

statement about limited
detector sensitivity.

Stochastic GW Backgrounds

- Consider a plane-wave expansion of the GW metric perturbation.

$$h_{ij}(t, \vec{x}) = \sum_A \int_{f_0}^{\infty} df \int d\Omega \cdot h_A(f, \hat{\omega}) e^{2\pi i f(t - \hat{\omega} \cdot \vec{x})} e_A^A(\hat{\omega})$$

←
spatial
indices

where $\hat{\omega} =$ GW propagation direction.

$e_{ij}^A =$ polarization basis tensor.

h_A = polarization amplitude.

REMEMBER
THIS?

$$z(b, \hat{\omega}) = \frac{1}{2} \frac{p_i p_j}{(1 + \hat{\omega} \cdot \hat{p})} \cdot \Delta h_{ij}$$

$$\Rightarrow \Delta h_{ij} = \int_{-\infty}^{\infty} df [e^{2\pi i ft} (e^{-2\pi i fL(1 + \hat{\omega} \cdot \hat{p})} - 1) \times \sum_A h_A(f, \hat{\omega}) e_A^A(\hat{\omega})]$$

FOURIER DOMAIN ↳ $\hat{h}_{ij}(f, \hat{\omega}) = (e^{-2\pi i fL(1 + \hat{\omega} \cdot \hat{p})} - 1) \times \sum_A h_A(f, \hat{\omega}) e_A^A(\hat{\omega})$

$$\tilde{z}(f, \hat{\omega}) = (e^{-2\pi i fL(1 + \hat{\omega} \cdot \hat{p})} - 1) \times \sum_A h_A(f, \hat{\omega}) F^A(\hat{\omega})$$

... where $F^A(\hat{\omega}) = \frac{1}{2} \frac{p_i p_j}{(1 + \hat{\omega} \cdot \hat{p})} \times e_{ij}^A(\hat{\omega})$

antenna response function, $\{+, \times\}$ ✓

also holds for stochastic
GW signals across spectrum

- We are sensitive to GW signals from the entire population of radiating SM PBHs (or cosmological processes), producing a stochastic (GW) background.

\rightarrow If $\tilde{h}_A(f, \hat{\omega})$ can be treated as zero-mean Gaussian random fields, then signal is fully described by its second moments i.e., VARIANCE

$$\langle \tilde{h}_A(f, \hat{\omega}) \tilde{h}_A^*(f', \hat{\omega}') \rangle = \delta(f-f') \delta^2(\hat{\omega}, \hat{\omega}') P_{AA}(f, \hat{\omega})$$

VARIANCE (MEAN = 0) STATIONARY DIRECTIONS UNCORRELATED

Polarization Tensor

$$P_{AA'}(f, \hat{\omega}) = \begin{pmatrix} I(f, \hat{\omega}) + Q(f, \hat{\omega}) & U(f, \hat{\omega}) - iV(f, \hat{\omega}) \\ U(f, \hat{\omega}) + iV(f, \hat{\omega}) & I(f, \hat{\omega}) - Q(f, \hat{\omega}) \end{pmatrix}$$

{ I, Q, U, V } = STOKES PARAMETERS

$$I(f, \hat{\omega}) = \frac{1}{2} \langle |\tilde{h}_+|^2 + |\tilde{h}_x|^2 \rangle$$

INTENSITY

$$Q(f, \hat{\omega}) = \frac{1}{2} \langle |\tilde{h}_+|^2 - |\tilde{h}_x|^2 \rangle$$

LINEAR POLARIZATION

$$U(f, \hat{\omega}) = \operatorname{Re} \langle \tilde{h}_+ \tilde{h}_x^* \rangle$$

$$V(f, \hat{\omega}) = \operatorname{Im} \langle \tilde{h}_+ \tilde{h}_x^* \rangle$$

CIRCULAR POLARIZATION

→ Astrophysical GWBs should be unpolarized
 i.e., $Q = U = V = O$

$$\langle \tilde{h}_A(f, \hat{\alpha}) \tilde{h}_A^*(f, \hat{\alpha}') \rangle = \frac{\delta_{AA'}}{2} \frac{\delta(f-f')}{2} \frac{\delta(\hat{\alpha}, \hat{\alpha}')}{4\pi} S_h(f) P(\hat{\alpha})$$

