6. EVALUATING TRIPLE INTEGRALS

Aim: Evaluating triple integrals (Cartesian, Cylindrical and Spherical coordinates) and visualizing regions using Matlab.

`int(f,v)` → symbolic integration of function `f` with respect to variable `v`.

- **`fsurf(f)`** → plots 3D surfaces of functions/regions.
- **`viewSolid(...)`** → used for visualizing 3D regions bounded by surfaces.

MATLAB Syntax used

int(f,v)	uses the symbolic object v as the variable of integration, rather than the variable determined by symvar
fsurf(f)	fsurf(f) creates a surface plot of the function z = f(x,y) over the default interval [-5 5] for x and y.
fsurf(f,xyinterval)	fsurf(f,xyinterval) plots over the specified interval. To use the same interval for both x and y, specify xyinterval as a two-element vector of the form [min max]. To use different intervals, specify a four-element vector of the form [xmin xmax ymin ymax].

Example 1

Evaluate the iterated integral $\int_0^1 \int_0^z \int_0^{x+z} 6\,xz\,dy\,dx\,dz$

Matlab code

```
syms x y z
sol = int(int(int(6*x*z,y,0,x+z),x,0,z),z,0,1)
```

Command window

```
sol = 1
```

Example 2

Evaluate the triple integral $\iiint_E 6\,xy\,dV$**, where E lies under the plane z =1+x+y and above the region in the xy-pane bounded by the curves** $y=\sqrt{x}$**, y=0 and x=1.**

Sol

Here $E=\{(x,y,z)|0\le x\le 1, 0\le y\le \sqrt{x}, 0\le z\le 1+x+y\}$

$$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx$$

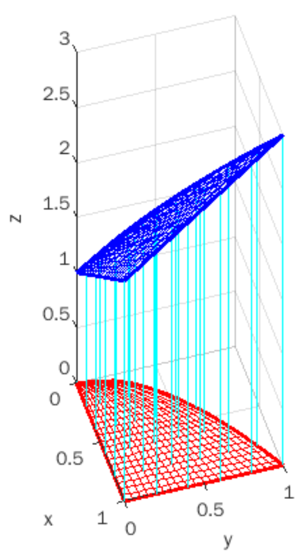
Matlab code

```
syms x y z
sol = int(int(int(6*x*y, z,0,1+x+y),y,0,sqrt(x)),x,0,1)
viewSolid(z,0+0*x*y,1+x+y,y,0+0*x,sqrt(x),x,0,1);
viewSolid(z, zmin, zmax, y, ymin, ymax, x, xmin, xmax)

    z, zmin, zmax → lower and upper surfaces for z
(vertical bounds).

    y, ymin, ymax → bounds for y variable.

    x, xmin, xmax → bounds for x variable.
axis equal; grid on;
```

Command window

sol = 65/28

The region E is shown below (between two surfaces)



Example 3

Evaluate the triple integral $\iiint_E y\,dV$, where E is bounded by the planes x =0, y = 0, z=0, and 2x+2y+z = 4.

Sol:

$$\iiint_E y\,dV = \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} y\,dz\,dy\,dx$$

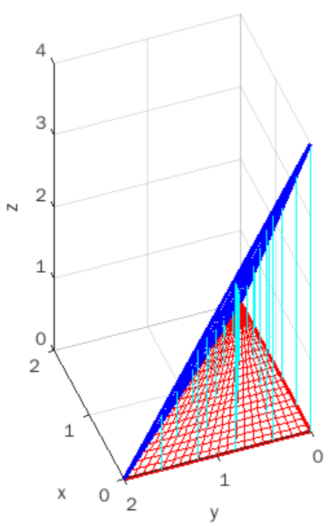
Matlab code

```
syms x y z
sol = int(int(int(y,z,0,4-2*x-2*y),y,0,2-x),x,0,2)
viewSolid(z,0+0*x*y,4-2*x-2*y,y,0+0*x,2-x,x,0,2);
axis equal; grid on;
```

Output in the command window

```
sol = 4/3
```

The region E is shown below



Example 4

A solid E lies within the cylinder $x^2+y^2=1$, below the plane z =4, and above the paraboloid $z=1-(x¿¿2+y^2)¿$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Sol

In cylindrical coordinates the cylinder is r =1 and the paraboloid is z = 1-r^2, so we can write

$$E = \{(r,\theta,z)\mid 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant r\leqslant 1,\ 1-r^2\leqslant z\leqslant 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z-axis, the density function is $f(x,y,z)=K\sqrt{x^2+y^2}=Kr$ where K is the proportionality constant.

