Eigenvalues and Eigenvectors

Definitions Let $\underset{\sim}{A} = [a_{ij}] \in \mathbb{C}^{n \times n}$ be an $n \times n$ matrix with entries $a_{ij} \in \mathbb{C}$. Consider the vector equation

$$\underset{\sim}{A}\,\underset{\sim}{x} = \lambda\,\underset{\sim}{x},$$

which is equivalent to the equation

$$(\underset{\sim}{A} - \lambda \underset{\sim}{I})\,\underset{\sim}{x} = \underset{\sim}{0}. \qquad (1)$$

This equation always admits the trivial solution $\underset{\sim}{x} = \underset{\sim}{0}$. Any $\lambda \in \mathbb{C}$ for which this equation has a solution $\underset{\sim}{x} \neq \underset{\sim}{0}$ is called an <u>eigenvalue</u> or a <u>characteristic value</u> of $\underset{\sim}{A}$; the corresponding solution $\underset{\sim}{x} \neq \underset{\sim}{0}$ of (1) is called an <u>eigenvector</u> or a <u>characteristic vector</u> of $\underset{\sim}{A}$.

Let us note that $\lambda \in \mathbb{C}$ is an eigenvalue of $\underset{\sim}{A}$ if and only if $\underset{\sim}{A} - \lambda \underset{\sim}{I}$ is not invertible.

The set $\{ \lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } \underset{\sim}{A} \}$, of all eigenvalues of $\underset{\sim}{A}$ is called the spectrum of $\underset{\sim}{A}$, denoted by $\text{spectrum}(\underset{\sim}{A})$, or sometimes simply by $\sigma(\underset{\sim}{A})$. The number

$$\rho(\underset{\sim}{A}) := \max \{ |\lambda| : \lambda \in \text{spectrum}(\underset{\sim}{A}) \}$$

is called the spectral radius of $\underset{\sim}{A}$.

Theorem 1 Let $\underset{\sim}{A} = [a_{ij}]$ be an $n \times n$ matrix with entries in $\mathbb{C}$. Then $\underset{\sim}{A}$ has at least one and at most n distinct eigenvalues.

Proof. Let us note that

$\lambda \in \mathbb{C}$ is an eigenvalue of $\underset{\sim}{A}$

$\Longleftrightarrow$ The matrix $\underset{\sim}{A} - \lambda \underset{\sim}{I}$ is not invertible.

$\Longleftrightarrow$ The homogeneous system of linear equations $(\underset{\sim}{A} - \lambda \underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ has a non-trivial solution $\underset{\sim}{x}$.

$\Longleftrightarrow$ $\operatorname{rank}(\underset{\sim}{A} - \lambda \underset{\sim}{I}) < n$.

$\Longleftrightarrow$ $\boxed{\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0}$ (2)

Since

$$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \\ \vdots & \vdots \quad \ddots & \\ a_{n1} & a_{n2} & a_{nn} - \lambda \end{vmatrix}$$

is a polynomial in λ of degree $\leq n$, Equation (2) has at least one and at most n distinct solutions by The Fundamental Theorem of Algebra. ∎

Definition. The determinant $D(\lambda) := \det(\underset{\sim}{A} - \lambda \underset{\sim}{I})$ is called the characteristic determinant (or characteristic polynomial) and the equation (2) is called the characteristic equation corresponding to the matrix $\underset{\sim}{A}$.

Remark. $\text{Spectrum}(\underset{\sim}{A}) = \{ \lambda \in \mathbb{C} : \lambda \text{ is a sol}^n \text{ of the characteristic eq}^n (2) \}$.

Definition Associated with each eigenvalue λ of the matrix $\underset{\sim}{A}$ (i.e., for each $\lambda \in \text{spectrum}(\underset{\sim}{A})$), the set

$$E_\lambda := \{ \underset{\sim}{x} \in \mathbb{C}^n : \underset{\sim}{A}\underset{\sim}{x} = \lambda \underset{\sim}{x} \} = \mathcal{N}(\underset{\sim}{A} - \lambda \underset{\sim}{I})$$

which is the set of all eigenvectors of $\underset{\sim}{A}$ corresponding to the eigenvalue λ, together with $\underset{\sim}{0}$, is a subspace of $\mathbb{C}^n$ called the eigenspace of $\underset{\sim}{A}$ corresponding to the eigenvalue λ.

