# Assignment 5

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## 1 PROBLEM 103 FROM CSIR UGC NET - DEC 2012

Let $\{X_n : n > 0\}$ and X be random variables defined on a common probability space. Further assume that $X_n$'s are non-negative and X takes values 0 and 1 with probabilities $p$ and $1 - p$ respectively where, $0 \leq p \leq 1$. Which of the following statements are necessarily true?

1) If $p = 0$ and $X_n$ converges to X in distribution, $X_n$ converges to X in probability
2) If $p = 1$ and $X_n$ converges to X in distribution, then $X_n$ converges to X in probability
3) If $0 < p < 1$ and $X_n$ converges to X in distribution, then $X_n$ converges to X in probability
4) If $X_n$ converges to X in probability, then $X_n$ converges to X almost surely

## 2 SOLUTION

For any given sequence of real-valued random variables $X_1, X_2, X_3, \ldots, X_n$ and a random variable X, let us look at some definitions

1) **Convergence in Distribution or Weak convergence**:

$$\lim_{n \to \infty} F_n(X_n) = F(X) \qquad (2.0.1)$$

where F is the cumulative probability distribution function.

2) **Convergence in Probability**:

$$\lim_{n \to \infty} \Pr\left(|X_n - X| > \epsilon\right) = 0, \text{ for all } \epsilon > 0 \qquad (2.0.2)$$

This is stronger than the convergence in distribution but weaker than Almost sure convergence.

3) **Almost sure Convergence**:

$$\Pr\left(\lim_{n \to \infty} X_n = X\right) = 1 \qquad (2.0.3)$$

This type of convergence is stronger than both convergence in distribution and probability.

- In general, stronger statements imply weaker statements but not vice versa, i.e. Convergence in probability implies convergence in distribution and Almost sure convergence implies convergence in probability.

**Lemma 1.** *Convergence in distribution implies convergence in probability if X is a constant.*

**Proof**:

- Let $\epsilon > 0$. Let X = c and the sequence $X_1, X_2, X_3, \ldots, X_n$ converges to X in distribution.
- Let

$$B_c(\epsilon) \to \text{open ball about c of radius } \epsilon \qquad (2.0.4)$$

$$B_c(\epsilon)' \to \text{complement of } B_c(\epsilon) \qquad (2.0.5)$$

$$\implies \Pr\left(|X_n - c| > \epsilon\right) = \Pr\left(X_n \in B_c(\epsilon)'\right) \qquad (2.0.6)$$

- From the Portmanteaus lemma,

**Lemma 2.** *The sequence $X_1, X_2, X_3, \ldots, X_n$ converges in distribution to X if and only if*

$$\limsup \Pr\left(X_n \in F\right) \leq \Pr\left(X \in F\right) \qquad (2.0.7)$$

*for every closed set F;*

$$\text{Since } \lim_{n \to \infty} X_n = c \qquad (2.0.8)$$

$$\implies \limsup_{n \to \infty} \Pr\left(X_n \in B_c(\epsilon)'\right) \leq \Pr\left(c \in B_c(\epsilon)'\right) \qquad (2.0.9)$$

$$\Pr\left(c \in B_c(\epsilon)'\right) = 0 \text{ (By defn)} \qquad (2.0.10)$$

- Thus,

$$\lim_{n\to\infty} \Pr\left(|X_n - c| > \epsilon\right) = \lim_{n\to\infty} \Pr\left(X_n \in B_c(\epsilon)'\right)$$
(2.0.11)

$$\leq \limsup_{n\to\infty} \Pr\left(X_n \in B_c(\epsilon)'\right)$$
(2.0.12)

$$\leq \Pr\left(c \in B_c(\epsilon)'\right)$$
(2.0.13)

$$\leq 0 \qquad \qquad (2.0.14)$$

$$= 0 \qquad \qquad (2.0.15)$$

(Since probability cannot be negative)
- Thus, by definition,

$$\Pr\left(|X_n - c| > \epsilon\right) = 0 \text{ for any } \epsilon > 0 \text{ given,}$$
$$\lim_{n\to\infty} F_n(X_n) = F(X) \text{ and X is constant}$$
(2.0.16)

Let us look at each option one after another.

1) Given,

$$p = 0 \implies X = 1$$

Since X is a constant, from Lemma 1, we can say that option 1 is true.

2) Given,

$$p = 1 \implies X = 0$$

Since X is a constant, from Lemma 1, we can say that option 2 is true.

3) Given,

$$0 < p < 1 \implies X \neq 0, 1$$

Since X is not a constant, we can say that option 3 is false.

4) Since Convergence in probability is weaker than Almost sure convergence, we can say that option 4 is false as a weaker statement does not imply a stronger statement.