Jonathan Chan

May 7, 2022

$$
\begin{array}{rl}
i, j, k, m, n ::= \text{naturals} & r, s ::= \alpha \mid \widehat{s} \mid \circ \mid \infty \qquad \Gamma ::= \cdot \mid \Gamma(x : \tau) \mid \Gamma(x \coloneqq e) \mid \Gamma(\alpha) \\
f, g, w, x, y, z, \alpha, \beta ::= \text{variables} & U_i ::= \mathsf{Prop}_i \mid \mathsf{Type}_i \qquad \Delta ::= \cdot \mid \Delta(x : \tau) \qquad \Phi ::= \cdot \mid \Phi(\alpha) \\
e, d, p, P, \tau, \sigma ::= & x \mid U_i \mid \Pi x : \tau . \tau \mid \lambda x : \tau . e \mid e\, e \mid \forall \alpha . \tau \mid \Lambda \alpha . e \mid e\,[s] \mid \mathsf{let}\; x \coloneqq e \;\mathsf{in}\; e \\
& \mid \Sigma x : \sigma . \tau \mid \langle e, e \rangle_{\Sigma x : \sigma . \tau} \mid \pi_{\mathsf{L}}\, e \mid \pi_{\mathsf{R}}\, e \mid \exists \alpha . \tau \mid (s, e)_{\exists \alpha . \tau} \mid \mathsf{let}\; (\alpha, x) \coloneqq e \;\mathsf{in}\; e \\
& \mid e \overset{\tau}{==} e \mid \mathsf{refl}_e \mid \mathsf{J}_P\, d\, p
\end{array}
$$

Figure 1: Syntax

$$
\begin{array}{lll}
s + 0 \equiv s & \tau_1 \to \tau_2 \equiv \Pi \_ : \tau_1 . \tau_2 & \tau_1 \times \tau_2 \equiv \Sigma \_ : \tau_1 . \tau_2 \\
s + n \equiv \widehat{s} + (n - 1) & \Delta \to \tau \equiv \Pi x_n : \tau_1 . \dots \Pi x_n : \tau_n . \tau & \Delta \times \tau \equiv \Sigma x_1 : \tau_1 . \dots \Sigma x_n : \tau_n . \tau \\
\top \equiv \Pi \tau : \mathsf{Prop}_0 . \Pi \_ : \tau . \tau & \forall \Phi . \tau \equiv \forall \alpha_1 . \dots \forall \alpha_n . \tau & \exists \Phi . \tau \equiv \exists \alpha_1 . \dots \exists \alpha_n . \tau \\
\langle\rangle \equiv \lambda \tau : \mathsf{Prop}_0 . \lambda x : \tau . x & \lambda \Delta . e \equiv \lambda x_1 : \tau_1 . \dots \lambda x_n : \tau_n . e & \langle e_1, \dots, e_n \rangle \equiv \langle e_1, \dots \langle e_n, \langle\rangle \rangle \rangle \\
\bot \equiv \Pi \tau : \mathsf{Prop}_0 . \tau & \Lambda \Phi . e \equiv \Lambda \alpha_1 . \dots \Lambda \alpha_n . e & (s_1, \dots, s_n, e) \equiv (s_1, \dots (s_n, e))
\end{array}
$$

$$
\textit{where } \Delta \equiv (x_1 : \tau_1) \dots (x_n : \tau_n), \Phi \equiv (\alpha_1) \dots (\alpha_n)
$$

$$
\begin{aligned}
e\, \Delta &\equiv e\, x_1 \dots\, x_n \\
e[z_1 \dots z_n \mapsto \Delta] &\equiv e[z_1 \mapsto x_1] \dots [z_n \mapsto x_n] \\
e[x_1 \dots x_n \mapsto e_1 \dots e_n] &\equiv e[x_1 \mapsto e_1] \dots [x_n \mapsto e_n] \\
\textit{where } \Delta &\equiv (x_1 : \_) \dots (x_n : \_)
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{let}\; \langle\rangle \coloneqq \_ \;\mathsf{in}\; e_2 &\equiv e_2 \\
\mathsf{let}\; \langle x_1, x_2, \dots, x_n \rangle \coloneqq e_1 \;\mathsf{in}\; e_2 &\equiv \mathsf{let}\; x_1 \coloneqq \pi_{\mathsf{L}}\, e_1 \;\mathsf{in} \\
&\qquad \mathsf{let}\; \langle x_2, \dots, x_n \rangle \coloneqq \pi_{\mathsf{R}}\, e_1 \;\mathsf{in}\; e_2 \\
\mathsf{let}\; (\alpha_1, \alpha_2, \dots, \alpha_n, x) \coloneqq e_1 \;\mathsf{in}\; e_2 &\equiv \mathsf{let}\; (\alpha_1, x_1) \coloneqq e_1 \;\mathsf{in} \\
&\qquad \mathsf{let}\; (\alpha_2, \dots, \alpha_n, x) \coloneqq x_1 \;\mathsf{in}\; e_2 \\
\textit{where } x_1, \dots, x_n \;\text{fresh}
\end{aligned}
$$

Figure 2: Syntactic sugar

$$
\begin{aligned}
\mathsf{Prop}_i \sqcup \mathsf{Prop}_j &\equiv \mathsf{Prop}_{\max(i,j)} \\
\mathsf{Type}_i \sqcup U_j &\equiv \mathsf{Type}_{\max(i,j)} \\
U_i \sqcup \mathsf{Type}_j &\equiv \mathsf{Type}_{\max(i,j)}
\end{aligned}
$$

