MA 102: LINEAR ALGEBRA

Tutorial Sheet No. 2

1. Solve the following system of linear equations in the unknowns $x_1, x_2, \ldots, x_5$ by the Gauss Elimination Method.

(a)
$$\begin{array}{rrrrrrrrrr} & & 2x_3 & - & 2x_4 & + & x_5 & = & 2 \\ 2x_2 & - & 8x_3 & + & 14x_4 & - & 5x_5 & = & 2 \\ x_2 & + & 3x_3 & & & + & x_5 & = & 8 \end{array}$$

(b)
$$\begin{array}{rrrrrrrrr} 2x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & 1 \\ & - & 2x_2 & + & x_3 & - & x_4 & = & 2 \\ 3x_1 & - & x_2 & + & 4x_3 & - & 2x_4 & = & -2 \end{array}$$

(c)
$$\begin{array}{rrrrrrrrr} & & & - & 2x_4 & + & x_5 & = & 2 \\ 2x_2 & - & 2x_3 & + & 14x_4 & - & x_5 & = & 2 \\ 2x_2 & + & 3x_3 & + & 13x_4 & + & x_5 & = & 3 \end{array}$$

(d)
$$\begin{array}{rrrrr} x_1 & & +2x_3 & -2x_4 & =0 \\ 2x_1 & -x_2 & & -x_4 & =0 \\ x_1 & & +2x_3 & -x_4 & =0 \\ 4x_1 & -x_2 & +3x_3 & -x_4 & =0 \end{array}$$

2. Prove that every invertible 2×2 matrix is a product of at most four elementary matrices.

3. Let A be a square matrix. Prove that there is a set of elementary matrices $E_1, E_2, \ldots, E_n$ such that $E_n \ldots E_1 A$ is either the identity matrix or its bottom row is zero.

4. If A is symmetric, and $a_{11} \neq 0$, then show that the matrix $\begin{pmatrix} a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \cdots & & \\ a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{pmatrix}$ obtained after first step of the Gauss Elimination Method is symmetric.

5. The nth *Hilbert matrix* H_n is defined as the $n \times n$ matrix whose (i,j)th entry is $\frac{1}{i+j-1}$. Obtain H_3^{-1} by GJEM.

6. Find the inverse of the following matrix using elementary row-operations:

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -7 \\ 0 & 1 & -4 \end{pmatrix}$$

7. Compute the last row of the inverse of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 6 \\ 4 & -24 & 8 & 11 \\ 0 & -7 & 2 & 2 \end{bmatrix} \xleftarrow{\text{Pl. correct}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -24 & 8 & -5 \\ 0 & -7 & 2 & 2 \end{pmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 6 \\ 4 & -24 & 8 & 11 \\ 0 & -7 & 2 & 2 \end{bmatrix}$$

8. Obtain the Gauss form of the following matrices. Use them to find rank and nullity of the matrix. Also write down a basis for the range. Finally obtain the Gauss-Jordan form and use it to write down a basis for the null space.

(a) $\begin{pmatrix} 1 & -2 & 1 \\ 3 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 3 & 4 & 0 \\ 5 & 1 & 1 \end{pmatrix}$

9. Prove that under elementary row operations, the linear dependance and independence of columns and hence the column rank of a matrix are unaffected. Do similar assertions hold for linear dependence of row and for the row rank? Justify your answer. Finally does this apply to the rank of a matrix also?

10. Give an example of two square matrices A, B (of equal orders) such that $rk(A) = rk(B)$ but $rk(A^2) \neq rk(B^2)$.

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Addendum to Tutorial Sheet No 2

Q I : An $m \times n$ matrix $\underset{\sim}{M}$ is said to be in row-echelon form if it satisfies:

(i) There is a number r, $0 \leq r \leq \min\{m, n\}$ such that each of the first r rows of $\underset{\sim}{M}$ has a nonzero entry and the remaining $m-r$ rows are identically zero;

(ii) For $1 \leq i \leq r$, suppose the first nonzero entry occurs in the p_i^{th} column, Then

$$1 \leq p_1 < p_2 \cdots < p_r \leq n.$$

The first nonzero entry in a nonzero row is called a pivot and the column in which it appears is called the pivot column.

Further, we say that the matrix $\underset{\sim}{M}$ is in reduced row-echelon form or normalized row-echelon form if in addition to the properties (i) and (ii), $\underset{\sim}{M}$ satisfies the property:

(iii) For $1 \leq i \leq r$, each pivot is 1 and in that pivot column, every other entry is 0.

Which of the following matrices are in row-echelon form, reduced row-echelon form, both, or neither.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 0 & 2 & 0 \\ 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Q II Solve the following systems by Gauss elimination:

(a)
$$\begin{aligned} x - y + 2z - u &= -1 \\ 2x + y - 2z - 2u &= -2 \\ -x + 2y - 4z + u &= 1 \\ 3x \qquad\qquad - 3u &= -3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Q III Solve the following systems by Gauss-Jordan elimination:

(a)
$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 \qquad + 15x_6 &= 5 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 + 2x_2 - x_3 \qquad + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 \qquad - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

Q. IV Determine the values of $\lambda, \mu \in \mathbb{R}$ such that the system of linear equations

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + \lambda z &= 3 \\ x + 3y + 3z &= \mu \end{aligned}$$

admits (i) no solution (ii) unique solution (iii) infinitely many solutions.