$\widehat{\mathtt{TT}}$

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$$
\begin{aligned}
i, j, k, m, n &::= \textsf{naturals} & r, s &::= \alpha \mid \hat{s} \mid \circ \mid \infty \\
f, g, x, y, z &::= \textsf{term variables} & U &::= \textsf{Prop} \mid \textsf{Type}_i \\
\alpha, \beta &::= \textsf{size variables} \\
e, p, P, \tau, \sigma &::= x \mid U_i \mid \Pi x : \tau.\, \tau \mid \lambda x : \tau.\, e \mid e\; e \mid \textsf{let}\; x \coloneqq e \;\textsf{in}\; e \\
&\mid \forall \alpha.\, \tau \mid \forall \alpha < s.\, \tau \mid \Lambda \alpha.\, e \mid \Lambda \alpha < s.\, e \mid e\; [s] \\
&\mid \Sigma x : \sigma.\, \tau \mid \langle e, e \rangle_{\Sigma x : \sigma.\tau} \mid \pi_1\; e \mid \pi_2\; e \\
&\mid e \overset{\tau}{=\!=} e \mid \textsf{refl}_e \mid \textsf{J}_P\; d\; p \\
\Delta &::= \cdot \mid \Delta(x : \tau) & \Phi &::= \cdot \mid \Phi(\alpha) \mid \Phi(\alpha < s) \\
\Gamma &::= \cdot \mid \Gamma(x : \tau) \mid \Gamma(x \coloneqq e) & \Theta &::= \cdot \mid \Theta(x : \tau) \mid \Theta(\alpha) \mid \Theta(\alpha < s)
\end{aligned}
$$

Figure 1: Syntax

$$
\begin{aligned}
\tau_1 \to \tau_2 &= \Pi \_ : \tau_1.\, \tau_2 \\
\Delta \to \tau &= \Pi x_n : \tau_1.\, \dots \Pi x_n : \tau_n.\, \tau \\
\tau_1 \times \tau_2 &= \Sigma \_ : \tau_1.\, \tau_2 \\
\Delta \times \tau &= \Sigma x_1 : \tau_1.\, \dots \Sigma x_n : \tau_n.\, \tau
\end{aligned}
$$

$$
\begin{aligned}
e\; \Delta &= e\; x_1 \dots x_n \\
e[z_1, \dots, z_n \mapsto \Delta] &= e[z_1 \mapsto x_1, \dots, z_n \mapsto x_n] \\
e[x_1, \dots, x_n \mapsto e_1, \dots, e_n] &= e[x_1 \mapsto e_1, \dots, x_n \mapsto e_n]
\end{aligned}
$$

$$\textit{where}\; \Delta = (x_1 : \tau_1) \dots (x_n : \tau_n)$$

Figure 2: Syntactic sugar

$$
\begin{aligned}
\textsf{axiom}(\textsf{Prop}) &= \textsf{Type}_1 \\
\textsf{axiom}(\textsf{Type}_i) &= \textsf{Type}_{i+1} \\
\textsf{rule}(U, \textsf{Prop}) &= \textsf{Prop} \\
\textsf{rule}(\textsf{Prop}, \textsf{Type}_i) &= \textsf{Type}_i \\
\textsf{rule}(\textsf{Type}_i, \textsf{Type}_j) &= \textsf{Type}_{\max(i,j)}
\end{aligned}
$$

Figure 3: Metafunctions and metarelations

$$\boxed{\Phi; \Gamma \vdash e \rhd e} \qquad \boxed{\Phi; \Gamma \vdash e \rhd^* e}$$

$$\frac{\begin{array}{c}(x : \tau) \in \Gamma \\ (x \coloneqq e) \in \Gamma\end{array}}{\Phi; \Gamma \vdash x \rhd_\delta e} \qquad \frac{}{\Phi; \Gamma \vdash (\lambda x : \tau.\, e)\ e' \rhd_\beta e[x \mapsto e']} \qquad \frac{}{\Phi; \Gamma \vdash \pi_i\ \langle e_1, e_2 \rangle \rhd_\pi e_i} \qquad \frac{}{\Phi; \Gamma \vdash \mathsf{J}\ d\ \mathsf{refl} \rhd_\rho d}$$

$$\frac{}{\Phi; \Gamma \vdash \mathsf{let}\ x \coloneqq e'\ \mathsf{in}\ e \rhd_\zeta e[x \mapsto e']} \qquad \frac{}{\Phi; \Gamma \vdash (\Lambda \alpha.\, e)\ [s] \rhd_\varsigma e[\alpha \mapsto s]} \qquad \frac{}{\Phi; \Gamma \vdash (\Lambda \alpha < r.\, e)\ [s] \rhd_\varsigma e[\alpha \mapsto s]}$$

$$\textsc{$\rhd^*$-refl}\ \frac{}{\Phi; \Gamma \vdash e \rhd^* e} \qquad \textsc{$\rhd^*$-trans}\ \frac{\begin{array}{c}\Phi; \Gamma \vdash e_1 \rhd e_2 \\ \Phi; \Gamma \vdash e_2 \rhd^* e_3\end{array}}{\Phi; \Gamma \vdash e_1 \rhd^* e_3}$$

$$\textsc{$\rhd^*$-cong}\ \frac{\text{For every } 1 \leq i \leq n: \qquad \Phi\Phi'; \Gamma\Gamma' \vdash e_i \rhd^* e_i'}{\Phi; \Gamma \vdash e[x_1, \ldots, x_n \mapsto e_1, \ldots, e_n] \rhd^* e[x_1, \ldots, x_n \mapsto e_1', \ldots, e_n']}$$

Figure 4: Reduction rules

$$\boxed{\Phi; \Gamma \vdash \tau \preccurlyeq \tau} \qquad \boxed{e \sqsubseteq e}$$

$$\textsc{$\preccurlyeq$-red}\ \frac{\Phi; \Gamma \vdash \tau_1 \rhd^* \sigma_1 \qquad \Phi; \Gamma \vdash \tau_2 \rhd^* \sigma_2 \qquad \sigma_1 \sqsubseteq \sigma_2}{\Phi; \Gamma \vdash \tau_1 \preccurlyeq \tau_2} \qquad \textsc{$\sqsubseteq$-refl}\ \frac{}{e \sqsubseteq e} \qquad \textsc{$\sqsubseteq$-prop}\ \frac{}{\mathsf{Prop} \sqsubseteq \mathsf{Type}_i} \qquad \textsc{$\sqsubseteq$-type}\ \frac{i \leq j}{\mathsf{Type}_i \sqsubseteq \mathsf{Type}_j}$$

$$\textsc{$\sqsubseteq$-pi}\ \frac{\tau_1 \sqsubseteq \tau_2}{\Pi x : \sigma.\, \tau_1 \sqsubseteq \Pi x : \sigma.\, \tau_2} \qquad \textsc{$\sqsubseteq$-forall}\ \frac{\tau_1 \sqsubseteq \tau_2}{\forall \alpha.\, \tau_1 \sqsubseteq \forall \alpha.\, \tau_2} \qquad \textsc{$\sqsubseteq$-forall$<$}\ \frac{\tau_1 \sqsubseteq \tau_2}{\forall \alpha < s.\, \tau_1 \sqsubseteq \forall \alpha < s.\, \tau_2} \qquad \textsc{$\sqsubseteq$-sigma}\ \frac{\sigma_1 \sqsubseteq \sigma_2 \qquad \tau_1 \sqsubseteq \tau_2}{\Sigma x : \sigma_1.\, \tau_1 \sqsubseteq \Sigma x : \sigma_2.\, \tau_2}$$

