MA 102:PART I: LINEAR ALGEBRA

Spring, 2016-17

Instructor:
Devidas Pai

Discipline of Mathematics

Indian Institute of Technology Gandhinagar

India

Name:

Tutorial Batch:

MA 102: Part I: Linear Algebra
(Course contents)

- Vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$, notions of linear dependence and independence, linear span of a set of vectors, vector subspaces of $\mathbb{R}^n$ and $\mathbb{C}^n$, the basis of a vector subspace.
- Systems of linear equations, matrices and Gauss elimination, row space, null space, and column space, rank of a matrix.
- Determinants and rank of a matrix in terms of determinants.
- Abstract vector spaces, linear transformations, matrix of a linear transformation, change of basis and similarity, rank-nullity theorem.
- Inner product space, the Gram-Schmidt process, orthonormal bases, projections, and the least squares approximation.
- Eigenvalues and eigenvectors, characteristic polynomials, the eigenvalue of special matrices(orthogonal, unitary, symmetric, hermitian, skew-symmetric, normal).
- Algebraic and geometric multiplicities, diagonalisation by similarity transformations, Spectral theorem for real symmetric matrices and applications to quadratic forms.
- Singular value decomposition of matrices.

Texts/References

1. E. Kreyszig, *Advanced Engineering Mathematics* (8th Edition), John Wiley(1999).
2. G. Strang, *Linear Algebra and its applications* (4th Edition), Thomson (2006).
3. H. Anton C. Rorres, *Elementary Linear Algebra:Applications Version* (9th Edition), Wiley-India(2008).
4. B. Kolman, D.R. Hill, *Introductory Linear Algebra: An Applied First Course* (8th edition), Pearson Education(2009).

Lectures and Tutorials:

Every week there will be three lectures and one tutorial as shown in the Time Table. For tutorials, the class will be divided into four batches, and each batch will be assigned a Course-Associate. The aim of the tutorials is to have a closer interaction with you in clearing your doubts, if any, and also to give you practise in problem solving. Based on the topics covered, a tutorial sheet will be given to you each week. You are expected to try some of the problems before hand for each tutorial class. You may feel free to seek the help of your course associate in case you have any doubts. In addition, the Instructors and the Course Associates will assign an *Office Hour* each week to enable you to meet them in case you have any course related difficulties.

Policy for Attendance:

Attendance in lectures and tutorials is compulsory. This will be carefully monitored. 5% weightage will be assigned for attendance.
In case you miss lectures/tutorials for valid (medical reasons), you may get a medical certificate from the Institute Medical Officer. You can produce a xerox copy of it to the Instructor in case you fall short of the attendance.

Evaluation Plan:

There will one Quiz of weightage 5% and one Class Test of weightage 10% of one hour's duration before Mid-Semester Examination. The dates and the timings of the Quiz and the Test will be announced subsequently in the class. The Mid-Semester Test will carry 30% weightage. This will be held during the Mid-Semester Examination week as prescribed by the Academic Calender.

Course Plan

(This is only indicative of the broad coverage.) In the following, [K] refers to the text book by E. Kreyzig, *Advanced Engineering Mathematices*, 8th Edition, John wiley and Sons (1999).

Sr. No.	Topic	Section from [K]	No. of Lec.
1.	Vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$, notions of linear dependence and independence, linear span of a set of vectors, vector subspaces of $\mathbb{R}^n$ and $\mathbb{C}^n$, the basis of a vector subspace.		1.5
2.	Generalities about matrices, matrix multiplication, special kinds of matrices.	6.1- 6.2	1.5
3.	System of linear equations, Gauss elimination, row and column operations.	6.3	1.5
4.	The null space, column space, rank of a matrix: row rank, column rank, the invariance of rank, existence and uniqueness of solutions.	6.4- 6.5	1.5
5.	Determinants, rank of a matrix in terms of determinants, Cramer's rule and inverse of matrices.	6.6 - 6.7	2
6.	Abstract vector spaces, linear transformations, matrix of a linear transformation, change of basis and similarity, the nullity-rank theorem.	6.8	2
7.	Inner product space, the Gram-Schmidt process, orthonormal bases, projections, the least squares approximation.	6.8	1.5
8.	Generalities on eigenvalues and eigenvectors, characteristic polynomials, the eigenvalues of special matrices(orthogonal, unitary, symmetric, hermitian, skew-symmetric, normal).	7.1, 7.3 -7.4	1.5
9.	Algebraic and geometric multiplicities, diagonalisation by similarity transformations.	7.5	1.5
10.	Spectral theorem for real symmetric matrices, applications to quadratic forms.	7.5	1.5
11.	Singular Value Decomposition (SVD) of matrices,		2

A separate handout will be given for Part II: Differential Equations, the second half of the course, which will commence after the Mid-Semester break. The evaluation for this part will consist of one Class Test of weightage 10% and the End-semester Examination of weightage 30 %. The application projects for both the parts: Linear Algebra and ODE will be carefully planned for presentations during the second half of the Semester. These will be co-ordinated by Prof. Chetan Pehelajani and carry a weightage of 10%.

Tutorial Sheet No. 1

1. Show that the only possible subspaces of $\mathbb{R}^3$ are the zero space $\{0\}$, lines passing through origin, planes passing through origin and all $\mathbb{R}^3$.

2. If $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$ are vectors in $\mathbb{R}^n$, we define the *inner product* of v and w by $v.w = v_1w_1 + v_2w_2 + \ldots + v_nw_n$. This is the generalisation of the familiar dot product in $\mathbb{R}^3$. Is the given set of vectors a vector space?

