statement about limited detector sensitivity

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$\overbrace{\text{Stochastic}}$ GW Backgrounds

- Consider a plane-wave expansion of the GW metric perturbation.

$$h_{ij}(t,\vec{x}) = \sum_{+,\times} \int_{-\infty}^{\infty} df \int_{S^2} d^2\Omega \, . \, \tilde{h}_A(f,\hat{\Omega})\, e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x})}\, e^A_{ij}(\hat{\Omega})$$

$$e^+_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

spatial indices

... where $\hat{\Omega}$ = GW propagation direction.

e^A_{ij} = polarization basis tensor.

$\tilde{h}_A$ = polarization amplitude.

REMEMBER THIS? $\left\{ \; z(t,\hat{\Omega}) = \frac{1}{2}\frac{p^i p^j}{(1+\hat{\Omega}\cdot\hat{p})} \cdot \Delta h_{ij} \right.$

$$\Rightarrow \Delta h_{ij} = \int_{-\infty}^{\infty} df \left[e^{2\pi i f t}\left(e^{-2\pi i f L(1+\hat{\Omega}\cdot\hat{p})} - 1\right) \times \sum_A \tilde{h}_A(f,\hat{\Omega})\, e^A_{ij}(\hat{\Omega}) \right]$$

FOURIER DOMAIN $\hookrightarrow$ $\Delta\tilde{h}_{ij}(f,\hat{\Omega}) = \left(e^{-2\pi i f L(1+\hat{\Omega}\cdot\hat{p})} - 1\right) \times \sum_A \tilde{h}_A(f,\hat{\Omega})\, e^A_{ij}(\hat{\Omega})$

$$\tilde{z}(f,\hat{\Omega}) = \left(e^{-2\pi i f L(1+\hat{\Omega}\cdot p)} - 1\right) \times \sum_A \tilde{h}_A(f,\hat{\Omega})\, F^A(\hat{\Omega})$$

$$\text{... where } \underbrace{F^A(\hat{\Omega}) = \frac{1}{2}\frac{p^i p^j}{(1+\hat{\Omega}\cdot\hat{p})} \times e^A_{ij}(\hat{\Omega})}_{\text{antenna response function, } \{+,\times\}}$$

also holds for stochastic GW signals across spectrum

- We are sensitive to GW signals from the entire population of radiating SMBBHs (or cosmological processes), producing a stochastic GW background.

$\Rightarrow$ If $\tilde{h}_A(f, \hat{\Omega})$ can be treated as zero-mean Gaussian random fields, then signal is fully described by its second moment, i.e., VARIANCE

$$\underbrace{\langle \tilde{h}_A(f,\hat{\Omega}) \tilde{h}^*_{A'}(f',\hat{\Omega}') \rangle}_{\text{VARIANCE (MEAN = 0)}} = \underbrace{\delta(f-f')}_{\text{STATIONARY}} \underbrace{\delta^2(\hat{\Omega},\hat{\Omega}')}_{\text{DIRECTIONS UNCORRELATED}} \mathbb{P}_{AA'}(f,\hat{\Omega})$$

POLARIZATION TENSOR

$$\mathbb{P}_{AA'}(f,\hat{\Omega}) = \begin{pmatrix} I(f,\hat{\Omega}) + Q(f,\hat{\Omega}) & U(f,\hat{\Omega}) - iV(f,\hat{\Omega}) \\ U(f,\hat{\Omega}) + iV(f,\hat{\Omega}) & I(f,\hat{\Omega}) - Q(f,\hat{\Omega}) \end{pmatrix}$$

$\{I, Q, U, V\}$ = STOKES PARAMETERS

$$I(f,\hat{\Omega}) = \frac{1}{2}\langle |\tilde{h}_+|^2 + |\tilde{h}_\times|^2 \rangle \rightarrow \text{INTENSITY}$$

$$\left.\begin{aligned} Q(f,\hat{\Omega}) &= \frac{1}{2}\langle |\tilde{h}_+|^2 - |\tilde{h}_\times|^2 \rangle \\ U(f,\hat{\Omega}) &= \mathrm{Re}\langle \tilde{h}_+ \tilde{h}^*_\times \rangle \end{aligned}\right\} \text{LINEAR POLARIZATION}$$

$$V(f,\hat{\Omega}) = \mathrm{Im}\langle \tilde{h}_+ \tilde{h}^*_\times \rangle \rightarrow \text{CIRCULAR POLARIZATION}$$

