(1) For a physical dipole with $-q$ at $\vec{r}$ and $+q$ at $\vec{r}+\vec{d}$

$$U = qV(\vec{r}+\vec{d}) - V(\vec{r})$$

$$= q\left[-\int_{\vec{r}}^{\vec{r}+\vec{d}} \vec{E}\cdot d\vec{l}\right]$$

$$= -q\,\vec{E}\cdot\vec{d}$$

$$\boxed{U = -\vec{P}\cdot\vec{E}}$$

$$U = -\vec{P_1}\cdot\vec{E_2}$$

But $\vec{E_2} = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left[3(\vec{P_2}\cdot\hat{r})\hat{r} - \vec{P_2}\right]$

$$\therefore\ U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[(\vec{P_1}\cdot\vec{P_2}) - 3(\vec{P_1}\cdot\hat{r})(\vec{P_2}\cdot\hat{r})\right]$$

(2)

$$\rho_v = \begin{cases} \rho_0 & , \; 0<r<a \\ 0 & , \; r>a \end{cases}$$

For $\quad r<a, \qquad \varepsilon E\,(4\pi r^2) = \rho_0 \frac{4}{3}\pi r^3$

$$\Rightarrow E = \frac{\rho_0 r}{3\epsilon}$$

$$\therefore V = -\int \vec{E}\cdot d\vec{l}$$

$$= -\int \frac{\rho_0 r}{3\epsilon}\, dr$$

$$= -\frac{\rho_0}{3\epsilon}\left[\frac{r^2}{2}\right]_{\infty}^{r} + C_1$$

$$= -\frac{\rho_0 r^2}{6\epsilon} + C_1$$

For $\quad r>a, \qquad \epsilon_0 E\,(4\pi r^2) = \rho_0 \frac{4}{3}\pi a^3$

$$\Rightarrow E = \frac{\rho_0 a^3}{3\epsilon_0 r^2}$$

$$\therefore V = -\int \vec{E}\cdot d\vec{l}$$

$$= -\int \frac{\rho_0 a^3}{3\epsilon_0 r^2}\, dr$$

$$= -\frac{\rho_0 a^3}{3\epsilon_0}\left[\frac{-1}{r}\right] + C_2$$

$$= \frac{\rho_0 a^3}{3\epsilon_0 r} + C_2$$

Boundary condition

As $\quad r \longrightarrow \infty \qquad V \longrightarrow 0 \qquad \therefore C_2 = 0$

Also $\quad$ at $\quad r = a \qquad V(a^+) = V(a^-)$

$$-\frac{\rho_0 a^2}{6\epsilon} + C_1 = \frac{\rho_0 a^3}{3\epsilon_0 a}$$

$$\Rightarrow -\frac{\rho_0 a^2}{6\epsilon} + C_1 = \frac{\rho_0 a^2}{3\epsilon_0}$$

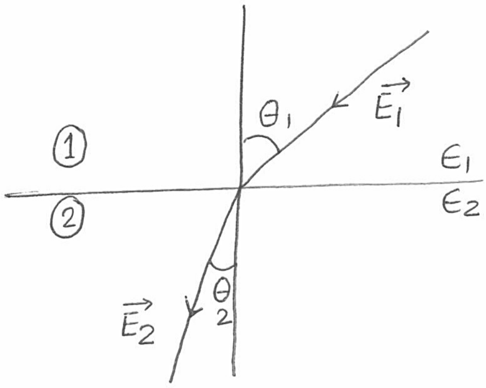
$$\Rightarrow C_1 = \frac{\rho_0 a^2}{6\epsilon}\left[1 + \frac{2\epsilon}{\epsilon_0}\right]$$

$$= \frac{\rho_0 a^2}{6\epsilon_0 \epsilon_r}(1 + 2\epsilon_r)$$

$$\boxed{\therefore V(0) = C_1 = \frac{\rho_0 a^2}{6\epsilon_0 \epsilon_r}(1 + 2\epsilon_r)}$$

Also, $\boxed{V(a) = \frac{\rho_0 a^2}{3\epsilon_0}}$

(3)



$E_{\parallel}$ is continuous $\Rightarrow E_{1x} = E_{2x}$

$D_{\perp}$ is continuous (with $\sigma_f = 0$)

$$\therefore D_{1y} = D_{2y}$$

$$\Rightarrow \varepsilon_1 E_{1y} = \varepsilon_2 E_{2y}$$

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\cancel{E_{1x}/E_{1y}}}{\cancel{E_{2x}/E_{2y}}} \quad \frac{E_{2x}/E_{2y}}{E_{1x}/E_{1y}}$$

$$= \frac{E_{1y}}{E_{2y}}$$

$$= \varepsilon_2/\varepsilon_1$$

(4)

Applying Ampere's law -

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$

$$\Rightarrow B \cdot 2\pi\rho = \mu_0 I \frac{\pi\rho^2}{\pi a^2}$$

$$\Rightarrow B \cdot 2\pi\rho = \mu_0 I \rho^2/a^2$$

$$\Rightarrow B = \frac{\mu_0 I \rho}{2\pi a^2}$$

$$\boxed{\vec{B} = \frac{\mu_0 I \rho}{2\pi a^2}\,\hat{\phi}}$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$\Rightarrow \vec{J} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})$$

$$= \frac{1}{\mu_0}\,\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\,\frac{\mu_0 I \rho}{2\pi a^2}\right)\right]\hat{z}$$

$$= \frac{1}{\mu_0}\,\frac{1}{\rho}\,\frac{\mu_0 I}{2\pi a^2}\,2\rho\;\hat{z}$$

$$= \frac{I}{\pi a^2}\,\hat{z}$$

⑤ ⓐ $\vec{A} = x^2y\,\hat{\imath} + y^2x\,\hat{\jmath} - 4xyz\,\hat{k}$ $\frac{Wb}{m}$

