Vector Spaces, Linear Transformations

First of all we begin to look at some additional examples of vector spaces. As before, we denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$ which has to be fixed a priori. ($\mathbb{K} = \mathbb{R}$ for a real vector space and $\mathbb{K} = \mathbb{C}$ for a complex vector space.)

Example 1 Let S be a nonempty set and $\mathcal{F}(S, \mathbb{K})$ denote the set of all functions from S to $\mathbb{K}$:

$$\mathcal{F}(S, \mathbb{K}) := \{ f : S \to \mathbb{K} \; : \; f \text{ a } \mathbb{K}\text{-valued function} \}$$

The set $\mathcal{F}(S, \mathbb{K})$ is a vector space with the operations of addition and scalar multiplication defined pointwise:

$$(f+g)(s) = f(s) + g(s), \quad (\alpha f)(s) = \alpha (f(s))$$

for $f, g \in \mathcal{F}(S, \mathbb{K})$ and $\alpha \in \mathbb{K}$.

It is easily verified that $X = \mathcal{F}(S, \mathbb{R})$ (resp. $X = \mathcal{F}(S, \mathbb{C})$) satisfies all the axioms of a real (resp. complex) vector space

Example 2 A polynomial with coefficients from $\mathbb{K}$ with indeterminate t is an expression of the form

$$p(t) = a_0 + a_1 t + \dots + a_n t^n$$

where $n \in \mathbb{N}$ and $a_i \in \mathbb{K}$, $i = 1, \dots, n$.
The degree of p is the largest exponent of t.
Let $\mathcal{P}(\mathbb{K}) := \{ p(t) : p$ is a polynomial with coefficients from $\mathbb{K}$ of arbitrary degree $\}$

$\mathcal{P}_n(\mathbb{K}) := \{ p(t) : p$ is a polynomial of degree n with coefficients from $\mathbb{K} \}$

Let $X = \mathcal{P}(\mathbb{K})$ (resp. $\mathcal{P}_n(\mathbb{K})$) with addition and scalar multiplication defined pointwise, it is easily verified that X is a vector space over $\mathbb{K}$. $\mathcal{P}(\mathbb{K})$ is infinite dimensional while $\mathcal{P}_n(\mathbb{K})$ is finite dimensional with dimension $n+1$.

As a particular case of Example 1, we have:

Example 3 Take $S = \mathbb{N}$ and let

$X = \mathbb{K}^{\mathbb{N}}$ denote the set of all sequences with entries in $\mathbb{K}$:

$$\mathbb{K}^{\mathbb{N}} := \left\{ x = (x_1, x_2, \cdots, x_n, \cdots) : x_i \in \mathbb{K},\ i = 1, 2, \cdots \right\}$$

If $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{K} = \mathbb{C}$), $\mathbb{R}^{\mathbb{N}}$ (resp. $\mathbb{C}^{\mathbb{N}}$) consists of real (resp. complex) sequences. With addition and scalar multiplication defined 'componentwise':

$$x + y = (x_1, x_2, \cdots, x_n, \cdots) + (y_1, y_2, \cdots, y_n, \cdots)$$
$$= (x_1 + y_1, \cdots, x_n + y_n, \cdots)$$

$$\alpha x = \alpha (x_1, x_2, \cdots, x_n, \cdots) = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n, \cdots)$$

$\mathbb{K}^{\mathbb{N}}$ becomes a vector space. Clearly, this vector space is infinite dimensional.

The following example would be useful to us while studying differential equations.

