A Social Choice Analysis of Retroactive Funding

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Abstract

Retrofunding by the Optimism Foundation involves a community wishing to grant public goods retroactively, i.e., to reward projects that, in hindsight, are deemed valuable for the community. Currently, we observe that what the community needs is a majority-based strategy-proof portioning rule rather than proportional allocation. Therefore, we investigate the relevant rules for these circumstances, particularly from the Computational Social Choice (COMSOC) literature, conducting both theoretical analyses and practical evaluations using generated and real data from Optimism. Based on these findings, we recommended the implementation of a type of moving phantoms voting rule [9] for the upcoming funding rounds.

1 Introduction

In this paper, we study Retro Funding from the perspective of computational social choice. Retro Funding is a unique process aimed at rewarding public goods *ex post*, meaning that it prizes projects which, in retrospect, have proven to be beneficial for the community. Retro Funding has been employed in decentralized ecosystems like Optimism to incentivize and reward contributors who have created valuable public goods. This approach is gaining traction as an innovative funding mechanism for communities where it is often difficult to evaluate contributions in real-time. Other examples of similar retroactive funding models can be found in blockchain ecosystems that seek to reward public goods and positive externalities in areas such as governance, open-source software, and decentralized infrastructure.

Our analysis focuses on the social choice mechanisms employed in the Optimism Retro Funding system, a decentralized framework that aims to allocate funds to impactful projects in a fair and transparent manner. Optimism's Retro Funding is part of a broader economic model for public goods, designed to incentivize innovation and reward positive contributions to the ecosystem. The initiative has undergone several iterations, with the current study taking place during the fifth round of Retro Funding.[1].

A distinguishing feature of the Retro Funding procedure is that the decisions on fund allocation are made by a group of certified badge holders who have both stake and reputation within the ecosystem. These badge holders are selected by the broader collective and tasked with distributing tokens to working projects based on their evaluation of each project's needs. Each funding round targets specific contexts; for example, the upcoming Round 6 is focused on governance-oriented projects.

A general theme that we highlight about Optimism's Retro Funding is its reliance on a small group of expert voters. The process aims for a form of majoritarian decision-making, where the goal is to capture the "ground truth" of what contributions have had the most impact on the community. Additionally, the badge holders have a substantial responsibility in determining how the tokens are allocated to the projects they evaluate, ensuring that the community's overall interests are prioritized.

In this paper, we are particularly interested in comparing various voting rules that govern the retrofunding process. Our goal is to evaluate how these rules behave across several key dimensions, outlined in the governance document [2]:

- **Iteration**, where the system iteratively decentralizes to learn and improve decision-making processes over time, much like adaptive mechanisms in social choice where feedback and adjustments refine collective decisions.

[1] Retro 1: `https://medium.com/ethereum-optimism/retropgf-experiment`, Retro 2: `https://optimism.mirror.xyz/7v1DehEY3dpRcYFhqWrVNc9Qj94H2L976LKlWH1FX-8`, Retro 3: `https://gov.optimism.io/t/retropgf-3-round-design/6802`, Retro 4: `https://optimism.mirror.xyz/7v1DehEY3dpRcYFhqWrVNc9Qj94H2L976LKlWH1FX-8`, Retro 5: `https://gov.optimism.io/t/retro-funding-5-op-stack-round-details/8612`

[2] `https://gov.optimism.io/t/the-future-of-optimism-governance/6471`

- **Balance**, which emphasizes extending governance influence beyond financial power, integrating concepts of equality and fairness—core themes in social choice theory aimed at preventing plutocracy and ensuring a balanced representation of voices.
- **Impact = profit**, the idea that individuals should be rewarded based on their positive contributions, mirroring social choice concepts that emphasize matching rewards to contributions for fairness and optimal collective outcomes.

The central objective of this study is to provide a comparative analysis of the social choice mechanisms used in Optimism's Retro Funding and recommended mechanisms we believe would make a good fit (according to Optimism's goals), with a focus on evaluating their robustness, strategy proofness (incentive compatibility), representing the majority of voters, and effectiveness in promoting fair and efficient outcomes. By exploring these mechanisms through a computational lens, we aim to contribute to a deeper understanding of how decentralized decision-making processes can be optimized for public good funding in a blockchain-based environment.

1.1 Paper Structure

Technically, we discuss the following:

- A formal model of retrofunding (Section 2).
- Related work (Section 3).
- Voting rules used for retrofunding (Section 4).
- Concrete evaluation metrics (Section 5).
- Theoretical analysis (Section 6).
- Experimental analysis (Section 7).
- Results and Discussion (Section 8)

2 Formal Model

We provide a formal model for (token-based) retrofunding.

2.1 Definitions

- $N = \{1, 2, \ldots, n\}$: Set of voters.
- $P = \{1, 2, \ldots, m\}$: Set of projects.
- $B \in \mathbb{N}$: Total budget available for distribution, normalized to 1.
- $c \in \mathbb{N}$: The amount of tokens given to each voter to distribute among the projects (under the constraint of $c \cdot n \leq B$), normalized to 1.

2.2 Budget Allocations

An allocation $\mathbf{x} = \{a_1, \cdots, a_m\}$ represents the distribution of tokens among projects. Each project p receives a_p tokens, where $a_p \in [0, 1]$. We denote A as a the group of all feasible allocations $\mathbf{x} = \{a_1, \cdots, a_m\}$, such that $\sum_{p \in P} a_p \leq 1$.

2.3 Ballot Design and Vote Aggregation

Each voter is given k tokens to distribute among the projects, a mechanism referred to in the literature as *cumulative ballot.* Formally we denote $x_{i,p} \in \mathbb{R}$

to represent the amount of tokens voter i allocates to project p, subject to the constraint:

$$\sum_{p \in P} x_{i,p} \leq 1$$

and $X_i \in \mathbb{R}^m$ is the vector of funds distribution provided by voter i. The aggregation of votes (i.e., funds distribution) is given by a voting rule now presented formally as a function:

$$f : (\{X_1, \cdots, X_n\}) \rightarrow \{a_1, \cdots a_m\} = \mathbf{x}$$

ensuring that the total budget constraint is respected:

$$\sum_{p \in P} a_p \leq 1,$$

Remark 1. *Notice that we model the case where the entire budget is allocated:*

$$\sum_{p \in P} a_p = 1,$$

but voters are not required to allocate all their c tokens:

$$\sum_{p \in P} x_{i,p} \leq 1.$$

However, there is a strategic advantage for voters to allocate all their tokens, as this maximizes their voting power. Therefore, we can assume:

$$\sum_{p \in P} x_{i,p} = 1.$$

3 Related Work

3.1 Public Good Funding

Public good funding refers to mechanisms for allocating resources to services or projects that benefit an entire community. In decentralized environments, these models have been adapted to ensure fair and efficient resource distribution. One notable example is quadratic funding [6], which is designed to amplify the influence of smaller individual contributions, preventing wealthier stakeholders from disproportionately controlling funding decisions. The goal is to enhance fairness in funding decisions, which is particularly important in decentralized systems where communities collectively govern resource allocation.

Participatory budgeting [2, 8], another democratic process closely related to public good funding, allows voters to decide directly on how to allocate resources before projects are implemented. However, retrofunding, as used in Optimism, is fundamentally different: it allocates resources retrospectively to projects that have already been completed, rewarding them based on their demonstrated impact rather than on anticipated outcomes.

3.2 Retrofunding and Portioning

Retrofunding, as used in Optimism, represents an innovative shift in public good funding models. Instead of distributing funds upfront, retrofunding evaluates the impact of completed projects and rewards those that have had the most positive effects on the community. This distinguishes it from traditional participatory budgeting and crowdfunding models, where projects are funded in advance [17]. Optimism's retrofunding model provides a unique way to encourage accountability, as only projects that deliver value to the community are rewarded.

Portioning [1], another relevant concept, involves determining how to divide a limited pool of resources among competing projects. In retrofunding, this is done based on an ex post evaluation of which projects had the most significant positive impact. Optimism leverages portioning by relying on a select group of expert voters (badge holders) who evaluate the projects and determine the distribution of funds.

3.3 Optimism's Retrofunding

Optimism's retrofunding model operates within the framework of a Decentralized Autonomous Organization (DAO), which allows decentralized decision-making through smart contracts. While DAOs like Optimism employ blockchain to ensure transparency and immutability, the retrofunding mechanism itself aligns with principles from social choice theory [16]. In particular, Optimism's governance system focuses on majoritarian decision-making, where badge holders with expertise aim to reflect the "ground truth" of what projects contributed the most to the community.
The governance structure in Optimism's DAO draws from both blockchain governance studies [11] and social choice theory [10], bridging these two fields by applying computational social choice mechanisms to decentralized resource allocation. Optimism's retrofunding has undergone several iterations, with the goal of refining decision-making processes that are fair, transparent, and efficient. Badge holders evaluate projects, ensuring that resources are allocated to those with the most demonstrable impact on the community.

