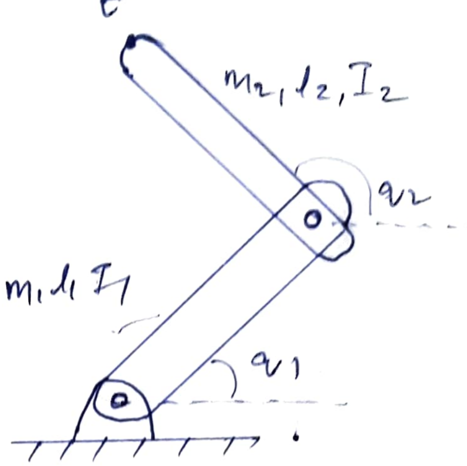
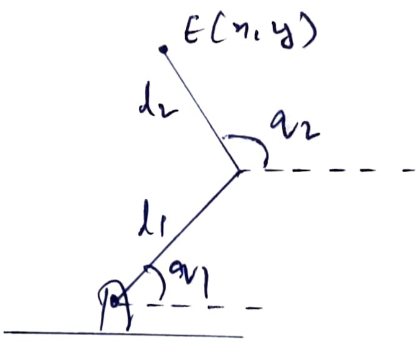
lets take the 2R elbow manipulator with two limits of masses m_1 & m_2 lengths l_1 & l_2 and moments of inertia I_1 & I_2 respectively. That is represented in the below fig.

q_1 q_2 are angles w.r.t to x-axis and. E is End effector.



As this is a Robot and a manipulator it has motors in it with different Tourques. ie τ_1, τ_2. which inturn control the angles q_1 q_2.

~~Folling~~ Below is the FBD of the above diagram.



The end effector has the following coordinates.

E(x, y)

$$x = l_1 \cos q_1 + l_2 \cos q_2$$
$$y = l_1 \sin q_1 + l_2 \sin q_2 \quad \text{——— (1)}$$

the time derivative of (1) is :-

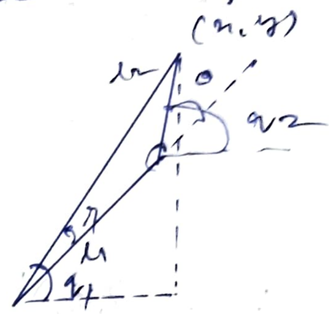
$$\dot{x} = -l_1 \sin q_1 \, \dot{q}_1 - l_2 \sin q_2 \, \dot{q}_2$$

$$\dot{y} = l_1 \cos q_1 \, \dot{q}_1 + l_2 \cos q_2 \, \dot{q}_2 .$$

which can also be written as :-

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin q_1 & -l_2 \sin q_2 \\ l_1 \cos q_1 & l_2 \cos q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad \text{—— (2)}$$

using Inverse kinematics we can calculate q_1, q_2



using cosine rule :-

$$\theta = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1 l_2}\right)$$

$$q_1 = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right)$$

$$q_2 = q_1 + \theta = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1 l_2}\right) + \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right)$$

—— (3)

To make a relationship between Tourques and forces we have to draw the FBD first.

$$\Sigma M_{O_1} = 0$$

$$\tau_1 = -F_x l_1 \sin q_1 + F_y l_1 \cos q_1$$

$$\Sigma M_{O_2} = 0$$

$$\tau_2 = -F_y l_2 \sin q_2 + F_y l_2 \cos q_2$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 s q_1 & l_1 c q_1 \\ -l_2 s q_2 & l_2 c q_2 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \quad \text{—— (4)}$$

By using Lagrangian : $L = K - V$ we can find the dynamics of the Manipulator.

$$\frac{d}{dt}\left(\frac{dL}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = Q_i' \qquad i = 1, 2, 3 \ldots n$$ (5)

Q_i = all the generalized forces.

$$K = \frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right)\dot{q}_2^2 + \frac{1}{2} m_2 V_2^2$$

Rotation of link 1 — Rotation of link 2 — translation of CM of link 2

$$V_{c_2}^2 = (l_1 \dot{q}_1)^2 + (l_2/2\, \dot{q}_2)^2 + 2 l_1 \dot{q}_1 l_2/2\, \dot{q}_2 \cos(q_2 - q_1)$$

Including Body forces. We get.

$$V = m_1 g l_1/2 \sin q_1 + m_2 g (l_1 \sin q_1 + l_2/2 \sin q_2)$$

$$\frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + \frac{m_2 l_1 l_2}{2} \ddot{q}_2 \cos(q_2 - q_1) - \frac{m_2 l_1 l_2}{2} \dot{q}_2 (\dot{q}_2 - \dot{q}_1) \times \sin(q_2 - q_1)$$
$$+ m_1 g \frac{l_1}{2} \cos q_2 + m_2 g l_1 \cos q_1 = Q_1$$

$$\frac{1}{3} m_1 l_2^2 \ddot{q}_2 + m_2 \frac{l_2^2}{4} \ddot{q}_2 + \frac{m_2 l_1 l_2}{2} \ddot{q}_1 \cos(q_2 - q_1) - \frac{m_2 l_1 l_2}{2} \dot{q}_1 (\dot{q}_2 - \dot{q}_1)$$
$$\times \sin q_2 - q_1 + m_2 g \frac{l_2}{2} \sin q_2 = Q_2$$

$\longrightarrow$ (6)

to Include stiffness we have to bring in the equation ④

$f_x = kx$

$F_y = ky$

$F_x = k_x(x - x_0)$ ∴ with reference

$F_y = k_y(y - y_0)$

∴ $F_x = k(l_1 cq_1 + l_2 cq_2)$

$F_y = k(l_1 sq_1 + l_2 sq_2)$

and from ④ we get

$$k(l_1 sq_1 + l_2 sq_2) l_2 cq_2 - k(l_1 cq_1 + l_2 cq_2) l_2 q_2 = \tau_{2s}$$

$$k(l_1 sq_1 + l_2 sq_2) l_1 cq_1 - k(l_1 cq_1 + l_2 cq_2) l_1 q_1 = \tau_{1s}$$

——— ⑦