$$
\begin{aligned}
\|\Pi x : \sigma.\, \tau\| &\equiv \Pi x : \|\sigma\|.\, \|\tau\| \\
\|\lambda x : \sigma.\, e\| &\equiv \lambda x : \|\sigma\|.\, \|e\| \\
\|e_1\ e_2\| &\equiv \|e_1\|\ \|e_2\| \\
\|\forall \alpha.\, \tau\| &\equiv \|\tau\| \\
\|\Lambda \alpha.\, e\| &\equiv \|e\| \\
\|e\ [s]\| &\equiv \|e\| \\
\|\mathsf{let}\ x \coloneqq e_1\ \mathsf{in}\ e_2\| &\equiv \mathsf{let}\ x \coloneqq \|e_1\|\ \mathsf{in}\ \|e_2\| \\
\|\Sigma x : \sigma.\, \tau\| &\equiv \Sigma x : \|\sigma\|.\, \|\tau\| \\
\|\langle e_1, e_2\rangle_{\Sigma x : \sigma.\tau}\| &\equiv \langle \|e_1\|, \|e_2\|\rangle_{\Sigma x : \|\sigma\|.\|\tau\|} \\
\|\pi_{\mathsf{L}}\ e\| &\equiv \pi_{\mathsf{L}}\ \|e\| \\
\|\pi_{\mathsf{R}}\ e\| &\equiv \pi_{\mathsf{R}}\ \|e\| \\
\|\exists \alpha.\, \tau\| &\equiv \|\tau\| \\
\|(s, e)_{\exists \alpha.\tau}\| &\equiv \|e\| \\
\|\mathsf{let}\ (\alpha, x) \coloneqq e_1\ \mathsf{in}\ e_2\| &\equiv \mathsf{let}\ x \coloneqq \|e_1\|\ \mathsf{in}\ \|e_2\| \\
\|e_1 \overset{\tau}{=\!=} e_2\| &\equiv \|e_1\| \overset{\|\tau\|}{=\!=} \|e_2\| \\
\|\mathsf{refl}_e\| &\equiv \mathsf{refl}_{\|e\|} \\
\|\mathsf{J}_P\ d\ p\| &\equiv \mathsf{J}_{\|P\|}\ \|d\|\ \|p\| \\
\|e\| &\equiv e
\end{aligned}
$$

Figure 3: Metafunctions

$\boxed{\Gamma \vdash e \rhd e}$

$$
\frac{(x \coloneqq e) \in \Gamma}{\Gamma \vdash x \rhd_\delta e}
\qquad
\frac{}{\Gamma \vdash (\lambda x : \tau.\, e)\ e' \rhd_\beta e[x \mapsto e']}
\qquad
\frac{}{\Gamma \vdash (\Lambda \alpha.\, e)\ [s] \rhd_\beta e[\alpha \mapsto s]}
$$

$$
\frac{}{\Gamma \vdash \mathsf{let}\ x \coloneqq e'\ \mathsf{in}\ e \rhd_\zeta e[x \mapsto e']}
\qquad
\frac{}{\Gamma \vdash \pi_{\mathsf{L}}\ \langle e_1, e_2\rangle \rhd_\pi e_1}
\qquad
\frac{}{\Gamma \vdash \pi_{\mathsf{R}}\ \langle e_1, e_2\rangle \rhd_\pi e_2}
$$

$$
\frac{}{\Gamma \vdash \mathsf{let}\ (\alpha, x) \coloneqq (s, e')\ \mathsf{in}\ e \rhd_\varepsilon e[\alpha \mapsto s][x \mapsto e']}
\qquad
\frac{}{\Gamma \vdash \mathsf{J}\ d\ \mathsf{refl} \rhd_\iota d}
$$

Figure 4: Reduction rules

$\boxed{\Gamma \vdash s \approx s}$

$$\frac{(\alpha) \in \Gamma}{\Gamma \vdash \alpha \approx \alpha} \qquad \frac{\Gamma \vdash s_1 \approx s_2}{\Gamma \vdash \hat{s}_1 \approx \hat{s}_2} \qquad \frac{}{\Gamma \vdash \circ \approx \circ} \qquad \frac{}{\Gamma \vdash \infty \approx \infty} \qquad \frac{\Gamma \vdash s \approx \infty}{\Gamma \vdash \hat{s} \approx \infty} \qquad \frac{\Gamma \vdash \infty \approx s}{\Gamma \vdash \infty \approx \hat{s}}$$

Figure 5: Convertibility rules (sizes)

$\boxed{\Gamma \vdash e \rhd^* e}$ $\qquad$ $\boxed{\Gamma \vdash e \approx e}$

Figure 6: Implicit judgements

$\boxed{\vdash \Gamma}$

$$\textsc{Γ-nil}\ \frac{\vdash}{\vdash \cdot} \qquad \textsc{Γ-cons-size}\ \frac{\vdash \Gamma}{\vdash \Gamma(\alpha)} \qquad \textsc{Γ-cons-ass}\ \frac{\vdash \Gamma \qquad \Gamma \vdash \tau \Uparrow U_i}{\vdash \Gamma(x : \tau)} \qquad \textsc{Γ-cons-def}\ \frac{\vdash \Gamma \qquad \Gamma \vdash e \Uparrow \tau}{\vdash \Gamma(x \coloneqq e)}$$

$\boxed{\vdash s}$

$$\textsc{svar}\ \frac{(\alpha) \in \Gamma}{\Gamma \vdash \alpha} \qquad \textsc{ssucc}\ \frac{\Gamma \vdash s}{\Gamma \vdash \widehat{s}} \qquad \textsc{so}\ \frac{}{\Gamma \vdash \circ} \qquad \textsc{sinf}\ \frac{}{\Gamma \vdash \infty}$$

Figure 7: Well-formedness rules

$$\boxed{\Gamma \vdash e \Updownarrow \tau}$$

$$\textsc{Conv}\ \frac{\Gamma \vdash \tau \Uparrow U_i \qquad \Gamma \vdash e \Uparrow \tau' \qquad \Gamma \vdash \tau \approx \tau'}{\Gamma \vdash e \Downarrow \tau}$$

$$\textsc{Var}\ \frac{\vdash \Gamma \qquad (x : \tau) \in \Gamma}{\Gamma \vdash x \Uparrow \tau}$$

$$\textsc{Type}\ \frac{\vdash \Gamma}{\Gamma \vdash U_i \Uparrow \mathsf{Type}_{i+1}}$$

$$\textsc{Pi}\ \frac{\Gamma \vdash \sigma \Uparrow U_i \qquad \Gamma(x : \sigma) \vdash \tau \Uparrow U'_j}{\Gamma \vdash \Pi x : \sigma.\, \tau \Uparrow U'_{\max(i,j)}}$$

$$\textsc{Lam}\ \frac{\Gamma \vdash \sigma \Uparrow U_i \qquad \Gamma(x : \sigma) \vdash e \Uparrow \tau}{\Gamma \vdash \lambda x : \sigma.\, e \Uparrow \Pi x : \sigma.\, \tau}$$

$$\textsc{App}\ \frac{\Gamma \vdash e_1 \Uparrow \Pi x : \sigma.\, \tau \qquad \Gamma \vdash e_2 \Downarrow \tau}{\Gamma \vdash e_1\ e_2 \Uparrow \tau[x \mapsto e_1]}$$

$$\textsc{Forall}\ \frac{\Gamma(\alpha) \vdash \tau \Uparrow U_i}{\Gamma \vdash \forall \alpha.\, \tau \Uparrow U_i}$$

$$\textsc{SLam}\ \frac{\Gamma(\alpha) \vdash e \Uparrow \tau}{\Gamma \vdash \Lambda \alpha.\, e \Uparrow \forall \alpha.\, \tau}$$