3.4 Preference Aggregation and Ground Truth

A key challenge in retrofunding is aggregating preferences in a way that uncovers the true impact of each project. This is closely related to epistemic social choice, where the objective is to aggregate individual preferences to reveal an accurate understanding of reality. Methods such as the Bayesian Truth Serum [14, 13] and other preference aggregation techniques [7] have been explored in this context, aiming to ensure that the projects with the most genuine support and the greatest positive impact receive funding.
In Optimism's retrofunding, the goal is to ensure that the projects that have had the most positive contributions to the community are rewarded. By combining preference aggregation techniques with blockchain technology, the process ensures transparency, accountability, and resistance to manipulation. This builds on the foundations of public good funding and participatory budgeting while introducing new mechanisms for ex post evaluation and resource distribution.

4 Voting Rules

In this section, we consider both the voting rules currently used by Optimism (OP) in their RetroPGF rounds, as well as some of our suggested adaptations based on discussions with them.

4.1 OP RetroPGF Round 1 - Quadratic Voting

Quadratic Voting (QV) allows voters to express the intensity of their preferences by purchasing votes, where the cost of votes increases quadratically. In OP RetroPGF Round 1, voters receive a predefined amount of tokens for voting. The voting distribution is calculated to meet the properties of the Quadratic Voting rule.

- Each voter $i \in N$ distributes $x_{i,p}$ tokens to project p.
- The true vote value is $\sqrt{x_{i,p}}$.
- The final vote distribution to project p is:

$$a_p = \frac{\sum_{i \in N} \sqrt{x_{i,p}}}{\sum_{i \in N} \sum_{p' \in P} \sqrt{x_{i,p'}}}$$

4.2 OP RetroPGF Round 2 - R2 Mean Rule

In OP RetroPGF Round 2, a weighted average (Mean Rule) was used. Each voter's allocation to a project is weighted by their total allocation.

- Each voter $i \in N$ allocates $x_{i,p}$ tokens to project p.
- The weight of project p is calculated as:

$$a_p = \frac{\sum_{i \in N} x_{i,p}}{\sum_{i \in N} \sum_{p' \in P} x_{i,p'}}$$

4.3 OP RetroPGF Round 3 - Quorum Median Rule

In Round 3, Optimism used the Quorum Median Rule. Here, the median of voters' allocations is used, with additional constraints on quorum (minimal amount of votes or tokens for funding).

- Each voter $i \in N$ allocates $x_{i,p}$ tokens to project p.
- Quorum is enforced via q_1 (minimum tokens) and q_2 (minimum votes).
- The median allocation for each project is calculated as:

$$b_p = \frac{\text{median}\{x_{i,p} | x_{i,p} > 0, i \in N\}}{\sum_{p' \in P} \text{median}\{x_{i,p'} | x_{i,p'} > 0, i \in N\}}$$

- The quorum condition is applied as:

$$d_p = \begin{cases} b_p & \text{if } b_p \geq q_1 \text{ and } \sum_{i \in N} c_{i,p} > q_2 \\ 0 & \text{else} \end{cases}$$

- Finally, the allocation is normalized:

$$a_p = \frac{d_p}{\sum_{p' \in P} d_{p'}}$$

4.4 OP RetroPGF Round 4 - Capped Median Rule

In the simplified Round 4 rule (since the full version of R4 includes voting on metrics, which we discuss in the outlook section as it is out of this work's scope), the median allocation for each project is capped, and further adjustments are made to redistribute tokens.

- Each voter $i \in N$ allocates $x_{i,p}$ tokens to project p having $x_{i,p} \leq K_1$ for all $i \in [n]$ and $p \in P$, and the initial allocation is:

$$c_p = \frac{\text{median}\{x_{1,p}, x_{2,p}, \ldots, x_{n,p}\}}{\sum_{p' \in P} \text{median}\{x_{1,p'}, x_{2,p'}, \ldots, x_{n,p'}\}} \cdot B$$

- The capped allocation is adjusted:

$$d_p = \min(c_p, K_2) + \frac{(\sum_{j \in P} \max(0, c_j - K_2)) \cdot c_p}{\sum_{j \in P} c_j}$$

- Projects receiving less than K_3 tokens are omitted, and their allocations are redistributed.

$$b_p = \begin{cases} 0, & \text{if } d_p < K_3 \\ d_p + \frac{\sum_{p' \in P, d_{p'} < K_3} d_{p'}}{\sum_{p' \in P, d_{p'} \geq K_3} d_{p'}} \cdot d_p, & \text{otherwise} \end{cases}$$

- The final allocation is normalized:

$$a_p = \frac{b_p}{\sum_{p' \in P} b_{p'}}$$

4.5 Adapted Voting Rules

Based on our discussions with OP, we propose two adapted rules to address potential vulnerabilities or weaknesses observed in the above rules.

4.5.1 Midpoint Rule

The Midpoint Rule selects the allocation (ballot) that minimizes the ℓ_1 distance from all other voters' allocations [12].

$$\mathrm{MR}(X_i) = \sum_{j=1}^{n} \|X_i - X_j\|_1$$

The chosen allocation is:

$$\mathbf{x} = X_{i^*} \quad \text{where} \quad i^* = \arg\min_{i \in N} \mathrm{MR}(X_i)$$

The ℓ_1 distance between two allocations X_i and X_j is:

$$\|X_i - X_j\|_1 = \sum_{p \in P} |x_{i,p} - x_{j,p}|$$

4.5.2 Moving Phantoms Rules

The Moving Phantoms family of voting rules is designed to address a common problem in median voting, where the outcome does not necessarily sum to the total available funds. In median voting, the allocations are determined by the median of voters' preferences, but this can lead to inconsistencies when the total allocation exceeds or falls short of the total funds. Optimism uses a simple normalization to fix this but we show this causes many problems. To resolve this, artificial phantom voters (often $n+1$ phantoms) are introduced to ensure that the final allocation sums correctly.

The phantoms act as additional voters, whose preferences help adjust the final allocation to maintain the desired total distribution. Each phantom's vote is designed to balance the influence of the real voters and ensure that the outcome is normalized correctly. Some mechanisms in this family satisfy specific qualities related to fairness and efficiency, as explored in the the previous works [9].

Formally phantom system $\mathcal{F}$ is a family of functions $\{f_k : k \in \{0, \dots, n\}\}$, where each $f_k : [0,1] \to [0,1]$ is continuous and satisfies:

- $f_k(0) = 0$
- $f_k(1) = 1$
- $f_0(t) \geq f_1(t) \geq \cdots \geq f_n(t)$ for all $t \in [0,1]$

The final allocation is computed using:

$$\mathcal{A}^F(a_p) = \mathrm{med}(f_0(t^*), \dots, f_n(t^*), x_{1,p}, \dots, x_{n,p})$$

where t^* is chosen such that [3]:

$$t^* \in \left\{ t : \sum_{p \in P} \mathrm{med}(f_0(t), f_1(t), \dots, f_n(t), x_{1,p}, \dots, x_{n,p}) = 1 \right\}.$$

We consider two types of phantom based mechanisms [9]:

1. **The Independent Markets Algorithm-** The Independent Market algorithm denoted by $\mathcal{A}^{IM}$ is a moving phantom algorithm given by:

$$f_k(t) = \min\{t(n-k), 1\}$$

2. **Majoritarian Phantoms Algorithm-** denoted by $\mathcal{A}^{F*}$ is given by:

$$f_k(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{k}{n+1}, \\ t(n+1) - k & \text{if } \frac{k}{n+1} < t \leq \frac{k+1}{n+1}, \\ 1 & \text{if } \frac{k+1}{n+1} \leq t \leq 1. \end{cases}$$

[3] t^* can be found using a simple binary search

5 Evaluation Metrics and Properties

We define several evaluation metrics and theoretical properties to compare the voting rules both theoretically and via simulations. These metrics correspond to the goals presented in the introduction, ensuring that we can measure important aspects such as fairness, robustness, and resistance to manipulation. Below, we provide formal definitions of the metrics and explain their relevance.

5.1 Resistance to Malicious Behavior

Voting systems need to be resistant to manipulation attempts, whether through bribery or strategic control. Two key metrics capture this:

5.1.1 Bribery

Bribery refers to the ability of an external party to influence the outcome by paying voters to change their preferences. The formal objective is to determine the minimum budget required to increase the allocation to a specific project p by X tokens. Let $V = \{X_1, \ldots, X_n\}$ be the initial vote profile, and $\mathbf{x} = \{a_1, \ldots, a_m\}$ the corresponding outcome under voting rule f. The cost to move one token from one project to another is 1. The minimum budget b to modify the profile V' such that the allocation to project p is increased by X is given by:

$$b = \sum_{i=1}^{n} \sum_{q \in P} |x_{i,q} - x'_{i,q}|$$

This metric is crucial because it measures how resilient a voting rule is to outside interference, ensuring that rules which require higher bribery costs are more robust.

5.1.2 Cost of Control

The cost of control measures the minimal effort required to change the outcome by adding or removing voters. This metric evaluates how resistant the system is to manipulation through changes in the electorate. Let $V = \{X_1, \ldots, X_n\}$ be the initial vote profile, and $\mathbf{x} = \{a_1, \ldots, a_m\}$ the corresponding outcome. The goal is to determine how many voters need to be added or removed to increase the allocation to project p by a specific percentage r. This concept has been widely studied in election control scenarios [3].

