Posteria distribution deriviation

Qetsiyah Wang and Wenhao Gou

2021.4.22

1 The model:

The main model is:

$$Y_i(t+6) = \beta_{0,i} + x_{i,1}\beta_{1,i} + x_{i,2}\beta_{2,i} + x_{i,3}\beta_{3,i} + \beta_{4,i}Y_i(t)$$
$$+\beta_{5,i}\Delta_{i,1}(t) + \beta_{6,i}\Delta_{i,2}(t) + \beta_{7,i}\Delta_{i,3} + \epsilon_i(t)$$

(NOTE: omit j as we dont know what it is for now)

1. $x_{i,1}$ is the month of year when the hurricane started
2. $x_{i,2}$ is the calendar year of the hurricane
3. $x_{i,3}$ is the type of hurricane
4. $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ is the change of latitude longitude, and wind speed between $t-6$ and t
5. $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t

2 Distributon and priors:

Marginal distributions for $Y_{i,t+6}$:

$$f(Y_{i,t+6}|\boldsymbol{\beta}_i, \sigma) \sim N(\mu_{i,t}, \sigma^2)$$
$$\mu_{i,t} = Y_i(t+6) - \epsilon_i(t)$$
$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma})$$

Priors:

$$f(\sigma^2) \propto \frac{1}{\sigma^2}$$
$$f(\boldsymbol{\beta}) \propto 1$$
$$f(\boldsymbol{\Sigma}^{-1}) \propto |\boldsymbol{\Sigma}|^{-d-1} \exp\{-\frac{1}{2}tr(\boldsymbol{\Sigma}^{-1})\}$$

3 Hierarchical model structure:

Denote $\boldsymbol{Y} = \{Y_{i,t}\}, i = 1, 2, \dots, n, t = 1, 2, \dots, t_i$.
Hierarchical model will be:

$$f(\boldsymbol{\beta}, \boldsymbol{\beta_i}, \boldsymbol{\Sigma}^{-1}, \sigma^2|\boldsymbol{Y}) \propto f(\boldsymbol{Y}|\boldsymbol{\beta_i}, \sigma^2) f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) f(\boldsymbol{\beta}) f(\boldsymbol{\Sigma}^{-1}) f(\sigma^2)$$

$$f(\boldsymbol{Y}|\boldsymbol{\beta_i}, \sigma) \propto \prod_{i=1}^{n} \prod_{t=1}^{t_i} \sigma^{-1} \exp\{-\frac{(Y_{i,t} - \mu_{i,t})^2}{2\sigma^2}\}$$

$$f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) \propto \prod_{i=1}^{n} \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma^{-1}} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\}$$

After put all element into the formula, the full posteria will be the mutiplication of all terms

4 Posterior Distribution Inference:

Suppose for hurricane i, we have t_i observations which can be used to build model, denote following matrixs for response and predictors for hurricane i:

$$\boldsymbol{Y_i} = (Y_{i,1}, Y_{i,2}, \dots, Y_{i,t_i})^T$$

$$\boldsymbol{X_i} = \begin{bmatrix} x_{i,1,1} & x_{i,2,1} & \dots & x_{i,8,1} \\ x_{i,1,2} & x_{i,2,2} & \dots & x_{i,8,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i,1,t_i} & x_{i,2,t_i} & \dots & x_{i,8,t_i} \end{bmatrix}_{t_i \times 8}$$

1. For $\pi(\boldsymbol{\beta_i}|.)$:

$$\begin{aligned}
\pi(\boldsymbol{\beta_i}|.) &\propto f(\boldsymbol{Y_i}|\boldsymbol{\beta_i}, \sigma^2) f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1}) \\
&\propto \left(\prod_{t=1}^{t_i} \sigma^{-1} \exp\{-\frac{(Y_{i,t} - \mu_{i,t})^2}{2\sigma^2}\} \right) \left(\frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma^{-1}} (\boldsymbol{\beta_i} - \boldsymbol{\beta})\} \right) \\
&\propto \exp\{-\frac{1}{2} \left(\sum_{t=1}^{t_i} \sigma^{-2} (Y_{i,t} - \boldsymbol{X_{i,t}\beta_i})^2 + (\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma^{-1}} (\boldsymbol{\beta_i} - \boldsymbol{\beta}) \right) \} \\
&\propto \exp\{-\frac{1}{2} \left((\boldsymbol{Y_i} - \boldsymbol{X_i\beta_i})^T (\sigma^{-2} \boldsymbol{I}_{t_i \times t_i}) (\boldsymbol{Y_i} - \boldsymbol{X_i\beta_i}) + (\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}_{8\times 8}^{-1} (\boldsymbol{\beta_i} - \boldsymbol{\beta}) \right) \}
\end{aligned}$$

