Posteria distribution deriviation

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1 The model:

The main model is:

$$Y_i(t+6) = \beta_{0,i} + x_{i,1}\beta_{1,i} + x_{i,2}\beta_{2,i} + x_{i,3}\beta_{3,i} + \beta_{4,i}Y_i(t)$$

$$+\beta_{5,i}\Delta_{i,1}(t) + \beta_{6,i}\Delta_{i,2}(t) + \beta_{7,i}\Delta_{i,3} + \epsilon_i(t)$$

(NOTE: omit j as we dont know what it is for now)

1. $x_{i,1}$ is the month of year when the hurricane started

2. $x_{i,2}$ is the calendar year of the hurricane

3. $x_{i,3}$ is the type of hurricane

4. $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ is the change of latitude longitude, and wind speed between $t-6$ and t

5. $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t

2 Distributon and priors:

Marginal distributions for $Y_{i,t+6}$:

$$f(Y_{i,t+6}|\boldsymbol{\beta}_i, \sigma) \sim N(\mu_{i,t}, \sigma^2)$$
$$\mu_{i,t} = Y_i(t+6) - \epsilon_i(t)$$
$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma})$$

Priors:

$$f(\sigma^2) \propto \frac{1}{\sigma^2}$$
$$f(\boldsymbol{\beta}) \propto 1$$
$$f(\boldsymbol{\Sigma}^{-1}) \propto |\boldsymbol{\Sigma}|^{-(d-1)/2} \exp\{-\frac{1}{2}tr(\boldsymbol{\Sigma}^{-1})\}$$

3 Hierarchical model structure:

Denote $\boldsymbol{Y} = \{Y_{i,t}\}, i = 1, 2, \ldots, n, t = 1, 2, \ldots, t_i$.
Hierarchical model will be:

$$f(\boldsymbol{\beta}, \boldsymbol{\beta_i}, \boldsymbol{\Sigma}^{-1}, \sigma^2 | \boldsymbol{Y}) \propto f(\boldsymbol{Y}|\boldsymbol{\beta_i}, \sigma^2) f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) f(\boldsymbol{\beta}) f(\boldsymbol{\Sigma}^{-1}) f(\sigma^2)$$

$$f(\boldsymbol{Y}|\boldsymbol{\beta_i}, \sigma) \propto \prod_{i=1}^{n} \prod_{t=1}^{t_i} \sigma^{-1} \exp\{-\frac{(Y_{i,t} - \mu_{i,t})^2}{2\sigma^2}\}$$

$$f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) \propto \prod_{i=1}^{n} \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\}$$

After put all element into the formula, the full posteria will be the mutiplication of all terms

4 Posterior Distribution Inference:

Suppose for hurricane i, we have t_i observations which can be used to build model, denote following matrixs for response and predictors for hurricane i:

$$\boldsymbol{Y_i} = (Y_{i,1}, Y_{i,2}, \ldots, Y_{i,t_i})^T$$

$$\boldsymbol{X_i} = \begin{bmatrix} x_{i,1,1} & x_{i,2,1} & \cdots & x_{i,8,1} \\ x_{i,1,2} & x_{i,2,2} & \cdots & x_{i,8,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i,1,t_i} & x_{i,2,t_i} & \cdots & x_{i,8,t_i} \end{bmatrix}_{t_i \times 8}$$

1. For $\pi(\boldsymbol{\beta_i}|.)$:

$$\pi(\boldsymbol{\beta_i}|.) \propto f(\boldsymbol{Y_i}|\boldsymbol{\beta_i}, \sigma^2) f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1})$$

$$\propto \left(\prod_{t=1}^{t_i} \sigma^{-1} \exp\{-\frac{(Y_{i,t} - \mu_{i,t})^2}{2\sigma^2}\} \right) \left(\frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\} \right)$$

$$\propto \exp\{-\frac{1}{2}\left(\sum_{t=1}^{t_i} \sigma^{-2}(Y_{i,t} - \boldsymbol{X_{i,t}}\boldsymbol{\beta_i})^2 + (\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i} - \boldsymbol{\beta}) \right)\}$$

$$\propto \exp\{-\frac{1}{2}\left((\boldsymbol{Y_i} - \boldsymbol{X_i}\boldsymbol{\beta_i})^T (\sigma^{-2}\boldsymbol{I}_{t_i \times t_i})(\boldsymbol{Y_i} - \boldsymbol{X_i}\boldsymbol{\beta_i}) + (\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}_{8 \times 8}^{-1}(\boldsymbol{\beta_i} - \boldsymbol{\beta}) \right)\}$$

