Name : ....

Roll No. : ....

Invigilator's Signature : ....

CS/B.Tech/SEM-1/M-101/2009-10

2009

MATHEMATICS

Time Allotted : 3 Hours *Full Marks* : 70

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

GROUP – A
(Multiple Choice Type Questions)

1. Choose the correct alternatives for any *ten* of the following : $10 \times 1 = 10$

i) If $f(x+1) + f(x-1) = 2f(x)$ and $f(0) = 0$, then the value of $f(4)$ is

a) $3f(1)$ b) $4f(1)$

c) $5f(1)$ d) $6f(1)$.

ii) If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \infty$, then $\frac{dy}{dx}$ is equal to

a) x b) 1

c) y d) none of these.

iii) Maclaurin's expansion of $(1-x)^{-1}$ about $x = 0$ in infinite series is valid if and only if

a) $x > -1$ b) $-x < x < 1$

c) $x \le 1$ d) none of these.

iv) The sequence $\{(-1)^n\}$ is

a) convergent b) oscillatory

c) divergent d) none of these.

v) The point (0, 3, 0) lies on

a) yz plane b) x-axis

c) y-axis d) xz plane.

vi) If $y = 5 . x^{100} + 3$, then y_{100} is

a) 100 ! b) 5 × 100 !

c) 0 d) none of these.

vii) The centre of the sphere $x^2 + y^2 + z^2 - 6x - 4y - 6z + 16 = 0$ is

a) (3, 3, 3) b) (3, 2, 3)

c) (– 3, 2, – 3) d) (– 3, – 2, 3).

viii) If $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \hat{b} = \hat{i} - \hat{j}$, then (a, b) . a is equal to

a) 1 b) – 1

c) 3 d) – 3.

ix) If a function f (x, y) has critical point at (2, 2), then f_x (2, 2) is equal to

a) 2 b) 0

c) any non-zero value d) none of these.

x) If $x = (u + v)$ and $y = (u - v)$, then $\frac{\partial(u,v)}{\partial(x,y)}$ is equal to

a) -1 b) $-1/2$

c) 1 d) 2.

xi) $\lim_{x \to \pi/2} \tan x$ is equal to

a) ∞ b) 0

c) $-\infty$ d) none of these.

xii) If $f(x) = 2|x| + |x-2|$, then $f'(1)$ is equal to

a) 1 b) -1

c) 2 d) 0.

xiii) The value of $\iint_R x^3 y \, dx \, dy$ where $R : \{0 \le x \le 1, 0 \le y \le 2\}$ is

a) $1/3$ b) $1/2$

c) $2/3$ d) $1/4$.

xiv) Which of the following functions does not satisfy the conditions of the Rolle's theorem ?

a) x^2 b) $\frac{1}{x^4 + 2}$

c) $\frac{1}{x}$ d) $\sqrt{x^2 + 3}$.

xv) If $y = \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$, then y_4 is equal to

a) $-\frac{1}{2}$ b) 0

c) $\frac{1}{1+x^2}$ d) $\frac{\pi}{4} - \frac{x}{2}$.

GROUP – B
(Short Answer Type Questions)

Answer any *three* of the following. $3 \times 5 = 15$

2. Test the convergence of the following series (Mention the tests you are using) :

   $2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \ldots\ldots \infty$, for $x > 0$

3. If $y_n = \frac{d^n}{dx^n}\left(x^n \log_e x\right)$, show that

   $y_n = ny_{n-1} + (n-1)!$.

4. If $f(x) = \tan^{-1} \frac{x^2 + y^2}{x - y}$, prove that by using

   Euler's Theorem $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}\sin 2f$.

5. Prove that $\sin 46^\circ \approx \frac{\sqrt{2}}{2}\left(1 + \frac{\pi}{180}\right)$. Is the estimate high or low ?

6. Obtain the reduction formula of $I_n = \int_0^{\frac{\pi}{2}} \sin^m x \, dx$.

   Hence evaluate $\int_0^1 \frac{(1-x)^{\frac{5}{2}}}{\sqrt{x}} \, dx$.

