homework 5

(2) moon lander with state $[h, v, m]^T$ so have the following dynamics:

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \dfrac{a(t)}{m(t)} \\ \dot{m}(t) = -ka(t) \end{cases}$$

where in h is the altitute, v is the velocity and m is the mass of the moon lander $a(t) \in [0, 1]$ is the thrust and k is a constant fuel burning rate. Let the initial state be $[h_0, v_0, m_0]^T$ and the target be $h(t^*) = 0$ and $v(t^*) = 0$ at terminal time t^*, Derive the optimal control policy for minimal fuel consumption

→ solution for minimal fuel consumption.

we say that the trajectory / control quadruple. (h, v, m, α) solving

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \dfrac{\alpha(t)}{m(t)} \\ \dot{m}(t) = -k\alpha(t) \end{cases}$$

is admissible if it satisfies the end point constrain

$(h(T), v(T), m(T)) \in C = \{0\} \times \{0\} \times [m_s, +\infty)$

we say that for a given $m_0 \geq m_s$, the initial data $(v_0, h_0) \in R \times R_+$ are admissible if there exists an admissible quadruple (h, v, m, α) satisfying $v(0) = v_0$ and $h(0) = h_0$.

solutions satisfying the state constraints $(m(t)) \geq m_s$ and $h(t) \geq 0$, for any $t \in [0, T]$, In order to land safely, the control $\alpha(t) = 1$ for any $t \in (t_*, T]$ and some $t_* \in [0, T)$

$gm_0 < 1$ is a sufficient condition to have $h(t) > 0$ in (t_*, T) whenever $h(T) = v(T) = 0$

The condition $gm(T) \geq 1$ would imply that $gm(t) > 1$ in (t_*, T) hence $\dot{v}(t) < 0$ in case of soft landing, $v(t) > 0$, yielding $h(t) < 0$ the conditions ensure that $m(T) \geq m_s$

if $gm_s \geq 1$ then the only admissible initial data are the trivial ones $(h_0, v_0) = (0, 0)$

In order to obtain an optimal control for the problem, we use pontryagin minimum principle, Given an optimal solution (h, v, m, α)

there exists

$(P, \mu) \in R^3 \times R$, $(P, \mu) \neq 0$ such that $\mu \geq 0$

and such that $P = (P_1, P_2, P_3)$ solves the adjoint equation

$$\begin{cases} \dot{P}_1(t) = 0 \\ \dot{P}_2(t) = -P_1(t) \\ \dot{P}_3(t) = \dfrac{P_2(t)\alpha(t)}{m(t)^2} \end{cases}$$

and satisfies the transversality condition $P(T) \in N_c = \mathbb{R} \times \mathbb{R} \times (-\infty, 0]$, this means that

$$P_3(T) = 0 \text{ if } m(T) > m_s, \quad P_3(T) \leq 0 \text{ if } m(T) = m_s$$

considering the hamiltonian

$$H(h, v, m, u, P_1, P_2, P_3, \alpha) = P_1 v - g P_2 + \alpha\left(\frac{P_2}{m} - K P_3 + u\right)$$

It vanishes along the optimal trajectory

$$P_1(t) \cdot v(t) - g P_2(t) + \left(\frac{P_2(t)}{m(t)} - K P_3(t) + u\right)\alpha(t) = 0$$

in $[0, T]$;

moreover the optimal control fulfills the following minimality conditions.

$$p_1(t)v(t) - p_2(t)g + \alpha\left(\frac{p_2(t)}{m(t)} - Kp_3(t) + u\right)$$

$$= p_1(t)v(t) - p_2(t)g + \min_{0\leq a\leq 1}\left\{a\left(\frac{p_2(t)}{m(t)} - Kp_3(t) + u\right)\right\}$$

for a.e. $t \in [0, \gamma]$

To determine for optimal control we proceed in different steps:

Step 1 (1) if it exists, then it is ~~at~~ with almost one switch and $\alpha(\gamma) = 1$

Considering the function

$$P(t) := \frac{p_2(t)}{m(t)} - Kp_3(t) + u$$

so that the minimality condition characterizes the optimal control α through the rule

$$\alpha(t) = \begin{cases} 0 & \text{if } P(t) > 0 \\ 1 & \text{if } P(t) < 0 \end{cases}$$

$P(t) = 0$ occurs at most in one point.

