ & K_{41} & K_{42} & K_{43} & K_{44} & 4 \end{array}$$

Lets. substitute all the value, then we get.

$$[K] = \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \left[\begin{array}{cc|cc} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ \hline -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array}\right] & & & & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

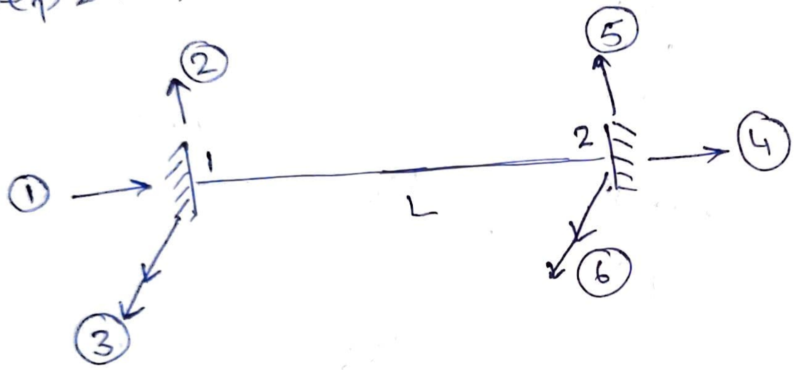
[K] is the final stiffness matrix of the beam element

Plane Frame Element Matrix :-



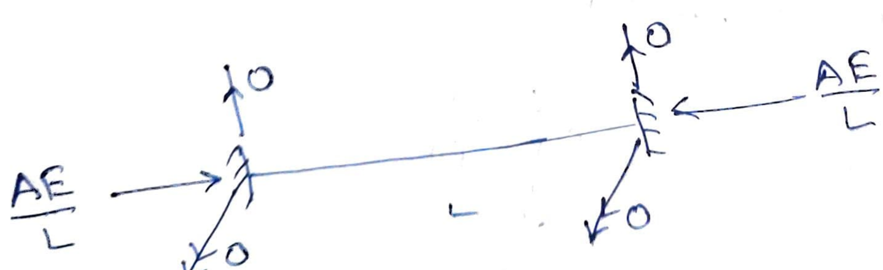
Step 1: Draw +ve Forces and Moments.

Step 2: Number the forces, Moments & Points



<u>Displacement due to force 1:</u>

unit Displacement at point 1 due to force 1 :-

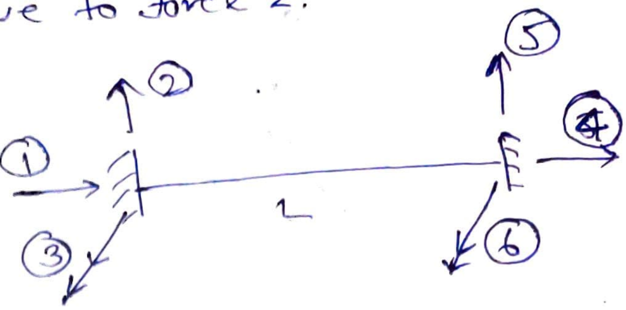


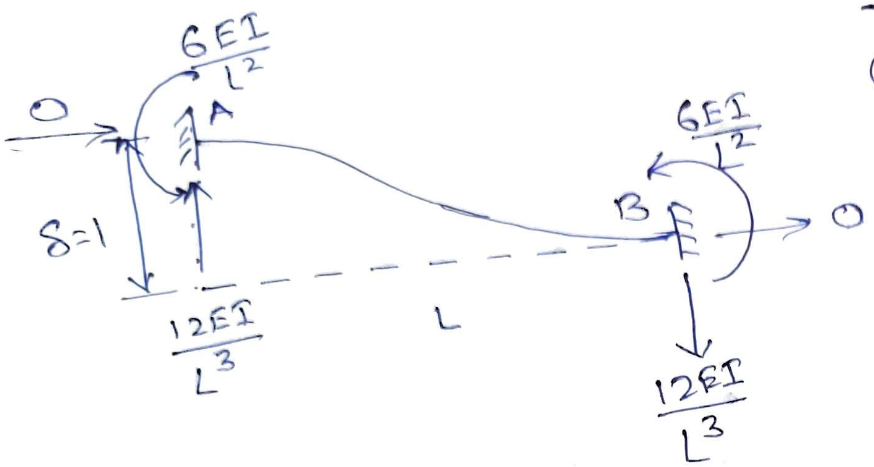
$K_{11} = \frac{AE}{L}$ $\quad K_{21} = 0$ $\quad K_{31} = 0$ $\quad K_{41} = -\frac{AE}{L}$

$K_{51} = 0$ $\quad K_{61} = 0$

Displacement due to Force 2:-

unit displacement at point 1 due to force 2.



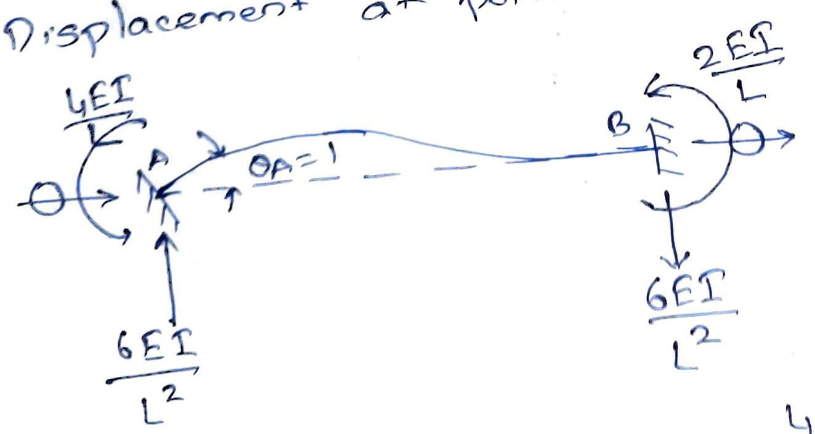


$K_{12} = 0$, $K_{22} = \frac{12EI}{L^3}$, $K_{32} = \frac{6EI}{L^2}$

$K_{42} = 0$, $K_{52} = -\frac{12EI}{L^3}$ $K_{62} = \frac{6EI}{L^2}$

Displacement due to Force 3:

unit Displacement at point 1 due to force 3!

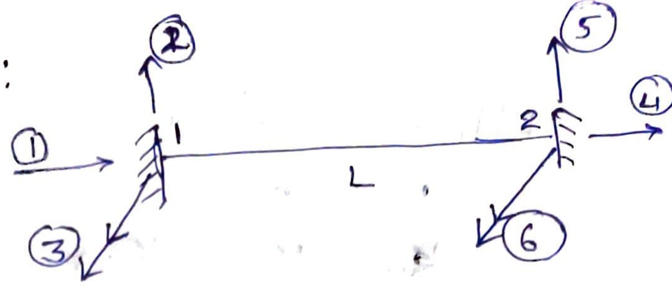


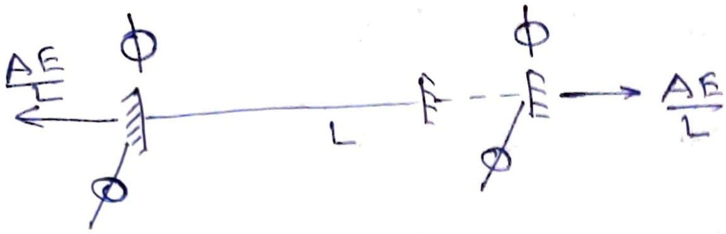
$K_{13} = 0$ $K_{23} = \frac{6EI}{L^2}$ $K_{33} = \frac{4EI}{L}$

$K_{53} = -\frac{6EI}{L^2}$ $K_{63} = \frac{2EI}{L}$

$K_{43} = 0$

Displacement due to Force 4:



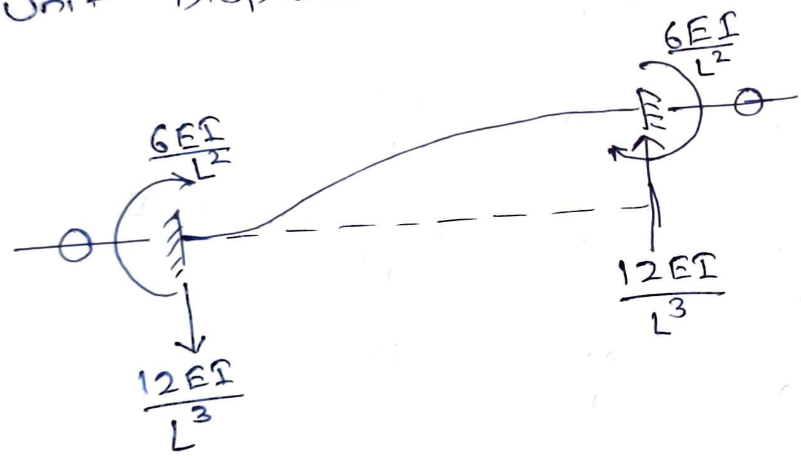


$K_{14} = \frac{-AE}{L}$ $K_{24} = 0$ $K_{34} = 0$

$K_{44} = \frac{AE}{4}$ $K_{54} = 0$ $K_{64} = 0$

Displacement due to force 5:

unit Displacement at point 2 due to force 5:



$K_{15} = 0$

$K_{25} = -\frac{12EI}{L^3}$

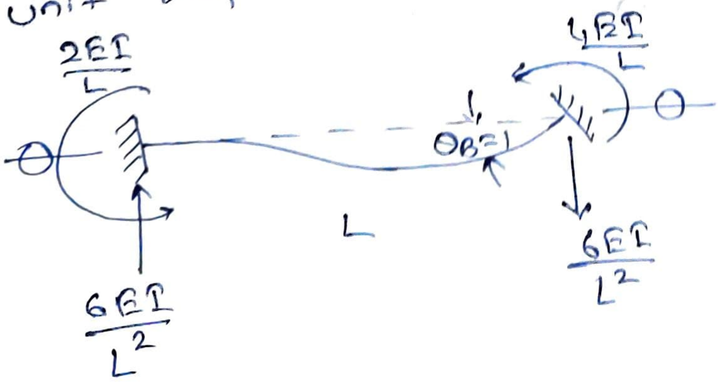
$K_{35} = -\frac{6EI}{L^2}$

$K_{45} = 0$ $K_{55} = \frac{12EI}{L^3}$

$K_{65} = -\frac{6EI}{L^2}$

Displacement due to force 6:

unit Displacement at point 2 due to force 6:



$K_{16} = 0$ $K_{46} = 0$

$K_{26} = \frac{6EI}{L^2}$ $K_{56} = -\frac{6EI}{L^2}$

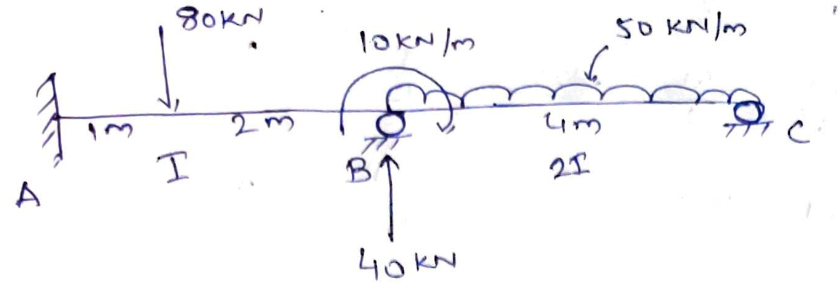
$K_{36} = \frac{2EI}{L}$ $K_{66} = \frac{4EI}{L}$

Stiffness matrix:-

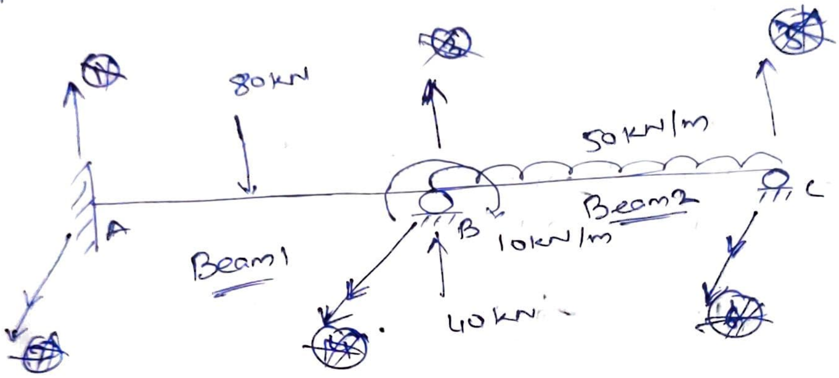
$$[K] = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \end{array}$$

$$[K] = \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

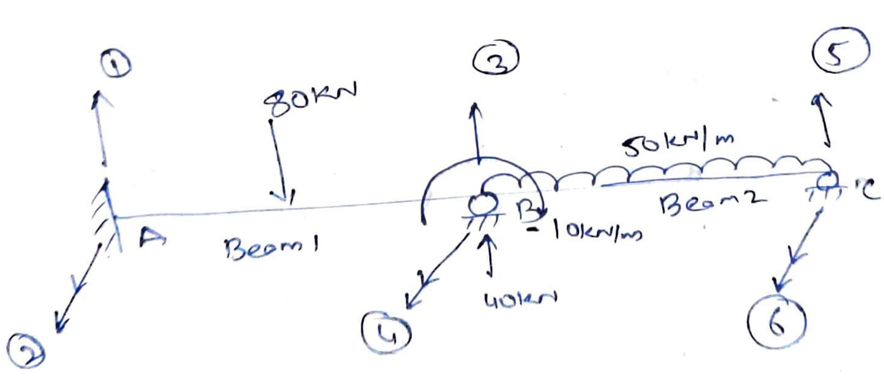
Stiffness matrix method Example



Step 1: Draw +ve forces & moments at each joint.



Step 2: Number each force and beam appropriately



Step 3:- Assemble the total Stiffness Matrix.

Total stiffness matrix = stiffness matrix of Element 1 (AB) + stiffness matrix of Element 2 (BC)

Stiffness matrix of Element 1

$$
[k_1] = \begin{array}{c c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\
\begin{bmatrix}
\frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

$$
= \begin{array}{c c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\
\begin{bmatrix}
\frac{12EI}{27} & \frac{6EI}{9} & -\frac{12EI}{27} & \frac{6EI}{9} \\
\frac{6EI}{9} & \frac{4EI}{3} & -\frac{6EI}{9} & \frac{2EI}{3} \\
-\frac{12EI}{27} & -\frac{6EI}{9} & \frac{12EI}{27} & -\frac{6EI}{9} \\
\frac{6EI}{9} & \frac{2EI}{3} & -\frac{6EI}{9} & \frac{4EI}{3}
\end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

Stiffness matrix of Element 2

$$
[K_2] = \begin{array}{c c}
\begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} & \\
\begin{bmatrix}
\frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix} & \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

$$
= \begin{array}{c} \\ \\ \\ \\ \\ \end{array}
\begin{array}{cccc|c}
3 & 4 & 5 & 6 & \\
\frac{24EI}{64} & \frac{12EI}{16} & -\frac{24EI}{64} & \frac{12EI}{16} & 3 \\
\frac{12EI}{16} & \frac{8EI}{4} & -\frac{12EI}{16} & \frac{4EI}{4} & 4 \\
-\frac{24EI}{64} & -\frac{12EI}{16} & \frac{24EI}{64} & -\frac{12EI}{16} & 5 \\
\frac{12EI}{16} & \frac{4EI}{4} & -\frac{12EI}{16} & \frac{8EI}{4} & 6
\end{array}
$$