Example 4. We take $S = [a,b]$ in Example 1. Let $n \in \mathbb{N}$ and let $\mathcal{C}^{(n)}[a,b]$ denote the set of all real-valued functions $f : [a,b] \to \mathbb{R}$ whose n^{th} derivative $f^{(n)}$ is continuous on $[a,b]$.

$$\mathcal{C}^{(n)}[a,b] := \left\{ f : [a,b] \to \mathbb{R} \ : \ f^{(n)} \text{ continuous on } [a,b] \right\}.$$

Let $\mathcal{C}[a,b]$ denote the set of all real-valued functions $f : [a,b] \to \mathbb{R}$ that are continuous.

$$\mathcal{C}[a,b] := \left\{ f : [a,b] \to \mathbb{R} \ : \ f \text{ continuous} \right\}.$$

If we take $X = \mathcal{C}[a,b]$ or $\mathcal{C}^{(n)}[a,b]$, and in X define the operations of addition and scalar multiplication 'pointwise':

$$(f+g)(t) = f(t) + g(t)$$
$$(\alpha f)(t) = \alpha f(t), \quad \text{for } f, g \in X,$$

Then X is a vector space, which is infinite dimensional.

Topic 20 : Linear Transformations

Let $\underset{\sim}{A}$ be an $m \times n$ matrix with real entries. Then $\underset{\sim}{A}$ "acts" on the n-dimensional space $\mathbb{R}^n$ by left multiplication : If $\underset{\sim}{v} \in \mathbb{R}^n$ then $\underset{\sim}{A}\underset{\sim}{v} \in \mathbb{R}^m$.

In other words A defines a function

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad T_A(\underset{\sim}{v}) = \underset{\sim}{A}\underset{\sim}{v}.$$

By properties of matrix multiplication, T_A satisfies the following conditions :

$$(i) \quad T_{\underset{\sim}{A}}(\underset{\sim}{v} + \underset{\sim}{w}) = T_{\underset{\sim}{A}}(\underset{\sim}{v}) + T_{\underset{\sim}{A}}(\underset{\sim}{w})$$
$$(ii) \quad T_{\underset{\sim}{A}}(c\underset{\sim}{v}) = cT_{\underset{\sim}{A}}(\underset{\sim}{v})$$

where $c \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$.

We say that

T_A respects the two operations in the vector space $\mathbb{R}^n$.

In this section we study such maps between vector spaces.

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

$\mathbb{K}$

Definition Let X, Y be vector spaces over $\mathbb{K}$.
A linear transformation $T: X \longrightarrow Y$ is a function satisfying
$$T(x+y) = T(x) + T(y) \quad \text{and} \quad T(\alpha x) = \alpha T(x)$$
for all $x, y \in X$ and $\alpha \in \mathbb{K}$.

Note the following properties which are easy to verify.

1. If $T: X \to Y$ is linear, then $T(0) = 0$;
2. $T: X \to Y$ is linear if and only if
$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$
for all $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$

Remark $T: X \longrightarrow Y$ is linear if and only if, for $x_1, x_2, \ldots, x_n \in X$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$, we have
$$T\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i T(x_i)$$

Examples (1) If X is any vector space over $\mathbb{K}$, the identity transformation I defined by
$$Ix = x, \quad \text{for all } x \in X$$
and the zero transformation 0, defined by
$$0x = 0 \quad \text{for all } x \in X$$
are linear transformations.

(2) Let $c \in \mathbb{R}$, $X = Y = \mathbb{R}^2$. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c\,x_1 \\ c\,x_2 \end{bmatrix} = c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

T 'stretches' each vector $\underset{\sim}{x} = (x_1, x_2)^T$ in $\mathbb{R}^2$ to $c\underset{\sim}{x}$.

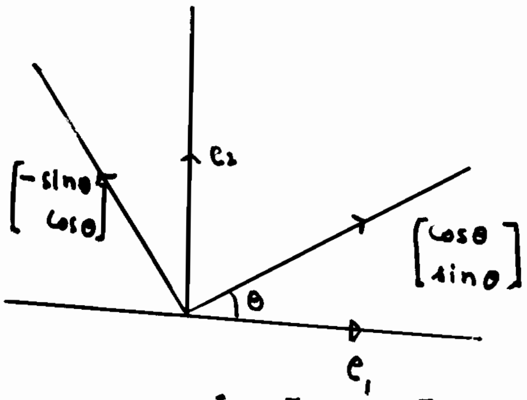
Hence $T(\underset{\sim}{x} + \underset{\sim}{y}) = c(\underset{\sim}{x} + \underset{\sim}{y}) = c\underset{\sim}{x} + c\underset{\sim}{y} = T(\underset{\sim}{x}) + T(\underset{\sim}{y})$

