

# Assignment 5

S Goutham Sai - CS20BTECH11042

Download all python codes from

<https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/Codes>

and latex-tikz codes from

<https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/main.tex>

## 1 PROBLEM 103 FROM CSIR UGC NET - DEC 2012

Let  $\{X_n : n > 0\}$  and  $X$  be random variables defined on a common probability space. Further assume that  $\{X_n \geq 0 \forall n > 0\}$  and

$$\Pr(X = x) = \begin{cases} p, & x = 0 \\ 1 - p, & x = 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.0.1)$$

where,  $0 \leq p \leq 1$ . Which of the following statements are necessarily true?

- 1) If  $p = 0$  and  $X_n \xrightarrow{d} X$ ,  $X_n \xrightarrow{p} X$
- 2) If  $p = 1$  and  $X_n \xrightarrow{d} X$ , then  $X_n \xrightarrow{p} X$
- 3) If  $0 < p < 1$  and  $X_n \xrightarrow{d} X$ , then  $X_n \xrightarrow{p} X$
- 4) If  $X_n \xrightarrow{p} X$ , then  $X_n \xrightarrow{a.s.} X$

## 2 SOLUTION

For any given sequence of real-valued random variables  $X_1, X_2, X_3, \dots, X_n$  and a random variable  $X$ , let us look at some definitions

**Definition 1. Convergence in Distribution or Weak convergence:**

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x) \quad (2.0.1)$$

where  $F_{X_n}$  and  $F_X$  are the cumulative probability distribution functions of  $X_n$  and  $X$  respectively.

**Definition 2. Convergence in Probability:**

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| > \epsilon) = 0 \forall \epsilon > 0 \quad (2.0.2)$$

This is stronger than the convergence in distribution but weaker than Almost sure convergence.

**Definition 3. Almost sure Convergence:**

$$\Pr\left(\lim_{n \rightarrow \infty} X_n = X\right) = 1 \quad (2.0.3)$$

This type of convergence is stronger than both convergence in distribution and probability.

- In general, stronger statements imply weaker statements but not vice versa, i.e. Convergence in probability implies convergence in distribution and Almost sure convergence implies convergence in probability.

**Lemma 1. Convergence in distribution implies convergence in probability if  $X$  is a constant.**

**Proof:**

- Let  $\epsilon > 0$ . Let  $X = c$  and the sequence  $X_1, X_2, X_3, \dots, X_n$  converges to  $X$  in distribution.
- In mathematics, a ball represents the volume space bounded by a solid sphere. It can either be a closed ball (including the boundary points constituting the sphere) or be an open ball (excluding the boundary points). Let

$$B_c(\epsilon) \rightarrow \text{open ball about } c \text{ of radius } \epsilon \quad (2.0.4)$$

$$B_c(\epsilon)' \rightarrow \text{complement of } B_c(\epsilon) \quad (2.0.5)$$

$$\Rightarrow \Pr(|X_n - c| > \epsilon) = \Pr(X_n \in B_c(\epsilon)') \quad (2.0.6)$$

- From the Portmanteaus lemma,

**Lemma 2. The sequence  $X_1, X_2, X_3, \dots, X_n$  converges in distribution to  $X$  if and only if**

$$\limsup \Pr(X_n \in F) \leq \Pr(X \in F) \quad (2.0.7)$$

for every closed set  $F$ ;

$$\because \lim_{n \rightarrow \infty} X_n = c \quad (2.0.8)$$

$$\implies \limsup_{n \rightarrow \infty} \Pr(X_n \in B_c(\epsilon)') \leq \Pr(c \in B_c(\epsilon)') \quad (2.0.9)$$

$$\Pr(c \in B_c(\epsilon)') = 0 \text{ (By defn)} \quad (2.0.10)$$

• Thus,

$$\lim_{n \rightarrow \infty} \Pr(|X_n - c| > \epsilon) = \lim_{n \rightarrow \infty} \Pr(X_n \in B_c(\epsilon)') \quad (2.0.11)$$

$$\leq \limsup_{n \rightarrow \infty} \Pr(X_n \in B_c(\epsilon)') \quad (2.0.12)$$

$$\leq \Pr(c \in B_c(\epsilon)') \quad (2.0.13)$$

$$\leq 0 \quad (2.0.14)$$

$$= 0 \quad (2.0.15)$$

(Since probability cannot be negative)

• Thus, by definition,

$$\Pr(|X_n - c| > \epsilon) = 0 \text{ for any } \epsilon > 0 \text{ given,}$$

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x) \text{ and } X \text{ is constant} \quad (2.0.16)$$

Let us look at each option one after another.

1) Given,

$$p = 0 \implies X = 1$$

Since  $X$  is a constant, from Lemma 1, we can say that option 1 is true.

2) Given,

$$p = 1 \implies X = 0$$

Since  $X$  is a constant, from Lemma 1, we can say that option 2 is true.

3) Given,

$$0 < p < 1 \implies X \neq 0, 1$$

Since  $X$  is not a constant, we can say that option 3 is false.

4) Since Convergence in probability is weaker than Almost sure convergence, we can say that option 4 is false as a weaker statement does not imply a stronger statement.