$\dot{P}_1$ is constant, $P_1(t) = \lambda_1$.

$$\Rightarrow \dot{p} = \frac{\dot{P}_2}{m} - \frac{\dot{m} P_2}{m^2} - K\dot{P}_3 = -\frac{\lambda_1}{m} + K\alpha \frac{P_2}{m_2} - K\alpha \frac{P_2}{m^2} = -\frac{\lambda}{m}$$

This equation states that p is monotone and is possibly constant. Hence the ambigous situation $p(t)=0$ may occur at ~~on~~ most one point or either ~~at~~ on the whole interval $[0, T]$. Assume for contradiction that $p(t) = 0$, the λ_1 which yields $P_1(t) = 0$. It implies that $P_2(t) = 0$ this implies P_3 is also constant i.e $P_3(t) = C \leq 0 \forall t$

$\therefore$ $(P, u) = (0, 0)$, Therefore it cannot be $p(t) = 0$ and $p(t)$ vanishes at most one point.

Therefore it shows that $\alpha(T) = 1$, since $\alpha(T) = 0$ would mean $\dot{v}(T) = -g < 0$ with crash of spacecraft, to make this fully rigorous we notice that if we assume $v(T) = 0$ then $v(t) < 0$ in a left neighbourhood of $t = T$ which implies $\dot{V}(T) \geq 0$

Step 2: explicit form of the solution of the optimal control α exists.

from step 1 we know that optimal control exists, then it has at most one switch on time $t_* \in (0, T)$. In the interval $(0, t_*)$ when the thrust is switched off ($\alpha = 0$; with spacecraft in free fall) the system with initial conditions has the solution

$$m(t) = m_0, \quad v(t) = -gt + v_0, \quad h(t) = -\frac{g}{2} t^2 + v_0 t + h_0$$

Therefore we get the constraint

$$h(t) = -\frac{v(t)^2}{2g} + h_0 + \frac{v_0^2}{2g} \quad (0 \le t \le t_*)$$

whose representation in the phase plane $\{v, h\}$ is a concave parabola having the vertex at the point $(v, h) = (0, h_0 + v_0^2/2g)$

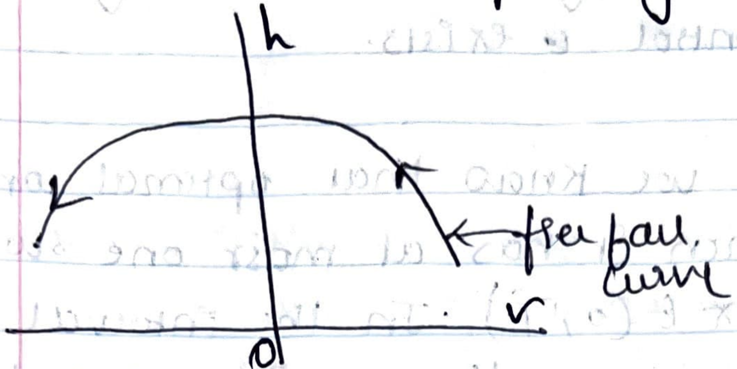
Therefore if the optimal control α exists the solution (h, v, m) of

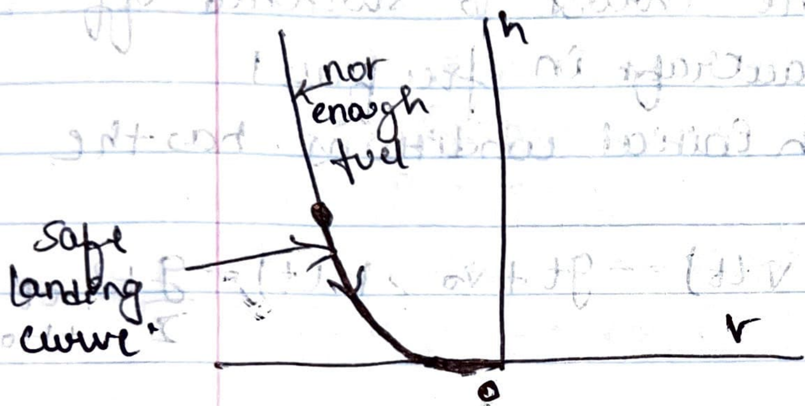
$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \dfrac{\alpha(t)}{m(t)} \\ \dot{m}(t) = -k\alpha(t) \end{cases}$$

is given by

$$m(t) = m_0, \quad v(t) = -gt + v_0, \quad h(t) = -\frac{g}{2} t^2 + v_0 t + h_0 \quad (0 \le t \le t_*)$$

This states that possibly $t_* = 0$





Step 3: Geometric interpretation of the existence of a switch on time.