5.2 Incentive Compatibility

Incentive compatibility ensures that voters cannot benefit from misreporting their preferences. We measure this through two metrics:

5.2.1 Strategyproofness

A voting rule is strategyproof if no voter can increase their utility by lying about their preferences. Let $u(X_i)$ denote the utility of voter i when voting truthfully, and $u(X'_i)$ denote the utility when voting strategically. A rule is strategyproof if:

$$u(X_i) \geq u(X'_i) \quad \forall i \in N$$

This metric is important because it ensures that voters are incentivized to report their true preferences, which is critical for the integrity of any voting system.

5.2.2 Group-strategyproofness

A voting rule is group-strategyproof if no coalition of voters can misreport their preferences in a way that benefits all its members. Let $C = \{X_1, \ldots, X_k\}$ be the coalition, and $u'(X_i)$ the utility after misreporting. The rule is group-strategyproof if:

$$\exists X_i \in C \quad \text{such that} \quad u(X_i) > u'(X_i)$$

Group-strategyproofness is vital for preventing coordinated manipulations in systems where groups might attempt to skew the outcome.

5.3 Participation

A voting rule satisfies participation if a voter is never worse off by participating in the election. Let $\mathbf{x}$ be the outcome without voter X_i, and $\mathbf{x}'$ the outcome when they participate. The participation constraint is:

$$u(\mathbf{x}') \geq u(\mathbf{x}) \quad \forall i \in N$$

This metric is key to ensuring that voters have no disincentive to participate in the voting process, thus maintaining the legitimacy of the system.

5.4 Expected Outcome

This category evaluates the robustness and predictability of the voting outcome.

5.4.1 Robustness

Robustness measures the sensitivity of the voting outcome to small changes in individual votes. For a profile $V = \{X_1, \ldots, X_n\}$ with outcome $\mathbf{x}$, let X_i' be a modified vote, and $\mathbf{x}'$ the corresponding new outcome. The robustness is the expected distance between $\mathbf{x}$ and $\mathbf{x}'$:

$$\text{Robustness} = \mathbb{E}[d(\mathbf{x}, \mathbf{x}')]$$

where d is a distance metric such as ℓ_1 or ℓ_2. This metric is critical to ensure that small deviations in votes do not result in drastically different outcomes, which is especially important in systems with high voter turnout.

5.4.2 Voter Extractable Value (VEV)

VEV quantifies the potential influence of a single voter on the outcome. For each voter i and project k, let $X_i^{(k)}$ be the vote where $r\%$ of the voter's tokens are allocated to k. Let $\mathbf{x}^{(i,k)}$ be the new outcome. The VEV is the maximum distance between the original outcome and the new outcome:

$$\text{VEV}(I) = \max_{i \in [n], k \in [m]} d(\mathbf{x}, \mathbf{x}^{(i,k)})$$

This metric helps identify whether any voter can disproportionately skew the allocation by concentrating their votes on a single project.

5.5 Pareto Efficiency

A voting rule is Pareto efficient if no alternative outcome can make at least one voter strictly better off without making another voter worse off. Formally, for an outcome $\mathbf{x}$ and an alternative $\mathbf{x}'$: $\forall v_i \in V, \quad U(v_i, \mathbf{x}') \geq U(v_i, \mathbf{x})$,
$\exists v_j \in V$ such that $U(v_j, \mathbf{x}') > U(v_j, \mathbf{x})$

Pareto efficiency ensures that the voting outcome maximizes social welfare without unfairly disadvantaging any voter.

5.6 Monotonicity

Monotonicity ensures that increasing the support for a project does not decrease its allocation. Let V' be a new profile where a voter reallocates more tokens to project p. A voting rule satisfies monotonicity if:

$$f(V', p) \geq f(V, p)$$

This metric is essential for fairness, as it guarantees that additional support for a project leads to an increase in its funding.

5.7 Reinforcement

Reinforcement ensures that if two voting profiles produce the same outcome individually, combining their votes will produce the same outcome. For two profiles V_1 and V_2 where $f(V_1) = f(V_2)$, the rule satisfies reinforcement if:

$$f(V_1 \cup V_2) = f(V_1) = f(V_2)$$

This property is important in decentralized systems, where subgroups of voters might aggregate their votes.

5.8 Proportionality

In cumulative voting, proportionality ensures that if a group of voters represents at least $\frac{1}{m}$ of the electorate and votes exclusively for a single project, that project will receive at least $\frac{1}{m}$ of the total budget. This is formalized as:

$$f(j) = \frac{|\text{Appr}(j)|}{n} \times B$$

Proportionality is critical to ensure that minority preferences are represented fairly.

5.9 Utilitarian Social Welfare

This metric measures the average ℓ_1 distance between the outcome and the voters' preferences, capturing how well the voting outcome reflects the collective preferences of the electorate. Formally:

$$\frac{1}{n}\sum_{i=1}^{n} \|\mathbf{x} - X_i\|_1$$

5.10 Diversity

- **Gini Index of Projects** – This metric measures the inequality of the allocation among projects. A higher Gini index indicates greater inequality, while a lower index suggests a more balanced distribution. Formally:

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} \|a_i - a_j\|_1}{2m \cdot \sum_{i=1}^{m} a_i}$$

- **Egalitarian Social Welfare** – This metric evaluates the utility of the least satisfied voter, ensuring that the worst-off voter is considered in the final outcome:

$$\arg\max_{i \in N} \|\mathbf{x} - X_i\|_1$$

5.11 Alignment with the Ground Truth

This metric evaluates how closely the voting outcome matches the "true values" of the projects. Let $\mathbf{x}^* = (a_1^*, \ldots, a_m^*)$ be the vector of true values for the projects, and $\mathbf{x} = (a_1, \ldots, a_m)$ be the outcome. The alignment is measured by:

$$\sum_{i=1}^{m} |a_i^* - a_i|$$

This is especially relevant in retrofunding, where the goal is to allocate funds based on the actual impact of projects.

6 Theoretical Results

We provide Table 1 and Table 2 that summarize the theoretical results. For space considerations proofs are in the Appendix in section 9

7 Experimental Design

First we discuss the artificial cumulative vote generation model we are using.

Voting Rule	**Reinforcement**	**Pareto-Efficiency**	**Monotonicity**	**Participation**	**Proportionality**
R2 Mean Rule	✓ (theorem 1)	✓ (theorem 4)	✓ (theorem 7)	✓ (theorem 12)	✓ (theorem 15)
Normalized Median Rule	✓ (theorem 2)	✓ (theorem 5)	✓ (theorem 8)	✓ (theorem 13)	× (example 6)
Midpoint Rule	✓ (theorem 3)	✓ (theorem 6)	✓ (theorem 9)	✓ (theorem 14)	× (example 7)
Independent Market	✓ [9][Theorem 5.9]	× for $m \geq n^2$ [9][Theorem 6.1]	✓ [9][Theorem 4.6]	✓ [9][Theorem 5.8]	✓ [9][Theorem 5.7]
Majoritarian Phantom	✓ [9][Theorem 6.6]	✓ [9][Theorem 6.1]	✓ [9][Theorem 4.6]	✓ [9][Theorem 6.5]	× [9][Theorem 5.7]
R1 Quadratic	✓ (corollary 1)	× (example 2)	✓ (corollary 4)	× (example 4)	× (example 8)
R3 Quorum Median	× (example 1)	× (example 3)	✓ (theorem 10)	× (example 5)	× (example 9)
R4 Capped Median	× (corollary 2)	× (corollary 3)	✓ (theorem 11)	× (corollary 5)	× (corollary 6)

Table 1: Properties of Voting Rules (Part 1)

Voting Rule	**Social Welfare Maximization**	**Strategyproofness**	**Group Strategyproofness**
R2 Mean Rule	× (theorem 25)	× (theorem 16)	× (corollary 8)
Normalized Median Rule	× (theorem 26)	× (theorem 17)	× (corollary 9)
Midpoint Rule	× (theorem 27)	× (theorem 18)	× (corollary 10)
Independent Market	× [9][Theorem 6.1]	✓ [9][Theorem 4.8] for $m > 2$	× (example 15)
Majoritarian Phantom	✓ [9][Theorem 6.1]	✓ [9][Theorem 4.8] for $m > 2$	× (example 16)
R1 Quadratic	× (theorem 30)	× (theorem 21)	× (corollary 11)
R3 Quorum Median	× (corollary 14)	× (theorem 22)	× (corollary 12)
R4 Capped Median	× (corollary 15)	× (corollary 7)	× (corollary 13)

Table 2: Properties of Voting Rules (Part 2)

7.1 Mallow's Model For Vote Generation

Inspired by earlier works [4, 15], where Mallow's model generates rankings based on a central reference ranking, often called the base vote, with a dispersion parameter controlling variability we distribute tokens across projects.

In this model, we generate a matrix of cumulative votes for n voters and m projects. Each voter's total vote sums to K (in our case $K = 1$), and the votes are generated with some randomness controlled by the parameter α.

We begin by generating a *base vote* for all projects using the Dirichlet distribution. This ensures that the sum of votes for each voter is exactly K. Noise is then introduced into each voter's vote by blending the base vote with an independent Dirichlet sample, with α controlling the degree of noise.

