WIKIPEDIA
The Free Encyclopedia

Main page
Contents
Featured content
Current events
Random article
Donate to Wikipedia
Wikipedia store

Interaction

Help
About Wikipedia
Community portal
Recent changes
Contact page

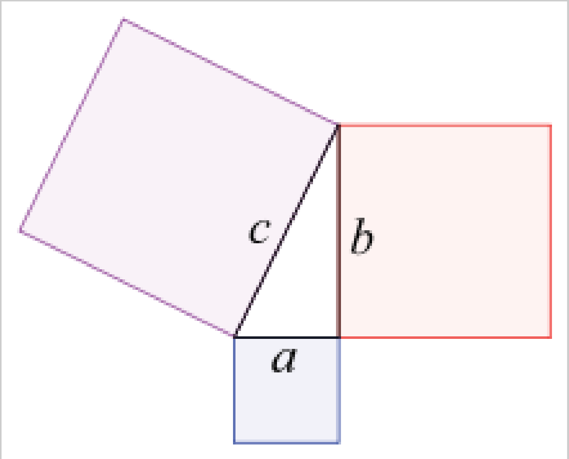
Tools

What links here
Related changes
Upload file
Special pages
Permanent link
Page information

Pythagorean triple

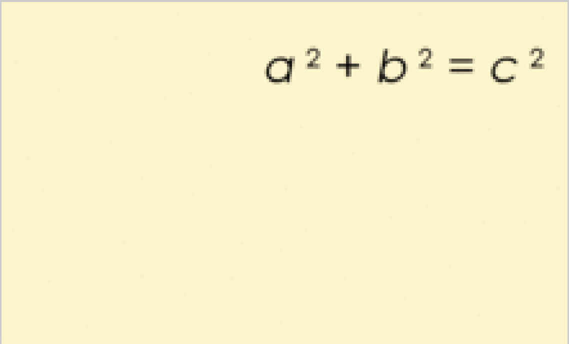
From Wikipedia, the free encyclopedia

A **Pythagorean triple** consists of three positive integers *a*, *b*, and *c*, such that $a^2 + b^2 = c^2$. Such a triple is commonly written (*a*, *b*, *c*), and a well-known example is (3, 4, 5). If (*a*, *b*, *c*) is a Pythagorean triple, then so is (*ka*, *kb*, *kc*) for any positive integer *k*. A **primitive Pythagorean triple** is one in which *a*, *b* and *c* are coprime. A right triangle whose sides form a Pythagorean triple is called a **Pythagorean triangle**.



The Pythagorean theorem: $a^2 + b^2 = c^2$

The name is derived from the Pythagorean theorem, stating that every right triangle has side lengths satisfying the formula $a^2 + b^2 = c^2$; thus, Pythagorean triples describe the three integer side lengths of a right triangle. However, right triangles with non-integer sides do not form Pythagorean triples. For instance, the triangle with sides *a* = *b* = 1 and *c* = √2 is right, but (1, 1, √2) is not a Pythagorean triple because √2 is not an integer. Moreover, 1 and √2 do not have an integer common multiple because √2 is irrational.



Contents [hide]

Wikidata item
Cite this page

Print/export
Create a book
Download as PDF
Printable version

Languages
العربية
Беларуская
Беларуская (тарашкевіца)
Български
Català
Čeština
Dansk
Deutsch
Ελληνικά
Español
Esperanto
فارسی
Français
한국어
Íslenska
Italiano
עברית
ქართული
Latviešu
Magyar
മലയാളം
Nederlands

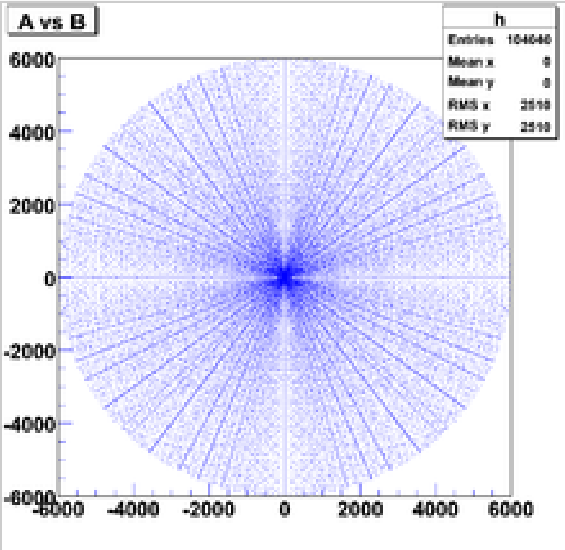
Animation demonstrating the simplest case of the Pythagorean Triple: $3^2 + 4^2 = 5^2$.

Norsk bokmål
Piemontèis
Polski
Português
Română
Русский
Sicilianu
Simple English
Slovenščina
Српски / srpski
Suomi
Svenska
தமிழ்
Türkçe
Українська
Tiếng Việt
中文
Edit links

Examples [edit]

There are 16 primitive Pythagorean triples with *c* ≤ 100:

(3, 4, 5)	(5, 12, 13)	(8, 15, 17)	(7, 24, 25)
(20, 21, 29)	(12, 35, 37)	(9, 40, 41)	(28, 45, 53)
(11, 60, 61)	(16, 63, 65)	(33, 56, 65)	(48, 55, 73)
(13, 84, 85)	(36, 77, 85)	(39, 80, 89)	(65, 72, 97)



A scatter plot of the legs (*a*,*b*) of the Pythagorean triples with *c* less than 6000. Negative values are included to illustrate the parabolic patterns in the plot more clearly.

Note, for example, that (6, 8, 10) is *not* a primitive Pythagorean triple, as it is a multiple of (3, 4, 5). Each one of these low-c points forms one of the more easily recognizable radiating lines in the scatter plot.

Additionally these are all the primitive Pythagorean triples with 100 < *c* ≤ 300:

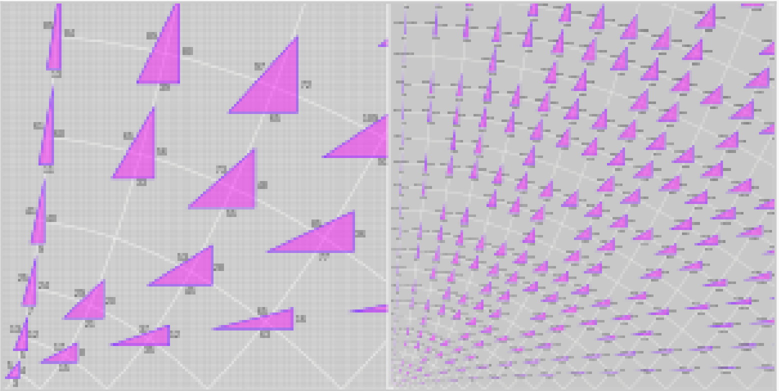
(20, 99, 101)	(60, 91, 109)	(15, 112, 113)	(44, 117, 125)
(88, 105, 137)	(17, 144, 145)	(24, 143, 145)	(51, 140, 149)

(85, 132, 157)	(119, 120, 169)	(52, 165, 173)	(19, 180, 181)
(57, 176, 185)	(104, 153, 185)	(95, 168, 193)	(28, 195, 197)
(84, 187, 205)	(133, 156, 205)	(21, 220, 221)	(140, 171, 221)
(60, 221, 229)	(105, 208, 233)	(120, 209, 241)	(32, 255, 257)
(23, 264, 265)	(96, 247, 265)	(69, 260, 269)	(115, 252, 277)
(160, 231, 281)	(161, 240, 289)	(68, 285, 293)	

