Jonathan Chan

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$$
\begin{aligned}
i, j, k, m, n &::= \textsf{naturals} & r, s &::= \alpha \mid \hat{s} \mid \circ \mid \infty \\
f, g, x, y, z &::= \textsf{term variables} & U &::= \textsf{Prop} \mid \textsf{Set} \mid \textsf{Type}_i \\
\alpha, \beta &::= \textsf{size variables} \\
e, p, P, \tau, \sigma &::= x \mid U_i \mid \Pi x : \tau.\, \tau \mid \lambda x : \tau.\, e \mid e\, e \mid \textsf{let}\ x \coloneqq e\ \textsf{in}\ e \\
&\mid \forall \alpha.\, \tau \mid \forall \alpha < s.\, \tau \mid \Lambda \alpha.\, e \mid \Lambda \alpha < s.\, e \mid e\, [s] \\
\Delta &::= \cdot \mid \Delta(x : \tau) & \Phi &::= \cdot \mid \Phi(\alpha) \mid \Phi(\alpha < s) \\
\Gamma &::= \cdot \mid \Gamma(x : \tau) \mid \Gamma(x \coloneqq e) & \Theta &::= \cdot \mid \Theta(x : \tau) \mid \Theta(\alpha) \mid \Theta(\alpha < s)
\end{aligned}
$$

Figure 1: Syntax

$$
\begin{aligned}
\textsf{axiom}(\textsf{Prop}) &= \textsf{Type}_1 \\
\textsf{axiom}(\textsf{Set}) &= \textsf{Type}_1 \\
\textsf{axiom}(\textsf{Type}_i) &= \textsf{Type}_{i+1} \\
\textsf{rule}(U, \textsf{Prop}) &= \textsf{Prop} \\
\textsf{rule}(U_1, U_2) &= \max_{\sqsubseteq}(U_1, U_2)
\end{aligned}
$$

$$
\textsf{Prop} \sqsubseteq \textsf{Set} \sqsubseteq \textsf{Type}_i \sqsubseteq \textsf{Type}_{i+j}
$$

$$
\begin{aligned}
\textsf{dom}(\cdot) &\equiv \cdot \\
\textsf{dom}(\Theta(x : \tau)) &\equiv \textsf{dom}(\Theta)\ x \\
\textsf{dom}(\Theta(\alpha)) &\equiv \textsf{dom}(\Theta)\ [\alpha] \\
\textsf{dom}(\Theta(\alpha < s)) &\equiv \textsf{dom}(\Theta)\ [\alpha]
\end{aligned}
$$

Figure 2: Metafunctions and metarelations

$$
\begin{aligned}
\tau_1 \to \tau_2 &\equiv \Pi \_ : \tau_1.\, \tau_2 \\
\cdot \to \tau &\equiv \tau \\
\Theta(x : \sigma) \to \tau &\equiv \Theta \to \Pi x : \sigma.\, \tau \\
\Theta(\alpha) \to \tau &\equiv \Theta \to \forall \alpha.\, \tau \\
\Theta(\alpha < s) \to \tau &\equiv \Theta \to \forall \alpha < s.\, \tau
\end{aligned}
$$

$$
\begin{aligned}
e\ \Theta &\equiv e\ \textsf{dom}(\Theta) \\
e[[\alpha] \mapsto [s]] &\equiv e[\alpha \mapsto s] \\
e[\Theta \mapsto e_1, \ldots, e_n] &\equiv e[\textsf{dom}(\Theta) \mapsto e_1, \ldots, e_n] \\
e[x_1, \ldots, x_n \mapsto e_1, \ldots, e_n] &\equiv e[x_1 \mapsto e_1, \ldots, x_n \mapsto e_n]
\end{aligned}
$$

Figure 3: Syntactic sugar

$\boxed{\Phi; \Gamma \vdash e \rhd e}$ $\boxed{\Phi; \Gamma \vdash e \rhd^* e}$

$$\frac{(x \coloneqq e) \in \Gamma}{\Phi; \Gamma \vdash x \rhd_\delta e} \qquad \frac{}{\Phi; \Gamma \vdash (\lambda x : \tau.\, e)\ e' \rhd_\beta e[x \mapsto e']} \qquad \frac{}{\Phi; \Gamma \vdash \mathsf{let}\ x \coloneqq e'\ \mathsf{in}\ e \rhd_\zeta e[x \mapsto e']}$$

$$\frac{}{\Phi; \Gamma \vdash (\Lambda \alpha.\, e)\ [s] \rhd_\varsigma e[\alpha \mapsto s]} \qquad \frac{}{\Phi; \Gamma \vdash (\Lambda \alpha < r.\, e)\ [s] \rhd_\varsigma e[\alpha \mapsto s]}$$

$$\textsc{$\rhd^*$-Refl}\ \frac{}{\Phi; \Gamma \vdash e \rhd^* e} \qquad \textsc{$\rhd^*$-Trans}\ \frac{\Phi; \Gamma \vdash e_1 \rhd e_2 \quad \Phi; \Gamma \vdash e_2 \rhd^* e_3}{\Phi; \Gamma \vdash e_1 \rhd^* e_3}$$

$$\textsc{$\rhd^*$-Cong}\ \frac{\text{For every } i\text{:} \qquad \Phi; \Gamma\Gamma' \vdash e_i \rhd^* e_i'}{\Phi; \Gamma \vdash e[x_1, \ldots, x_n \mapsto e_1, \ldots, e_n] \rhd^* e[x_1, \ldots, x_n \mapsto e_1', \ldots, e_n']}$$

Figure 4: Reduction rules

$\boxed{\Phi; \Gamma \vdash e \approx e}$

$$\textsc{$\approx$-Red}\ \frac{\Phi; \Gamma \vdash e_1 \rhd^* e \quad \Phi; \Gamma \vdash e_2 \rhd^* e}{\Phi; \Gamma \vdash e_1 \approx e_2}$$

$$\textsc{$\approx$-Lam-η_L}\ \frac{\Phi; \Gamma \vdash e_1 \rhd^* \lambda x : \tau.\, e \quad \Phi; \Gamma \vdash e_2 \rhd^* e_2' \quad \Phi; \Gamma(x : \tau) \vdash e \approx e_2'\ x}{\Phi; \Gamma \vdash e_1 \approx e_2}$$

$$\textsc{$\approx$-Lam-η_R}\ \frac{\Phi; \Gamma \vdash e_1 \rhd^* e_1' \quad \Phi; \Gamma \vdash e_2 \rhd^* \lambda x : \tau.\, e \quad \Phi; \Gamma(x : \tau) \vdash e_1'\ x \approx e}{\Phi; \Gamma \vdash e_1 \approx e_2}$$

