5. EVALUATION OF VOLUME UNDER A SURFACE BY DOUBLE INTEGRAL

Aim: **To evaluate the volume under surface using double integral and to visualize the same using MatLab.**

Statement of the problem: Evaluate and visualize the volume represented by the double integral

$$\int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)} f(x,y)dydx. \tag{1}$$

Above integral represents volume of the region below the surface $\mathrm{z = f(x, y)}$ and above the plane $\mathrm{z} = 0$. This integral can also be setup in the following way (by changing the order of integration of x and y):

$$\int_{y_1}^{y_2}\int_{x_1(y)}^{x_2(y)} f(x,y)dxdy. \tag{2}$$

Solution Approach: We evaluate the double integrals by repeated application of the symbolic toolbox command for integration that, on applying twice, will read as**:**

$$\mathrm{volume} = \mathrm{int}\big(\mathrm{int}\big(\mathrm{f(x,y)}, y, y_1(x), y_2(x)\big), x, x_1, x_2\big).$$

$$\mathrm{volume} = \mathrm{int}\big(\mathrm{int}\big(\mathrm{f(x,y)}, x, x_1(y), x_2(y)\big), y, y_1, y_2\big)$$

Further, to visualize the volume in MatLab we make use of two additional MatLab functions (provided by MathWorks) viz. “viewSolid” and “viewSolidone”. These supporting function files can be downloaded from the link **ftp://10.30.2.53/MATLAB/**.

The first function “viewSolid” is used to visualize the integrals in which the order of integration is as given in (1) and “viewSolidone” is for the integrals of the form (2).

What follows is the syntax for using “viewSolid” and “viewSolidone” commands:

viewSolid(z,0,f(x,y),y,y1(x),y2(x),x,x1,x2)

viewSolidone(z,0,f(x,y),x,x1(y),x2(y),y,y1,y2)

It should be observed that the "viewSolid" command is used when y1 and y2 are functions of x whereas x1 and x2 are constants. The "viewSolidone" command is used in the reverse case. Now we consider few examples for illustration of the approach mentioned above.

Example 1:

Set up a double integral to find the volume of a sphere of unit radius.

Solution:

Let the sphere be $x^2 + y^2 + z^2 = 1$. We know that due to the symmetry the volume of the sphere is 8 times its volume in the first octant. Thus we setup a double integral to find the volume below the surface of the sphere in the first octant only and write the total volume as:

$$V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy\, dx\,.$$

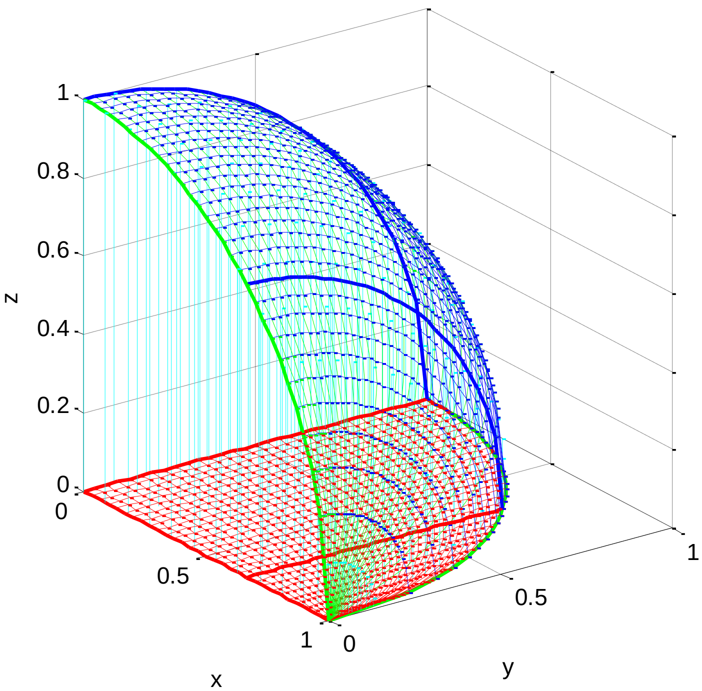
MATLAB Code:

```
clc
clear all
syms x y z
vol=8*int(int(sqrt(1-x^2-y^2),y,0,sqrt(1-x^2)),x,0,1)
double(vol)
viewSolid(z,0+0*x*y,sqrt(1-x^2-y^2),y,0+0*x,sqrt(1-x^2),x,0,1);
axis equal; grid on;
viewSolid(z,-sqrt(1-x^2-y^2),sqrt(1-x^2-y^2),y, -sqrt(1-x^2),sqrt(1-x^2),x,-1,1);
```

```
clc
clear all
syms x y z
vol=8*int(int(sqrt(1-x^2-y^2),y,0,sqrt(1-x^2)),x,0,1)
figure(1)
viewSolid(z,0+0*x*y,sqrt(1-x^2-y^2),y,0+0*x,sqrt(1-x^2),x,0,1); axis equal; grid
on;
figure(2)
viewSolid(z,-sqrt(1-x^2-y^2),sqrt(1-x^2-y^2),y, -sqrt(1-x^2),sqrt(1-x^2),x,-1,1);
```

