Assignment CS _ 15Q

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Question

The Lucas sequence $L_n$is defined by the recurrence relation:

$$L_n = L_{n-1} + L_{n-2}, forn \geq 3$$

with L_1=1 and L_2=3
Which one of the option given is TRUE?

1) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$
2) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{3}\right)^n$
3) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{3}\right)^n$
4) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$

(GATE 2023 CS 15)

Solution:
Initial condition L_1=1 and L_2=3

$$L_n = L_{n-1} + L_{n-2} \tag{1}$$

Assume $L_{n+1} = x(n)$

$$x(n) = [x(n-1) + x(n-2) - 3]u(n-2) + u(n) + 2u(n-1) \tag{2}$$

$$X(z) = z^{-1}(X(z) - 1) + z^{-2}X(z) - 3\frac{z^{-2}}{1-z^{-1}} + \frac{1}{1-z^{-1}} + 2\frac{z^{-1}}{1-z^{-1}} \tag{3}$$

$$X(z)(1 - z^{-1} - z^{-2})(1 - z^{-1}) = 1 + z^{-1} - 2z^{-2} \tag{4}$$

$$X(z) = \frac{1 + z^{-1} - 2z^{-2}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \tag{5}$$

$$X(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-\alpha z^{-1}} + \frac{C}{1-\beta z^{-1}} \tag{6}$$

Where, $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$

using partial fractions,

$$X(z) = \frac{\alpha+2}{(\alpha-\beta)(1-\alpha z^{-1})} + \frac{\beta+2}{(\beta-\alpha)(1-\beta z^{-1})} \tag{7}$$

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}} \quad |z| > |a|$$

Substituting this result,

$$x(n) = \frac{\alpha+2}{(\alpha-\beta)}(\alpha^n u(n)) - \frac{\beta+2}{(\alpha-\beta)}(\beta^n u(n)) \tag{8}$$

$$x(n) = \frac{(5+\sqrt{5})(1+\sqrt{5})^n - (5-\sqrt{5})(1-\sqrt{5})^n}{2^{n+1}\sqrt{5}}u(n) \tag{9}$$

$$x(n) = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}}u(n) \tag{10}$$

$\therefore L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$ option 1 is correct.

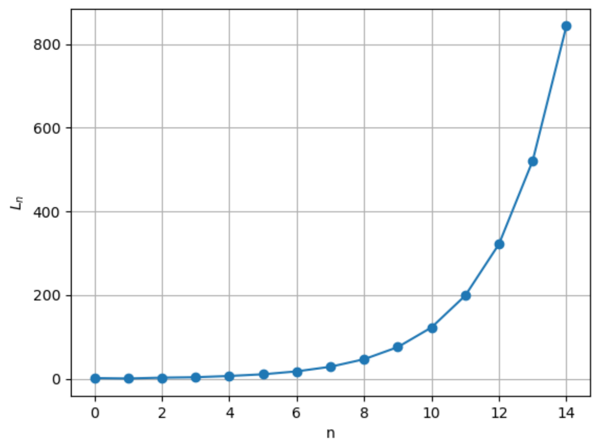

Fig. 1. $X(s) = L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$