-- where $S_h(f)$ = 1-sided power spectral density
 of Fourier modes
 $P(\hat{\alpha})$ = distribution of intensity over sky ($= 1 A \hat{\alpha}$ if isotropic)

→ Let's assume isotropy for now --

$$\underbrace{\langle \tilde{z}_a(f) \tilde{z}_b^*(f) \rangle}_{\text{pulsars}} = \iint_{S^2 S^2} d\hat{\alpha} d\hat{\alpha}' \left[e^{-2\pi i f L_a(\hat{\alpha}, \hat{p}_a)} \right] \left[e^{2\pi i f L_b(\hat{\alpha}, \hat{p}_b)} \right]$$

$$\times \left\langle \sum_A h_A(f, \hat{\alpha}) F_A^A(\hat{\alpha}) \sum_{A'} h_{A'}^*(f, \hat{\alpha}') F_{A'}^A(\hat{\alpha}') \right\rangle$$

$$\begin{aligned} \langle \tilde{z}_a(f) \tilde{z}_b^*(f) \rangle &= \frac{1}{2} \delta(f-f') S_z(f)_{ab} \\ &= \frac{1}{2} \delta(f-f') S_h(f) \int_{S^2} \frac{d^2 \hat{\alpha}}{8\pi} K_{ab}(f, \hat{\alpha}) \\ &\quad \times \sum_{A=+, \times} F_A^A(\hat{\alpha}) F_b^A(\hat{\alpha}) \end{aligned}$$

$$K_{ab}(f, \hat{\alpha}) \equiv \left[e^{-2\pi i f L_a(f, \hat{\alpha}, \hat{p}_a)} \right] \left[e^{2\pi i f L_b(f, \hat{\alpha}, \hat{p}_b)} \right]$$

$$\text{Thus, } S_z(f)_{ab} = \frac{1}{2} S_h(\ell) \int_{S^2} \frac{d^2\hat{\ell}}{4\pi} K_{ab}(f, \hat{\ell}) \sum_{A=t,x} F_a^A(\hat{\ell}) F_b^A(\hat{\ell})$$

UNITS = [time]

WHAT IS $K_{ab}(f, \hat{\ell})$?

\Rightarrow controls how rapidly the pulsar terms spatially decorrelate.

$$f \sim 10^{-9} \text{ Hz}; L \sim 100 \text{ pc} \quad \left. \begin{array}{l} fL \\ \end{array} \right\} fL > 10$$

$\therefore e^{(....)}$ oscillate rapidly on sky, contributing negligibly to integral.
--- except when pulsars are IDENTICAL

$$\Rightarrow K_{ab}(f, \hat{\ell}) \rightarrow 2 \text{ when } a=b \\ \rightarrow 4 \text{ when } a \neq b$$

WHAT IS $\int_{S^2} \frac{d^2\hat{\ell}}{4\pi} \sum_A F_a^A F_b^A$?

\Rightarrow overlap reduction function (ORF)

$$\tilde{F}_{ab} = \int_{S^2} \frac{d^2\hat{\ell}}{4\pi} \sum_{A=t,x} F_a^A(\hat{\ell}) F_b^A(\hat{\ell})$$

contour integral

$$= x_{ab} \ln(x_{ab}) - \frac{1}{6} x + \frac{1}{3}$$

... where $x_{ab} = \frac{1}{2}(1 - \cos \theta_{ab})$

Let's re-normalize this so that $\Gamma_{ab} = 1$ for $a=b$

$$\Rightarrow \boxed{\Gamma_{ab} = \frac{3}{2} x_{ab} \ln(x_{ab}) - \frac{1}{4} x_{ab} + \frac{1}{2} + \frac{1}{2} \sum_{ab}}$$

HELLINGS & DAWNS CURVE.

SHOW HD CURVE

- ↳ * $a \neq b$, $\Gamma_{ab} \leq 0.5$
- * quadrupolar, but not perfectly
- * $\sqrt{\Gamma_{ab}} (\theta = 90^\circ) = 0.25$
- * $\langle +\hat{e}_p \rangle$ introduces preferred direction.

