Illustration

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Re-state the question:

Climate researchers are interested in modeling the hurricane trajectories to forecast the wind speed. Let t be time (in hours) since a hurricane began, and For each hurricane i, we denote $Y_i(t)$ be the wind speed of the ith hurricane at time t. The following Bayesian model was suggested.

$$Y_i(t+6) = \beta_{0,i} + x_{i,1}\beta_{1,i} + x_{i,2}\beta_{2,i} + x_{i,3}\beta_{3,i} + \beta_{4,i,j}Y_{i,j}(t) + +\beta_{5,i,j}\Delta_{i,1}(t) + \beta_{6,i,j}\Delta_{i,2}(t) + +\beta_{7,i}\Delta_{i,3} + \epsilon_i(t)$$

where $x_{i,1}$ is the month of year when the hurricane started, $x_{i,2}$ is the calendar year of the hurricane, and $x_{i,3}$ is the type of hurricane, $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ is the change of latitude longitude, and wind speed between $t-6$ and t, and $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t.

In the model, $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, ..., \beta_{7,i})$ are the random coefficients associated the ith hurricane, we assume that

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma})$$

follows a multivariate normal distributions with mean $\boldsymbol{\beta}$ and covariance matrix Σ.

Prior distributions We assume the following non-informative or weak prior distributions for σ^2, $\boldsymbol{\beta}$ and Σ.

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\boldsymbol{\beta}) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp(-\frac{1}{2}\Sigma^{-1})$$

d is dimension of β.

Find out the posterial distribution:

It is easy to see that:

$$f(Y_i(t+6)|\boldsymbol{\beta}, \sigma) \sim N(\mu, \sigma^2)$$

Where μ is the linear combination of all coefficients.

Our objective is to derive $f(\boldsymbol{\beta}, \sigma|Y_i(t+6))$

$$
\begin{aligned}
f(\boldsymbol{\beta}, \boldsymbol{\beta_i}, \boldsymbol{\Sigma}^{-1}, \sigma^2 | Y_i(t+6)) &\propto f(Y_i(t+6)|\boldsymbol{\beta_i}, \sigma^2) f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) f(\boldsymbol{\beta}) f(\boldsymbol{\Sigma}^{-1}) f(\sigma^2) \\
f(Y_i(t+6)|\boldsymbol{\beta_i}, \sigma) &\propto \prod_{i=1}^{N} \frac{1}{\sigma} \exp\{-\frac{(x_i - \mu_i)^2}{2\sigma^2}\} \text{ ,where } \mu_i = X_i^T \boldsymbol{\beta_i} \\
f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) &\propto \frac{1}{\sqrt{|\boldsymbol{\Sigma^{-1}}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\} \\
f(\sigma^2) &\propto \frac{1}{\sigma^2} \\
f(\boldsymbol{\beta}) &\propto 1 \\
f(\boldsymbol{\Sigma}^{-1}) &\propto |\Sigma|^{-8} \exp(-\frac{1}{2}\Sigma^{-1})
\end{aligned}
$$

The full posterior, then, is simply the product of these three terms—the likelihood, prior, and the distribution for Σ.

$$
\prod_{i=1}^{N} \frac{1}{\sigma} \exp\{-\frac{(x_i - \mu_i)^2}{2\sigma^2}\} * \frac{1}{\sqrt{|\boldsymbol{\Sigma^{-1}}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\} * |\Sigma|^{-8} \exp(-\frac{1}{2}\Sigma^{-1})
$$

```r
data = read.csv("hurrican356.csv")
df = data %>%
  mutate(Month = factor(Month, levels = c("January", "April", "May", "June", "July", "August",
                                          "September", "October", "November", "December")),
         Month = as.numeric(Month),
         Nature = as.numeric(as.factor(Nature)))
id = df %>% group_by(Season, ID) %>% summarize(n = n())
```

```
## `summarise()` has grouped output by 'Season'. You can override using the `.groups` argument.
```

```r
upd.bi = function(df, epis, beta, sigma, n){
  sd = solve(epis + diag(n*sigma, 8, 8)) # posterior distribution for beta.i
  mu = sd %*% (epis %*% beta + sigma*sum(df %*% beta)) # posterior distribution for beta.i
  return(list(sd = sd, mu = mu))}
mc = function(df, a, b, ini.sigma, niter = 2){
  delta1 = c(0, sapply(2:nrow(df), function(i){df[i,7] - df[i-1, 7]}))
  delta2 = c(0, sapply(2:nrow(df), function(i){df[i,8] - df[i-1, 8]}))
  delta3 = c(0, sapply(2:nrow(df), function(i){df[i,9] - df[i-1, 9]}))
  dt = cbind(df[,1], df[,4], df[,3], df[,5], df[,9], delta1, delta2, delta3) # final dataset for each i
  n = nrow(dt)
  epis = vector("list", niter)
  beta = matrix(NA, nrow = 8, ncol = niter)
  beta.i = matrix(NA, nrow = 8, ncol = niter)
  sigma = rep(NA, niter)
  epis[[1]] = cor(matrix(1/extraDistr::rinvgamma(64, alpha = a, beta = b), 8, 8)) # initial epsilon^-1
  beta[,1] = rep(1, 8) # initial beta (8x1)
  sigma[1] = ini.sigma # initial sigma
  beta.i[,1] = mvrnorm(1, mu = upd.bi(dt, epis[[1]], beta[,1], sigma[1], n)$mu,
  Sigma = upd.bi(dt, epis[[1]], beta[,1], sigma[1], n)$sd)
  # mvrnorm generates the beta_i for each subgroup(8x1) - beta.i ~ N(beta, epsilon)
  for (i in 2:niter){
    epis[[i]] = rinvwishart(n + a, diag(sum((beta.i[,i-1] - beta[,i-1])^2), 8, 8) + 0.001)
    # use beta.i and beta update epsilon, pi(epsilon) ~ Inviwishart(n + a, beta_variance + b)!!!!!!!!!!
    beta[,i] = mvrnorm(1, mu = n*beta.i[,i-1], Sigma = n*epis[[i]])  # beta ~ N(n*beta_i, n*epsilon)
```

```r
    sigma[i] = extraDistr::rinvgamma(1, alpha = n + 1, sum((dt[,5] - dt %*% beta.i[,i-1])^2))
    # sigma ~ inverse.gamma(n + 1, sample residual of Yi)
    beta.i[,i] = mvrnorm(1, mu = upd.bi(dt, epis[[i]], beta[,i], sigma[i], n)$mu,
  Sigma = upd.bi(dt, epis[[i]], beta[,i], sigma[i], n)$sd)}
  return(list(beta.i =  beta.i))}

whole.mc = function(df, id){
  sub.grp = vector("list", nrow(id))
  id.n = c(0, id$n)
  for (i in 1:length(sub.grp)){
    sub.grp[[i]] = df[(id.n[i]+1):id.n[i+1],]}
  ith.mc = vector("list", length(sub.grp))
  for (ii in 1:length(ith.mc)){
    ith.mc[[ii]] = sapply(1: length(sub.grp),
                          function(iii){mc(sub.grp[[iii]],a = 0.001, b = 0.001, ini.sigma = 1)})}
  return(ith.mc)}
```