$$\textsc{$\approx$-SLam-η_L}\ \frac{\Phi; \Gamma \vdash e_1 \rhd^* \Lambda \alpha.\, e \quad \Phi; \Gamma \vdash e_2 \rhd^* e_2' \quad \Phi(\alpha); \Gamma \vdash e \approx e_2'\ [\alpha]}{\Phi; \Gamma \vdash e_1 \approx e_2}$$

$$\textsc{$\approx$-SLam-η_R}\ \frac{\Phi; \Gamma \vdash e_1 \rhd^* e_1' \quad \Phi; \Gamma \vdash e_2 \rhd^* \Lambda \alpha.\, e \quad \Phi(\alpha); \Gamma \vdash e_1'\ [\alpha] \approx e}{\Phi; \Gamma \vdash e_1 \approx e_2}$$

$$\textsc{$\approx$-SLam$<$-η_L}\ \frac{\Phi; \Gamma \vdash e_1 \rhd^* \Lambda \alpha < s.\, e \quad \Phi; \Gamma \vdash e_2 \rhd^* e_2' \quad \Phi(\alpha < s); \Gamma \vdash e \approx e_2'\ [\alpha]}{\Phi; \Gamma \vdash e_1 \approx e_2}$$

$$\textsc{$\approx$-SLam$<$-η_R}\ \frac{\Phi; \Gamma \vdash e_1 \rhd^* e_1' \quad \Phi; \Gamma \vdash e_2 \rhd^* \Lambda \alpha < s.\, e \quad \Phi(\alpha < s); \Gamma \vdash e_1'\ [\alpha] \approx e}{\Phi; \Gamma \vdash e_1 \approx e_2}$$

Figure 5: Convertibility rules

$\boxed{\Phi; \Gamma \vdash \tau \preccurlyeq \tau}$

$$\textsc{$\preccurlyeq$-Conv}\ \frac{\Phi; \Gamma \vdash \tau_1 \approx \tau_2}{\Phi; \Gamma \vdash \tau_1 \preccurlyeq \tau_2} \qquad \textsc{$\preccurlyeq$-Trans}\ \frac{\Phi; \Gamma \vdash \tau_1 \preccurlyeq \tau_2 \qquad \Phi; \Gamma \vdash \tau_2 \preccurlyeq \tau_3}{\Phi; \Gamma \vdash \tau_1 \preccurlyeq \tau_3} \qquad \textsc{$\preccurlyeq$-Univ}\ \frac{U_1 \sqsubseteq U_2}{\Phi; \Gamma \vdash U_1 \preccurlyeq U_2}$$

$$\textsc{$\preccurlyeq$-Pi}\ \frac{\Phi; \Gamma \vdash \sigma_2 \approx \sigma_1 \quad \Phi; \Gamma(x_2 : \sigma_2) \vdash \tau_1[x_1 \mapsto x_2] \preccurlyeq \tau_2}{\Phi; \Gamma \vdash \Pi x_1 : \sigma_1.\, \tau_1 \preccurlyeq \Pi x_2 : \sigma_2.\, \tau_2}$$

$$\textsc{$\preccurlyeq$-Forall}\ \frac{\Phi(\alpha_2); \Gamma \vdash \tau_1[\alpha_1 \mapsto \alpha_2] \preccurlyeq \tau_2}{\Phi; \Gamma \vdash \forall \alpha_1.\, \tau_1 \preccurlyeq \forall \alpha_2.\, \tau_2} \qquad \textsc{$\preccurlyeq$-Forall$<$}\ \frac{\Phi(\alpha_2 < s); \Gamma \vdash \tau_1[\alpha_1 \mapsto \alpha_2] \preccurlyeq \tau_2}{\Phi; \Gamma \vdash \forall \alpha_1 < s.\, \tau_1 \preccurlyeq \forall \alpha_2 < s.\, \tau_2}$$

Figure 6: Subtyping rules

$\boxed{\Phi \vdash s}$ $\quad$ $\boxed{\Phi \vdash s \leqslant s}$

$$
\frac{\begin{array}{l}\vdash \Phi \\ (\alpha) \in \Phi \ \mathit{or} \\ (\alpha < s) \in \Phi\end{array}}{\Phi \vdash \alpha}
\qquad
\frac{\Phi \vdash s}{\Phi \vdash \hat{s}}
\qquad
\frac{\vdash \Phi}{\Phi \vdash \circ}
\qquad
\frac{\begin{array}{c}\vdash \Phi \\ (\alpha < s) \in \Phi\end{array}}{\Phi \vdash \hat{\alpha} \leqslant s}
\qquad
\frac{\Phi \vdash s}{\Phi \vdash s \leqslant s}
\qquad
\frac{\Phi \vdash s}{\Phi \vdash \circ \leqslant s}
\qquad
\frac{\Phi \vdash s}{\Phi \vdash s \leqslant \hat{s}}
\qquad
\frac{\begin{array}{c}\Phi \vdash s_1 \leqslant s_2 \\ \Phi \vdash s_2 \leqslant s_3\end{array}}{\Phi \vdash s_1 \leqslant s_3}
$$

Figure 7: Wellformedness (sizes) and subsizing rules

$\boxed{\vdash \Phi}$ $\quad$ $\boxed{\Phi \vdash \Gamma}$

$$
\overset{\text{NIL}}{\frac{}{\vdash \cdot}}
\qquad
\overset{\text{CONS-SIZE}}{\frac{\vdash \Phi}{\vdash \Phi(\alpha)}}
\qquad
\overset{\text{CONS-SIZE}<}{\frac{\vdash \Phi \qquad \Phi \vdash s}{\vdash \Phi(\alpha < s)}}
\qquad
\overset{\text{NIL}}{\frac{\vdash \Phi}{\Phi \vdash \cdot}}
\qquad
\overset{\text{CONS-ASS}}{\frac{\Phi \vdash \Gamma \qquad \Gamma \vdash \tau \Uparrow U}{\Phi \vdash \Gamma(x : \tau)}}
\qquad
\overset{\text{CONS-DEF}}{\frac{\Phi \vdash \Gamma \qquad \Gamma \vdash e \Uparrow \tau}{\Phi \vdash \Gamma(x \coloneqq e)}}
$$