$$\textsc{SApp}\ \frac{\Gamma \vdash e \Uparrow \forall \alpha.\, \tau \qquad \Gamma \vdash s}{\Gamma \vdash e\,[s] \Uparrow \tau[\alpha \mapsto s]}$$

$$\textsc{Let}\ \frac{\Gamma \vdash e_1 \Uparrow \sigma \qquad \Gamma(x \coloneqq e_1)(x : \sigma) \vdash e_2 \Uparrow \tau}{\Gamma \vdash \mathsf{let}\ x \coloneqq e_1\ \mathsf{in}\ e_2 \Uparrow \tau[x \mapsto e_1]}$$

$$\textsc{Sigma}\ \frac{\Gamma \vdash \sigma \Uparrow U_i \qquad \Gamma(x : \sigma) \vdash \tau \Uparrow U'_j}{\Gamma \vdash \Sigma x : \sigma.\, \tau \Uparrow U_i \sqcup U'_j}$$

$$\textsc{Pair}\ \frac{\Gamma \vdash e_1 \Downarrow \sigma \qquad \Gamma \vdash e_2 \Downarrow \tau[x \mapsto e_1]}{\Gamma \vdash \langle e_1, e_2 \rangle_{\Sigma x : \sigma.\tau} \Uparrow \Sigma x : \sigma.\, \tau}$$

$$\textsc{Left}\ \frac{\Gamma \vdash e \Uparrow \Sigma x : \sigma.\, \tau}{\Gamma \vdash \pi_{\mathsf{L}}\ e \Uparrow \sigma}$$

$$\textsc{Right}\ \frac{\Gamma \vdash e \Uparrow \Sigma x : \sigma.\, \tau}{\Gamma \vdash \pi_{\mathsf{R}}\ e \Uparrow \tau[x \mapsto \pi_{\mathsf{L}}\ e]}$$

$$\textsc{Exists}\ \frac{\Gamma(\alpha) \vdash \tau \Uparrow U_i}{\Gamma \vdash \exists \alpha.\, \tau \Uparrow U_i}$$

$$\textsc{SPair}\ \frac{\Gamma \vdash s \qquad \Gamma \vdash e \Downarrow \tau[\alpha \mapsto s]}{\Gamma \vdash (s, e)_{\exists \alpha.\tau} \Uparrow \exists \alpha.\, \tau}$$

$$\textsc{Unpair}\ \frac{\begin{array}{c}\Gamma \vdash e_1 \Uparrow \exists \alpha.\, \sigma \\ \Gamma(\alpha)(x : \sigma) \vdash e_2 \Uparrow \tau \qquad \Gamma \vdash \tau \Uparrow U_i\end{array}}{\Gamma \vdash \mathsf{let}\ (\alpha, x) \coloneqq e_1\ \mathsf{in}\ e_2 \Uparrow \tau}$$

$$\textsc{Eq}\ \frac{\begin{array}{c}\Gamma \vdash \tau \Uparrow U_i \\ \Gamma \vdash e_1 \Downarrow \tau \qquad \Gamma \vdash e_2 \Downarrow \tau\end{array}}{\Gamma \vdash e_1 \overset{\tau}{=} e_2 \Uparrow U_i}$$

$$\textsc{Refl}\ \frac{\Gamma \vdash e \Uparrow \tau}{\Gamma \vdash \mathsf{refl}_e \Uparrow e \overset{\tau}{=} e}$$

$$\textsc{J}\ \frac{\begin{array}{c}\Gamma \vdash p \Uparrow e_1 \overset{\tau}{=} e_2 \qquad y, z\ \mathsf{fresh} \\ \Gamma(y : \tau)(z : e_1 \overset{\tau}{=} y) \vdash P\ y\ z \Uparrow U_i \qquad \Gamma \vdash d \Downarrow P\ e_1\ \mathsf{refl}_{e_1}\end{array}}{\Gamma \vdash \mathsf{J}_P\ d\ p \Uparrow P\ e_2\ p}$$

Figure 8: Typing rules

$$c ::= \mathsf{zero} \mid \mathsf{succ} \mid \mathsf{nil} \mid \mathsf{cons}$$

$$e ::= \cdots \mid \mathbb{N}\,[s] \mid \mathsf{zero}\,[s] \mid \mathsf{succ}\,[s]\,e \mid \mathbb{V}\,[s]\,\tau\,e \mid \mathsf{nil}\,[s]\,\tau \mid \mathsf{cons}\,[s]\,e\,e\,e$$
$$\mid \mathsf{match}\ e\ \mathsf{return}\ (y_1 \ldots y_n).x.P\ \mathsf{with}\ (c\ z_1 \ldots z_m \Rightarrow e) \ldots \mid \mathsf{fix}\ f : \tau \coloneqq e$$

$\boxed{\Gamma \vdash e \rhd e}$ $\cdots$

$$\frac{}{\Gamma \vdash \mathsf{match}\ \mathsf{zero}\,[\_]\ \mathsf{return}\ ().\_.\_\ \mathsf{with}\ (\mathsf{zero} \Rightarrow e_z)(\mathsf{succ}\ \_ \Rightarrow \_) \rhd_\iota e_z}$$

$$\frac{}{\Gamma \vdash \mathsf{match}\ \mathsf{succ}\,[\_]\,e\ \mathsf{return}\ ().\_.\_\ \mathsf{with}\ (\mathsf{zero} \Rightarrow \_)(\mathsf{succ}\ z \Rightarrow e_s) \rhd_\iota e_s[z \mapsto e]}$$

$$\frac{}{\Gamma \vdash \mathsf{match}\ \mathsf{nil}\,[\_]\,\_\ \mathsf{return}\ (\_).\_.\_\ \mathsf{with}\ (\mathsf{nil} \Rightarrow e_n)(\mathsf{cons}\ \_\ \_\ \_ \Rightarrow \_) \rhd_\iota e_n}$$

$$\frac{}{\Gamma \vdash \mathsf{match}\ \mathsf{cons}\,[\_]\,e_1\,e_2\,e_3\ \mathsf{return}\ (\_).\_.\_\ \mathsf{with}\ (\mathsf{nil} \Rightarrow \_)(\mathsf{cons}\ z_1\ z_2\ z_3 \Rightarrow e_c) \rhd_\iota e_c[z_1\ z_2\ z_3 \mapsto e_1\ e_2\ e_3]}$$

$\boxed{\Gamma \vdash e \Updownarrow \tau}$ $\cdots$

NAT
$$\frac{\vdash \Gamma \qquad \Gamma \vdash s}{\Gamma \vdash \mathbb{N}\,[s] \Uparrow \mathsf{Type}_0}$$