The mass of E is

$$m=\iiint_E K\sqrt{x^2+y^2}\,dV=\int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (Kr)\,r\,dz\,dr\,d\theta$$

Matlab code

syms r z theta K %syms r z theta K → declares symbolic variables.

Ma= int(int(int((K*r)*r, z, 1-r^2,4), r ,0, 1),theta,0,2*pi) % integration (density × Jacobian).

x = r*cos(theta), y = r*sin(theta),

s = sym(4) %symbolic constant for plane height.

fsurf(x,y,1-r^2, [0 1 0 2*pi], 'g', 'EdgeColor', 'none'); % plotting paraboloid Plots the **paraboloid** z=1−r2 (green surface).

hold on

fsurf(1*cos(theta), 1*sin(theta), z , 'y', [0 2*pi 0 4], 'EdgeColor', 'none') % plotting cylinder of radius 1 with height z = 4 (yellow).

fsurf(x,y,s, [0 1 0 2*pi], 'b', 'EdgeColor', 'none'); % plotting circular plane z=4.

axis equal; xlabel('x'); ylabel('y'); zlabel('z');

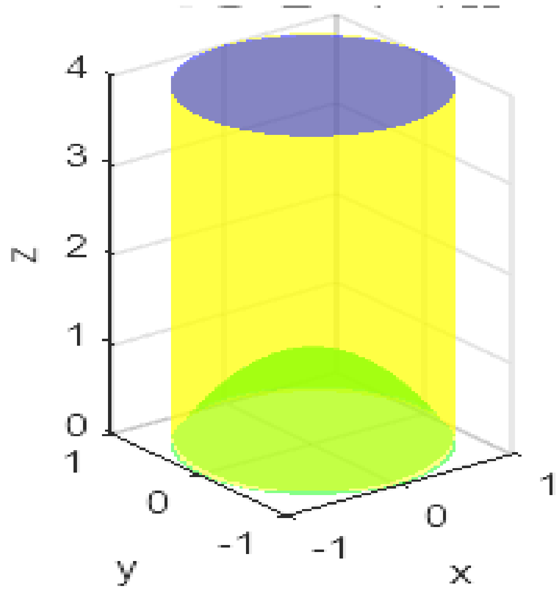
alpha 0.5 %makes surfaces semi-transparent so you can see inside.

Output; In the command window

```
Ma =(12*pi*K)/5
```

In the figure window

The region E is shown below(above the paraboloid and below the surface z=4 inside the cylinder)



Example 5

Evaluate $\iiint_E e^z \, dV$**, where E is enclosed by the paraboloid** $z=1+x^2+y^2$ **, the cylinder** $x^2+y^2=5$ **, and the xy-plane.**

Sol

By Converting Cartesian to Cylindrical coordinates we get

$$\iiint_E e^z \, dV = \int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{1+r^2} e^z \, r dz dr d\theta$$

Matlab code

```
clc
clear all
syms x y r z theta
Sol= int(int(int(exp(z)*r,z,0,1+r^2),r,0,sqrt(5)),theta,0,2*pi) % integration
f=1+(x^2+y^2);
fsurf(f,[-sqrt(5) sqrt(5) -sqrt(5) sqrt(5)],’r’, 'EdgeColor', 'none')
```

Defines paraboloid surface: z=1+(x2+y2).

