Illustration

Wenhao Gou | wg2364

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Re-state the question:

Climate researchers are interested in modeling the hurricane trajectories to forecast the wind speed. Let t be time (in hours) since a hurricane began, and For each hurricane i, we denote $Y_i(t)$ be the wind speed of the ith hurricane at time t. The following Bayesian model was suggested.

$$Y_i(t+6) = \beta_{0,i} + x_{i,1}\beta_{1,i} + x_{i,2}\beta_{2,i} + x_{i,3}\beta_{3,i} + \beta_{4,i,j}Y_{i,j}(t) + +\beta_{5,i,j}\Delta_{i,1}(t) + \beta_{6,i,j}\Delta_{i,2}(t) + +\beta_{7,i}\Delta_{i,3} + \epsilon_i(t)$$

where $x_{i,1}$ is the month of year when the hurricane started, $x_{i,2}$ is the calendar year of the hurricane, and $x_{i,3}$ is the type of hurricane, $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ is the change of latitude longitude, and wind speed between $t-6$ and t, and $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t.

In the model, $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, ..., \beta_{7,i})$ are the random coefficients associated the ith hurricane, we assume that

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma})$$

follows a multivariate normal distributions with mean $\boldsymbol{\beta}$ and covariance matrix Σ.

Prior distributions We assume the following non-informative or weak prior distributions for σ^2, $\boldsymbol{\beta}$ and Σ.

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\boldsymbol{\beta}) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp(-\frac{1}{2}\Sigma^{-1})$$

d is dimension of β.

Find out the posterial distribution:

It is easy to see that:

$$f(Y_i(t+6)|\boldsymbol{\beta}, \sigma) \sim N(\mu, \sigma^2)$$

Where μ is the linear combination of all coefficients.

Our objective is to derive $f(\boldsymbol{\beta}, \sigma|Y_i(t+6))$

$$f(\boldsymbol{\beta}, \boldsymbol{\beta_i}, \boldsymbol{\Sigma}^{-1}, \sigma^2 | Y_i(t+6)) \propto f(Y_i(t+6)|\boldsymbol{\beta_i}, \sigma^2) f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) f(\boldsymbol{\beta}) f(\boldsymbol{\Sigma}^{-1}) f(\sigma^2)$$

$$f(Y_i(t+6)|\boldsymbol{\beta_i}, \sigma) \propto \prod_{i=1}^{N} \frac{1}{\sigma} \exp\{-\frac{(x_i - \mu_i)^2}{2\sigma^2}\} \text{ ,where } \mu_i = X_i^T \boldsymbol{\beta_i}$$

$$f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) \propto \frac{1}{\sqrt{|\boldsymbol{\Sigma^{-1}}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\}$$

$$f(\sigma^2) \propto \frac{1}{\sigma^2}$$

$$f(\boldsymbol{\beta}) \propto 1$$

$$f(\boldsymbol{\Sigma}^{-1}) \propto |\Sigma|^{-8} \exp(-\frac{1}{2}\Sigma^{-1})$$

The full posterior, then, is simply the product of these three terms—the likelihood, prior, and the distribution for Σ.