For the expnential term:

$$\begin{aligned}
&(\boldsymbol{Y_i} - \boldsymbol{X_i\beta_i})^T (\sigma^{-2} \boldsymbol{I}_{t_i \times t_i}) (\boldsymbol{Y_i} - \boldsymbol{X_i\beta_i}) + (\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}_{8\times 8}^{-1} (\boldsymbol{\beta_i} - \boldsymbol{\beta}) \\
&= \boldsymbol{Y_i}^T \sigma^{-2} \boldsymbol{I} \boldsymbol{Y_i}^T + \boldsymbol{\beta_i}^T \boldsymbol{X_i}^T \sigma^{-2} \boldsymbol{I} \boldsymbol{X_i\beta_i} - 2\boldsymbol{Y_i}^T \sigma^{-2} \boldsymbol{I} \boldsymbol{X_i\beta_i} \\
&\quad + \boldsymbol{\beta_i}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\beta_i} + \boldsymbol{\beta}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\beta} - 2\boldsymbol{\beta}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\beta_i} \\
&= \boldsymbol{Y_i}^T \sigma^{-2} \boldsymbol{I} \boldsymbol{Y_i}^T + \boldsymbol{\beta}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\beta} + \boldsymbol{\beta_i}^T (\boldsymbol{\Sigma}^{-1} + \boldsymbol{X_i}^T \sigma^{-2} \boldsymbol{I} \boldsymbol{X_i}) \boldsymbol{\beta_i} \\
&\quad - 2(\boldsymbol{Y_i}^T \sigma^{-2} \boldsymbol{I} \boldsymbol{X_i} + \boldsymbol{\beta}^T \boldsymbol{\Sigma}^{-1}) \boldsymbol{\beta_i} \\
&= \boldsymbol{R} + \boldsymbol{\beta_i}^T \boldsymbol{V} \boldsymbol{\beta_i} - 2\boldsymbol{M} \boldsymbol{\beta_i}
\end{aligned}$$

Where:

$$\begin{aligned}\boldsymbol{R} &= \boldsymbol{Y_i}^T\sigma^{-2}\boldsymbol{I}\boldsymbol{Y_i}^T + \boldsymbol{\beta}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\beta}\\ \boldsymbol{V} &= \boldsymbol{\Sigma}^{-1} + \sigma^{-2}\boldsymbol{X_i}^T\boldsymbol{X_i}\\ \boldsymbol{M} &= \sigma^{-2}\boldsymbol{Y_i}^T\boldsymbol{X_i} + \boldsymbol{\beta}^T\boldsymbol{\Sigma}^{-1}\end{aligned}$$

Then, the exponential term can be reduced to:

$$(\boldsymbol{\beta_i} - \boldsymbol{V^{-1}M})^T\boldsymbol{V}(\boldsymbol{\beta_i} - \boldsymbol{V^{-1}M}) - \boldsymbol{M}^T\boldsymbol{V^{-1}M}^T + \boldsymbol{R}$$

We can ignore the latter 2 term as it is not related to $\boldsymbol{\beta_i}$. That indicate:

$$\pi(\boldsymbol{\beta_i}|.) \sim N(\boldsymbol{V^{-1}M}, \boldsymbol{V^{-1}})$$

2.For $\pi(\sigma^2|.)$:

$$\begin{aligned}\pi(\sigma^2|.) &\propto f(\boldsymbol{Y}|\boldsymbol{\beta_i}, \sigma^2)\cdot\pi(\sigma^2)\\ &\propto \left(\prod_{i=1}^{n}\prod_{t=1}^{t_i}\sigma^{-1}\exp\{-\frac{(Y_{i,t}-\mu_{i,t})^2}{2\sigma^2}\}\right)\frac{1}{\sigma^2}\\ &\propto (\sigma^2)^{-1-\frac{\sum t_i}{2}}\prod_{i=1}^{n}\prod_{t=1}^{t_i}\exp\{-\frac{(Y_{i,t}-\mu_{i,t})^2}{2\sigma^2}\}\\ &\propto \sigma^{-2-\sum t_i}\exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{t=1}^{ti}(Y_{i,t}-\mu_{i,t})^2\}\end{aligned}$$

