

# Assignment 5

S Goutham Sai - CS20BTECH11042

Download all python codes from

<https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/Codes>

and latex-tikz codes from

<https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/main.tex>

## 1 PROBLEM 103 FROM CSIR UGC NET - DEC 2012

Let  $\{X_n : n > 0\}$  and  $X$  be random variables defined on a common probability space. Further assume that  $X_n$ 's are non-negative and  $X$  takes values 0 and 1 with probabilities  $p$  and  $1-p$  respectively where,  $0 \leq p \leq 1$ . Which of the following statements are necessarily true?

- 1) If  $p=0$  and  $X_n$  converges to  $X$  in distribution,  $X_n$  converges to  $X$  in probability
- 2) If  $p=1$  and  $X_n$  converges to  $X$  in distribution, then  $X_n$  converges to  $X$  in probability
- 3) If  $0 < p < 1$  and  $X_n$  converges to  $X$  in distribution, then  $X_n$  converges to  $X$  in probability
- 4) If  $X_n$  converges to  $X$  in probability, then  $X_n$  converges to  $X$  almost surely

## 2 SOLUTION

For any given sequence of real-valued random variables  $X_1, X_2, X_3, \dots, X_n$ , Let us look at some definitions

- 1) **Convergence in Distribution or Weak convergence:**

$$\lim_{n \rightarrow \infty} F_n(X_n) = F(X) \quad (2.0.1)$$

where  $F$  is the cumulative probability distribution function.

- 2) **Convergence in Probability:**

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| > \epsilon) = 0, \text{ for all } \epsilon > 0 \quad (2.0.2)$$

This is stronger than the convergence in distribution but weaker than Almost sure convergence.

- 3) **Almost sure Convergence:**

$$\Pr\left(\lim_{n \rightarrow \infty} X_n = X\right) = 1 \quad (2.0.3)$$

This type of convergence is stronger than both convergence in distribution and probability.

- In general, stronger statements imply weaker statements but not vice versa, i.e. Convergence in probability implies convergence in distribution and Almost sure convergence implies convergence in probability.

- However, convergence in distribution implies convergence in probability if  $X$  is a constant.

**Proof:**

- Let  $\epsilon > 0$ . Let  $X = c$  and the sequence  $X_1, X_2, X_3, \dots, X_n$  converges to  $X$  in distribution.

- Let

$$B_c(c) \rightarrow \text{open ball about } c \text{ of radius } \epsilon \quad (2.0.4)$$

$$B_c(c)^c \rightarrow \text{complement of } B_c(c) \quad (2.0.5)$$

$$\implies \Pr(|X_n - c| > \epsilon) = \Pr(X_n \in B_c(c)^c) \quad (2.0.6)$$

- By the Portmanteaus lemma(Part C),

$$\text{Since } \lim_{n \rightarrow \infty} X_n = c \quad (2.0.7)$$

$$\implies \Pr(X_n \in B_c(c)^c) \leq \limsup_{n \rightarrow \infty} \Pr(c \in B_c(c)^c) \quad (2.0.8)$$

$$\Pr(c \in B_c(c)^c) = 0 \text{ (By definition)} \quad (2.0.9)$$

– Thus,

$$\lim_{n \rightarrow \infty} \Pr(|X_n - c| > \epsilon) = \lim_{n \rightarrow \infty} \Pr(X_n \notin B_\epsilon(c)) \quad (2.0.10)$$

$$\leq \limsup_{n \rightarrow \infty} \Pr(X_n \notin B_\epsilon(c)) \quad (2.0.11)$$

$$\leq 0 \quad (2.0.12)$$

$$= 0 \quad (2.0.13)$$

(Since probability cannot be negative)

– Thus, by definition,

$$\Pr(|X_n - c| > \epsilon) = 0 \text{ for any } \epsilon > 0 \quad (2.0.14)$$

- Using this property, we can say that options 1 and 2 of the given question are correct because  $X$  is constant.
- However, in option 3 when  $0 < p < 1$  since  $X$  does not take any value with probability 1, i.e  $X$  is a constant, convergence in distribution does not imply convergence in probability.
- The fourth option is not true since a weaker statement does not necessarily imply a stronger statement.