Figure 5: Subtyping rules

$$\boxed{\Phi \vdash s} \qquad \boxed{\Phi \vdash s \leqslant s}$$

$$\frac{\begin{array}{c}\vdash \Phi \\ (\alpha) \in \Phi\ \ or \\ (\alpha < s) \in \Phi\end{array}}{\Phi \vdash \alpha} \qquad \frac{\Phi \vdash s}{\Phi \vdash \hat{s}} \qquad \frac{\vdash \Phi}{\Phi \vdash \circ} \qquad \frac{\begin{array}{c}\vdash \Phi \\ (\alpha < s) \in \Phi\end{array}}{\Phi \vdash \hat{\alpha} \leqslant s} \qquad \frac{\Phi \vdash s}{\Phi \vdash s \leqslant s} \qquad \frac{\Phi \vdash s}{\Phi \vdash \circ \leqslant s} \qquad \frac{\Phi \vdash s}{\Phi \vdash s \leqslant \hat{s}} \qquad \frac{\begin{array}{c}\Phi \vdash s_1 \leqslant s_2 \\ \Phi \vdash s_2 \leqslant s_3\end{array}}{\Phi \vdash s_1 \leqslant s_3}$$

Figure 6: Wellformedness (sizes) and subsizing rules

$\boxed{\vdash \Phi}$ $\boxed{\Phi \vdash \Gamma}$

$$\text{NIL}\ \frac{}{\vdash \cdot} \qquad \text{CONS-SIZE}\ \frac{\vdash \Phi}{\vdash \Phi(\alpha)} \qquad \text{CONS-SIZE}{<}\ \frac{\vdash \Phi \qquad \Phi \vdash s}{\vdash \Phi(\alpha < s)}$$

$$\text{NIL}\ \frac{\vdash \Phi}{\Phi \vdash \cdot} \qquad \text{CONS-ASS}\ \frac{\Phi \vdash \Gamma \qquad \Phi;\Gamma \vdash \tau \Uparrow U}{\Phi \vdash \Gamma(x : \tau)} \qquad \text{CONS-DEF}\ \frac{\Phi \vdash \Gamma \qquad \Phi;\Gamma \vdash e \Uparrow \tau}{\Phi \vdash \Gamma(x \coloneqq e)}$$

Figure 7: Environment wellformedness rules

$\boxed{\Phi;\Gamma \vdash e \Updownarrow \tau}$

$$\text{CONV}\ \frac{\Phi;\Gamma \vdash e \Uparrow \tau' \qquad \Phi;\Gamma \vdash \tau \Uparrow U \qquad \Phi;\Gamma \vdash \tau' \Downarrow U \qquad \Phi;\Gamma \vdash \tau' \preccurlyeq \tau}{\Phi;\Gamma \vdash e \Downarrow \tau} \qquad \text{VAR}\ \frac{\Phi \vdash \Gamma \qquad (x : \tau) \in \Gamma}{\Phi;\Gamma \vdash x \Uparrow \tau}$$

$$\text{UNIV}\ \frac{\Phi \vdash \Gamma}{\Phi;\Gamma \vdash U \Uparrow \mathsf{axiom}(U)} \qquad \text{PI}\ \frac{\Phi;\Gamma \vdash \sigma \Uparrow U_1 \qquad \Phi;\Gamma(x : \sigma) \vdash \tau \Uparrow U_2}{\Gamma \vdash \Pi x : \sigma.\, \tau \Uparrow \mathsf{rule}(U_1, U_2)} \qquad \text{LAM}\ \frac{\Phi;\Gamma \vdash \sigma \Uparrow U \qquad \Phi;\Gamma(x : \sigma) \vdash e \Uparrow \tau}{\Phi;\Gamma \vdash \lambda x : \sigma.\, e \Uparrow \Pi x : \sigma.\, \tau}$$

$$\text{APP}\ \frac{\Phi;\Gamma \vdash e_1 \Uparrow \Pi x : \sigma.\, \tau \qquad \Phi;\Gamma \vdash e_2 \Downarrow \sigma}{\Phi;\Gamma \vdash e_1\, e_2 \Uparrow \tau[x \mapsto e_1]} \qquad \text{LET}\ \frac{\Phi;\Gamma \vdash e_1 \Uparrow \sigma \qquad \Phi;\Gamma(x : \sigma)(x \coloneqq e_1) \vdash e_2 \Uparrow \tau}{\Phi;\Gamma \vdash \mathsf{let}\ x \coloneqq e_1\ \mathsf{in}\ e_2 \Uparrow \tau[x \mapsto e_1]} \qquad \text{FORALL}\ \frac{\Phi(\alpha);\Gamma \vdash \tau \Uparrow U}{\Phi;\Gamma \vdash \forall \alpha.\, \tau \Uparrow U}$$

$$\text{FORALL}{<}\ \frac{\Phi \vdash s \qquad \Phi(\alpha < s);\Gamma \vdash \tau \Uparrow U}{\Phi;\Gamma \vdash \forall \alpha < s.\, \tau \Uparrow U} \qquad \text{SLAM}\ \frac{\Phi(\alpha);\Gamma \vdash e \Uparrow \tau}{\Phi;\Gamma \vdash \Lambda \alpha.\, e \Uparrow \forall \alpha.\, \tau} \qquad \text{SLAM}{<}\ \frac{\Phi \vdash s \qquad \Phi(\alpha < s);\Gamma \vdash e \Uparrow \tau}{\Phi;\Gamma \vdash \Lambda \alpha < s.\, e \Uparrow \forall \alpha < s.\, \tau}$$

$$\text{SAPP}\ \frac{\Phi;\Gamma \vdash e \Uparrow \forall \alpha.\, \tau \qquad \Phi \vdash s}{\Phi;\Gamma \vdash e\,[s] \Uparrow \tau[\alpha \mapsto s]} \qquad \text{SAPP}{<}\ \frac{\Phi;\Gamma \vdash e \Uparrow \forall \alpha < r.\, \tau \qquad \Phi \vdash \hat{s} \leqslant r}{\Phi;\Gamma \vdash e\,[s] \Uparrow \tau[\alpha \mapsto s]} \qquad \text{SIGMA}\ \frac{\Phi;\Gamma \vdash \sigma \Uparrow U_1 \qquad \Phi;\Gamma(x : \sigma) \vdash \tau \Uparrow U_2}{\Phi;\Gamma \vdash \Sigma x : \sigma.\, \tau \Uparrow \max_{\sqsubseteq}(U_1, U_2)}$$