Figure 8: Environment wellformedness rules

$\boxed{\Phi; \Gamma \vdash e \Updownarrow \tau}$

$$
\overset{\text{CONV}}{\frac{\Phi; \Gamma \vdash \tau \Uparrow U \qquad \Phi; \Gamma \vdash e \Uparrow \tau' \qquad \Phi; \Gamma \vdash \tau' \preccurlyeq \tau}{\Phi; \Gamma \vdash e \Downarrow \tau}}
\qquad
\overset{\text{VAR}}{\frac{\Phi \vdash \Gamma \qquad (x : \tau) \in \Gamma}{\Phi; \Gamma \vdash x \Uparrow \tau}}
\qquad
\overset{\text{UNIV}}{\frac{\Phi \vdash \Gamma}{\Phi; \Gamma \vdash U \Uparrow \mathsf{axiom}(U)}}
$$

$$
\overset{\text{PI}}{\frac{\Phi; \Gamma \vdash \sigma \Uparrow U_1 \qquad \Phi; \Gamma(x : \sigma) \vdash \tau \Uparrow U_2}{\Gamma \vdash \Pi x : \sigma.\, \tau \Uparrow \mathsf{rule}(U_1, U_2)}}
\qquad
\overset{\text{LAM}}{\frac{\Phi; \Gamma \vdash \sigma \Uparrow U \qquad \Phi; \Gamma(x : \sigma) \vdash e \Uparrow \tau}{\Phi; \Gamma \vdash \lambda x : \sigma.\, e \Uparrow \Pi x : \sigma.\, \tau}}
$$

$$
\overset{\text{APP}}{\frac{\Phi; \Gamma \vdash e_1 \Uparrow \Pi x : \sigma.\, \tau \qquad \Phi; \Gamma \vdash e_2 \Downarrow \sigma}{\Phi; \Gamma \vdash e_1\ e_2 \Uparrow \tau[x \mapsto e_1]}}
\qquad
\overset{\text{LET}}{\frac{\Phi; \Gamma \vdash e_1 \Uparrow \sigma \qquad \Phi; \Gamma(x : \sigma)(x \coloneqq e_1) \vdash e_2 \Uparrow \tau}{\Phi; \Gamma \vdash \mathsf{let}\ x \coloneqq e_1\ \mathsf{in}\ e_2 \Uparrow \tau[x \mapsto e_1]}}
$$

$$
\overset{\text{FORALL}}{\frac{\Phi(\alpha); \Gamma \vdash \tau \Uparrow U}{\Phi; \Gamma \vdash \forall \alpha.\, \tau \Uparrow U}}
\qquad
\overset{\text{FORALL}<}{\frac{\Phi \vdash s \qquad \Phi(\alpha < s); \Gamma \vdash \tau \Uparrow U}{\Phi; \Gamma \vdash \forall \alpha < s.\, \tau \Uparrow U}}
\qquad
\overset{\text{SLAM}}{\frac{\Phi(\alpha); \Gamma \vdash e \Uparrow \tau}{\Phi; \Gamma \vdash \Lambda \alpha.\, e \Uparrow \forall \alpha.\, \tau}}
$$

$$
\overset{\text{SLAM}<}{\frac{\Phi \vdash s \qquad \Phi(\alpha < s); \Gamma \vdash e \Uparrow \tau}{\Phi; \Gamma \vdash \Lambda \alpha < s.\, e \Uparrow \forall \alpha < s.\, \tau}}
\qquad
\overset{\text{SAPP}}{\frac{\Phi; \Gamma \vdash e \Uparrow \forall \alpha.\, \tau \qquad \Phi \vdash s}{\Phi; \Gamma \vdash e\ [s] \Uparrow \tau[\alpha \mapsto s]}}
\qquad
\overset{\text{SAPP}<}{\frac{\Phi; \Gamma \vdash e \Uparrow \forall \alpha < r.\, \tau \qquad \Phi \vdash \hat{s} \leqslant r}{\Phi; \Gamma \vdash e\ [s] \Uparrow \tau[\alpha \mapsto s]}}
$$

Figure 9: Typing rules

$$X ::= \text{inductive type names} \qquad c ::= \text{inductive constructor names}$$

$$D ::= \mathsf{data}\ X\ [\alpha]\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta \qquad t ::= e \mid [s]$$

$$e ::= \cdots \mid X \mid c \mid \mathsf{match}\ e\ \mathsf{return}\ \lambda(y \ldots).x.\, P\ \mathsf{with}\ (c\ [\alpha]\ z \ldots \Rightarrow e) \ldots \mid \mathsf{fix}\ f\ [\alpha] : \tau \coloneqq e$$

$\boxed{\Phi; \Gamma \vdash e \rhd e}$ $\cdots$

$$\frac{}{\Phi; \Gamma \vdash \mathsf{match}\ c\ [s]\ p \ldots\ [r]\ e_1 \ldots e_n\ \mathsf{return}\ \_\ \mathsf{with}\ \ldots (c\ [\alpha]\ z_1 \ldots z_n \Rightarrow e) \ldots \rhd_\iota e[\alpha \mapsto r][z_1, \ldots, z_n \mapsto e_1, \ldots, e_n]}$$

$$\frac{\sigma \equiv \Theta \to (x : \tau) \to \tau' \qquad \tau \equiv X\ [\alpha]\ \_ \ldots \qquad |\Theta| = n \qquad \beta\ \text{fresh}}{\Phi; \Gamma \vdash (\mathsf{fix}\ f\ [\alpha] : \sigma \coloneqq e)\ [s]\ t_1 \ldots\ t_n\ (c\ \_ \ldots) \rhd_\mu e[\alpha \mapsto s][f \mapsto \Lambda\beta < s.\, (\mathsf{fix}\ f\ [\alpha] : \sigma \coloneqq e)\ [\beta]]\ t_1 \ldots\ t_n\ (c\ \_ \ldots)}$$

$\boxed{\Phi; \Gamma \vdash e \Updownarrow \tau}$ $\cdots$

IND

$$\frac{\Phi \vdash \Gamma \qquad \mathsf{data}\ X\ [\alpha]\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta}{\Phi; \Gamma \vdash X \Uparrow \forall\alpha.\, \Delta_P \Delta_I \to U}$$

CONSTR

$$\frac{\Phi \vdash \Gamma \qquad \mathsf{data}\ X\ [\alpha]\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta \qquad (c : \tau) \in \Delta}{\Phi; \Gamma \vdash c \Uparrow \Delta_P \to \tau}$$