Formally Let V_i represent the vote of voter i, where $i \in \{1, 2, \ldots, n\}$, and let m be the number of projects. The goal is to ensure that the sum of votes for each voter satisfies:

$$\sum_{j=1}^{m} V_{i,j} = 1, \quad \text{for each } i = 1, 2, \ldots, n.$$

The model is based on two components:

1. A **base vote** B generated from a Dirichlet distribution:

$$\mathit{Base} \sim \text{Dirichlet}(\mathbf{1}_m),$$

   where $\mathbf{1}_m$ is a vector of ones of size m, and the base vote is scaled so that it sums to K.

2. A **noisy vote** V_i for each voter i is generated by taking a weighted combination of the base vote B and an independent sample from another Dirichlet distribution, controlled by α:

$$V_i = (1 - \alpha) \cdot \mathit{Base} + \alpha \cdot \text{Dirichlet}(\mathbf{1}_m).$$

Here, $\alpha \in [0, 1]$ controls the level of noise:

- If $\alpha = 0$, each voter's vote is exactly the base vote B (homogeneous).
- If $\alpha = 1$, each voter's vote is independently generated (heterogeneous).
- For values of α between 0 and 1, the votes are a mixture of homogeneity (shared base vote) and heterogeneity (individual noise).

We use $\alpha = 0.5$ which represents a balance between **homogeneity** (all voters being similar in their vote distributions) and **heterogeneity** (each voter having individual differences).

Next we outline the experimental setup and parameters used in a series of simulations conducted [4]. These experiments were designed to evaluate the performance of various voting rules across multiple metrics, including bribery susceptibility, cost of control by adding or removing voters, robustness,

[4]All Implementations can be viewed at:
`https://github.com/GovXS/OP-Evaluating-Voting-Design-Tradeoffs-for-Retro-Funding-RESEARCH-`

Voter Extractable Value (VEV), Gini index, Egalitarian and Utilitarian Social Welfare (measured via ℓ_1 distance), and alignment to the ground truth.

Given that the results of this work were also delivered to Optimism, we focused on simulations that involved voting rules from Rounds 1-4 and the recommended majoritarian voting rule, which maximizes ℓ_1-based social welfare and is strategyproof.

7.2 General Setup

For efficiency, the experiments on bribery susceptibility, cost of control, robustness, and VEV share the following parameters:

- Number of Voters: 40
- Number of Projects: 145
- Total OP Tokens: 8 million

For experiments assessing the Gini index, Egalitarian and Utilitarian Social Welfare, and alignment to the ground truth, the following parameters were used:

- Number of Voters: 145
- Number of Projects: 600
- Total OP Tokens: 30 million
- Number of different instances tested: 100

7.3 Cost of Control Experiment

The Cost of Control Experiment assesses the resistance of the voting system to manipulation by adding or removing voters, where the cost of adding or deleting a voter is set to 1 currency unit. The goal is to evaluate how costly it is to adjust the funding for a target project by adding or removing voters. The target project is defined as the one that achieves the desired increase in funds with the minimum cost. The primary parameter swept in this experiment is the *desired increase* in project funding.

Setup:

- Number of different instances per desired increase: 10
- Parameters Swept: Desired increase in funding, ranging from 1% to 30%, across 30 iterations.

Output: The system calculates the average number of voters that need to be added (voting all their tokens on a project) or removed from existing voters to achieve the desired percentage increase in project funding.

7.4 Bribery Experiment

The Bribery Experiment evaluates the susceptibility of each voting rule to bribery, measuring the cost required to increase the allocation of a target project by a specified percentage.

Setup:

- Rounds per Experiment: 10
- Parameters Swept: Desired increase percentage, ranging from 1% to 30%, across 30 iterations.

Output: For each iteration, the bribery cost is calculated, assuming the cost of moving a single token from one project to another is 1 currency unit.

7.5 Robustness Experiment

The Robustness Experiment assesses the stability of the voting system by introducing random changes to voters' preferences and analyzing the resulting impact on the allocation outcomes. The experiment uses the ℓ_1 distance between the original and modified allocations to quantify this effect.

Output: The system computes the sensitivity of each voting rule to random changes across 100 rounds, logging the ℓ_1 distances for each voting rule.

7.6 Voter Extractable Value (VEV) Experiment

The VEV Experiment investigates how much influence a single voter can exert on the allocation outcome of a specific project. This experiment measures the extent to which a single voter can skew the allocation in favor of their preferred project.

Setup:

- Parameters Swept: $r \in \{90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$, representing the percentage of tokens allocated by a voter to a particular project, with the remaining tokens evenly distributed across the other $m - 1$ projects.
- Number of different instances per Experiment: 50

Output: The system computes the maximum Voter Extractable Value (VEV) for each project in each round, showing the degree to which a single voter can manipulate the allocation towards a project.

7.7 Alignment with the Ground Truth

To measure the alignment with the ground truth, we define the base vote generated using Mallow's model as the ground truth. The alignment is quantified by calculating the ℓ_1 distance between the output of a voting rule and the base vote. This distance is then compared across all voting rules to determine which rule aligns most closely with the ground truth.

8 Results and Discussion

Work in progress- soon to be published

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9 Appendix- Proofs

9.1 Reinforcement

Theorem 1. *The Mean Rule satisfies the reinforcement property.*

Proof. Consider two voting profiles V_1 and V_2, each with n_1 and n_2 voters, respectively. The allocation to project p under V_1 is:

$$a_p^1 = \frac{\sum_{i \in N_1} x_{i,p}}{\sum_{i \in N_1} \sum_{p' \in P} x_{i,p'}}$$

and under V_2 is:

$$a_p^2 = \frac{\sum_{i \in N_2} x_{i,p}}{\sum_{i \in N_2} \sum_{p' \in P} x_{i,p'}}$$

When combining the two profiles into $V_3 = V_1 \cup V_2$, the allocation is:

$$a_p^3 = \frac{\sum_{i \in N_1 \cup N_2} x_{i,p}}{\sum_{i \in N_1 \cup N_2} \sum_{p' \in P} x_{i,p'}}$$

Since the sum of allocations is linear, the relative proportions between projects p and q are preserved, assuming the individual allocations in V_1 and V_2 are consistent in ranking p over q. Therefore, $a_p^3 \geq a_q^3$ if $a_p^1 \geq a_q^1$ and $a_p^2 \geq a_q^2$.

Thus, the Mean Rule satisfies the reinforcement property. □

Theorem 2. *The Normalized Median Rule satisfies the reinforcement property.*

Proof. Consider two voting profiles V_1 and V_2, each with n_1 and n_2 voters respectively. Let a_p^1 be the median allocation to project p under V_1, and a_p^2 the median allocation under V_2. The combined profile $V_3 = V_1 \cup V_2$ has $n_1 + n_2$ voters.

Given that $a_p^1 \geq a_q^1$ and $a_p^2 \geq a_q^2$, the median allocation in the combined profile V_3 will still satisfy $a_p^3 \geq a_q^3$, because the median, as a central tendency, will reflect the relative positions from both V_1 and V_2.

Thus, the Normalized Median Rule satisfies the reinforcement property, as combining profiles does not change the relative ordering of projects. □

Theorem 3. *The Midpoint Rule satisfies the reinforcement property.*

Proof. Consider the voter v that minimizes the ℓ_1 distance in V_1 and in V_2. Thus, $v = \text{argmin} \sum_{V_1} \ell_1$ and also $v = \text{argmin} \sum_{V_2} \ell_1$; therefore, $v = \text{argmin} \sum_{V_1 \cup V_2} \ell_1$.

Thus, the Midpoint Rule satisfies the reinforcement property. □

Corollary 1. *The Quadratic Rule satisfies the reinforcement property.*

Proof. Follows from Theorem 1 as it is a mean after taking the square root. □

Example 1. *The R3 Quorum Median Rule does not satisfy the reinforcement property.*

Consider two projects, p_1 and p_2, and let V_1 contain only v_1, voting $[0.9, 0.1]$, and let V_2 contain only v_2, voting $[0.9, 0.1]$. Let the quorum be 0.2. Now, the winner of V_1 is $[1, 0]$ and the winner of V_2 is $[1, 0]$, but the winner of $V_1 \cup V_2$ is $[0.8, 0.2]$.

Corollary 2. *The R4 Capped Median rule does not satisfy reinforcement since it includes a Quorum K3.*

9.2 Pareto Efficiency

Theorem 4. *The Mean Rule satisfies Pareto efficiency.*

Proof. The Mean Rule distributes the budget B proportionally to the sum of individual allocations.

Suppose an alternative allocation a'_p makes one voter better off by increasing the allocation for project p. Given the budget constraint B, increasing a'_p would require decreasing the allocation to another project q, thus making voters who prefer q worse off.

Since any deviation from the mean allocation would reduce the utility of at least one voter, no Pareto improvement is possible.

Therefore, the Mean Rule satisfies Pareto efficiency. □

Theorem 5. *The Normalized Median Rule satisfies Pareto efficiency.*

Proof. Consider a voting profile $V = \{x_{1,p}, x_{2,p}, \ldots, x_{n,p}\}$ for a project p. The median allocation a_p is the median value among all voters' allocations for that project.

To prove Pareto efficiency, suppose there exists an alternative allocation a' that makes at least one voter better off without making any other voter worse off.

If we try to increase the allocation for one project p to make a particular voter better off, we must decrease the allocation for another project q due to the budget constraint.

Since the median allocation minimizes the sum of absolute deviations from voters' allocations, any reallocation that deviates from this median would necessarily make some voters worse off.

