

# Project 3: Bayesian Modeling of Hurricane Trajectories

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## Introduction

Hierarchical Bayesian Model is the combination of two methods: Linear Regression Model in Hierarchical

Hierarchical form model is the linear regression model containing both Within-group analysis and Between-group analysis. Observations for each subject, from hierarchical form model, are usually collected with natural heterogeneity across the whole population over research time periods. This natural heterogeneity can be regarded as subject-specific mean response trajectories (i.e., random effects) for each individual group. Nested data for each individual group violates the independent assumption for the linear regression model. Holding these individual-specific effects, the overall mean response over time across the whole research population is still considered as linearly over time, which is called Population-level effects (i.e., fixed effects).

$$\underbrace{\pi(\theta|X)}_{\text{posterior distribution}} \propto \underbrace{\pi(X|\theta)}_{\text{likelihood}} \times \underbrace{\pi(\theta)}_{\text{prior distribution}}$$

Bayesian Inference is a statistical inference method about parameter. Before data collection, a proper prior distribution of parameter  $\theta$  is set based on our belief about  $\theta$ . Then after data collection  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , the belief of parameter  $\theta$  would be updated by exploring the posterior distribution of  $\theta$  based on observed data and its pre-assumed likelihood function  $L(X; \theta)$ . The linear regression model in hierarchical form incorporating with Bayesian inference is implemented with Markov Chain Monte Carlo Integration algorithm for updating the parameter estimation in the final MCMC stationary phase.

## Objective

In this project, we are going to explore whether the population-level changing trend of hurricane wind speed over years through a total of 356 groups of hurricanes. Each hurricane contained its own individual-level-specific effects. The hierarchical Bayesian model for the  $i$ th hurricane is shown as

$$Y_i(t+6) = \beta_{0,i} + X_{i_{Month}}\beta_{1,i} + X_{i_{Year}}\beta_{2,i} + X_{i_{Type}}\beta_{3,i} + X_{i_{Y_{i,t}(t)}}\beta_{4,i,t} + \Delta_{i,t_{lat}}\beta_{5,i,t} + \Delta_{i,t_{lon}}\beta_{6,i,t} + \Delta_{i,t_{Speed}}\beta_{7,i,t} + \epsilon_i(t),$$

where  $i = 1, 2, \dots, m$  standing for each hurricane group and  $t = 1, 2, \dots, t_i$  standing for each recorded time point within  $i$ th hurricane group.

The provided hierarchical Bayesian model for hurricane trajectories for  $i$ th hurricane contains 4 population-level effects (i.e., fixed):  $X_{i_{Month}}$  - the month of year when hurricane started,  $X_{i_{Year}}$  - the calendar year of the hurricane,  $X_{i_{Type}}$  - the type of hurricane, and 4 individual-level-effects (i.e., random):  $X_{i_{Y_{i,t}(t)}}$  - the  $i$ th wind speed at  $t-1$  time point for  $t$  time point,  $\Delta_{i,t_{lat}}$ ,  $\Delta_{i,t_{lon}}$ ,  $\Delta_{i,t_{Speed}}$  - the change of latitudes, longitudes and wind speeds between two recorded time points. 4 prior information are provided as following:

$$\beta_i \sim N(\beta, \Sigma^{-1}), \quad \pi(\sigma^2) \propto \frac{1}{\sigma^2},$$
$$\pi(\beta) \propto 1, \quad \pi(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)/2} \exp\left(-\frac{1}{2}\Sigma^{-1}\right),$$

Then our Hierarchical Bayesian Model for analyzing the trend of hurricane wind speed across years with considering all other covariates would be performed firstly by exploring estimated parameters from posterior distributions based on Bayes Theorem.

## Hurricane Data

hurricane356.csv collected the track data of 356 hurricanes in the North Atlantic area since 1989. For all the storms, their location (longitude & latitude) and maximum wind speed were recorded every 6 hours. The data includes the following variables

1. **ID**: ID of the hurricanes
2. **Season**: In which **year** the hurricane occurred
3. **Month**: In which **month** the hurricane occurred
4. **Nature**: Nature of the hurricane
  - ET: Extra Tropical
  - DS: Disturbance
  - NR: Not Rated
  - SS: Sub Tropical
  - TS: Tropical Storm
5. **time**: dates and time of the record
6. **Latitude** and **Longitude**: The location of a hurricane check point
7. **Wind.kt** Maximum wind speed (in Knot) at each check point

## Posterior Distribution Inference and MCMC Methods

### Posterior Distribution Inference

We have  $\beta_i = (\beta_{0,i}, \beta_{1,i}, \dots, \beta_{7,i,t})$  associated with the  $i$ th hurricane. According to the knowledge of Multi-variate Linear Regression Model, we know that  $Y_{i,t} \sim N(X_{i,t}\beta_i, \sigma^2 I_{t_i})$ , denoting,  $\mu_{i,t} = \mathbf{X}_{i,t}\beta_i$ ,