MATCH

$$\frac{\begin{array}{c} \mathsf{data}\ X\ [\alpha]\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta \\ \Phi; \Gamma \vdash e \Uparrow X\ [s]\ p_1 \ldots p_k\ e_1 \ldots e_n \qquad \Phi; \Gamma(\Delta_I[\alpha \mapsto s][\Delta_P \mapsto p_1, \ldots, p_k])(x : X\ [s]\ p_1 \ldots p_k\ \Delta_I) \vdash P \Uparrow U' \\ \text{For each constructor } (c : \tau_c) \in \Delta\text{:} \qquad \tau_c[\alpha \mapsto s][\Delta_P \mapsto p_1, \ldots, p_k] \equiv \forall\beta < s.\, \Delta_c \to X\ [s]\ p_1 \ldots p_k\ e'_1 \ldots e'_n \\ \Phi(\beta < \alpha); \Gamma\Delta_c \vdash e_c \Downarrow P[\Delta_I \mapsto e'_1, \ldots, e'_n][x \mapsto c\ [s]\ p_1 \ldots p_k\ [\beta]\ \Delta_c] \end{array}}{\Phi; \Gamma \vdash \mathsf{match}\ e\ \mathsf{return}\ \lambda(\mathsf{dom}(\Delta_I)).x.\, P\ \mathsf{with}\ (c\ [\beta]\ \Delta_c \Rightarrow e_c) \ldots \Uparrow P[\Delta_I \mapsto e_1, \ldots, e_n][x \mapsto e]}$$

FIX

$$\frac{\begin{array}{c} \Phi(\alpha); \Gamma \vdash \sigma \Uparrow U \qquad \Phi(\alpha); \Gamma \vdash \sigma \rhd^* \Theta \to (x : \tau) \to \tau' \qquad \beta\ \text{fresh} \\ \tau \equiv X\ [\alpha]\ \_ \ldots \qquad \Phi(\alpha); \Gamma(f : \forall\beta < \alpha.\, \sigma[\alpha \mapsto \beta]) \vdash e \Downarrow \sigma \end{array}}{\Phi; \Gamma \vdash \mathsf{fix}\ f\ [\alpha] : \sigma \coloneqq e \Uparrow \forall\alpha.\, \sigma}$$

Figure 10: Sized inductives and fixpoints

$$\Gamma ::= \cdots \mid \Gamma\blacktriangledown$$

$$e ::= \cdots \mid \mathbb{N} \mid \mathbb{W}x : \tau.\, \tau \mid \mathsf{zero} \mid \mathsf{succ}\ e \mid \mathsf{sup}_{\mathbb{W}x:\tau.\tau}\ e\ e \mid \mathsf{erase}\ e$$
$$\mid \mathsf{match}\ e\ \mathsf{return}\ \lambda x.\, P\ \mathsf{with}\ (c\ z_1 \ldots\ z_2 \Rightarrow e) \ldots$$

$\boxed{\Gamma \vdash e \rhd e}$ $\cdots$

$$\frac{}{\Gamma \vdash \mathsf{match}\ \mathsf{zero}\ \mathsf{return}\ \_\ \mathsf{with}\ (\mathsf{zero} \Rightarrow e_z)(\mathsf{succ}\ \_ \Rightarrow \_) \rhd_\iota e_z}$$

$$\frac{}{\Gamma \vdash \mathsf{match}\ \mathsf{succ}\ e\ \mathsf{return}\ \_\ \mathsf{with}\ (\mathsf{zero} \Rightarrow \_)(\mathsf{succ}\ z \Rightarrow e_s) \rhd_\iota e_s[z \mapsto e]}$$

$$\frac{}{\Gamma \vdash \mathsf{match}\ \mathsf{sup}_{\_}\ e_1\ e_2\ \mathsf{return}\ \_\ \mathsf{with}\ (\mathsf{sup}\ z_1\ z_2 \Rightarrow e) \rhd_\iota e[z_1 \mapsto e_1][z_2 \mapsto e_2]}$$

$$\frac{\|\forall\alpha.\, \sigma\| \equiv \Delta \to (x : \tau) \to \tau' \qquad \tau \equiv \mathbb{N}\ \mathit{or}\ \mathbb{W}\_ : \_.\, \_ \qquad |\Delta| = n}{\Gamma \vdash (\mathsf{erase}\ (\mathsf{fix}\ f\ [\alpha] : \sigma \coloneqq e))\ e_1 \ldots\ e_n\ (c\ \_ \ldots) \rhd_\mu \|\Lambda\alpha.\, e[f \mapsto \mathsf{fix}\ f\ [\alpha] : \sigma \coloneqq e]\|\ e_1 \ldots\ e_n\ (c\ \_ \ldots)}$$

$$\frac{}{\Gamma \vdash \mathsf{erase}\ e \rhd_\epsilon \|e\|}$$

$\boxed{\vdash \Gamma}$ $\cdots$

CONS-FILTER
$$\frac{\vdash \Gamma}{\vdash \Gamma\blacktriangledown}$$

$\boxed{\Gamma \vdash e \Updownarrow \tau}$ $\cdots$

VAR
$$\frac{\blacktriangledown \notin \Gamma_2 \qquad \vdash \Gamma_1(x : \tau)\Gamma_2}{\Gamma_1(x : \tau)\Gamma_2 \vdash x \Uparrow \tau}$$

VAR$^\infty$-ANN
$$\frac{\|\tau\| \equiv \tau \qquad \blacktriangledown \in \Gamma_2 \qquad \vdash \Gamma_1(x : \tau)\Gamma_2}{\Gamma_1(x : \tau)\Gamma_2 \vdash x \Uparrow \tau}$$

VAR$^\infty$-DEF
$$\frac{\blacktriangledown \in \Gamma_2 \qquad \Gamma_1\blacktriangledown \vdash e \Uparrow \tau}{\Gamma_1(x \coloneqq e)\Gamma_2 \vdash x \Uparrow \tau}$$

ERASE
$$\frac{\Gamma\blacktriangledown \vdash e \Uparrow \tau}{\Gamma \vdash \mathsf{erase}\ e \Uparrow \|\tau\|}$$

NAT$^\infty$
$$\frac{\vdash \Gamma}{\Gamma \vdash \mathbb{N} \Uparrow \mathsf{Type}_0}$$

ZERO$^\infty$
$$\frac{\vdash \Gamma}{\Gamma \vdash \mathsf{zero} \Uparrow \mathbb{N}}$$

SUCC$^\infty$
$$\frac{\Gamma \vdash e \Downarrow \mathbb{N}}{\Gamma \vdash \mathsf{succ}\ e \Uparrow \mathbb{N}}$$