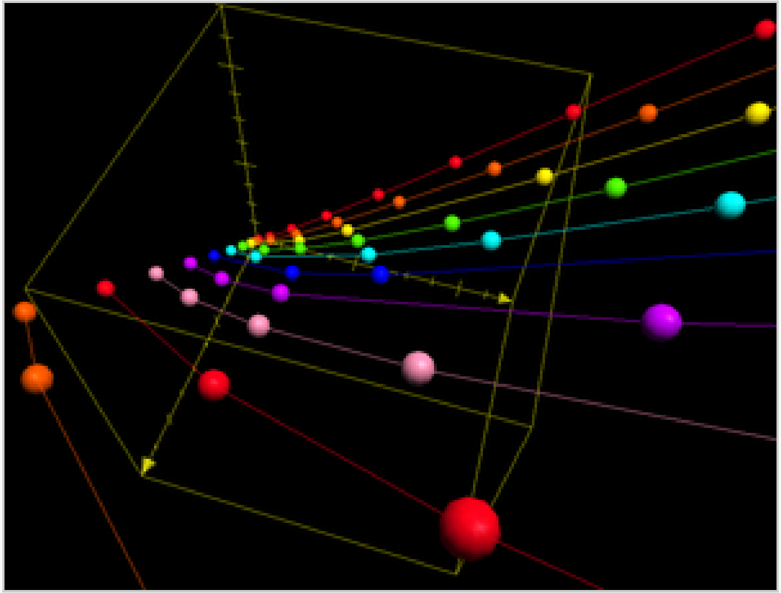
Generating a triple [edit]

Main article: Formulas for generating Pythagorean triples

Euclid's formula[1] is a fundamental formula for generating Pythagorean triples given an arbitrary pair of positive integers *m* and *n* with $m > n$. The formula states that the integers



The primitive Pythagorean triples. The odd leg *a* is plotted on the horizontal axis, the even leg *b* on the vertical. The curvilinear grid is composed of curves of constant *m* - *n* and of constant *m* + *n* in Euclid's formula.



A plot of triples generated by Euclid's formula map out part of the $z^2 = x^2 + y^2$ cone. A constant m or n traces out part of a parabola on the cone.

$$a = m^2 - n^2, \ \ b = 2mn, \ \ c = m^2 + n^2$$

form a Pythagorean triple. The triple generated by Euclid's formula is primitive if and only if m and n are coprime and $m - n$ is odd. If both m and n are odd, then a, b, and c will be even, and so the triple will not be primitive; however, dividing a, b, and c by 2 will yield a primitive triple if m and n are coprime.[2]

Every primitive triple arises from a *unique pair* of coprime numbers m, n, one of which is even. It follows that there are infinitely many primitive Pythagorean triples. This relationship of a, b and c to m and n from Euclid's formula is referenced throughout the rest of this article.

Despite generating all primitive triples, Euclid's formula does not produce all triples—for example, (9, 12, 15) cannot be generated using integer m and n. This can be remedied by inserting an additional parameter *k* to the formula. The following will generate all Pythagorean triples uniquely:

$$a = k\cdot(m^2 - n^2),\ \ b = k\cdot(2mn),\ \ c = k\cdot(m^2 + n^2)$$

where *m*, *n*, and *k* are positive integers with $m > n$, $m - n$ odd, and with *m* and *n* coprime.

That these formulas generate Pythagorean triples can be verified by expanding $a^2 + b^2$ using elementary algebra and verifying that the result coincides with c^2. Since every Pythagorean triple can be divided through by some integer *k* to obtain a primitive triple, every triple can be generated uniquely by using the formula with *m* and *n* to generate its primitive counterpart and then multiplying through by *k* as in the last equation.

Many formulas for generating triples with particular properties have been developed since the time of Euclid.

Proof of Euclid's formula [edit]

That satisfaction of Euclid's formula by *a, b, c* is sufficient for the triangle to be Pythagorean is apparent from the fact that for positive integers *m* and *n*, $m > n$, the *a, b,* and *c* given by the formula are all positive integers, and from the fact that

$$a^2 + b^2 = (m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2 = c^2.$$

A simple proof of the *necessity* that *a, b, c* be expressed by Euclid's formula for any primitive Pythagorean triple is as follows.[3] All such triples can be written as (*a*, *b*, *c*) where $a^2 + b^2 = c^2$ and *a*, *b*, *c* are coprime, and where *b* and *c* have opposite parities (one is even and one is odd). (If *c* had the same parity as both legs, then if all were even the parameters would not be coprime, and if all were odd then $a^2 + b^2 = c^2$ would equate an even to an odd.) From $a^2 + b^2 = c^2$ we obtain $c^2 - a^2 = b^2$ and hence $(c - a)(c + a) = b^2$. Then $\frac{(c+a)}{b} = \frac{b}{(c-a)}$. Since $\frac{(c+a)}{b}$ is rational, we set it equal to $\frac{m}{n}$ in lowest terms. We also observe that $\frac{(c-a)}{b}$ equals the reciprocal of $\frac{b}{(c-a)}$ and hence equals the reciprocal of $\frac{(c+a)}{b}$, and thus equals $\frac{n}{m}$. Then solving

$$\frac{c}{b}+\frac{a}{b}=\frac{m}{n}, \qquad \frac{c}{b}-\frac{a}{b}=\frac{n}{m}$$

for $\frac{c}{b}$ and $\frac{a}{b}$ gives

$$\frac{c}{b}=\frac{1}{2}\left(\frac{m}{n}+\frac{n}{m}\right)=\frac{m^2+n^2}{2mn}, \qquad \frac{a}{b}=\frac{1}{2}\left(\frac{m}{n}-\frac{n}{m}\right)=\frac{m^2-n^2}{2mn}.$$

Since $\frac{c}{b}$ and $\frac{a}{b}$ are fully reduced by assumption, the numerators can be equated and the denominators can be equated if and only if the right side of each equation is fully reduced; given the previous specification that $\frac{m}{n}$ is fully reduced, implying that *m* and *n* are coprime, the right sides are fully reduced if and only if *m* and *n* have opposite parity (one is even and one is odd) so that the numerators are not divisible by 2. (And *m* and *n* *must* have opposite parity: if both were odd then dividing through $\frac{m^2+n^2}{2mn}$ by 2 would give the ratio of two odd numbers; equating this ratio to $\frac{c}{b}$, which is a ratio of two numbers with opposite parities, would give different 2-adic orders for numbers supposedly equal.) So, equating numerators and equating denominators, we have Euclid's formula $a=m^2-n^2,\ \ b=2mn,\ \ c=m^2+n^2$ with *m* and *n* coprime and of opposite parities.

A longer but more commonplace proof is given in Maor (2007)[4] and Sierpiński (2003).[5]:4–7

Interpretation of parameters in Euclid's formula [edit]

Suppose the sides of a Pythagorean triangle are m^2-n^2, $2mn$, and m^2+n^2, and suppose the angle between the leg m^2-n^2 and the hypotenuse m^2+n^2 is denoted as θ. Then $\tan\theta=\frac{2mn}{m^2-n^2}$ and $\tan\frac{\theta}{2}=\frac{n}{m}$.[6]

Elementary properties of primitive Pythagorean triples [edit]

General properties [edit]

The properties of a primitive Pythagorean triple (a, b, c) with $a < b < c$ (without specifying which of a or b is even and which is odd) include:

- $\frac{(c-a)(c-b)}{2}$ is always a perfect square.[7] This is particularly useful in checking if a given triple of numbers is a Pythagorean triple, but it is only a necessary condition, not a sufficient one. The triple {6, 12, 18} passes the test that (c − a)(c − b)/2 is a perfect square, but it is not a Pythagorean triple. When a triple of numbers a, b and c forms a primitive Pythagorean triple, then (c minus the even leg) and one-half of (c minus the odd leg) are both perfect squares; however this is not a sufficient condition, as the triple {1, 8, 9} is a counterexample since $1^2 + 8^2 \neq 9^2$.
- At most one of a, b, c is a square.[8]
- The area of a Pythagorean triangle cannot be the square[9]:p. 17 or twice the square[9]:p. 21 of a natural number.
- Exactly one of a, b is odd; c is odd.[10]
- Exactly one of a, b is divisible by 3.[5]
- Exactly one of a, b is divisible by 4.[5]
- Exactly one of a, b, c is divisible by 5.[5]
- The largest number that always divides abc is 60.[11]
- All prime factors of c are primes of the form $4n + 1$.[12] Therefore c is of the form $4n + 1$.
- The area ($K = ab/2$) is an even congruent number.[13]
- In every Pythagorean triple, the radius of the incircle and the radii of the three excircles are natural numbers. Specifically, for a primitive triple the radius of the incircle is $r = n(m - n)$, and the radii of the excircles opposite the sides $m^2–n^2$, $2mn$, and the hypotenuse m^2+n^2 are respectively $m(m - n)$, $n(m + n)$, and $m(m + n)$.[14]
- As for any right triangle, the converse of Thales' theorem says that the diameter of the circumcircle equals the hypotenuse; hence for primitive triples the circumdiameter is m^2+n^2, and the

circumradius is half of this and thus is rational but non-integer (since *m* and *n* have opposite parity).

