

Assignment 5

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Download all python codes from

<https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/Codes>

and latex-tikz codes from

<https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/main.tex>

1 PROBLEM 103 FROM CSIR UGC NET - DEC 2012

Let $\{X_n : n > 0\}$ and X be random variables defined on a common probability space. Further assume that $\{X_n \geq 0 \forall n > 0\}$ and

$$\Pr(X = x) = \begin{cases} p, & x = 0 \\ 1 - p, & x = 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.0.1)$$

where, $0 \leq p \leq 1$. Which of the following statements are necessarily true?

- 1) If $p = 0$ and $X_n \xrightarrow{d} X$, $X_n \xrightarrow{p} X$
- 2) If $p = 1$ and $X_n \xrightarrow{d} X$, then $X_n \xrightarrow{p} X$
- 3) If $0 < p < 1$ and $X_n \xrightarrow{d} X$, then $X_n \xrightarrow{p} X$
- 4) If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{a.s} X$

2 SOLUTION

For any given sequence of real-valued random variables $X_1, X_2, X_3, \dots, X_n$ and a random variable X , let us look at some definitions

Definition 1. Convergence in Distribution or Weak convergence:

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x) \quad (2.0.1)$$

where F_{X_n} and F_X are the cumulative probability distribution functions of X_n and X respectively.

Definition 2. Convergence in Probability:

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| > \epsilon) = 0 \forall \epsilon > 0 \quad (2.0.2)$$

This is stronger than the convergence in distribution but weaker than Almost sure convergence.

Definition 3. Almost sure Convergence:

$$\Pr\left(\lim_{n \rightarrow \infty} X_n = X\right) = 1 \quad (2.0.3)$$

This type of convergence is stronger than both convergence in distribution and probability.

- In general, stronger statements imply weaker statements but not vice versa, i.e. Convergence in probability implies convergence in distribution and Almost sure convergence implies convergence in probability.

Lemma 1. Convergence in distribution implies convergence in probability if X is a constant.

Proof:

- Let $\epsilon > 0$. Let $X = c$ and the sequence $X_1, X_2, X_3, \dots, X_n$ converges to X in distribution.
- Let

$$B_c(\epsilon) \rightarrow \text{open ball about } c \text{ of radius } \epsilon \quad (2.0.4)$$

$$B_c(\epsilon)' \rightarrow \text{complement of } B_c(\epsilon) \quad (2.0.5)$$

$$\Rightarrow \Pr(|X_n - c| > \epsilon) = \Pr(X_n \in B_c(\epsilon)') \quad (2.0.6)$$

- From the Portmanteaus lemma,

Lemma 2. The sequence $X_1, X_2, X_3, \dots, X_n$ converges in distribution to X if and only if

$$\limsup \Pr(X_n \in F) \leq \Pr(X \in F) \quad (2.0.7)$$

for every closed set F ;

$$\because \lim_{n \rightarrow \infty} X_n = c \quad (2.0.8)$$

$$\Rightarrow \limsup_{n \rightarrow \infty} \Pr(X_n \in B_c(\epsilon)') \leq \Pr(c \in B_c(\epsilon)') \quad (2.0.9)$$

$$\Pr(c \in B_c(\epsilon)') = 0 \text{ (By defn)} \quad (2.0.10)$$

- Thus,

$$\lim_{n \rightarrow \infty} \Pr(|X_n - c| > \epsilon) = \lim_{n \rightarrow \infty} \Pr(X_n \in B_c(\epsilon)') \quad (2.0.11)$$

$$\leq \limsup_{n \rightarrow \infty} \Pr(X_n \in B_c(\epsilon)') \quad (2.0.12)$$

$$\leq \Pr(c \in B_c(\epsilon)') \quad (2.0.13)$$

$$\leq 0 \quad (2.0.14)$$

$$= 0 \quad (2.0.15)$$

(Since probability cannot be negative)

- Thus, by definition,

$$\begin{aligned} \Pr(|X_n - c| > \epsilon) &= 0 \text{ for any } \epsilon > 0 \text{ given,} \\ \lim_{n \rightarrow \infty} F_{X_n}(x) &= F_X(x) \text{ and } X \text{ is constant} \end{aligned} \quad (2.0.16)$$

Let us look at each option one after another.

- 1) Given,

$$p = 0 \implies X = 1$$

Since X is a constant, from Lemma 1, we can say that option 1 is true.

- 2) Given,

$$p = 1 \implies X = 0$$

Since X is a constant, from Lemma 1, we can say that option 2 is true.

- 3) Given,

$$0 < p < 1 \implies X \neq 0, 1$$

Since X is not a constant, we can say that option 3 is false.

- 4) Since Convergence in probability is weaker than Almost sure convergence, we can say that option 4 is false as a weaker statement does not imply a stronger statement.