⇒ Astrophysical GWBs should be unpolarized

i.e., $Q = U = V = 0$

$A = \{+, \times\}$

$$\langle \tilde{h}_A(f,\hat{\Omega})\tilde{h}^*_{A'}(f',\hat{\Omega}')\rangle = \frac{\delta_{AA'}}{2}\frac{\delta(f-f')}{2}\frac{\delta^2(\hat{\Omega},\hat{\Omega}')}{4\pi}S_h(f)P(\hat{\Omega})$$

... where $S_h(f)$ = 1-sided power spectral density of Fourier modes

$P(\hat{\Omega})$ = distribution of intensity over sky ($= 1\ \forall\ \hat{\Omega}$ if isotropic)

⇒ Let's assume isotropy for now...

$$\langle \tilde{z}_a(f)\tilde{z}^*_b(f)\rangle = \iint_{S^2\,S^2} d^2\hat{\Omega}\,d^2\hat{\Omega}'\left[e^{-2\pi i f L_a(1+\hat{\Omega}\cdot\hat{p}_a)}-1\right]\left[e^{2\pi i f L_b(1+\hat{\Omega}\cdot\hat{p}_b)}-1\right]$$

$$\times\left\langle \sum_A \tilde{h}_A(f,\hat{\Omega})F^A_a(\hat{\Omega})\sum_{A'}\tilde{h}^*_{A'}(f',\hat{\Omega}')F^{A'}_b(\hat{\Omega}')\right\rangle$$

pulsars

$$\langle \tilde{z}_a(f)\tilde{z}^*_b(f')\rangle = \frac{1}{2}\delta(f-f')\underbrace{S_z(f)_{ab}}_{\text{cross-power spectral density of redshift to arrival rates.}}$$

$$= \frac{1}{2}\delta(f-f')S_h(f)\int_{S^2}\frac{d^2\hat{\Omega}}{8\pi}\kappa_{ab}(f,\hat{\Omega})$$

$$\times\sum_{A=+,\times}F^A_a(\hat{\Omega})F^A_b(\hat{\Omega})$$

$$\kappa_{ab}(f,\hat{\Omega}) \equiv \left[e^{-2\pi i f L_a(1+\hat{\Omega}\cdot\hat{p}_a)}-1\right]\left[e^{2\pi i f L_b(1+\hat{\Omega}\cdot\hat{p}_b)}-1\right]$$

UNITS = [time]

Thus, $S_z(f)_{ab} = \frac{1}{2} S_h(f) \int_{S^2} \frac{d^2\hat{\Omega}}{4\pi} K_{ab}(f,\hat{\Omega}) \sum_{A=+,\times} F_a^A(\hat{\Omega}) F_b^A(\hat{\Omega})$

$\boxed{\text{WHAT IS } K_{ab}(f,\hat{\Omega})\text{?}}$ $\Rightarrow$ controls how rapidly the pulsar terms spatially decorrelate!

$f \sim 10^{-9}$ Hz ; $L \sim 100$ pc $\}$ $fL > 10$

$\therefore$ $e^{(\ldots)}$ oscillate rapidly on sky, contributing $\underbrace{\text{negligibly}}$ to integral.

.... except when pulsars are IDENTICAL

$$\Rightarrow \quad K_{ab}(f,\hat{\Omega}) \to 2 \quad \text{when } a=b$$
$$\to 4 \quad \text{when } a \neq b$$

$\boxed{\text{WHAT IS } \int_{S^2} \frac{d^2\hat{\Omega}}{4\pi} \sum_A F_a^A F_b^A \text{ ?}}$ $\Rightarrow$ overlap reduction function (ORF)

$$\tilde{\Gamma}_{ab} = \int_{S^2} \frac{d^2\hat{\Omega}}{4\pi} \sum_{A=+,\times} F_a^A(\hat{\Omega}) F_b^A(\hat{\Omega})$$

contour integral

$$= x_{ab}\ln(x_{ab}) - \frac{1}{6}x + \frac{1}{3}$$

... where $x_{ab} = \frac{1}{2}(1-\cos\theta_{ab})$

Let's re-normalize this so that $\Gamma_{ab} = 1$ for $a = b$

$$\Rightarrow \boxed{\Gamma_{ab} = \frac{3}{2} x_{ab} \ln(x_{ab}) - \frac{1}{4} x_{ab} + \frac{1}{2} + \frac{1}{2}\delta_{ab}}$$

HELLINGS & DOWNS CURVE.

$\Rightarrow$ SHOW H+D CURVE

↳ * $a \neq b$, $\Gamma_{ab} \leq 0.5$
* quadrupolar, but not perfectly
($\Gamma_{ab}(\theta = 180°) = 0.25$)
* $1/(1+\hat{\ell}\cdot\hat{p})$ introduces preferred direction.

