MSAS – Assignment #2: Modeling

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1 Questions

Question 1

1) List the stages of dynamic investigation and their meaning. 2) When going from the *real system* to the *physical model* a number of assumptions are made; report the most important ones along with their mathematical implications. 3) For each of the assumptions below, shortly state what sort of simplification may result: i) The gravity torque on a pendulum is taken proportional to the pendulum angle θ; ii) Only wind forces and gravity are assumed in studying the motion of an aircraft; iii) A temperature sensor is assumed to report the temperature exactly; iv) The pressure in a hydraulic actuator is assumed uniform throughout the chamber. 4) List the *effort* and *flow* variables for the domains treated and discuss their similarity.

The procedure known as dynamic investigation consists in the generation of a model of a real system and later in its simulation in order to obtain a system response to a particular input, the knowledge of the system behaviour can then be used to apply modification to match some engineering requirements. In particular it can be structured in four stages:

1. The real system is clearly identified with respect to the environment and a physical model which features all the major characteristics is produced.
2. The physical model is used to determine without ambiguity a mathematical model of the system, which consists of a system of differential equations.
3. The response is simulated thus obtaining information about the behaviour of the system.
4. if the behaviour is considered adequate for our needs the analysis comes to an end, otherwise modification to the system are introduced to verify some requirements or optimize the response.

The assumptions used in the transformation from real system to physical model are:

1. Ignoring negligible effects, this makes the system less complex although maintaining the important parts of the dynamics, this means that the system of equations will be smaller and with less variables.
2. Considering the environment independent from the system, this makes it possible to uncouple the dynamics of the system and the environment and treat them separately.
3. Lumping continuous states or elements, this makes the mathematical description simpler by considering ordinary differential equations instead of partial.
4. Linearization of elements response, with this approximation analytical results can be found, even though its importance is secondary when computer simulation is available.
5. Time invariance of the parameters has implication similar to linearization, in the sense that it is easier to write an analytical solution.
6. Considering the system in a deterministic framework. Every system will be perturbed by effects that can be modeled as random processes, with this approximation the analysis can be carried out without use of statistical tools.

It is important to state all the assumptions carried out when modeling a system, to avoid making structural mistakes by not checking their validity over the whole response. For the reported examples the main approximations and their effects are:

- A pendulum torque proportional to the angle leads to linearized dynamics around the equilibrium position $\theta = 0$, then it is important to check that the oscillation remains contained around the equilibrium point
- The motion of an aircraft is due to 3 main forces: wind forces, gravity and thrust, so the study would lack this last one, and a structural error is made.
- A temperature sensor which reports an exact temperature is the approximation of a real sensor, for which the measurement is corrupted by a noise, that for example could be modeled as a white gaussian process. A real sensor will also determine the temperature at a particular point in the system, involving a lumped approximation, a better description of the state requires multiple temperature sensors.
- A uniform pressure inside the chamber of an hydraulic actuator is an example of lumping of a system, the real system chamber pressure will have in general values different point by point.

The analogy treatment of various physical domain can be carried out with effort and flow variables, reported in Table 1.

Table 1: Systems analogy

Domain	Effort variable	Flow variable
Mechanical (translation)	Force F	Velocity v
Mechanical (rotation)	Torque T	Angular velocity ω
Electrical	Voltage ΔV	Current i
Fluid	Pressure Δp	Volume flow rate Q
Thermal	Temperature ΔT	Heat flow Q_h

This analogy table is not the only one that can be constructed, in this case the starting element for the analogy are dissipative algebraic elements such as dampers, electric and heat resistors and head and concentrated pressure losses, then the effort and flow variable relates the 3 main kind of elements as shown in Eq. (1).

$$\text{Resistors : effort} = \text{resistance} \times \text{flow} \tag{1a}$$

$$\text{Capacitors : flow} = \text{capacitance} \times \frac{d}{dt}\text{effort} \tag{1b}$$

$$\text{Inductors : effort} = \text{inductance} \times \frac{d}{dt}\text{flow} \tag{1c}$$

Question 2

1) Derive from scratch the mathematical model for RC and RL circuits and express the system response in closed form. 2) Consider a real DC motor and a) sketch its physical model (list the assumptions made); b) derive its mathematical model; c) show how the motor constant depends on the physical parameters.

For both RL and RC circuits the differential equations for the systems can be obtained by applying Kirchhoff voltage law and via the assumptions of lumped passive elements, and using linear laws for resistors, capacitors and inductors. The ode equations are reported in Eq. (2) and Eq. (3).

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{V(t)}{RC} \tag{2}$$

$$\frac{di_l}{dt} + \frac{R}{L}i_l = \frac{V(t)}{L} \tag{3}$$

By using the method of variation of the constants the response to the non-homogeneous system can be written analytically as respectively Eq. (4) and Eq. (5), the response to the homogeneous case with non zero initial condition is simply the first term of the sum.

$$V_c(t) = V_c(0)e^{-\frac{1}{RC}t} + e^{-\frac{1}{RC}t}\frac{1}{RC}\int_0^t V(\tau)e^{\frac{1}{RC}\tau}\,d\tau \tag{4}$$

$$i_l(t) = i_l(0)e^{-\frac{R}{L}t} + e^{-\frac{R}{L}t}\frac{1}{L}\int_0^t V(\tau)e^{\frac{R}{L}\tau}\,d\tau \tag{5}$$

The physical model of a DC motor is shown in Fig. 1, the assumptions made are the following:

1. The coils inside the magnetic field are modeled as a serie of a resistor and an inductor. In particular the resistance models the dissipative effect of the current in a conductor, the inductor models the self inductance of the coil as variable current in the coil generate a variable magnetic flow through the coil, and because of Faraday law this is equivalent to a fem opposite to the cause of change. The approximation used for this part of the model are lumping of the characteristics of the circuit in discrete elements, neglecting the capacity that is always present when conductors are placed near each others, and use of linearized laws for the elements.

2. The battery power source is modeled as a dc ideal voltage, this source it's ideal because it is considered capable of maintaining a certain voltage for any current passing through it, this is an idealization that in nominal conditions should not generate modeling errors but it's important to check since it can lead to infinite power generation.

3. Since the coils are rotating inside of a constant magnetic field the effect of the variation of the magnetic flux due to this rotation is modeled as a back electromotive force element which depends proportionally on the speed of rotation, from Faraday law of induction.

4. The mechanical part of the motor is modeled with a lumped approach where the main axis consists of a inertia and a damping effect, modeling the friction of the bearings, The torque of the motor is considered linear with the current using Lorentz force law on the electrons moving inside the coils.

Then the mathematical model is derived with Kirchhoff law and Newton's law for the mechanical part, the model is reported in Eq. (6)

$$L\frac{di}{dt} + Ri + K_m\omega_m = v(t) \tag{6a}$$

$$J\frac{d\dot{\omega}_m}{dt} + b\omega_m - K_m i = -T_L \tag{6b}$$

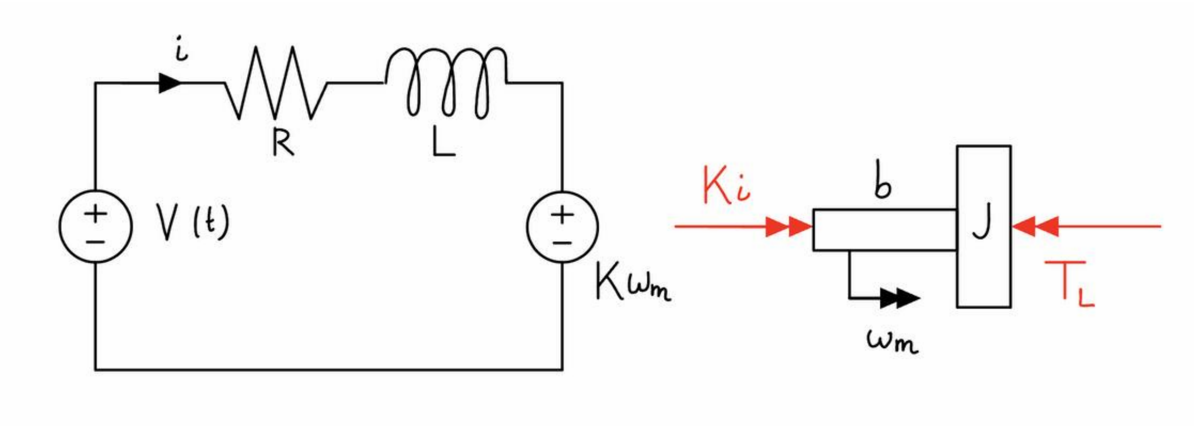


Figure 1: Physical model of an electric motor

By using Lorentz force law the constant K_m can be expressed in terms of the physical parameters, first the force on the wire in the magnetic field can be expressed as $F_e = q(\mathbf{v}\times\mathbf{B}) = ilB$ where l is the length of the wire in the magnetic field, B is the magnetic fiels and i the current in the wire. Considering the number of coils, and the distance r from the axis, the expression in Eq. (7) is found. Alternatively the constant can be obtained by Faraday law.