Then we have,

$$\pi(\mathbf{Y}_{i,t} | \mu_{i,t}, \sigma^2, \beta, \Sigma^{-1}) \propto \frac{1}{\sqrt{\sigma^2}} \exp\left\{-\frac{1}{2}(\mathbf{Y}_{i,t} - \mu_{i,t})^T (\sigma^2 I_{t_i})^{-1} (\mathbf{Y}_{i,t} - \mu_{i,t})\right\}$$

1. For  $\pi(\beta_i | \cdot)$ :

$$\pi(\beta_i | \cdot) \propto f(\mathbf{Y}_i | \beta_i, \sigma^2) f(\beta_i | \beta, \Sigma^{-1}) \tag{1}$$

$$\propto \left( \prod_{t=1}^{t_i} \frac{1}{\sqrt{\sigma^2}} \exp\left\{-\frac{1}{2}(\mathbf{Y}_{i,t} - \mu_{i,t})^T (\sigma^2 I_{t_i})^{-1} (\mathbf{Y}_{i,t} - \mu_{i,t})\right\} \right) \left( |\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(\beta_i - \beta)^T \Sigma^{-1} (\beta_i - \beta)\right\} \right) \tag{2}$$

$$\propto \exp\left\{-\frac{1}{2} \left( (\mathbf{Y}_i - \mathbf{X}_i \beta_i)^T (\sigma^{-2} \mathbf{I}_{t_i \times t_i}) (\mathbf{Y}_i - \mathbf{X}_i \beta_i) + (\beta_i - \beta)^T \Sigma_{8 \times 8}^{-1} (\beta_i - \beta) \right)\right\} \tag{3}$$

For the exponential term:

$$(\mathbf{Y}_i - \mathbf{X}_i \beta_i)^T (\sigma^{-2} \mathbf{I}_{t_i \times t_i}) (\mathbf{Y}_i - \mathbf{X}_i \beta_i) + (\beta_i - \beta)^T \Sigma_{8 \times 8}^{-1} (\beta_i - \beta) \quad (4)$$

$$= \mathbf{Y}_i^T \sigma^{-2} \mathbf{I} \mathbf{Y}_i^T + \beta_i^T \mathbf{X}_i^T \sigma^{-2} \mathbf{I} \mathbf{X}_i \beta_i - 2 \mathbf{Y}_i^T \sigma^{-2} \mathbf{I} \mathbf{X}_i \beta_i \quad (5)$$

$$+ \beta_i^T \Sigma^{-1} \beta_i + \beta^T \Sigma^{-1} \beta - 2 \beta^T \Sigma^{-1} \beta_i \quad (6)$$

$$= \mathbf{Y}_i^T \sigma^{-2} \mathbf{I} \mathbf{Y}_i^T + \beta^T \Sigma^{-1} \beta + \beta_i^T (\Sigma^{-1} + \mathbf{X}_i^T \sigma^{-2} \mathbf{I} \mathbf{X}_i) \beta_i \quad (7)$$

$$- 2 (\mathbf{Y}_i^T \sigma^{-2} \mathbf{I} \mathbf{X}_i + \beta^T \Sigma^{-1}) \beta_i \quad (8)$$

$$= \mathbf{R} + \beta_i^T \mathbf{V} \beta_i - 2 \mathbf{M} \beta_i \quad (9)$$

Where:

$$\mathbf{R} = \mathbf{Y}_i^T \sigma^{-2} \mathbf{I}_{t_i \times t_i} \mathbf{Y}_i + \beta^T \Sigma^{-1} \beta \quad (10)$$

$$\mathbf{V} = \Sigma^{-1} + \sigma^{-2} \mathbf{X}_i^T \mathbf{X}_i \quad (11)$$

$$\mathbf{M} = \sigma^{-2} \mathbf{X}_i^T \mathbf{Y}_i + \Sigma^{-1} \beta \quad (12)$$

Then, the exponential term can be reduced to:

$$(\beta_i - \mathbf{V}^{-1} \mathbf{M})^T \mathbf{V} (\beta_i - \mathbf{V}^{-1} \mathbf{M}) - \mathbf{M}^T \mathbf{V}^{-1} \mathbf{M}^T + \mathbf{R}$$