- When the area of a Pythagorean triangle is multiplied by the curvatures of its incircle and 3 excircles, the result is four positive integers *w* > *x* > *y* > *z*, respectively. Integers -*w*, *x*, *y*, *z* satisfy Descartes’s Circle Equation.[15] Equivalently, the radius of the outer Soddy circle of any right triangle is equal to its semiperimeter. The outer Soddy center is located at *D*, where *ACBD* is a rectangle, *ACB* the right triangle and *AB* its hypotenuse.[15]:p. 6
- There are no Pythagorean triples in which the hypotenuse and one leg are the legs of another Pythagorean triple; this is one of the equivalent forms of Fermat's right triangle theorem.[9]:p. 14
- Each primitive Pythagorean triangle has a ratio of area to squared semiperimeter that is unique to itself and is given by[16]

$$\frac{K}{s^2} = \frac{n(m-n)}{m(m+n)} = 1 - \frac{c}{s}.$$

- No primitive Pythagorean triangle has an integer altitude from the hypotenuse; that is, every primitive Pythagorean triangle is indecomposable.[17]
- The set of all primitive Pythagorean triples forms a rooted ternary tree in a natural way; see Tree of primitive Pythagorean triples.

Special cases [edit]

In addition, special Pythagorean triples with certain additional properties can be guaranteed to exist:

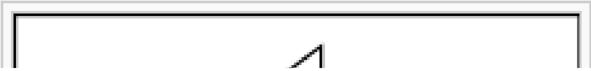
- Every integer greater than 2 that is not congruent to 2 mod 4 (in other words, every integer greater than 2 which is *not* of the form 4*n* + 2) is part of a primitive Pythagorean triple.
- Every integer greater than 2 is part of a primitive or non-primitive Pythagorean triple. For example, the integers 6, 10, 14, and 18 are not part of primitive triples, but are part of the non-primitive triples (6, 8, 10), (14, 48, 50) and (18, 80, 82).
- There exist infinitely many Pythagorean triples in which the hypotenuse and the longer of the two legs differ by exactly one (such triples are necessarily primitive). One method to generate such

triples is the relation $(2n+1)^2 + [2n(n+1)]^2 = [2n(n+1) + 1]^2$, leading to triples (3,4,5), (5,12,13), (7,24,25), etc. More generally, for every odd integer j, there exist infinitely many primitive Pythagorean triples in which the hypotenuse and the even leg differ by j^2.

- There exist infinitely many primitive Pythagorean triples in which the hypotenuse and the longer of the two legs differ by exactly two. Generalization: For every integer $k > 0$, there exist infinitely many primitive Pythagorean triples in which the hypotenuse and the odd leg differ by $2k^2$.
- There exist infinitely many Pythagorean triples in which the two legs differ by exactly one. For example, $20^2 + 21^2 = 29^2$.
- For each natural number n, there exist n Pythagorean triples with different hypotenuses and the same area.
- For each natural number n, there exist at least n different Pythagorean triples with the same leg a, where a is some natural number
- For each natural number n, there exist at least n different Pythagorean triples with the same hypotenuse.[5]:31
- There exist infinitely many Pythagorean triples with square numbers for both the hypotenuse c and the sum of the legs $a+b$. According to Fermat, the **smallest** such triple[18] has sides a = 4,565,486,027,761; b = 1,061,652,293,520; and c = 4,687,298,610,289. Here $a+b = 2{,}372{,}159^2$ and $c = 2{,}165{,}017^2$. This is generated by Euclid's formula with parameter values m = 2,150,905 and n = 246,792.
- There exist non-primitive Pythagorean triangles with integer altitude from the hypotenuse.[19][20] Such Pythagorean triangles are known as decomposable since they can be split along this altitude into two separate and smaller Pythagorean triangles.[17]

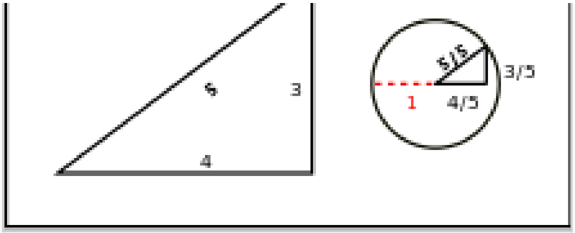
Geometry of Euclid's formula [edit]

Euclid's formulae for a Pythagorean triple



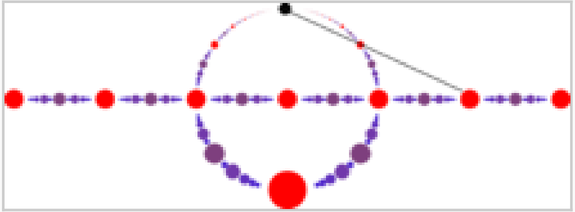
$$a = 2mn, \quad b = m^2 - n^2, \quad c = m^2 + n^2$$

can be understood in terms of the geometry of rational number points on the unit circle (Trautman 1998). To motivate this, consider a right triangle with legs *a* and *b*, and hypotenuse *c*, where *a*, *b*, and *c* are positive integers. By the Pythagorean theorem, $a^2 + b^2 = c^2$ or, dividing both sides by c^2,



3,4,5 maps to x,y point (4/5,3/5) on the unit circle

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$



The rational points on a circle correspond, under stereographic projection, to the rational points of the line.

Geometrically, the point in the Cartesian plane with coordinates

$$x = \frac{a}{c}, \quad y = \frac{b}{c}$$

is on the unit circle $x^2 + y^2 = 1$. In this equation, the coordinates *x* and *y* are given by rational numbers. Conversely, any point on the unit circle whose coordinates *x*, *y* are rational numbers gives rise to a primitive Pythagorean triple. Indeed, write *x* and *y* as fractions in lowest terms:

$$x = \frac{a}{c}, \quad y = \frac{b}{c}$$

where the greatest common divisor of *a*, *b*, and *c* is 1. Then, since *x* and *y* are on the unit circle,

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \implies a^2 + b^2 = c^2,$$

as claimed.

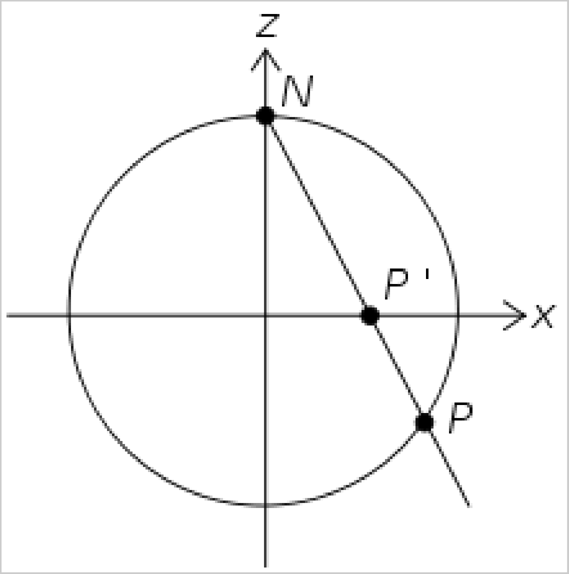
There is therefore a correspondence between points on the

There is therefore a correspondence between points on the unit circle with rational coordinates and primitive Pythagorean triples. At this point, Euclid's formulae can be derived either by methods of trigonometry or equivalently by using the stereographic projection.