$$T_m = 2NF_e r = 2NlBri = Ki \tag{7a}$$

$$K = 2NlBr \tag{7b}$$

Question 3

1) Write down the Fourier law and show how it is specialized in the case of conduction through a thin plate; discuss the concept of thermal resistance. 2) Report the equation for thermal radiation in case of a) black body and b) real body and discuss them.

The general Fourier law of conduction for a three dimensional framework is expressed in Eq. (8), where $\mathbf{q}$ is the local heat flux density, k the material conductivity and ∇T the temperature gradient.

$$\mathbf{q} = -k\nabla T \tag{8}$$

It can be specialized for thin plates by considering only a change in the x axis perpendicular to the plate, this approximation stems from the fact that the heat rate lateral to the plates is negligible since the lateral area is much smaller than the frontal. Then transforming the heat flux density to heat rate $q = Q_h/A$, by considering the flow constant with x and by integrating, Eq. (9) is obtained.

$$Q_h = -\frac{kA}{l}\Delta T \tag{9}$$

Since in this equation Q_h is a flow variable and ΔT is an effort variable we can introduce the concept of thermal resistance as $R = l/(kA)$.

The equations of thermal radiation for a black body and a real body are expressed in Eq. (10) and Eq. (11), where σ is the Stefan-Boltzmann constant, T the temperature of the body and Q_h the heat rate. The main peculiarities of the real body model with respect to the black body are:

1. the introduction of emissivity ε, this is required as a black body is an ideal object with the maximum possible heat rate at a certain temperature, real surfaces instead are worse emitter that can reach emission rate only a fraction of those of black bodies.
2. A body which is not isolated in space will exchange heat as a function of both the objects temperatures. If the other object is an environment with high enough capacitance, it can be considered independent from the thermal behaviour of the object and its temperature will be constant.
3. An object emission between two bodies must account for the fact that part of the radiation is lost in free space, geometrically a viewing factor F_v can be computed.

$$Q_h = \sigma A T^4 \tag{10}$$

$$Q_h = \varepsilon F_v \sigma A (T_H^4 - T_L^4) \tag{11}$$

2 Exercises

Exercise 1

A miniaturized reaction wheel can be modeled as a couple of massive disks, connected with a flexible shaft (Figure 2). The first disk is driven by a rigid shaft, linked to an electric motor. The motor provides a given torque, while the rigid shaft is subjected to viscous friction, due to motor internal mechanisms. At $t_0 = 0$, the motor provides the torque $T(t_0) = T_0$.
1) Write down the mathematical model from first principles. 2) Using the data given in the figure caption, and guessing a value for the flexible shaft stiffness k and the viscous friction coefficient b, compute the system response from t_0 to $t_f = 10$ s. 3) Two accelerometers placed on the two disks recorded samples at 100 Hz, which were saved in the file `samples.txt`; the samples are affected by measurement noise. Determine the values of k and b that allow retracing the experimental data, so avoiding parametric errors.

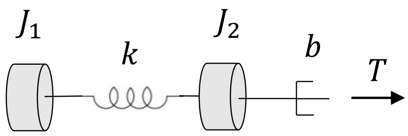


Figure 2: Physical model ($J_1 = 0.2\,\text{kg·m}$; $J_2 = 0.1\,\text{kg·m}$; $T_0 = 0.1\,\text{Nm}$).

The mathematical model is found by applying the balance of angular momentum separately to both the bodies in the system, the free body diagrams are visible in Fig. 3, the resulting equations are contained in Eq. (12), for all the elements a linear characteristic equation was used.

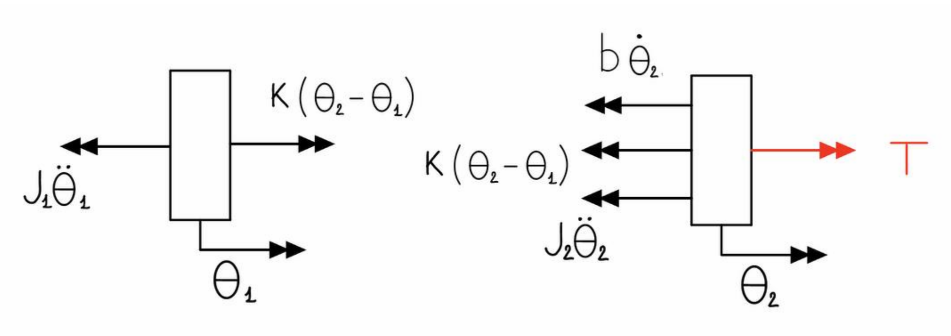
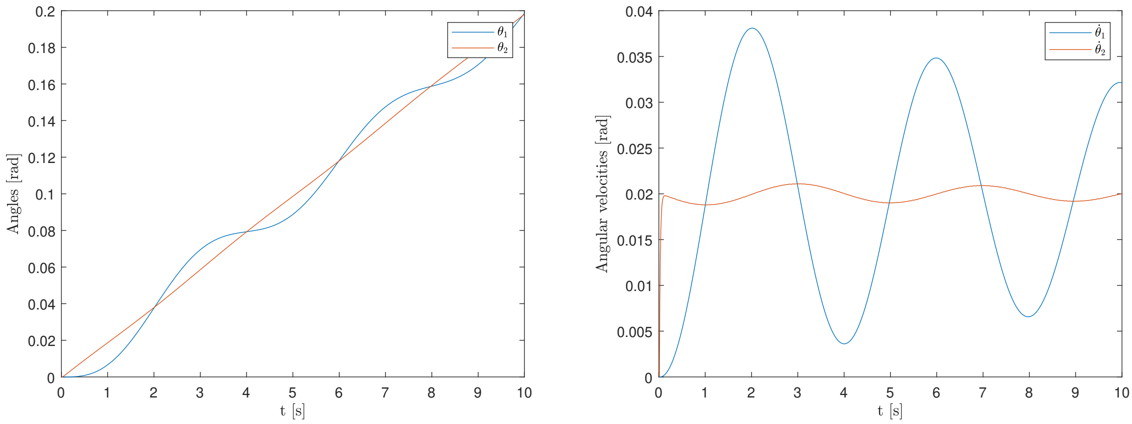


Figure 3: Free body diagrams

$$\begin{cases} J_1\ddot{\theta}_1 - k(\theta_2 - \theta_1) = 0 \\ J_2\ddot{\theta}_2 + b\,\dot{\theta}_2 + k(\theta_2 - \theta_1) = T(t) \end{cases} \tag{12}$$

As a guess the values $k = 0.5$ and $b = 5$ where used, the response is visible in Fig. 12a for the angular positions and in Fig. 12b for the velocities.

It is evident how the disk directly connected to the motor has almost a linear angular response and after a quick transient it rapidly assumes a steady state value for the angular velocity with small oscillations. While the other disk due to the spring has a delayed response and has continuous oscillations.



(a) Angular positions of guess **(b)** Angular velocities of guess

Figure 4

The problem of finding estimates of the real values of k and b is an example of system identification problem. To solve it a least square estimator was chosen, it is based on minimizing the sum of the squares of the differences between simulated and experimental response, the matlab function lsqcurvefit was used. Since the dataset had an initial point which is not compatible with the physical model and its an outlier in the dataset it was decided to avoid its contribution by considering the system response starting from 1E-2 seconds. The results of the estimation are contained in Table 2 and the differences in the response are visible in Fig. 5. Even if the steady state is considerable a very good fitting, the first part of the response drops in accuracy.