Hence, no alternative allocation can make one voter better off without making another voter worse off. Therefore, the Median Rule satisfies Pareto efficiency. □

Theorem 6. *The Midpoint Rule satisfies Pareto efficiency.*

Proof. Consider the output p; as p equals the vote of some voter v, it follows that any other p' will not equal v, thus v will be less satisfied.

Thus, the Midpoint Rule satisfies Pareto efficiency. □

Example 2. *The Quadratic Rule does not satisfy Pareto efficiency.*

Consider two projects and one voter voting $[0.7, 0.3]$. As the result is the mean of the square roots, it is not equal to $[0.7, 0.3]$, which is better for the voter.

Example 3. *The R3 Quorum Median Rule does not satisfy Pareto efficiency.*

Consider two projects and one voter voting $[0.7, 0.3]$ and let the quorum be 0.4. The output will be $[1, 0]$, which is worse for the voter than $[0.7, 0.3]$.

Corollary 3. *The R4 Capped Median rule does not satisfy Pareto efficiency since it includes a Quorum K3.*

9.3 Monotonicity

Theorem 7. *The Mean Rule satisfies monotonicity.*

Proof. Consider a voter i who increases their allocation to project p from $x_{i,p}$ to $x'_{i,p} > x_{i,p}$. The mean allocation to project p is given by:

$$a_p = \frac{\sum_{i \in N} x_{i,p}}{\sum_{i \in N} \sum_{p' \in P} x_{i,p'}}$$

Increasing $x_{i,p}$ increases the numerator, leading to an increase in a_p, while the denominator also increases, but not enough to reduce a_p.

Thus, increasing $x_{i,p}$ cannot decrease the final allocation a_p.

Therefore, the Mean Rule satisfies monotonicity. □

Theorem 8. *The Normalized Median Rule satisfies monotonicity.*

Proof. Consider a voting profile $V = \{x_{1,p}, \ldots, x_{n,p}\}$ for a project p, and suppose voter i increases their allocation from $x_{i,p}$ to $x'_{i,p} > x_{i,p}$.

If $x_{i,p}$ is below the median, increasing it can shift the median upward, increasing a_p. If $x_{i,p}$ is above the median, increasing it will not decrease the median and hence will not decrease the allocation a_p. If $x_{i,p}$ is exactly at the median, increasing it will either leave the median unchanged or shift it upwards.

In all cases, increasing a voter's allocation to a project does not decrease the final median allocation. Therefore, the Normalized Median Rule satisfies monotonicity. □

Theorem 9. *The Midpoint Rule satisfies monotonicity.*

Proof. Consider v, the winner. Consider a bribery that moves to some project p and consider some voter v' that gives less to p and, towards contradiction, will be the next winner. However, the distance it has is at least increasing for p (because ℓ_1 is linear); thus, contradiction.

Therefore, the Midpoint Rule satisfies monotonicity. □

Corollary 4. *The Quadratic Rule satisfies monotonicity following Theorem 7.*

Theorem 10. *The R3 Quorum Median Rule satisfies monotonicity.*

Proof. There are several cases to consider:

1. If the current project is below quorum, it can only move above it.
2. If the project being reduced was above quorum but now falls below it, the project receiving additional votes can only increase since there are fewer projects above quorum to split the votes.

Therefore, the R3 Quorum Median Rule satisfies monotonicity. □

Theorem 11. *The R4 Capped Median rule satisfies monotonicity.*

Proof. Considering capping with $K2$ If the current allocation of a project is below $K2$, then adding more tokens from a voter can only increase its allocation, benefiting the project. If the project's allocation is already at or above $K2$, adding more tokens will not change the allocation, as it is capped at $K2$. Therefore, the project's allocation neither decreases nor increases beyond $K2$.

Considering quorum $K3$:

1. If the current allocation of a project is below the quorum $K3$, adding more tokens can only raise the project's allocation above the quorum.
2. If a project that was above quorum $K3$ receives fewer tokens and falls below quorum, the project receiving additional votes will see its allocation increase. This occurs because there are now fewer projects above the quorum, allowing for a higher distribution of tokens to those still receiving votes.

Thus, the R4 Capped Median rule ensures that adding tokens to a project either increases or maintains its allocation, satisfying the monotonicity property. □

9.4 Participation

Theorem 12. *The Mean Rule satisfies the participation property.*

Proof. Consider a voter i who abstains from voting. The allocation to each project p is recalculated as:

$$a'_p = \frac{\sum_{j \in N \setminus \{i\}} x_{j,p}}{\sum_{j \in N \setminus \{i\}} \sum_{p' \in P} x_{j,p'}}$$

If voter i abstains, their influence on the allocation is removed, which generally results in a less favorable outcome for them, as their preferred projects might receive less allocation.

Thus, abstaining does not lead to a higher utility.

Therefore, the Mean Rule satisfies the participation property. □

Theorem 13. *The Normalized Median Rule the participation property.*

Proof. Consider a voting profile $V = \{x_{1,p}, \ldots, x_{n,p}\}$ and let voter i consider abstaining. The current median allocation to project p is $a_p = \text{median}(x_{1,p}, \ldots, x_{n,p})$.

If $x_{i,p}$ is below the median, abstaining does not change the median or could increase it, leading to a worse or neutral outcome for the voter. If $x_{i,p}$ is above the median, abstaining could lower the median, reducing the voter's preferred allocation. If $x_{i,p}$ is exactly at the median, abstaining would shift the median, potentially lowering the allocation for the voter's preferred project.

In all scenarios, abstaining from voting does not result in a better outcome for the voter. Therefore, the Normalized Median Rule satisfies the participation property. □

Theorem 14. *The Midpoint Rule satisfies the participation property.*

Proof. There are two cases: if, by participating, the voter becomes the winner, then the voter will obviously benefit; otherwise, it is similar to the proof for the Mean Rule, where the voter's participation helps minimize the sum of ℓ_1 distances.

Therefore, the Midpoint Rule satisfies the participation property. □

Example 4. *The Quadratic Rule does not satisfy the participation property.*

Consider an election without a voter v where the output without v is $[0.8, 0.2]$. If the true vote of v is $[0.8, 0.2]$, then by voting, v actually pushes the mean allocation away from the preferred allocation due to the square roots.

Example 5. *The R3 Quorum Median Rule does not satisfy the participation property.*

Consider an election without a voter v where, without v, some project p_2 is not funded (but is very close to being funded). Now, if voter v votes $[1-\epsilon, \epsilon]$ so that project p_2 receives funding, this actually gives a lot of funding to p_2, pushing the winning allocation away from what v prefers.

Corollary 5. *The R4 Capped Median rule does not satisfy participation since it includes a Quorum K3.*

9.5 Proportionality

Theorem 15. *The Mean Rule satisfies proportionality.*

Proof. The Mean Rule aggregates votes by summing the values across all voters. This method ensures that each project receives funding proportional to the total amount of tokens allocated to it. Since the allocation directly reflects the proportion of votes each project receives, the rule satisfies proportionality. □

Example 6. *The Normalized Median Rule does not satisfy proportionality.*

Consider a scenario with three voters whose vote distributions are $[1, 0]$, $[1, 0]$, and $[0, 1]$. The median allocation would be $[1, 0]$. However, proportionality requires that the allocation reflects the distribution of preferences among voters, which would result in an allocation closer to $[2/3, 1/3]$. Thus, the Normalized Median Rule fails to satisfy proportionality.

Example 7. *The Midpoint Rule does not satisfy proportionality.*

Using the same example with three voters voting $[1, 0]$, $[1, 0]$, and $[0, 1]$, the Midpoint Rule selects the input allocation that minimizes the ℓ_1 distance, hence $[1, 0]$ instead of the desired allocation of $[2/3, 1/3]$. Thus, the Midpoint Rule does not satisfy proportionality.

Example 8. *The Quadratic Rule does not satisfy proportionality.*

Consider a scenario with three voters whose vote distributions are $[9, 1]$*,* $[5, 5]$*, and* $[4, 6]$*. The rule would output* $[5.6, 4.4]$*, where the proportional output should be* $[6, 4]$*.*

Example 9. *The R3 Quorum Median Rule does not satisfy proportionality.*

Using the same example with three voters voting $[1, 0]$*,* $[1, 0]$*, and* $[0, 1]$*, and a quorum of a minimum of 2 voters and 1.1 tokens, we get an output of* $[1, 0]$ *that does not satisfy proportionality.*

Corollary 6. *The R4 Capped Median rule does not satisfy proportionality since it includes a Quorum K3 and is median based.*

9.6 Strategyproofness

Theorem 16. *The Mean Rule does not satisfy strategyproofness.*

Example 10. *Consider a scenario with two voters whose vote distributions are* $[0.75, 0.25]$ *and* $[0, 1]$*. The Mean Rule will output* $[0.375, 0.625]$*, meaning the utility of voter 1 (measured using the* ℓ_1 *distance between the output and the vote) would be* 0.75*. If voter 1 votes untruthfully with* $[1, 0]$*, the Mean Rule would output* $[0.5, 0.5]$*, reducing voter 1's* ℓ_1 *distance to* 0.5*.*