ZERO
$$\frac{\vdash \Gamma \qquad \Gamma \vdash s}{\Gamma \vdash \mathsf{zero}\,[s] \Uparrow \mathbb{N}\,[\hat{s}]}$$

SUCC
$$\frac{\Gamma \vdash e \Downarrow \mathbb{N}\,[s]}{\Gamma \vdash \mathsf{succ}\,[s]\,e \Uparrow \mathbb{N}\,[\hat{s}]}$$

VEC
$$\frac{\Gamma \vdash s \qquad \Gamma \vdash \tau \Uparrow U_i \qquad \Gamma \vdash e \Downarrow \mathbb{N}\,[s]}{\Gamma \vdash \mathbb{V}\,[s]\,\tau\,e \Uparrow U_i}$$

NIL
$$\frac{\Gamma \vdash s \qquad \Gamma \vdash \tau \Uparrow U_i}{\Gamma \vdash \mathsf{nil}\,[s]\,\tau \Uparrow \mathbb{V}\,[\hat{s}]\,\tau\,(\mathsf{zero}\,[s])}$$

CONS
$$\frac{\Gamma \vdash e_1 \Downarrow \mathbb{N}\,[s] \qquad \Gamma \vdash e_2 \Uparrow \tau \qquad \Gamma \vdash e_3 \Downarrow \mathbb{V}\,[s]\,\tau\,e_1}{\Gamma \vdash \mathsf{cons}\,[s]\,e_1\,e_2\,e_3 \Uparrow \mathbb{V}\,[\hat{s}]\,\tau\,(\mathsf{succ}\,[s]\,e_1)}$$

MATCH-NAT
$$\frac{\begin{array}{c}\Gamma \vdash e \Uparrow \mathbb{N}\,[\hat{s}] \qquad z \notin \mathsf{FV}(P) \qquad \Gamma(x : \mathbb{N}\,[\hat{s}]) \vdash P \Uparrow U_i \\ \Gamma \vdash e_z \Downarrow P[x \mapsto \mathsf{zero}\,[s]] \qquad \Gamma(z : \mathbb{N}\,[s]) \vdash e_s \Downarrow P[x \mapsto \mathsf{succ}\,[s]\,z]\end{array}}{\Gamma \vdash \mathsf{match}\ e\ \mathsf{return}\ ().x.P\ \mathsf{with}\ (\mathsf{zero} \Rightarrow e_z)(\mathsf{succ}\ z \Rightarrow e_s) \Uparrow P[x \mapsto e]}$$

MATCH-VEC
$$\frac{\begin{array}{c}\Gamma \vdash e \Uparrow \mathbb{V}\,[\hat{s}]\,\tau\,e_N \qquad z_1, z_2, z_3 \notin \mathsf{FV}(P) \\ \Gamma(y : \mathbb{N}\,[\hat{s}])(x : \mathbb{V}\,[\hat{s}]\,\tau\,y) \vdash P \Uparrow U_i \qquad \Gamma \vdash e_n \Downarrow P[y \mapsto \mathsf{zero}\,[s]][x \mapsto \mathsf{nil}\,[s]\,\tau] \\ \Gamma(z_1 : \mathbb{N}\,[s])(z_2 : \tau)(z_3 : \mathbb{V}\,[s]\,\tau\,z_1) \vdash e_c \Downarrow P[y \mapsto \mathsf{succ}\,[s]\,z_1][x \mapsto \mathsf{cons}\,[s]\,z_1\,z_2\,z_3]\end{array}}{\Gamma \vdash \mathsf{match}\ e\ \mathsf{return}\ (y).x.P\ \mathsf{with}\ (\mathsf{nil} \Rightarrow e_n)(\mathsf{cons}\ z_1\ z_2\ z_3 \Rightarrow e_c) \Uparrow P[y \mapsto e_N][x \mapsto e]}$$

FIX
$$\frac{\begin{array}{c}\Gamma \vdash \sigma \Uparrow U_i \qquad \Gamma \vdash \sigma \rhd^* \forall \alpha.\,\sigma' \qquad \sigma' \equiv \forall \Phi.\,\Delta \to (x : \tau) \to \tau' \\ \tau \equiv \mathbb{N}\,[\alpha]\ \mathit{or}\ \mathbb{V}\,[\alpha]\,\_\,\_ \qquad \Gamma(\alpha)(f : \sigma) \vdash e \Downarrow \sigma[\alpha \mapsto \hat{\alpha}]\end{array}}{\Gamma \vdash \mathsf{fix}\ f : \sigma \coloneqq e \Uparrow \sigma}$$

Figure 9: Naturals and vectors

$$e ::= \cdots \mid \mathbb{N} \mid \mathsf{zero} \mid \mathsf{succ}\; e$$

$$\begin{array}{l}
\mathsf{let}\ \mathsf{shiftN1} : \forall\alpha.\, \mathbb{N}\,[\alpha] \to \mathbb{N}\,[\hat{\alpha}] \coloneqq \\
\quad \mathsf{fix}\ \mathit{shiftN1} : \forall\alpha.\, \mathbb{N}\,[\alpha] \to \mathbb{N}\,[\hat{\alpha}] \coloneqq \\
\quad\quad \lambda n : \mathbb{N}\,[\hat{\alpha}].\ \mathsf{match}\ n\ \mathsf{return}\ ()._.\mathbb{N}\,[\hat{\hat{\alpha}}]\ \mathsf{with} \\
\quad\quad\quad (\mathsf{zero} \Rightarrow \mathsf{zero}\,[\hat{\alpha}]) \\
\quad\quad\quad (\mathsf{succ}\ m \Rightarrow \mathsf{succ}\,[\hat{\alpha}]\,(\mathit{shiftN1}\ m))
\end{array}$$

$$\begin{array}{l}
\mathsf{let}\ \mathsf{shiftN} : \forall\alpha.\, \mathbb{N}\,[\alpha] \to \mathbb{N} \coloneqq \\
\quad \mathsf{fix}\ \mathit{shiftN} : \forall\alpha.\, \mathbb{N}\,[\alpha] \to \mathbb{N} \coloneqq \\
\quad\quad \lambda n : \mathbb{N}\,[\hat{\alpha}].\ \mathsf{match}\ n\ \mathsf{return}\ ()._.\mathbb{N}\ \mathsf{with} \\
\quad\quad\quad (\mathsf{zero} \Rightarrow \mathsf{zero}) \\
\quad\quad\quad (\mathsf{succ}\ m \Rightarrow \mathsf{succ}\,(\mathit{shiftN}\ m))
\end{array}$$

