

Assignment 5

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Download all python codes from

<https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/Codes>

and latex-tikz codes from

<https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/main.tex>

1 PROBLEM 103 FROM CSIR UGC NET - DEC 2012

Let $\{X_n : n > 0\}$ and X be random variables defined on a common probability space. Further assume that $\{X_n \geq 0 \forall n > 0\}$ and

$$\Pr(X = x) = \begin{cases} p, & x = 0 \\ 1 - p, & x = 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.0.1)$$

where, $0 \leq p \leq 1$. Which of the following statements are necessarily true?

- 1) If $p = 0$ and $X_n \xrightarrow{d} X$, $X_n \xrightarrow{p} X$
- 2) If $p = 1$ and $X_n \xrightarrow{d} X$, then $X_n \xrightarrow{p} X$
- 3) If $0 < p < 1$ and $X_n \xrightarrow{d} X$, then $X_n \xrightarrow{p} X$
- 4) If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{a.s} X$

2 SOLUTION

For any given sequence of real-valued random variables $X_1, X_2, X_3, \dots, X_n$ and a random variable X , let us look at some definitions

- 1) **Convergence in Distribution or Weak convergence:**

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x) \quad (2.0.1)$$

where F_{X_n} and F_X are the cumulative probability distribution functions of X_n and X respectively.

- 2) **Convergence in Probability:**

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| > \epsilon) = 0 \forall \epsilon > 0 \quad (2.0.2)$$

This is stronger than the convergence in distribution but weaker than Almost sure convergence.

- 3) **Almost sure Convergence:**

$$\Pr\left(\lim_{n \rightarrow \infty} X_n = X\right) = 1 \quad (2.0.3)$$

This type of convergence is stronger than both convergence in distribution and probability.

- In general, stronger statements imply weaker statements but not vice versa, i.e. Convergence in probability implies convergence in distribution and Almost sure convergence implies convergence in probability.

Lemma 1. *Convergence in distribution implies convergence in probability if X is a constant.*

Proof:

- Let $\epsilon > 0$. Let $X = c$ and the sequence $X_1, X_2, X_3, \dots, X_n$ converges to X in distribution.
- Let

$$B_c(\epsilon) \rightarrow \text{open ball about } c \text{ of radius } \epsilon \quad (2.0.4)$$

$$B_c(\epsilon)' \rightarrow \text{complement of } B_c(\epsilon) \quad (2.0.5)$$

$$\Rightarrow \Pr(|X_n - c| > \epsilon) = \Pr(X_n \in B_c(\epsilon)') \quad (2.0.6)$$

- From the Portmanteaus lemma,

Lemma 2. *The sequence $X_1, X_2, X_3, \dots, X_n$ converges in distribution to X if and only if*

$$\limsup \Pr(X_n \in F) \leq \Pr(X \in F) \quad (2.0.7)$$

for every closed set F ;

$$\because \lim_{n \rightarrow \infty} X_n = c \quad (2.0.8)$$

$$\Rightarrow \limsup_{n \rightarrow \infty} \Pr(X_n \in B_c(\epsilon)') \leq \Pr(c \in B_c(\epsilon)') \quad (2.0.9)$$

$$\Pr(c \in B_c(\epsilon)') = 0 \text{ (By defn)} \quad (2.0.10)$$

- Thus,

$$\lim_{n \rightarrow \infty} \Pr(|X_n - c| > \epsilon) = \lim_{n \rightarrow \infty} \Pr(X_n \in B_c(\epsilon)') \quad (2.0.11)$$

$$\leq \limsup_{n \rightarrow \infty} \Pr(X_n \in B_c(\epsilon)') \quad (2.0.12)$$

$$\leq \Pr(c \in B_c(\epsilon)') \quad (2.0.13)$$

$$\leq 0 \quad (2.0.14)$$

$$= 0 \quad (2.0.15)$$

(Since probability cannot be negative)

- Thus, by definition,

$$\begin{aligned} \Pr(|X_n - c| > \epsilon) &= 0 \text{ for any } \epsilon > 0 \text{ given,} \\ \lim_{n \rightarrow \infty} F_{X_n}(x) &= F_X(x) \text{ and } X \text{ is constant} \end{aligned} \quad (2.0.16)$$

Let us look at each option one after another.

- 1) Given,

$$p = 0 \implies X = 1$$

Since X is a constant, from Lemma 1, we can say that option 1 is true.

- 2) Given,

$$p = 1 \implies X = 0$$

Since X is a constant, from Lemma 1, we can say that option 2 is true.

- 3) Given,

$$0 < p < 1 \implies X \neq 0, 1$$

Since X is not a constant, we can say that option 3 is false.

- 4) Since Convergence in probability is weaker than Almost sure convergence, we can say that option 4 is false as a weaker statement does not imply a stronger statement.