Proposition Let $\underset{\sim}{A} \in \mathbb{R}^{m\times n}$ and $\underset{\sim}{U}$ be the row echelon form of $\underset{\sim}{A}$. Then we have:

(i) $r(\underset{\sim}{A}) = r(\underset{\sim}{U})$, Hence $\dim r(\underset{\sim}{A}) = \dim r(\underset{\sim}{U})$;

(ii) $\dim c(\underset{\sim}{A}) = \dim c(\underset{\sim}{U})$.

Proof. (i) We first note that if u, v are vectors in a vector space X then

$$\text{span}\{u, v\} = \text{span}\{\alpha u, \beta v\} = \text{span}\{u, v+\alpha u\}$$

where α, β are nonzero real numbers. Indeed, let $w = au + b(v+\alpha u)$ where $a, b \in \mathbb{R}$. Then

$$w = (a+b\alpha)u + bv \in \text{span}\{u, v\}.$$

On the other hand, if $c, d \in \mathbb{R}$, Then

$$\begin{aligned} cu + dv &= cu + d(v+\alpha u) - d\alpha u \\ &= (c-d\alpha)u + d(v+\alpha u) \in \text{span}\{u, v+\alpha u\}. \end{aligned}$$

It is easy to see that the first two linear spans are the same. Thus the elementary row operations do not alter the row space of $\underset{\sim}{A}$.

(ii) Suppose $\dim c(\underset{\sim}{A}) = r$.

Without loss of generality, we may assume that the first r columns $\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_r$ of $\underset{\sim}{A}$ are L.I.

Let $\underset{\sim}{B}$ be an invertible $m \times m$ matrix such that

$$\underset{\sim}{U} = \underset{\sim}{B}\,\underset{\sim}{A}.$$

By the previous lemma, the columns

$\underset{\sim}{B}\,\underset{\sim}{c}_1, \ldots\ldots, \underset{\sim}{B}\,\underset{\sim}{c}_r$ of $\underset{\sim}{B}\,\underset{\sim}{A}$ are L.I.

Hence $\dim c(\underset{\sim}{U}) \geq r = \dim c(\underset{\sim}{A})$.

On the other hand, since $\underset{\sim}{A} = \underset{\sim}{B}^{-1}\underset{\sim}{U}$,

$\dim c(\underset{\sim}{A}) \geq \dim c(\underset{\sim}{U})$.

Thus, $\dim c(\underset{\sim}{U}) = \dim c(\underset{\sim}{A})$. ∎

Theorem Let $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$. Then

$$\dim r(\underset{\sim}{A}) = \dim c(\underset{\sim}{A}).$$

<u>Proof</u>. We give an argument in a special case. The general argument is similar.

Suppose that the row echelon form $\underset{\sim}{U}$ of $\underset{\sim}{A}$ looks like:

$$\underset{\sim}{U} = \begin{bmatrix} d_1 & * & * & * & * & * \\ 0 & 0 & 0 & d_2 & * & * \\ 0 & 0 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where d_1, d_2, d_3 are the pivots.

Take a linear combination of the columns containing the pivots:

$$\alpha_1 \begin{bmatrix} d_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} * \\ d_2 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} * \\ * \\ d_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 d_1 + \cdots \\ \alpha_2 d_2 + \cdots \\ \alpha_3 d_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By back substitution, we obtain $\alpha_3 = \alpha_2 = \alpha_1 = 0$.

Thus the columns containing the pivots are L.I.

In a like manner, the nonzero rows of U (rows containing the pivots) are L.I.

Hence $\dim r(\underset{\sim}{U}) = \dim c(\underset{\sim}{U})$.

In conjunction with the previous proposition, we obtain

$$\dim r(\underset{\sim}{A}) = \dim r(\underset{\sim}{U}) = \dim c(\underset{\sim}{U}) = \dim c(\underset{\sim}{A}). \quad \blacksquare$$

Definition The rank of an $m\times n$ matrix $\underset{\sim}{A}$ denoted by rank$(\underset{\sim}{A})$ is $\dim \mathcal{R}(\underset{\sim}{A}) = \dim \mathcal{C}(\underset{\sim}{A})$. The nullity of $\underset{\sim}{A}$ denoted by nullity $(\underset{\sim}{A})$ is the dimension of the null space $\mathcal{N}(\underset{\sim}{A})$ of $\underset{\sim}{A}$

The rank-nullity Theorem

Theorem :

Let A be an $m \times n$ matrix. Then

$$\text{rank}\, A + \text{ nullity } A = n.$$

Proof :

Suppose that rank $A = r$. Hence there are r linearly independent columns in A. For simplicity of argument we assume that the first r columns of A are linearly independent.

The columns of A are $Ae_1, Ae_2, \ldots, Ae_n$.