$\boxed{\Gamma \vdash e \Updownarrow \tau \rightsquigarrow e^*}$ $\cdots$

$$\textsc{nat}^* \quad \frac{\vdash \Gamma}{\Gamma \vdash \mathbb{N} \Uparrow \mathsf{Type}_0 \rightsquigarrow \exists\alpha.\, \mathbb{N}\,[\alpha]}
\qquad\qquad
\textsc{zero}^* \quad \frac{\vdash \Gamma}{\Gamma \vdash \mathsf{zero} \Uparrow \mathbb{N} \rightsquigarrow (\circ, \mathsf{zero}\,[\circ])}$$

$$\textsc{succ}^* \quad \frac{\Gamma \vdash e \Downarrow \mathbb{N} \rightsquigarrow e^*}{\Gamma \vdash \mathsf{succ}\ e \Uparrow \mathbb{N} \rightsquigarrow \mathsf{let}\ (\alpha, x) \coloneqq e^*\ \mathsf{in}\ (\hat{\alpha}, \mathsf{succ}\,[\alpha]\ x)}$$

$$\textsc{match-nat}^* \quad \frac{\begin{array}{c} \Gamma \vdash e \Downarrow \mathbb{N} \rightsquigarrow e^* \qquad z \notin \mathsf{FV}(P) \\ \Gamma(x : \mathbb{N}) \vdash P \Uparrow U_i \rightsquigarrow P^* \qquad \Gamma \vdash e_z \Downarrow P[x \mapsto \mathsf{zero}] \rightsquigarrow e_z^* \qquad \Gamma(z : \mathbb{N}) \vdash e_s \Downarrow P[x \mapsto \mathsf{succ}\ z] \rightsquigarrow e_s^* \end{array}}{\Gamma \vdash \begin{array}{l} \mathsf{match}\ e\ \mathsf{return}\ ().x.P\ \mathsf{with} \\ \quad (\mathsf{zero} \Rightarrow e_z) \\ \quad (\mathsf{succ}\ z \Rightarrow e_s) \end{array} \Uparrow P[x \mapsto e] \rightsquigarrow \begin{array}{l} \mathsf{let}\ (\alpha, y) \coloneqq e^*\ \mathsf{in} \\ \quad \mathsf{match}\ \mathsf{shiftN1}\ y\ \mathsf{return}\ ().x.\mathsf{let}\ x \coloneqq (\hat{\alpha}, x)\ \mathsf{in}\ P^*\ \mathsf{with} \\ \quad\quad (\mathsf{zero} \Rightarrow e_z^*) \\ \quad\quad (\mathsf{succ}\ z \Rightarrow \mathsf{let}\ z \coloneqq (\alpha, z)\ \mathsf{in}\ e_s^*) \end{array}}$$

Figure 10: Sugared full naturals

$$
\begin{aligned}
X &::= \text{inductive type names}\\
c &::= \text{inductive constructor names}\\
D &::= \mathsf{data}\ X : \Delta_I \to U_i\ \mathsf{where}\ \Delta_c\\
e &::= \cdots \mid \mathsf{match}\ e\ \mathsf{return}\ (y_1 \dots y_n).x.P\ \mathsf{with}\ (c\, z_1 \dots z_m \Rightarrow e) \dots \mid \mathsf{fix}_{\,i} f : \tau \coloneqq e
\end{aligned}
$$

An inductive data definition $\mathsf{data}\ X : \Delta_I \to U_i\ \mathsf{where}\ \Delta_c$ is well-formed if the following hold:

1. X does not appear in Δ_I;
2. For every constructor $(c : \tau) \in \Delta_c$, τ has the shape $\Delta_a \to X\ \_ \dots$;
3. For every constructor argument $(x : \tau) \in \Delta_a$,
   - The inductive type X does not appear in τ; or
   - τ has the shape $\Delta \to X\ \_ \dots$, where X does not appear in Δ;
4. $\cdot \vdash \Delta_I \to U_i \Uparrow \mathsf{Type}_j$ for some universe level j;
5. For every constructor $(c : \tau) \in \Delta_c$, $(X : \Delta_I \to U_i) \vdash \tau \Downarrow U_i$.

$\boxed{\Gamma \vdash e \rhd e}$ $\cdots$

$$
\frac{}{\Gamma \vdash \mathsf{match}\ c\, e_1 \dots e_m\ \mathsf{return}\ (\_ \dots).\_\ \mathsf{with}\ \dots (c\, z_1 \dots z_m \Rightarrow e) \dots \rhd_\iota e[z_1 \dots z_m \mapsto e_1 \dots e_m]}
$$

$\boxed{\Gamma \vdash e \Updownarrow \tau}$

CONV VAR TYPE PI LAM APP LET *as above*

MATCH

$$
\frac{\begin{array}{c}\mathsf{data}\ X : \Delta_I \to U_i\ \mathsf{where}\ \Delta_c\\ \Gamma \vdash e \Uparrow X\ \Delta_i \qquad \Gamma\Delta_I(x : X\ \Delta_I) \vdash P[y_1 \dots y_n \mapsto \Delta_I] \Uparrow U_j'\\ \text{For each constructor } c\text{:} \qquad (c : \Delta_a \to X\ e_1' \dots e_n') \in \Delta_c\\ \Gamma\Delta_a \vdash e_c[z_1 \dots z_m \mapsto \Delta_a] \Downarrow P[y_1 \dots y_n \mapsto e_1' \dots e_n'][x \mapsto c\ \Delta_a]\end{array}}{\Gamma \vdash \mathsf{match}\ e\ \mathsf{return}\ (y_1 \dots y_n).x.P\ \mathsf{with}\ (c\, z_1 \dots z_m \Rightarrow e_c) \dots \Uparrow P[y_1 \dots y_n \mapsto e_1 \dots e_n][x \mapsto e]}
$$

FIX

$$
\frac{\Gamma \vdash \sigma \Uparrow U_i \qquad \Gamma(f : \sigma) \vdash e \Downarrow \sigma}{\Gamma \vdash \mathsf{fix}_{\,i} f : \sigma \coloneqq e \Uparrow \sigma}
$$