$$\text{PAIR}\ \frac{\begin{array}{c}\Phi;\Gamma \vdash e_1 \Downarrow \sigma \\ \Phi;\Gamma \vdash e_2 \Downarrow \tau[x \mapsto e_1]\end{array}}{\Phi;\Gamma \vdash \langle e_1, e_2 \rangle_{\Sigma x : \sigma.\tau} \Uparrow \Sigma x : \sigma.\, \tau} \qquad \text{FST}\ \frac{\Phi;\Gamma \vdash e \Uparrow \Sigma x : \sigma.\, \tau}{\Phi;\Gamma \vdash \pi_1\, e \Uparrow \sigma} \qquad \text{SND}\ \frac{\Phi;\Gamma \vdash e \Uparrow \Sigma x : \sigma.\, \tau}{\Phi;\Gamma \vdash \pi_2\, e \Uparrow \tau[x \mapsto \pi_1\, e]} \qquad \text{EQ}\ \frac{\begin{array}{c}\Phi;\Gamma \vdash \tau \Uparrow U \\ \Phi;\Gamma \vdash e_1 \Downarrow \tau \\ \Phi;\Gamma \vdash e_2 \Downarrow \tau\end{array}}{\Phi;\Gamma \vdash e_1 \overset{\tau}{=\!=} e_2 \Uparrow U}$$

$$\text{REFL}\ \frac{\Phi;\Gamma \vdash e \Uparrow \tau}{\Phi;\Gamma \vdash \mathsf{refl}_e \Uparrow e \overset{\tau}{=\!=} e} \qquad \text{J}\ \frac{\begin{array}{c}\Phi;\Gamma \vdash p \Uparrow e_1 \overset{\tau}{=\!=} e_2 \qquad y, z\ \mathsf{fresh} \\ \Phi;\Gamma(y : \tau)(z : e_1 \overset{\tau}{=\!=} y) \vdash P\, y\, z \Uparrow U_i \qquad \Phi;\Gamma \vdash d \Downarrow P\, e_1\, \mathsf{refl}_{e_1}\end{array}}{\Phi;\Gamma \vdash \mathsf{J}_P\, d\, p \Uparrow P\, e_2\, p}$$

Figure 8: Typing rules

$$c ::= \mathsf{zero} \mid \mathsf{succ} \mid \mathsf{sup} \qquad\qquad t ::= e \mid [s]$$

$$e ::= \cdots \mid \mathbb{N}\ [s] \mid \mathsf{zero}_{\mathbb{N}\ [s]}\ [s] \mid \mathsf{succ}_{\mathbb{N}\ [s]}\ [s]\ e \mid \mathsf{W} x : \tau.\, \tau\ [s] \mid \mathsf{sup}_{\mathsf{W} x:\tau.\tau\ [s]}\ [s]\ e\ e$$

$$\mid \mathsf{match}\ e\ \mathsf{return}\ \lambda x.\, P\ \mathsf{with}\ (c\ [\alpha]\ z_1 \ldots z_m \Rightarrow e) \ldots \mid \mathsf{fix}\ f\ [\alpha] : \tau \coloneqq e$$

$$\boxed{\Phi; \Gamma \vdash e \rhd e} \ \cdots$$

$$\frac{}{\Phi; \Gamma \vdash \mathsf{match}\ \mathsf{zero}_{\_}\ [s]\ \mathsf{return}\ \_\ \mathsf{with}\ (\mathsf{zero}\ [\alpha] \Rightarrow e_z)(\mathsf{succ}\ [\_]\ \_ \Rightarrow \_) \rhd_\iota e_z[\alpha \mapsto s]}$$

$$\frac{}{\Phi; \Gamma \vdash \mathsf{match}\ \mathsf{succ}_{\_}\ [s]\ e\ \mathsf{return}\ \_\ \mathsf{with}\ (\mathsf{zero}\ [\_] \Rightarrow \_)(\mathsf{succ}\ [\alpha]\ z \Rightarrow e_s) \rhd_\iota e_s[\alpha \mapsto s][z \mapsto e]}$$

$$\frac{}{\Phi; \Gamma \vdash \mathsf{match}\ \mathsf{sup}_{\_}\ [s]\ e_1\ e_2\ \mathsf{return}\ \_\ \mathsf{with}\ (\mathsf{sup}\ [\alpha]\ z_1\ z_2 \Rightarrow e) \rhd_\iota e[\alpha \mapsto s][z_1 \mapsto e_1][z_2 \mapsto e_2]}$$

$$\frac{\sigma = \Theta \to (x : \tau) \to \tau' \qquad \tau = \mathbb{N}\ [\alpha]\ \mathit{or}\ \mathsf{W}_{\_} : \_.\,\_\ [\alpha] \qquad |\Theta| = n \qquad \beta\ \mathsf{fresh}}{\Phi; \Gamma \vdash (\mathsf{fix}\ f\ [\alpha] : \sigma \coloneqq e)\ [s]\ t_1 \ldots t_n\ (c_\tau\ \_\ldots) \rhd_\mu e[\alpha \mapsto s][f \mapsto \Lambda\beta < s.\, (\mathsf{fix}\ f\ [\alpha] : \sigma \coloneqq e)\ [\beta]]\ t_1 \ldots t_n\ (c_\tau\ \_\ldots)}$$

$$\boxed{\Phi; \Gamma \vdash e \Updownarrow \tau} \ \cdots$$

$$\textsc{Nat}\ \frac{\Phi \vdash \Gamma \qquad \Phi \vdash s}{\Phi; \Gamma \vdash \mathbb{N}\ [s] \Uparrow \mathsf{Type}_1} \qquad \textsc{Zero}\ \frac{\Phi \vdash \Gamma \qquad \Phi \vdash \hat{r} \leqslant s}{\Phi; \Gamma \vdash \mathsf{zero}_{\mathbb{N}\ [s]}\ [r] \Uparrow \mathbb{N}\ [s]} \qquad \textsc{Succ}\ \frac{\Phi \vdash \hat{r} \leqslant s \qquad \Phi; \Gamma \vdash e \Downarrow \mathbb{N}\ [r]}{\Phi; \Gamma \vdash \mathsf{succ}_{\mathbb{N}\ [s]}\ [r]\ e \Uparrow \mathbb{N}\ [s]}$$

$$\textsc{W}\ \frac{\begin{array}{c}\Phi \vdash s \qquad \Phi; \Gamma \vdash \sigma \Uparrow U \\ \Phi; \Gamma(x : \sigma) \vdash \tau \Uparrow U\end{array}}{\Phi; \Gamma \vdash \mathsf{W} x : \sigma.\, \tau\ [s] \Uparrow \mathsf{axiom}(U)} \qquad \textsc{Sup}\ \frac{\begin{array}{c}\Phi \vdash \hat{r} \leqslant s \qquad \Phi; \Gamma \vdash \sigma \Uparrow U_1 \qquad \Phi; \Gamma(x : \sigma) \vdash \tau \Uparrow U_2 \\ \Phi; \Gamma \vdash e_1 \Downarrow \sigma \qquad \Phi; \Gamma \vdash e_2 \Downarrow \tau[x \mapsto e_1] \to \mathsf{W} x : \sigma.\, \tau\ [r]\end{array}}{\Phi; \Gamma \vdash \mathsf{sup}_{\mathsf{W} x:\sigma.\tau\ [s]}\ [r]\ e_1\ e_2 \Uparrow \mathsf{W} x : \sigma.\, \tau\ [s]}$$