$K = K_1 + K_2$

$$
= \begin{array}{cccc|c}
1 & 2 & 3 & 4 & \\
\frac{12EI}{27} & \frac{6EI}{9} & -\frac{12EI}{27} & \frac{6EI}{9} & 1 \\
\frac{6EI}{9} & \frac{4EI}{3} & -\frac{6EI}{9} & \frac{2EI}{3} & 2 \\
-12EI & -\frac{6EI}{9} & \frac{12EI}{27} & -\frac{6EI}{9} & 3 \\
\frac{6EI}{9} & \frac{2EI}{3} & -\frac{6EI}{9} & \frac{4EI}{3} & 4
\end{array}
+
\begin{array}{cccc|c}
3 & 4 & 5 & 6 & \\
\frac{24EI}{64} & \frac{12EI}{16} & -\frac{24EI}{64} & \frac{12EI}{16} & 3 \\
\frac{12EI}{16} & \frac{8EI}{4} & -\frac{12EI}{16} & \frac{4EI}{4} & 4 \\
-\frac{24EI}{64} & -\frac{12EI}{16} & \frac{24EI}{64} & -\frac{12EI}{16} & 5 \\
\frac{12EI}{16} & \frac{4EI}{4} & -\frac{12EI}{16} & \frac{8EI}{4} & 6
\end{array}
$$

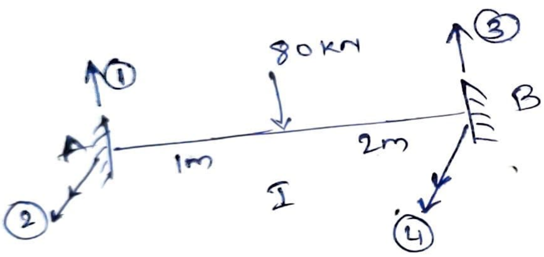
$$
K = \begin{array}{cccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & \\
\frac{12EI}{27} & \frac{6EI}{9} & -\frac{12EI}{27} & \frac{6EI}{9} & 0 & 0 & 1 \\
-\frac{12EI}{9} & \frac{4EI}{3} & -\frac{6EI}{9} & \frac{2EI}{3} & 0 & 0 & 2 \\
-\frac{12EI}{27} & -\frac{6EI}{9} & \frac{59EI}{73} & \frac{EI}{12} & -\frac{24EI}{64} & \frac{12EI}{16} & 3 \\
\frac{6EI}{9} & \frac{2EI}{3} & \frac{EI}{12} & \frac{10EI}{3} & -\frac{12EI}{16} & \frac{4EI}{4} & 4 \\
0 & 0 & -\frac{24EI}{64} & -\frac{12EI}{16} & \frac{24EI}{64} & -\frac{12EI}{16} & 5 \\
0 & 0 & \frac{12EI}{16} & \frac{4EI}{4} & -\frac{12EI}{16} & \frac{8EI}{4} & 6
\end{array}
$$

Step 4: Joint Load Matrix.

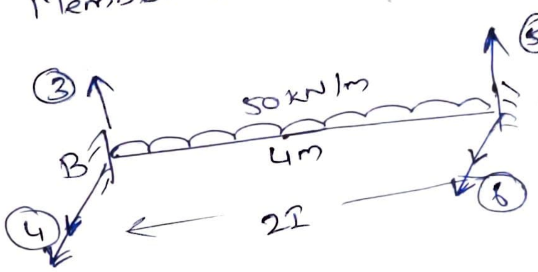
$$[A_j] = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 40 \\ -10 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Step 5: Member Reaction Matrix $[A_r]$

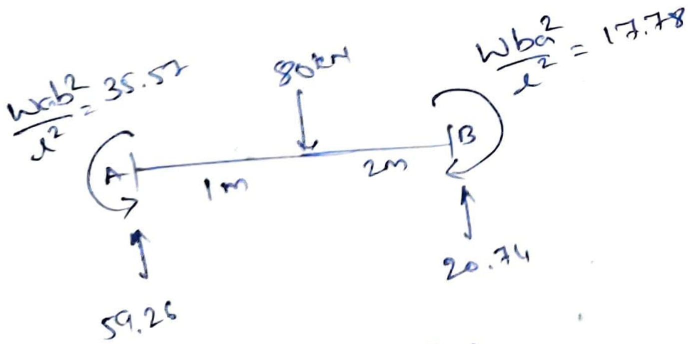
Member 1 $[A_{r1}]$

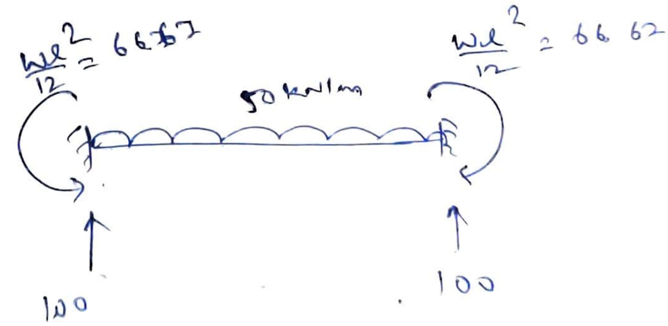


Member 2 $[A_{r2}]$



Find the reaction b/c of load.





$$A_{r_1} = \begin{bmatrix} 59.26 \\ 35.57 \\ 20.74 \\ -17.78 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad A_{r2} = \begin{bmatrix} 100 \\ 66.67 \\ 100 \\ -66.67 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$[A_r] = [A_{r1}] + [A_{r2}]$$

$$= \begin{array}{c c} \begin{bmatrix} 59.26 \\ 35.57 \\ 20.74 \\ -17.78 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} + \begin{array}{c c} \begin{bmatrix} 100 \\ 66.67 \\ 100 \\ -66.67 \end{bmatrix} & \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

$$= \begin{bmatrix} 59.26 \\ 35.57 \\ 120.74 \\ 48.89 \\ 100 \\ -66.67 \end{bmatrix}$$

Step 6 : Combined load matrix. $[A_c]$

$$[A_c] = [A_j] - [A_r] = \begin{bmatrix} -\cancel{100}\ 59.26 \\ -35.57 \\ -80.74 \\ -58.89 \\ -100 \\ +66.67 \end{bmatrix}$$

Step 7 :

Assemble stiffness Equation [K] * [δ] = [A_c]

$$
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
\frac{12EI}{27} & \frac{6EI}{9} & -\frac{12EI}{27} & \frac{6EI}{9} & 0 & 0 \\
\frac{6EI}{9} & \frac{4EI}{3} & -\frac{6EI}{9} & \frac{2EI}{3} & 0 & 0 \\
-\frac{12EI}{27} & -\frac{6EI}{9} & \frac{59EI}{72} & \frac{EI}{12} & -\frac{24EI}{64} & \frac{12EI}{16} \\
\frac{6EI}{9} & \frac{2EI}{3} & \frac{EI}{12} & \boxed{\frac{10EI}{3}} & -\frac{12EI}{16} & \boxed{\frac{4EI}{4}} \\
0 & 0 & -\frac{24EI}{64} & -\frac{12EI}{16} & \frac{24EI}{64} & -\frac{12EI}{16} \\
0 & 0 & \frac{12EI}{16} & \boxed{\frac{4EI}{4}} & -\frac{12EI}{16} & \boxed{\frac{8EI}{4}}
\end{bmatrix}
\end{array}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\times
\begin{bmatrix} \delta_A \\ \theta_A \\ \delta_B \\ \theta_B \\ \delta_C \\ \theta_C \end{bmatrix}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

$$
= \begin{bmatrix} -59.26 \\ -35.86 \\ -80.79 \\ \boxed{58.89} \\ -100 \\ \boxed{66.67} \end{bmatrix}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

$$
\begin{array}{cc|c}
4 & 6 & \\
\end{array}
\begin{array}{c}
\\
\end{array}
$$

$$
\begin{array}{cc}
4 & 6 \\
\end{array}
$$

$$
\left[\begin{array}{cc} \frac{10EI}{3} & \frac{4EI}{4} \\ \frac{4EI}{4} & \frac{8EI}{4} \end{array}\right]\begin{array}{c} 4 \\ 6 \end{array} \times \left[\begin{array}{c} \theta_B \\ \theta_C \end{array}\right]\begin{array}{c} 4 \\ 6 \end{array} = \left[\begin{array}{c} -58.89 \\ 66.67 \end{array}\right]\begin{array}{c} 4 \\ 6 \end{array}
$$

$$\theta_B = \frac{-32.55}{EI} \text{ rad} \quad , \quad \theta_C = \frac{49.61}{EI} \text{ rad}$$