NOW $\Gamma_{ab} = \sum_{l=0}^{\infty} a_l P_l(\cos \theta_{ab})$

... where $a \neq b$, $a_0 = 0 = a_1$

$$a_l = \frac{3}{2} \frac{(l-2)!}{(l+2)!} (2l+1)$$

SHOW LEGENDRE SPECTRUM

- ↳ * HD curve orthogonal to monopole/dipole.
- * achieves \propto precision + uniform pulsar coverage.
- * $a_2 / \sum_l a_l = 0.63$, $a_3 / \sum_l a_l = 0.17$

→ back to spectrum: $S_z(f)_{ab} = \frac{1}{3} \Gamma_{ab} S_h(f)$

→ we don't measure pulse arrival rates, we measure pulse arrival times.

$$R(t) = \int_0^t z(t') dt'$$

↳ brings down factors of $1/2\pi i f$

Thus $S_t(f)_{ab} = \frac{S_z(f)_{ab}}{4\pi^2 f^2}$

$$\downarrow = \Gamma_{ab} \frac{S_h(f)}{12\pi^2 f^2}$$

$$S_t(f)_{ab} = \Gamma_{ab} \frac{h_c^2(f)}{12\pi^2 f^3}$$

UNITS [time]³

→ $h_c(f) = A (f/f_{yr})^\alpha$

$\alpha = -2/3$
SMBBts

$\alpha = 0$
RELIC GWS

Finally $S_t(f)_{ab} = \Gamma_{ab} \frac{A^2}{12\pi^2} \times \left(\frac{f}{f_{yr}}\right)^{-\gamma}$... $\gamma \equiv 3 - 2\alpha$

$\begin{cases} 1/3 & \text{SMBBts} \\ 5 & \text{RELIC} \end{cases}$

Spectrum of GW Source populations

$$\Omega_{\text{sewb}}(f) = \frac{1}{\rho_c} \frac{dp}{df} \quad \leftarrow \rho_{\text{GW}} = \frac{1}{32\pi} \langle h_{ab} h^{ab} \rangle$$

fractional energy density of Universe in GWs per log frequency bin.

$$\left(\frac{3H_0^2}{8\pi} \right) \rightarrow \frac{2\pi^2}{3H_0^2} f^2 h_c^2(f)$$

proportional to GW energy flux.

- Primordial GWB

$$\Rightarrow \text{scale-invariant } \Omega_{\text{sewb}}(f) = \text{constant.}$$

$$\therefore h_c \propto 1/f$$

$$= A \left(\frac{f}{f_{\text{gr}}} \right)^{\alpha} \quad \alpha = 0$$

- Compact-binary population

→ integrate over continuous distribution of sources.

$$\frac{dp}{df} = \int_0^\infty dz \cdot \frac{dn}{dz} \times \frac{1}{(1+z)} \times \frac{dE}{d\ln f_r} \Big|_{f_r = f(1+z)}$$

+ density in z redshifting energy source form $f_r = f(1+z)$

$$\frac{\partial E}{\partial h_{fr}} = f_r \frac{\partial E}{\partial b_r} \times \frac{\partial b_r}{\partial f_r}$$

$$\propto f_r \times f_r^{1/3} \times f_r^{-1/3}$$

$$\propto f_r^{2/3}$$

Thus $\mathcal{L}_{SGWB}(f) \propto f^{2/3}$

$$h_c(f) \propto f^{-2/3}$$

$$= A (f/f_{gyr})^{\alpha} \quad \text{--- } \alpha = -2/3$$

IN TENSITY
 $= \frac{1}{2} (h_x^2 + h_y^2)$

$$\mathcal{L}_{SGWB}(f) = \frac{32\pi^3 f^3}{3H_0^2} I(f)$$

$$S_h(f) = 16\pi I(f)$$

$$h_c(f) = \sqrt{16\pi f I(f)}$$

$$S_z(f) = \frac{16\pi}{3} I(f)$$

$$S_t(f) = \frac{4I(f)}{3\pi f^2}$$