7. In what direction from the point (1, 1, – 1) is the directional derive of $f = x^2 - 2y^2 + 4z^2$ a maximum ? What is the magnitude of this directional derivative ?

GROUP – C

(Long Answer Type Questions)

Answer any *three* of the following. $3 \times 15 = 45$

8. a) State Ratio Test for convergence of an infinite series of positive terms. Test the convergence of any one of the following series.

i) $\sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$

ii) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1} x^n$

b) If $y = \sin(m \sin^{-1} x)$, then show that

i) $\left(1-x^2\right) y_2 - x y_1 + m_2\, y = 0$

ii) $\left(1-x^2\right) y_{n+2} - (2n+1)\, x y_{n+1} + m^2 - n^2 = 0$.

Also obtain $y_n(0)$.

c) Expand a^x in an infinite series by using Maclaurin's Theorem, mentioning the conditions required.

9. a) State Lagrange's mean value theorem. Prove that

$\frac{x}{1+x} < \log(1+x) < x, x > 0$

b) If u be an homoeneous function of x and y on n dimensions, then prove that

$$x^2 \frac{\partial^2 u}{\partial u^2} + 2xy \frac{\partial^2 u}{\partial x \, \mathrm{d}\partial} y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u .$$

c) Find the extrema of the function $f(xy) = xy + a^3 \left(\frac{1}{x} + \frac{1}{y} \right)$.

10. a) Evaluate : $\iint xy(x+y) \mathrm{d}x \, \mathrm{d}y$, where R is the region enclosed by the curves $y = x^2$ and $y = x$.

b) Obtain the reduction formula for $\int_0^{\frac{\pi}{4}} \tan^n x \mathrm{d}x$. Hence $\int_0^{\frac{\pi}{4}} \tan^7 x \mathrm{d}x$.

c) The smaller of the two arcs into which the parabola $y^2 = 8ax$ divides the circle $x^2 + y^2 = 9a^2$ is rotate x-axis. Show that the volume of the solid generated is $\frac{28}{3} \pi a^3$.

11. a) Find the equation of the plane through the interaction of the planes $x + 2y + 3z = 4$ and $2x + y + z = 5$ is perpendicular to the plane $5x + 3y + 6z + 8 = 0$.

b) Show that the pair of lines whose direction cosines are given by $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}=0$ and $2l - m + 2n = 0$ are perpendicular.

c) Show that the straight lines $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$ are coplanar. Find the point of interaction of the two lines and also find the equation of the plane on which they lie.

12. a) Show that normal to the surface $\Phi(x,y,z): 2xz^2 - 3xy - 4x = 7$ at (1, – 1, 2) is $7\hat{i} - 3\hat{j} + 8\hat{k}$.

b) Show that $\vec{V} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x+2z-y)\hat{k}$ is irrotational. Obtain a scalar function Φ (x, y, z), such that $\vec{V} = \vec{\nabla}\Phi$.

c) Verify Green's theorem for $\oint_C \vec{F}.\vec{dr}$, where $\vec{F} = x^2 y\hat{i} + x^2 \hat{j}$ and C is the boundary described in counter-clockwise direction of the triangle with vertices (0, 0), (1, 0), (1, 1).

13. a) Find the centroid of the area of one loop of the lemniscate $r^2 = a^2 \cos 2\theta$.

b) Evaluate $\iiint z^2 \, dx \, dy \, dz$ over the region defined by $z \geq 0, x^2 + y^2 + z^2 \leq a^2$, using suitable transformation.

c) Is the following series absolutely convergent ? (any one)

i) $\sum u_n = \sum (-1)^{n-1} \left(\frac{n^n}{n^{n+1}} - \frac{n+1}{n} \right)^{-n}$

ii) $\frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \frac{5}{26} - \ldots\ldots \infty.$