Table 2: Results: Linear Damper

Quantity	Value	Units
k	3.069042	$[Nm]$
b	0.534494	$[N\ m\ s]$
J_{loss}	0.909989	$[-]$

An alternative model considered for the system is one with a nonlinear damper due to turbulent flow $F_d = b\,\text{sign}(\dot\theta_2)\dot\theta_2^2$, the results are in Table 3 and in Fig. 6. This element permits to achieve a better response even in the first part and should be considered as the correct choice for the modeling of the damper.

Table 3: Results: Nonlinear Damper

Quantity	Value	Units
k	3.2575165	$[Nm]$
b	2.9277827	$[N\ m\ s^2]$
J_{loss}	0.0315679	$[-]$

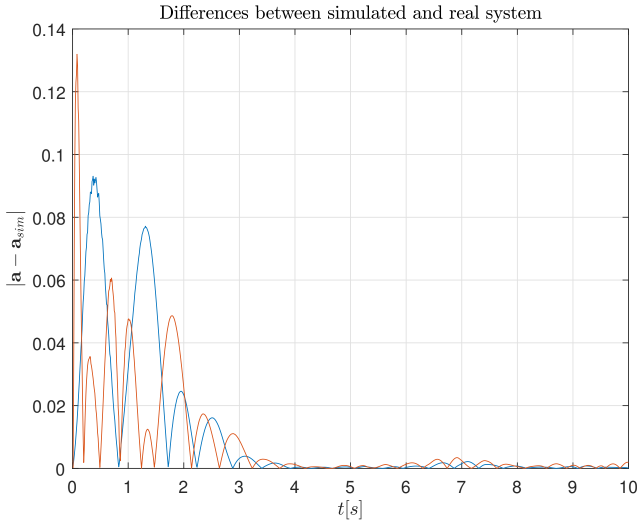


Figure 5: Absolute errors: Linear

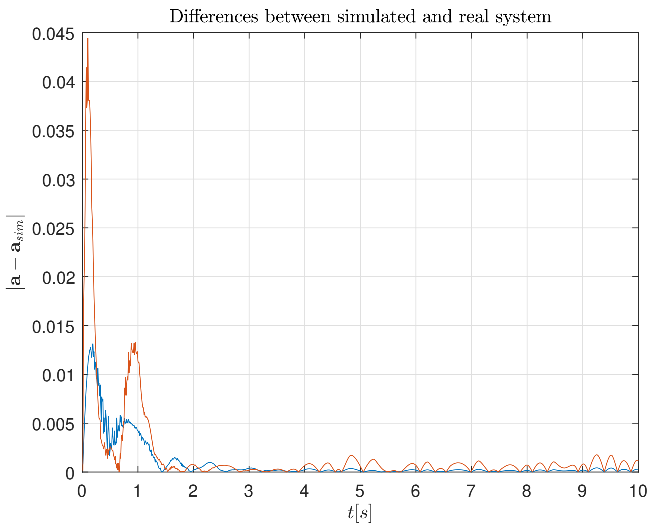


Figure 6: Absolute errors: Nonlinear

Exercise 2

Consider the ideal physical model network shown in Figure 7. The resistance R_2 varies its value with the value of the current, i.e., $R_2 = R_{2k}i$.The switch has been open for a long time. The capacitor is charged and has a voltage drop between its ends equal to 1 V. Then, at $t = 0$, the switch is closed. 1) Plot the subsequent time history of the voltage V_C across the capacitor. 2) Assume a voltage source characterized by $v(t) = \sin(2\pi f t)\arctan(t)$ having the positive terminal downward inserted in place of the switch. What is in this case the voltage history across the capacitor?

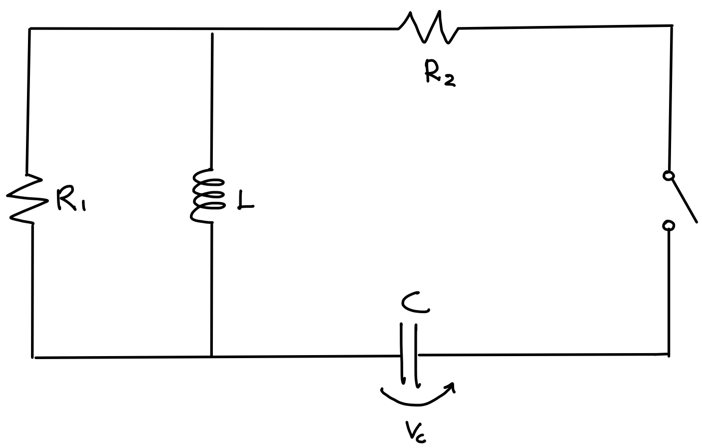


Figure 7: Circuit physical model ($R_1 = 100\,\Omega$; $R_{2k} = 10\,\Omega/\mathrm{A}$; $L = 10\,\mathrm{H}$; $C = 1\,\mathrm{mF}$; $f = 5\,\mathrm{Hz}$.)

The objective of this exercise is the determination of the system response of two different circuits. The first circuit equation was obtained via KVL for the three main loops and KCL, and by isolating the state variable V_c. The main assumptions from the already simplified and available physical model scheme are those of linearized characteristic equations.

The final result is the mathematical model in Eq. (13), expressed in state space form, where $x_1 = V_c$.

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ LC\left(2\dfrac{R_2}{R_1}x_2 + 1\right)\dfrac{dx_2}{dt} + \left(\dfrac{L}{R_1} + R_2 C\right)x_2 + x_1 = 0 \end{cases} \tag{13}$$

After integration using Runge-Kutta ode45 matlab integrator the result is visible in Fig. 8.

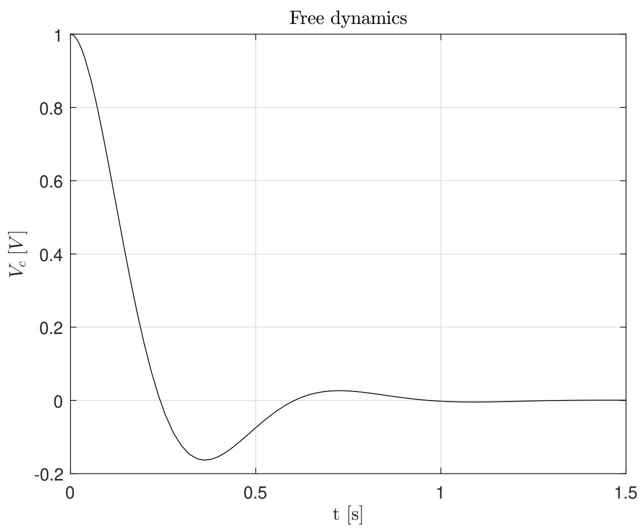


Figure 8: Free response

Now by adding instead a voltage generator the system will act under a forced input, the model is Eq. (14) and the integration leads to the response in Fig. 9.