$$\prod_{i=1}^{N} \frac{1}{\sigma} \exp\{-\frac{(x_i - \mu_i)^2}{2\sigma^2}\} * \frac{1}{\sqrt{|\boldsymbol{\Sigma^{-1}}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\} * |\Sigma|^{-8} \exp(-\frac{1}{2}\Sigma^{-1})$$

```r
data = read.csv("hurrican356.csv")
df = data %>%
  mutate(Month = factor(Month, levels = c("January", "April", "May", "June", "July", "August",
                                          "September", "October", "November", "December")),
         Month = as.numeric(Month),
         Nature = as.numeric(as.factor(Nature)))
id = df %>% group_by(Season, ID) %>% summarize(n = n())

upd.bi = function(df, epis, beta, sigma, n){
  sd = solve(epis + diag(n*sigma, 8, 8)) # posterior distribution for beta.i
  mu = sd %*% (epis %*% beta + sigma*sum(df %*% beta)) # posterior distribution for beta.i
  return(list(sd = sd, mu = mu))}
mc = function(df, a, b, ini.sigma, niter = 2){
<<<<<<< HEAD
  d.lat = sapply(2:nrow(df), function(i){df[i,7] - df[i-1, 7]})
  d.lon = sapply(2:nrow(df), function(i){df[i,8] - df[i-1, 8]})
  d.spd = sapply(2:nrow(df), function(i){df[i,9] - df[i-1, 9]})
  d.yi = df[-1,9]
  dt = cbind(d.yi, df[-1,1], df[-1,4], df[-1,3], df[-1,5], df[-nrow(df),9], d.lat, d.lon, d.spd)
  # final dataset for each individual grp
=======
  delta1 = c(0, sapply(2:nrow(df), function(i){df[i,7] - df[i-1, 7]}))
  delta2 = c(0, sapply(2:nrow(df), function(i){df[i,8] - df[i-1, 8]}))
  delta3 = c(0, sapply(2:nrow(df), function(i){df[i,9] - df[i-1, 9]}))
  dt = cbind(df[,1], df[,4], df[,3], df[,5], df[,9], delta1, delta2, delta3) # final dataset for each i
>>>>>>> 198e79a4f5745ebd7448378ee32e24267cc86359
  n = nrow(dt)
  epis = vector("list", niter)
  beta = matrix(NA, nrow = 8, ncol = niter)
  beta.i = matrix(NA, nrow = 8, ncol = niter)
  sigma = rep(NA, niter)
  epis[[1]] = cor(matrix(1/extraDistr::rinvgamma(64, alpha = a, beta = b), 8, 8))
  # initial epsilon^-1 ~ inverse.gamma(0.001, 0.001) (8x8)
```

```
  beta[,1] = rep(1, 8) # initial beta (8x1)
  sigma[1] = ini.sigma # initial sigma
  beta.i[,1] = mvrnorm(1, mu = upd.bi(dt, epis[[1]], beta[,1], sigma[1], n)$mu,
  Sigma = upd.bi(dt, epis[[1]], beta[,1], sigma[1], n)$sd)
  # mvrnorm generates the beta_i for each subgroup(8x1) - beta.i ~ N(beta, epsilon)
  for (i in 2:niter){
    epis[[i]] = rinvwishart(n + a, diag(sum((beta.i[,i-1] - beta[,i-1])^2), 8, 8) + 0.001)
    # use beta.i and beta update epsilon, pi(epsilon) ~ Inviwishart(n + a, beta_variance + b)!!!!!!!!(M
    beta[,i] = mvrnorm(1, mu = n*beta.i[,i-1], Sigma = n*epis[[i]])  # beta ~ N(n*beta_i, n*epsilon)
    sigma[i] = extraDistr::rinvgamma(1, alpha = n + 1, sum((dt[,5] - dt %*% beta.i[,i-1])^2))
    # sigma ~ inverse.gamma(n + 1, sample residual of Yi)
    beta.i[,i] = mvrnorm(1, mu = upd.bi(dt, epis[[i]], beta[,i], sigma[i], n)$mu,
  Sigma = upd.bi(dt, epis[[i]], beta[,i], sigma[i], n)$sd)}
  return(list(beta.i =  beta.i))}

grp = function(df, id){
  sub.grp = vector("list", nrow(id))
  id.n = c(0, id$n)
  for (i in 1:length(sub.grp)){
    sub.grp[[i]] = df[(id.n[i]+1):id.n[i+1],]}
  return(sub.grp)}
```

Posterior Distribution Inference:

$Y_{it} = X_i\beta_i + \epsilon_{it}$, with $i = 1, 2, ..., n,\ t = 1, 2, ...t_i$ 1. $\beta_i|.$:

$$Y_{it} \sim N(X_i\beta_i, \sigma^2),\ \beta_i \sim N(\beta, \Sigma^{-1})$$

$$\pi(Y_{it}|X_i\beta_i, \sigma^2) \propto \prod_{i=1}^{n}\prod_{t=1}^{t_i} \sigma^{-2} exp\{-\frac{(Y_{it} - X_i\beta_i)^2}{2\sigma^2}\}$$

$$\pi(X_i\beta_i|Y_{it}, \sigma^2) \propto \pi(Y_{it}|X_i\beta_i, \sigma^2) \cdot \pi(\beta_i)$$