Since the column space $\mathcal{C}(A)$ of A is the linear span of $Ae_1, Ae_2, \ldots, Ae_r$,

we can express $Ae_{r+1}, Ae_{r+2}, \ldots, Ae_n$, as linear combinations of $Ae_1, Ae_2, \ldots, Ae_r$. We write

$$\begin{aligned} Ae_{r+1} &= \alpha_{r+11}Ae_1 + \alpha_{r+12}Ae_2 + \cdots + \alpha_{r+1r}Ae_r \\ Ae_{r+2} &= \alpha_{r+21}Ae_1 + \alpha_{r+22}Ae_2 + \cdots + \alpha_{r+2r}Ae_r \\ A_{e_n} &= \alpha_{n1}Ae_1 + \alpha_{n2}Ae_2 + \cdots + \alpha_{nr}Ae_r. \end{aligned}$$

rank$(\underset{\sim}{A})$ = maximum no. of L.I. row vectors of $\underset{\sim}{A}$

= maximum no. of L.I. column vectors of $\underset{\sim}{A}$

If $\underset{\sim}{A}$ is an $m\times n$ matrix, rank$(\underset{\sim}{A}) \le \min\{m, n\}$.

Hence for $j = r+1, r+2, \ldots, n$,

$$A\left[e_j - \sum_{k=1}^{r} \alpha_{jk} e_k\right] = 0$$

Hence for $j = r+1, \ldots, n$,

$$v_j =: e_j - \sum_{k=1}^{r} \alpha_{jk} e_k \in \mathcal{N}(A)$$

We now show that $\{v_{r+1}, v_{r+2}, \ldots, v_n\}$ is a basis of $\mathcal{N}(A)$.

First we show that

any vector v satisfying $Av = 0$ is a linear combination of $v_{r+1}, v_{r+2}, \ldots, v_n$.

Let $\beta_1, \beta_2, \ldots, \beta_n$ be scalars such that
$v = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n$.

Then
$$\begin{aligned}
Av &= \beta_1 Ae_1 + \beta_2 Ae_2 + \cdots + \beta_n Ae_n \\
&= \beta_1 Ae_1 + \cdots + \beta_r Ae_r + \sum_{j=r+1}^{n} \beta_j Ae_j \\
&= \sum_{k=1}^{r} \beta_k Ae_k + \sum_{j=r+1}^{n} \beta_j \left[\sum_{k=1}^{r} \alpha_{jk} Ae_k\right] \\
&= \sum_{k=1}^{r} \left[\beta_k + \sum_{j=r+1}^{n} \beta_j \alpha_{jk}\right] Ae_k
\end{aligned}$$

Since $Ae_1, Ae_2, \ldots, Ae_r$ are linearly independent,

$$\beta_k = -\sum_{j=r+1}^{n} \beta_j \alpha_{jk} \quad \text{for } k = 1, 2, \ldots, r.$$

Substitute this into the expression for v

$$\begin{aligned} v &= \sum_{k=1}^{r} \beta_k e_k + \sum_{k=r+1}^{n} \beta_k e_k \\ &= \sum_{k=1}^{r} \left(-\sum_{j=r+1}^{n} \beta_j \alpha_{jk} \right) e_k + \sum_{j=r+1}^{n} \beta_j e_j \\ &= \sum_{j=r+1}^{n} \beta_j \left[e_j - \sum_{k=1}^{r} \alpha_{jk} e_k \right] \\ &= \sum_{j=r+1}^{n} \beta_j v_j \end{aligned}$$

It remains to show $v_{r+1}, v_{r+2}, \ldots, v_n$ are linearly independent. Suppose that $a_{r+1}, \ldots, a_n \in I\!R$ and

$$a_{r+1} \left[e_{r+1} - \sum_{k=1}^{r} \alpha_{r+1k} e_k \right] + \cdots + a_n \left[e_n - \sum_{k=1}^{r} \alpha_{nk} e_k \right] = 0.$$

Hence

$$a_{r+1} e_{r+1} + a_{r+2} e_{r+2} + \cdots + a_n e_n + (\ )e_1 + \cdots + (\ )e_r = 0.$$

Therefore $a_{r+1} = a_{r+2} = \cdots = a_n = 0$ by linear independence of $e_1, e_2, \ldots, e_n$. Hence $v_{r+1}, \ldots, v_n$ are linearly independent. ∎

Fundamental Theorem for Systems of Linear Equations

Theorem Consider the following system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \text{or } \underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b} \qquad (1)$$

(i) (Existence) The system admits a solution iff

$$\operatorname{rank}(\underset{\sim}{A}) = \operatorname{rank}(\tilde{\underset{\sim}{A}})$$

where $\tilde{\underset{\sim}{A}} = (\underset{\sim}{A}, \underset{\sim}{b})$ denotes the augmented matrix of the system (1).