Output:
vol =

(4*pi)/3



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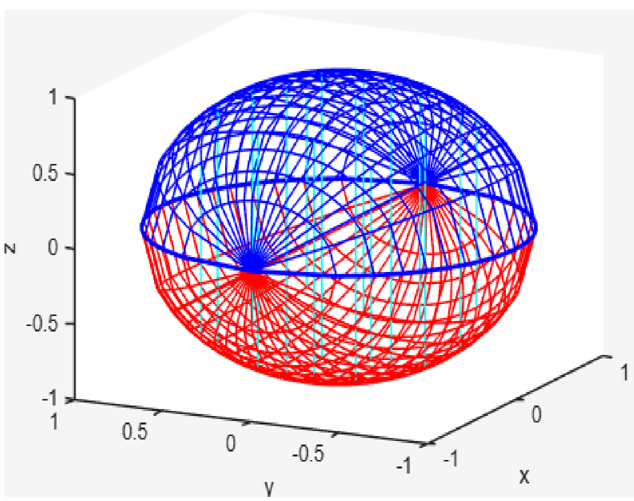


Example 2:

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the lines $y = 2x$ and the parabola $y = x^2$

Solution:

The double integral for this problem can be setup as:

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

```
clc
clear all
syms x y z
vol = int(int(x^2+y^2, x,y/2,sqrt(y)), y, 0, 4)

viewSolidone(z,0+0*x*y,x^2+y^2,x,y/2,sqrt(y),y,0,4);

grid on;
```

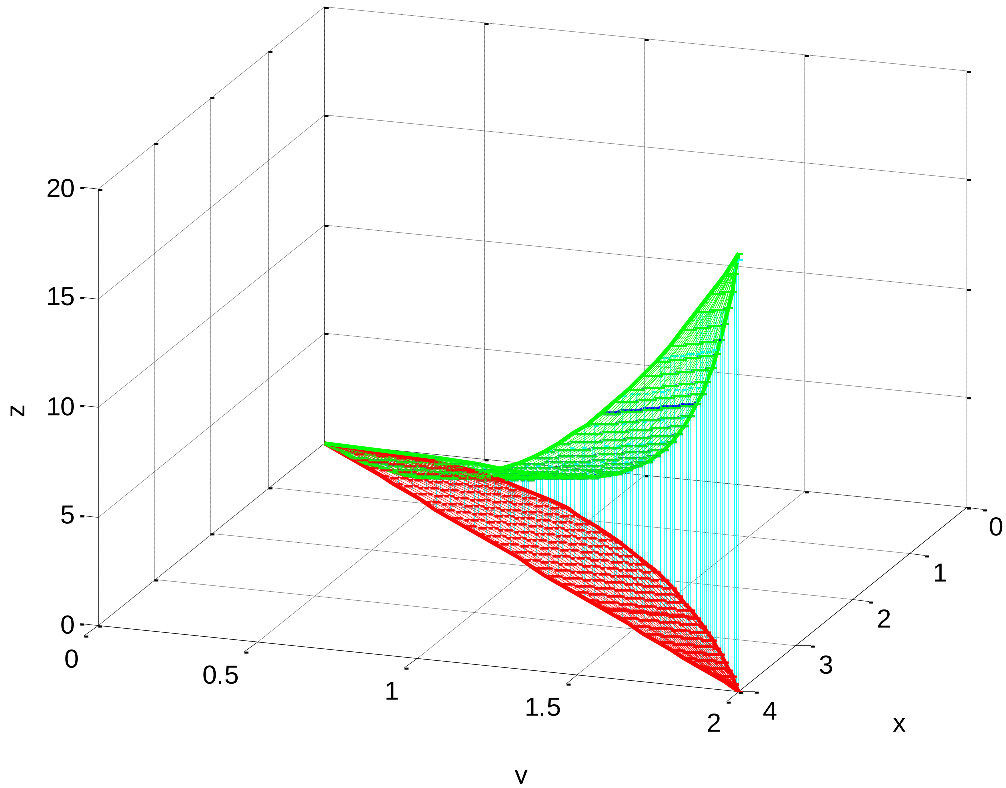
MATLAB Code:

```
Clc
clear all
syms x y z
vol = int(int(x^2+y^2, x,y/2,sqrt(y)), y, 0, 4)
viewSolidone(z,0+0*x*y,x^2+y^2,x,y/2,sqrt(y),y,0,4); grid on
```

Output:

vol =

216/35



Example 3:

Consider the following mathematical problem

$$\text{Evaluate } \iint_R (x-3y^2)\,dA \text{ where } R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$$

Given below is the Matlab code for the above problem.