$$\propto \prod_{i=1}^{n}\prod_{t=1}^{t_i} \sigma^{-1}\Sigma^{-1/2} exp\{-\frac{1}{2}(Y_{it} - X_i\beta_i)^T(I\sigma_i^{-2})(Y_{it} - X_i\beta_i)\} \cdot exp\{-\frac{1}{2}(\beta_i - \beta)^T\Sigma^{-1}(\beta_i - \beta)\}$$

$$\propto exp\{-\frac{1}{2}((\sigma_i^2(X_i^TX_i)^{-1}) + \Sigma^{-1})^{-1}\sum_{i=1}^{n}((X_i^TX_i)^{-1}\Sigma^{-1}\beta + I\sigma_i^{-2}X_i^TY_{it}))^T((\sigma_i^2(X_i^TX_i)^{-1} + \Sigma^{-1})((X_i^TX_i)^{-1}\Sigma^{-1}\beta + I\sigma_i^{-2}X_i^T$$

$$\beta_i \sim N(K((X_i^TX_i)^{-1}\Sigma^{-1}\beta + \sigma^{-2}X_i^TY_{it}), K), where\ K = ((X_i^TX_i)^{-1}\Sigma^{-1} + \sigma^{-2}(X_i^TX_i)^{-1})^{-1}$$

2.$\sigma^2|.$:

$$\pi(\sigma_i^2|.) = \pi(Y_{it}|X_i\beta_i) \cdot \pi(\sigma_i^2)$$

$$\propto \prod_{i=1}^{n}\prod_{t=1}^{t_i} \frac{1}{\sqrt{\sigma_i^2}} exp\{-\frac{1}{2}(Y_{it} - X_i\beta_i)^T(I\sigma_i^{-2})(Y_{it} - X_i\beta_i)\} \cdot \frac{1}{\sigma_i^2}$$

$$\propto \sigma_i^{-(n-2)} exp\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{t=1}^{ti}(Y_{it} - X_i\beta_i)^T(Y_{it} - X_i\beta_i)n\sigma_i^{-2}\}$$

$$\sigma_i^2 \sim InvGamma(\frac{n}{2}, \frac{nSSR}{2}),\ where\ SSR = \sum_{i=1}^{n}\sum_{t=1}^{ti}(Y_{it} - X_i\beta_i)^T(Y_{it} - X_i\beta_i)$$

3. $\Sigma^{-1}|.$:

$$\pi(\Sigma^{-1}|.) = \prod_{i=1}^{n} \pi(Y_{it}|\beta_i, \beta, \Sigma^{-1}, \sigma^2) \cdot \pi(\beta_i|\beta, \Sigma^{-1}) \cdot \pi(\Sigma^{-1})$$

$$\propto \Sigma^{-n} exp\{-\frac{1}{2}\sum_{i=1}^{n}(\beta_i - \beta)^T \Sigma^{-1}(\beta_i - \beta)\} \cdot |\Sigma|^{-(d+1)} exp\{-\frac{1}{2}\Sigma^{-1}\}$$

$$\propto \Sigma^{-(n/2+d+1)} exp\{-\frac{1}{2}(\sum_{i=1}^{n}(\beta_i - \beta)^T(\beta_i - \beta) + 1)n\Sigma^{-1}\}$$

$$\Sigma^{-1} \sim InvWhishart(\frac{n}{2}, nSSR_{\beta_i}),\ where\ SSR_{\beta_i} = \sum_{i=1}^{n}((X_i^T X_i)^{-1}\beta_i - (X_i^T X_i)^{-1}\beta)^2$$

4. $\beta|.$:

$$\pi(\beta|.) = \pi(Y_{it}|.) \cdot \pi(\beta_i|.) \cdot \pi(\beta) \propto \pi(\beta_i|.)$$

$$\propto exp\{\frac{1}{2}\sum_{i=1}^{n}((X_i^T X_i)^{-1}(\beta_i - \beta))^T((X_i^T X_i)^{-1}(\beta_i - \beta))((X_i^T X_i)^{-1}\Sigma^{-1})\}$$

$$\beta \sim N(\hat{\beta}_i, \Sigma^{-1}((X_i^T X_i)^{-1}),\ where\ \hat{\beta} = (X_i^T X_i)^{-1} X_i^T Y_{it}$$

Or, according to the Central Limit Theorem,

$$\frac{\beta_i - \beta}{\sqrt{\Sigma^{-1}/n}},\ \beta = \bar{\beta}_i \text{ for } i = 1, 2, ..., n$$