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ LC\left(2\dfrac{R_2}{R_1}x_2 + 1\right)\dfrac{dx_2}{dt} + \left(\dfrac{L}{R_1} + R_2 C\right)x_2 + x_1 = V(t) + \dfrac{L}{R_1}\dfrac{dV}{dt} \end{cases} \tag{14}$$

The main steps of the derivation are reported in Eq. (15), Eq. (16) and Eq. (17). The resistance which varies with the current will introduce nonlinearities in the system.

$$\begin{cases} i_3 + i_1 + i_2 = 0 \\ \\ R_1 i_1 = L\dfrac{di_3}{dt} \\ \\ R_1 i_1 + V(t) = R_2 i_2 + V_c \\ \\ i_2 = C\dfrac{dV_c}{dt} \end{cases} \tag{15}$$

$$
\begin{cases}
\dfrac{di_3}{dt} + \dfrac{di_1}{dt} + \dfrac{di_2}{dt} = 0 \\[2ex]
\dfrac{di_3}{dt} = \dfrac{1}{L}\left(R_2 C \dfrac{dV_c}{dt} + V_c - V(t)\right) \\[2ex]
i_1 = \dfrac{1}{R_1}\left(R_2 C \dfrac{dV_c}{dt} + V_c - V(t)\right) \\[2ex]
i_2 = C\dfrac{dV_c}{dt}
\end{cases}
\tag{16}
$$

$$
\frac{di_1}{dt} = \frac{1}{R_1}\left(\frac{dR_2}{dt}C\frac{dV_c}{dt} + R_2 C \frac{dV_c^2}{dt^2}\right) + \frac{1}{R_1}\left(\frac{dV_c}{dt} - \frac{dV(t)}{dt}\right) \tag{17a}
$$

$$
\frac{dR_2}{dt}C\frac{dV_c}{dt} = C^2 R_{2k}\frac{d^2V_c}{dt^2}\frac{dV_c}{dt} = R_2 C\frac{d^2V_c}{dt^2} \tag{17b}
$$

$$
LC\left(2\frac{R_2}{R_1} + 1\right)\frac{d^2V_c}{dt^2} + \left(R_2 C + \frac{L}{R_1}\right)\frac{dV_c}{dt} + V_c = \frac{L}{R_1}\frac{dV(t)}{dt} + V(t) \tag{17c}
$$

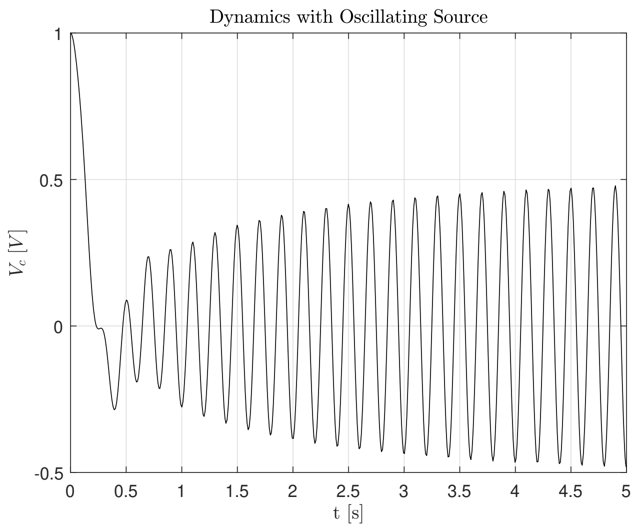


Figure 9: Forced response

Exercise 3

The rocket engine in Figure 17a is fired in laboratory conditions. With reference to Figure 17a, the nozzle is made up of an inner lining (k_1), an inner layer having specific heat c_2 and high conductivity k_2, an insulating layer having specific heat c_4 and low conductivity k_4, and an outer coating (k_5). The interface between the conductor and the insulator layers has thermal conductivity k_3. 1) Select the materials of which the nozzle is made of[1], and therefore determine the values of k_i ($i = 1, \ldots, 5$), c_2, and c_4. Assign also the values of ℓ_i (i=1,. . . ,5), L, and A in Figure 17a. 2) Derive a physical model and the associated mathematical model using one node per each of the five layers and considering that only the conductor and insulator layers have thermal capacitance. The inner wall temperature, T_i, as well as the outer wall temperature, T_o, are assigned. 3) Using the mathematical model at point 2), carry out a dynamic simulation to show the temperature profiles across the different sections. At initial time, $T_i(t_0) = T_o(t) = 20$ C°. When the rocket is fired, $T_i(t) = 1000$ C°, $t \in [t_1, t_f]$, following a ramp profile in $[t_0, t_1]$. Integrate the system using $t_1 = 1$ s and $t_f = 60$ s. 4) Repeat the simulation in point 3) using a mathematical model implementing two nodes for the conductor and insulator layers.

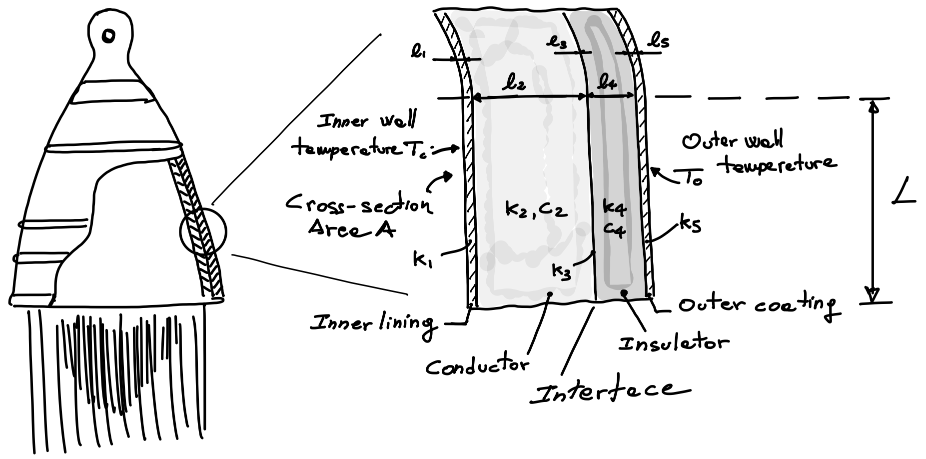


Figure 10: Real thermal system.

The determination of the system characteristic properties was made through a literature search about heat absorption nozzles. These kind of nozzles do not try to actively limit the temperatures but rely on heat absorbing capacity and slow heat transfer to deal with thermal stresses on the materials, enough at least to maintain the structural integrity of the nozzle. In particular as detailed in Sutton[2] there are 4 main materials with different functions in a nozzle:

1. A structure and pressure container, is the outermost layer, described in Fig. 17a as outer coating, it is usually made of Aluminium or Low carbon steel, its temperatures are limited to 515 °C

2. A heat sink and heat-resistant material, it is an element with high thermal capacity, since the thermal load in this nozzle is not very high molded graphite can be chosen as it is relatively inexpensive.

3. An insulator layer behind the heat sink, usually made of ablative plastics with fillers of silica, it has low conductivity and good adhesion to the other layers.

[1] The interface layer is not made of a physically existing material, though it produces a thermal resistance. For this layer, the value of the thermal resistance R_3 can be direcly assumed, so avoiding to choose k_3 and ℓ_3.

[2] Sutton GP Biblarz O. Rocket Propulsion Elements. 7th ed. New York: John Wiley & Sons; 2001.

4. A flame barrier, here reported as inner lining, it is a layer exposed to the hot gases, it is usually made of ablative plastics but with less fillers than the insulator, it has better erosion resistance and can reach higher temperatures than the other layers.

The main thermal characteristics with relative values chosen for the various layers are contained in Table 5. The interface layer was assumed directly as a small resistance with $R_{int} = 0.02K/W$.

Table 4: Nozzle Materials

Layer	Length [mm]	k [W/(mK)]	ρ [kg/m^3]	c [J/(kgK)]
1. Inner Layer	1	50	-	-
2. Conductor	24	100	8000	500
3. Insulator	12	0.7	2500	1500
4. Outer Layer	1	150	-	-

The construction of a physical model is based on the electrical analogy of circuits, the biggest assumption here is that of lumped elements, every layer is concentrated onto a central lump with a certain thermal capacitance and lateral resistances for half of the layer. The resulting physical model is depicted in Fig. 11.