$$\textsc{Match-Nat}\ \frac{\begin{array}{c}\Phi; \Gamma \vdash e \Uparrow \mathbb{N}\ [s] \qquad z \notin \mathsf{FV}(P) \qquad \Phi; \Gamma(x : \mathbb{N}\ [s]) \vdash P \Uparrow U \\ \Phi(\alpha < s); \Gamma \vdash e_z \Downarrow P[x \mapsto \mathsf{zero}_{\mathbb{N}\ [s]}\ [\alpha]] \qquad \Phi(\beta < s); \Gamma(z : \mathbb{N}\ [\beta]) \vdash e_s \Downarrow P[x \mapsto \mathsf{succ}_{\mathbb{N}\ [s]}\ [\beta]\ z]\end{array}}{\Phi; \Gamma \vdash \mathsf{match}\ e\ \mathsf{return}\ \lambda x.\, P\ \mathsf{with}\ (\mathsf{zero}\ [\alpha] \Rightarrow e_z)(\mathsf{succ}\ [\beta]\ z \Rightarrow e_s) \Uparrow P[x \mapsto e]}$$

$$\textsc{Match-Sup}\ \frac{\begin{array}{c}\Phi; \Gamma \vdash e \Uparrow \mathsf{W} y : \sigma.\, \tau\ [s] \qquad z_1, z_2 \notin \mathsf{FV}(P) \qquad \Phi; \Gamma(x : \mathsf{W} y : \sigma.\, \tau\ [s]) \vdash P \Uparrow U \\ \Phi(\alpha < s); \Gamma(z_1 : \sigma)(z_2 : \tau[y \mapsto z_1] \to \mathsf{W} y : \tau.\, \sigma\ [\alpha]) \vdash e_s \Downarrow P[x \mapsto \mathsf{sup}_{\mathsf{W} y:\sigma.\tau\ [s]}\ [\alpha]\ z_1\ z_2]\end{array}}{\Phi; \Gamma \vdash \mathsf{match}\ e\ \mathsf{return}\ \lambda x.\, P\ \mathsf{with}\ (\mathsf{sup}\ [\alpha]\ z_1\ z_2 \Rightarrow e_s) \Uparrow P[x \mapsto e]}$$

$$\textsc{Fix}\ \frac{\begin{array}{c}\Phi(\alpha); \Gamma \vdash \sigma \Uparrow U \qquad \Phi(\alpha); \Gamma \vdash \sigma \rhd^* \Theta \to (x : \tau) \to \tau' \qquad \beta\ \mathsf{fresh} \\ \tau = \mathbb{N}\ [\alpha]\ \mathit{or}\ \mathsf{W}_{\_} : \_.\,\_\ [\alpha] \qquad \Phi(\alpha); \Gamma(f : \forall \beta < \alpha.\, \sigma[\alpha \mapsto \beta]) \vdash e \Downarrow \sigma\end{array}}{\Phi; \Gamma \vdash \mathsf{fix}\ f\ [\alpha] : \sigma \coloneqq e \Uparrow \forall \alpha.\, \sigma}$$

Figure 9: Sized naturals and W types

$$X ::= \text{inductive type names} \qquad c ::= \text{inductive constructor names}$$

$$D ::= \mathsf{data}\ X\ \Delta_P : \Delta_I \to U_i\ \mathsf{where}\ \Delta_c$$

$$e ::= \cdots \mid \mathsf{match}\ e\ \mathsf{return}\ \lambda(y_1 \ldots y_n).x.\ P\ \mathsf{with}\ (c\, z_1 \ldots z_m \Rightarrow e) \ldots \mid \mathsf{fix}_n f : \tau \coloneqq e$$

$\boxed{\Gamma \vdash e \equiv e : \tau}$

$\equiv$-REFL
$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash e \equiv e : \tau}$$

$\equiv$-SYM
$$\frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$\equiv$-TRANS
$$\frac{\Gamma \vdash e_1 \equiv e_2 : \tau \qquad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau}$$

$\equiv$-CONV
$$\frac{\Gamma \vdash e_1 \equiv e_2 : \sigma \qquad \Gamma \vdash \sigma \preccurlyeq \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$\equiv$-CONG
$$\frac{\text{For every } 1 \leq i \leq n\text{:} \qquad \Gamma\Gamma' \vdash e_i \equiv e_i' : \tau'}{\Gamma \vdash e[x_1, \ldots, x_n \mapsto e_1, \ldots, e_n] \equiv e[x_1, \ldots, x_n \mapsto e_1', \ldots, e_n'] : \tau}$$

$\equiv$-REFLECT
$$\frac{\Gamma \vdash p : e_1 \overset{\tau}{=\!=} e_2}{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$\equiv$-δ
$$\frac{(x \coloneqq e) \in \Gamma \qquad (x : \tau) \in \Gamma}{\Gamma \vdash x \equiv e : \tau}$$

$\equiv$-β
$$\frac{\Gamma \vdash \sigma : U \qquad \Gamma(x : \sigma) \vdash e : \tau \qquad \Gamma \vdash e' : \sigma}{\Gamma \vdash (\lambda x : \sigma.\, e)\ e' \equiv e[x \mapsto e'] : \tau[x \mapsto e']}$$

$\equiv$-η
$$\frac{\Gamma(x : \sigma) \vdash e_1\ x \equiv e_2\ x : \tau}{\Gamma \vdash e_1 \equiv e_2 : \Pi x : \sigma.\, \tau}$$

$\equiv$-ζ
$$\frac{\Gamma \vdash e' : \sigma \qquad \Gamma(x : \sigma)(x \coloneqq e') \vdash e : \tau}{\Gamma \vdash \mathsf{let}\ x \coloneqq e'\ \mathsf{in}\ e \equiv e[x \mapsto e'] : \tau[x \mapsto e']}$$

$\equiv$-ρ
$$\frac{\begin{array}{c}\Gamma \vdash e : \tau \qquad y, z\ \text{fresh} \\ \Gamma(y : \tau)(z : e \overset{\tau}{=\!=} y) \vdash P\ y\ z : U \qquad \Gamma \vdash d : P\ e\ \mathsf{refl}_e\end{array}}{\Gamma \vdash \mathsf{J}_P\ d\ \mathsf{refl}_e \equiv d : P\ e\ \mathsf{refl}_e}$$

$\equiv$-π_1
$$\frac{\Gamma \vdash e_1 : \sigma \qquad \Gamma \vdash e_2 : \tau[x \mapsto e_1]}{\Gamma \vdash \pi_1\ \langle e_1, e_2 \rangle_{\Sigma x : \sigma.\tau} \equiv e_1 : \sigma}$$