Figure 11: Model: CIC

$$
\begin{array}{l}
\mathsf{data}\ \mathsf{Size} : \mathsf{Type}\ \mathsf{where} \\
\quad \mathsf{base} : \mathsf{Size} \\
\quad \mathsf{next} : \mathsf{Size} \to \mathsf{Size}
\end{array}
$$

$$
\begin{array}{l}
\mathsf{data}\ \mathsf{Eq} : (A : \mathsf{Type}) \to (a, b : A) \to \mathsf{Type}\ \mathsf{where} \\
\quad \mathsf{refl} : (A : \mathsf{Type}) \to (a : A) \to \mathsf{Eq}\ A\ a\ a
\end{array}
$$

$$
\begin{array}{l}
\mathsf{data}\ \mathsf{Pair} : (A : \mathsf{Type}) \to (B : A \to \mathsf{Type}) \to \mathsf{Type}\ \mathsf{where} \\
\quad \mathsf{pair} : (A : \mathsf{Type}) \to (B : A \to \mathsf{Type}) \to (a : A) \to B\ a \to \mathsf{Pair}\ A\ B
\end{array}
$$

$$
\begin{array}{l}
\mathsf{data}\ \mathsf{Nat} : \mathsf{Size} \to \mathsf{Type}\ \mathsf{where} \\
\quad \mathsf{zero} : (\alpha : \mathsf{Size}) \to \mathsf{Nat}\ (\mathsf{next}\ \alpha) \\
\quad \mathsf{succ} : (\alpha : \mathsf{Size}) \to \mathsf{Nat}\ \alpha \to \mathsf{Nat}\ (\mathsf{next}\ \alpha)
\end{array}
$$

$$
\begin{array}{l}
\mathsf{data}\ \mathsf{Vec} : \mathsf{Type} \to (\alpha : \mathsf{Size}) \to \mathsf{Nat}\ \alpha \to \mathsf{Type}\ \mathsf{where} \\
\quad \mathsf{nil} : (A : \mathsf{Type}) \to (\alpha : \mathsf{Size}) \to \mathsf{Vec}\ A\ (\mathsf{next}\ \alpha)\ (\mathsf{zero}\ \alpha) \\
\quad \mathsf{cons} : (A : \mathsf{Type}) \to (\alpha : \mathsf{Size}) \to (n : \mathsf{Nat}\ \alpha) \to A \to \mathsf{Vec}\ A\ \alpha\ n \to \mathsf{Vec}\ A\ (\mathsf{next}\ \alpha)\ (\mathsf{succ}\ \alpha\ n)
\end{array}
$$

Figure 12: Preliminary CIC inductive data

$$
\begin{aligned}
&\textsf{let subst} : (A : \textsf{Type}) \to (P : A \to \textsf{Type}) \to (x, y : A) \to x = y \to P\,x \to P\,y \coloneqq \\
&\quad \lambda A : \textsf{Type}.\ \lambda P : A \to \textsf{Type}.\ \lambda x : A.\ \lambda y : A.\ \lambda p : x = y. \\
&\qquad \textsf{match}\ p\ \textsf{return}\ (\_\ x\ y).\_.P\,x \to P\,y\ \textsf{with} \\
&\qquad\quad (\textsf{refl}\ \_\ x \Rightarrow \lambda px : P\,x.\ px)
\end{aligned}
$$

$$
\begin{aligned}
&\textsf{let ap} : (A, B : \textsf{Type}) \to (f : A \to B) \to (x, y : A) \to x = y \to f\,x = f\,y \coloneqq \\
&\quad \lambda A : \textsf{Type}.\ \lambda B : \textsf{Type}.\ \lambda f : A \to B.\ \lambda x : A.\ \lambda y : A.\ \lambda p : x = y. \\
&\qquad \textsf{match}\ p\ \textsf{return}\ (\_\ x\ y).\_.f\,x = f\,y\ \textsf{with} \\
&\qquad\quad (\textsf{refl}\ \_\ x \Rightarrow \textsf{refl}\ B\ (f\ x))
\end{aligned}
$$

$$
\begin{aligned}
&\textsf{let absurd} : (A : \textsf{Type}) \to (\alpha : \textsf{Size}) \to \textsf{base} = \textsf{next}\ \alpha \to A \coloneqq \\
&\quad \lambda A : \textsf{Type}.\ \textsf{let}\ \mathit{discr} \coloneqq \\
&\qquad \lambda \alpha : \textsf{Size}.\ \textsf{match}\ \alpha\ \textsf{return}\ ().\_.\textsf{Type}\ \textsf{with} \\
&\qquad\quad (\textsf{base} \Rightarrow \textsf{Size}) \\
&\qquad\quad (\textsf{next}\ \_ \Rightarrow A)\ \textsf{in} \\
&\quad \lambda \alpha : \textsf{Size}.\ \lambda p : \textsf{base} = \textsf{next}\ \alpha. \\
&\qquad \textsf{subst Size}\ \mathit{discr}\ \textsf{base}\ (\textsf{next}\ \alpha)\ p\ \textsf{base}
\end{aligned}
$$

$$
\begin{aligned}
&\textsf{let inj} : (\alpha, \beta : \textsf{Size}) \to \textsf{next}\ \alpha = \textsf{next}\ \beta \to \alpha = \beta \coloneqq \\
&\quad \textsf{let}\ \mathit{pred} \coloneqq \\
&\qquad \lambda \beta : \textsf{Size}.\ \textsf{match}\ \beta\ \textsf{return}\ ().\_.\textsf{Size}\ \textsf{with} \\
&\qquad\quad (\textsf{base} \Rightarrow \textsf{base}) \\
&\qquad\quad (\textsf{next}\ \beta' \Rightarrow \beta')\ \textsf{in} \\
&\quad \lambda \alpha : \textsf{Size}.\ \lambda \beta : \textsf{Size}.\ \lambda p : \textsf{next}\ \alpha = \textsf{next}\ \beta. \\
&\qquad \textsf{ap Size Size}\ \mathit{pred}\ (\textsf{next}\ \alpha)\ (\textsf{next}\ \beta)\ p
\end{aligned}
$$

where $e_1 = e_2 \equiv \textsf{Eq}\ \_\ e_1\ e_2$, type inferred from context

Figure 13: Preliminary CIC definitions (1/2)

$$\begin{array}{l}
\textsf{let substInj} : (P : \textsf{Size} \to \textsf{Type}) \to (\alpha, \beta : \textsf{Size}) \to \textsf{next}\ \alpha == \textsf{next}\ \beta \to P\ \alpha \to P\ \beta \coloneqq \\
\quad \lambda P : \textsf{Size} \to \textsf{Type}.\ \lambda \alpha : \textsf{Size}.\ \lambda \beta : \textsf{Size}.\ \lambda p : \textsf{next}\ \alpha == \textsf{next}\ \beta. \\
\qquad \textsf{subst Size}\ P\ \alpha\ \beta\ (\textsf{inj}\ \alpha\ \beta\ p)
\end{array}$$