For the stereographic approach, suppose that *P*′ is a point on the *x*-axis with rational coordinates

$$P' = \left(\frac{m}{n}, 0\right).$$

Then, it can be shown by basic algebra that the point *P* has coordinates



Stereographic projection of the unit circle onto the *x*-axis. Given a point *P* on the unit circle, draw a line from *P* to the point *N* = (0, 1) (the *north pole*). The point *P*′ where the line intersects the *x*-axis is the stereographic projection of *P*. Inversely, starting with a point *P*′ on the *x*-axis, and drawing a line from *P*′ to *N*, the inverse stereographic projection is the point *P* where the line intersects the unit circle.

$$P = \left(\frac{2\left(\frac{m}{n}\right)}{\left(\frac{m}{n}\right)^2+1}, \frac{\left(\frac{m}{n}\right)^2-1}{\left(\frac{m}{n}\right)^2+1}\right) = \left(\frac{2mn}{m^2+n^2}, \frac{m^2-n^2}{m^2+n^2}\right).$$

This establishes that each rational point of the *x*-axis goes over to a rational point of the unit circle. The converse, that every rational point of the unit circle comes from such a point of the *x*-axis, follows by applying the inverse stereographic projection. Suppose that *P*(*x*, *y*) is a point of the unit circle with *x* and *y* rational numbers. Then the point *P*′ obtained by stereographic projection onto the *x*-axis has coordinates

$$\left(\frac{x}{1-y}, 0\right)$$

which is rational.

In terms of algebraic geometry, the algebraic variety of rational points on the unit circle is birational to the affine line over the rational numbers. The unit circle is thus called a rational curve, and it is this fact which enables an explicit parameterization of the (rational number) points on it by means of rational functions.

Pythagorean triangles in a 2D lattice [edit]

A 2D lattice is a regular array of isolated points where if any one point is chosen as the Cartesian origin (0, 0), then all the other points are at (*x*, *y*) where *x* and *y* range over all positive and negative integers. Any Pythagorean triangle with triple (*a*, *b*, *c*) can be drawn within a 2D lattice with vertices at coordinates (0, 0), (*a*, 0) and (0, *b*). The count of lattice points lying strictly within the bounds of the triangle is given by $\frac{(a-1)(b-1)-\gcd(a,b)+1}{2}$;[21] for primitive Pythagorean triples this interior lattice count is $\frac{(a-1)(b-1)}{2}$. The area (by Pick's theorem equal to one less than the interior lattice count plus half the boundary lattice count) equals $\frac{ab}{2}$.

The first occurrence of two primitive Pythagorean triples sharing the same area occurs with triangles with sides (20, 21, 29), (12, 35, 37) and common area 210 (sequence A093536 in OEIS). The first occurrence of two primitive Pythagorean triples sharing the same interior lattice count occurs with (18108, 252685, 253333), (28077, 162964, 165365) and interior lattice count 2287674594 (sequence A225760 in OEIS). Three primitive Pythagorean triples have been found sharing the same area: (4485, 5852, 7373), (3059, 8580, 9109), (1380, 19019, 19069) with area 13123110. As yet, no set of three primitive Pythagorean triples have been found sharing the same interior lattice count.

Spinors and the modular group [edit]

Pythagorean triples can likewise be encoded into a matrix of the form

$$X = \begin{bmatrix} c+b & a \\ a & c-b \end{bmatrix}.$$

A matrix of this form is symmetric. Furthermore, the determinant of *X* is

$$\det X = c^2 - a^2 - b^2$$

which is zero precisely when (*a*,*b*,*c*) is a Pythagorean triple. If *X* corresponds to a Pythagorean triple, then as a matrix it must have rank 1.

Since *X* is symmetric, it follows from a result in linear algebra that there is a column vector ξ = [*m* *n*]^T such that the outer product

$$X = 2\begin{bmatrix} m \\ n \end{bmatrix}[m\ n] = 2\xi\xi^T \qquad (1)$$

holds, where the *T* denotes the matrix transpose. The vector ξ is called a spinor (for the Lorentz group SO(1, 2)). In abstract terms, the Euclid formula means that each primitive Pythagorean triple can be written as the outer product with itself of a spinor with integer entries, as in (**1**).

The modular group Γ is the set of 2×2 matrices with integer entries

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

with determinant equal to one: αδ − βγ = 1. This set forms a group, since the inverse of a matrix in Γ is again in Γ, as is the product of two matrices in Γ. The modular group acts on the collection of all integer spinors. Furthermore, the group is transitive on the collection of integer spinors with relatively prime entries. For if $[m\ n]^T$ has relatively prime entries, then

$$\begin{bmatrix} m & -v \\ n & u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$$

where *u* and *v* are selected (by the Euclidean algorithm) so that $mu + nv = 1$.

By acting on the spinor ξ in (**1**), the action of Γ goes over to an action on Pythagorean triples, provided one allows for triples with possibly negative components. Thus if *A* is a matrix in Γ, then

$$2(A\xi)(A\xi)^T = AXA^T \qquad (2)$$

gives rise to an action on the matrix *X* in (**1**). This does not give a well-defined action on primitive triples, since it may take a primitive triple to an imprimitive one. It is convenient at this point (per Trautman 1998) to call a triple (*a*,*b*,*c*) **standard** if *c* > 0 and either (*a*,*b*,*c*) are relatively prime or (*a*/2,*b*/2,*c*/2) are relatively prime with *a*/2 odd. If the spinor $[m\ n]^T$ has relatively prime entries, then the associated triple (*a*,*b*,*c*) determined by (**1**) is a standard triple. It follows that the action of the modular group is transitive on the set of standard triples.

Alternatively, restrict attention to those values of *m* and *n* for which *m* is odd and *n* is even. Let the subgroup Γ(2) of Γ be the kernel of the group homomorphism

$$\Gamma = \mathrm{SL}(2, \mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}_2)$$

where $\mathrm{SL}(2,\mathbf{Z}_2)$ is the special linear group over the finite field $\mathbf{Z}_2$ of integers modulo 2. Then Γ(2) is the group of unimodular transformations which preserve the parity of each entry. Thus if the first entry of ξ is odd and the second entry is even, then the same is true of *A*ξ for all $A \in \Gamma(2)$. In fact, under the action (**2**), the group Γ(2) acts transitively on the collection of primitive Pythagorean triples (Alperin 2005).

The group Γ(2) is the free group whose generators are the matrices

$$U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Consequently, every primitive Pythagorean triple can be obtained in a unique way as a product of copies of the matrices *U* and *L*.

Parent/child relationships [edit]

Main article: Tree of Pythagorean triples

By a result of Berggren (1934), all primitive Pythagorean triples can be generated from the (3, 4, 5) triangle by using the three linear transformations T1, T2, T3 below, where *a*, *b*, *c* are sides of a triple:

	new side *a*	new side *b*	new side *c*
T1:	$a - 2b + 2c$	$2a - b + 2c$	$2a - 2b + 3c$
T2:	$a + 2b + 2c$	$2a + b + 2c$	$2a + 2b + 3c$
T3:	$-a + 2b + 2c$	$-2a + b + 2c$	$-2a + 2b + 3c$

If one begins with 3, 4, 5 then all other primitive triples will eventually be produced. In other words, every primitive triple will be a “parent” to 3 additional primitive triples. Starting from the initial node with *a* = 3, *b* = 4, and *c* = 5, the next generation of triples is

new side a	new side b	new side c
3 − (2×4) + (2×5) = 5	(2×3) − 4 + (2×5) = 12	(2×3) − (2×4) + (3×5) = 13
3 + (2×4) + (2×5) = 21	(2×3) + 4 + (2×5) = 20	(2×3) + (2×4) + (3×5) = 29
−3 + (2×4) + (2×5) = 15	−(2×3) + 4 + (2×5) = 8	−(2×3) + (2×4) + (3×5) = 17

The linear transformations T1, T2, and T3 have a geometric interpretation in the language of quadratic forms. They are closely related to (but are not equal to) reflections generating the orthogonal group of $x^2 + y^2 - z^2$ over the integers. A different set of three linear transformations is discussed in Pythagorean triples by use of matrices and linear transformations. For further discussion of parent-child relationships in triples, see: Pythagorean triple (Wolfram) and (Alperin 2005).