Definition $\dim E_\lambda$ is defined as the $\boxed{\text{geometric multiplicity}}$ of the eigenvalue λ of $\underset{\sim}{A}$. An eigenvalue λ of $\underset{\sim}{A}$ is said to be of $\boxed{\text{algebraic multiplicity } m}$ if λ is a root of multiplicity m of the characteristic eq^n $D(\lambda) = 0$.

Definition. The polynomial $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I})$ for a square matrix $\underset{\sim}{A}$ is called the characteristic polynomial of $\underset{\sim}{A}$.

Example. (1) $\underset{\sim}{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. To find the eigenvalues of $\underset{\sim}{A}$ we solve the equation

$$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = \det \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) = 0.$$

Hence the eigenvalues of A are 1 and 3.

Let us calculate the eigenspaces $\overset{E_1}{E(1)}$ and $\overset{E_3}{E(3)}$.

$\overset{E_1}{E(1)} = \{v \mid (A - I)v = 0\}$ and $\overset{E_3}{E(3)} = \{v \mid (A - 3I)v = 0\}$

$A - I = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$. Hence $\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Hence $\underset{E_1}{E(1)} = \underset{\text{span}}{L}\{(1, 0)\}$.

$$\underset{\sim}{A} - 3\underset{\sim}{I} = \begin{bmatrix} 1-3 & 2 \\ 0 & 3-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}.$$

Suppose $\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Then $\begin{bmatrix} -2x+2y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence $x = y$.

Thus $E(3) = L(\{(1,1)\})$. $\quad E_3 = \text{span}\{(1,1)\}$

(2) Let $\underset{\sim}{A} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix}$. Then $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = (3-\lambda)^2(6-\lambda)$.

Hence eigenvalues of $\underset{\sim}{A}$ are 3 and 6. The eigenvalue $\lambda = 3$ is a double root of the characteristic polynomial of $\underset{\sim}{A}$.

We say that $\lambda = 3$ has **algebraic multiplicity** 2.

$$\lambda = 3 : \underset{\sim}{A} - 3\underset{\sim}{I} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 1 & 2 \end{bmatrix}. \quad \text{Hence} \quad \text{rank}(\underset{\sim}{A} - 3\underset{\sim}{I}) = 1.$$

Thus nullity $(A - 3I) = 2$. By solving the system $(\underset{\sim}{A} - 3\underset{\sim}{I})v = 0$ we find that

$$\mathcal{N}(\underset{\sim}{A} - 3\underset{\sim}{I}) = E_3 = \text{span}\{(1,0,1), (1,2,0)\}$$

$$N(A - 3I) = E(3) = L(\{(1,0,1), (1,2,0)\}).$$

The dimension of $\overset{E_\lambda}{E(\lambda)}$ is called the <u>geometric multiplicity</u> of λ.

Hence geometric multiplicity of $\lambda = 3$ is 2.

$$\lambda = 6 : A - 6I = \begin{bmatrix} -3 & 0 & 0 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}. \quad \text{Hence rank}(A - 6I) = 2.$$

Thus $\dim E(6) = 1$.

It can be shown that $\{(0, 1, 1)\}$ is a basis of $\overset{E_6}{E(6)}$.

Thus the algebraic and geometric multiplicity of $\lambda = 6$ are one.

Example 3

[redacted]3 Let $\underset{\sim}{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. we have

$$D(\lambda) = \det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}$$

$$= (-2-\lambda)(-\lambda+\lambda^2-12) - 2(-2\lambda-6) - 3(-4+1-\lambda)$$

$$= -(\lambda^3 + \lambda^2 - 21\lambda - 45) = -(\lambda+3)^2(\lambda-5).$$

Thus the eigenvalues of $\underset{\sim}{A}$ are $5, -3, -3$. Let us find the eigenvectors for $\lambda = 5$. These are sol$^{\underset{..}{ns}}$ of the homogeneous system $(\underset{\sim}{A} - 5\underset{\sim}{I})\underset{\sim}{x} = 0$, i.e $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

We have rank $(\underset{\sim}{A} - 5\underset{\sim}{I}) = 2$. Hence nullity $(\underset{\sim}{A} - 5\underset{\sim}{I}) = 1$

Solving the system, we get $\underset{\sim}{x}_1 = (1, 2, -1)^t$ as an eigenvector.

