More on Linear Transformations

One fundamental aspect of a linear transformation is that it is completely specified by its action on a basis. This is revealed by the next proposition and its corollary.

Proposition Let X, Y be vector spaces over $\mathbb{K}$. Suppose X is finite dimensional with a basis $\{u_1, u_2, \dots, u_n\}$. For every prescription of elements $v_1, \dots, v_n$ in Y, there is one and only one linear transformation $T : X \longrightarrow Y$ such that

$$T(u_i) = v_i \quad \text{for } i = 1, 2, \dots, n.$$

<u>Proof</u> Let $x \in X$. Then

$$x = \sum_{i=1}^{n} \alpha_i u_i = \alpha_1 u_1 + \dots + \alpha_n u_n,$$

for unique scalars $\alpha_1, \dots, \alpha_n$.

Define $T : X \longrightarrow Y$ by setting

$$T(x) = \sum_{i=1}^{n} \alpha_i v_i \quad , \quad x = \sum_{i=1}^{n} \alpha_i u_i \quad , x \in X$$

It is easily checked that T is linear. In fact, if $x = \sum_{i=1}^{n} \alpha_i u_i$ and $y = \sum_{i=1}^{n} \beta_i u_i$, then for $\alpha, \beta \in \mathbb{k}$, $\alpha x + \beta y = \sum_{i=1}^{n} (\alpha \alpha_i + \beta \beta_i) u_i$. Hence

$$\begin{aligned} T(\alpha x + \beta y) &= \sum_{i=1}^{n} (\alpha \alpha_i + \beta \beta_i) v_i \\ &= \alpha \sum_{i=1}^{n} \alpha_i v_i + \beta \sum_{i=1}^{n} \beta_i v_i \\ &= \alpha T(x) + \beta T(y). \end{aligned}$$

Hence, T is linear and clearly by definition

$$T(u_i) = v_i \quad , \quad i = 1, 2, \cdots, n$$

<u>Uniqueness</u>: Suppose $S : X \longrightarrow Y$ is linear and satisfies $S(u_i) = v_i$, $i = 1, \cdots, n$.

Then for $x \in X$ with $x = \sum_{i=1}^{n} \alpha_i u_i$, we have

$$S(x) = \sum_{i=1}^{n} \alpha_i S(u_i) = \sum_{i=1}^{n} \alpha_i v_i = T(x).$$

Thus $T = S$. ∎

Corollary Let X, Y be vector spaces over $\mathbb{K}$, and suppose X is finite dimensional with a basis $\{u_1, u_2, \cdots, u_n\}$. If $T: X \to Y$, $S: X \to Y$ are linear and

$$T(u_i) = S(u_i), \quad i = 1, 2, \cdots, n,$$

then $T = S$.

Definition Let X be a finite dimensional vector space. By an $\quad B$ of X, we mean a basis of X with a specific order, i.e., it is a finite sequence of L.I. elements of X which spans X.

<u>Examples</u> For $X = \mathbb{R}^3$, $B = \{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ is an ordered basis, $B_1 = \{\underset{\sim}{e}_3, \underset{\sim}{e}_2, \underset{\sim}{e}_1\}$ is also an ordered basis; but $B \neq B_1$. Similarly, for $X = \mathcal{P}_n(\mathbb{R})$, $B = \{1, t, \cdots, t^n\}$ is a standard ordered basis.

Definition (coordinatization). Let X be a finite dimensional vector space, and $B = \{x_1, \cdots, x_n\}$ be an ordered basis X. For $x \in X$, let $\alpha_1, \cdots, \alpha_n$ be the unique scalars such that

$$x = \sum_{i=1}^{n} \alpha_i x_i .$$

The coordinate vector of x relative to B, denoted by $[x]_B$, is defined by

$$[x]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Clearly, $[x]_B$ is an element of $\mathbb{K}^n$ and it is easily seen that $[x_i]_B = \underset{\sim}{e}_i$, $i = 1, \cdots, n$.

It is an easy exercise to show that

$$\begin{aligned} X &\longrightarrow \mathbb{K}^n \\ x &\longrightarrow [x]_B, \quad x \in X \end{aligned}$$

gives a linear transformation from X to $\mathbb{K}^n$.