W$^\infty$
$$\frac{\Gamma \vdash \sigma \Uparrow U \qquad \Gamma(x : \sigma) \vdash \tau \Uparrow U}{\Gamma \vdash \mathbb{W}x : \sigma.\, \tau \Uparrow \max_{\sqsubseteq}(U, \mathsf{Set})}$$

SUP$^\infty$
$$\frac{\Gamma \vdash \mathbb{W}x : \sigma.\, \tau \Uparrow U \qquad \Gamma \vdash e_1 \Downarrow \sigma \qquad \Gamma \vdash e_2 \Downarrow \tau[x \mapsto e_1] \to \mathbb{W}x : \sigma.\, \tau}{\Gamma \vdash \mathsf{sup}_{\mathbb{W}x:\sigma.\tau}\ e_1\ e_2 \Uparrow \mathbb{W}x : \sigma.\, \tau}$$

MATCH-NAT$^\infty$
$$\frac{\Gamma \vdash e \Downarrow \mathbb{N} \qquad z \notin \mathsf{FV}(P) \qquad \Gamma(x : \mathbb{N}) \vdash P \Uparrow U \qquad \Gamma \vdash e_z \Downarrow P[x \mapsto \mathsf{zero}] \qquad \Gamma(z : \mathbb{N}) \vdash e_s \Downarrow P[x \mapsto \mathsf{succ}\ z]}{\Gamma \vdash \mathsf{match}\ e\ \mathsf{return}\ \lambda x.\, P\ \mathsf{with}\ (\mathsf{zero} \Rightarrow e_z)(\mathsf{succ}\ z \Rightarrow e_s) \Uparrow P[x \mapsto e]}$$

MATCH-SUP$^\infty$
$$\frac{\Gamma \vdash e \Uparrow \mathbb{W}y : \sigma.\, \tau \qquad z_1, z_2 \notin \mathsf{FV}(P) \qquad \Gamma(x : \mathbb{W}y : \sigma.\, \tau) \vdash P \Uparrow U \qquad \Gamma(z_1 : \sigma)(z_2 : \tau[y \mapsto z_1] \to \mathbb{W}y : \tau.\, \sigma) \vdash e_s \Downarrow P[x \mapsto \mathsf{sup}_{\mathbb{W}y:\sigma.\tau}\ z_1\ z_2]}{\Gamma \vdash \mathsf{match}\ e\ \mathsf{return}\ \lambda x.\, P\ \mathsf{with}\ (\mathsf{sup}\ z_1\ z_2 \Rightarrow e_s) \Uparrow P[x \mapsto e]}$$

Figure 11: Full naturals and W types

$X ::=$ inductive type names $\qquad c ::=$ inductive constructor names

$D ::= \mathsf{data}\ X\ \Delta_P : \Delta_I \to U_i\ \mathsf{where}\ \Delta$

$e ::= \cdots \mid \mathsf{match}\ e\ \mathsf{return}\ \lambda(y_1 \ldots y_n).x.\,P\ \mathsf{with}\ (c\,z_1 \ldots z_m \Rightarrow e) \ldots \mid \mathsf{fix}_n f : \tau \coloneqq e \mid e \overset{\tau}{=\!=} e \mid \mathsf{refl}\ e$

$\boxed{\Gamma \vdash e \rhd e}$ $\qquad \rhd_\delta \qquad \rhd_\beta \qquad \rhd_\zeta \qquad$ *as above*

$$\frac{}{\Gamma \vdash \mathsf{match}\ c\,e_1 \ldots e_m\ \mathsf{return}\ \_\ \mathsf{with}\ \ldots (c\,z_1 \ldots z_m \Rightarrow e) \ldots \rhd_\iota e[z_1, \ldots, z_m \mapsto e_1, \ldots, e_m]}$$

$$\frac{}{\Gamma \vdash (\mathsf{fix}_n f : \tau \coloneqq e)\ e'_1 \ldots e'_n\ (c\,e_1 \ldots e_m) \rhd_\mu e[f \mapsto \mathsf{fix}_n f : \tau \coloneqq e]\ e'_1 \ldots e'_n\ (c\,e_1 \ldots e_m)}$$

$\boxed{\Gamma \vdash e \rhd^* e}$ $\qquad \rhd^*$-REFL $\qquad \rhd^*$-TRANS $\qquad \rhd^*$-CONG $\qquad$ *as above*

$\boxed{\Gamma \vdash e \approx e}$ $\qquad \approx$-RED $\qquad \approx$-LAM-η_L $\qquad \approx$-LAM-η_R $\qquad$ *as above*

$\approx$-TRANS
$$\frac{\Gamma \vdash e_1 \approx e_2 \qquad \Gamma \vdash e_2 \approx e_3}{\Gamma \vdash e_1 \approx e_3}$$

$\approx$-MATCH-η_L
$$\frac{\begin{array}{c}\Gamma \vdash e_1 \rhd^* \mathsf{match}\ e\ \mathsf{return}\ \_\ \mathsf{with}\ (c\,z_1 \ldots\ z_m \Rightarrow e_1^*\ (c\,z_1\ z_m)) \ldots \\ \Gamma \vdash e_2 \rhd^* e_2^* \qquad \Gamma \vdash e_1^*\ e \approx e_2^*\end{array}}{\Gamma \vdash e_1 \approx e_2}$$

$\approx$-MATCH-η_R
$$\frac{\begin{array}{c}\Gamma \vdash e_1 \rhd^* e_1^* \qquad \Gamma \vdash e_1^* \approx e_2^*\ e \\ \Gamma \vdash e_2 \rhd^* \mathsf{match}\ e\ \mathsf{return}\ \_\ \mathsf{with}\ (c\,z_1 \ldots\ z_m \Rightarrow e_2^*\ (c\,z_1\ z_m)) \ldots\end{array}}{\Gamma \vdash e_1 \approx e_2}$$

$\approx$-REFLECT
$$\frac{\begin{array}{c}\Gamma \vdash e_1 \rhd^* e'_1 \qquad \Gamma \vdash e_2 \rhd^* e'_2 \\ (x : e'_1 \overset{\tau}{=\!=} e'_2) \in \Gamma\ \textit{or}\ (x : e'_2 \overset{\tau}{=\!=} e'_1) \in \Gamma\end{array}}{\Gamma \vdash e_1 \approx e_2}$$

