Illustration

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Re-state the question:

Climate researchers are interested in modeling the hurricane trajectories to forcase the windspeed. Let t be time (in hours) since a hurricane began, and For each hurrican i, we denote $Y_i(t)$ be the wind speed of the ith hurrican at time t. The following Baysian model was suggested.

$$Y_i(t+6) = \beta_{0,i} + x_{i,1}\beta_{1,i} + x_{i,2}\beta_{2,i} + x_{i,3}\beta_{3,i} + \beta_{4,i,j}Y_{i,j}(t) + +\beta_{5,i,j}\Delta_{i,1}(t) + \beta_{6,i,j}\Delta_{i,2}(t) + +\beta_{7,i}\Delta_{i,3} + \epsilon_i(t)$$

where $x_{i,1}$ is the month of year when the hurricane started, $x_{i,2}$ is the calendar year of the hurricane, and $x_{i,3}$ is the type of hurricane, $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ is the change of latitude longitude, and wind speed between $t-6$ and t, and $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t.

In the model, $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, ..., \beta_{7,i})$ are the random coefficients associated the ith hurricane, we assume that

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma})$$

follows a multivariate normal distributions with mean $\boldsymbol{\beta}$ and covariance matrix Σ.

Prior distributions We assume the following non-informative or weak prior distributions for σ^2, $\boldsymbol{\beta}$ and Σ.

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\boldsymbol{\beta}) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp(-\frac{1}{2}\Sigma^{-1})$$

d is dimension of β.

Find out the posterial distribution:

It is easy to see that:

$$f(Y_i(t+6)|\boldsymbol{\beta}, \sigma) \sim N(\mu, \sigma^2)$$

Where μ is the linear combination of all coefficients.

Our objective is to derive $f(\boldsymbol{\beta}, \sigma | Y_i(t+6))$

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\begin{aligned}
f(\boldsymbol{\beta}, \boldsymbol{\beta_i}, \boldsymbol{\Sigma}^{-1}, \sigma^2 | Y_i(t+6)) &\propto f(Y_i(t+6)|\boldsymbol{\beta_i}, \sigma^2) f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) f(\boldsymbol{\beta}) f(\boldsymbol{\Sigma}^{-1}) f(\sigma^2) \\
f(Y_i(t+6)|\boldsymbol{\beta_i}, \sigma) &\propto \prod_{i=1}^{N} \frac{1}{\sigma} \exp\{-\frac{(x_i - \mu_i)^2}{2\sigma^2}\} \text{ ,where } \mu_i = X_i^T \boldsymbol{\beta_i} \\
f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) &\propto \frac{1}{\sqrt{|\boldsymbol{\Sigma^{-1}}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\} \\
f(\sigma^2) &\propto \frac{1}{\sigma^2} \\
f(\boldsymbol{\beta}) &\propto 1 \\
f(\boldsymbol{\Sigma}^{-1}) &\propto |\Sigma|^{-8} \exp(-\frac{1}{2}\Sigma^{-1})
\end{aligned}
$$