Relation to Gaussian integers [edit]

Alternatively, Euclid's formulae can be analyzed and proven using the Gaussian integers.[22] Gaussian integers are complex numbers of the form α = $u + vi$, where u and v are ordinary integers and i is the square root of negative one. The units of Gaussian integers are ±1 and ±i. The ordinary integers are called the rational integers and denoted as **Z**. The Gaussian integers are denoted as **Z**[*i*].. The right-hand side of the Pythagorean theorem may be factored in Gaussian integers:

$$c^2 = a^2 + b^2 = (a+bi)\overline{(a+bi)} = (a+bi)(a-bi).$$

A primitive Pythagorean triple is one in which a and b are coprime, i.e., they share no prime factors in the integers. For such a triple, either a or b is even, and the other is odd; from this, it follows that c is also odd.

The two factors $z := a + bi$ and $z^* := a - bi$ of a primitive Pythagorean triple each equal the square of a Gaussian integer. This can be proved using the property that every Gaussian integer can be factored uniquely into Gaussian primes up to units.[23] (This unique factorization follows from the fact that, roughly speaking, a version of the Euclidean algorithm can be defined on them.) The proof has three steps. First, if a and b share no prime factors in the integers, then they also share no prime factors in the Gaussian integers. (Assume $a = gu$ and $b = gv$ with Gaussian integers g, u and v and g not a unit. Then u and v lie on the same line through the origin. All Gaussian integers on such a line are integer multiples of some Gaussian integer h. But then the integer $gh \neq \pm 1$ divides both a and b.) Second, it follows that z and z^* likewise share no prime factors in the Gaussian integers. For if they did, then their common divisor δ would also divide $z + z^* = 2a$ and $z - z^* = 2ib$. Since a and b are coprime, that implies that δ divides $2 = (1 + i)(1 - i) = i(1 - i)^2$. From the formula $c^2 = zz^*$, that in turn would imply that c is even, contrary to the hypothesis of a primitive Pythagorean triple. Third, since c^2 is a square, every Gaussian prime in its factorization is doubled, i.e., appears an even number of times. Since z and z^* share no prime factors, this doubling is also true for them. Hence, z and z^* are squares.

Thus, the first factor can be written

$$a + bi = \varepsilon\left(m + ni\right)^2, \quad \varepsilon \in \{\pm 1, \pm i\}.$$

The real and imaginary parts of this equation give the two formulas:

$$\begin{cases} \varepsilon = +1, & a = +\left(m^2 - n^2\right), \quad b = +2mn; \\ \varepsilon = -1, & a = -\left(m^2 - n^2\right), \quad b = -2mn; \\ \varepsilon = +i, & a = -2mn, \quad b = +\left(m^2 - n^2\right); \\ \varepsilon = -i, & a = +2mn, \quad b = -\left(m^2 - n^2\right). \end{cases}$$

For any primitive Pythagorean triple, there must be integers m and n such that these two equations are satisfied. Hence, every Pythagorean triple can be generated from some choice of these integers.

As perfect square Gaussian integers [edit]

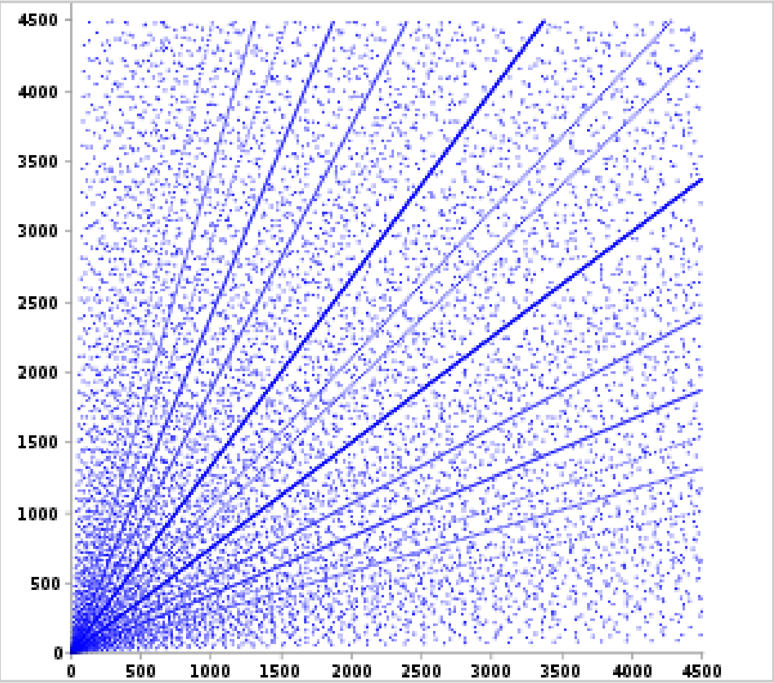
If we consider the square of a Gaussian integer we get the following direct interpretation of Euclid's formulae as representing a perfect square Gaussian integers.

$$(m + ni)^2 = (m^2 - n^2) + 2mni.$$

Using the facts that the Gaussian integers are a Euclidean domain and that for a Gaussian integer p $|p|^2$ is always a square it is possible to show that a Pythagorean triples correspond to the square of a prime Gaussian integer if the hypotenuse is prime.

If the Gaussian integer is not prime then it is the product of two Gaussian integers p and q with $|p|^2$ and $|q|^2$ integers. Since magnitudes multiply in the Gaussian integers, the product must be $|p||q|$, which when squared to find a Pythagorean triple must be composite. The contrapositive completes the proof.

Distribution of triples [edit]



A scatter plot of the legs (*a*,*b*) of the first Pythagorean triples with *a* and *b* less than 4500.

There are a number of results on the distribution of Pythagorean triples. In the scatter plot, a number of obvious patterns are already apparent. Whenever the legs (*a*,*b*) of a primitive triple appear in the plot, all integer multiples of (*a*,*b*) must also appear in the plot, and this property produces the appearance of lines radiating from the origin in the diagram.

Within the scatter, there are sets of parabolic patterns with a high density of points and all their foci at the origin, opening up in all four directions. Different parabolas intersect at the axes and appear to reflect off the axis with an incidence angle of 45 degrees, with a third parabola entering in a perpendicular fashion. Within this quadrant, each arc centered on the origin shows that section of the parabola that lies between its tip and its intersection with its semi-latus rectum.

These patterns can be explained as follows. If $a^2/4n$ is an integer, then (*a*, $|n - a^2/4n|$, $n + a^2/4n$) is a Pythagorean triple. (In fact every Pythagorean triple (*a*, *b*, *c*) can be written in this way with integer *n*, possibly after exchanging *a* and *b*, since $n = (b + c)/2$ and *a* and *b* cannot both be odd.) The Pythagorean triples thus lie on curves given by $b = |n - a^2/4n|$, that is, parabolas reflected at the *a*-axis, and the corresponding curves with *a* and *b* interchanged. If *a* is varied for a given *n* (i.e. on a given parabola), integer values of *b* occur relatively frequently if *n* is a square or a small multiple of a square. If several such values happen to lie close together, the corresponding parabolas approximately coincide, and the triples cluster in a narrow parabolic strip. For instance, 38^2 = 1444, 2 × 27^2 = 1458, 3 × 22^2 = 1452, 5 × 17^2 = 1445 and 10 × 12^2 = 1440; the corresponding parabolic strip around *n* ≈ 1450 is clearly visible in the scatter plot.