$\approx$-CONG
$$\frac{\begin{array}{c}\text{For every } 1 \le i \le n\text{:} \qquad \Gamma\Gamma' \vdash e_i \approx e'_i \\ \Gamma \vdash e_1 \rhd^* e[x_1, \ldots, x_n \mapsto e_1, \ldots, e_n] \qquad \Gamma \vdash e_2 \rhd^* e[x_1, \ldots, x_n \mapsto e'_1, \ldots, e'_n]\end{array}}{\Gamma \vdash e_1 \approx e_2}$$

$\boxed{\Gamma \vdash \tau \preccurlyeq \tau}$ $\qquad \preccurlyeq$-CONV $\qquad \preccurlyeq$-TRANS $\qquad \preccurlyeq$-PROP $\qquad \preccurlyeq$-UNIV $\qquad \preccurlyeq$-PI $\qquad$ *as above*

$\boxed{\vdash \Gamma}$ $\qquad$ NIL $\qquad$ CONS-ASS $\qquad$ CONS-DEF $\qquad$ *as above*

Figure 12: CIC syntax and judgements

$$\begin{aligned}\mathsf{rule}(U, \mathsf{Prop}) &\equiv \mathsf{Prop} \\ \mathsf{rule}(U, \mathsf{Set}) &\equiv \mathsf{Set} \\ \mathsf{rule}(U_1, U_2) &\equiv \max_{\sqsubseteq}(U_1, U_2)\end{aligned}$$

Figure 13: Impredicative Set

$\boxed{\Gamma \vdash e \Updownarrow \tau}$ CONV VAR UNIV PI LAM APP LET *as above*

IND
$$\frac{\vdash \Gamma \qquad \mathsf{data}\ X\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta}{\Gamma \vdash X \Uparrow \Delta_P \Delta_I \to U}$$

CONSTR
$$\frac{\vdash \Gamma \qquad \mathsf{data}\ X\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta \qquad (c : \tau) \in \Delta}{\Gamma \vdash c \Uparrow \Delta_P \to \tau}$$

MATCH
$$\frac{\begin{array}{c} \mathsf{data}\ X\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta \\ \Gamma \vdash e \Uparrow X\ p_1 \ldots p_k\ e_1 \ldots e_n \qquad \Gamma(\Delta_I[\Delta_P \mapsto p_1,\ \ldots, p_k])(x : X\ p_1 \ldots p_k\ \Delta_I) \vdash P \Uparrow U' \\ \text{For each constructor } (c : \tau_c) \in \Delta\text{:} \qquad \tau[\Delta_P \mapsto p_1,\ \ldots, p_k] \equiv \Delta_c \to X\ p_1 \ldots p_k\ e'_1 \ldots e'_n \\ \Gamma\Delta_c \vdash e_c \Downarrow P[\Delta_I \mapsto e'_1,\ \ldots, e'_n][x \mapsto c\, p_1 \ldots p_k\ \Delta_c] \end{array}}{\Gamma \vdash \mathsf{match}\ e\ \mathsf{return}\ \lambda(\mathsf{dom}(\Delta_I)).x.\ P\ \mathsf{with}\ (c\ \Delta_c \Rightarrow e_c) \ldots \Uparrow P[\Delta_I \mapsto e_1,\ \ldots, e_n][x \mapsto e]}$$

FIX
$$\frac{\Gamma \vdash \sigma \Uparrow U \qquad \Gamma \vdash \sigma \rhd^* \Delta \to X\ \_ \ldots \to \_ \qquad |\Delta| = n \qquad \Gamma(f : \sigma) \vdash e \Downarrow \sigma}{\Gamma \vdash \mathsf{fix}_n f : \sigma \coloneqq e \Uparrow \sigma}$$

EQ
$$\frac{\Gamma \vdash \tau \Uparrow U \qquad \Gamma \vdash e_1 \Downarrow \tau \qquad \Gamma \vdash e_2 \Downarrow \tau}{\Gamma \vdash e_1 \overset{\tau}{=\!=} e_2 \Uparrow U}$$

REFL
$$\frac{\Gamma \vdash e \Uparrow \tau}{\Gamma \vdash \mathsf{refl}\ e \Uparrow e \overset{\tau}{=\!=} e}$$

Figure 14: CIC typing judgement

$\mathsf{data}\ \bot : \mathsf{Prop}\ \mathsf{where}$
$\mathsf{let}\ \mathsf{elim}\bot : (P : \mathsf{Type}_i) \to \bot \to P \coloneqq$
$\quad \lambda P : \mathsf{Type}_i.\ \lambda b : \bot.\ \mathsf{match}\ b\ \mathsf{return}\ \lambda().\_.\ P\ \mathsf{with}$

$\mathsf{data}\ \mathsf{Size} : \mathsf{Set}\ \mathsf{where}$
$\quad \mathsf{suc} : \mathsf{Size} \to \mathsf{Size}$
$\quad \mathsf{lim} : (A : \mathsf{Type}_i) \to (A \to \mathsf{Size}) \to \mathsf{Size}$
$\mathsf{let}\ \mathsf{base} : \mathsf{Size} \coloneqq \mathsf{lim}\ \bot\ (\mathsf{elim}\bot\ \mathsf{Size})$

$\mathsf{data}\ \_ \leq \_ : \mathsf{Size} \to \mathsf{Size} \to \mathsf{Set}\ \mathsf{where}$
$\quad \mathsf{mono} : (\alpha, \beta : \mathsf{Size}) \to \alpha \leq \beta \to \mathsf{suc}\ \alpha \leq \mathsf{suc}\ \beta$
$\quad \mathsf{cocone} : (A : \mathsf{Type}_i) \to (\beta : \mathsf{Size}) \to (f : A \to \mathsf{Size}) \to (a : A) \to \beta \leq f\ a \to \beta \leq \mathsf{lim}\ A\ f$
$\quad \mathsf{limit} : (A : \mathsf{Type}_i) \to (\beta : \mathsf{Size}) \to (f : A \to \mathsf{Size}) \to ((a : A) \to f\ a \leq \beta) \to \mathsf{lim}\ A\ f \leq \beta$
$\mathsf{let}\ \_ < \_ : \mathsf{Size} \to \mathsf{Size} \to \mathsf{Set} \coloneqq \lambda\alpha : \mathsf{Size}.\ \lambda\beta : \mathsf{Size}.\ \mathsf{suc}\ \alpha \leq \beta$