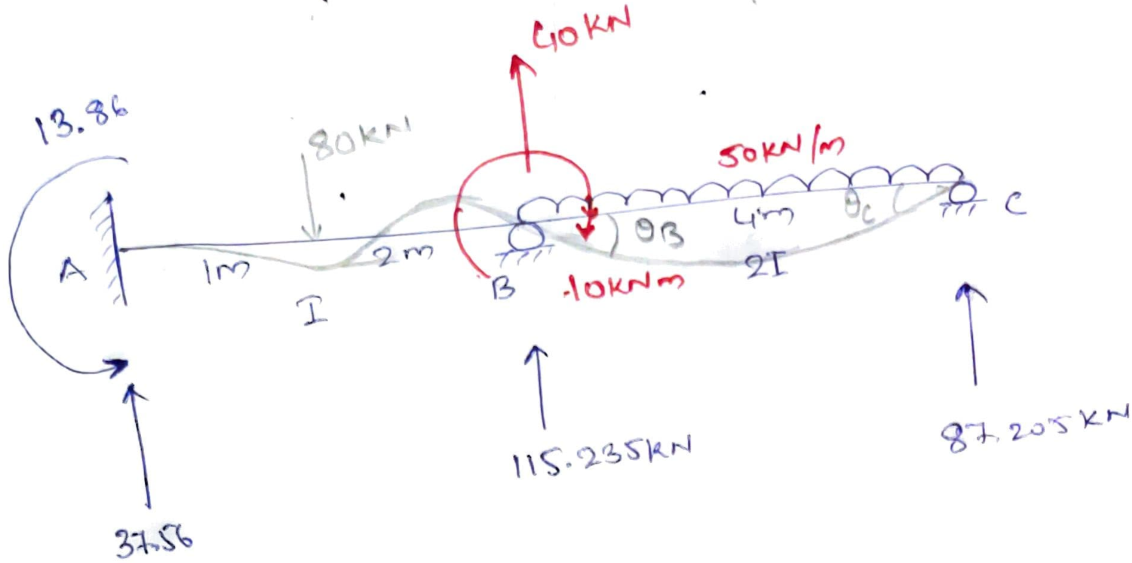
Step 8: find Reactions

$$[R] = [k] \times [\delta] - [A_c]$$

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

$$
\left[\begin{array}{cccccc}
\frac{12EI}{27} & \frac{6EI}{9} & -\frac{12EI}{27} & \frac{6EI}{9} & 0 & 0 \\
\frac{6EI}{9} & \frac{4EI}{3} & -\frac{6EI}{9} & \frac{2EI}{3} & 0 & 0 \\
-\frac{12EI}{27} & -\frac{6EI}{9} & \frac{59EI}{72} & \frac{EI}{12} & -\frac{24EI}{64} & \frac{12EI}{16} \\
\frac{6EI}{9} & \frac{2EI}{3} & \frac{EI}{12} & \frac{10EI}{3} & -\frac{12EI}{16} & \frac{4EI}{4} \\
0 & 0 & -\frac{24EI}{64} & -\frac{12EI}{16} & \frac{24EI}{64} & -\frac{12EI}{16} \\
0 & 0 & \frac{12EI}{16} & \frac{4EI}{4} & -\frac{12EI}{16} & \frac{8EI}{16}
\end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\times
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -\frac{32.55}{EI} \\ 0 \\ \frac{49.61}{EI} \end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
-
\left[\begin{array}{c} -59.26 \\ -35.56 \\ -80.74 \\ -58.89 \\ -100 \\ 66.67 \end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
=
\left[\begin{array}{c} V_A \\ M_A \\ V_B \\ M_B \\ V_C \\ M_C \end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

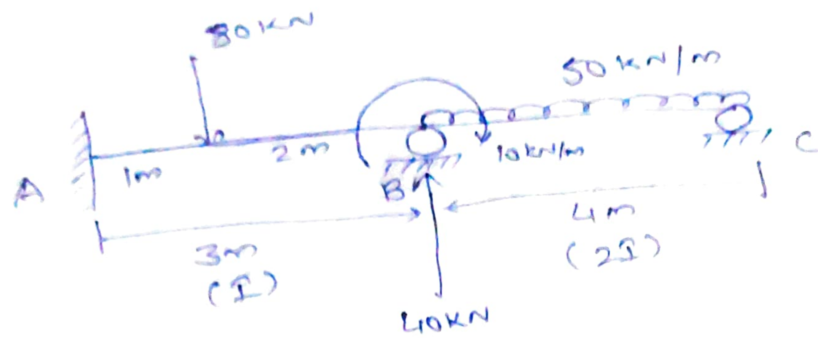
$$
= \left[\begin{array}{c} 37.56 \\ 13.86 \\ 115.235 \\ 0 \\ 87.205 \\ 0 \end{array}\right]
\begin{array}{l} kN \\ kNm \\ kN \\ kNm \\ kN \\ kNm \end{array}
$$



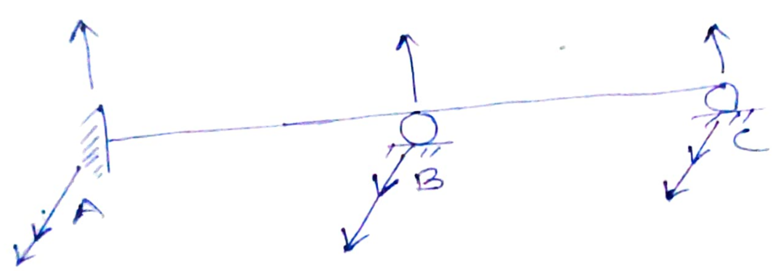
40KN
13.86
80KN
50KN/m
A
1m
I
2m
B
θ_B
4m
θ_C
C
10KNm
2I
115.235KN
87.205KN
37.56

Example 1:

① Analyse the beam using stiffness matrix method

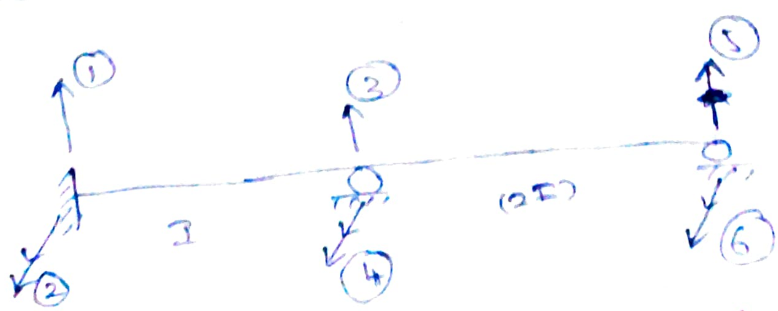


Step 1: Draw +ve force and moments at each joint



Step 2: Number each force and beam appropriately

Numbering is very important because the stiffness matrix that will be forming will depend upon how the numbering has been done while deriving the stiffness matrix itself.



The numbering in beam element was done as the upward force are having the first number and then the moment.

The numbering has to be done in this manner only. The numbering cannot be interchanged, as ~~then~~ the matrix will get interchanged. ~~and~~

<u>Step 8:</u> Assemble the total stiffness matrix

Total stiffness matrix = ~~total~~ stiffness matrix of Element 1 + stiffness matrix of element 2.

Here, Element AB is considered as element 1 and element BC is considered as element 2.