MATLAB Code:

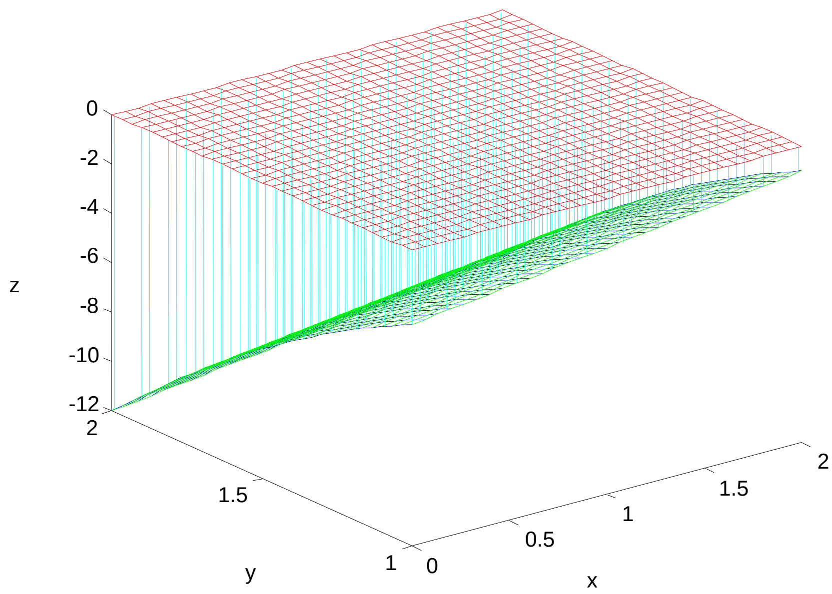
```
clc
clear all
syms x y z
viewSolid(z,0+0*x+0*y,x-3*y^2+0*y,y,1+0*x,2+0*y,x,0,2)
int(int(x-3*y^2+0*y,y,1,2),x,0,2)
```

Output:

In the Command window:

```
>> ans
       -12
```

In the Figure window:



Inference:

In this figure the required volume is below the plane z=o (shown in red) and above the surface z=(x-3y^2)(shown in green) . The reason why the answer is negative is that the surface z=(x-3y^2) is below z=0 for the given domain of integration.

Example 4:

Evaluate $\iint_R y\sin(xy)\, dA$ where $R=[1,2]\times[0,\pi]$

MATLAB Code:

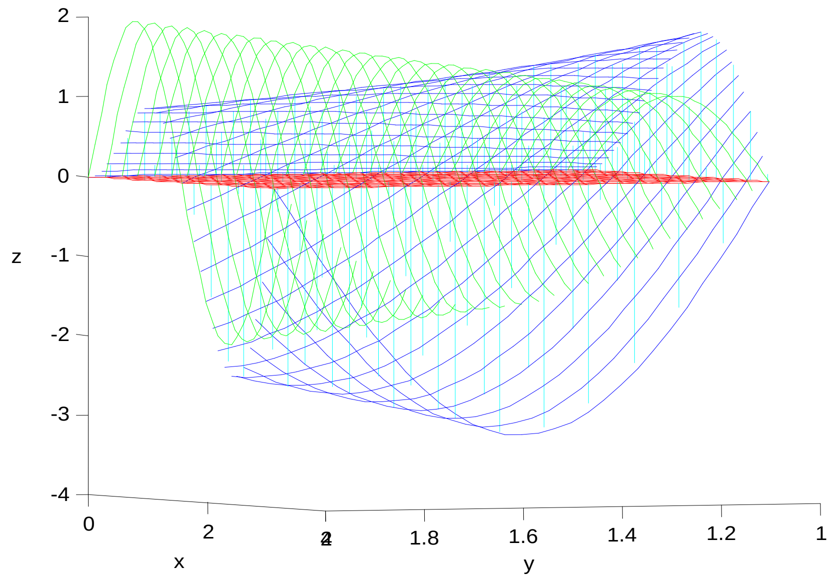
```
clc
clear all
syms x y z
viewSolidone(z,0+0*x+0*y,y*sin(x*y),x,1+0*y,2+0*y,y,0,pi)
int(int(y*sin(x*y),x,1,2),y,0,pi)
```