The angular properties described above follow immediately from the functional form of the parabolas. The parabolas are reflected at the *a*-axis at *a* = 2*n*, and the derivative of *b* with respect to *a* at this point is –1; hence the incidence angle is 45°. Since the clusters, like all triples, are repeated at integer multiples, the value 2*n* also corresponds to a cluster. The corresponding parabola intersects the *b*-axis at right angles at *b* = 2*n*, and hence its reflection upon interchange of *a* and *b* intersects the *a*-axis at right angles at *a* = 2*n*, precisely where the parabola for *n* is reflected at the *a*-axis. (The same is of course true for *a* and *b* interchanged.)

Albert Fässler and others provide insights into the significance of these parabolas in the context of conformal mappings.[24][25]

Special cases [edit]

The Platonic sequence [edit]

The case *n* = 1 of the more general construction of Pythagorean triples has been known for a long time. Proclus, in his commentary to the 47th Proposition of the first book of Euclid's Elements, describes it as follows:

> Certain methods for the discovery of triangles of this kind are handed down, one which they refer to Plato, and another to Pythagoras. (The latter) starts from odd numbers. For it makes the odd number the smaller of the sides about the right angle; then it takes the square of it, subtracts unity and makes half the difference the greater of the sides about the right angle; lastly it adds unity to this and so forms the remaining side, the hypotenuse. ...For the method of Plato argues from even numbers. It takes the given even number and makes it one of the sides about the right angle; then, bisecting this number and squaring the half, it adds unity to the square to form the hypotenuse, and subtracts unity from the square to form the other side about the right angle. ... Thus it has formed the same triangle that which was obtained by the other method.

In equation form, this becomes:

a is odd (Pythagoras, c. 540 BC):

$$\text{side } a : \text{side } b = \frac{a^2 - 1}{2} : \text{side } c = \frac{a^2 + 1}{2}.$$

a is even (Plato, c. 380 BC):

$$\text{side } a : \text{side } b = \left(\frac{a}{2}\right)^2 - 1 : \text{side } c = \left(\frac{a}{2}\right)^2 + 1$$

It can be shown that all Pythagorean triples can be obtained, with appropriate rescaling, from the basic Platonic sequence (a, $(a^2 - 1)/2$ and $(a^2 + 1)/2$) by allowing a to take non-integer rational values. If a is replaced with the fraction m/n in the sequence, the result is equal to the 'standard' triple generator ($2mn$, $m^2 - n^2$,$m^2 + n^2$) after rescaling. It follows that every triple has a corresponding rational a value which can be used to generate a similar triangle (one with the same three angles and with sides in the same proportions as the original). For example, the Platonic equivalent of (56, 33, 65) is generated by $a = m/n = 7/4$ as (a, $(a^2 -1)/2$, $(a^2+1)/2$) = (56/32, 33/32, 65/32). The Platonic sequence itself can be derived[clarification needed] by following the steps for 'splitting the square' described in Diophantus II.VIII.

The Jacobi-Madden equation [edit]

Main article: Jacobi-Madden equation

The equation,

$$a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$$

is equivalent to the special Pythagorean triple,

$$(a^2 + ab + b^2)^2 + (c^2 + cd + d^2)^2 = ((a + b)^2 + (a + b)(c + d) + (c + d)^2)^2$$

There is an infinite number of solutions to this equation as solving for the variables involves an elliptic curve. Small ones are,

$$a, b, c, d = -2634, 955, 1770, 5400$$

$$a, b, c, d = -31764, 7590, 27385, 48150$$

Equal sums of two squares [edit]

One way to generate solutions to $a^2 + b^2 = c^2 + d^2$ is to parametrize *a, b, c, d* in terms of integers *m, n, p, q* as follows:[26]

$$(m^2 + n^2)(p^2 + q^2) = (mp - nq)^2 + (np + mq)^2 = (mp + nq)^2 + (np - mq)^2.$$

Equal sums of two fourth powers [edit]

Given two sets of Pythagorean triples,

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2$$

$$(c^2 - d^2)^2 + (2cd)^2 = (c^2 + d^2)^2$$

the problem of finding equal products of a non-hypotenuse side and the hypotenuse,

$$(a^2-b^2)(a^2+b^2)=(c^2-d^2)(c^2+d^2)$$

is easily seen to be equivalent to the equation,

$$a^4-b^4=c^4-d^4$$

and was first solved by Euler as $a,b,c,d=133,59,158,134$. Since he showed this is a rational point in an elliptic curve, then there is an infinite number of solutions. In fact, he also found a 7th degree polynomial parameterization.

Descartes' Circle Theorem [edit]

For the case of Descartes' circle theorem where all variables are squares,

$$2(a^4+b^4+c^4+d^4)=(a^2+b^2+c^2+d^2)^2$$

Euler showed this is equivalent to three simultaneous Pythagorean triples,

$$(2ab)^2+(2cd)^2=(a^2+b^2-c^2-d^2)^2$$
$$(2ac)^2+(2bd)^2=(a^2-b^2+c^2-d^2)^2$$
$$(2ad)^2+(2bc)^2=(a^2-b^2-c^2+d^2)^2$$

There is also an infinite number of solutions, and for the special case when $a+b=c$, then the equation simplifies to,

$$4(a^2+ab+b^2)=d^2$$

with small solutions as $a,b,c,d=3,5,8,14$ and can be solved as binary quadratic forms.

Almost-isosceles Pythagorean triples [edit]

No Pythagorean triples are isosceles, because the ratio of the hypotenuse to either other side is √2, but √2 cannot be expressed as the ratio of 2 integers.

There are, however, right-angled triangles with integral sides for which the lengths of the non-hypotenuse sides differ by one, such as,

$$3^2+4^2=5^2$$

$$20^2+21^2=29^2$$

and an infinite number of others. They can be completely parameterized as,

$$\left(\tfrac{x-1}{2}\right)^2+\left(\tfrac{x+1}{2}\right)^2=y^2$$

where {*x, y*} are the solutions to the Pell equation $x^2-2y^2=-1$.

When it is the longer non-hypotenuse side and hypotenuse that differ by one, such as in

$$5^2+12^2=13^2$$

$$7^2+24^2=25^2$$

then the complete solution is

$$(2m+1)^2+(2m^2+2m)^2=(2m^2+2m+1)^2$$

which also shows that all odd numbers (greater than 1) appear in a primitive Pythagorean triple.

Generalizations [edit]

There are several ways to generalize the concept of Pythagorean triples.

Pythagorean quadruple [edit]

Main article: Pythagorean quadruple

A set of four positive integers *a*, *b*, *c* and *d* such that $a^2 + b^2 + c^2 = d^2$ is called a Pythagorean quadruple. The simplest example is (1, 2, 2, 3), since $1^2 + 2^2 + 2^2 = 3^2$. The next simplest (primitive) example is (2, 3, 6, 7), since $2^2 + 3^2 + 6^2 = 7^2$.

