Assignment 5

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Download all python codes from

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https://github.com/GouthamSai22/AI1103/blob/
    main/Assignment5/Codes
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and latex-tikz codes from

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https://github.com/GouthamSai22/AI1103/blob/
    main/Assignment5/main.tex
```

1 Problem 103 from CSIR UGC NET - Dec 2012

Let $\{X_n : n > 0\}$ and X be random variables defined on a common probability space. Further assume that X_n's are non-negative and X takes values 0 and 1 with probabilities p and 1-p respectively where, $0 \leq p \leq 1$. Which of the following statements are necessarily true?

1) If p=0 and X_n converges to X in distribution, X_n converges to X in probability
2) If p=1 and X_n converges to X in distribution, then X_n converges to X in probability
3) If $0 < p < 1$ and X_n converges to X in distribution, then X_n converges to X in probability
4) If X_n converges to X in probability, then X_n converges to X almost surely

2 Solution

For any given sequence of real-valued random variables $X_1, X_2, X_3, \ldots, X_n$ and a random variable X, let us look at some definitions

1) **Convergence in Distribution or Weak convergence**:
$$\lim_{n\to\infty} F_n(X_n) = F(X) \tag{2.0.1}$$
where F is the cumulative probability distribution function.
2) **Convergence in Probability**:
$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0, \text{ for all } \epsilon > 0 \tag{2.0.2}$$
This is stronger than the convergence in distribution but weaker than Almost sure convergence.
3) **Almost sure Convergence**:
$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1 \tag{2.0.3}$$
This type of convergence is stronger than both convergence in distribution and probability.

- In general, stronger statements imply weaker statements but not vice versa, i.e. Convergence in probability implies convergence in distribution and Almost sure convergence implies convergence in probability.

- However, convergence in distribution implies convergence in probability if X is a constant.
Proof:
  - Let $\epsilon > 0$. Let X = c and the sequence $X_1, X_2, X_3, \ldots, X_n$ converges to X in distribution.
  - Let
$$B_c(c) \to \text{open ball about c of radius } \epsilon \tag{2.0.4}$$
$$B_c(c)^c \to \text{complement of } B_c(c) \tag{2.0.5}$$
$$\implies \Pr(|X_n - c| > \epsilon) = \Pr(X_n \epsilon B_c(c)^c) \tag{2.0.6}$$
  - From the Portmanteaus lemma, the sequence $X_1, X_2, X_3, \ldots, X_n$ converges in distribution to X if and only if $\limsup \Pr(X_n \in F) \leq \Pr(X \in F)$ for every closed set F;
$$\text{Since } \lim_{n\to\infty} X_n = c \tag{2.0.7}$$
$$\implies \limsup_{n\to\infty} \Pr(X_n \epsilon B_c(c)^c) \leq \Pr(c \epsilon B_c(c)^c) \tag{2.0.8}$$
$$\Pr(c \epsilon B_c(c)^c) = 0 \text{ (By defn)} \tag{2.0.9}$$

- Thus,

$$\lim_{n\to\infty} \Pr\left(|X_n - c| > \epsilon\right) = \lim_{n\to\infty} \Pr\left(X_n \epsilon B_c(c)^c\right) \tag{2.0.10}$$

$$\leq \limsup_{n\to\infty} \Pr\left(X_n \epsilon B_c(c)^c\right) \tag{2.0.11}$$

$$\leq \Pr\left(c \epsilon B_c(c)^c\right) \tag{2.0.12}$$

$$\leq 0 \tag{2.0.13}$$

$$= 0 \tag{2.0.14}$$

(Since probability cannot be negative)

- Thus, by definition,

$$\Pr\left(|X_n - c| > \epsilon\right) = 0 \text{ for any } \epsilon > 0 \tag{2.0.15}$$

- Using this property, we can say that options 1 and 2 of the given question are correct because X is constant.
- However, in option 3 when 0 < p < 1 since X does not take any value with probability 1 ,i.e X is a constant, convergence in distribution does not imply convergence in probability.
- The fourth option is not true since a weaker statement does not necessarily imply a stronger statement.