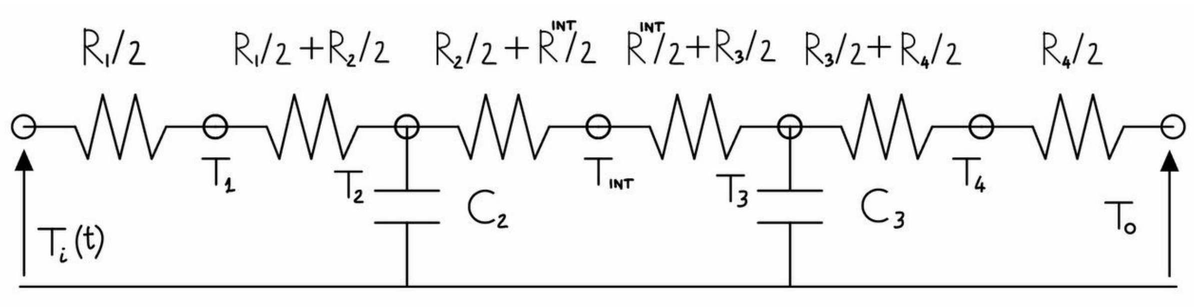
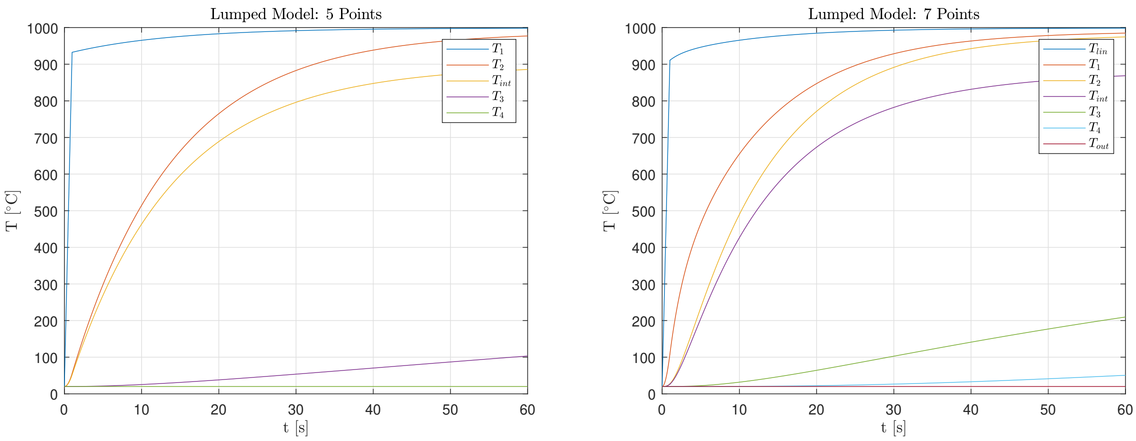


Figure 11: Physical Model of a Nozzle

The associated mathematical model is obtained by solving this circuit considering linear characteristic equations, it is reported in Eq. (18). The state is comprised of only the layer 2 and 3 temperatures but the remaining layers temperatures can be easily found with the thermal flow rate and the characteristic equations of the resistances.

$$\begin{cases} \dfrac{dT_2}{dt} = \dfrac{1}{C_2}\left(\dfrac{T_i - T_2}{R_1 + R_2/2} - \dfrac{T_2 - T_3}{R_2/2 + R_{int} + R_3/2}\right) \\ \dfrac{dT_3}{dt} = \dfrac{1}{C_3}\left(\dfrac{T_2 - T_3}{R_2/2 + R_{int} + R_3/2} - \dfrac{T_3 - T_o}{R_3/2 + R_4}\right) \end{cases} \tag{18}$$

The model is then simulated using a runge-kutta integrator since no stiffness problems are expected, the results are then confronted with an alternative model where the insulator and conductor layers are modeled as two main capacitative lumps with the layer resistance divided in three parts, two lateral and a in-between lumps resistance. The results are reported in Fig. 12 and in Fig. 13, increasing the lumps helps with the discretization of the temperatures, which tends to make the behaviour more similar to the continuous model described by partial differential equations. The simulation findings are coherent of what is expected of the model, as time passes all the temperatures rise, and across the conductor there is a low temperature gradient. Since the temperatures of the outer layers are kept under the critical safety points the safety requirements are satisfied and the choice of the materials is approved.



(a) 5 lump response **(b)** 7 lump response

Figure 12

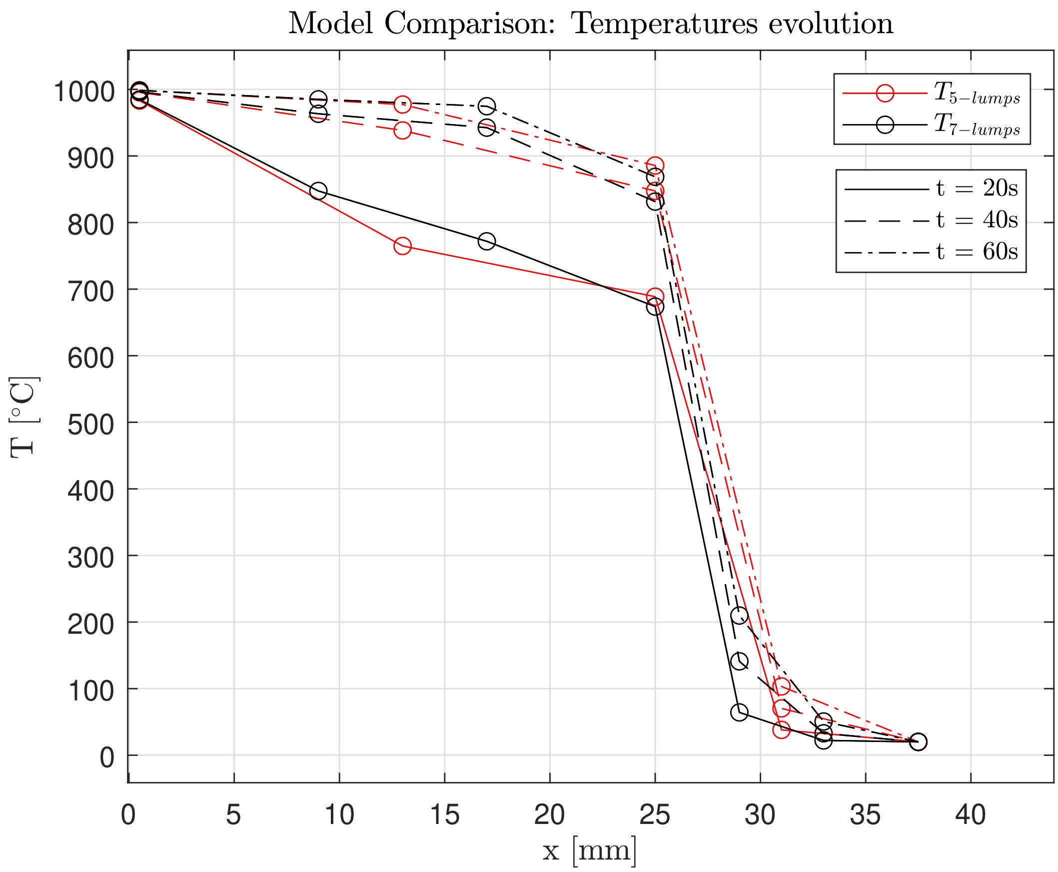


Figure 13: Temperatures evolution

Exercise 4

The electro-mechanical system in Figure 14 is constituted by an electric circuit that drives a mechanical part. A voltage source $v(t) = v_0\ \cos(\omega t)\ e^{-\beta t}$ is activated at $t = 0$. Assume a coefficient K_m such that $T(t) = K_m\ i(t)$ and $e(t) = K_m\ \Omega_1$, where T is the torque acting on the mechanical part, i the current in the circuit, and Ω_1 the angular velocity of the first disk. Considering $K_m = 20$; $R = 200$; $v_0 = 2$ V; $\omega = 5$ Hz; $\beta = 0.2$ Hz; $L = 2$ mH; $J_1 = 0.5$ kg m^2; $J_2 = 0.3$ kg m^2; $b = 0.1$ kg m^2 s^{-1}; $k = 0.5$ Nm, it is asked to:

1) Derive the mathematical model of the system;
2) Determine the system eigenvalues;
3) Select and motivate the most appropriate integration scheme;
4) Show the system response until $t = 30$ s;
5) Setup, discuss, and run a procedure to find the values of K_m and R that allow matching the second disk angular velocity profile sampled at 10 Hz given in the file `Profile.txt`

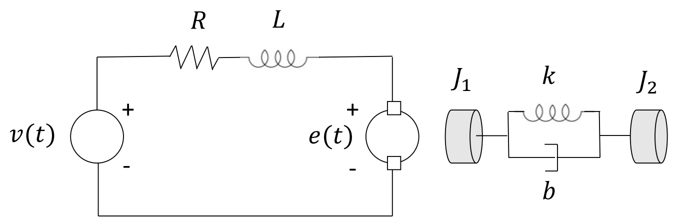


Figure 14: Electro-mechanical system.