<u>Stiffness matrix for element 1.</u>

$$[K_1] = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{12EI}{27} & \frac{6EI}{9} & -\frac{12EI}{27} & \frac{6EI}{9} \\ \frac{6EI}{9} & \frac{4EI}{3} & -\frac{6EI}{9} & \frac{2EI}{3} \\ -\frac{12EI}{27} & -\frac{6EI}{9} & \frac{12EI}{27} & -\frac{6EI}{9} \\ \frac{6EI}{9} & \frac{2EI}{3} & -\frac{6EI}{9} & \frac{4EI}{3} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Stiffness matrix of element 2

$$[K_2] = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc|c} 3 & 4 & 5 & 6 & \\ \end{array}$$

$$[K_2] = \begin{array}{c@{}c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} & \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

After substituting the values of 'I' and 'L', we get

$$[K_2] = \begin{array}{c@{}c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} \frac{24EI}{64} & \frac{12EI}{16} & -\frac{24EI}{64} & \frac{12EI}{16} \\ \frac{12EI}{16} & \frac{8EI}{4} & -\frac{12EI}{16} & \frac{4EI}{4} \\ -\frac{24EI}{64} & -\frac{12EI}{16} & \frac{24EI}{64} & -\frac{12EI}{16} \\ \frac{12EI}{16} & \frac{4EI}{4} & -\frac{12EI}{16} & \frac{8EI}{4} \end{bmatrix} & \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

∴ Total stiffness matrix = stiffness matrix of element 1 + stiffness matrix of element 2

$$[K] = [K_1] + [K_2]$$

$$\therefore [K] = \begin{array}{c@{}c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} \frac{12EI}{27} & \frac{6EI}{9} & -\frac{12EI}{27} & \frac{6EI}{9} \\ \frac{6EI}{9} & \frac{4EI}{3} & -\frac{6EI}{9} & \frac{2EI}{3} \\ -\frac{12EI}{27} & -\frac{6EI}{9} & \frac{12EI}{27} & -\frac{6EI}{9} \\ \frac{6EI}{9} & \frac{2EI}{3} & -\frac{6EI}{9} & \frac{4EI}{3} \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} + \begin{array}{c@{}c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} \frac{24EI}{64} & \frac{12EI}{16} & -\frac{24EI}{64} & \frac{12EI}{16} \\ \frac{12EI}{16} & \frac{8EI}{4} & -\frac{12EI}{16} & \frac{4EI}{4} \\ -\frac{24EI}{64} & -\frac{12EI}{16} & \frac{24EI}{64} & -\frac{12EI}{16} \\ \frac{12EI}{16} & \frac{4EI}{4} & -\frac{12EI}{16} & \frac{8EI}{4} \end{bmatrix} & \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

$$
[K] = \begin{array}{cc}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\
\left[\begin{array}{cccccc}
\frac{12EI}{27} & \frac{6EI}{9} & -\frac{12EI}{27} & \frac{6EI}{9} & 0 & 0 \\
\frac{6EI}{9} & \frac{4EI}{3} & -\frac{6EI}{9} & \frac{2EI}{3} & 0 & 0 \\
-\frac{12EI}{27} & -\frac{6EI}{9} & \frac{59EI}{72} & \frac{EI}{12} & -\frac{24EI}{64} & \frac{12EI}{16} \\
\frac{6EI}{9} & \frac{2EI}{3} & \frac{EI}{12} & \frac{10EI}{3} & -\frac{12EI}{16} & \frac{4EI}{4} \\
0 & 0 & -\frac{24EI}{64} & -\frac{12EI}{16} & \frac{24EI}{64} & -\frac{12EI}{16} \\
0 & 0 & \frac{12EI}{16} & \frac{4EI}{4} & -\frac{12EI}{16} & \frac{8EI}{4}
\end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

The total stiffness matrix will be of 6x6 elements. As there are 6 forces, thus it will be a 6x6 element matrix.

Step 4.

Joint load Matrix.

$$
A_j = \left[\begin{array}{c} 0 \\ 0 \\ 40 \\ -10 \\ 0 \\ 0 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

This is the matrix at every joint that is A, B and C. and if there exists any load at any of the joint.

There is no load at joint A and at joint C. where as there is load as well as a moment at joint B.

So the value at '1' is zero because zero force Q

→ At force '2' the value is zero because there is zero moment in the direction.

→ At force '3' there is an upward force of 40KN. So this is +40 KN.

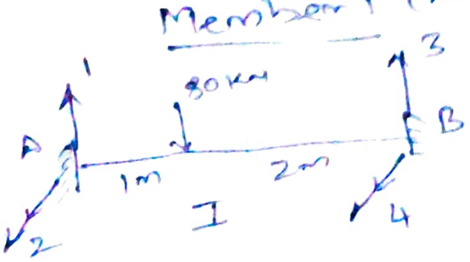
→ At force '4' there is again a clockwise moment of 10 KN-m at joint B. So the value has to be taken as -10KNm because we consider anticlockw moments are positive

→ Similarly at joint C there are no forces so these will be 0.

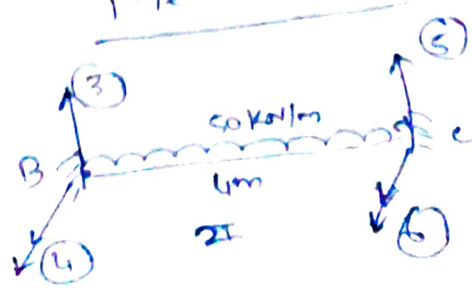
Step 5:

Member reaction matrix (A_r).

Member 1 (A_{r1})



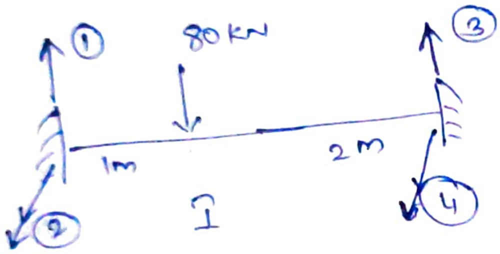
Member 2 (A_{r2})



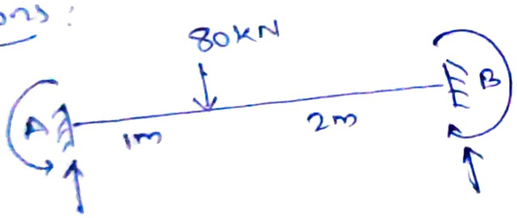
$$[A_r] = [A_{r1}] + [A_{r2}]$$

Need to find the reactions because of the load

Member 1 $[A_{r1}]$



Reactions:



Moment at A $= \dfrac{Wab^2}{l^2} = 35.56$ KN-m

Moment at B $= \dfrac{Wab^2}{l^2} = 17.78$ KN-m

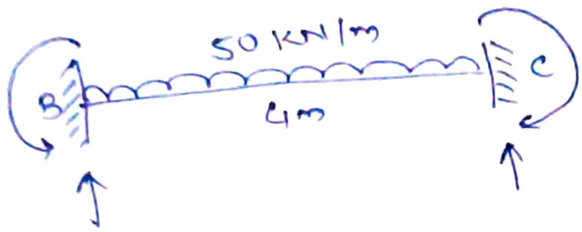
Reaction at A and B

$R_A = 59.26$

$R_B = 20.74$

$$[A_{r1}] = \begin{array}{c} \begin{bmatrix} 59.26 \\ 35.56 \\ 20.74 \\ -17.78 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Similarly for member 2 $[A_{r2}]$



$$\text{Moment at } B = \frac{wl^2}{12} = 66.67 \text{ kN-m}$$

$$\text{Moment at } C = \frac{wl^2}{12} = 66.67 \text{ kN-m}$$

Reactions at B and C

$R_B = 100$ kN

$R_C = 100$ kN.

