# Assignment 5

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Download all python codes from

```
https://github.com/GouthamSai22/AI1103/blob/
    main/Assignment5/Codes
```

and latex-tikz codes from

```
https://github.com/GouthamSai22/AI1103/blob/
    main/Assignment5/main.tex
```

## 1 Problem 103 from CSIR UGC NET - Dec 2012

Let $\{X_n : n > 0\}$ and X be random variables defined on a common probability space. Further assume that $\{X_n \geq 0 \; \forall n > 0\}$ and

$$\Pr(X = x) = \begin{cases} p, & x = 0 \\ 1 - p, & x = 1 \\ 0, & \text{otherwise} \end{cases} \tag{1.0.1}$$

where, $0 \leq p \leq 1$. Which of the following statements are necessarily true?

1) If $p = 0$ and $X_n \xrightarrow{\text{d}} X$, $X_n \xrightarrow{\text{p}} X$
2) If $p = 1$ and $X_n \xrightarrow{\text{d}} X$, then $X_n \xrightarrow{\text{p}} X$
3) If $0 < p < 1$ and $X_n \xrightarrow{\text{d}} X$, then $X_n \xrightarrow{\text{p}} X$
4) If $X_n \xrightarrow{\text{p}} X$, then $X_n \xrightarrow{\text{a.s}} X$

## 2 Solution

For any given sequence of real-valued random variables $X_1, X_2, X_3, \ldots, X_n$ and a random variable X, let us look at some definitions

**Definition 1** (Convergence in Distribution or Weak convergence)**.** *:*

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{2.0.1}$$

*where $F_{X_n}$ and $F_X$ are the cumulative probability distribution functions of $X_n$ and X respectively.*

**Definition 2** (Convergence in Probability)**.** *:*

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0 \forall \epsilon > 0 \tag{2.0.2}$$

This is stronger than the convergence in distribution but weaker than Almost sure convergence.

**Definition 3** (Almost sure Convergence)**.** *:*

$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1 \tag{2.0.3}$$

This type of convergence is stronger than both convergence in distribution and probability.

- In general, stronger statements imply weaker statements but not vice versa, i.e. Convergence in probability implies convergence in distribution and Almost sure convergence implies convergence in probability.
- We shall use the following statement from Portmanteau's Lemma in the following proof:

**Lemma 1** (Portmanteau's Lemma)**.** *:*
*The sequence $X_1, X_2, X_3, \ldots, X_n$ converges in distribution to X if and only if*

$$\limsup \Pr(X_n \in F) \leq \Pr(X \in F) \tag{2.0.4}$$

*for every closed set F;*

**Lemma 2.** *Convergence in distribution implies convergence in probability if X is a constant.*

**Proof**:

- Let $\epsilon > 0$. Let X = c and the sequence $X_1, X_2, X_3, \ldots, X_n$ converges to X in distribution.
- Let

$$S = \{X : |X - c| > \epsilon\} \tag{2.0.5}$$
$$\implies \Pr(|X_n - c| > \epsilon) = \Pr(X_n \in S) \tag{2.0.6}$$

- From Lemma 1,

$$\because \lim_{n\to\infty} X_n = c \tag{2.0.7}$$
$$\implies \limsup_{n\to\infty} \Pr(X_n \in S) \leq \Pr(c \in S) \tag{2.0.8}$$
$$\because \Pr(c \in S) = 0 \text{ (By defn)} \tag{2.0.9}$$
$$\implies \limsup_{n\to\infty} \Pr(X_n \in S) \leq 0 \tag{2.0.10}$$
$$\implies \lim_{n\to\infty} \Pr(X_n \in S) \leq 0 \tag{2.0.11}$$
$$\implies \lim_{n\to\infty} \Pr(X_n \in S) = 0 \text{ (Probability} \geq 0) \tag{2.0.12}$$

- Thus, by definition,

$$\Pr(|X_n - c| > \epsilon) = 0 \text{ for any } \epsilon > 0 \text{ given,}$$
$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \text{ and X is constant} \tag{2.0.13}$$

Let us look at each option one after another.

1) Given,
$$p = 0 \implies X = 1$$
Since X is a constant, from Lemma 1, we can say that option 1 is true.
2) Given,
$$p = 1 \implies X = 0$$
Since X is a constant, from Lemma 1, we can say that option 2 is true.
3) Given,
$$0 < p < 1 \implies X \neq 0, 1$$
Since X is not a constant, we can say that option 3 is false.
4) Since Convergence in probability is weaker than Almost sure convergence, we can say that option 4 is false as a weaker statement does not imply a stronger statement.

Therefore, the true statements from the options are options 1 and 2.