Assignment 5

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Download all python codes from

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https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/Codes
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and latex-tikz codes from

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https://github.com/GouthamSai22/AI1103/blob/main/Assignment5/main.tex
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1 Problem 103 from CSIR UGC NET - Dec 2012

Let $\{X_n : n > 0\}$ and X be random variables defined on a common probability space. Further assume that X_n's are non-negative and X takes values 0 and 1 with probabilities p and 1-p respectively where, $0 \le p \le 1$. Which of the following statements are necessarily true?

1) If p=0 and X_n converges to X in distribution, X_n converges to X in probability
2) If p=1 and X_n converges to X in distribution, then X_n converges to X in probability
3) If $0 < p < 1$ and X_n converges to X in distribution, then X_n converges to X in probability
4) If X_n converges to X in probability, then X_n converges to X almost surely

2 Solution

For any given sequence of real-valued random variables $X_1, X_2, X_3, \dots, X_n$

1) **Convergence in Distribution**: The sequence is said to converge to X in distribution, or converge weakly if
$$\lim_{n\to\infty} F_n(X_n) = F(X) \tag{2.0.1}$$
where F is the cumulative probability distribution function.

2) **Convergence in Probability**: The sequence is said to converge to X in probability if
$$\lim_{n\to\infty} \Pr\left(|X_n - X| > \epsilon\right) = 0, \text{ for all } \epsilon > 0 \tag{2.0.2}$$
This is stronger than the convergence in distribution but weaker than Almost sure convergence.

3) **Almost sure Convergence**: The sequence is said to converge to X almost surely if
$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1 \tag{2.0.3}$$
This type of convergence is stronger than both convergence in distribution and probability.

- In general, stronger statements imply weaker statements but not vice versa, i.e. Convergence in probability implies convergence in distribution and Almost sure convergence implies convergence in probability.

- However, convergence in distribution implies convergence in probability if X is a constant.
Proof:
  – Let $\epsilon > 0$. Let X = c and the sequence $X_1, X_2, X_3, \dots, X_n$ converges to X in distribution.
  – Let $B_c(c)$ denote the open ball of radius ϵ and $B_c(c)^c$ denote it's complement. Then, $\Pr\left(|X_n - c| > \epsilon\right) = \Pr\left(X_c \epsilon B_c(c)^c\right)$.
  – By the Portmanteaus lemma(Part C), if X_n converges to c, then
$$\Pr\left(X_c \epsilon B_c(c)^c\right) \le \limsup_{n\to\infty} \Pr\left(c \epsilon B_c(c)^c\right)$$
, the latter of which is obviously zero as c does not belong to the complement of the open ball space around itself.
  – Thus,
$$\lim_{n\to\infty} \Pr\left(|X_n - c| > \epsilon\right) = \lim_{n\to\infty} \Pr\left(X_n \epsilon B_c(c)^c\right) \tag{2.0.4}$$
$$\le \limsup_{n\to\infty} \Pr\left(c \epsilon B_c(c)^c\right) \tag{2.0.5}$$
$$\le 0 \tag{2.0.6}$$
$$= 0 \tag{2.0.7}$$

(Since probability cannot be negative)
  - This, by definition implies that the sequence $X_1, X_2, X_3, \ldots, X_n$ converges to X in probability.

- Using this property, we can say that options 1 and 2 of the given question are correct because when p = 0 X takes the value 1 with probability 1, i.e. X = 1(a constant) and when p = 1, X takes the value 0 with probability 1, i.e. X = 0(a constant).
- Hence, in both these cases if X_n converges to X in distribution, then X_n converges to X in probability.
- However, in option 3 when 0 ¡ p ¡ 1 since X does not take any value with probability 1 ,i.e X is a constant, convergence in distribution does not imply convergence in probability.
- The fourth option is not true since a weaker statement does not necessarily imply a stronger statement.