$\equiv$-π_2
$$\frac{\Gamma \vdash e_1 : \sigma \qquad \Gamma \vdash e_2 : \tau[x \mapsto e_1]}{\Gamma \vdash \pi_2\ \langle e_1, e_2 \rangle_{\Sigma x : \sigma.\tau} \equiv e_2 : \tau[x \mapsto e_1]}$$

$\equiv$-π_η
$$\frac{\begin{array}{c}\Gamma \vdash \pi_1\ e_1 \equiv \pi_1\ e_2 : \sigma \\ \Gamma \vdash \pi_2\ e_1 \equiv \pi_2\ e_2 : \tau[x \mapsto \pi_1\ e_1]\end{array}}{\Gamma \vdash e_1 \equiv e_2 : \Sigma x : \sigma.\, \tau}$$

$\equiv$-ι
$$\frac{\Gamma \vdash \mathsf{match}\ c\, e_1 \ldots e_m\ \mathsf{return}\ \_\ \mathsf{with}\ \ldots (c\, z_1 \ldots z_m \Rightarrow e) \ldots : \tau}{\Gamma \vdash \mathsf{match}\ c\, e_1 \ldots e_m\ \mathsf{return}\ \_\ \mathsf{with}\ \ldots (c\, z_1 \ldots z_m \Rightarrow e) \ldots \equiv e[z_1, \ldots, z_m \mapsto e_1, \ldots, e_m] : \tau}$$

$\equiv$-μ
$$\frac{\begin{array}{c}\Gamma \vdash \tau : U \qquad \Gamma(f : \tau) \vdash e : \tau \qquad \Gamma \vdash \tau \rhd^* \Delta \to \sigma \to \_ \qquad \Gamma \vdash c\, e_1 \ldots e_m : \sigma \\ \Delta = (x_1 : \sigma_1) \ldots (x_n : \sigma_n) \qquad \text{For every } 1 \leq \mathrm{i} \leq \mathrm{n}\text{:} \qquad \Gamma \vdash e_i' : \sigma_i\end{array}}{\Gamma \vdash (\mathsf{fix}_n f : \tau \coloneqq e)\ e_1' \ldots e_n'\ (c\, e_1 \ldots e_m) \equiv e[f \mapsto \mathsf{fix}_n f : \tau \coloneqq e]\ e_1' \ldots e_n'\ (c\, e_1 \ldots e_m) : \tau}$$

$\boxed{\Gamma \vdash \tau \preccurlyeq \tau}$

$\preccurlyeq$-CONV
$$\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : U}{\Gamma \vdash \tau_1 \preccurlyeq \tau_2}$$

$\preccurlyeq$-TRANS
$$\frac{\Gamma \vdash \tau_1 \preccurlyeq \tau_2 \qquad \Gamma \vdash \tau_2 \preccurlyeq \tau_3}{\Gamma \vdash \tau_1 \preccurlyeq \tau_3}$$

$\preccurlyeq$-PROP
$$\frac{}{\Gamma \vdash \mathsf{Prop} \preccurlyeq \mathsf{Type}_i}$$

$\preccurlyeq$-TYPE
$$\frac{i \leq j}{\Gamma \vdash \mathsf{Type}_i \preccurlyeq \mathsf{Type}_j}$$

$\preccurlyeq$-PI
$$\frac{\Gamma \vdash \sigma_1 \equiv \sigma_2 : U \qquad \Gamma(x : \sigma_1) \vdash \tau_1 \preccurlyeq \tau_2}{\Gamma \vdash \Pi x : \sigma_1.\, \tau_1 \preccurlyeq \Pi x : \sigma_2.\, \tau_2}$$

$\preccurlyeq$-SIGMA
$$\frac{\Gamma \vdash \sigma_1 \preccurlyeq \sigma_2 \qquad \Gamma(x : \sigma_1) \vdash \tau_1 \preccurlyeq \tau_2}{\Gamma \vdash \Sigma x : \sigma_1.\, \tau_1 \preccurlyeq \Sigma x : \sigma_2.\, \tau_2}$$

$\boxed{\vdash \Gamma}$ NIL CONS-ASS CONS-DEF *as above*

Figure 10: CIC syntax and judgements

$\boxed{\Gamma \vdash e \Updownarrow \tau}$

CONV VAR UNIV PI LAM APP LET

SIGMA PAIR LEFT RIGHT EQ REFL J *as above*

IND

$$\frac{\vdash \Gamma \qquad \mathsf{data}\ X\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta_c}{\Gamma \vdash X \Uparrow \Delta_P \Delta_I \to U}$$

CONSTR

$$\frac{\vdash \Gamma \qquad \mathsf{data}\ X\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta_c \qquad (c : \tau) \in \Delta_c}{\Gamma \vdash c \Uparrow \Delta_P \to \tau}$$

MATCH

$$\frac{\begin{array}{c} \mathsf{data}\ X\ \Delta_P : \Delta_I \to U\ \mathsf{where}\ \Delta_c \\ \Gamma \vdash e \Uparrow X\ p_1 \ldots p_k\ e_1 \ldots e_n \qquad \Delta'_I = \Delta_I[\Delta_P \mapsto p_1, \ldots, p_k] \\ \Gamma\Delta'_I(x : X\ p_1 \ldots p_k\ \Delta'_I) \vdash P[y_1, \ldots, y_n \mapsto \Delta'_I] \Uparrow U' \\ \text{For each constructor } c\text{:} \qquad (c : \Delta_a \to X\ \_{}_1 \ldots \_{}_k\ e'_1 \ldots e'_n) \in \Delta_c \\ \Delta'_a = \Delta_a[\Delta_P \mapsto p_1, \ldots, p_k] \qquad e''_1, \ldots e''_n = e'_1[\Delta_P \mapsto p_1, \ldots, p_k] \ldots e'_n[\Delta_P \mapsto p_1, \ldots, p_k] \\ \Gamma\Delta'_a \vdash e_c[z_1, \ldots, z_m \mapsto \Delta'_a] \Downarrow P[y_1, \ldots, y_n \mapsto e''_1, \ldots, e''_n][x \mapsto c\ \Delta'_a] \end{array}}{\Gamma \vdash \mathsf{match}\ e\ \mathsf{return}\ \lambda(y_1 \ldots y_n).x.\ P\ \mathsf{with}\ (c\ z_1 \ldots z_m \Rightarrow e_c) \ldots \Uparrow P[y_1, \ldots, y_n \mapsto e_1, \ldots, e_n][x \mapsto e]}$$

FIX

$$\frac{\Gamma \vdash \sigma \Uparrow U \qquad \Gamma \vdash \sigma \rhd^* \Delta \to X\ \_ \ldots \to \_ \qquad |\Delta| = n \qquad \Gamma(f : \sigma) \vdash e \Downarrow \sigma}{\Gamma \vdash \mathsf{fix}_n f : \sigma \coloneqq e \Uparrow \sigma}$$

Figure 11: CIC typing judgement

$\mathsf{data}\ \bot : \mathsf{Prop}\ \mathsf{where}$

$\mathsf{let}\ \mathsf{elim}\bot : (P : \mathsf{Type}_i) \to \bot \to P \coloneqq$