$$\begin{array}{l}
\textsf{let absurdNatBase} : (x : \textsf{Nat base}) \to (A : \textsf{Type}) \to A \coloneqq \\
\quad \lambda x : \textsf{Nat base}.\ \lambda A : \textsf{Type}. \\
\qquad \begin{bmatrix} \textsf{match}\ x\ \textsf{return}\ (\alpha)._.\textsf{base} == \alpha \to A\ \textsf{with} \\ \quad (\textsf{zero}\ \alpha \Rightarrow \textsf{absurd}\ A\ \alpha) \\ \quad (\textsf{succ}\ \alpha\ _ \Rightarrow \textsf{absurd}\ A\ \alpha) \end{bmatrix} (\textsf{refl Size base})
\end{array}$$

$$\begin{array}{l}
\textsf{let elim} : (P : \textsf{Size} \to \textsf{Type}) \to P\ \textsf{base} \to ((\alpha : \textsf{Size}) \to P\ \alpha \to P\ (\textsf{next}\ \alpha)) \to (\alpha : \textsf{Size}) \to P\ \alpha \coloneqq \\
\quad \lambda P : \textsf{Size} \to \textsf{Type}.\ \lambda pb : P\ \textsf{base}.\ \lambda pn : (\alpha : \textsf{Size}) \to P\ \alpha \to P\ (\textsf{next}\ \alpha). \\
\qquad \textsf{fix}\ \mathit{elim} : (\alpha : \textsf{Size}) \to P\ \alpha \coloneqq \\
\qquad\quad \lambda \alpha : \textsf{Size}.\ \textsf{match}\ \alpha\ \textsf{return}\ ().\alpha.P\ \alpha\ \textsf{with} \\
\qquad\qquad (\textsf{base} \Rightarrow pb) \\
\qquad\qquad (\textsf{next}\ \alpha' \Rightarrow pn\ (\mathit{elim}\ \alpha'))
\end{array}$$

where $e_1 == e_2 \equiv \textsf{Eq}\ \_\ e_1\ e_2$, type inferred from context

Figure 14: Preliminary CIC definitions (2/2)

$$\boxed{[\![s]\!] = e^*}$$

$$
\begin{aligned}
[\![\alpha]\!] &= \alpha \\
[\![\circ]\!] &= \mathsf{base} \\
[\![\hat{s}]\!] &= \mathsf{next}\ [\![s]\!]
\end{aligned}
$$

$$\boxed{[\![e]\!] = e^*}\ \textit{where}\ \Gamma \vdash e \Uparrow \_ \rightsquigarrow e^*$$

$$
\begin{aligned}
[\![x]\!] &= x \\
[\![U_i]\!] &= U_i \\
[\![\Pi x : \sigma.\, \tau]\!] &= \Pi x : [\![\sigma]\!].\, [\![\tau]\!] \\
[\![\lambda x : \sigma.\, e]\!] &= \lambda x : [\![\sigma]\!].\, [\![e]\!] \\
[\![e_1\ e_2]\!] &= [\![e_1]\!]\ [\![e_2]\!] \\
[\![\forall \alpha.\, \tau]\!] &= \Pi \alpha : \mathsf{Size}.\, [\![\tau]\!] \\
[\![\Lambda \alpha.\, e]\!] &= \lambda \alpha : \mathsf{Size}.\, [\![e]\!] \\
[\![e\ [s]]\!] &= [\![e]\!]\ [\![s]\!] \\
[\![\mathsf{let}\ x \coloneqq e_1\ \mathsf{in}\ e_2]\!] &= \mathsf{let}\ x \coloneqq [\![e_1]\!]\ \mathsf{in}\ [\![e_2]\!] \\
[\![\Sigma x : \sigma.\, \tau]\!] &= \mathsf{Pair}\ [\![\sigma]\!]\ (\lambda x : [\![\sigma]\!].\, [\![\tau]\!]) \\
[\![\langle e_1, e_2 \rangle_{\Sigma x : \sigma.\tau}]\!] &= \mathsf{pair}\ [\![\sigma]\!]\ (\lambda x : [\![\sigma]\!].\, [\![\tau]\!])\ [\![e_1]\!]\ [\![e_2]\!] \\
[\![\pi_{\mathsf{L}}\ e]\!] &= \mathsf{match}\ [\![e]\!]\ \mathsf{return}\ (x\ \_).\_.x\ \mathsf{with}\ (\mathsf{pair}\ \_\ \_\ x\ \_ \Rightarrow x) \\
[\![\pi_{\mathsf{R}}\ e]\!] &= \mathsf{match}\ [\![e]\!]\ \mathsf{return}\ (\_\ y).\_.y\ [\![\pi_{\mathsf{L}}\ e]\!]\ \mathsf{with}\ (\mathsf{pair}\ \_\ \_\ \_\ y \Rightarrow y) \\
[\![\exists \alpha.\, \tau]\!] &= \mathsf{Pair}\ \mathsf{Size}\ (\lambda x : \mathsf{Size}.\, [\![\tau]\!]) \\
[\![(s, e)]\!] &= \mathsf{pair}\ \mathsf{Size}\ (\lambda x : \mathsf{Size}.\, [\![\tau]\!])\ [\![s]\!]\ [\![e]\!] \\
[\![e_1 \overset{\tau}{=\!=} e_2]\!] &= \mathsf{Eq}\ [\![\tau]\!]\ [\![e_1]\!]\ [\![e_2]\!]
\end{aligned}
$$

$$\boxed{\Gamma \vdash e \Updownarrow \tau \rightsquigarrow e^*}$$

CONV*

$$\frac{\Gamma \vdash e \Uparrow \tau' \rightsquigarrow e^*}{\Gamma \vdash e \Downarrow \tau \rightsquigarrow e^*}$$

UNPAIR*

$$\frac{\Gamma \vdash e_1 \Uparrow \exists \alpha.\, \sigma \rightsquigarrow e_1^* \qquad \Gamma(\alpha)(x : \sigma) \vdash e_2 \Uparrow \tau \rightsquigarrow e_2^* \qquad \Gamma \vdash \tau \Uparrow U_i \rightsquigarrow \tau^*}{\Gamma \vdash \mathsf{let}\ (\alpha, x) \coloneqq e_1\ \mathsf{in}\ e_2 \Uparrow \tau \rightsquigarrow \mathsf{match}\ e_1^*\ \mathsf{return}\ (\_\ \_).\_.\tau^*\ \mathsf{with}\ (\mathsf{pair}\ \_\ \_\ \alpha\ x \Rightarrow e_2^*)}$$

