

# Assignment CS \_ 15Q

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## QUESTION

The Lucas sequence  $L_n$  is defined by the recurrence relation:

$$L_n = L_{n-1} + L_{n-2}, \text{ for } n \geq 3$$

with  $L_1=1$  and  $L_2=3$

Which one of the option given is TRUE?

- 1)  $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$
- 2)  $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{3}\right)^n$
- 3)  $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{3}\right)^n$
- 4)  $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$

(GATE 2023 CS 15)

## Solution:

Initial condition  $L_1=1$  and  $L_2=3$

$$L_n = L_{n-1} + L_{n-2} \quad (1)$$

Assume  $L_{n+1} = x(n)$

$$x(n) = [x(n-1) + x(n-2) - 3]u(n-2) + u(n) + 2u(n-1) \quad (2)$$

$$X(z) = z^{-1}(X(z) - 1) + z^{-2}X(z) - 3\frac{z^{-2}}{1-z^{-1}} + \frac{1}{1-z^{-1}} + 2\frac{z^{-1}}{1-z^{-1}} \quad (3)$$

$$X(z)(1 - z^{-1} - z^{-2})(1 - z^{-1}) = 1 + z^{-1} - 2z^{-2} \quad (4)$$

$$X(z) = \frac{1 + z^{-1} - 2z^{-2}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \quad (5)$$

$$X(z) = \frac{A}{1 - z^{-1}} + \frac{B}{1 - \alpha z^{-1}} + \frac{C}{1 - \beta z^{-1}} \quad (6)$$

$$\text{Where, } \alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2}$$

using partial fractions,

$$X(z) = \frac{\alpha + 2}{(\alpha - \beta)(1 - \alpha z^{-1})} + \frac{\beta + 2}{(\beta - \alpha)(1 - \beta z^{-1})} \quad (7)$$

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1 - az^{-1}} \quad |z| > |a|$$

Substituting this result,

$$x(n) = \frac{\alpha + 2}{(\alpha - \beta)}(\alpha^n u(n)) - \frac{\beta + 2}{(\alpha - \beta)}(\beta^n u(n)) \quad (8)$$

$$x(n) = \frac{(5 + \sqrt{5})(1 + \sqrt{5})^n - (5 - \sqrt{5})(1 - \sqrt{5})^n}{2^{n+1} \sqrt{5}} u(n) \quad (9)$$

$$x(n) = \frac{(1 + \sqrt{5})^{n+1} + (1 - \sqrt{5})^{n+1}}{2^{n+1}} u(n) \quad (10)$$

$$\therefore L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \text{ option 1 is correct.}$$

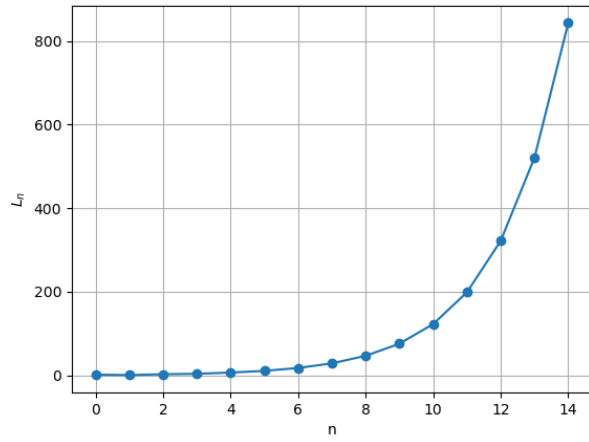


Fig. 1.  $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$