   (a) The set of vectors $\{v \mid v.x = 0\}$, for a fixed vector x in $\mathbb{R}^n$.

   (b) The set of vectors $\{v \mid v.x = 1\}$, for a fixed vector x in $\mathbb{R}^n$.

   (c) All vectors $(v_1, v_2, v_3)^T$ in $\mathbb{R}^3$ such that $3v_1 - 2v_2 + v_3 = 0$ and $4v_1 + 5v_2 = 0$.

   (d) All vectors in $\mathbb{R}^2$ with components less than 1 in absolute value.

3. Let $\alpha_1, \alpha_2, \alpha_3$ be fixed real numbers. Show that the vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_4 = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3$ form a subspace, which is spanned by $(1, 0, 0, \alpha_1), (0, 1, 0, \alpha_2)$ and $(0, 0, 1, \alpha_3)$. Find the dimension of this subspace.

4. Examine whether the following sets of vectors constitute vector subspaces of $\mathbb{R}^4$. If so, find the dimension and a basis of that vector space.
   The set of all (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ such that
   (a) $x_4 = 0$; (b) $x_1 \leq x_2$; (c) $x_1^2 - x_2^2 = 0$; (d) $x_1 = x_2 = x_3 = x_4$; (e) $x_1x_2 = 0$.

5. (a) Determine whether the vectors $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ are linearly independent in $\mathbb{C}^3$.

   (b) Show that the vectors $(1, 0)$ and $(i, 0)$ in $\mathbb{C}^2$ are linearly independent over $\mathbb{R}$ but not linearly independent over $\mathbb{C}$.

   (c) In $\mathbb{C}^n$ we define the *inner product* of two vectors v and w by $v.w = v_1\overline{w_1} + v_2\overline{w_2} + \ldots + v_n\overline{w_n}$, where $\overline{z}$ denotes the complex conjugate of z for z in $\mathbb{C}$. Show that the set of vectors $\{v \mid v.x = 0\}$, for a fixeed vector x in $\mathbb{C}^n$ is a vector space. What is its complex dimension?

6. An $n \times n$ matrix $A = (a_{ij})$ is called *symmetric* if $a_{ij} = a_{ji}$ for all pairs of $(i, j), 1 \leq i, j \leq n$. Let A and B be symmetric matrices of same size. Show that AB is a symmetric matrix if and only if $AB = BA$.

7. A square matrix A is called *nilpotent* if $A^m = 0$ for some positive integer m.

   (a) Show that an $n \times n$ matrix $A = (a_{ij})$ in which $a_{ij} = 0$ for $i \geq j$, is nilpotent. In fact $A^n = 0$.

   (b) Let A and B be nilpotent matrices of the same order.

      i. Show by an example that $A + B$ need not be nilpotent.

      ii. However, prove that this is the case A and B commute with other (that is $AB = BA$).

8. A square matrix $A = (a_{ij})$ is called *upper triangular* if $a_{ij} = 0$ for all $j < i$. A *lower triangular* matrix is defined similarly.(or equivalently, as the transpose of an upper triangular matrix).

   (a) Prove that the sum as well as the product of two upper triangular matrices (of equal orders) is upper triangular.

   (b) Show that $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$ for all $\lambda \in \mathbb{R}, n \geq 1$.

9. The *conjugate transpose* (also called the *Hermitian adjoint*) A^* of a complex $m \times n$ matrix A is defined as the transpose of its conjugate (or equivalently, the conjugate of its transpose). Prove that the properties of the conjugate transpose are analogous to that of the transpose: e.g. $(A + B)^* = A^* + B^*$ and $(AB)^* = B^*A^*$. Note, however, that $(\alpha A)^* = \overline{\alpha}A^*$.

10. A square matrix A is called *Hermitian* if $A^* = A$ and *skew-Hermitian* if $A^* = -A$.

    (a) Show that every square matrix can be uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrix.

    (b) If A and B are $n \times n$ Hermitian matrices and α and β are any real numbers, show that $C = \alpha A + \beta B$ is a Hermitian matrix.

    (c) Let A and B be skew-Hermitian matrices. For arbitrary complex scalars a and b, under what conditions is the matrix $C = aA + bB$ skew-Hermitian?

11. The *trace* of a square matrix $A = (a_{ij})$ is defined as the sum of its diagonal elements, that is, $tr(A) = \sum_i a_{ii}$. Prove that if A, B are square matrices of the same order and α, β are scalars then

    (a) $tr(\alpha A + \beta B) = \alpha tr(A) + \beta tr(B)$.

    (b) $tr(AB) = tr(BA)$.

    (c) If A is invertible then $tr(ABA^{-1}) = tr(B)$.

12. Given a polynomial $p(x)$ and a square matrix A, let $p(A)$ denote the matrix obtained by 'substituting' A for the variable x. Thus, if $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_kx^k$, then $p(A) = a_0 + a_1A + a_2A^2 + \ldots + a_kA^k$. Given polynomials p and q, show that

    (a) (p + q)(A) = p(A) + q(A) and

    (b) (pq)(A) = P(A)q(A).

13. (a) Show that any diagonal matrix D whose only possible entries are 0 and 1 is idempotent, that is, $D^2 = D$.

    (b) If $A^2 = A$, prove that $(A + I)^n = I + (2^n - 1)A$.

14. A *Markov* (or *stochastic*) matrix is a square matrix (a_{ij}) such that $0 \leq a_{ij} \leq 1$ and $\sum_{j=1}^{n} a_{ij} = 1$, for all $i = 1, 2, \ldots, n$. Prove that the product of two Markov matrices is a Markov matrix.