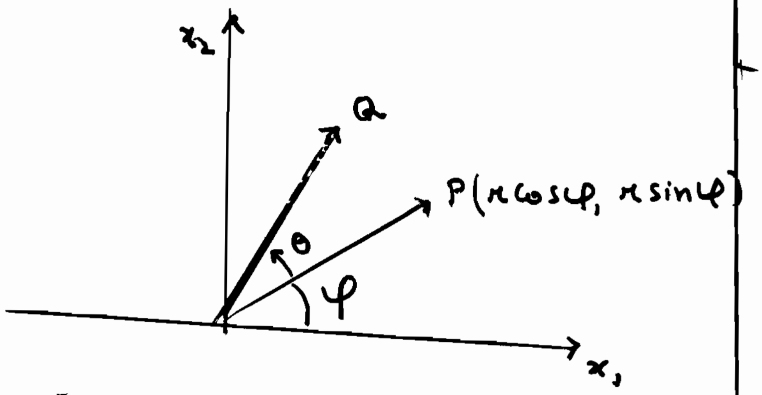
$$T(\alpha \underset{\sim}{x}) = c(\alpha \underset{\sim}{x}) = \alpha(c\underset{\sim}{x}) = \alpha T(\underset{\sim}{x}),$$

and this shows that T is a linear transformation.

(3) Rotation. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$





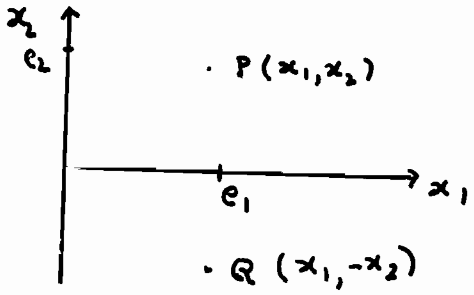
$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix} = \begin{bmatrix} r\cos(\theta+\varphi) \\ r\sin(\theta+\varphi) \end{bmatrix}$$

$T(\underset{\sim}{e_1}) = (\cos\theta, \sin\theta)^t$ and $T(\underset{\sim}{e_2}) = (-\sin\theta, \cos\theta)^t$.

T rotates the whole space by θ.

(4) (Reflection) Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$



T is a reflection about x_1-axis. Note that

$T\underset{\sim}{e_1} = \underset{\sim}{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T\underset{\sim}{e_2} = -\underset{\sim}{e_2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

T is a linear transformation.

(5) (Projection) Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

T is a linear transformation called projection onto x_1-axis. Note that $T\underset{\sim}{e_1} = \underset{\sim}{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T\underset{\sim}{e_2} = \underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

5. Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

As before we consider the vectors in $\mathbb{R}^n$, $\mathbb{R}^m$ as column vectors. We have already seen that any matrix $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ induces a linear transformation $T_{\underset{\sim}{A}} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by

$$T_{\underset{\sim}{A}}(\underset{\sim}{x}) = \underset{\sim}{A}\,\underset{\sim}{x}, \quad \underset{\sim}{x} \in \mathbb{R}^n.$$

Conversely, we show that every linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ can be written as $T_{\underset{\sim}{A}}$ for some $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ (uniquely determined by the linear transformation)

Let us choose for $\mathbb{R}^n$, the 'standard basis' $\{\underset{\sim}{e}_1, \ldots\ldots, \underset{\sim}{e}_n\}$, where $\underset{\sim}{e}_i := (0, 0, \ldots, \underbrace{1}_{i^{th}\ \text{place}}, \ldots 0)^t$

Then any $\underset{\sim}{x} \in \mathbb{R}^n$ has a unique representation

$$\underset{\sim}{x} = x_1 \underset{\sim}{e}_1 + \cdots\cdot + x_n \underset{\sim}{e}_n .$$

