Problem 1:

~~to~~ calculating gradient Descent for f and g_i at the corner points

$$\frac{\partial f}{\partial x}\Big|_{x_1=0,\, x_2=0} = \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}\Big|_{x_1=0,\, x_2=1} = \begin{bmatrix} 2(x_1+1) \\ 2(x_2-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}\Big|_{x_1=2,\, x_2=0} = \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}\Big|_{x_1=2,\, x_2=1} = \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$\frac{\partial g_1}{\partial x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \frac{\partial g_2}{\partial x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \frac{\partial g_3}{\partial x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\frac{\partial g_4}{\partial x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

using KKT conditions

Lagrange exp is

$$\mathcal{L}(x_1, x_2, \lambda) = (x_1+1)^2 + (x_2-2)^2 + u_1(x_1-2) + u_2(x_2-1) - u_3 x_1 - u_4 x_2$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 2(x_1+1) + u_1 - u_3 \\ 2(x_2-2) + u_2 - u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Assume

$u_2 > 0, u_3 > 0, u_1 = u_4 = 0$

$x_1 = 0$

$x_2 = 1$

$u_3 = 2 > 0$

$u_2 = 2 > 0$

(∵ since it is an convex function and feasible domain the problem is convex problem

$x_1 = 0$, $x_2 = 1$, $u_1 = 0$, $u_2 = 0$, $u_3 = 2$, $u_4 = 0$

The min value is 1

which u_1 and u_4 contradicts, so $x_1 = 2$, $x_2 = 0$ is not the min soln. from the figure we know that this point should be maximum solution for the problem

Assume $u_1 > 0$, $u_2 > 0$, $u_3 = u_4 = 0$

then

$x_1 = 2$, $x_2 = 1$, $u_1 = -6$, $u_2 = 2$

u_1 does not obey the Assumption

so $x_1 = 0$, $x_2 = 0$ is not the soln

next Assume $u_3 > 0$, $u_4 > 0$, $u_1 = u_2 = 0$ then

$x_1 = 0$, $x_2 = 0$, $u_3 = 2 > 0$, $u_4 = -4 < 0$

u_4 does not obey the Assumption, so the min value is $1 (x_1 = 0, x_2 = 1)$

Problem 2:

KKT conditions :

using obj function and Lagrange expression with four parameter μ_1, μ_2 $(\mu_1 \geq 0, \mu_2 \geq 0)$

$$L(x_1, x_2, \mu) = -x_1 + \mu_1(x_2 - (1-x_1)^3) - \mu_2 x_2$$

The derivation of Lagrange exp is

$$\frac{\partial L}{\partial x} = \begin{bmatrix} -1 + 3\mu_1(1-x_1)^2 \\ \mu_1 - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

⇒ Assume $\mu_1 > 0, \mu_2 > 0$ then

$$x_1 = 1, \quad x_2 = 0$$

So $-1 + 3\mu_1(1-1)^2 = -1 = 0$ which is wrong

⇒ Assume $\mu_1 > 0, \mu_2 > 0$ then we will have $-1 + 3 \times 0 + (1-x_1)^2$ $= -1 = 0$ which is also wrong

from above results we can find that $\frac{\partial g_1}{\partial x}$ and $\frac{\partial g_2}{\partial x}$ is linearly dependent, which cannot meet regularity of KKT condition, so the KKT method fails.

∴ There is not optimal solution for this problem.

Problem 3:

$$\min f = -x_1x_2 + x_2x_3 + x_1x_3$$

Subject to: $h = x_1 + x_2 + x_3 - 3 = 0$.

Find the local solution to the problem,

using reduced gradient method:

$\Rightarrow$

Assume, X_1 = state variable

X_3, X_2 = decision variable.

$$\frac{\partial f}{\partial d} = \begin{bmatrix} -x_1 - x_3 \\ -x_1 - x_2 \end{bmatrix}, \quad \frac{\partial f}{\partial s} = -x_2 - x_3, \quad \frac{\partial h}{\partial d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \frac{\partial h}{\partial s} = 1$$

W.K.T $= \frac{\partial s}{\partial d} = -\left(\frac{\partial h}{\partial s}\right)^{-1}\left(\frac{\partial h}{\partial d}\right) \Rightarrow \frac{\partial s}{\partial h} \times \frac{\partial h}{\partial d} = -\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial d}\right) \times \frac{\partial s}{\partial h} = \begin{bmatrix} -x_1 - x_3 \\ -x_1 - x_2 \end{bmatrix} - (x_2 - x_3)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$[x_1 = x_2 = x_3]$ Substitute the solution into the constraint

$h = x_1 + x_2 + x_3 - 3 = 0$, then $x_1 = x_2 = x_3 = 1$

Second order condition is as follows.

$$\frac{\partial^2 h}{\partial d^2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\frac{\partial^2 h}{\partial s^2} = 0$$

$$\frac{\partial^2 h}{\partial s \partial d} = [0 \quad 0]$$

$$\frac{\partial^2 h}{\partial d\, \partial s} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{d^2 h}{dd^2} = \begin{bmatrix} I & \left(\frac{ds}{dd}\right)^T \end{bmatrix} \begin{bmatrix} \frac{\partial^2 h}{\partial d^2} & \frac{\partial^2 h}{\partial d\, \partial s} \\ \frac{\partial^2 h}{\partial s\, \partial d} & \frac{\partial^2 h}{\partial s^2} \end{bmatrix} \begin{bmatrix} I \\ \frac{ds}{dd} \end{bmatrix} + \frac{\partial h}{\partial s} \frac{d^2 s}{dd^2} = 0$$

$$\Rightarrow \frac{d^2 s}{dd^2} = 0$$

$$\frac{\partial^2 f}{\partial d^2} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \frac{\partial^2 f}{\partial s^2} = 0, \quad \frac{\partial^2 f}{\partial s\, \partial d} = \begin{bmatrix} -1 & -1 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial d\, \partial s} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \frac{d^2 f}{dd^2} = \begin{bmatrix} I & \left(\frac{ds}{dd}\right)^T \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial d^2} & \frac{\partial^2 f}{\partial d\, \partial s} \\ \frac{\partial^2 f}{\partial s\, \partial d} & \frac{\partial^2 f}{\partial s^2} \end{bmatrix} \begin{bmatrix} I \\ \frac{ds}{dd} \end{bmatrix} + \frac{\partial f}{\partial s} \frac{d^2 s}{dd^2} =$$

$$= \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 2 > 0$$

when $x_1 = x_2 = x_3$, the function has the maximum value

Now L-multiplier:

$$L(x_1, x_2, x_3, \lambda) = -(x_1 x_2 + x_2 x_3 + x_1 x_3) + \lambda(x_1 + x_2 + x_3 - 3)$$