REFL*

$$\frac{\Gamma \vdash e \Uparrow \tau \rightsquigarrow e^* \qquad \Gamma \vdash \tau \Uparrow U_i \rightsquigarrow \tau^*}{\Gamma \vdash \mathsf{refl}_e \Uparrow e \overset{\tau}{=\!=} e \rightsquigarrow \mathsf{refl}\ \tau^*\ e^*}$$

J*

$$\frac{\begin{array}{c}\Gamma \vdash p \Uparrow e_1 \overset{\tau}{=\!=} e_2 \rightsquigarrow p^* \qquad \Gamma \vdash e_1 \Downarrow \tau \rightsquigarrow e_1^* \qquad x, y, z\ \mathsf{fresh} \\ \Gamma(y : \tau)(z : e_1 \overset{\tau}{=\!=} y) \vdash P\ y\ z \Uparrow U_i \rightsquigarrow P^* \qquad \Gamma \vdash d \Downarrow P\ e_1\ \mathsf{refl}_{e_1} \rightsquigarrow d^*\end{array}}{\Gamma \vdash \mathsf{J}_P\ d\ p \Uparrow P\ e_2\ p \rightsquigarrow \mathsf{match}\ p^*\ \mathsf{return}\ (\_\ x\ y).z.\mathsf{let}\ x \coloneqq e_1^*\ \mathsf{in}\ P\ y\ z\ \mathsf{with}\ (\mathsf{refl}\ \_\ \_ \Rightarrow d^*)}$$

Figure 15: Model in CIC (1/2)

$$\boxed{[\![e]\!] = e^*} \quad \textit{where } \Gamma \vdash e \Uparrow \_ \rightsquigarrow e^*$$

$$\begin{aligned}
[\![\mathbb{N}\,[s]]\!] &= \mathsf{Nat}\ [\![s]\!] \\
[\![\mathsf{zero}\,[s]]\!] &= \mathsf{zero}\ [\![s]\!] \\
[\![\mathsf{succ}\,[s]\ e]\!] &= \mathsf{succ}\ [\![s]\!]\ [\![e]\!] \\
[\![\mathbb{V}\,[s]\ \tau\ e]\!] &= \mathsf{Vec}\ [\![s]\!]\ [\![\tau]\!]\ [\![e]\!] \\
[\![\mathsf{nil}\,[s]\ \tau]\!] &= \mathsf{nil}\ [\![\tau]\!]\ [\![s]\!]
\end{aligned}$$

$$\boxed{\Gamma \vdash e \Updownarrow \tau \rightsquigarrow e^*}$$

$$\textsc{cons}^*\ \frac{\Gamma \vdash e_1 \Downarrow \mathbb{N}\,[s] \rightsquigarrow e_1^* \qquad \Gamma \vdash e_2 \Uparrow \tau \rightsquigarrow e_2^* \qquad \Gamma \vdash e_3 \Downarrow \mathbb{V}\,[s]\ \tau\ e_1 \rightsquigarrow e_3^* \qquad \Gamma \vdash \tau \Uparrow U_i \rightsquigarrow \tau^*}{\Gamma \vdash \mathsf{cons}\,[s]\ e_1\ e_2\ e_3 \Uparrow \mathbb{V}\,[\hat{s}]\ \tau\ (\mathsf{succ}\,[s]\ e_1) \rightsquigarrow \mathsf{cons}\ \tau^*\ [\![s]\!]\ e_1^*\ e_2^*\ e_3^*}$$

$$\textsc{match-nat}^*\ \frac{\begin{array}{c}\Gamma \vdash e \Uparrow \mathbb{N}\,[\hat{s}] \rightsquigarrow e^* \qquad \Gamma(x : \mathbb{N}\,[\hat{s}]) \vdash P \Uparrow U_i \rightsquigarrow P^* \\ \Gamma \vdash e_z \Downarrow P[x \mapsto \mathsf{zero}\,[s]] \rightsquigarrow e_z^* \qquad \Gamma(z : \mathbb{N}\,[s]) \vdash e_s \Downarrow P[x \mapsto \mathsf{succ}\,[s]\ z] \rightsquigarrow e_s^* \beta, q \text{ fresh}\end{array}}{\Gamma \vdash \begin{array}{l}\mathsf{match}\ e\ \mathsf{return}\ ().x.P\ \mathsf{with} \\ \quad (\mathsf{zero} \Rightarrow e_z) \\ \quad (\mathsf{succ}\ z \Rightarrow e_s)\end{array} \Uparrow P[x \mapsto e] \rightsquigarrow \begin{bmatrix}(\mathsf{match}\ e^*\ \mathsf{return}\ (\beta).x.\beta == \mathsf{next}\ [\![s]\!] \to P^*\ \mathsf{with} \\ \quad (\mathsf{zero}\ \beta \Rightarrow \lambda\_ : \mathsf{next}\ \beta == \mathsf{next}\ [\![s]\!].\ e_z^*) \\ \quad (\mathsf{succ}\ \beta\ z \Rightarrow \lambda q : \mathsf{next}\ \beta == \mathsf{next}\ [\![s]\!]. \\ \qquad \mathsf{let}\ z \coloneqq \mathsf{substInj}\ \mathsf{Nat}\ \beta\ [\![s]\!]\ q\ z\ \mathsf{in}\ e_s^*))\end{bmatrix} (\mathsf{refl}\ \mathsf{Size}\ [\![\hat{s}]\!])}$$

$$\textsc{fix-nat}^*\ \frac{\Gamma(\alpha) \vdash \sigma \Uparrow U_i \rightsquigarrow \sigma^* \qquad \Gamma(\alpha) \vdash \sigma^* \rhd^* \forall \Phi.\, \Delta \to (x : \mathbb{N}\,[\alpha]) \to \tau \qquad \Gamma(\alpha)(f : \sigma) \vdash e \Downarrow \sigma[\alpha \mapsto \hat{\alpha}] \rightsquigarrow e^*}{\Gamma \vdash \mathsf{fix}\ f : \forall \alpha.\, \sigma \coloneqq e \Uparrow \sigma \rightsquigarrow \begin{array}{l}\mathsf{elim}\ (\lambda \alpha : \mathsf{Size}.\, \sigma^*) \\ \quad (\lambda \Phi.\, \lambda \Delta[\alpha \mapsto \mathsf{base}].\, \lambda x : \mathsf{Nat}\ \mathsf{base}. \\ \qquad \mathsf{absurdNatBase}\ x\ \tau[\alpha \mapsto \mathsf{base}]) \\ \quad (\lambda \alpha : \mathsf{Size}.\, \lambda f : \sigma^*.\, e^*)\end{array}}$$

Figure 16: Model in CIC (2/2)