[redacted] consider the eigenvalue -3. To find the eigenvectors [redacted]rresponding to this eigenvalue, we must solve

$(\underset{\sim}{A} + 3\underset{\sim}{I})\underset{\sim}{x} = 0$, i.e., $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 + 2x_3 = 0$, $x_1 + 2x_2 + 5x_3 = 0 \Rightarrow (x_1, x_2, x_3)^t = -(1, 2, -1)^t x_3$

$x_1 + 2x_2 - 3x_3 = 0 \Rightarrow (x_1, x_2, x_3)^t = (-2x_2 + 3x_3, x_2, x_3)^t$
$= (-2, 1, 0)^t x_2 + (3, 0, 1)^t x_3$

We have : rank $(\underset{\sim}{A} + 3\underset{\sim}{I}) = 1$, nullity $(\underset{\sim}{A} + 3\underset{\sim}{I}) = \underline{2}$

Thus the solutions of $(\underset{\sim}{A} + 3\underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ is a vector space of dimension 2. A basis for $\mathcal{N}(\underset{\sim}{A} + 3\underset{\sim}{I})$ consists of the eigenvectors

$$\underset{\sim}{x}_2 = (-2, 1, 0)^t, \quad \underset{\sim}{x}_3 = (3, 0, 1)^t.$$

corresp. to the eigenvalue $\lambda = -3$.

Definitions For an eigenvalue λ_0 of the matrix $\underset{\sim}{A}$, the set

$$E_{\lambda_0} := \{ \underset{\sim}{x} \in \mathbb{C}^n : \underset{\sim}{A}\underset{\sim}{x} = \lambda_0 \underset{\sim}{x} \} = \mathcal{N}(\underset{\sim}{A} - \lambda_0 \underset{\sim}{I})$$

which is the set of all eigenvectors of $\underset{\sim}{A}$ corresponding to the eigenvalue λ_0, together with $\underset{\sim}{0}$, is a subspace of $\mathbb{C}^n$ called the eigenspace corresponding to the eigenvalue λ_0.

$\dim E_{\lambda_0} =:$ the geometric multiplicity of λ_0.

An eigenvalue λ_0 of $\underset{\sim}{A}$ is said to have algebraic multiplicity m, if λ_0 is a root of multiplicity m of the char. eq.^{n} $D(\lambda) = 0$

Example In the previous example, the eigenvalue $\lambda = 5$ has both the algebraic multiplicity and the geometric multiplicity $= \underline{1}$

The eigenvalue $\lambda = -3$ has algebraic multiplicity 2.
Its geometric multiplicity is also $= 2$.

Example 4

(3) $\underset{\sim}{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = (1-\lambda)^2$.

Thus $\lambda = 1$ has algebraic multiplicity 2. $A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Hence

$E_1 = L\{e_1\} = \text{span}\{e_1\}$

nullity $(A - I) = 1$ and $E(1) = \{e_1\}$. In this case the geometric multiplicity of $\lambda = 1 \boxed{<}$ algebraic multiplicity of $\lambda = 2$.

Basic properties of eigenvalues and eigenvectors

Proposition. Let $\underset{\sim}{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

(i) $\text{tr}(\underset{\sim}{A}) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

(ii) $\det \underset{\sim}{A} = \lambda_1 \lambda_2 \ldots \lambda_n$.

Proof. The characteristic polynomial of $\underset{\sim}{A}$ is

$$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

$$= (-1)^n \lambda^n + (-1)^{n-1} \lambda^{n-1} (a_{11} + \ldots + a_{nn}) + \ldots$$

Put $\lambda = 0$ to get $\det \underset{\sim}{A}$ = constant term of $\det(A - \lambda I)$.

Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are roots of $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$ we have

$$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n).$$

$$= (-1)^n [\lambda^n - (\lambda_1 + \lambda_2 + \ldots + \lambda_n) \lambda^{n-1} + \ldots + (-1)^n \lambda_1 \ldots \lambda_n].$$

Hence constant term of $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = \lambda_1 \lambda_2 \ldots \lambda_n = \det \underset{\sim}{A}$ and $tr(\underset{\sim}{A}) = a_{11} + a_{22} + \ldots + a_{nn} = \lambda_1 + \lambda_2 + \ldots + \lambda_n$. ∎

Proposition : Let $\underset{\sim}{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then eigenvalues of $\underset{\sim}{A}^k$ for $k \in \mathbb{N}$ are precisely $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$. Every eigenvector of $\underset{\sim}{A}$ is an eigenvector of $\underset{\sim}{A}^k$.

Proof : Suppose $\underset{\sim}{A}\underset{\sim}{v} = \lambda \underset{\sim}{v}$ for a non-zero vector $\underset{\sim}{v}$ and λ a scalar.

Then $\underset{\sim}{A}^2\underset{\sim}{v} = \underset{\sim}{A}(\lambda \underset{\sim}{v}) = \lambda(\underset{\sim}{A}\underset{\sim}{v}) = \lambda^2 \underset{\sim}{v}$.

Thus λ^2 is an eigenvalue of $\underset{\sim}{A}^2$ with eigenvector $\underset{\sim}{v}$.

Apply induction to show that $\underset{\sim}{A}^k\underset{\sim}{v} = \lambda^k \underset{\sim}{v} \; \forall \, k \in \mathbb{N}$.

For the converse, let μ be an eigenvalue of $\underset{\sim}{A}^k$.

By the Fundamental Theorem of Algebra the polynomial

$x^k - \mu = (x - z_1)(x - z_2) \ldots (x - z_k)$ for some complex numbers

$z_1, z_2, \ldots, z_k$. This gives a factorization

$$\underset{\sim}{A}^k - \mu \underset{\sim}{I} = (\underset{\sim}{A} - z_1 \underset{\sim}{I})(\underset{\sim}{A} - z_2 \underset{\sim}{I}) \ldots (\underset{\sim}{A} - z_k \underset{\sim}{I}).$$

$$\text{Hence,} \qquad \det(\underset{\sim}{A}^k - \mu \underset{\sim}{I}) = 0 = \prod_{i=1}^{i=k} \det(\underset{\sim}{A} - z_i \underset{\sim}{I}).$$

Hence there is an i so that $\det(\underset{\sim}{A} - z_i \underset{\sim}{I}) = 0$. Hence z_i is an eigenvalue of A. But $\mu = z_i^k$. ∎

Corollary
If $\lambda \in \sigma(\underset{\sim}{A})$, then $g(\lambda) = a_0 + a_1\lambda + \cdots a_m \lambda^m \in \sigma(g(\underset{\sim}{A}))$

where $g(\underset{\sim}{A}) = a_0 + a_1 \underset{\sim}{A} + \ldots + a_m \underset{\sim}{A}^m$

$\boxed{\text{Recall: } \sigma(\underset{\sim}{A}) = \text{spectrum}(\underset{\sim}{A})}$

Example : Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; and \; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then 0 is the only eigenvalue of A and B . The product of A and B is

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence eigenvalues of AB are 1 and 0. Hence eigenvalues of AB are not products of eigenvalues of A and B.

Proposition : Let $v_1, v_2, \ldots, v_k$ be eigenvectors of a matrix A associated to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then $v_1, v_2, \ldots, v_k$ are linearly independent.

Proof : Apply induction on k. It is clear for $k = 1$.

Suppose $k \geq 2$ and

$c_1 v_1 + \ldots + c_k v_k = 0$ for some scalars $c_1, c_2, \ldots, c_k$.

Hence $c_1 A v_1 + c_2 A v_2 + \ldots + c_k A v_k = 0$

$$\Longrightarrow c_1\lambda_1 v_1 + c_2\lambda_2 v_2 + \ldots + c_k\lambda_k v_k = 0$$

$$\lambda_1(c_1v_1 + c_2v_2 + \ldots + c_kv_k) - (\lambda_1c_1v_1 + \lambda_2c_2v_2 + \ldots + \lambda_kc_kv_k)$$

$$= (\lambda_1 - \lambda_2)c_2v_2 + (\lambda_1 - \lambda_3)c_3v_3 + \ldots + (\lambda_1 - \lambda_k)c_kv_k = 0$$

By induction, $v_2, v_3, \ldots, v_k$ are linearly independent.