For the expnential term:

$$(\boldsymbol{Y_i} - \boldsymbol{X_i}\boldsymbol{\beta_i})^T (\sigma^{-2}\boldsymbol{I}_{t_i \times t_i})(\boldsymbol{Y_i} - \boldsymbol{X_i}\boldsymbol{\beta_i}) + (\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}_{8 \times 8}^{-1}(\boldsymbol{\beta_i} - \boldsymbol{\beta})$$

$$= \boldsymbol{Y_i}^T \sigma^{-2}\boldsymbol{I}\boldsymbol{Y_i}^T + \boldsymbol{\beta_i}^T \boldsymbol{X_i}^T \sigma^{-2}\boldsymbol{I}\boldsymbol{X_i}\boldsymbol{\beta_i} - 2\boldsymbol{Y_i}^T \sigma^{-2}\boldsymbol{I}\boldsymbol{X_i}\boldsymbol{\beta_i}$$

$$\quad + \boldsymbol{\beta_i}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta_i} + \boldsymbol{\beta}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta} - 2\boldsymbol{\beta}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta_i}$$

$$= \boldsymbol{Y_i}^T \sigma^{-2}\boldsymbol{I}\boldsymbol{Y_i}^T + \boldsymbol{\beta}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta} + \boldsymbol{\beta_i}^T (\boldsymbol{\Sigma}^{-1} + \boldsymbol{X_i}^T \sigma^{-2}\boldsymbol{I}\boldsymbol{X_i})\boldsymbol{\beta_i}$$

$$\quad - 2(\boldsymbol{Y_i}^T \sigma^{-2}\boldsymbol{I}\boldsymbol{X_i} + \boldsymbol{\beta}^T \boldsymbol{\Sigma}^{-1})\boldsymbol{\beta_i}$$

$$= \boldsymbol{R} + \boldsymbol{\beta_i}^T \boldsymbol{V}\boldsymbol{\beta_i} - 2\boldsymbol{M}\boldsymbol{\beta_i}$$

Where:

$$\boldsymbol{R} = \boldsymbol{Y_i}^T \sigma^{-2} \boldsymbol{I} \boldsymbol{Y_i}^T + \boldsymbol{\beta}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\beta}$$
$$\boldsymbol{V} = \boldsymbol{\Sigma}^{-1} + \sigma^{-2} \boldsymbol{X_i}^T \boldsymbol{X_i}$$
$$\boldsymbol{M} = \sigma^{-2} \boldsymbol{Y_i}^T \boldsymbol{X_i} + \boldsymbol{\beta}^T \boldsymbol{\Sigma}^{-1}$$

Then, the exponential term can be reduced to:

$$(\boldsymbol{\beta_i} - \boldsymbol{V^{-1}M})^T \boldsymbol{V}(\boldsymbol{\beta_i} - \boldsymbol{V^{-1}M}) - \boldsymbol{M}^T \boldsymbol{V^{-1}} \boldsymbol{M}^T + \boldsymbol{R}$$

We can ignore the latter 2 term as it is not related to $\boldsymbol{\beta_i}$. That indicate:

$$\pi(\boldsymbol{\beta_i}|.) \sim N(\boldsymbol{V^{-1}M}, \boldsymbol{V^{-1}})$$