$$[A_{r2}] = \begin{array}{c} \begin{bmatrix} 100 \\ 66.67 \\ 100 \\ -66.67 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$\therefore \; [A_r] = [A_{r_1}] + [A_{r_2}]$

$$[A_r] = \begin{array}{c} \begin{bmatrix} 59.26 \\ 35.56 \\ 20.74 \\ -17.78 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} + \begin{array}{c} \begin{bmatrix} 100 \\ 66.67 \\ 100 \\ -66.67 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$= \begin{array}{c} \begin{bmatrix} 59.26 \\ 35.56 \\ 120.74 \\ 48.89 \\ 100 \\ -66.67 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Step 6: Combined Load Matrix $[A_c]$

$$[A_c] = [A_j] - [A_r]$$

where,

$[A_j] \Rightarrow$ Joint load matrix

$[A_r] \rightarrow$ Member Reaction matrix

$$[A_c] = \begin{array}{c}[A_J']\\ \begin{bmatrix} 0 \\ 0 \\ 40 \\ -10 \\ 0 \\ 0 \end{bmatrix}\end{array} \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} - \begin{array}{c}[A_r]\\ \begin{bmatrix} 59.26 \\ 35.56 \\ 120.74 \\ 48.89 \\ 100.00 \\ -66.67 \end{bmatrix}\end{array} \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}$$

$$= \begin{bmatrix} -59.26 \\ -35.56 \\ -80.74 \\ -58.89 \\ -100.00 \\ 66.67 \end{bmatrix} \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}$$

Step 7: Assemble stiffness equation $[K] \times [\delta] = [A_c]$

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$\begin{bmatrix} \frac{12EI}{27} & \frac{6EI}{9} & -\frac{12EI}{27} & \frac{6EI}{9} & 0 & 0 \\ \frac{6EI}{9} & \frac{4EI}{3} & -\frac{6EI}{9} & \frac{2EI}{3} & 0 & 0 \\ -\frac{12EI}{27} & -\frac{6EI}{9} & \frac{59EI}{72} & \frac{EI}{12} & -\frac{24EI}{64} & \frac{12EI}{16} \\ \frac{6EI}{9} & \frac{2EI}{3} & \frac{EI}{12} & \frac{10EI}{3} & -\frac{12EI}{16} & \frac{4EI}{4} \\ 0 & 0 & -\frac{24EI}{64} & -\frac{12EI}{16} & \frac{24EI}{64} & -\frac{12EI}{16} \\ 0 & 0 & \frac{12EI}{16} & \frac{4EI}{4} & -\frac{12EI}{16} & \frac{8EI}{4} \end{bmatrix} \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} \times \begin{bmatrix} \delta_A \\ \theta_A \\ \delta_B \\ \theta_B \\ \delta_C \\ \theta_C \end{bmatrix} \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} = \begin{bmatrix} -59.26 \\ -35.56 \\ -80.74 \\ -58.89 \\ -100.00 \\ 66.67 \end{bmatrix} \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}$$

where,

$[K]$ = Total stiffness matrix

$[\delta]$ = deflection matrix

$[A]$ = Combined load matrix

→ As the joint A is fixed. So the deflections (δ_A, θ_A) are 'zero'. i.e $[\delta_A = 0, \theta_A = 0]$.

→ At the joint B is a roller support, there won't be any displacement in the vertical direction $(\delta_B = 0)$ where as θ_B will occur because this is a roller support and once it is deflected there will be some rotation at this point. Therefore θ_B will remain.

→ At the joint C is a roller support, again there cannot occur an upward deflection at C. So $[\delta_C = 0]$

θ_C will occur because it is a roller support and there will be some rotation.

Now, the left out matrix will be

$$\begin{array}{cc} 4 & 6 \end{array}$$

$$\begin{bmatrix} \frac{10EI}{3} & \frac{4EI}{4} \\ \frac{4EI}{4} & \frac{8EI}{4} \end{bmatrix} \begin{array}{c} 4 \\ 6 \end{array} \times \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} \begin{array}{c} 4 \\ 6 \end{array} = \begin{bmatrix} -58.89 \\ 66.67 \end{bmatrix} \begin{array}{c} 4 \\ 6 \end{array}$$

$\therefore$ From above, $\theta_B = -\dfrac{32.55}{EI}$ radians

$\theta_C = \dfrac{49.61}{EI}$ radians

Step 8: Find Reactions

$$[R] = [K] \times [\delta] - [A_c]$$

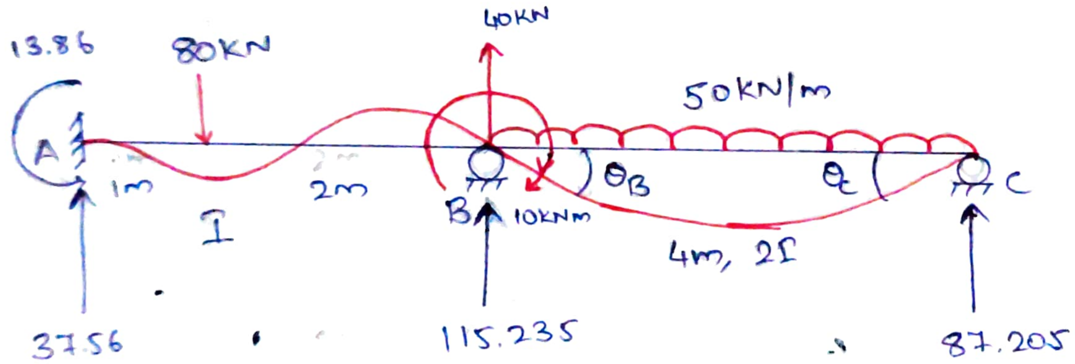
$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$\begin{bmatrix} \frac{12EI}{27} & \frac{6EI}{9} & -\frac{12EI}{27} & \frac{6EI}{9} & 0 & 0 \\ \frac{6EI}{9} & \frac{4EI}{3} & -\frac{6EI}{9} & \frac{2EI}{3} & 0 & 0 \\ -\frac{12EI}{27} & -\frac{6EI}{9} & \frac{59EI}{72} & \frac{EI}{12} & -\frac{24EI}{64} & \frac{12EI}{16} \\ \frac{6EI}{9} & \frac{2EI}{3} & \frac{EI}{12} & \frac{10EI}{3} & -\frac{12EI}{16} & \frac{4EI}{4} \\ 0 & 0 & -\frac{24EI}{64} & -\frac{12EI}{16} & \frac{24EI}{64} & -\frac{12EI}{16} \\ 0 & 0 & \frac{12EI}{16} & \frac{4EI}{4} & -\frac{12EI}{16} & \frac{8EI}{4} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{32.55}{EI} \\ 0 \\ \frac{49.61}{EI} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} - \begin{bmatrix} -59.26 \\ -35.56 \\ -80.74 \\ -58.89 \\ -100 \\ 66.67 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

After simplifing the reachons will be

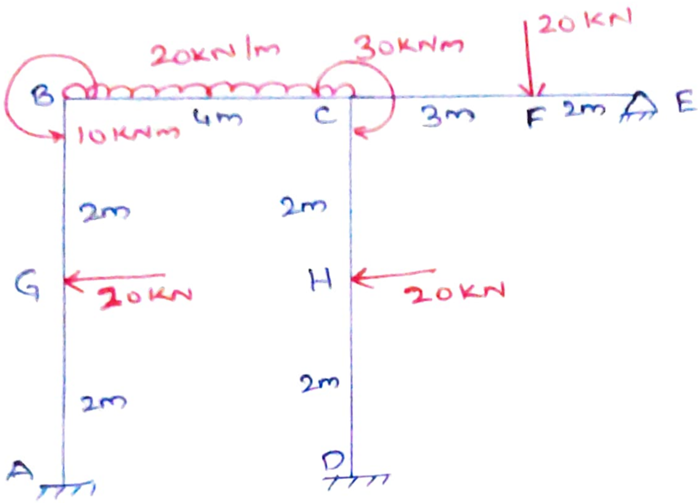
$$\Rightarrow [R] = \begin{array}{c} \begin{bmatrix} V_A \\ M_A \\ V_B \\ M_B \\ V_C \\ M_C \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} = \begin{bmatrix} 37.56 \\ 13.86 \\ 115.235 \\ 0 \\ 87.205 \\ 0 \end{bmatrix} \begin{array}{l} kN \\ kNm \\ kN \\ kNm \\ kN \\ kNm \end{array}$$