Hence $(\lambda_1 - \lambda_j)c_j = 0$ for $j = 2, 3, \ldots, k$.

Since $\lambda_1 \neq \lambda_j$ for $j = 2, 3, \ldots, k$, $\;c_j = 0$ for $j = 2, 3, \ldots, k$.

Hence c_1 is also zero.

Thus $v_1, v_2, \ldots, v_k$ are linearly independent. ∎

Proposition : Suppose A is an $n \times n$ matrix.

Let A have n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Let E be the matrix whose column vectors are $v_1, v_2, \ldots, v_n$ where v_i is an eigenvector for λ_i for $i = 1, 2, \ldots, n$. Then

$$E^{-1}AE = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

Proof : It is enough to prove $AE = ED$.

Write $E = [v_1\ v_2\ \ldots\ v_n]$. Then for $i = 1, 2, \ldots, n$:

i^{th} column of $AE = A[i^{th}$ column of $E] = Av_i = \lambda_i v_i$.

Similarly i^{th} a column of $ED = E[i^{th}$ column of $D] = \lambda_i v_i$.

Hence $E^{-1}AE = D$. ∎

→ Go to 12(a)

Relation between algebraic and geometric multiplicities

Proposition : Let A be an $n \times n$ matrix. Then the geometric multiplicity of an eigenvalue μ of A is less than or equal to the algebraic multiplicity of μ.

Proof : If μ is real then $(A - \mu I)v = 0$ has real solutions.

But in general $\mu \in \mathbb{C}$.

Hence we work in $\mathbb{C}^n$. Suppose that the algebraic multiplicity of μ is k.

Hence $(\lambda - \mu)^k$ divides $\det(A - \lambda I)$ but $(\lambda - \mu)^{k+1}$ does not.

Let g = geometric multiplicity of μ.

Hence $E(\mu)$ has a basis of g eigenvectors $v_1, v_2, \ldots, v_g$.

Defintion. An $n \times n$ matrix A is called **diagonalizable** if there is an invertible $n \times n$ matrix E such that

$$E^{-1}AE = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

Proposition.

1. $E^{-1}AE = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ iff the column vectors of E are eigenvectors of A

2. The set of eigenvalues of $A = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

Proof. Let $T : C^n \longrightarrow C^n$ be the linear transformation associated to the matrix A. Let $E = [v_1\ v_2\ \ldots\ v_n]$
Thus $E^{-1}AE$ = matrix of T with respect to the basis $\{v_1, v_2, \ldots, v_n\}$.
Thus $Tv_i = \lambda v_i \quad$ for $i = 1, 2, \ldots, n$.
Hence $v_1, v_2, \ldots, v_n$ are eigenvalues of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
(2) Since eigenvalues of $E^{-1}AE$ are same as those of A, and eigenvalues of $E^{-1}AE$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$. ∎

Proposition 2 Let $\underset{\sim}{A} \in \mathbb{K}^{n\times n}$, then for each $\lambda \in \sigma(\underset{\sim}{A})$
geometric multiplicity of $\lambda \leq$ algebraic multiplicity of λ.

 Proof. This is left to the reader.

Theorem 2: An $n \times n$ matrix A is diagonalizable if and only if the algebraic and geometric multiplicities of each eigenvalue of A are equal.

Proof. Suppose A is diagonalizable.

Thus there exists an invertible matrix E

whose column vectors are eigenvectors and

$E^{-1}AE = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Hence the sum of geometric multiplicities is n = sum of algebraic multiplicities;

it follows that the geometric and algebraic multiplicities coincide for each eigenvalue.

Conversely suppose that the geometric and algebraic multiplicities coincide for each eigenvalue.

Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all the eigenvalues of A

with algebraic multiplicities $n_1, n_2, \ldots, n_k$.