$\quad \lambda P : \mathsf{Type}_i.\ \lambda b : \bot.\ \mathsf{match}\ b\ \mathsf{return}\ \lambda ().\_.\ P\ \mathsf{with}$

$\mathsf{data}\ \mathsf{Size} : \mathsf{Type}_{i+1}\ \mathsf{where}$

$\quad \mathsf{suc} : \mathsf{Size} \to \mathsf{Size}$

$\quad \mathsf{lim} : (A : \mathsf{Type}_i) \to (A \to \mathsf{Size}) \to \mathsf{Size}$

$\mathsf{let}\ \mathsf{base} : \mathsf{Size} \coloneqq \mathsf{lim}\ \bot\ (\mathsf{elim}\bot\ \mathsf{Size})$

$\mathsf{data}\ \_ \leq \_ : \mathsf{Size} \to \mathsf{Size} \to \mathsf{Type}_{i+1}\ \mathsf{where}$

$\quad \mathsf{mono} : (\alpha, \beta : \mathsf{Size}) \to \alpha \leq \beta \to \mathsf{suc}\ \alpha \leq \mathsf{suc}\ \beta$

$\quad \mathsf{cocone} : (A : \mathsf{Type}_i) \to (\beta : \mathsf{Size}) \to (f : A \to \mathsf{Size}) \to (a : A) \to \beta \leq f\ a \to \beta \leq \mathsf{lim}\ A\ f$

$\quad \mathsf{limit} : (A : \mathsf{Type}_i) \to (\beta : \mathsf{Size}) \to (f : A \to \mathsf{Size}) \to ((a : A) \to f\ a \leq \beta) \to \mathsf{lim}\ A\ f \leq \beta$

$\mathsf{let}\ \_ < \_ : \mathsf{Size} \to \mathsf{Size} \to \mathsf{Type}_{i+1} \coloneqq \lambda\alpha : \mathsf{Size}.\ \lambda\beta : \mathsf{Size}.\ \mathsf{suc}\ \alpha \leq \beta$

$\mathsf{data}\ \mathsf{Acc}\ (\alpha : \mathsf{Size}) : \mathsf{Prop}\ \mathsf{where}$

$\quad \mathsf{acc} : ((\beta : \mathsf{Size}) \to \beta < \alpha \to \mathsf{Acc}\ \beta) \to \mathsf{Acc}\ \alpha$

$\mathsf{let}\ \mathsf{accIsProp} : (\alpha : \mathsf{Size}) \to (\mathit{acc1}, \mathit{acc2} : \mathsf{Acc}\ \alpha) \to \mathit{acc1} \overset{\mathsf{Acc}\ \alpha}{=\!=} \mathit{acc2} \coloneqq \_$

$\mathsf{let}\ \mathsf{accessible} : (\alpha : \mathsf{Size}) \to \mathsf{Acc}\ \alpha \coloneqq \_$

$\mathsf{data}\ \mathsf{Pair}\ (A : U)(B : A \to U) : U\ \mathsf{where}$

$\quad \mathsf{pair} : (a : A) \to B\ a \to \mathsf{Pair}\ A\ B$

$\mathsf{let}\ \mathsf{fst} : (A : U) \to (B : A \to U) \to \mathsf{Pair}\ A\ B \to A \coloneqq \_$

$\mathsf{let}\ \mathsf{snd} : (A : U) \to (B : A \to U) \to (p : \mathsf{Pair}\ A\ B) \to B\ (\mathsf{fst}\ p) \coloneqq \_$

$\mathsf{data}\ \mathsf{Nat}\ (\alpha : \mathsf{Size}) : \mathsf{Type}_1\ \mathsf{where}$

$\quad \mathsf{zero} : (\beta : \mathsf{Size}) \to \beta < \alpha \to \mathsf{Nat}\ \alpha$

$\quad \mathsf{succ} : (\beta : \mathsf{Size}) \to \beta < \alpha \to \mathsf{Nat}\ \beta \to \mathsf{Nat}\ \alpha$

$\mathsf{data}\ \mathsf{W}\ (\alpha : \mathsf{Size})(A : \mathsf{Type}_i)(B : A \to \mathsf{Type}_i) : \mathsf{Type}_{i+1}\ \mathsf{where}$

$\quad \mathsf{sup} : (\beta : \mathsf{Size}) \to \beta < \alpha \to (a : A) \to (B\ a \to \mathsf{W}\ A\ B\ \beta) \to \mathsf{W}\ A\ B\ \alpha$

$\mathsf{let}\ \mathsf{mkW} : (A : \mathsf{Type}_i) \to (B : A \to \mathsf{Type}_i) \to (a : A) \to$

$\qquad (B\ a \to (\alpha : \mathsf{Size}) \times \mathsf{W}\ A\ B\ \alpha) \to (\alpha : \mathsf{Size}) \times \mathsf{W}\ A\ B\ \alpha \coloneqq \_$

Figure 12: Preliminary CIC definitions

$\mathsf{let}\ \mathsf{base}{\leq} : (\alpha : \mathsf{Size}) \to \mathsf{base} \leq \alpha \coloneqq \lambda\alpha : \mathsf{Size}.$

$\quad \mathsf{let}\ f \coloneqq \mathsf{elim}\bot\ \mathsf{Size}\ \mathsf{in}\ \mathsf{limit}\ \bot\ \alpha\ f\ (\lambda b : \bot.\ \mathsf{elim}\bot\ (f\ b \leq \alpha)\ b)$

$\mathsf{let}\ \mathsf{refl}{\leq} : (\alpha : \mathsf{Size}) \to \alpha \leq \alpha \coloneqq \_$

$\mathsf{let}\ \mathsf{trans}{\leq} : (\alpha, \beta, \gamma : \mathsf{Size}) \to \alpha \leq \beta \to \beta \leq \gamma \to \alpha \leq \gamma \coloneqq \_$

$\mathsf{let}\ \mathsf{suc}{\leq} : (\alpha : \mathsf{Size}) \to \alpha \leq \mathsf{suc}\ \alpha \coloneqq \_$

$\mathsf{let}\ \mathsf{elim} : (P : \mathsf{Size} \to \mathsf{Type}_i) \to ((\alpha : \mathsf{Size}) \to ((\beta : \mathsf{Size}) \to \beta < \alpha \to P\ \beta) \to P\ \alpha) \to (\alpha : \mathsf{Size}) \to P\ \alpha \coloneqq$

$\quad \lambda P : \mathsf{Size} \to \mathsf{Type}.\ \lambda f : (\alpha : \mathsf{Size}) \to ((\beta : \mathsf{Size}) \to \beta < \alpha \to P\ \beta) \to P\ \alpha.\ \lambda\alpha : \mathsf{Size}.$

$\quad\quad \mathsf{let}\ \mathit{elimAcc} \coloneqq \mathsf{fix}_1\ \mathit{elimAccRec} : (\alpha : \mathsf{Size}) \to \mathsf{Acc}\ \alpha \to P\ \alpha \coloneqq$