Finally:



$$\theta_B = \frac{-32.55}{EI} \text{ rad} \qquad \theta_c = \frac{49.61}{EI} \text{ rad}$$

Stiffness matrix method Example (Portal Frame)



Step 1: Determine the indeterminacy

$D_k = 3j - (R+m)$ $\quad \theta_B \rightarrow 1$

$D_k = 3(5) - (8+4)$ $\quad \theta_C \rightarrow 2$

$= 15 - 12$ $\quad \theta_E \rightarrow 3$

$D_k = 3$

Step 2: Fixed end moments.

$$M_{FAB} = \frac{Wb^2a}{l^2} = \frac{20 \times 2^2 \times 2}{4^2} = 10 KNm$$

$$M_{FBA} = -\frac{Wa^2b}{l^2} = \frac{-20 \times 2^2 \times 2}{4^2} = -10 KNm$$

$$M_{FBC} = -\frac{Wl^2}{12} = \frac{-20 \times 4^2}{12} = -\frac{80}{3} KN\text{-}m$$

$$M_{FCB} = \frac{Wl^2}{12} = \frac{20\times4^2}{12} = \frac{80}{3} \text{ KNm}$$

$$M_{FCD} = \frac{-Wb^2a}{l^2} = \frac{-20\times2^2\times2}{4^2} = -10 \text{ KNm}$$

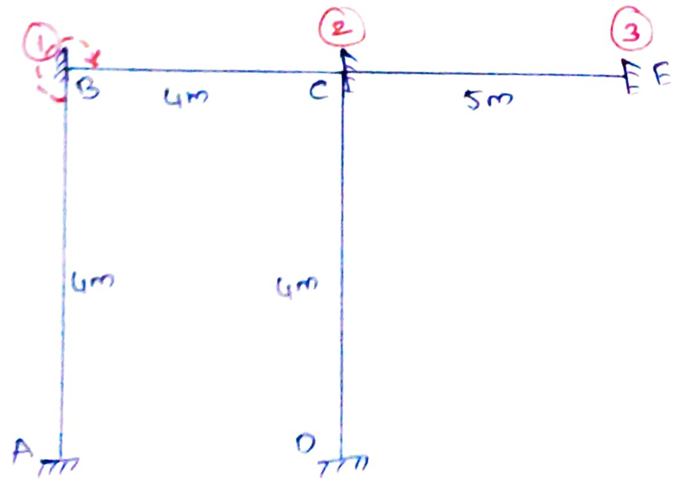
$$M_{FDC} = \frac{Wa^2b}{l^2} = \frac{20\times2^2\times2}{4^2} = 10 \text{ KNm}$$

$$M_{FCE} = \frac{-Wb^2a}{l^2} = \frac{-20\times2^2\times3}{5^2} = -9.6 \text{ KNm}$$

$$M_{FEC} = \frac{Wa^2b}{l^2} = \frac{20\times3^2\times2}{5^2} = 14.4 \text{ KNm}$$

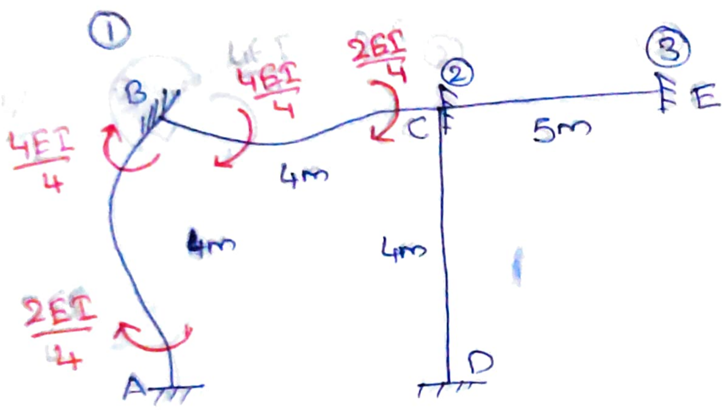
Step 3: Unit Deformation at D_K

i) Giving unit rotation for 1st $D_K \rightarrow \theta_B$.



All the points i.e., A, B, C, D and E have been associated with a fixed support.

Therefore, whenever a rotation is given at any joint only the members connected to that joint will rotate, and all the other member will remain uneffected.



- The moment that is required to rotate the joint B is given as $\frac{4EI}{L}$ $(L=4)$. The reaction formed at the opposite joint A is $\frac{2EI}{L}$ $(L=4)$.

- Again the ~~ma~~ for the member BC. The force required for creating unit rotation is $\frac{4EI}{L}$. The reaction formed ~~at~~ the opposite joint C is $\frac{2EI}{L}$ $(L=4)$

K_{11} is a force/moment formed at joint 1 due to unit deflection/rotation given at joint 1.

$$K_{11} = \frac{4EI}{4} + \frac{4EI}{4} = 2EI$$

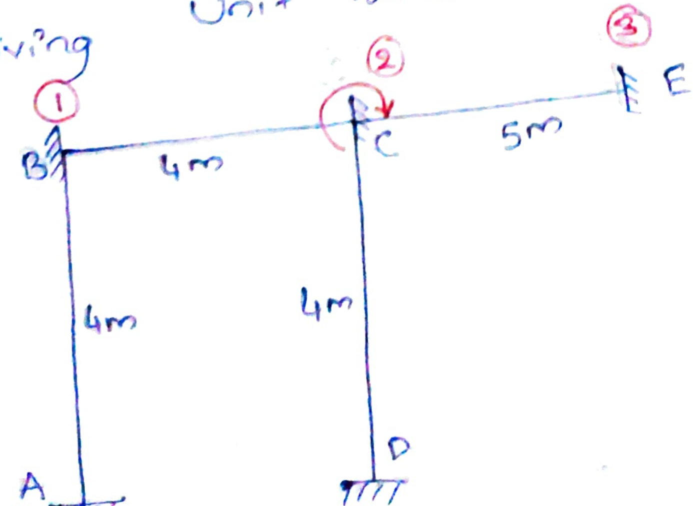
$$K_{21} = \frac{2EI}{4} = 0.5EI$$

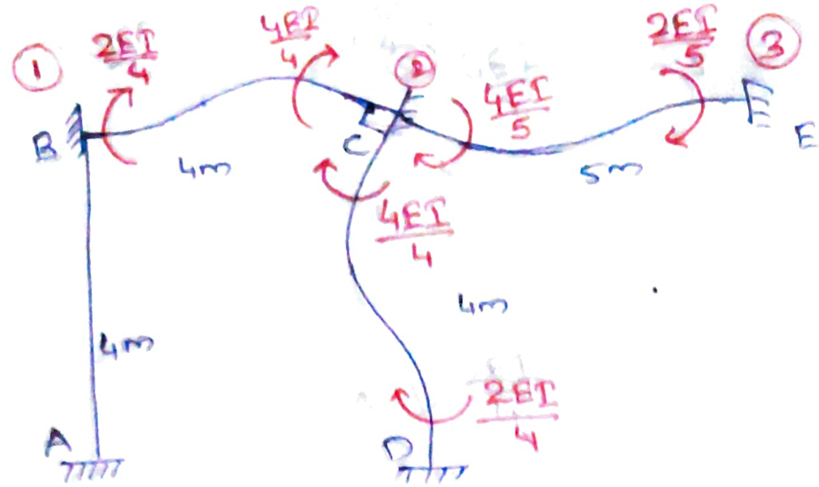
K_{21} is force/moment formed at joint 2 due to unit deflection/rotation given at joint 1.