$\vec{B} = \vec{\nabla}\times\vec{A}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2x & -4xyz \end{vmatrix}$$

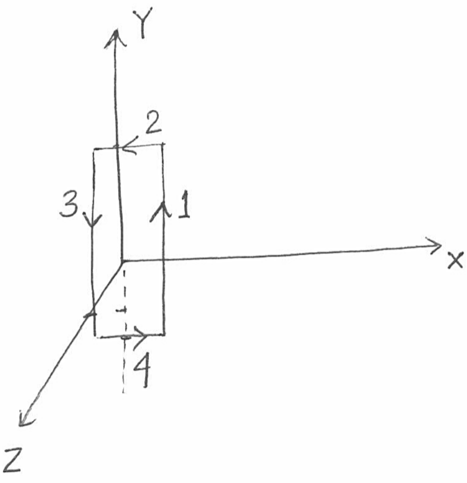
$$= \hat{\imath}\,(-4xz - 0) - \hat{\jmath}\,(-4yz - 0) + \hat{k}\,(y^2 - x^2)$$

$$= -4xz\,\hat{\imath} + 4yz\,\hat{\jmath} + (y^2 - x^2)\,\hat{k}$$

$$\therefore \vec{B}(-1,2,5) = -4(-5)\hat{\imath} + 4\times 10\,\hat{\jmath} + (4-1)\hat{k}$$

$$= 20\hat{\imath} + 40\hat{\jmath} + 3\hat{k} \quad \frac{Wb}{m^2}$$

ⓑ Flux through the area is given by



$$\oint_S \vec{B}\cdot d\vec{s}$$

$$= \oint_l \vec{A}\cdot d\vec{l} = \phi_1 + \phi_2 + \phi_3 + \phi_4$$

Now $\phi_1 = \left[\int_{y=-1}^{4} (x^2y\,\hat{\imath} + y^2x\,\hat{\jmath} - 4xyz\,\hat{k})\cdot dy\,\hat{\jmath} \right]_{\substack{x=1\\ z=1}}$

$$= \int_{y=-1}^{4} y^2x\,dy \Big|_{\substack{x=1\\ z=1}}$$

$$= \frac{y^3}{3}\Big|_{-1}^{4}$$

$$= \frac{64}{3} + \frac{1}{3} = \frac{65}{3}$$

$$\phi_2 = \left[\int_{x=1}^{0} (x^2y\hat{i} + y^2x\hat{j} - 4xyz\hat{k}) \cdot -dx\,\hat{i} \right]_{\substack{y=4\\z=1}}$$

$$= -\int_{x=1}^{0} x^2y\,dx \Big|_{\substack{y=4\\z=1}}$$

$$= -4\left[\frac{x^3}{3}\right]_1^0$$

$$= +4 \times \frac{1}{3}$$

$$= +\frac{4}{3}$$

$$\phi_3 = \left[\int_{4}^{-1} (x^2y\,\hat{i} + y^2x\hat{j} - 4xyz\hat{k}) \cdot -dy\,\hat{j} \right]_{x=0,\, z=1}$$

$$= 0$$

$$\phi_4 = \left[\int_{x=0}^{x=1} (x^2y\hat{i} + y^2x\hat{j} - 4xyz\hat{k}) \cdot (+dx\hat{i}) \right]_{\substack{y=-1\\z=1}}$$

$$= \left. \int_0^1 x^2y\,dx \right]_{\substack{y=-1\\z=1}}$$

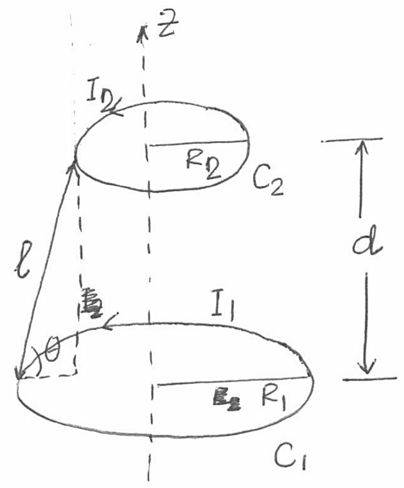
$$= -\left[\frac{x^3}{3}\right]_0^1$$

$$= -\frac{1}{3}$$

$\therefore$ Flux through the surface $= \phi = \frac{65}{3} + \frac{4}{3} + 0 - \frac{1}{3}$

$$= \frac{68}{3} \text{ Wb}.$$

(6)



We consider two coils C_1 and C_2 of radii R_1 and R_2 respectively, carrying currents I_1 and I_2 respectively. Let d be the separation between the centers of the coils.

Force per unit length on either coil is

$$F_0 = \frac{\mu_0 I_1 I_2}{2\pi l}$$

By symmetry, the component of F_0 perpendicular to the z-axis will cancel.

$\therefore$ The total force along z-axis

$$F = \frac{\mu_0 I_1 I_2}{2\pi l} \sin\theta \times 2\pi R_2 \qquad R_1 \simeq R_2$$

$$= 2\pi R_2 \times \frac{\mu_0 I_1 I_2}{2\pi l} \times \frac{d}{l}$$

$$\boxed{F = \mu_0 I_1 I_2 R_2 \frac{d}{d^2 + (R_1 - R_2)^2}}$$

Force is maximum when,

$$\frac{dF}{dd} = 0$$

$$\Rightarrow \frac{d^2 + (R_1 - R_2)^2 - d \cdot 2d}{(d^2 + (R_1 - R_2)^2)^2} = 0$$

$$\Rightarrow (R_1 - R_2)^2 = d^2$$

$$\boxed{\Rightarrow d = R_1 - R_2}$$