$\quad\quad\quad \lambda\alpha : \mathsf{Size}.\ \lambda \mathit{access} : \mathsf{Acc}\ \alpha.\ f\ \alpha\ (\lambda\beta : \mathsf{Size}.\ \lambda\beta^* : \beta < \alpha.\ \mathit{elimAccRec}\ \beta$

$\quad\quad\quad\quad (\mathsf{match}\ \mathit{access}\ \mathsf{return}\ \lambda().\_.\ \mathsf{Acc}\ \beta\ \mathsf{with}\ (\mathsf{acc}\ p \Rightarrow p\ \beta\ \beta^*)))\ \mathsf{in}$

$\quad\quad \mathit{elimAcc}\ \alpha\ (\mathsf{accessible}\ \alpha)$

Figure 13: Properties of the order on sizes

$\boxed{[\![s]\!] = e^*}$

$$[\![\alpha]\!] = \alpha$$
$$[\![\circ]\!] = \mathsf{base}$$
$$[\![\hat{s}]\!] = \mathsf{suc}\ [\![s]\!]$$

$\boxed{\Phi \vdash s \leqslant s \rightsquigarrow e^*}$ *over* $\Phi \vdash s \leqslant s$

$$\frac{(\alpha < s) \in \Phi}{\Phi \vdash \hat{\alpha} \leqslant s \rightsquigarrow \alpha^*} \qquad \frac{}{\Phi \vdash s \leqslant s \rightsquigarrow \mathsf{refl}{\leq}\ [\![s]\!]}$$

$$\frac{}{\Phi \vdash \circ \leqslant s \rightsquigarrow \mathsf{base}{\leq}\ [\![s]\!]} \qquad \frac{}{\Phi \vdash s \leqslant \hat{s} \rightsquigarrow \mathsf{suc}{\leq}\ [\![s]\!]}$$

$$\frac{\Phi \vdash s_1 \leqslant r \rightsquigarrow e_1^* \qquad \Phi \vdash r \leqslant s_2 \rightsquigarrow e_2^*}{\Phi \vdash s_1 \leqslant s_2 \rightsquigarrow \mathsf{trans}{\leq}\ [\![s_1]\!]\ [\![r]\!]\ [\![s_2]\!]\ e_1^*\ e_2^*}$$

Figure 14: Model in CIC (sizes)

$\boxed{[\![\Phi]\!] = \Gamma}$ *over* $\vdash \Phi$

$$[\![\cdot]\!] = \cdot$$
$$[\![\Phi(\alpha)]\!] = [\![\Phi]\!](\alpha : \mathsf{Size})$$
$$[\![\Phi(\alpha < s)]\!] = [\![\Phi]\!](\alpha : \mathsf{Size})(\alpha^* : \alpha < [\![s]\!])$$

$\boxed{[\![\Gamma]\!] = \Gamma^*}$ *over* $\Phi \vdash \Gamma$

$$[\![\cdot]\!] = \cdot$$
$$[\![\Gamma(x : \tau)]\!] = [\![\Gamma]\!](x : [\![\tau]\!])$$
$$[\![\Gamma(x \coloneqq e)]\!] = [\![\Gamma]\!](x \coloneqq [\![e]\!])$$

Figure 15: Model in CIC (environments)

$$\boxed{[\![e]\!]_{\Phi';\Gamma'} = e^*} \quad \textit{where } \Phi\Phi';\Gamma\Gamma' \vdash e \Uparrow \_ \rightsquigarrow e^*$$

$$
\begin{aligned}
[\![x]\!] &= x \\
[\![U]\!] &= U \\
[\![\Pi x : \sigma.\, \tau]\!] &= \Pi x : [\![\sigma]\!].\, [\![\tau]\!]_{\cdot;(x:\sigma)} \\
[\![\lambda x : \sigma.\, e]\!] &= \lambda x : [\![\sigma]\!].\, [\![e]\!]_{\cdot;(x:\sigma)} \\
[\![e_1\ e_2]\!] &= [\![e_1]\!]\ [\![e_2]\!] \\
[\![\forall \alpha.\, \tau]\!] &= \Pi \alpha : \mathsf{Size}.\, [\![\tau]\!]_{(\alpha);\cdot} \\
[\![\forall \alpha < s.\, \tau]\!] &= \Pi \alpha : \mathsf{Size}.\, \Pi \alpha^* : \alpha < [\![s]\!].\, [\![\tau]\!]_{(\alpha<s);\cdot} \\
[\![\Lambda \alpha.\, e]\!] &= \lambda \alpha : \mathsf{Size}.\, [\![e]\!]_{(\alpha);\cdot} \\
[\![\Lambda \alpha < s.\, e]\!] &= \lambda \alpha : \mathsf{Size}.\, \lambda \alpha^* : \alpha < [\![s]\!].\, [\![e]\!]_{(\alpha<s);\cdot} \\
[\![e\ [s]]\!] &= [\![e]\!]\ [\![s]\!] \\
[\![\mathsf{let}\ x \coloneqq e_1\ \mathsf{in}\ e_2]\!] &= \mathsf{let}\ x \coloneqq [\![e_1]\!]\ \mathsf{in}\ [\![e_2]\!]_{(x:\sigma)(x:=e_1)} \\
[\![\Sigma x : \sigma.\, \tau]\!] &= \Sigma x : [\![\sigma]\!].\, [\![\tau]\!]_{\cdot;(x:\sigma)} \\
[\![\langle e_1, e_2 \rangle_{\Sigma x:\sigma.\tau}]\!] &= \langle [\![e_1]\!], [\![e_2]\!] \rangle_{[\![\Sigma x:\sigma.\tau]\!]} \\
[\![\pi_1\ e]\!] &= \pi_1\ [\![e]\!] \\
[\![\pi_2\ e]\!] &= \pi_2\ [\![e]\!] \\
[\![e_1 \overset{\tau}{=\!=} e_2]\!] &= [\![e_1]\!] \overset{[\![\tau]\!]}{=\!=} [\![e_2]\!] \\
[\![\mathsf{refl}_e]\!] &= \mathsf{refl}_{[\![e]\!]} \\
[\![\mathsf{J}_P\ d\ p]\!] &= \mathsf{J}_{[\![P]\!]}\ [\![d]\!]\ [\![p]\!]
\end{aligned}
$$

$$\boxed{\Phi;\Gamma \vdash e \Updownarrow \tau \rightsquigarrow e^*} \quad \textit{over } \Phi;\Gamma \vdash e \Updownarrow \tau$$

CONV*

$$\frac{\Phi;\Gamma \vdash e \Uparrow \tau' \rightsquigarrow e^* \qquad \Phi;\Gamma \vdash \tau' \preccurlyeq \tau}{\Phi;\Gamma \vdash e \Downarrow \tau \rightsquigarrow e^*}$$

SAPP*<

$$\frac{\Phi;\Gamma \vdash e \Uparrow \forall \alpha < r.\, \tau \rightsquigarrow e^* \qquad \Phi \vdash \hat{s} \leqslant r \rightsquigarrow e^*_{\leq}}{\Phi;\Gamma \vdash e\ [s] \Uparrow \tau[\alpha \mapsto s] \rightsquigarrow e^*\ [\![s]\!]\ e^*_{\leq}}$$