- `fsurf` → plots surface over square region [−√5,√5] for both x and y.
- `'r'` → red color surface.
- `'EdgeColor','none'` → hides mesh lines, gives smooth surface.

```
hold on
fsurf(sqrt(5)*cos(theta), sqrt(5)*sin(theta), z, 'y', [0 2*pi, 0  8 ], 'EdgeColor', 'none')
```

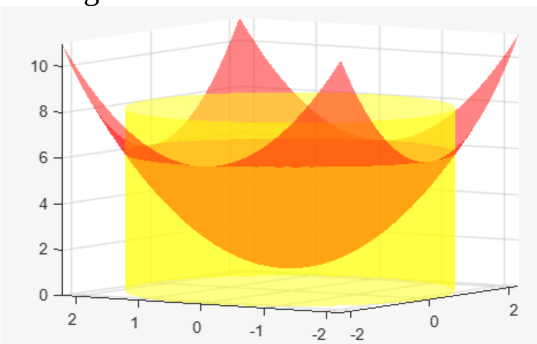
Parametric cylinder equation:

- x=√5cosθ,

- y=√5sinθ,
- 0≤z≤8.
- Plots cylinder wall in **yellow** ('y').

alpha 0.5

Output
In the command window
Sol = -pi*(exp(1) - exp(6) + 5)
The region E is shown below



Example 6

Evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV$**, where E is enclosed by the sphere** $x^2+y^2+z^2=9$ **in the first octant.**

Sol: By Changing Cartesian to Spherical coordinates we can get

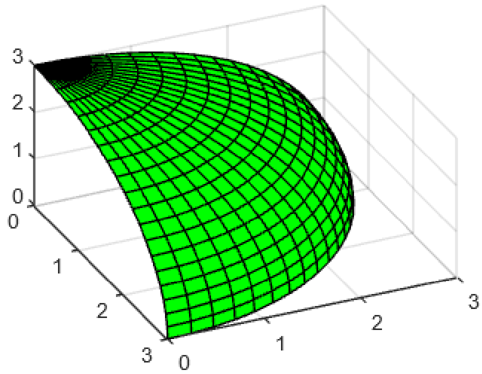
$$\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{3}\rho^2 e^{\rho}\sin(\phi)\,d\rho\,d\phi\,d\theta$$

Matlab code

```
syms r  phi rho theta

Sol=int(int(int((exp(rho))*(rho)^2*sin(phi), rho,0,3),
phi ,0, pi/2),theta,0,pi/2)
rho=3
x = rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta),
z = rho*cos(phi) ;
fsurf(x,y,z, [0 pi/2 0 pi/2], 'g', 'MeshDensity', 20);
```

Output: In the command window
Sol =(pi*(5*exp(3) - 2))/2

In the Figure window



Example 7

Evaluate $\iiint_E z\,dV$**, where E is enclosed by the spheres** $x^2+y^2+z^2=1$ **and** $x^2+y^2+z^2=4$ **in the first octant.**

Sol: By Changing Cartesian to Spherical coordinates we can get

$$\iiint_E z\,dV=\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos(\phi))\,\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

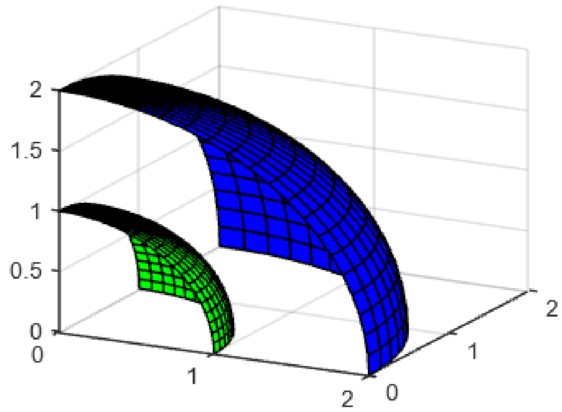
Matlab code

```
clc
clear all
syms r  phi rho theta
Sol=int(int(int((rho*cos(phi))*(rho)^2*sin(phi), rho,1,2), phi ,0, pi/2),theta,0,pi/2)
rho=1;
x = rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z = rho*cos(phi) ;
fsurf(x,y,z, [0 pi/2 0 pi/2], 'g', 'MeshDensity', 20);
hold on
rho=2;
x = rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z = rho*cos(phi) ;
fsurf(x,y,z, [0 pi/2 0 pi/2], 'b', 'MeshDensity', 20);
```

Output: In the command window

Sol = (15*pi)/16

In the figure window



Exercise

1. Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy –plane, and below the cone $z=\sqrt{x^2+y^2}$.
2. Sketch the solid whose volume is given by the integral and evaluate the integral $\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$
3. Evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$ and $z=4$.
4. Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by the paraboloid $y=x^2+z^2$ and $y=4$.