Theorem 17. *The Normalized Median Rule does not satisfy strategyproofness.*

Example 11. *Consider a scenario with three voters whose vote distributions are* $[0.57, 0.24, 0.19]$*,* $[0.39, 0.48, 0.13]$*, and* $[0.44, 0.09, 0.48]$*. The Normalized Median Rule will output* $[0.506, 0.276, 0.218]$*, meaning the utility of voter 1 (measured using the* ℓ_1 *distance between the output and the vote) would be* 0.1285*. If voter 1 votes untruthfully with* $[0.6, 0.2, 0.2]$*, the Normalized Median Rule would output* $[0.524, 0.238, 0.238]$*, reducing voter 1's* ℓ_1 *distance to* 0.0962*.*

Theorem 18. *The Midpoint Rule does not satisfy strategyproofness.*

Example 12. *Consider a scenario with three voters whose vote distributions are* $[0.9, 0.1]$*,* $[0.4, 0.6]$*, and* $[0.2, 0.8]$*. The Midpoint Rule will output* $[0.4, 0.6]$*, meaning the utility of voter 1 (measured using the* ℓ_1 *distance between the output and the vote) would be* 1*. If voter 1 votes untruthfully with* $[0.5, 0.5]$*, the Midpoint Rule would output* $[0.5, 0.5]$*, reducing voter 1's* ℓ_1 *distance to* 0.8*.*

Theorem 19. *The Independent Markets Rule satisfies strategyproofness.*

Proof. The proof follows from [9][Theorem 4.8] for $m > 2$. □

Theorem 20. *The Majoritarian Phantom Rule satisfies strategyproofness.*

Proof. The proof follows from [9][Theorem 4.8] for $m > 2$. □

Theorem 21. *The Quadratic Rule does not satisfy strategyproofness.*

Example 13. *Consider a scenario with three voters whose vote distributions are* $[0.7, 0.3]$*,* $[0.4, 0.6]$*, and* $[0.3, 0.7]$*. The Quadratic Rule will output* $[0.483, 0.517]$*, meaning the utility of voter 1 (measured using the* ℓ_1 *distance between the output and the vote) would be* 0.434*. If voter 1 votes untruthfully with* $[0.8, 0.2]$*, the Quadratic Rule would output* $[0.502, 0.498]$*, reducing voter 1's* ℓ_1 *distance to* 0.396*.*

Theorem 22. *The R3 Quorum Median Rule does not satisfy strategyproofness.*

Example 14. *Consider a scenario with three voters whose vote distributions are* $[0.2, 0.8]$*,* $[0.1, 0.9]$*, and* $[0.5, 0.5]$ *with a quorum of 0.3 tokens and 2 voters for a project. The R3 Quorum Median Rule will output* $[0, 1]$*, meaning the utility of voter 1 (measured using the* ℓ_1 *distance between the output and the vote) would be* 0.4*. If voter 1 votes untruthfully with* $[0, 1]$*, the Quadratic Rule that filters out all 0 votes would output* $[0.25, 0.75]$*, reducing voter 1's* ℓ_1 *distance to* 0.1*.*

Corollary 7. *The R4 Capped Median rule rule does not satisfy Strategy proofness since it includes a Quorum K3 and is median based.*

9.7 Group Strategyproofness

Corollary 8. *The Mean Rule does not satisfy group strategyproofness, as it does not satisfy strategyproofness.*

Corollary 9. *The Normalized Median Rule does not satisfy group strategyproofness, as it does not satisfy strategyproofness.*

Corollary 10. *The Midpoint Rule does not satisfy group strategyproofness, as it does not satisfy strategyproofness.*

Theorem 23. *The Independent Markets Rule does not satisfy group strategyproofness.*

Example 15. *Consider a scenario where a group of three voters cast their votes as* $[0.01, 0.99]$, $[0.97, 0.03]$, *and* $[0.46, 0.54]$.

The Independent Markets algorithm computes an allocation of approximately $[0.4781, 0.5219]$. *The* ℓ_1 *distances (which represent the dissatisfaction of the voters with this outcome) between each voter's preference and the computed allocation are* 0.9398, 0.9803, *and* 0.0439, *respectively.*

However, if the first and third voters decide to misreport their preferences as $[0.4, 0.6]$ *and* $[0.4, 0.6]$, *while the second voter remains truthful, the algorithm would compute a new allocation of* $[0.45, 0.55]$. *This new allocation decreases their respective* ℓ_1 *distances to* 0.8836, 1.0364, *and* 0.0123.

Intuitive Explanation: *- The Independent Markets Rule tries to balance the preferences of all voters by considering "phantom" voters. - In this example, the true preferences are very skewed: one voter heavily favors the first project, another favors the second, and the third voter is more balanced. - Initially, the rule finds a compromise between these preferences, but it isn't perfect for anyone. - When the first and third voters misreport their preferences to show more moderate support, the algorithm shifts the allocation slightly closer to their new reported preferences. This reduces the overall dissatisfaction (as measured by* ℓ_1 *distance) for the first and third voters, even though the second voter becomes slightly more dissatisfied. - This shows that groups of voters can strategically manipulate their reports to achieve a more favorable outcome, demonstrating that the Independent Markets Rule is not group strategyproof.*

Theorem 24. *The Majoritarian Phantom Rule does not satisfy group strategyproofness.*

Example 16. *Consider a scenario where three voters cast their votes as* $[0.93, 0.07]$, $[0.73, 0.27]$, *and* $[0.99, 0.01]$.

The Majoritarian Phantom algorithm computes an allocation of approximately $[0.8303, 0.1697]$. *The* ℓ_1 *distances between each voter's preference and the computed allocation are* 0.2049, 0.2049, *and* 0.3287, *respectively.*

However, if the first and third voters decide to misreport their preferences as $[1, 0]$ *and* $[1, 0]$, *while the second voter remains truthful, the algorithm would compute a new allocation of* $[0.8639, 0.1361]$. *This new allocation decreases their respective* ℓ_1 *distances to* 0.1377, 0.2722, *and* 0.2614.

Intuitive Explanation: *- The Majoritarian Phantom Rule tries to reflect the majority preferences while also considering the intensity of these preferences. - In this example, the original preferences are strongly in favor of the first project, but there is some support for the second project. - Initially, the rule gives a significant allocation to the first project but still provides some allocation to the second. - When the first and third voters misreport their preferences to show complete support for the first project, the algorithm shifts the allocation even more toward the first project. - This manipulation results in a slightly better outcome for the first and third voters, even though the second voter, who is more balanced, becomes more dissatisfied. - This example shows that groups of voters can influence the outcome by strategically misreporting their preferences, proving that the Majoritarian Phantom Rule is not group strategyproof.*

Corollary 11. *The Quadratic Rule does not satisfy group strategyproofness, as it does not satisfy strategyproofness.*

Corollary 12. *The R3 Quorum Median Rule does not satisfy group strategyproofness, as it does not satisfy strategyproofness.*

Corollary 13. *The R4 Capped Median rule does not satisfy Strategy proofness since it does not satisfy strategyproofness.*

9.8 Maximal Social Welfare (Minimal L1)

A solution to an instance satisfies maximum SWF if there is no other allocation that results in a smaller total ℓ_1 distance between the final allocation and the ballots.

Theorem 25. *The Mean Rule does not satisfy maximum SWF.*

Example 17. *Consider the following example:*
Setup:

- ***Projects:*** P_1, P_2
- ***Voters:*** *3 voters (A, B, C)*
- ***Votes:***
  - ***Voter A:*** $(1, 0)$
  - ***Voter B:*** $(1, 0)$
  - ***Voter C:*** $(0.2, 0.8)$

In this setup, Voters A and B strongly favor P_1, while Voter C strongly favors P_2.
Mean Rule Calculation:

- ***Mean Allocation:***

$$a = \left(\frac{1+1+0.2}{3}, \frac{0+0+0.8}{3}\right) = \left(\frac{2.2}{3}, \frac{0.8}{3}\right) \approx (0.7333, 0.2667)$$

L_1 ***Distance Calculation for Mean Rule:***

- ***Voter A:*** $|1 - 0.7333| + |0 - 0.2667| = 0.2667 + 0.2667 = 0.5334$
- ***Voter B:*** $|1 - 0.7333| + |0 - 0.2667| = 0.2667 + 0.2667 = 0.5334$
- ***Voter C:*** $|0.2 - 0.7333| + |0.8 - 0.2667| = 0.5333 + 0.5333 = 1.0666$
- ***Total*** L_1 ***Distance:*** $0.5334 + 0.5334 + 1.0666 = 2.1334$

Midpoint Rule as an Alternative:

- ***Midpoint Allocation:*** $(1, 0)$

L_1 ***Distance Calculation for Midpoint Allocation:***

- ***Voter A:*** $|1 - 1| + |0 - 0| = 0 + 0 = 0$
- ***Voter B:*** $|1 - 1| + |0 - 0| = 0 + 0 = 0$
- ***Voter C:*** $|0.2 - 1| + |0.8 - 0| = 0.8 + 0.8 = 1.6$
- ***Total*** L_1 ***Distance:*** $0 + 0 + 1.6 = 1.6$

Remark 2. *The Mean Rule satisfies maximum SWF if it is defined as minimizing the ℓ_2 distance between the aggregation output and all ballots [5].*

Theorem 26. *The Normalized Median Rule does not satisfy maximum SWF optimization.*