The matrix representation of a linear transformation

Let X, Y be finite dimensional vector spaces over $\mathbb{K}$ with ordered bases $B = \{x_1, x_2, \cdots, x_n\}$ and $C = \{y_1, y_2, \cdots, y_m\}$, respectively. Let $T : X \longrightarrow Y$ be a linear transformation. Then for each j, $1 \leq j \leq n$, there are unique scalars $a_{ij} \in \mathbb{K}$, $1 \leq i \leq m$, such that

$$T(x_j) = \sum_{i=1}^{m} a_{ij}\, y_i \quad , \quad \text{for } 1 \leq j \leq n$$

Definition With the notation as above, we call the matrix $\underset{\sim}{A} \in \mathbb{K}^{m \times n}$ defined by $\underset{\sim}{A} = [a_{ij}]$ the matrix of T with respect to the ordered bases B and C and write

$$\boxed{\underset{\sim}{A} = [a_{ij}] = [T]_C^B}.$$

In case $X = Y$ and $B = C$, we simply write $\underset{\sim}{A} = [T]_B$.

Note that the j^{th} column of $\underset{\sim}{A} = [T]_C^B$ is simply $[T(x_j)]_C$. Let us also observe that it follows the corollary to the last theorem that if $S : X \to Y$ is a linear transformation such that $[S]_C^B = [T]_C^B$, then $S = T$.

Examples: (1) Let $X = \mathcal{P}_3(\mathbb{R})$, and let $B = \{1, t, t^2, t^3\}$ be the standard ordered basis for X. Let $p(t) = 3 + 5t - 3t^2 + 7t^3$, then

$$[p]_B = \begin{bmatrix} 3 \\ 5 \\ -3 \\ 7 \end{bmatrix}.$$

(2) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear transformation given by

$$T((x_1, x_2)^t) = (x_1 + 3x_2, 0, 2x_1 - 4x_2)^t$$

Let B, C be the standard ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Now $T(\underset{\sim}{e_1}) = (1, 0, 2)^t = 1\underset{\sim}{e_1} + 0\underset{\sim}{e_2} + 2\underset{\sim}{e_3}$

$T(\underset{\sim}{e_2}) = (3, 0, -4)^t = 3\underset{\sim}{e_1} + 0\underset{\sim}{e_2} - 4\underset{\sim}{e_3}$

Hence $$[T]_C^B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}.$$

If we let $C' = \{e_3, e_2, e_1\}$, then $[T]_{C'}^B = \begin{bmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}$.

Example 3 Let $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ be defined by

$$T((x_1, x_2, x_3, x_4, x_5)^t)$$

$$= (2x_3 - 2x_4 + x_5,\ 2x_2 - 8x_3 + 14x_4 - 5x_5,\ x_2 + 3x_3 + x_5)^t$$

Then T is a linear transformation.

We have $T(\underset{\sim}{e_1}) = (0,0,0)^t$, $T(\underset{\sim}{e_2}) = (0,2,1)^t$

$T(\underset{\sim}{e_3}) = (2,-8,3)^t$, $T(\underset{\sim}{e_4}) = (-2,14,0)^t$

$T(\underset{\sim}{e_5}) = (1,-5,1)^t$. Thus $T = T_A$, where

$$\underset{\sim}{A} = (T(\underset{\sim}{e_1}), T(\underset{\sim}{e_2}), T(\underset{\sim}{e_3}), T(\underset{\sim}{e_4}), T(\underset{\sim}{e_5}))$$

$$= \begin{bmatrix} 0 & 0 & 2 & -2 & 1 \\ 0 & 2 & -8 & 14 & -5 \\ 0 & 1 & 3 & 0 & 1 \end{bmatrix}.$$

Example 4.2 Let $T : \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$ be defined by $T(p) = p'$ for $p \in \mathcal{P}_3(\mathbb{R})$.

Take $\{1, t, t^2, t^3\}$ as a basis for $\mathcal{P}_3(\mathbb{R})$ and $\{1, t, t^2\}$ as a basis for $\mathcal{P}_2(\mathbb{R})$.

Then $T(1) = 0$, $T(t) = 1$, $T(t^2) = 2t$, $T(t^3) = 3t^2$

Consequently, the Matrix $M(T)$ is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

If we take the basis in $\mathcal{P}_3(\mathbb{R})$ as before, but take the basis in $\mathcal{P}_2(\mathbb{R})$ as $\{1, 1+t, (1+t)^2\}$ instead, Then

$$\left.\begin{aligned} T(1) &= 0 = 0\cdot 1 + 0\cdot(1+t) + 0\cdot(1+t)^2 \\ T(t) &= 1 = 1\cdot 1 + 0\cdot(1+t) + 0\cdot(1+t)^2 \\ T(t^2) &= 2t = -2 + 2(1+t) + 0\cdot(1+t)^2 \\ T(t^3) &= 3t^2 = 3 - 6(1+t) + 3(1+t)^2 \end{aligned}\right\} \Rightarrow M(T) = \begin{bmatrix} 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Spaces of Linear Transformations