The engine is switched off until the free fall trajectory reaches a safe landing curve and this happens at a point of the switching curve, there exists $t_* > 0$ and $\bar{m} [m_s, m_0]$ such that $(v(t_*), h(t_*)) = (v, \Gamma_{\bar{m}}(v))$ for $v \in [v_{\bar{m}}(m_s), 0)$. Then the engine is switched on and then the optimal trajectory follows the safe landing curve corresponding to the final fuel mass equal to $\bar{m} - m_s$ i.e. $m(T) = \bar{m}$

In the extreme situation where $v = v_{mo}(m_s)$ the safe landing occurs with no fuel left.

This means that $(v(t_*), h(t_*)) = (v_{mo}(m_s), \Gamma_{mo}(v_{mo}(m_s)))$

And this the cornerpoint of the black line in the left picture.

Step 4: analytic formulation of the geometric pattern,

Safe landing region is the switching curve whose equation is $h = \Gamma_{mo}(v)$ for $v \in [v_{mo}(m_s), 0)$ its extremal point $(v_{mo}(m_s), \Gamma_{mo}(v_{mo}(m_s)))$ may also be expressed through parametric representation of the switching curve so that if t_* denotes the switching instant

$$v(t_*) = \frac{g}{K}(m_0 - m_s) - \frac{1}{K}\log\frac{m_0}{m_s}$$

$$h(t) = \frac{m_s - m_0}{K^2} - \frac{g}{2K^2}(m_0 - m_s)^2 + \frac{m_0}{K^2}\log\frac{m_0}{m_s}$$

It is clear and fully apparent fro that the corner point depends on m_0 with this extremal point we seek the critical parabola, at some time $t* > 0$ this gives

$$h_0 + \frac{v_0^2}{2g} = h(t) + \frac{v(t*)^2}{2g} = \frac{m_s - m_0}{K^2} + \frac{1}{2gK^2}\log^2\frac{m_0}{m_s} + \frac{m_s}{K^2}\log\frac{m_0}{m_s}$$

Step 5 conclusions.

Therefore there are three different conditions where optimal control does not exists.

i) The thrust is too weak, unable to stop the free fall
ii) there is not enough fuel to stop the spacecraft
iii) The spacecraft is too close to the surface of the moon with too large downwards velocity

if $gm_s \geq 1$ then the only admissible initial data are the trivial ones $(h_0, v_0) = (0, 0)$

if $v_0 \in [v_{m_0}(m_s), 0)$ and $h_0 = \Gamma_{m_0}(v_0)$ or ~~h0~~

$$h_0 + \frac{v_0^2}{2g} = \frac{m_s - m_0}{K^2} + \frac{1}{2gK^2}\log^2\frac{m_0}{m_s} + \frac{m_s}{K^2}\log\frac{m_0}{m_s}$$

Then safe landing is possible.

If $v_0 \le v_{m_0}(m_s)$ or $v_0 > v_{m_0}(m_s)$ and $h_0 < T_{m_0}(v_0)$ then safe landing is impossible

→ if $v_{m_0}(m_s) < v_0 \le 0$ and $T_{m_0}(v_0) \le h_0 < \frac{m_s - m_0}{K^2}$

$$+ \frac{1}{2gK^2} \log^2 \frac{m_0}{m_s} + \frac{m_s}{K^2} \log \frac{m_0}{m_s} - \frac{v_0^2}{2g} \quad \text{or}$$

$v_0 > 0$ and $0 \le h_0 < \frac{m_s - m_0}{K^2} + \frac{1}{2gK^2} \log^2 \frac{m_0}{m_s}$

$$+ \frac{m_s}{K^2} \log \frac{m_0}{m_s} - \frac{v_0^2}{2g}.$$

Then safe landing is possible and the optimal control has exactly one switch on time.