The mathematical model can be obtained by separate evaluation of the electrical and mechanical subparts, specifically for the electrical circuit the KVL is used and the linear characteristic equations are used, the state variable is chosen as the loop current. While for the mechanical subpart the conservation of angular momentum can be used. The state then can be identified as $\mathbf{x} = [i, \theta_1, \theta_2, \Omega_1, \Omega_2]$, where θ are the angular positions of the disks and Ω their angular velocities. The model is reported in Eq. (19)

$$\begin{cases} L\dfrac{di}{dt} + Ri + K_m\Omega_1 = v(t) \\[2ex] \dfrac{d\theta_1}{dt} = \Omega_1 \\[2ex] \dfrac{d\theta_2}{dt} = \Omega_2 \\[2ex] J_1\dfrac{d\Omega_1}{dt} + k(\theta_1 - \theta_2) + b(\Omega_1 - \Omega_2) - K_m i = 0 \\[2ex] J_2\dfrac{d\Omega_2}{dt} - k(\theta_1 - \theta_2) - b(\Omega_1 - \Omega_2) = 0 \end{cases} \tag{19}$$

Then the system can be written in state space form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, and the matrix $\mathbf{A}$ is identified, exploiting the matrix decomposition in $\mathbf{TDT}^{-1}$, the eigenvalues will be found as the diagonal elements of $\mathbf{D}$ and the general motion will be a combination of the modes represented by these eigenvalues through the eigenvectors, columns of the matrix **T**. The eigenvalues are then reported in Table 5, since **A** has rank 4 instead of 5, one of them will be zero.

Table 5: System Eigenvalues

Number	Value
λ_1	-99996
λ_2	-3.9480
λ_3	0.0
λ_4	-0.2927 + 1.2661i
λ_5	-0.2927 - 1.2661i

Since the system combine two domains with very different reaction speeds, the system eigenvalues will have very different order of magnitude, the system then can be defined as stiff, as an integrator where the length of the step is governed by stability constraints must proceed with very small steps for a non small integration time to depict the complete response of the slower part of the system. It is decided then to use Matlab ode15s integrator, this integrator employs a numerical differentiation formulas (NDFs) scheme of orders 1 to 5. This has similar characteristics to BDFs multi-step, implicit integrator family, with a stability region that can handle very stiff problem and good enough accuracy properties.

The system response obtained with the integrator described above is shown in Fig. 15.

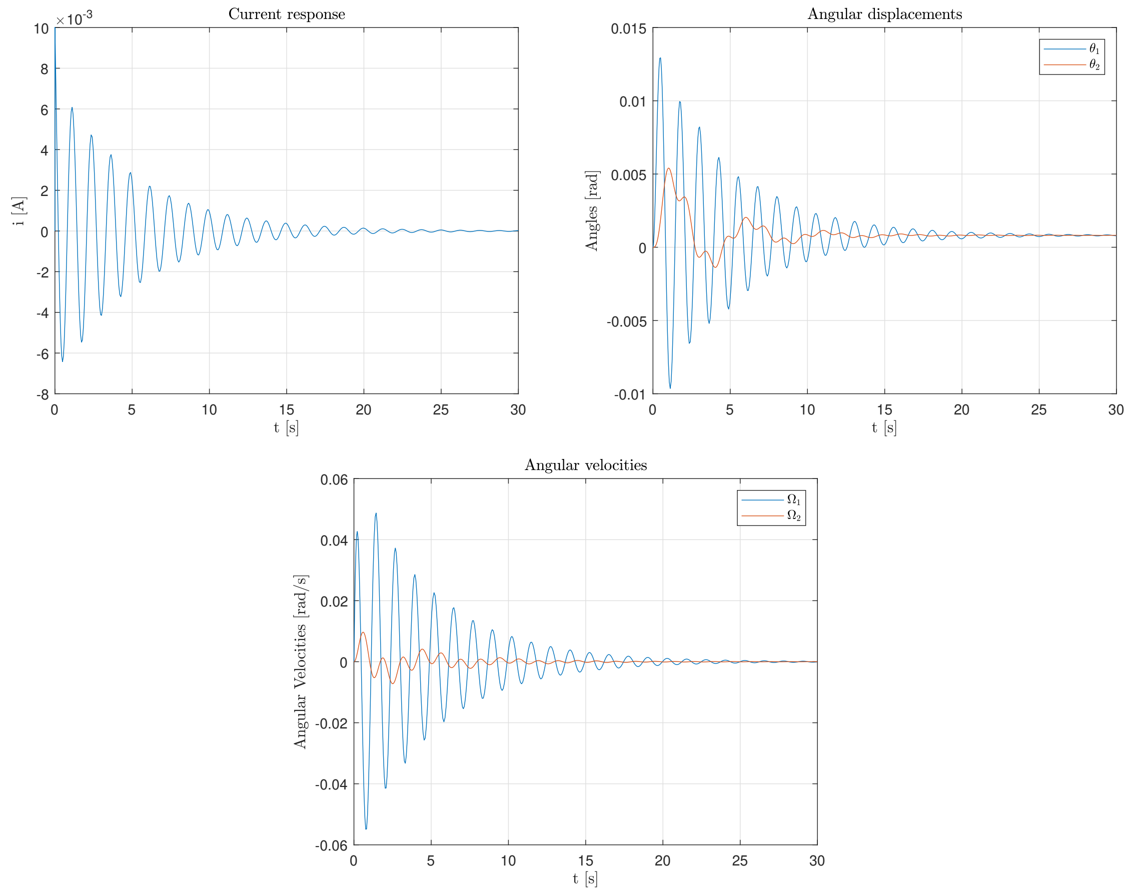


Figure 15: Electro-mechanical system response

Then the analysis can be extended with the objective of matching a real system response. This is the problem of system identification that uses real data from the true system to eliminate parametric errors on the choice of parameters of the mathematical model. This problem can be solved by an estimator, using the maximum likelihood approach, the parameters K_m and R are then chosen as to maximize the probability density function of the system model output. This is equivalent to a least squares procedure which minimizes the loss function in Eq. (20).

$$J(K_m, R) = \frac{1}{N} \sum_{k=1}^{K} (y_k - y(x_k, K_m, R))^2 \tag{20}$$

The procedure is implemented in the matlab function lsqcurvefit, the results of this method are reported in Table 6 and the residual errors are shown in Fig. 16.

Table 6: System Identification

Quantity	Value	Units
K_m	16.622556	$[-]$
R	225.235507	$[\Omega]$
J_{fin}	3.23122E-08	$[-]$

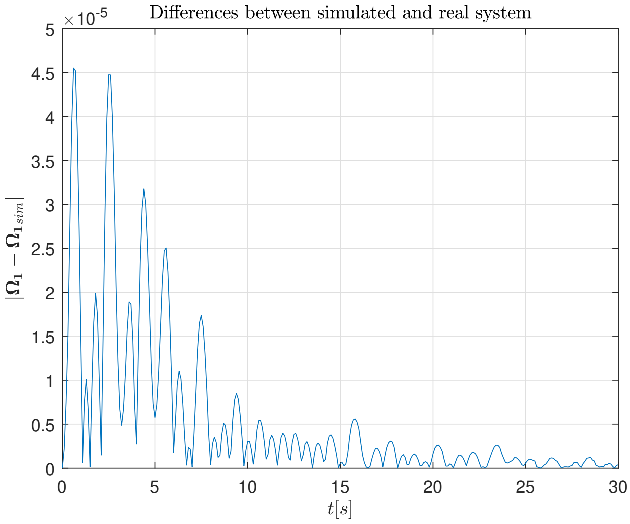


Figure 16: Residual of identification

Exercise 5

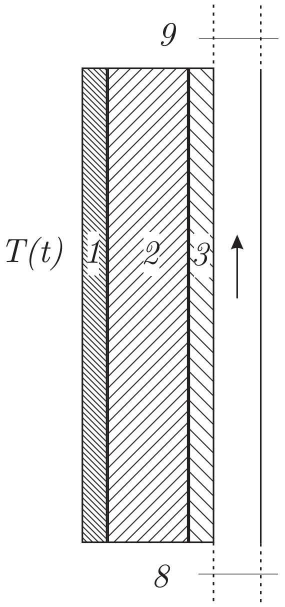
The hydraulic system in Figure 17b is made of a tank, a pump, a check valve, a distribution valve, a filter and an heat exchanger, plus the lines. The heat exchanger is used to cool down an external system, having a temperature profile $T(t) = T_0 + k_T \cos(\omega t)$, and its wall is made up of three layers with different thermal properties, as depicted in Figure 17a. The pressure drop inside the heat exchanger can be modelled as a simplified Rayleigh flow, such that $P_{out} = e^{\dot{Q}/\kappa} P_{in}$. Assuming:

- Fluid: Incompressible fluid, $\rho = 1000$ kg/m^3, specific heat $c_w = 4186$ J/(kg · K)
- Lines: Coefficient of pressure drop across the check valve $k_{cv} = 2$, diameter of the lines $D = 20$ mm;
  - Branch T–1: Length $L_{T1} = 0.5$ m, friction factor $f_{T1} = 0.032$;
  - Branch 3–4: Length $L_{34} = 1.5$ m, friction factor $f_{34} = 0.032$;
  - Branch 5–6: Length $L_{56} = 2.7$ m, friction factor $f_{56} = 0.040$;
  - Branch 7–8: Length $L_{78} = 2.5$ m, friction factor $f_{78} = 0.028$;
  - Branch 9–T: Length $L_{9T} = 1$ m, friction factor $f_{9T} = 0.032$.
- Tank: Adiabatic tank with constant pressure $P_T = 0.1$ MPa;
- Pump:

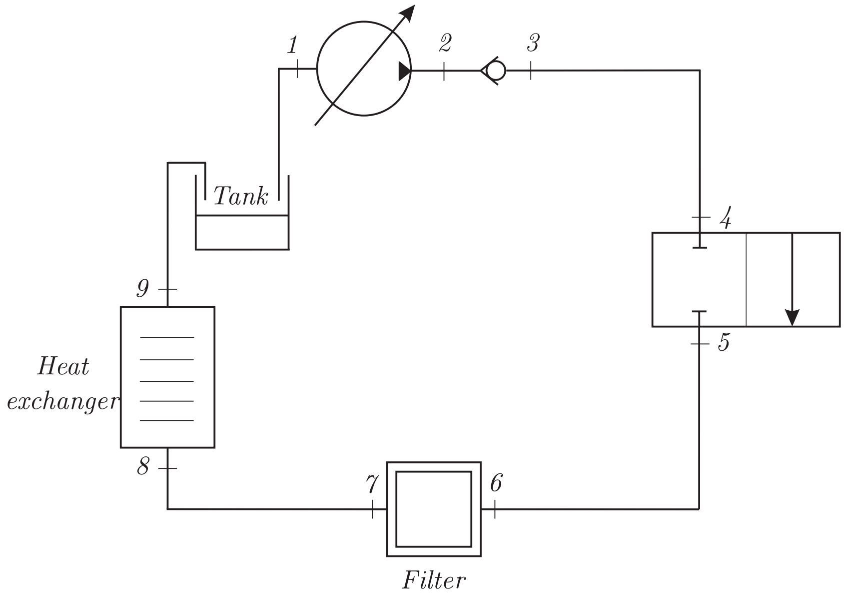
  Pistons:
  - Number: $N = 9$;
  - Diameter: $D_p = 0.7$ cm
  - Distance between the shaft of the shaft: $d_p = 1.5$ cm
  - Nominal pressure: 5 atm

  Pilot piston:
  - Diameter: $D_k = 1$ cm
  - Control lever length: $l_c = 10$ cm
  - Maximum angle of the control plate: $\theta_{\max} = 20$ deg
  - Rotation speed: $n = 4000$ rpm
  - Equivalent mass: $m_k = 2$ kg
  - Pre-loaded force: $F_0 = 5$ N
  - Friction coefficient: $r_k = 1$ Ns/m
  - Diameter of the pipe: $d_k = 1$ mm
  - Pilot pipe head loss: $k_p = 2.5$
- Distributor: Coefficient of pressure drop across the distributor $k_d = 15$, diameter $d_o = 10$ mm. At $t_0 = 0$ s the valve is half open; it is fully open after $\Delta t = 2$ s.
- Cooler: Diameter of the pipe $D = 20$ mm; Heat flux $\dot{Q}_c = 100$ W.
- Filter: Coefficient of pressure drop across the filter $k_f = 35$, leaking coefficient $k_l = 2.5\%$[2];

[2] $Q_{leak} = k_l Q_{in}$



(a) Heat exchanger model close-up.



(b) Hydraulic system physical model

Figure 17: Thermo-hydraulic system. Assume any other missing data.

- Heat exchanger: Diameter of the pipe $D = 20$ mm, length of the pipe inside the exchanger: $L_e = 0.5$ m; planar exchanger with exchange area $A_e = 1000$ cm^2; First layer thermal properties: $k_1 = 395$ W/(m · K), $\ell_1 = 1$ cm, second layer thermal properties: $k_2 = 310$ W/(m · K), $\rho_2 = 8620$ kg/m^3, $c_2 = 100$ J/(kg · K), $\ell_2 = 2.5$ cm, third layer thermal properties: $k_3 = 125$ W/(m · K), $\ell_3 = 1$ cm, heat transfer coefficient with the fluid $h = 20$ W/(m^2 · K);
- $T_0 = 350$K, $k_T = 20$, $\omega = 5$ s^{-1}, $\kappa = 1000$ W;
- The temperatures are propagated instantaneously along the pipes;
- The heat exchanger layers have an initial temperature of 320 K.

It is asked to:

1) Derive a lumped-approach physical model for the heat exchanger;
2) Derive the mathematical model of the whole system;
3) Select and motivate the most appropriate integration scheme;
4) Show the system response until $t = 25$ s;
5) Discuss at least one possible way to modify the system in order to keep the fluid temperature as close as possible to 20 °C.

Since the heat exchanger sub-system is independent from the rest of the model state a local analysis of the system can be developed. The fluid which at a certain instant is found in the control volume between 8 and 9 in Fig. 17a is modeled as a capacitance with value $C_{fluid} = c_w \rho_{fluid} A_{tube} L_{tube}$. The layers of the walls of the heat exchanger are modeled as resistances, the central has also a certain capacitance. Finally considering a thermal boundary layer, between the temperature of the fluid in contact with the walls and the fluid at a certain distance there is a natural convective resistance. The model is represented in Fig. 18.

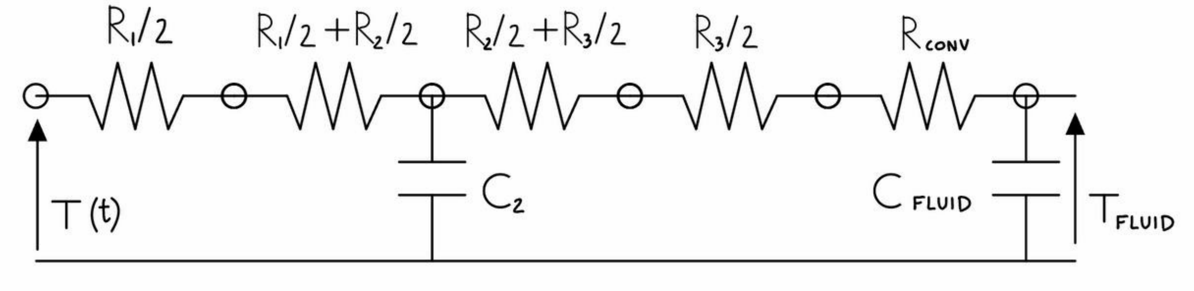


Figure 18: Physical model: Heat exchanger

The complete model of the system can be expressed in term of the state $\mathbf{x} = [x\ v\ V\ T_2\ T_{fluid}]$, where x and v are position and velocity of the control piston, V is the tank volume, since it is assumed that all leakages find their way back to this pressurized tank there will be no variation, nevertheless it is assumed a initial volume of 25L. The model can then be written as a system of 4 odes and multiple algebraic equations, these are not reported, preferring to illustrate only the main causalities governing the behaviour. As a starting point a simplified model of a swash-plate displacement pump generates a certain volumetric flow $Q_1 = nN_pA_ps(x)$ where the pistons stroke is determined by x. Then the pressure drops can be obtained as $\Delta p = 1/2k\rho Q^2/A^2$ since Q_1 is known and filter leakage is easily computed. The heat flow from the exchanger is also needed since it is approximately a Rayleigh flow which produce a pressure increase. Then having obtained the pressure at the control piston this can be used for a balance of force and the evaluation of the dynamics of the piston.