Let X, Y be vector spaces over $\mathbb{K}$ and let us denote $\mathcal{L}(X,Y)$ the set of all linear transformations from X to Y:

$$\mathcal{L}(X,Y) := \{ T: X \to Y : T \text{ a linear transformation} \}.$$

Suppose $S, T \in \mathcal{L}(X,Y)$ and α in $\mathbb{K}$ be a scalar. Define $S+T$ and αT pointwise:

$$(S+T)(x) = S(x) + T(x)$$

$$(\alpha T)(x) = \alpha T(x), \quad \text{for all } x \in X.$$

It is easily verified that $\mathcal{L}(X,Y)$ is a vector space under these operations. In the following we assume X, Y to be finite dimensional with ordered basis $B = \{x_1, \dots, x_n\}$, $C = \{y_1, \dots, y_m\}$, respectively.

Proposition For all $S, T \in \mathcal{L}(X,Y)$ and scalar α we have:

(i) $[S+T]_C^B = [S]_C^B + [T]_C^B$;

(ii) $[\alpha T]_C^B = \alpha [T]_C^B$;

(iii) $[T]_C^B = [S]_C^B \iff T = S$.

Proof Exercise. ∎

From (iii) of the last proposition, we conclude that

$$\dim \mathcal{L}(X,Y) = \dim \mathbb{K}^{m \times n} = m \times n.$$

Composition of linear transformations

Suppose X, Y, Z are vector spaces of dimensions n, p, m respectively, and let

Composition of linear Transformations

Let X, Y and Z be vector spaces of dimensions n, p and m, respective. Suppose

$B = \{x_1, x_2, \cdots, x_n\}$, $C = \{y_1, \cdots, y_p\}$ and $D = \{z_1, \cdots, z_m\}$

be ordered bases of X, Y and Z, respectively

If $T : X \longrightarrow Y$ and $S : Y \longrightarrow Z$ are linear transformations then it is easy to check that $S \circ T : X \longrightarrow Z$ is a linear transformation. (The reader is asked to verify this fact.) Let us write (X, B) to indicate the fact that X is considered with the ordered basis. Then we have the maps:

$T : (X, B) \longrightarrow (Y, C)$ and $S : (Y, C) \longrightarrow (Z, D)$.

So the composition is the map $S \circ T : (X, B) \longrightarrow (Z, D)$.

Schematically,

$$\begin{tikzcd}
(X, B) \arrow[r, "T"] \arrow[rr, bend right, "S \circ T"'] & (Y, C) \arrow[r, "S"] & (Z, D)
\end{tikzcd}$$

We ask what is the matrix representation of this composition.

Proposition With the notations as above, we have

$$[S \circ T]^{B}_{D} = [S]^{C}_{D} \, [T]^{B}_{C}.$$

Proof. It is convenient to write the ordered bases B, C, D of X, Y, Z respectively as

$B = \{ x_j : 1 \leq j \leq n \}$, $C = \{ y_i : 1 \leq i \leq p \}$ and $D = \{ z_k : 1 \leq k \leq m \}$.

Now we write the matrices of S and T. Suppose

$$S y_i = \sum_{k=1}^{m} a_{ki} z_k \, , \qquad i = 1, 2, \cdots\cdots p$$

$$T x_j = \sum_{i=1}^{p} b_{ij} \, y_i \, , \qquad j = 1, 2, \cdots\cdots, n .$$

Let us write $\underset{\sim}{A} = [a_{ki}]_{m \times p}$ for the matrix $[S]^{C}_{D}$ and $\underset{\sim}{B} = [b_{ij}]_{p \times n}$ for the matrix $[T]^{B}_{C}$.

Now consider

$$
\begin{aligned}
(S \circ T)(x_j) &= S\left(\sum_{i=1}^{p} b_{ij}\, y_i\right) \\
&= \sum_{i=1}^{p} b_{ij}\, S(y_i) \\
&= \sum_{i=1}^{p} b_{ij} \sum_{k=1}^{m} a_{ki}\, z_k \\
&= \sum_{k=1}^{m} \sum_{i=1}^{p} a_{ki}\, b_{ij}\, z_k \quad \text{(interchanging the order of the summations)} \\
&= \sum_{k=1}^{m} \left(\sum_{i=1}^{p} a_{ki}\, b_{ij}\right) z_k \\
&= \sum_{k=1}^{m} (\underset{\sim}{A}\, \underset{\sim}{B})_{kj}\, z_k .
\end{aligned}
$$

Here $(\underset{\sim}{A}\, \underset{\sim}{B})_{kj}$ denotes the $(k,j)^{th}$ entry of the matrix $\underset{\sim}{A}\, \underset{\sim}{B}$.

