Posterial distribution generation

1 The model:

The main model is:

$$Y_i(t+6) = \beta_{0,i} + x_{i,1}\beta_{1,i} + x_{i,2}\beta_{2,i} + x_{i,3}\beta_{3,i} + \beta_{4,i}Y_i(t)$$

$$+\beta_{5,i}\Delta_{i,1}(t) + \beta_{6,i}\Delta_{i,2}(t) + \beta_{7,i}\Delta_{i,3} + \epsilon_i(t)$$

(NOTE: omit j as we dont know what it is for now)

1. $x_{i,1}$ is the month of year when the hurricane started
2. $x_{i,2}$ is the calendar year of the hurricane
3. $x_{i,3}$ is the type of hurricane
4. $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ is the change of latitude longitude, and wind speed between $t-6$ and t
5. $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t

2 Distributon and priors:

Marginal distributions:

$$f((Y_{i,t+6})|\mu_i, \sigma) \sim N(\mu_{i,t}, \sigma^2)$$
$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma})$$

Priors:

$$f(\sigma^2) \propto \frac{1}{\sigma^2}$$
$$f(\boldsymbol{\beta}) \propto 1$$
$$f(\boldsymbol{\Sigma}^{-1}) \propto |\Sigma|^{-d-1} \exp\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\}$$

3 Hierarchical model structure:

Denote $\boldsymbol{Y} = \{Y_{i,t}\}, i = 1, 2, \ldots, n, t = 1, 2, \ldots, t_i$.
Hierarchical model will be:

$$
\begin{aligned}
f(\boldsymbol{\beta}, \boldsymbol{\beta_i}, \boldsymbol{\Sigma}^{-1}, \sigma^2|\boldsymbol{Y}) &\propto f(\boldsymbol{Y}|\boldsymbol{\beta_i}, \sigma^2) f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) f(\boldsymbol{\beta}) f(\boldsymbol{\Sigma}^{-1}) f(\sigma^2) \\
f(\boldsymbol{Y}|\boldsymbol{\beta_i}, \sigma) &\propto \prod_{i=1}^{n} \prod_{t=1}^{t_i} \sigma^{-1} \exp\{-\frac{(Y_{i,t} - \mu_{i,t})^2}{2\sigma^2}\} \\
f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) &\propto \prod_{i=1}^{n} \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma^{-1}} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\}
\end{aligned}
$$

After put all element into the formula, the full posteria will be the mutiplication of all terms

4 Posterior Distribution Inference:

1. For $\pi(\boldsymbol{\beta_i}|.)$:
Let:

$$
\boldsymbol{Y_i} = (Y_{i,1}, Y_{i,2}, \ldots, Y_{i,t_i})^T
$$

$$
\boldsymbol{X_i} = \begin{bmatrix} x_{i,1,1} & x_{i,2,1} & \ldots & x_{i,8,1} \\ x_{i,1,2} & x_{i,2,2} & \ldots & x_{i,8,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i,1,t_i} & x_{i,2,t_i} & \ldots & x_{i,8,t_i} \end{bmatrix}_{t_i \times 8}
$$

The posteria will be:

$$
\begin{aligned}
\pi(\boldsymbol{\beta_i}|.) &\propto f(\boldsymbol{Y_i}|\boldsymbol{\beta_i}, \sigma^2) f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) \\
&\propto \prod_{t=1}^{t_i} \left(\sigma^{-1} \exp\{-\frac{(Y_{i,t} - \mu_{i,t})^2}{2\sigma^2}\}\right) \left(\frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma^{-1}} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\}\right) \\
&\propto \exp\{-\frac{1}{2}\left(\sum_{t=1}^{t_i} \sigma^{-2}(Y_{i,t} - \boldsymbol{X_{i,t}\beta_i})^2 + (\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma^{-1}} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\right)\} \\
&\propto \exp\{-\frac{1}{2}\left((\boldsymbol{Y_i} - \boldsymbol{X_i\beta_i})^T (\sigma^{-2}\boldsymbol{I}_{t_i \times t_i})(\boldsymbol{Y_i} - \boldsymbol{X_i\beta_i}) + (\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}_{8\times 8}^{-1} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\right)\} \\
&\propto \exp\{-\frac{1}{2}((\sigma_i^2(X_i^T X_i)^{-1}) + \Sigma^{-1})^{-1}]((X_i^T X_i)^{-1}\Sigma^{-1}\boldsymbol{\beta} + I\sigma_i^{-2}X_i^T Y_{it}))^T((\sigma_i^2(X_i^T X_i)^{-1} + \Sigma^{-1}) \\
&((X_i^T X_i)^{-1}\Sigma^{-1}\beta + I\sigma_i^{-2}X_i^T Y_{it}))\}
\end{aligned}
$$