All quadruples are given by the formula

$$(m^2+n^2-p^2-q^2)^2+(2mq+2np)^2+(2nq-2mp)^2=(m^2+n^2+p^2+q^2)^2.$$

Pythagorean *n*-tuple [edit]

Using the simple algebraic identity,

$$(x_1^2-x_0)^2+(2x_1)^2x_0=(x_1^2+x_0)^2$$

for arbitrary x_0, x_1, it is easy to prove that the square of the sum of n squares is itself the sum of n squares by letting $x_0 = x_2^2 + x_3^2 + \ldots + x_n^2$ and then distributing terms.[27] One can see how Pythagorean triples and quadruples are just the particular cases $x_0 = x_2^2$ and $x_0 = x_2^2 + x_3^2$, respectively, and so on for other n, with quintuples given by

$$(a^2-b^2-c^2-d^2)^2+(2ab)^2+(2ac)^2+(2ad)^2=(a^2+b^2+c^2+d^2)^2.$$

Since the sum $F(k,m)$ of k consecutive squares beginning with m^2 is given by the formula,[28]

$$F(k,m)=km(k-1+m)+\frac{k(k-1)(2k-1)}{6}$$

one may find values (k, m) so that $F(k,m)$ is a square, such as one by Hirschhorn where the number of terms is itself a square,[29]

$$m=\tfrac{v^4-24v^2-25}{48},\ k=v^2,\ F(m,k)=\tfrac{v^5+47v}{48}$$

and $v \geq 5$ is any integer not divisible by 2 or 3. For the smallest case $v = 5$, hence $k = 25$, this yields the well-known cannonball-stacking problem of Lucas,

$$0^2+1^2+2^2+\ldots+24^2=70^2$$

a fact which is connected to the Leech lattice.

In addition, if in a Pythagorean n-tuple ($n \geq 4$) all addends are consecutive except one, one can use the equation,[30]

$$F(k,m)+p^2=(p+1)^2$$

Since the second power of p cancels out, this is only linear and easily solved for as $p=\frac{F(k,m)-1}{2}$ though k, m should be chosen so that p is an integer, with a small example being $k = 5$, $m = 1$ yielding,

$$1^2+2^2+3^2+4^2+5^2+27^2=28^2$$

Thus, one way of generating Pythagorean n-tuples is by using, for various x,[31]

$$x^2+(x+1)^2+\cdots+(x+q)^2+p^2=(p+1)^2,$$

where $q = n–2$ and where

$$p=\frac{(q+1)x^2+q(q+1)x+\frac{q(q+1)(2q+1)}{6}-1}{2}.$$

Fermat's Last Theorem [edit]

Main article: Fermat's Last Theorem

A generalization of the concept of Pythagorean triples is the search for triples of positive integers a, b, and c, such that $a^n + b^n = c^n$, for some n strictly greater than 2. Pierre de Fermat in 1637 claimed that no such triple exists, a claim that came to be known as Fermat's Last Theorem because it took longer than any other conjecture by Fermat to be proven or disproven. The first proof was given by Andrew Wiles in 1994.

n − 1 or *n* *n*th powers summing to an *n*th power [edit]

Main article: Euler's sum of powers conjecture

Another generalization is searching for sequences of $n + 1$ positive integers for which the n^{th} power of the last is the sum of the n^{th} powers of the previous terms. The smallest sequences for known values of n are:

- $n = 3$: {3, 4, 5; 6}.
- $n = 4$: {30, 120, 272, 315; 353}
- $n = 5$: {19, 43, 46, 47, 67; 72}
- $n = 7$: {127, 258, 266, 413, 430, 439, 525; 568}
- $n = 8$: {90, 223, 478, 524, 748, 1088, 1190, 1324; 1409}

For the n=3 case, in which $x^3 + y^3 + z^3 = w^3,$ called the Fermat cubic, a general formula exists giving all solutions.

A slightly different generalization allows the sum of $(k + 1)$ nth powers to equal the sum of $(n - k)$ nth powers. For example:

- ($n = 3$): $1^3 + 12^3 = 9^3 + 10^3$, made famous by Hardy's recollection of a conversation with Ramanujan about the number 1729 being the smallest number that can be expressed as a sum of two cubes in two distinct ways.

There can also exist $n - 1$ positive integers whose n^{th} powers sum to an n^{th} power (though, by Fermat's last theorem, not for $n = 3$); these are counterexamples to Euler's sum of powers conjecture. The smallest known counterexamples are[32][33][11]

- $n = 4$: (95800, 217519, 414560; 422481)
- $n = 5$: (27, 84, 110, 133; 144)

Heronian triangle triples [edit]

Main article: Heronian triangle

A **Heronian triangle** is commonly defined as one with integer sides whose area is also an integer, and we shall consider Heronian triangles with *distinct* integer sides. The lengths of the sides of such a triangle form a **Heronian triple** (a, b, c) provided $a < b < c$. Clearly, any Pythagorean triple is a Heronian triple, since in a Pythagorean triple at least one of the legs a, b must be even, so that the area $ab/2$ is an integer. Not every Heronian triple is a Pythagorean triple, however, as the example (4, 13, 15) with area 24 shows.

If (a, b, c) is a Heronian triple, so is (ma, mb, mc) where m is any positive integer greater than one. The Heronian triple (a, b, c) is **primitive** provided a, b, c are pairwise relatively prime (as with a Pythagorean triple). Here are a few of the simplest primitive Heronian triples that are not Pythagorean triples:

(4, 13, 15) with area 24
(3, 25, 26) with area 36
(7, 15, 20) with area 42
(6, 25, 29) with area 60
(11, 13, 20) with area 66
(13, 14, 15) with area 84
(13, 20, 21) with area 126

By Heron's formula, the extra condition for a triple of positive integers (a, b, c) with $a < b < c$ to be Heronian is that

$$(a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4)$$

or equivalently

$$2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$$

be a nonzero perfect square divisible by 16.

Application to cryptography [edit]

Primitive Pythagorean triples have been used in cryptography as random sequences and for the generation of keys.[34]

See also [edit]

- Congruum
- Diophantus II.VIII
- Eisenstein triple
- Euler brick
- Heronian triangle
- Hilbert's theorem 90
- Integer triangle
- Modular arithmetic
- Nonhypotenuse number
- Plimpton 322
- Pythagorean prime
- Pythagorean quadruple
- Tangent half-angle formula
- Trigonometric identity

Notes [edit]

1. ^ Joyce, D. E. (June 1997), "Book X , Proposition XXIX", *Euclid's Elements*, Clark University
2. ^ Mitchell, Douglas W. (July 2001), "An Alternative Characterisation of All Primitive Pythagorean Triples", *The Mathematical Gazette* **85** (503): 273–5, doi:10.2307/3622017, JSTOR 3622017

3. ^ Beauregard, Raymond A.; Suryanarayan, E. R. (2000), "Parametric representation of primitive Pythagorean triples", in Nelsen, Roger B., *Proofs Without Words: More Exercises in Visual Thinking* **II**, Mathematical Association of America, p. 120, ISBN 978-0-88385-721-2, OCLC 807785075
4. ^ Maor, Eli, *The Pythagorean Theorem*, Princeton University Press, 2007: Appendix B.
5. ^ *a b c d e* Sierpiński, Wacław (2003), *Pythagorean Triangles*, Dover, ISBN 978-0-486-43278-6
6. ^ Houston, David (1993), "Pythagorean triples via double-angle formulas", in Nelsen, Roger B., *Proofs Without Words: Exercises in Visual Thinking*, Mathematical Association of America, p. 141, ISBN 978-0-88385-700-7, OCLC 29664480
7. ^ Posamentier, Alfred S. (2010), *The Pythagorean Theorem: The Story of Its Power and Beauty*, Prometheus Books, p. 156, ISBN 9781616141813.
8. ^ For the nonexistence of solutions where *a* and *b* are both square, originally proved by Fermat, see Koshy, Thomas (2002), *Elementary Number Theory with Applications*, Academic Press, p. 545, ISBN 9780124211711. For the other case, in which *c* is one of the squares, see Stillwell, John (1998), *Numbers and Geometry*, Undergraduate Texts in Mathematics, Springer, p. 133, ISBN 9780387982892.
9. ^ *a b c* Carmichael, R. D., 1914, "Diophantine analysis," in second half of R. D. Carmichael, *The Theory of Numbers and Diophantine Analysis*, Dover Publ., 1959.
10. ^ Sierpiński 2003, pp. 4–6
11. ^ *a b* MacHale, Des; van den Bosch, Christian (March 2012), "Generalising a result about Pythagorean triples", *Mathematical Gazette* **96**: 91–96
12. ^ Sally, Judith D. (2007), *Roots to Research: A Vertical Development of Mathematical Problems*, American Mathematical Society, pp. 74–75, ISBN 9780821872673.
13. ^ This follows immediately from the fact that one of *a* or *b* is divisible by four, together with the definition of congruent numbers as the areas of rational-sided right triangles. See e.g. Koblitz, Neal (1993), *Introduction to Elliptic Curves and Modular Forms*, Graduate Texts in Mathematics **97**, Springer, p. 3, ISBN 9780387979663.
14. ^ Baragar, Arthur (2001), *A Survey of Classical and Modern Geometries: With Computer Activities*, Prentice Hall, Exercise 15.3, p. 301, ISBN 9780130143181