K_{31} is force/moment formed at joint 3 due to unit deflection/rotation given at joint 1.

$$K_{31} = 0$$

(2) Giving unit rotation for 2nd $D_k \rightarrow \theta_c$





Now, K_{12} is a force/moment formed at joint 1 due to unit deflection/rotation given at joint 2

$$K_{12} = \frac{2EI}{4} = 0.5EI$$

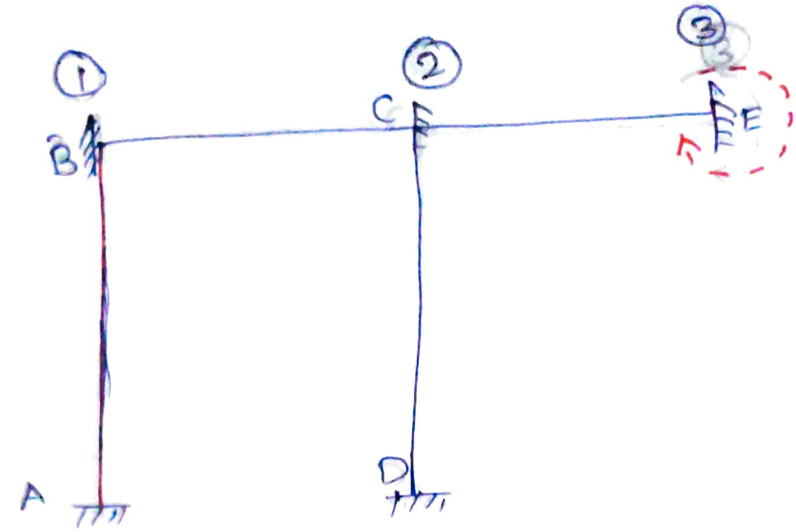
$$K_{22} = \frac{4EI}{4} + \frac{4EI}{5} + \frac{4EI}{4} = \frac{14EI}{5}$$

K_{22} is a force/moment formed at joint 2 due to unit deflection/rotation given at joint 2.

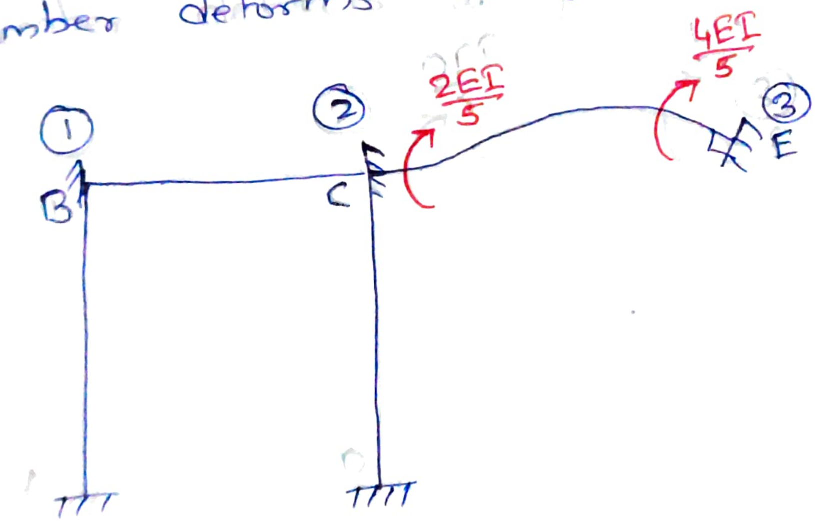
K_{32} is a force/moment formed at joint 3 due to unit deflection/rotation given at joint 2.

$$K_{32} = \frac{2EI}{5}$$

(3) Giving Unit deflection / rotation at 3rd Dk → θ_E



After giving unit rotation at 3rd Dk at joint 3 the member deforms in following manner.



K_{13} is the force / moment formed at joint 1 due to unit deflection / rotation given at joint 3.

$$K_{13} = 0$$

K_{23} is the force/moment formed at joint 2 due to unit deflection/rotation given at joint 3.

$$K_{23} = \frac{2EI}{5}$$

K_{33} is the force/moment formed at joint 3 due to unit deflection/rotation given at joint 3.

$$K_{33} = \frac{4EI}{5}$$

<u>Step 4:</u> Finding Unknowns.

$$[P']_{n\times1} + [K_{ij}]_{n\times n} * [\delta]_{n\times1} = [P]_{n\times1}$$

In this problem there are 3 unknowns i.e (θ_B, θ_C, θ_E). therefore $n=3$.

$[P']_{n\times1} \Rightarrow$ Fixed end moment at the corresponding joint.

i.e for the first D_K is at joint B. There are two fixed moments (M_{FBC}, M_{FBA}). These two values are to be added.

For the ~~2~~ 2nd D_K, ~~*~~ is at joint C and the unknown is θ_C. There are ~~two~~ three fixed end moments (M_{FCB}, M_{FCD}, M_{FCE}).

And, for ~~3d~~ 3rd D_K, is at joint E, the unknown is θ_E. There is only one fixed end moment i.e M_{FEC}

$[K_{ij}]_{n\times n}$ ⇒ stiffness ~~factor~~ matrix.

$[\delta]_{n\times 1}$ ⇒ unknown (θ_B, θ_C and θ_E)

$[P]_{n\times 1}$ ⇒ External force occuring at joint ~~ø~~ (unknown D_K)
i.e moment at joint (B, C, E)

$\therefore$ $[P']_{n\times 1} + [K_{ij}]_{n\times n} * [\delta_{n\times 1}] = [P]_{n\times 1}$ can be written as

$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{bmatrix} M_{FBA}+M_{FBC} \\ M_{FCB}+M_{FCD}+M_{FCE} \\ M_{FEC} \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} * \begin{bmatrix} \theta_B \\ \theta_C \\ \theta_E \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{bmatrix} -\frac{110}{3} \\ \frac{105}{15} \\ 144 \end{bmatrix} + EI \begin{bmatrix} 2 & 0.5 & 0 \\ 0.5 & \frac{14}{5} & \frac{2}{5} \\ 0 & \frac{2}{3} & \frac{4}{5} \end{bmatrix} \times \begin{bmatrix} \theta_B \\ \theta_C \\ \theta_E \end{bmatrix} = \begin{bmatrix} -10 \\ 30 \\ 0 \end{bmatrix}$$

After solving this equation we will get the unknown values $(\theta_B, \theta_c, \theta_E)$.

$$\theta_B = \frac{10.9632}{EI} \text{ rad.}$$

$$\theta_c = \frac{9.4813}{EI} \text{ rad.}$$

$$\theta_E = \frac{-22.7407}{EI} \text{ rad.}$$

For final moments.

Add all the moments caused due to the fixed end moment, Moment due to rotation θ_B, moment due to rotation θ_c and moment due to rotation θ_E.

$$M_{AB} = M_{FAB} + \frac{EI}{2}\theta_B$$

$$= 10 + \frac{EI}{2}\left(\frac{10.9632}{EI}\right)$$

$$= 15.4816 \text{ kNm}$$

$$M_{BA} = -10 + EI\,\theta_B = 0.9631 \text{ KNm}$$

$$M_{BC} = -\frac{80}{3} + EI\,\theta_B + 0.5\,EI\,\theta_C = -10.9628 \text{ kNm}$$

$$M_{CB} = \frac{80}{3} + 0.5\,EI\,\theta_B + EI\,\theta_C = 41.6296 \text{ KNm}$$

$$M_{CD} = -10 + EI\,\theta_C = -0.5187 \text{ KNm}$$

$$M_{DC} = 10 + \frac{EI}{2}(\theta_C) = 14.7404 \text{ KNm}$$

$$M_{CE} = -9.6 + \frac{4}{5}EI\theta_C + \frac{2}{5}EI\,\theta_E = -11.1112 \text{ KNm}$$

$$M_{EC} = 14.4 + \frac{2}{5}EI\,\theta_C + \frac{4}{5}EI\,\theta_E = 0$$