The ode equations of the system are reported in Eq. (21).

$$
\begin{cases}
\dfrac{dx}{dt} = v \\[2ex]
\dfrac{dv}{dt} = \dfrac{1}{m_k}(p_kA_k - F_0 - hx - r_kv) \\[2ex]
\dfrac{dV_{tank}}{dt} = Q_9 - Q_1 + Q_{leak} \\[2ex]
\dfrac{dT_2}{dt} = \dfrac{1}{C}\left(\dfrac{T - T_2}{R_1 + R_2/2} - \dfrac{T_2 - T_{fluid}}{R_2/2 + R_3 + R_{conv}}\right) \\[2ex]
\dfrac{dT_{fluid}}{dt} = \dfrac{1}{C_{fluid}}\dfrac{T_2 - T_{fluid}}{R_2/2 + R_3 + R_{conv}}
\end{cases}
\tag{21}
$$

The sizing of the pump control piston was obtained considering a fixed pre-loading term F_0 and by imposing the equilibrium at a nominal pressure of 5 atm. The result is reported in Eq. (22).

$$
h = \frac{p_{nom}A_k - F_0}{c} = 955.8538N/m \tag{22}
$$

The main approximations in the modeling of the systems are detailed in the following for the various components:

1. The displacement pump is modeled as having a constant volume flow rate for every cycle, a more realistic pump should show a variation due to the sinusoidal motion of the pistons, this is a reasonable assumption since for a high number of pistons, as 9 in this case, and more importantly for odd numbers of them the motion in the complex show small irregularities. Furthermore the pump is considered as having no leakage.

2. Since the liquid is considered as incompressible, its bulk modulus is infinity, no compressibility effects arise. This is a reasonable assumption if the pressures and temperatures of the liquid don't reach high enough values, which is the case of this system.

3. The distribution valve is considered to open under a linear varying command u, this is equivalent to the evolution described in Eq. (23) for the variation of angle α and the area of the distributor valve.

4. The pressure drops are only considered for the main distributed elements identified on the schematics, concentrated drops due to rapid changes in geometry of the tubes are considered negligible.

5. The heat exchanger is considered as composed of a unique heat exchange pipe, a more accurate analysis should account for a serpentine and a parallel capacitance model. The wall of the tube are considered also to have negligible thermal resistance.

$$u = \frac{t}{\Delta t} \text{ for t} < \Delta\text{t else } u = 1 \tag{23a}$$

$$\alpha = \pi + 2\arccos(1 - |u|) \tag{23b}$$

$$A = \frac{r_0^2}{2}(\alpha - \sin(\alpha)) \tag{23c}$$

The model was simulated with various integration scheme, using ode45, from the runge-kutta family which is neither A-stable nor F-stable, produced a swift integration. Then this was considered as appropriate and the following system response was obtained using ode45 and RelTol = 1e-12, AbsTol = 1e-12. The response of the system is shown in Fig. 19. From the response we can see that many physical signals exhibits similar behaviour:

1. After a quick initial transient an equilibrium is reached, for the pressure this happens almost instantaneusly, other signals such as position of the pilot piston exhibit an initial overshoot before equilibrium. The temperature of the fluid instead will get to equilibrium in a much larger time scale.

2. In all the responses of the hydraulic system there is a fast oscillating effect which comes from the rapid oscillation of the pilot piston, this effect decreases with time as the piston stabilizes.

3. In all the responses a slower oscillating effect is present and it's due to the object in proximity of the heat exchanger. Its slowly oscillating temperature, through Rayleigh flow pressure increase act as a disturbance on the whole system response.

To keep the fluid temperature close to 20°C as possible at least a temperature sink after the heat exchanger must be introduced. The temperature of this sink can be regulated such that the temperature of the fluid does not exceed 20°C. A possibility is to model it similarly to the heat exchanger with a temperature at the walls constant since the sink is assumed to have a high enough thermal capacitance to avoid temperature changes during operations.

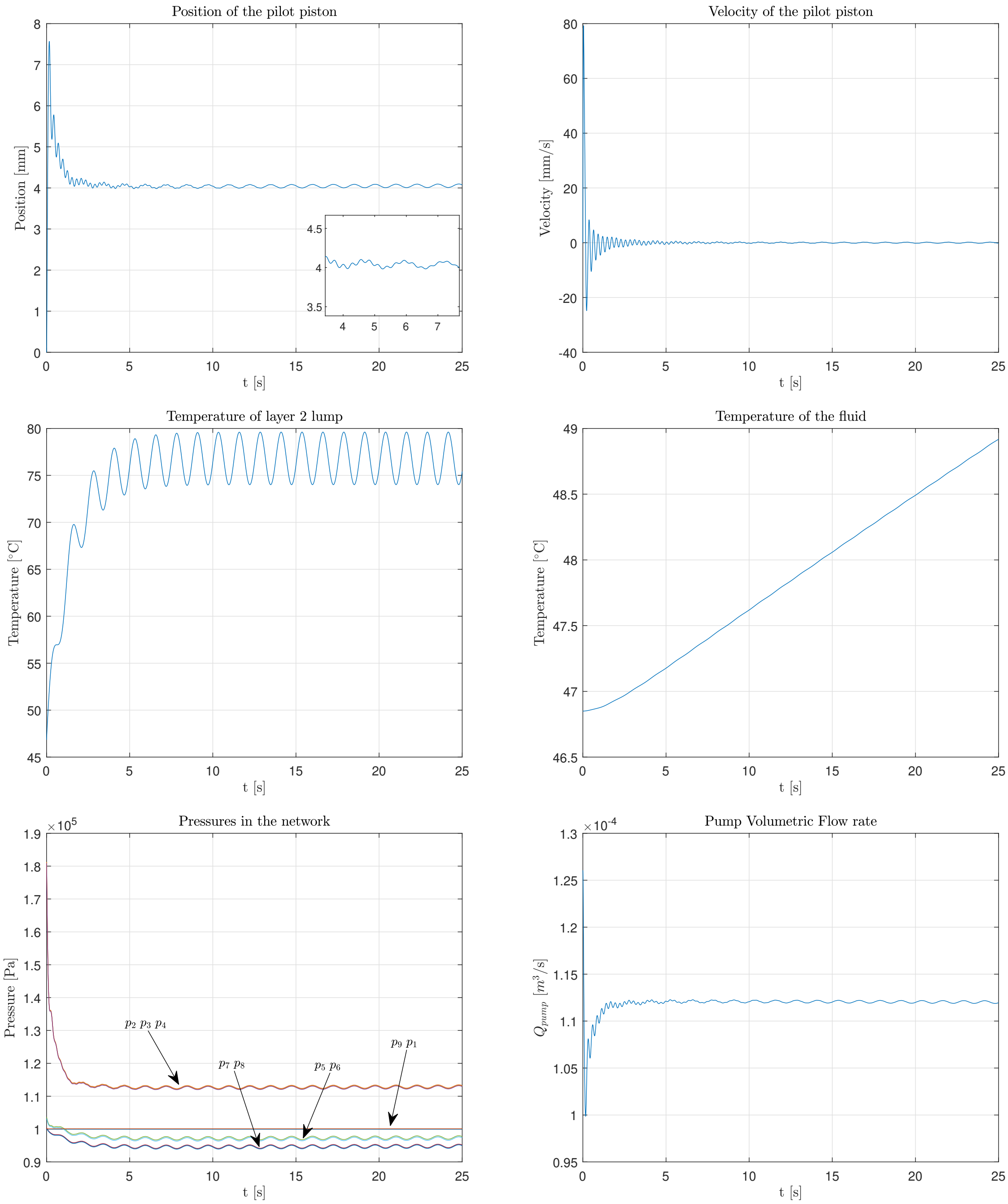


Figure 19: Thermo-hydraulic system response