

Assignment 5

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Download all python codes from

<https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/Codes>

and latex-tikz codes from

<https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/main.tex>

1 PROBLEM 103 FROM CSIR UGC NET - DEC 2012

Let $\{X_n : n > 0\}$ and X be random variables defined on a common probability space. Further assume that X_n 's are non-negative and X takes values 0 and 1 with probabilities p and $1-p$ respectively where, $0 \leq p \leq 1$. Which of the following statements are necessarily true?

- 1) If $p=0$ and X_n converges to X in distribution, X_n converges to X in probability
- 2) If $p=1$ and X_n converges to X in distribution, then X_n converges to X in probability
- 3) If $0 < p < 1$ and X_n converges to X in distribution, then X_n converges to X in probability
- 4) If X_n converges to X in probability, then X_n converges to X almost surely

2 SOLUTION

For any given sequence of real-valued random variables $X_1, X_2, X_3, \dots, X_n$ and a random variable X , let us look at some definitions

- 1) **Convergence in Distribution or Weak convergence:**

$$\lim_{n \rightarrow \infty} F_n(X_n) = F(X) \quad (2.0.1)$$

where F is the cumulative probability distribution function.

- 2) **Convergence in Probability:**

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| > \epsilon) = 0, \text{ for all } \epsilon > 0 \quad (2.0.2)$$

This is stronger than the convergence in distribution but weaker than Almost sure convergence.

- 3) **Almost sure Convergence:**

$$\Pr\left(\lim_{n \rightarrow \infty} X_n = X\right) = 1 \quad (2.0.3)$$

This type of convergence is stronger than both convergence in distribution and probability.

- In general, stronger statements imply weaker statements but not vice versa, i.e. Convergence in probability implies convergence in distribution and Almost sure convergence implies convergence in probability.

- However, convergence in distribution implies convergence in probability if X is a constant.

Proof:

- Let $\epsilon > 0$. Let $X = c$ and the sequence $X_1, X_2, X_3, \dots, X_n$ converges to X in distribution.

- Let

$$B_c(c) \rightarrow \text{open ball about } c \text{ of radius } \epsilon \quad (2.0.4)$$

$$B_c(c)^c \rightarrow \text{complement of } B_c(c) \quad (2.0.5)$$

$$\implies \Pr(|X_n - c| > \epsilon) = \Pr(X_n \in B_c(c)^c) \quad (2.0.6)$$

- By the Portmanteaus lemma(Part C),

$$\text{Since } \lim_{n \rightarrow \infty} X_n = c \quad (2.0.7)$$

$$\implies \Pr(X_n \in B_c(c)^c) \leq \limsup_{n \rightarrow \infty} \Pr(c \in B_c(c)^c) \quad (2.0.8)$$

$$\Pr(c \in B_c(c)^c) = 0 \text{ (By definition)} \quad (2.0.9)$$

– Thus,

$$\lim_{n \rightarrow \infty} \Pr(|X_n - c| > \epsilon) = \lim_{n \rightarrow \infty} \Pr(X_n \notin B_\epsilon(c)^c) \quad (2.0.10)$$

$$\leq \limsup_{n \rightarrow \infty} \Pr(c \notin B_\epsilon(c)^c) \quad (2.0.11)$$

$$\leq 0 \quad (2.0.12)$$

$$= 0 \quad (2.0.13)$$

(Since probability cannot be negative)

– Thus, by definition,

$$\Pr(|X_n - c| > \epsilon) = 0 \text{ for any } \epsilon > 0 \quad (2.0.14)$$

- Using this property, we can say that options 1 and 2 of the given question are correct because X is constant.
- However, in option 3 when $0 < p < 1$ since X does not take any value with probability 1, i.e X is a constant, convergence in distribution does not imply convergence in probability.
- The fourth option is not true since a weaker statement does not necessarily imply a stronger statement.