Example 18. *Consider the following example with two projects and three voters:*
Votes:

- ***Voter A:*** $(0.1, 0.9)$
- ***Voter B:*** $(0.4, 0.2)$
- ***Voter C:*** $(0.6, 0.1)$

Normalized Median Rule Calculation:

- *Normalized Median for P_1:* 0.4
- *Normalized Median for P_2:* 0.2

The sum of the medians is $0.4+0.2 = 0.6$. *To ensure the allocations sum to 1, normalization is required:*

$$(0.4, 0.2) \text{ is normalized to } \left(\frac{0.4}{0.6}, \frac{0.2}{0.6}\right) = (0.667, 0.333)$$

L_1 ***Distance for Normalized Median Rule:***

$$\begin{aligned}
\text{Voter A:} &\quad |0.1-0.667| + |0.9-0.333| = 0.567+0.567 = 1.134\\
\text{Voter B:} &\quad |0.4-0.667| + |0.2-0.333| = 0.267+0.133 = 0.4\\
\text{Voter C:} &\quad |0.6-0.667| + |0.1-0.333| = 0.067+0.233 = 0.3\\
\text{Total } L_1 \text{ Distance:} &\quad 1.134+0.4+0.3 = 1.834
\end{aligned}$$

Alternative Allocation: *Consider the allocation* $(0.5, 0.5)$.
L_1 ***Distance for Alternative Allocation:***

$$\begin{aligned}
\text{Voter A:} &\quad |0.1-0.5| + |0.9-0.5| = 0.4+0.4 = 0.8\\
\text{Voter B:} &\quad |0.4-0.5| + |0.2-0.5| = 0.1+0.3 = 0.4\\
\text{Voter C:} &\quad |0.6-0.5| + |0.1-0.5| = 0.1+0.4 = 0.5\\
\text{Total } L_1 \text{ Distance:} &\quad 0.8+0.4+0.5 = 1.7
\end{aligned}$$

Remark 3. *The Median Rule satisfies maximum SWF if it is defined as minimizing the ℓ_1 distance between the aggregation output and all ballots and when there is no normalization of the final allocation [5].*

Theorem 27. *The Midpoint Rule does not satisfy maximum SWF.*

Example 19. *Consider 4 projects (P_1, P_2, P_3, P_4) and 4 voters with the following votes:*

$$\begin{aligned}
&\text{Voter A: } (0.8, 0.1, 0.05, 0.05)\\
&\text{Voter B: } (0.1, 0.8, 0.05, 0.05)\\
&\text{Voter C: } (0.05, 0.05, 0.8, 0.1)\\
&\text{Voter D: } (0.05, 0.05, 0.1, 0.8)
\end{aligned}$$

The Midpoint Rule might select any voter allocation since all ballots have the same distance from one another. The total ℓ_1 distance would always be 4.6.

Consider the alternative allocation $(0.25, 0.25, 0.25, 0.25)$. *The total ℓ_1 distance will now be 4.4.*

Example 20. *Consider the following example: we have 3 voters, each giving a ballot for two projects:* $(0.99, 0.01), (0.01, 0.99), (0.2, 0.8)$.

Theorem 28. *The IM algorithm does not satisfy maximum SWF as following the proof of [9][Theorem 6.1].*

Theorem 29. *The Majoritarian Phantoms algorithm satisfies maximum SWF, following the proof of [9][Theorem 6.1].*

Theorem 30. *The Quadratic Voting rule does not satisfy Social Welfare.*

Example 21. ***Setup:***

- ***Projects:*** P_1, P_2
- ***Voters:*** *3 voters (A, B, C)*
- ***Votes:***
  - ***Voter A:*** $(0.01, 0.99)$
  - ***Voter B:*** $(0.01, 0.99)$
  - ***Voter C:*** $(0.99, 0.01)$

In this example, two voters strongly favor P_2, while one voter strongly favors P_1.
Quadratic Voting Calculation:

- ***True Votes:***
  - ***Voter A:*** $\sqrt{0.01} \approx 0.1$, $\sqrt{0.99} \approx 0.995$
  - ***Voter B:*** $\sqrt{0.01} \approx 0.1$, $\sqrt{0.99} \approx 0.995$
  - ***Voter C:*** $\sqrt{0.99} \approx 0.995$, $\sqrt{0.01} \approx 0.1$
- ***Aggregate Votes:***
  - *For P_1:* $0.1 + 0.1 + 0.995 = 1.195$
  - *For P_2:* $0.995 + 0.995 + 0.1 = 2.09$
- ***Final Allocation:***
$$a_1 = \frac{1.195}{3.285} \approx 0.364, \quad a_2 = \frac{2.09}{3.285} \approx 0.636$$

L_1 ***Distance Calculation for Quadratic Voting:***

- ***Voter A:*** $|0.01 - 0.364| + |0.99 - 0.636| = 0.354 + 0.354 = 0.708$
- ***Voter B:*** $|0.01 - 0.364| + |0.99 - 0.636| = 0.354 + 0.354 = 0.708$
- ***Voter C:*** $|0.99 - 0.364| + |0.01 - 0.636| = 0.626 + 0.626 = 1.252$
- ***Total** L_1 **Distance:*** $0.708 + 0.708 + 1.252 = 2.668$

Alternative Allocation:
Consider an alternative allocation that better aligns with the majority preference: $(0.2, 0.8)$.
L_1 ***Distance Calculation for Alternative Allocation:***

- ***Voter A:*** $|0.01 - 0.2| + |0.99 - 0.8| = 0.19 + 0.19 = 0.38$
- ***Voter B:*** $|0.01 - 0.2| + |0.99 - 0.8| = 0.19 + 0.19 = 0.38$
- ***Voter C:*** $|0.99 - 0.2| + |0.01 - 0.8| = 0.79 + 0.79 = 1.58$
- ***Total** L_1 **Distance:*** $0.38 + 0.38 + 1.58 = 2.34$

Corollary 14. *The R3 Quorum Median Rule does not satisfy maximum SWF since the Median does not satisfy maximum SWF (because of the normalization).*

Corollary 15. *The R4 Capped Median rule does not satisfy maximum SWF since the Median does not satisfy maximum SWF (because of the normalization).*

10 Appendix- Figures

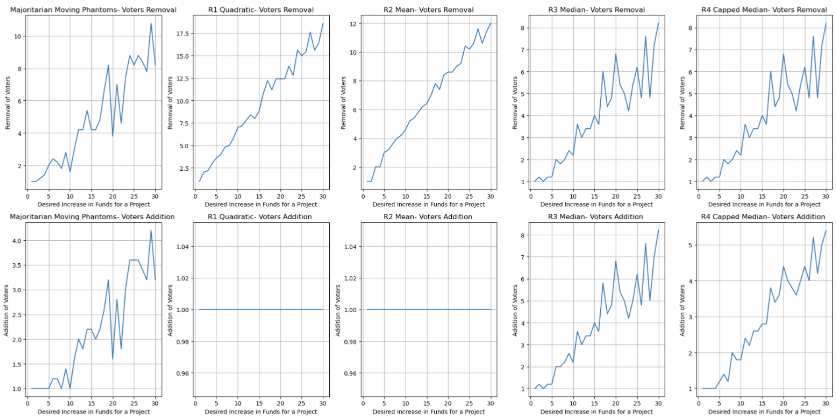

Figure 1: Cost of Adding and deleting voters as a function of the desired funding increase.

