CSIR UGC NET Dec 2012 Q.103

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Question

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Let $\{X_n : n > 0\}$ and X be random variables defined on a common probability space. Further assume that $\{X_n \geq 0 \; \forall n > 0\}$ and

$$\Pr(X = x) = \begin{cases} p, & x = 0 \\ 1-p, & x = 1 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where, $0 \leq p \leq 1$. Which of the following statements are necessarily true?

1. If $p = 0$ and $X_n \xrightarrow{\text{d}} X$, $X_n \xrightarrow{\text{p}} X$
2. If $p = 1$ and $X_n \xrightarrow{\text{d}} X$, then $X_n \xrightarrow{\text{p}} X$
3. If $0 < p < 1$ and $X_n \xrightarrow{\text{d}} X$, then $X_n \xrightarrow{\text{p}} X$
4. If $X_n \xrightarrow{\text{p}} X$, then $X_n \xrightarrow{\text{a.s}} X$

Definitions

Convergence in Distribution

For any given sequence of real-valued random variables $X_1, X_2, X_3, \ldots, X_n$ and a random variable X,

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{2}$$

where F_{X_n} and F_X are the cumulative probability distribution functions of X_n and X respectively.

Definitions

Convergence in Probability

For any given sequence of real-valued random variables $X_1, X_2, X_3, \ldots, X_n$ and a random variable X,

$$\lim_{n\to\infty} \Pr\left(|X_n - X| > \epsilon\right) = 0 \forall \epsilon > 0 \tag{3}$$

Almost sure Convergence

For any given sequence of real-valued random variables $X_1, X_2, X_3, \ldots, X_n$ and a random variable X,

$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1 \tag{4}$$

Portmanteaus Lemma

Portmanteaus Lemma

The sequence $X_1, X_2, X_3, \ldots, X_n$ converges in distribution to X if and only if

$$\limsup \Pr(X_n \in F) \leq \Pr(X \in F) \tag{5}$$

for every closed set F;

Solution

Lemma

Convergence in distribution implies convergence in probability if X is a constant.

Proof

- Let $\epsilon > 0$. Let X = c and the sequence $X_1, X_2, X_3, \ldots, X_n$ converges to X in distribution.
- Let

$$S = \{X : |X - c| > \epsilon\} \tag{6}$$

$$\implies \Pr\left(|X_n - c| > \epsilon\right) = \Pr\left(X_n \in S\right) \tag{7}$$

Solution

Proof

- From Portmanteau's Lemma,

$$\because \lim_{n\to\infty} X_n = c \tag{8}$$

$$\implies \limsup_{n\to\infty} \Pr\left(X_n \in S\right) \leq \Pr\left(c \in S\right) \tag{9}$$

$$\because \Pr\left(c \in S\right) = 0 \text{ (By defn)} \tag{10}$$

$$\implies \limsup_{n\to\infty} \Pr\left(X_n \in S\right) \leq 0 \tag{11}$$

$$\implies \lim_{n\to\infty} \Pr\left(X_n \in S\right) \leq 0 \tag{12}$$

$$\implies \lim_{n\to\infty} \Pr\left(X_n \in S\right) = 0 \text{ (Probability} \geq 0\text{)} \tag{13}$$

- Thus, by definition,

$$\Pr\left(|X_n - c| > \epsilon\right) = 0 \text{ for any } \epsilon > 0 \text{ given,}$$

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \text{ and X is constant} \tag{14}$$

Solution

Checking Options

1. Given,

$$p = 0 \implies X = 1$$

Since X is a constant, from Lemma, we can say that option 1 is true.

2. Given,

$$p = 1 \implies X = 0$$

Since X is a constant, from Lemma, we can say that option 2 is true.

3. Given,

$$0 < p < 1 \implies X \neq 0, 1$$

Since X is not a constant, we can say that option 3 is false.

Solution

Checking Options

1. Since Convergence in probability is weaker than Almost sure convergence, we can say that option 4 is false as a weaker statement does not imply a stronger statement.

Therefore, the true statements from the options are options 1 and 2.