Assignment 5

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Download all python codes from

https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/Codes

and latex-tikz codes from

https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/main.tex

1 Problem 103 from CSIR UGC NET - Dec 2012

Let $\{X_n : n > 0\}$ and X be random variables defined on a common probability space. Further assume that $\{X_n \geq 0 \ \forall n > 0\}$ and

$$\Pr(X = x) = \begin{cases} p, & x = 0 \\ 1 - p, & x = 1 \\ 0, & \text{otherwise} \end{cases} \tag{1.0.1}$$

where, $0 \leq p \leq 1$. Which of the following statements are necessarily true?

1) If $p = 0$ and $X_n \xrightarrow{\text{d}} X$, $X_n \xrightarrow{\text{p}} X$
2) If $p = 1$ and $X_n \xrightarrow{\text{d}} X$, then $X_n \xrightarrow{\text{p}} X$
3) If $0 < p < 1$ and $X_n \xrightarrow{\text{d}} X$, then $X_n \xrightarrow{\text{p}} X$
4) If $X_n \xrightarrow{\text{p}} X$, then $X_n \xrightarrow{\text{a.s}} X$

2 Solution

For any given sequence of real-valued random variables $X_1, X_2, X_3, \ldots, X_n$ and a random variable X, let us look at some definitions

Definition 1. ***Convergence in Distribution or Weak convergence***:

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{2.0.1}$$

where F_{X_n} and F_X are the cumulative probability distribution functions of X_n and X respectively.

Definition 2. ***Convergence in Probability***:

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0 \forall \epsilon > 0 \tag{2.0.2}$$

This is stronger than the convergence in distribution but weaker than Almost sure convergence.

Definition 3. ***Almost sure Convergence***:

$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1 \tag{2.0.3}$$

This type of convergence is stronger than both convergence in distribution and probability.

- In general, stronger statements imply weaker statements but not vice versa, i.e. Convergence in probability implies convergence in distribution and Almost sure convergence implies convergence in probability.

Lemma 1. *Convergence in distribution implies convergence in probability if X is a constant.*

Proof:

- Let $\epsilon > 0$. Let X = c and the sequence $X_1, X_2, X_3, \ldots, X_n$ converges to X in distribution.
- Let

$$B_c(\epsilon) \to \text{open ball about c of radius } \epsilon \tag{2.0.4}$$

$$B_c(\epsilon)' \to \text{complement of } B_c(\epsilon) \tag{2.0.5}$$

$$\implies \Pr(|X_n - c| > \epsilon) = \Pr(X_n \in B_c(\epsilon)') \tag{2.0.6}$$

- From the Portmanteaus lemma,

  Lemma 2. *The sequence $X_1, X_2, X_3, \ldots, X_n$ converges in distribution to X if and only if*

$$\limsup \Pr(X_n \in F) \leq \Pr(X \in F) \tag{2.0.7}$$

  for every closed set F;

$$\because \lim_{n\to\infty} X_n = c \tag{2.0.8}$$

$$\implies \limsup_{n\to\infty} \Pr(X_n \in B_c(\epsilon)') \leq \Pr(c \in B_c(\epsilon)') \tag{2.0.9}$$

$$\Pr(c \in B_c(\epsilon)') = 0 \text{ (By defn)} \tag{2.0.10}$$

- Thus,

$$
\begin{aligned}
\lim_{n\to\infty} \Pr\left(|X_n - c| > \epsilon\right) &= \lim_{n\to\infty} \Pr\left(X_n \in B_c(\epsilon)'\right) && (2.0.11) \\
&\le \limsup_{n\to\infty} \Pr\left(X_n \in B_c(\epsilon)'\right) && (2.0.12) \\
&\le \Pr\left(c \in B_c(\epsilon)'\right) && (2.0.13) \\
&\le 0 && (2.0.14) \\
&= 0 && (2.0.15)
\end{aligned}
$$

(Since probability cannot be negative)

- Thus, by definition,

$$
\begin{aligned}
\Pr\left(|X_n - c| > \epsilon\right) = 0 \text{ for any } \epsilon > 0 \text{ given,} \\
\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \text{ and X is constant}
\end{aligned} \tag{2.0.16}
$$

Let us look at each option one after another.

1) Given,

$$p = 0 \implies X = 1$$

Since X is a constant, from Lemma 1, we can say that option 1 is true.

2) Given,

$$p = 1 \implies X = 0$$

Since X is a constant, from Lemma 1, we can say that option 2 is true.

3) Given,

$$0 < p < 1 \implies X \neq 0, 1$$

Since X is not a constant, we can say that option 3 is false.

4) Since Convergence in probability is weaker than Almost sure convergence, we can say that option 4 is false as a weaker statement does not imply a stronger statement.