$\mathsf{data}\ \mathsf{Acc}\ (\alpha : \mathsf{Size}) : \mathsf{Set}\ \mathsf{where}$
$\quad \mathsf{acc} : ((\beta : \mathsf{Size}) \to \beta < \alpha \to \mathsf{Acc}\ \beta) \to \mathsf{Acc}\ \alpha$
$\mathsf{let}\ \mathsf{accIsProp} : (\alpha : \mathsf{Size}) \to (acc1, acc2 : \mathsf{Acc}\ \alpha) \to acc1 \overset{\mathsf{Acc}\ \alpha}{=\!=} acc2 \coloneqq \_$
$\mathsf{let}\ \mathsf{accessible} : (\alpha : \mathsf{Size}) \to \mathsf{Acc}\ \alpha \coloneqq \_$

Figure 15: Preliminary CIC definitions

$$
\begin{array}{l}
\mathsf{let}\ \mathsf{base}{\leq} : (\alpha : \mathsf{Size}) \to \mathsf{base} \leq \alpha \coloneqq \lambda\alpha : \mathsf{Size}.\\
\quad \mathsf{let}\ f \coloneqq \mathsf{elim}\bot\ \mathsf{Size}\ \mathsf{in}\ \mathsf{limit}\ \bot\ \alpha\ f\ (\lambda b : \bot.\ \mathsf{elim}\bot\ (f\ b \leq \alpha)\ b)\\
\mathsf{let}\ \mathsf{refl}{\leq} : (\alpha : \mathsf{Size}) \to \alpha \leq \alpha \coloneqq \_\\
\mathsf{let}\ \mathsf{trans}{\leq} : (\alpha, \beta, \gamma : \mathsf{Size}) \to \alpha \leq \beta \to \beta \leq \gamma \to \alpha \leq \gamma \coloneqq \_\\
\mathsf{let}\ \mathsf{suc}{\leq} : (\alpha : \mathsf{Size}) \to \alpha \leq \mathsf{suc}\ \alpha \coloneqq \_\\
\mathsf{let}\ \mathsf{elim} : (P : \mathsf{Size} \to \mathsf{Type}_i) \to ((\alpha : \mathsf{Size}) \to ((\beta : \mathsf{Size}) \to \beta < \alpha \to P\ \beta) \to P\ \alpha) \to (\alpha : \mathsf{Size}) \to P\ \alpha \coloneqq\\
\quad \lambda P : \mathsf{Size} \to \mathsf{Type}.\ \lambda f : (\alpha : \mathsf{Size}) \to ((\beta : \mathsf{Size}) \to \beta < \alpha \to P\ \beta) \to P\ \alpha.\ \lambda\alpha : \mathsf{Size}.\\
\qquad \mathsf{let}\ \mathit{elimAcc} \coloneqq \mathsf{fix}_1\ \mathit{elimAccRec} : (\alpha : \mathsf{Size}) \to \mathsf{Acc}\ \alpha \to P\ \alpha \coloneqq\\
\qquad\quad \lambda\alpha : \mathsf{Size}.\ \lambda \mathit{access} : \mathsf{Acc}\ \alpha.\ \mathsf{match}\ \mathit{access}\ \mathsf{return}\ \lambda().\_.\ P\ \alpha\ \mathsf{with}\\
\qquad\qquad (\mathsf{acc}\ p \Rightarrow f\ \alpha\ (\lambda\beta : \mathsf{Size}.\ \lambda\beta^* : \beta < \alpha.\ \mathit{elimAccRec}\ \beta\ (p\ \beta\ \beta^*)))\ \mathsf{in}\\
\qquad \mathit{elimAcc}\ \alpha\ (\mathsf{accessible}\ \alpha)
\end{array}
$$

Figure 16: Properties of the order on sizes

$\boxed{[\![s]\!] = e^*}$

$$
\begin{aligned}
[\![\alpha]\!] &= \alpha\\
[\![\circ]\!] &= \mathsf{base}\\
[\![\hat{s}]\!] &= \mathsf{suc}\ [\![s]\!]
\end{aligned}
$$

$\boxed{\Phi \vdash s \leqslant s \rightsquigarrow e^*}$ *over* $\Phi \vdash s \leqslant s$

$$
\frac{(\alpha < s) \in \Phi}{\Phi \vdash \hat{\alpha} \leqslant s \rightsquigarrow \alpha^*}
\qquad
\frac{}{\Phi \vdash s \leqslant s \rightsquigarrow \mathsf{refl}{\leq}\ [\![s]\!]}
$$

$$
\frac{}{\Phi \vdash \circ \leqslant s \rightsquigarrow \mathsf{base}{\leq}\ [\![s]\!]}
\qquad
\frac{}{\Phi \vdash s \leqslant \hat{s} \rightsquigarrow \mathsf{suc}{\leq}\ [\![s]\!]}
$$

$$
\frac{\Phi \vdash s_1 \leqslant r \rightsquigarrow e_1^* \qquad \Phi \vdash r \leqslant s_2 \rightsquigarrow e_2^*}{\Phi \vdash s_1 \leqslant s_2 \rightsquigarrow \mathsf{trans}{\leq}\ [\![s_1]\!]\ [\![r]\!]\ [\![s_2]\!]\ e_1^*\ e_2^*}
$$

Figure 17: Model in CIC (sizes)

$\boxed{[\![\Phi]\!] = \Gamma}$ *over* $\vdash \Phi$

$$
\begin{aligned}
[\![\cdot]\!] &= \cdot\\
[\![\Phi(\alpha)]\!] &= [\![\Phi]\!](\alpha : \mathsf{Size})\\
[\![\Phi(\alpha < s)]\!] &= [\![\Phi]\!](\alpha : \mathsf{Size})(\alpha^* : \alpha < [\![s]\!])
\end{aligned}
$$

$\boxed{[\![\Gamma]\!] = \Gamma^*}$ *over* $\Phi \vdash \Gamma$

$$
\begin{aligned}
[\![\cdot]\!] &= \cdot\\
[\![\Gamma(x : \tau)]\!] &= [\![\Gamma]\!](x : [\![\tau]\!])\\
[\![\Gamma(x \coloneqq e)]\!] &= [\![\Gamma]\!](x \coloneqq [\![e]\!])
\end{aligned}
$$