NOW $$\Gamma_{ab} = \sum_{\ell=0}^{\infty} a_\ell P_\ell(\cos\theta_{ab})$$

... where $a \neq b$, $a_0 = 0 = a_1$

$$a_\ell = \frac{3}{2} \frac{(\ell-2)!}{(\ell+2)!} (2\ell+1)$$

$\Rightarrow$ SHOW LEGENDRE SPECTRUM

↳ * HD curve orthogonal to monopole/dipole.
* Assumes ∞ precision + uniform pulsar coverage.
* $a_2 / \sum_\ell a_\ell = 0.63$, $a_3 / \sum_\ell a_\ell = 0.17$

$\Rightarrow$ back to spectrum : $S_z(f)_{ab} = \frac{1}{3}\Gamma_{ab} S_h(f)$

$\Rightarrow$ we don't measure pulse arrival rates, we measure pulse arrival times.

$$R(t) = \int_0^t z(t')\,dt'$$

$\hookrightarrow$ brings down factors of $1/2\pi i f$

Thus
$$S_t(f)_{ab} = \frac{S_z(f)_{ab}}{4\pi^2 f^2}$$
$$\Downarrow \quad = \Gamma_{ab}\frac{S_h(f)}{12\pi^2 f^2} \qquad \frac{h_c^2(f)}{f}$$
$$S_t(f)_{ab} = \Gamma_{ab}\frac{h_c^2(f)}{12\pi^2 f^3}$$

UNITS [time]^3

$\Rightarrow$
$$h_c(f) = A\left(f/f_{yr}\right)^{\alpha}$$

$\alpha = -2/3$ SMBBHs

$\alpha = 0$ RELIC GWS

Finally
$$S_t(f)_{ab} = \Gamma_{ab}\frac{A^2}{12\pi^2}\times\left(\frac{f}{f_{yr}}\right)^{-\gamma} \quad \dots \gamma \equiv 3-2\alpha$$

$13/3$ SMBBHs

5 RELIC

Spectrum of GW source populations

$$\underbrace{\Omega_{SGWB}(f)}_{\text{fractional energy density of Universe in GWs per log frequency bin.}} \equiv \frac{1}{\rho_c}\frac{d\rho}{d\ln f} \leftarrow \rho_{GW} = \underbrace{\frac{1}{32\pi}\langle \dot{h}_{ab}\dot{h}^{ab}\rangle}_{\text{proportional to GW energy flux.}}$$

$$\rho_c = \frac{3H_0^2}{8\pi}$$

$$\rightarrow \frac{2\pi^2}{3H_0^2} f^2 h_c^2(f)$$

- Primordial GWB

$\Rightarrow$ scale-invariant $\Omega_{SGWB}(f) = \text{constant}$.

$$\therefore \quad h_c \propto 1/f$$

$$= A\left(f/f_{yr}\right)^{\alpha} \quad \alpha = -1$$

- Compact-binary population

$\Rightarrow$ integrate over continuous distribution of sources.

$$\frac{d\rho}{d\ln f} = \int_0^\infty dz \cdot \underbrace{\frac{dn}{dz}}_{\#\text{ density in } z} \times \underbrace{\frac{1}{(1+z)}}_{\text{redshifting energy}} \times \underbrace{\frac{dE}{d\ln f_r}}_{\text{source frame}}\Bigg|_{f_r = f(1+z)}$$

$$\frac{dE}{d\ln f_r} = f_r \frac{dE}{dt_r} \times \frac{dt_r}{df_r}$$

$$\propto f_r \times f_r^{10/3} \times f_r^{-11/3}$$

$$\propto f_r^{2/3}$$

Thus $\underbrace{\Omega_{SGWB}(f)\Big|_{CB} \propto f^{2/3}}_{}$

$$h_c(f) \propto f^{-2/3}$$

$$= A\left(f/f_{yr}\right)^{\alpha} \quad \alpha = -2/3$$

INTENSITY $= \frac{1}{2}\langle|\tilde{h}_+|^2 + |\tilde{h}_\times|^2\rangle$

$$\boxed{\begin{aligned}
\Omega_{SGWB}(f) &= \frac{32\pi^3}{3H_0^2} f^3 I(f) \\
S_h(f) &= 16\pi I(f) \\
h_c(f) &= \sqrt{16\pi f I(f)} \\
S_z(f) &= \frac{16\pi}{3} I(f) \\
S_t(f) &= \frac{4 I(f)}{3\pi f^2}
\end{aligned}}$$