We see from above that the $(k,j)^{th}$ entry of the matrix $[S \circ T]^{B}_{D}$ equals the $(k,j)^{th}$ entry of the matrix $\underset{\sim}{A}\, \underset{\sim}{B}$.

This proves the result ∎

Example Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ and $S : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be the linear transformations defined respectively by

$$T(p(t)) = \int_0^t p(s)\,ds \quad \text{and} \quad S(p(t)) = p'(t).$$

Let B and C denote the standard bases of $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_3(\mathbb{R})$, respectively. It follows from Calculus, that $S \circ T = I$, the identity transformation on $\mathcal{P}_2(\mathbb{R})$.

One can easily see that

$$[T]_C^B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \quad \text{and} \quad [S]_B^C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

From the preceding theorem it follows that

$$[S \circ T]_B = [S]_B^C [T]_C^B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [I]_B.$$

This is just a verification of the result.

Let X and Y be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and C, respectively. If $T: X \to Y$ is a linear transformation, then we ask if there is a relation between the coordinatizations $[x]_B$ and $[Tx]_C$. The link is the matrix $[T]_C^B$ as seen from the next proposition.

Proposition With the notations as above, we have

$$[T(x)]_C = [T]_C^B [x]_B ,$$

for all $x \in X$.

Proof. Let us fix up $x \in X$, and consider the linear transformations $f: \mathbb{K} \to X$ and $g: \mathbb{K} \to Y$ defined by $f(\alpha) = \alpha x$ and $g(\alpha) = \alpha T(x)$ for all $\alpha \in \mathbb{K}$. Let us take $A = \{1\}$ as the standard ordered basis for $\mathbb{K}$. Noting that $g = T \circ f$ and identifying column vectors as matrices, we obtain from the previous theorem,

$$[T(x)]_C = [g(1)]_C = [g]_C^A = [T\circ f]_C^A = [T]_C^B\,[f]_B^A$$

$$= [T]_C^B\,[f(1)]_B = [T]_C^B\,[x]_B. \qquad \blacksquare$$

Rank and Nullity

Let $T: X \longrightarrow Y$ be a linear transformation between vector spaces X, Y over $\mathbb{k}$. There are two important subspaces associated with T.

- Null space of T = $\mathcal{N}(T) = \{x \in X : T(x) = 0\}$ (also called the kernel of T, denoted by $\ker(T)$).
- Range of T = $\mathcal{R}(T) = \{T(x) : x \in X\}$ (also called the image of T, denoted by $\operatorname{Im}(T)$).

It is easy to see that $\mathcal{N}(T)$ is a subspace of X. (Verify this fact as an easy exercise.) The dimension of $\mathcal{N}(T)$ is called the nullity of T and it is denoted by nullity (T).

Likewise, it is easy to verify that $\mathcal{R}(T)$ is a subspace of Y. Its dimension, denoted by $\operatorname{rank}(T)$ is called the rank of T.

Theorem (The Rank-Nullity Theorem) Let $T: X \to Y$ be a linear transformation of vector spaces X, Y over $\mathbb{k}$. Assume X is finite dimensional. Then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(X).$$

Proof. The proof follows from the rank-nullity theorem for matrices. However, we give a different proof here which is far simpler. Suppose $\dim X = n$ and let $B = \{x_1, x_2, \cdots, x_k\}$ be an ordered basis for $\mathcal{N}(T)$. We can extend B to an ordered basis

$$C = \{x_1, x_2, \cdots, x_k, w_1, w_2, \cdots, w_{n-k}\}$$

of X. We claim that

$$D = \{ T(w_1), T(w_2), \cdots, T(w_{n-k}) \}$$

is a basis of $\mathcal{R}(T)$. Indeed, any $x \in X$ can be expressed uniquely as

$$x = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k + \beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k}.$$

Hence $T(x) = \alpha_1 T(x_1) + \cdots + \alpha_k T(x_k) + \beta_1 T(w_1) + \cdots \cdots \beta_{n-k} T(w_{n-k})$

$$= \beta_1 T(w_1) + \cdots + \beta_{n-k} T(w_{n-k}).$$

Hence D spans $\mathcal{R}(T)$. Next, suppose

$$\beta_1 T(w_1) + \cdots \cdots + \beta_{n-k} T(w_{n-k}) = 0.$$

Then $T(\beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k}) = 0$.