Figure 18: Model in CIC (environments)

$$\mathsf{data}\ X\ [\alpha]\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ (c : \tau_c) \ldots \rightsquigarrow \mathsf{data}\ X\ (\alpha : \mathsf{Size})[\![\Delta_P]\!] : [\![\Delta_I \to U]\!]_{(\alpha);\Delta_P}\ \mathsf{where}\ (c : [\![\tau_c]\!]_{(\alpha);\Delta_P}) \ldots$$

$$\boxed{[\![e]\!]_{\Phi';\Gamma'} = e^*}\ \textit{where}\ \Phi\Phi';\Gamma\Gamma' \vdash e \Uparrow \_ \rightsquigarrow e^*$$

$$\begin{aligned}
[\![x]\!] &= x \\
[\![X]\!] &= X \\
[\![c]\!] &= c \\
[\![U]\!] &= U \\
[\![\Pi x : \sigma.\, \tau]\!] &= \Pi x : [\![\sigma]\!].\, [\![\tau]\!]_{\cdot;(x:\sigma)} \\
[\![\lambda x : \sigma.\, e]\!] &= \lambda x : [\![\sigma]\!].\, [\![e]\!]_{\cdot;(x:\sigma)} \\
[\![e_1\ e_2]\!] &= [\![e_1]\!]\ [\![e_2]\!] \\
[\![\forall \alpha.\, \tau]\!] &= \Pi \alpha : \mathsf{Size}.\, [\![\tau]\!]_{(\alpha);\cdot} \\
[\![\forall \alpha < s.\, \tau]\!] &= \Pi \alpha : \mathsf{Size}.\, \Pi \alpha^* : \alpha < [\![s]\!].\, [\![\tau]\!]_{(\alpha<s);\cdot} \\
[\![\Lambda \alpha.\, e]\!] &= \lambda \alpha : \mathsf{Size}.\, [\![e]\!]_{(\alpha);\cdot} \\
[\![\Lambda \alpha < s.\, e]\!] &= \lambda \alpha : \mathsf{Size}.\, \lambda \alpha^* : \alpha < [\![s]\!].\, [\![e]\!]_{(\alpha<s);\cdot}
\end{aligned}$$

$$\boxed{\Phi;\Gamma \vdash e \Updownarrow \tau \rightsquigarrow e^*}\ \textit{over}\ \Phi;\Gamma \vdash e \Updownarrow \tau$$

CONV*
$$\frac{\Phi;\Gamma \vdash e \Uparrow \tau' \rightsquigarrow e^* \qquad \Phi;\Gamma \vdash \tau' \approx \tau}{\Phi;\Gamma \vdash e \Downarrow \tau \rightsquigarrow e^*}$$

LET
$$\frac{\Phi;\Gamma \vdash e_1 \Uparrow \sigma \rightsquigarrow e_1^* \qquad \Phi;\Gamma(x : \sigma)(x \coloneqq e_1) \vdash e_2 \Uparrow \tau \rightsquigarrow e_2^*}{\Phi;\Gamma \vdash \mathsf{let}\ x \coloneqq e_1\ \mathsf{in}\ e_2 \Uparrow \tau[x \mapsto e_1] \rightsquigarrow \mathsf{let}\ x \coloneqq e_1^*\ \mathsf{in}\ e_2^*}$$

SAPP
$$\frac{\Phi;\Gamma \vdash e \Uparrow \forall \alpha.\, \tau \rightsquigarrow e^*}{\Phi;\Gamma \vdash e\ [s] \Uparrow \tau[\alpha \mapsto s] \rightsquigarrow e^*\ [\![s]\!]}$$

SAPP*<
$$\frac{\Phi;\Gamma \vdash e \Uparrow \forall \alpha < r.\, \tau \rightsquigarrow e^* \qquad \Phi \vdash \hat{s} \leqslant r \rightsquigarrow e^*_{\leq}}{\Phi;\Gamma \vdash e\ [s] \Uparrow \tau[\alpha \mapsto s] \rightsquigarrow e^*\ [\![s]\!]\ e^*_{\leq}}$$

MATCH*
$$\frac{\begin{array}{c}\mathsf{data}\ X\ [\alpha]\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta \\ \Phi;\Gamma \vdash e \Uparrow X\ [s]\ p_1 \ldots p_k\ e_1 \ldots e_n \rightsquigarrow e^* \\ \Phi;\Gamma(\Delta_I[\alpha \mapsto s][\Delta_P \mapsto p_1, \ldots, p_k])(x : X\ [s]\ p_1 \ldots p_k\ \Delta_I) \vdash P \Uparrow U' \rightsquigarrow P^* \\ \text{For each constructor } (c : \tau_c) \in \Delta\text{:} \quad \tau_c[\alpha \mapsto s][\Delta_P \mapsto p_1, \ldots, p_k] \equiv \forall \beta < s.\, \Delta_c \to X\ [s]\ p_1 \ldots p_k\ e'_1 \ldots e'_n \\ \Phi(\beta < \alpha);\Gamma\Delta_c \vdash e_c \Downarrow P[\Delta_I \mapsto e'_1, \ldots, e'_n][x \mapsto c\ [s]\ p_1 \ldots p_k\ [\beta]\ \Delta_c] \rightsquigarrow e_c^*\end{array}}{\Phi;\Gamma \vdash \begin{array}{l}\mathsf{match}\ e\ \mathsf{return}\ \lambda(\mathsf{dom}(\Delta_I)).x.\, P\ \mathsf{with} \\ \quad (c\ [\beta]\ \Delta_c \Rightarrow e_c) \ldots\end{array} \Uparrow P[\Delta_I \mapsto e_1, \ldots, e_n][x \mapsto e] \rightsquigarrow \begin{array}{l}\mathsf{match}\ e^*\ \mathsf{return}\ \lambda(\mathsf{dom}(\Delta_I)).x.\, P^*\ \mathsf{with} \\ \quad (c\ \beta\ \beta^*\ \Delta_c^* \Rightarrow e_c^*) \ldots\end{array}}$$

FIX*
$$\frac{\beta\ \mathsf{fresh} \qquad \Phi(\alpha);\Gamma \vdash \sigma \Uparrow U \rightsquigarrow \sigma^* \qquad \Phi(\alpha);\Gamma(f : \forall \beta < \alpha.\, \sigma[\alpha \mapsto \beta]) \vdash e \Downarrow \sigma \rightsquigarrow e^*}{\Phi;\Gamma \vdash \mathsf{fix}\ f\ [\alpha] : \sigma \coloneqq e \Uparrow \forall \alpha.\, \sigma \rightsquigarrow \begin{array}{l}\mathsf{elim}\ (\lambda \alpha : \mathsf{Size}.\, \sigma^*) \\ \quad (\lambda \alpha : \mathsf{Size}.\, \lambda f : (\beta : \mathsf{Size}) \to \beta < \alpha \to \sigma^*[\alpha \mapsto \beta].\, e^*)\end{array}}$$

Figure 19: Model in CIC (terms and types)