So:

$$\sigma^2 \sim \text{InvGamma}(\frac{1}{2}\sum_{i=1}^{n}t_i, \frac{1}{2}\sum_{i=1}^{n}\sum_{t=1}^{ti}(Y_{i,t}-\mu_{i,t})^2)$$

3. For $\pi(\boldsymbol{\Sigma}^{-1}|.)$:

$$\begin{aligned}\pi(\boldsymbol{\Sigma}^{-1}|.) &\propto f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1})f(\boldsymbol{\Sigma}^{-1})\\ &\propto \left(\prod_{i=1}^{n}\boldsymbol{\Sigma}^{-\frac{1}{2}}\exp\{-\frac{1}{2}(\boldsymbol{\beta_i}-\boldsymbol{\beta})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i}-\boldsymbol{\beta})\}\right)|\boldsymbol{\Sigma}|^{-d-1}\exp\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\}\\ &\propto \boldsymbol{\Sigma}^{-\frac{n}{2}-d-1}\exp\{-\frac{1}{2}\boldsymbol{\Sigma^{-1}} - \frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{\beta_i}-\boldsymbol{\beta})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i}-\boldsymbol{\beta})\}\\ &\propto \boldsymbol{\Sigma}^{-\frac{n}{2}-d-1}\exp\{\frac{1}{2}\left(tr(\boldsymbol{\Sigma^{-1}}) + \sum_{i=1}^{n}(\boldsymbol{\beta_i}-\boldsymbol{\beta})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i}-\boldsymbol{\beta})\right)\}\end{aligned}$$

That indicate:

$$\boldsymbol{\Sigma}^{-1} \sim \text{InvWhishart}\left(\frac{n}{2}, tr(\boldsymbol{\Sigma^{-1}}) + \sum_{i=1}^{n}(\boldsymbol{\beta_i}-\boldsymbol{\beta})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i}-\boldsymbol{\beta})\right)$$

4. For $\pi(\boldsymbol{\beta}|.)$:

$$\begin{aligned}
\pi(\boldsymbol{\beta}|.) &\propto f(\boldsymbol{\beta_i}|\boldsymbol{\beta}, \boldsymbol{\Sigma}^{-1})f(\boldsymbol{\beta}) \\
&\propto \left(\prod_{i=1}^{n} \exp\{-\frac{1}{2}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i} - \boldsymbol{\beta})\}\right) \\
&\propto \exp\{-\frac{1}{2}\left(\sum_{i=1}^{n}(\boldsymbol{\beta_i} - \boldsymbol{\beta})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta_i} - \boldsymbol{\beta})\right)\} \\
&\propto \exp\{-\frac{1}{2}\left(\sum_{i=1}^{n}\boldsymbol{\beta_i}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta_i} + \boldsymbol{\beta}^T n\boldsymbol{\Sigma}^{-1}\boldsymbol{\beta} - \sum_{i=1}^{n} 2\boldsymbol{\beta_i}^T \boldsymbol{\Sigma^{-1}}\boldsymbol{\beta}\right)\}
\end{aligned}$$

For the exponential term, if we set:

$$\begin{aligned}
\boldsymbol{V} &= n\boldsymbol{\Sigma}^{-1} \\
\boldsymbol{R} &= \sum_{i=1}^{n}\boldsymbol{\beta_i}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta_i} \\
\boldsymbol{M} &= \sum_{i=1}^{n}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\beta_i}\right)
\end{aligned}$$

Then, use the same technique when generating $\boldsymbol{\beta_i}$

$$R + \boldsymbol{\beta}\boldsymbol{V}\boldsymbol{\beta} - 2\boldsymbol{M}\boldsymbol{\beta} \propto (\boldsymbol{\beta} - \boldsymbol{V}^{-1}\boldsymbol{M})^T \boldsymbol{V}^{-1}(\boldsymbol{\beta} - \boldsymbol{V}^{-1}\boldsymbol{M})$$

(NOTE: This is the same as using OLS to estimate $\boldsymbol{\beta}$ using all $\boldsymbol{\beta_i}$)
That indicate:

$$\boldsymbol{\beta} \sim N\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{\beta_i}, \frac{1}{n}\boldsymbol{\Sigma}\right)$$