2.For $\pi(\sigma^2|.)$:

$$\pi(\sigma^2|.) \propto f(\boldsymbol{Y}|\boldsymbol{\beta_i}, \sigma^2) \cdot \pi(\sigma^2)$$

$$\propto \left(\prod_{i=1}^{n} \prod_{t=1}^{t_i} \sigma^{-1} \exp\{-\frac{(Y_{i,t} - \mu_{i,t})^2}{2\sigma^2}\} \right) \frac{1}{\sigma^2}$$

$$\propto (\sigma^2)^{-1-\frac{\sum t_i}{2}} \prod_{i=1}^{n} \prod_{t=1}^{t_i} \exp\{-\frac{(Y_{i,t} - \mu_{i,t})^2}{2\sigma^2}\}$$

$$\propto \sigma^{-2-\sum t_i} \exp\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} \sum_{t=1}^{ti} (Y_{i,t} - \mu_{i,t})^2\}$$

So:

$$\sigma^2 \sim \text{InvGamma}(\frac{1}{2}\sum_{i=1}^{n} t_i, \frac{1}{2}\sum_{i=1}^{n}\sum_{t=1}^{ti}(Y_{i,t} - \mu_{i,t})^2)$$

3. For $\pi(\boldsymbol{\Sigma}^{-1}|.)$:

$$\pi(\boldsymbol{\Sigma}^{-1}|.) \propto f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) f(\boldsymbol{\Sigma}^{-1})$$

$$\propto \left(\prod_{i=1}^{n} \boldsymbol{\Sigma}^{-\frac{1}{2}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i} - \boldsymbol{\beta})\} \right) |\boldsymbol{\Sigma}|^{-(d-1)/2} \exp\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\}$$

$$\propto |\boldsymbol{\Sigma}|^{-(n+d+1)/2} \exp\{-\frac{1}{2}tr(\boldsymbol{\Sigma}^{-1}) - \frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i} - \boldsymbol{\beta})\}$$

$$\propto |\boldsymbol{\Sigma}|^{-(n+d+1)/2} \exp\{-\frac{1}{2}tr\left(\boldsymbol{\Sigma}^{-1}(\boldsymbol{I} + \sum_{i=1}^{n}(\boldsymbol{\beta_i} - \boldsymbol{\beta})(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T) \right)\}$$

That indicate:

$$\boldsymbol{\Sigma}^{-1} \sim \text{InvWhishart}\left(n, \boldsymbol{I} + \sum_{i=1}^{n}(\boldsymbol{\beta_i} - \boldsymbol{\beta})(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \right)$$

4. For $\pi(\boldsymbol{\beta}|.)$:

$$\pi(\boldsymbol{\beta}|.) \propto f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) f(\boldsymbol{\beta})$$

$$\propto \left(\prod_{i=1}^{n} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i} - \boldsymbol{\beta})\} \right)$$

$$\propto \exp\{-\frac{1}{2}\left(\sum_{i=1}^{n}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i} - \boldsymbol{\beta}) \right)\}$$

$$\propto \exp\{-\frac{1}{2}\left(\sum_{i=1}^{n}\boldsymbol{\beta_i}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta_i} + \boldsymbol{\beta}^T n \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta} - \sum_{i=1}^{n} 2\boldsymbol{\beta_i}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta} \right)\}$$

For the exponential term, if we set:

$$\boldsymbol{V} = n\boldsymbol{\Sigma}^{-1}$$

$$\boldsymbol{R} = \sum_{i=1}^{n} \boldsymbol{\beta_i}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta_i}$$

$$\boldsymbol{M} = \sum_{i=1}^{n} \left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\beta_i} \right)$$

Then, use the same technique when generating $\boldsymbol{\beta_i}$

$$R + \boldsymbol{\beta V \beta} - 2\boldsymbol{M\beta} \propto (\boldsymbol{\beta} - \boldsymbol{V}^{-1}\boldsymbol{M})^T \boldsymbol{V}^{-1}(\boldsymbol{\beta} - \boldsymbol{V}^{-1}\boldsymbol{M})$$

(NOTE: This is the same as using OLS to estimate $\boldsymbol{\beta}$ using all $\boldsymbol{\beta_i}$)
That indicate:

$$\boldsymbol{\beta} \sim N\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{\beta_i}, \frac{1}{n}\boldsymbol{\Sigma} \right)$$