Output:

In the Command window:

```
>>  ans
      0
```

In the Figure window:



Inference:

For a function f(x,y) that takes on both positive and negative values $\iint_R f(x,y)dA$ is a difference of volumes V_1-V_2, V_1 is the volume above R and below the graph of f and V_2 is the volume below R and above the graph. The integral in this example is 0 means $V_1=V_2$

Converting Cartesian to polar coordinates

Example 5:

Find the volume of the solid bounded by the plane z=0 and the paraboloid
$z = 1 - x^2 - y^2$

Sol:

By changing the coordinates from Cartesian to Polar we get

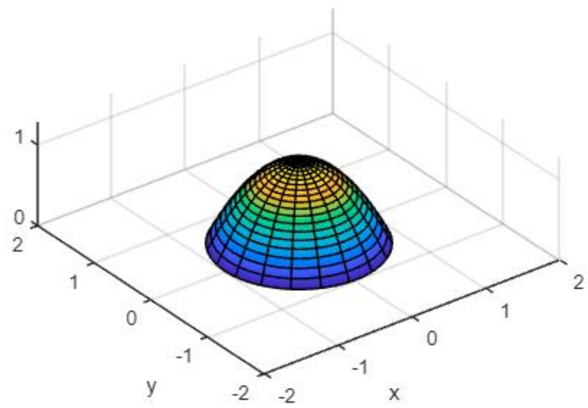
$$V = \iint_D (1 - x^2 - y^2)dA = \int_0^{2\pi} \int_0^1 (1 - r^2)r dr d\theta$$

```
clc
clear all
syms r theta
V = int(int((1-r^2)*r, r, 0, 1), theta, 0, 2*pi)
fsurf(r*cos(theta),r*sin(theta), 1-r^2, [0 1 0 2*pi], 'MeshDensity', 20)
axis equal; axis([-2 2 -2 2 0 1.3])
xticks(-2:2); yticks(-2:2); zticks(0:1.3)
xlabel('x'); ylabel('y')
```

Output

V = pi/2

Figure window:



Example 6

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the xy- plane, and inside the cylinder $x^2 + y^2 = 2x$

Sol:

By changing the coordinates from Cartesian to Polar we get

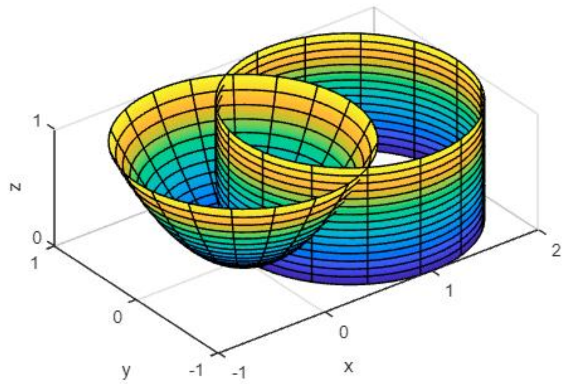
$$V = \iint_D (x^2 + y^2) dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (r^2) r dr\theta$$

Matlab code

```
clc
clear all
syms r theta z r1
V = int(int((r^2)*r, r, 0,2*cos(theta) ), theta, -pi/2, pi/2)
r = 2*cos(theta), x = r*cos(theta), y = r*sin(theta)
fsurf(x,y,z, [0 2*pi 0 1], 'MeshDensity', 16)
axis equal; xlabel('x'); ylabel('y'); zlabel('z')
zticks(0:1.5)
hold on
fsurf(r1*cos(theta),r1*sin(theta),r1^2, [0 1 0 2*pi], 'MeshDensity', 20)
```

Output:

V = (3*pi)/2



Exercise Problems:

1. Set up a double integral to find the volume of the *hoof of Archimedes*, which is the solid region bounded by the planes $z = y$, $z = 0$, and the cylinder $x^2 + y^2 = 1$.
2. Write an iterated integral to view the volume enclosed by the cone $\mathrm{z}^2 = \mathrm{x}^2 + \mathrm{y}^2$ and the plane $\mathrm{z} = 0$. Hence find the volume.
3. Find the volume of the solid bounded by the elliptic paraboloid $\mathrm{x}^2 + 2\mathrm{y}^2 + \mathrm{z} = 16$, the planes $\mathrm{x} = 2$ and $\mathrm{y} = 2$ and the three coordinate planes.
4. Use polar coordinates to find the volume of the solid that lies under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leq 4$