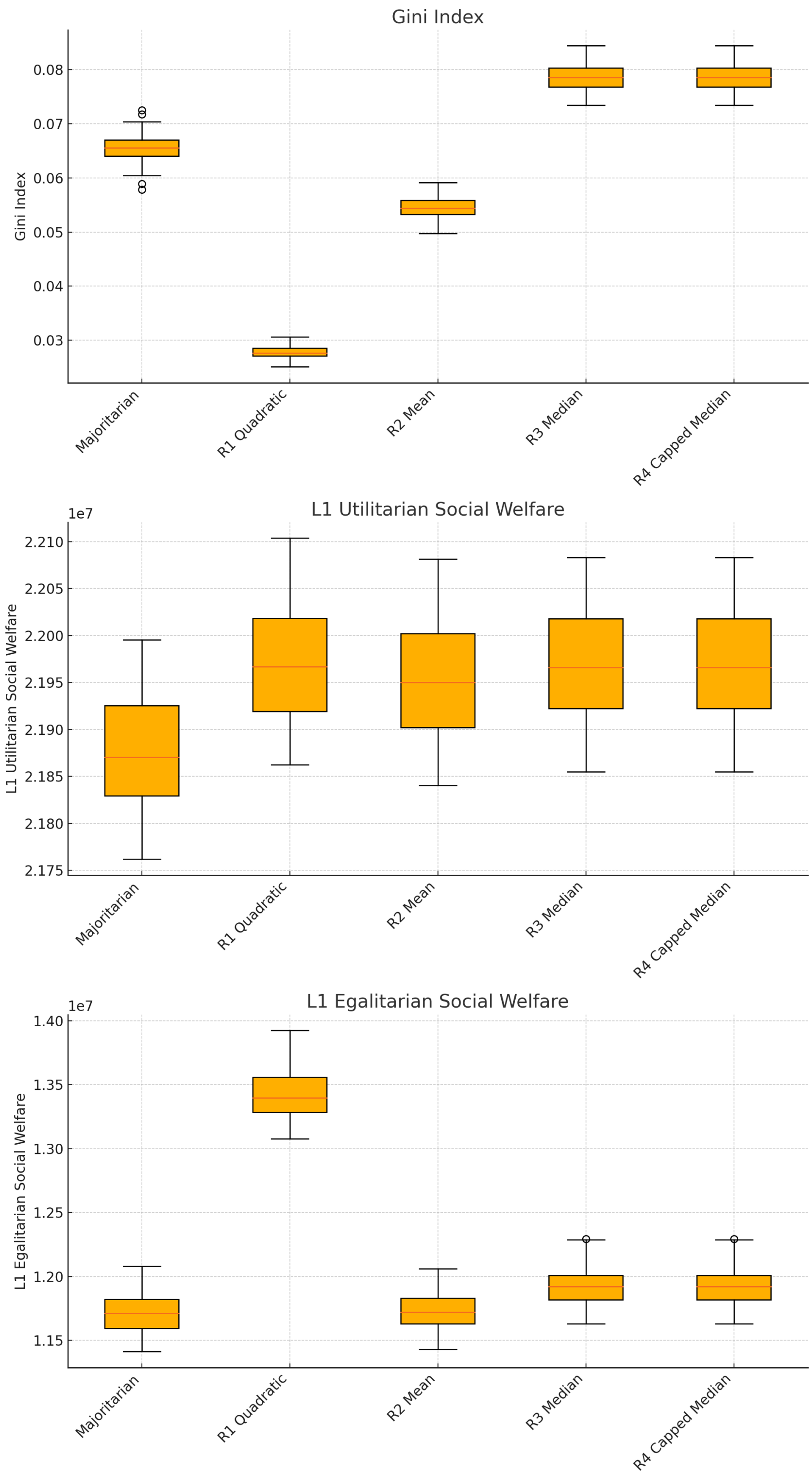

Figure 2: Comparison of Gini Index, L1 Utilitarian Social Welfare, and L1 Egalitarian Social Welfare across voting rules.

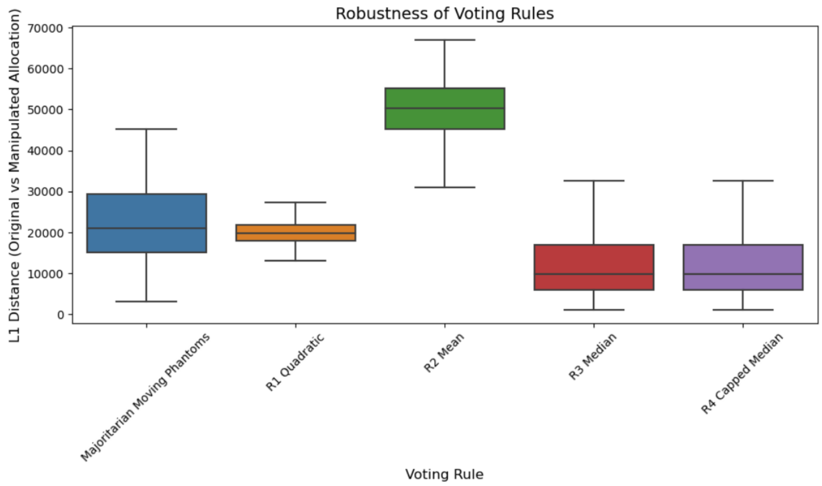

Figure 3: Robustness of voting rules under manipulation, represented by L1 distance between original and manipulated allocations.

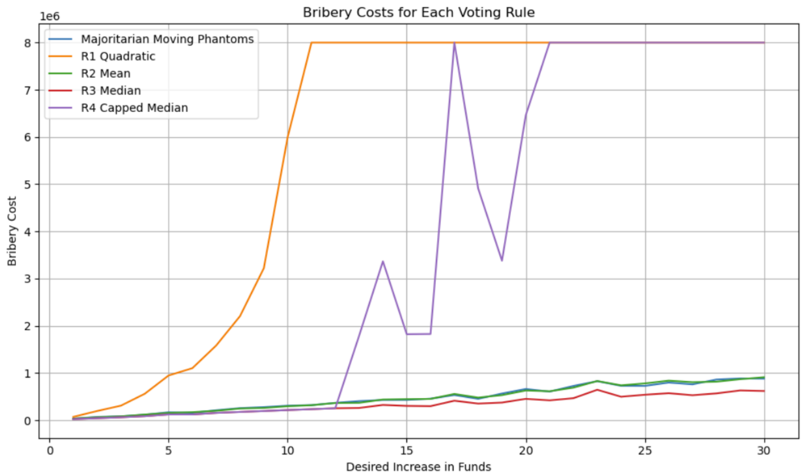

Figure 4: Bribery costs for different voting rules as a function of desired increase in funds for a project.

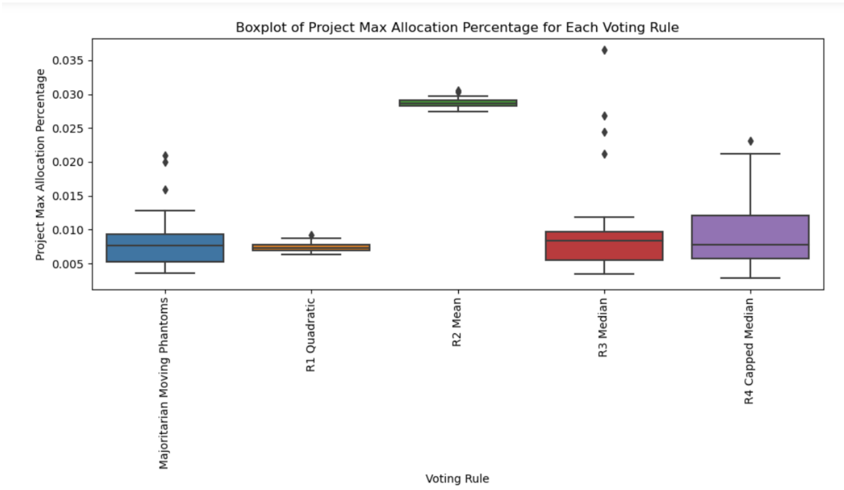

Figure 5: The maximal new project allocation caused by one voter skewing the outcome divided by the total number of tokens to be funded for each voting rule (average of each instance across all r %).

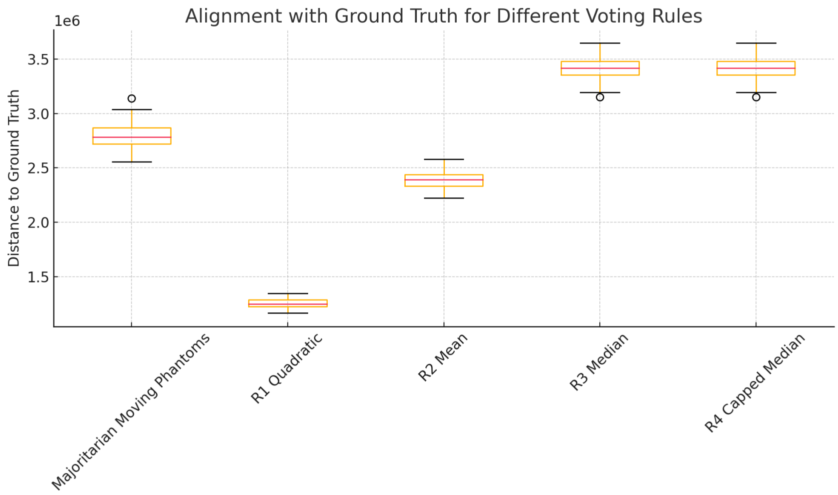

Figure 6: Alignment with ground truth for different voting rules, measured by distance to ground truth.

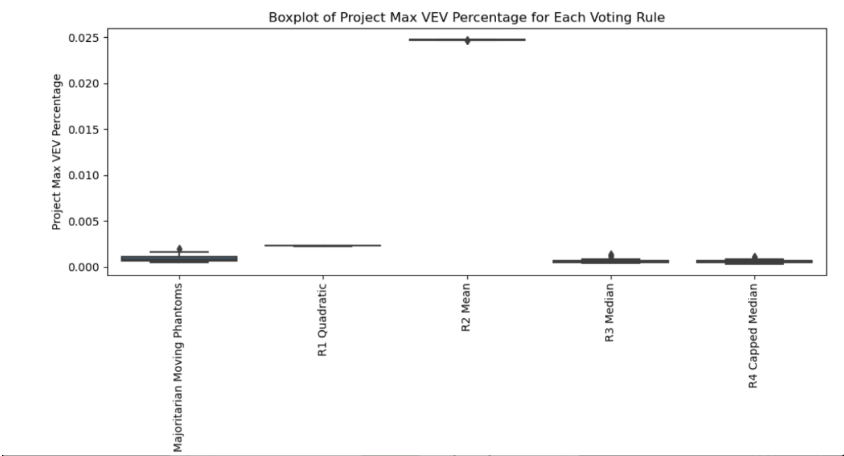

Figure 7: The maximal difference in token allocation caused by one voter skewing the outcome, divided by the total number of tokens to be funded for each voting rule (average of each instance across all r %).