15. ^ a b Bernhart, Frank R.; Price, H. Lee (2005). "Heron's formula, Descartes circles, and Pythagorean triangles". arXiv:math/0701624.
16. ^ Rosenberg, Steven; Spillane, Michael; Wulf, Daniel B. (May 2008), "Heron triangles and moduli spaces", *Mathematics Teacher* **101**: 656–663
17. ^ a b Yiu, Paul (2008), *Heron triangles which cannot be decomposed into two integer right triangles* (PDF), 41st Meeting of Florida Section of Mathematical Association of America, p. 17
18. ^ Pickover, Clifford A. (2009), "Pythagorean Theorem and Triangles", *The Math Book*, Sterling, p. 40, ISBN 1402757964
19. ^ Voles, Roger, "Integer solutions of $a^{-2}+b^{-2}=d^{-2}$," *Mathematical Gazette* 83, July 1999, 269–271.
20. ^ Richinick, Jennifer, "The upside-down Pythagorean Theorem," *Mathematical Gazette* 92, July 2008, 313–317.
21. ^ Yiu, Paul. "RecreationalMathematics" (PDF). *Course Notes, Chapter 2, page 110, Dept. of Mathematical Sciences, Florida Atlantic University (2003)*.
22. ^ Stillwell, John (2002), "6.6 Pythagorean Triples", *Elements of Number Theory*, Springer, pp. 110–2, ISBN 978-0-387-95587-2
23. ^ Gauss CF (1832), "Theoria residuorum biquadraticorum", *Comm. Soc. Reg. Sci. Gött. Rec.* **4**. See also *Werke*, **2**:67–148.
24. ^ 1988 Preprint See Figure 2 on page 3., later published as Fässler, Albert (June–July 1991), "Multiple Pythagorean number triples", *American Mathematical Monthly* **98** (6): 505–517, doi:10.2307/2324870, JSTOR 2324870
25. ^ Benito, Manuel; Varona, Juan L. (June 2002), "Pythagorean triangles with legs less than *n*", *Journal of Computational and Applied Mathematics* **143**: 117–126, doi:10.1016/S0377-0427(01)00496-4 as PDF
26. ^ Nahin, Paul. *An Imaginary Tale: The Story of* $\sqrt{-1}$, pp. 25–26.
27. ^ "A Collection of Algebraic Identities: Sums of n Squares".
28. ^ "Sum of consecutive cubes equal a cube".
29. ^ Hirschhorn, Michael (November 2011), "When is the sum of consecutive squares a square?", *The Mathematical Gazette* **95**: 511–2, ISSN 0025-5572, OCLC 819659848

30. ^ Goehl, John F. Jr. (May 2005), "Reader reflections", *Mathematics Teacher* **98** (9): 580
31. ^ Goehl, John F., Jr., "Triples, quartets, pentads", *Mathematics Teacher* 98, May 2005, p. 580.
32. ^ Kim, Scott (May 2002), "Bogglers", *Discover*: 82, "The equation $w^4 + x^4 + y^4 = z^4$ is harder. In 1988, after 200 years of mathematicians' attempts to prove it impossible, Noam Elkies of Harvard found the counterexample, $2{,}682{,}440^4 + 15{,}365{,}639^4 + 18{,}796{,}760^4 = 20{,}615{,}673^4$."
33. ^ Elkies, Noam (1988), "On $A^4 + B^4 + C^4 = D^4$", *Mathematics of Computation* **51**: 825–835, MR 930224
34. ^ Kak, S. and Prabhu, M. Cryptographic applications of primitive Pythagorean triples. Cryptologia, 38:215-222, 2014. [1]

References [edit]

- Alperin, Roger C. (2005), "The modular tree of Pythagoras" (PDF), *American Mathematical Monthly* (Mathematical Association of America) **112** (9): 807–816, doi:10.2307/30037602, JSTOR 30037602, MR 2179860
- Berggren, B. (1934), "Pytagoreiska trianglar", *Tidskrift för elementär matematik, fysik och kemi* (in Swedish) **17**: 129–139
- Barning, F.J.M. (1963), "Over pythagorese en bijna-pythagorese driehoeken en een generatieproces met behulp van unimodulaire matrices" (PDF), *Math. Centrum Amsterdam Afd. Zuivere Wisk.* (in Dutch), ZW-011: 37
- Eckert, Ernest (1992), "Primitive Pythagorean triples", *The College Mathematics Journal* (Mathematical Association of America) **23** (5): 413–7, doi:10.2307/2686417, JSTOR 2686417
- Elkies, Noam, *Pythagorean triples and Hilbert's theorem 90* (PDF)
- Heath, Thomas (1956), *The Thirteen Books of Euclid's Elements Vol. 1 (Books I and II)* (2nd ed.), Dover Publications, ISBN 0-486-60088-2
- Martin, Artemas (1875), "Rational right angled triangles nearly isosceles", *The Analyst* (Annals of Mathematics) **3** (2): 47–50, doi:10.2307/2635906, JSTOR 2635906

- McCullough, Darryl (2005), "Height and excess of Pythagorean triples" (PDF), *Mathematics Magazine* **78** (1)
- Romik, Dan (2004), *The dynamics of Pythagorean triples*, p. 6512, arXiv:math.DS/0406512, Bibcode:2004math......6512R
- Teigen, M. G.; Hadwin, D. W. (1971), "On Generating Pythagorean Triples", *The American Mathematical Monthly* (Mathematical Association of America) **78** (4): 378–9, doi:10.2307/2316903, JSTOR 2316903
- Trautman, Andrzej (1998), "Pythagorean spinors and Penrose twistors", in S.A. Hugget, L.J. Mason, K.P. Tod, S.T. Tsou, N.M.J. Woodhouse, *Geometric universe* (Postscript)

External links [edit]

- Weisstein, Eric W., "Pythagorean Triple", *MathWorld*.
- Pythagorean Triples at cut-the-knot Interactive Applet showing unit circle relationships to Pythagorean Triples
- The Trinary Tree(s) underlying Primitive Pythagorean Triples at cut-the-knot
- Theoretical properties of the Pythagorean Triples and connections to geometry
- Clifford Algebras and Euclid's Parameterization of Pythagorean triples
- Pythagorean Triplets
- Discussion of Properties of Pythagorean triples, Interactive Calculators, Puzzles and Problems
- Generating Pythagorean Triples Using Arithmetic Progressions
- Parameterization of Pythagorean Triples by a single triple of polynomials
- Curious Consequences of a Miscopied Quadratic
- Solutions to Quadratic Compatible Pairs in relation to Pythagorean Triples
- The negative Pell equation and Pythagorean triples
- The Remarkable Incircle of a Triangle

- Interactive Calculator for Pythagorean Triples
- Price, H. Lee (2008), *The Pythagorean Tree: A New Species* **0809**, p. 4324, arXiv:0809.4324, Bibcode:2008arXiv0809.4324P
- Pythagorean Triples and the Unit Circle, chap. 2-3, in "A Friendly Introduction to Number Theory" by Joseph H. Silverman, 3rd ed., 2006, Pearson Prentice Hall, Upper Saddle River, NJ, ISBN 0-13-186137-9

Categories: Arithmetic problems of plane geometry | Triangle geometry | Diophantine equations