We can ignore the latter 2 term as it is not related to  $\beta_i$ . That indicate:

$$\pi(\beta_i | \cdot) \sim N(\mathbf{V}^{-1} \mathbf{M}, \mathbf{V}^{-1})$$

2. For  $\pi(\sigma^2 | \cdot)$ :

$$\pi(\sigma^2 | \cdot) \propto f(\mathbf{Y} | \beta_i, \sigma^2) \cdot \pi(\sigma^2) \quad (13)$$

$$\propto \left( \prod_{i=1}^n \prod_{t=1}^{t_i} \sigma^{-1} \exp \left\{ -\frac{(\mathbf{Y}_{i,t} - \boldsymbol{\mu}_{i,t})^2}{2\sigma^2} \right\} \right) \frac{1}{\sigma^2} \quad (14)$$

$$\propto (\sigma^2)^{-1 - \sum_{i=1}^n \sum_{t=1}^{t_i}} \prod_{i=1}^n \prod_{t=1}^{t_i} \exp \left\{ -\frac{(\mathbf{Y}_{i,t} - \boldsymbol{\mu}_{i,t})^2}{2\sigma^2} \right\} \quad (15)$$

$$\propto \sigma^{-2 - \sum_{i=1}^n \sum_{t=1}^{t_i}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n \sum_{t=1}^{t_i} (\mathbf{Y}_{i,t} - \boldsymbol{\mu}_{i,t})^2 \right\} \quad (16)$$

So:

$$\sigma^2 \sim \text{Inverse Gamma} \left( \frac{1}{2} \sum_{i=1}^n t_i, \frac{1}{2} \sum_{i=1}^n \sum_{t=1}^{t_i} (\mathbf{Y}_{i,t} - \boldsymbol{\mu}_{i,t})^2 \right)$$

3. For  $\pi(\Sigma^{-1} | \cdot)$ : We have the prior of  $\pi(\Sigma^{-1}) \sim |\Sigma|^{-(d+1)/2} \exp(-\frac{1}{2} \Sigma^{-1})$ , which actually follows the Inverse Wishart distribution  $(\Sigma^{-1}) \sim \text{Inverse Wishart}(v_0, S_0)$ , where  $v_0 = 0, S_0 = 1$

$$\pi(\Sigma^{-1} | \cdot) \propto f(\beta_i | \beta, \Sigma^{-1}) f(\Sigma^{-1}) \quad (17)$$

$$\propto \left( \prod_{i=1}^n \Sigma^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\beta_i - \beta)^T \Sigma^{-1} (\beta_i - \beta) \right\} \right) |\Sigma|^{-(d-1)/2} \exp \left\{ -\frac{1}{2} \Sigma^{-1} \right\} \quad (18)$$

$$\propto |\Sigma|^{-(n+d+1)/2} \exp \left\{ -\frac{1}{2} \text{tr}(\Sigma^{-1}) - \frac{1}{2} \sum_{i=1}^n (\beta_i - \beta)^T \Sigma^{-1} (\beta_i - \beta) \right\} \quad (19)$$

$$\propto |\Sigma|^{-(n+d+1)/2} \exp \left\{ -\frac{1}{2} \text{tr} \left( \Sigma^{-1} \left( \mathbf{I} + \sum_{i=1}^n (\beta_i - \beta)(\beta_i - \beta)^T \right) \right) \right\} \quad (20)$$

That indicate:

$$\Sigma^{-1} \sim \text{Inverse Whishart} \left( n, \mathbf{I} + \sum_{i=1}^n (\beta_i - \beta)^T (\beta_i - \beta) \right)$$

4. For  $\pi(\beta|\cdot)$ :

$$\pi(\beta|\cdot) \propto f(\beta_i|\beta, \Sigma^{-1})f(\beta) \quad (21)$$

$$\propto \left( \prod_{i=1}^n \exp \left\{ -\frac{1}{2} (\beta_i - \beta)^T \Sigma^{-1} (\beta_i - \beta) \right\} \right) \quad (22)$$

$$\propto \exp \left\{ -\frac{1}{2} \left( \sum_{i=1}^n (\beta_i - \beta)^T \Sigma^{-1} (\beta_i - \beta) \right) \right\} \quad (23)$$

$$\propto \exp \left\{ -\frac{1}{2} \left( \sum_{i=1}^n \beta_i^T \Sigma^{-1} \beta_i + \beta^T n \Sigma^{-1} \beta - \sum_{i=1}^n 2 \beta_i^T \Sigma^{-1} \beta \right) \right\} \quad (24)$$

For the exponential term, if we set:

$$\mathbf{V} = n \Sigma^{-1} \quad (25)$$

$$\mathbf{R} = \sum_{i=1}^n \beta_i^T \Sigma^{-1} \beta_i \quad (26)$$

$$\mathbf{M} = \sum_{i=1}^n \left( \Sigma^{-1} \beta_i \right) \quad (27)$$

Then, use the same technique when generating  $\beta_i$

$$\mathbf{R} + \beta \mathbf{V} \beta - 2 \mathbf{M} \beta \propto (\beta - \mathbf{V}^{-1} \mathbf{M})^T \mathbf{V}^{-1} (\beta - \mathbf{V}^{-1} \mathbf{M})$$

(NOTE: This is the same as using OLS to estimate  $\beta$  using all  $\beta_i$ )

That indicate:

$$\beta \sim N \left( \text{Vec}(\hat{\beta}_i), \Sigma \beta_i^T \beta_i \right)$$