Proof of Proposition 2. Suppose μ is an eigenvalue of $\underset{\sim}{A}$ and let its algebraic multiplicity $a(\mu) = k$. Then $(\lambda - \mu)^k$ divides $D(\lambda) = \det(\underset{\sim}{A} - \lambda \underset{\sim}{I})$ but $(\lambda - \mu)^{k+1}$ does not. Let the geometric multiplicity $g(\mu)$ of μ be equal to r. Then E_μ has a basis of r eigenvectors $\underset{\sim}{x}_1, \dots, \underset{\sim}{x}_r$. Extend this basis of E_μ to a basis of $\mathbb{C}^n$, say, $B = \{\underset{\sim}{x}_1, \dots, \underset{\sim}{x}_r, \dots\dots, \underset{\sim}{x}_n\}$. Put $\underset{\sim}{S} = [\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n]$. Then $\underset{\sim}{S}$ is an invertible matrix. Letting $T_{\underset{\sim}{A}}$ denote the linear operator on $\mathbb{C}^n$ induced by $\underset{\sim}{A}$. We know that

$$[T_{\underset{\sim}{A}}]_B = \underset{\sim}{S}^{-1} \underset{\sim}{A} \underset{\sim}{S}$$

But $T_{\underset{\sim}{A}} x_i = \mu x_i = \underset{\sim}{A} x_i$ for $i = 1, 2, \dots r$. Hence

$$[T_{\underset{\sim}{A}}]_B = \left[\begin{array}{c|c} \mu \underset{\sim}{I}_r & \underset{\sim}{D} \\ \hline 0 & \underset{\sim}{C} \end{array}\right]$$

where $\underset{\sim}{D}$ is a $r \times (n-r)$ matrix and $\underset{\sim}{C}$ is a $(n-r) \times (n-r)$ matrix. Since

$$\det(\underset{\sim}{S}^{-1}\underset{\sim}{A}\underset{\sim}{S} - \lambda \underset{\sim}{I}) = \det(\underset{\sim}{S}^{-1}(\underset{\sim}{A}-\lambda\underset{\sim}{I})\underset{\sim}{S}) = \det(\underset{\sim}{A}-\lambda\underset{\sim}{I})$$

and from the form of $[T_{\underset{\sim}{A}}]_{\mathbb{B}}$, we see that $(\lambda-\mu)^r$ devides $\det(\underset{\sim}{A}-\lambda\underset{\sim}{I})$, we conclude that $r \leq k$. ∎

We now return to Theorem 2 (on p. 13) and examples following this theorem which illustrate easily verifiable conditions for diagonalizability of a matrix $\underset{\sim}{A}$ in $\mathbb{K}^{n \times n}$.

Example 1 $\underset{\sim}{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. We have verified that $D(\lambda) = \det(\underset{\sim}{A} - \lambda\underset{\sim}{I}) = (1-\lambda)(3-\lambda)$. Hence the eigenvalues of $\underset{\sim}{A}$ are 1 and 3. We have already seen that $E_1 = \text{span}\{(1,0)^t\}$ and $E_3 = \text{span}\{(1,1)^t\}$. This is the case of distinct eigenvalues. Here $\underset{\sim}{E} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

and $\underset{\sim}{E}^{-1}\underset{\sim}{A}\underset{\sim}{E} = \underset{\sim}{D} = \text{diag}(1,3) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

Example 2 Let $\underset{\sim}{A} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix}$. Here $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = (3-\lambda)^2(6-\lambda)$.

Hence $\lambda = 3$ is an eigenvalue of algebraic multiplicity 2, and $\lambda = 6$ is an eigenvalue of algebraic multiplicity 1. We have already computed the eigenspaces E_3 and E_6 and seen that $E_3 = \text{span}\left\{(1,0,1)^t, (1,2,0)^t\right\}$, $E_6 = \text{span}\left\{(0,1,1)^t\right\}$. Hence in this example

$$a(3) = g(3), \quad a(6) = g(6),$$

the matrix $\underset{\sim}{E}$ consisting of the eigenvectors of $\underset{\sim}{A}$ is

$$\underset{\sim}{E} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and $\underset{\sim}{E}$ diagonalizes $\underset{\sim}{A}$: $\underset{\sim}{E}^{-1} \underset{\sim}{A} \underset{\sim}{E} = \underset{\sim}{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

Example 3 Let $\underset{\sim}{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Here $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = (1-\lambda)^2$.

Thus $\lambda = 1$ has algebraic multiplicity 2. We have already computed the eigenspace E_1 and seen that $E_1 = \text{span}\left\{(1,0)^t\right\}$. Hence geometric multiplicity $g(1) <$ algebraic multiplicity $a(1)$. This shows that $\underset{\sim}{A}$ is <u>not diagonalizable</u>.