Now if $T: \mathbb{R}^n \to \mathbb{R}^m$ is any linear transformation, then

$$T(\underset{\sim}{x}) = x_1 T(\underset{\sim}{e}_1) + \cdots\cdot + x_n T(\underset{\sim}{e}_n) .$$

If we write

$$T(\underset{\sim}{e}_i) = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix}, \quad i = 1, \ldots\ldots, n$$

Then

$$T(\underset{\sim}{x}) = \left(T(\underset{\sim}{e}_1) \cdots\cdots T(\underset{\sim}{e}_n) \right) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underset{\sim}{A}\,\underset{\sim}{x}, \quad \underset{\sim}{x} \in \mathbb{R}^n,$$

where $\underset{\sim}{A} = (T\underset{\sim}{e}_1, \ldots\ldots, T\underset{\sim}{e}_n) = (a_{ij})$ is an $m \times n$ matrix.

Thus $T(\underset{\sim}{x}) = \underset{\sim}{A}\,\underset{\sim}{x} = T_{\underset{\sim}{A}}(\underset{\sim}{x})$, for all $\underset{\sim}{x} \in \mathbb{R}^n$.

Summerizing, we conclude that

$\boxed{\text{there is a one-to-one correspondence between the set of all linear transformations from } \mathbb{R}^n \text{ to } \mathbb{R}^m \text{ and the set } \mathbb{R}^{m\times n} \text{ of all } m\times n \text{ matrices with real entries, when standard basis are taken in } \mathbb{R}^n \text{ and } \mathbb{R}^m.}$

We give below some more examples of linear ~~spaces~~ transformation.

Example 6 Let $T : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{n\times m}$ be defined by $T(\underset{\sim}{A}) = \underset{\sim}{A}^t$, $\underset{\sim}{A} \in \mathbb{R}^{m\times n}$.

It is easily verified that T is a linear transformation.

Example 7 Let $T : \mathbb{K}^{n\times n} \longrightarrow \mathbb{K}$ be defined by

$$T(\underset{\sim}{A}) = \text{trace}(\underset{\sim}{A}) = \sum_{k=1}^{n} a_{kk}$$

for $\underset{\sim}{A} = (a_{ij}) \in \mathbb{K}^{n\times n}$. It is easy to verify that ~~A~~ T is a linear transformation.

Example 8 Let $X = \mathbb{P}_n(\mathbb{R})$, $Y = \mathbb{P}_{n-1}(\mathbb{R})$ and $D : X \longrightarrow Y$ be defined by

$D(p) = p'$, $p \in \mathbb{P}_n(\mathbb{R})$.

It is easy to verify that D is a linear transformation.

Example 9 Let $X = \mathcal{C}[a,b]$, $Y =$ the space of real-valued functions f defined on $[a,b]$ that are differentiable in (a,b). Define $T : X \longrightarrow Y$ by

$$T(f)(x) = \int_a^x f(t)\,dt \quad , \quad f \in \mathcal{C}[a,b]$$

By second fundamental thm of integral calculus $T(f) = g \in Y$. It is easy to verify that T is a linear transformation.

The next example occurs frequently in differential equations.

Example 10 Let $n \in \mathbb{N}$ and $X = \mathcal{C}^{(n)}[a,b]$.

Let $X = \mathcal{C}[a,b]$. Let

$D : X \longrightarrow Y$ be defined by

$D(f) = f'$, $f \in \mathcal{C}^{(n)}[a,b]$.

Then $D^2(f) = f''$, ... $D^{(n)}(f) = f^{(n)}$

and $D^2 : X \longrightarrow Y$, ... , $D^{(n)} : X \longrightarrow Y$

Let $T : X \longrightarrow Y$ be defined by

$$T = a_0 + a_1 D + \cdots + a_n D^n$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Then T maps $\mathcal{C}^{(n)}[a,b]$ into $\mathcal{C}[a,b]$. It is easily verified that T is a linear transformation. It is usually called a Linear differential operator with constant coefficients.