Figure 16: Model in CIC (sized types)

$$\boxed{[\![e]\!] = e^*} \textit{ where } \Phi; \Gamma \vdash e \Uparrow \_ \rightsquigarrow e^*$$

$$[\![\mathbb{N}\ [s]]\!] = \mathsf{Nat}\ [\![s]\!]$$

$$[\![\mathbb{W}x : \sigma. \tau\ [s]]\!] = \mathsf{W}\ [\![\sigma]\!]\ [\![\lambda x : \sigma. \tau]\!]\ [\![s]\!]$$

$$\boxed{\Phi; \Gamma \vdash e \Updownarrow \tau \rightsquigarrow e^*} \textit{ over } \Phi; \Gamma \vdash e \Updownarrow \tau$$

$$\textsc{zero}^* \quad \frac{\Phi \vdash \Gamma \qquad \Phi \vdash \hat{r} \leqslant s \rightsquigarrow e^*_\leq}{\Phi; \Gamma \vdash \mathsf{zero}_{\mathbb{N}\ [s]}\ [r] \Uparrow \mathbb{N}\ [s] \rightsquigarrow \mathsf{zero}\ [\![s]\!]\ [\![r]\!]\ e^*_\leq}$$

$$\textsc{succ}^* \quad \frac{\Phi \vdash \hat{r} \leqslant s \rightsquigarrow e^*_\leq \qquad \Phi; \Gamma \vdash e \Downarrow \mathbb{N}\ [r] \rightsquigarrow e^*}{\Gamma \vdash \mathsf{succ}_{\mathbb{N}\ [s]}\ [r]\ e \Uparrow \mathbb{N}\ [s] \rightsquigarrow \mathsf{succ}\ [\![s]\!]\ [\![r]\!]\ e^*_\leq\ e^*}$$

$$\textsc{sup}^* \quad \frac{\begin{array}{c}\Phi \vdash \hat{r} \leqslant s \rightsquigarrow e^*_\leq \qquad \Phi; \Gamma \vdash \sigma \Uparrow U_1 \rightsquigarrow \sigma^* \qquad \Phi; \Gamma(x : \sigma) \vdash \tau \Uparrow U_2 \rightsquigarrow \tau^* \\ \Phi; \Gamma \vdash e_1 \Downarrow \sigma \rightsquigarrow e^*_1 \qquad \Phi; \Gamma \vdash e_2 \Downarrow \tau[x \mapsto e_1] \to \mathbb{W}x : \sigma. \tau\ [r] \rightsquigarrow e^*_2\end{array}}{\Phi; \Gamma \vdash \mathsf{sup}_{\mathbb{W}x:\sigma.\tau\ [s]}\ [r]\ e_1\ e_2 \Uparrow \mathbb{W}x : \sigma. \tau\ [s] \rightsquigarrow \mathsf{sup}\ [\![s]\!]\ \sigma^*\ (\lambda x : \sigma^*. \tau^*)\ [\![r]\!]\ e^*_\leq\ e^*_1\ e^*_2}$$

$$\textsc{match-nat}^* \quad \frac{\begin{array}{c}\Phi; \Gamma \vdash e \Uparrow \mathbb{N}\ [s] \rightsquigarrow e^* \qquad \Phi; \Gamma(x : \mathbb{N}\ [s]) \vdash P \Uparrow U \rightsquigarrow P^* \\ \Phi(\alpha < s); \Gamma \vdash e_z \Downarrow P[x \mapsto \mathsf{zero}_{\mathbb{N}\ [s]}\ [\alpha]] \rightsquigarrow e^*_z \qquad \Phi(\beta < s); \Gamma(z : \mathbb{N}\ [\beta]) \vdash e_s \Downarrow P[x \mapsto \mathsf{succ}_{\mathbb{N}\ [s]}\ [\beta]\ z] \rightsquigarrow e^*_s\end{array}}{\Phi; \Gamma \vdash \begin{array}{l}\mathsf{match}\ e\ \mathsf{return}\ \lambda x. P\ \mathsf{with} \\ (\mathsf{zero}\ [\alpha] \Rightarrow e_z) \\ (\mathsf{succ}\ [\beta]\ z \Rightarrow e_s)\end{array} \Uparrow P[x \mapsto e] \rightsquigarrow \begin{array}{l}\mathsf{match}\ e^*\ \mathsf{return}\ \lambda().x. P^*\ \mathsf{with} \\ (\mathsf{zero}\ \alpha\ \alpha^* \Rightarrow e^*_z) \\ (\mathsf{succ}\ \beta\ \beta^*\ z \Rightarrow e^*_s)\end{array}}$$

$$\textsc{match-sup}^* \quad \frac{\begin{array}{c}\Phi; \Gamma \vdash e \Uparrow \mathbb{W}y : \sigma. \tau\ [s] \rightsquigarrow e^* \qquad \Phi; \Gamma(x : \mathbb{W}y : \sigma. \tau\ [s]) \vdash P \Uparrow U \rightsquigarrow P^* \\ \Phi(\alpha < s); \Gamma(z_1 : \sigma)(z_2 : \tau[y \mapsto z_1] \to \mathbb{W}y : \tau. \sigma\ [\alpha]) \vdash e_s \Downarrow P[x \mapsto \mathsf{sup}_{\mathbb{W}y:\sigma.\tau\ [s]}\ [\alpha]\ z_1\ z_2] \rightsquigarrow e^*_s\end{array}}{\Phi; \Gamma \vdash \begin{array}{l}\mathsf{match}\ e\ \mathsf{return}\ \lambda x. P\ \mathsf{with} \\ (\mathsf{sup}\ [\alpha]\ z_1\ z_2 \Rightarrow e_s)\end{array} \Uparrow P[x \mapsto e] \rightsquigarrow \begin{array}{l}\mathsf{match}\ e^*\ \mathsf{return}\ \lambda().x. P^*\ \mathsf{with} \\ (\mathsf{sup}\ \alpha\ \alpha^*\ z_1\ z_2 \Rightarrow e^*_s)\end{array}}$$

$$\textsc{fix}^* \quad \frac{\beta\ \mathsf{fresh} \qquad \Phi(\alpha); \Gamma \vdash \sigma \Uparrow U \rightsquigarrow \sigma^* \qquad \Phi(\alpha); \Gamma(f : \forall \beta < \alpha. \sigma[\alpha \mapsto \beta]) \vdash e \Downarrow \sigma \rightsquigarrow e^*}{\Phi; \Gamma \vdash \mathsf{fix}\ f\ [\alpha] : \sigma \coloneqq e \Uparrow \forall \alpha. \sigma \rightsquigarrow \begin{array}{l}\mathsf{elim}\ (\lambda \alpha : \mathsf{Size}. \sigma^*) \\ \quad (\lambda \alpha : \mathsf{Size}. \lambda f : (\beta : \mathsf{Size}) \to \beta < \alpha \to \sigma^*[\alpha \mapsto \beta]. e^*)\end{array}}$$

Figure 17: Model in CIC (naturals and W types)