Hence $\beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k} \in \mathcal{N}(T)$. Since B is a basis for $\mathcal{N}(T)$, there are scalars $\alpha_1, \cdots, \alpha_k$ such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = \beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k}$

By linear independence of $\{x_1, \cdots, x_k, w_1, \cdots, w_{n-k}\}$, we conclude $\beta_1 = \beta_2 \cdots = \beta_{n-k} = 0$. Hence D is a basis of $\mathcal{R}(T)$. Thus

$$\text{rank}(T) = n-k = \dim(X) - \text{nullity}(T). \quad \blacksquare$$

Proposition

If $T: X \longrightarrow Y$ is a linear transformation between vector spaces, then T is 1-1 if and only if $\mathcal{N}(T) = \{0\}$.

Indeed, if T is 1-1 and $T(x) = 0$, then $T(x) = T(0)$ which implies $x = 0$. Hence $\mathcal{N}(T) = \{0\}$. On the other hand if $\mathcal{N}(T) = \{0\}$ and $T(x) = T(y)$. Then $0 = T(x) - T(y)$ $= T(x-y)$. Therefore $x - y \in \mathcal{N}(T) = \{0\} \Rightarrow x - y = 0 \Rightarrow x = y$

In conjunction with Rank-Nullity Theorem, the above proposition gives:

Theorem Let X and Y be finite dimensional vector spaces of equal dimension, and $T: X \longrightarrow Y$ be a linear transformation. Then the following statements are equivalent.

(i) T is 1-1 (injective).

(ii) T is onto (surjective)

(iii) $\text{rank}(T) = \dim(X)$.

<u>Proof</u>. Exercise.

As in the previous theorem, let $T : X \longrightarrow Y$ be a linear transformation between finite dimensional vector spaces of equal dimension n. Then if either T is injective or surjective then T is bijective and hence it is invertible: there is a map $S : Y \longrightarrow X$ such that $S \circ T = I_X$ and $T \circ S = I_Y$. (Here I_X, I_Y denote the identity maps on X, Y, respectively.) In this case, we write $T^{-1} = S$. The important thing to observe here is that $T^{-1} : Y \longrightarrow X$ is a linear transformation.

Indeed, let $y_1, y_2 \in Y$ and $\alpha_1, \alpha_2 \in \mathbb{K}$. There are unique elements $x_1, x_2 \in X$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$. Then $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$; so

$$T^{-1}(\alpha_1 y_1 + \alpha_2 y_2) = T^{-1}(\alpha_1 T(x_1) + \alpha_2 T(x_2)) = T^{-1}(T(\alpha_1 x_1 + \alpha_2 x_2))$$
$$= \alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 T^{-1}(y_1) + \alpha_2 T^{-1}(y_2).$$

Proposition Let X, Y be finite dimensional vector spaces (of equal dimension) over $\mathbb{K}$ with ordered bases B and C, respectively. Let $T: X \to Y$ be a linear transformation. Then T is invertible if and only if $[T]^B_C$ is invertible. Moreover

$$[T^{-1}]^C_B = \left([T]^B_C\right)^{-1}.$$

Proof Suppose T is invertible. Then $T^{-1}: Y \to X$ satisfies $T \circ T^{-1} = I_Y$ and $T^{-1}T = I_X$.
let $\dim(X) = \dim(Y) = n$. Then

$$I_n = [I_X]_B = [T^{-1}T]_B = [T^{-1}]^C_B \, [T]^B_C$$

Similarly $[T]^B_C [T^{-1}]^C_B = I_n$. So the matrix $[T]^B_C$ is invertible and $\left([T]^B_C\right)^{-1} = [T^{-1}]^C_B$

Conversely, suppose $\underset{\sim}{A} = [T]^B_C$ is invertible. Then let $\underset{\sim}{E}$ be its inverse. There exists $S \in \mathcal{L}(Y, X)$ such that $S(y_j) = \sum_{i=1}^{n} E_{ij} x_i$, $j = 1, 2, \dots, n$

where $C = \{y_1, y_2, \dots y_n\}$ and $B = \{x_1, \dots, x_n\}$. It follows that $[S]^C_B = \underset{\sim}{E}$. Then $ST = I_X$, and similarly, $TS = I_Y$. ∎

Remark Let $T : X \to Y$ be a linear transformation between finite dimensional vector spaces of equal dimension n. Then

$$T \text{ is not invertible} \iff T \text{ is not injective}$$

$$\iff \operatorname{nullity}(T) > 0$$

$$\iff \operatorname{rank}(T) < n$$

$$\iff \det([T]_C^B) = 0$$

Here B, C are ordered bases in X, Y respectively