So:

$$
\boldsymbol{\beta_i} \sim N(\boldsymbol{K}((X_i^T X_i)^{-1}\Sigma^{-1}\boldsymbol{\beta} + \sigma^{-2}X_i^T Y_i), K)
$$

where:

$$
K = ((X_i^T X_i)^{-1}\Sigma^{-1} + \sigma^{-2}(X_i^T X_i)^{-1})^{-1}
$$

$$\propto \exp\{-\frac{1}{2}((\sigma_i^2(X_i^TX_i)^{-1})+\Sigma^{-1})^{-1}]((X_i^TX_i)^{-1}\Sigma^{-1}\boldsymbol{\beta}+I\sigma_i^{-2}X_i^TY_{it}))^T((\sigma_i^2(X_i^TX_i)^{-1}$$

$$+\Sigma^{-1})((X_i^TX_i)^{-1}\Sigma^{-1}\beta+I\sigma_i^{-2}X_i^TY_{it}))\}$$

2.For $\pi(\sigma^2|.)$:

$$\begin{aligned}\pi(\sigma^2|.) =&\propto f(\boldsymbol{Y}|\boldsymbol{\beta_i},\sigma^2)\cdot\pi(\sigma^2)\\ &\propto \left(\prod_{i=1}^{n}\prod_{t=1}^{t_i}\sigma^{-1}\exp\{-\frac{(Y_{i,t}-\mu_{i,t})^2}{2\sigma^2}\}\right)\frac{1}{\sigma^2}\\ &\propto (\sigma^2)^{-1-\frac{\sum t_i}{2}}\prod_{i=1}^{n}\prod_{t=1}^{t_i}\exp\{-\frac{(Y_{i,t}-\mu_{i,t})^2}{2\sigma^2}\}\\ &\propto \sigma^{-2-\sum t_i}exp\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{t=1}^{ti}(Y_{i,t}-\mu_{i,t})^2\sigma^{-2}\}\end{aligned}$$

where,

$$\sigma^2\sim InvGamma(\frac{\sum t_i}{2},\frac{SSR}{2})$$

$$SSR=\sum_{i=1}^{n}\sum_{t=1}^{ti}(Y_{it}-X_i\beta_i)^T(Y_{it}-X_i\beta_i)$$

3. $\Sigma^{-1}|.$:

$$\pi(\Sigma^{-1}|.)=\prod_{i=1}^{n}\pi(Y_{it}|\beta_i,\beta,\Sigma^{-1},\sigma^2)\cdot\pi(\beta_i|\beta,\Sigma^{-1})\cdot\pi(\Sigma^{-1})$$

$$\propto \Sigma^{-n}exp\{-\frac{1}{2}\sum_{i=1}^{n}(\beta_i-\beta)^T\Sigma^{-1}(\beta_i-\beta)\}\cdot|\Sigma|^{-(d+1)}exp\{-\frac{1}{2}\Sigma^{-1}\}$$

$$\propto \Sigma^{-(n/2+d+1)}exp\{-\frac{1}{2}(\sum_{i=1}^{n}(\beta_i-\beta)^T(\beta_i-\beta)+1)n\Sigma^{-1}\}$$

$$\Sigma^{-1}\sim InvWhishart(\frac{n}{2},nSSR_{\beta_i}),\ where\ SSR_{\beta_i}=\sum_{i=1}^{n}((X_i^TX_i)^{-1}\beta_i-(X_i^TX_i)^{-1}\beta)^2$$

4. $\beta|.$:

$$\pi(\beta|.)=\pi(Y_{it}|.)\cdot\pi(\beta_i|.)\cdot\pi(\beta)\propto\pi(\beta_i|.)$$

$$\propto exp\{\frac{1}{2}\sum_{i=1}^{n}((X_i^TX_i)^{-1}(\beta_i-\beta))^T((X_i^TX_i)^{-1}(\beta_i-\beta))((X_i^TX_i)^{-1}\Sigma^{-1})\}$$

$$\beta\sim N(\hat{\beta}_i,\Sigma^{-1}((X_i^TX_i)^{-1}),\ where\ \hat{\beta}=(X_i^TX_i)^{-1}X_i^TY_i$$

Or, according to the Central Limit Theorem,

$$\frac{\beta_i - \beta}{\sqrt{\Sigma^{-1}/n}},\ \beta = \bar{\beta}_i \text{ for } i = 1, 2, ..., n$$