(ii) (Uniqueness) The system admits a unique solution if and only if

$$\operatorname{rank}(\underset{\sim}{A}) = \operatorname{rank}(\tilde{\underset{\sim}{A}}) = n$$

(iii) If $\operatorname{rank}(\underset{\sim}{A}) = \operatorname{rank}(\tilde{\underset{\sim}{A}}) = r < n$, then the system admits infinitely many solutions

Proof. Let $\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_n$ be the columns of $\underset{\sim}{A}$. Then

$$\underset{\sim}{A}\,\underset{\sim}{x} = (\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_n)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\underset{\sim}{c}_1 + \cdots + x_n\underset{\sim}{c}_n$$

Hence if $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$ has a solution $x_1 = a_1, \ldots x_n = a_n$, Then $a_1\underset{\sim}{c}_1 + a_2\underset{\sim}{c}_2 + \cdots + a_n\underset{\sim}{c}_n = \underset{\sim}{b}$.

This implies that $\underset{\sim}{b} \in c(\underset{\sim}{A})$ so $\underset{\sim}{A}$ and $\underset{\sim}{\tilde{A}}$ have same column space: $c(\underset{\sim}{A}) = c(\underset{\sim}{\tilde{A}})$. Thus they have equal rank. Conversely, if $\text{rank}(\underset{\sim}{A}) = \text{rank}(\underset{\sim}{\tilde{A}})$, then $\underset{\sim}{b} \in c(\underset{\sim}{A})$. Hence $\underset{\sim}{b} = d_1\underset{\sim}{c}_1 + \cdots + d_n\underset{\sim}{c}_n$ for some scalars $d_1, d_2, \ldots, d_n$. Then

$$d_1\underset{\sim}{c}_1 + \cdots + d_n\underset{\sim}{c}_n = \underset{\sim}{A}\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \underset{\sim}{b}.$$

Hence $x_1 = d_1, \ldots x_n = d_n$ is a solution.

(ii) Let $\text{rank}(\underset{\sim}{A}) = \text{rank}(\underset{\sim}{\tilde{A}}) = n$. Then by the rank-nullity theorem,

$$\text{nullity}(\underset{\sim}{A}) = 0.$$

Hence $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{0}$ has unique solution, namely, $x_1 = \cdots = x_n = 0$.

If $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b} = \underset{\sim}{A}\,\underset{\sim}{y}$ then $\underset{\sim}{A}(\underset{\sim}{x}-\underset{\sim}{y}) = \underset{\sim}{0}$. Hence $\underset{\sim}{x}-\underset{\sim}{y} = \underset{\sim}{0} \Rightarrow \underset{\sim}{x} = \underset{\sim}{y}$. This establishes uniqueness.

(iii) Suppose $rank(\underset{\sim}{A}) = rank(\tilde{\underset{\sim}{A}}) = r < n$.

Then $n - r = \dim \mathcal{N}(\underset{\sim}{A}) > 0$.

Thus $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{0}$ has infinitely many solutions.

Let $\underset{\sim}{c} \in \mathbb{R}^n$ and $\underset{\sim}{A}\,\underset{\sim}{c} = \underset{\sim}{b}$.

If $\underset{\sim}{x} \in \mathbb{R}^n$ satisfies $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$ Then

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{A}\,\underset{\sim}{c} \Rightarrow \underset{\sim}{A}(\underset{\sim}{x}-\underset{\sim}{c}) = \underset{\sim}{0} \Rightarrow \underset{\sim}{x}-\underset{\sim}{c} \in \mathcal{N}(\underset{\sim}{A})$$

$$\Rightarrow \underset{\sim}{x} \in \underset{\sim}{c} + \mathcal{N}(\underset{\sim}{A}).$$

Thus any solution $\underset{\sim}{x}$ of $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$ is in the set

$$\underset{\sim}{c} + \mathcal{N}(\underset{\sim}{A}) = \left\{ \underset{\sim}{c} + \underset{\sim}{y} \;:\; \underset{\sim}{A}\,\underset{\sim}{y} = \underset{\sim}{0} \right\}.$$

Conversely, if we take an element $\underset{\sim}{c} + \underset{\sim}{y}$ of $\underset{\sim}{c} + \mathcal{N}(\underset{\sim}{A})$ where $\underset{\sim}{A}\,\underset{\sim}{y} = \underset{\sim}{0}$, then $\underset{\sim}{A}(\underset{\sim}{c}+\underset{\sim}{y}) = \underset{\sim}{A}\,\underset{\sim}{c} + \underset{\sim}{A}\,\underset{\sim}{y} = \underset{\sim}{b}$.

Hence $\underset{\sim}{c}+\underset{\sim}{y}$ is a solution of $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$. We conclude that the solution set of $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$ is the set $\underset{\sim}{c} + \mathcal{N}(\underset{\sim}{A})$ which contains infinitely many elements. ∎

Let $c \in I\!\!R^n$ and $Ac = b$.

Then we have seen before that

all the solutions of $Ax = b$ are in the set

$c + \mathcal{N}(A) = \{c + x \mid Ax = 0\}$.

Hence $Ax = b$ has infinitely many solutions. ∎

Corollary A homogeneous system of linear equations with fewer equations than unknowns always admits nontrivial solutions.

Example : Consider the system of linear equations

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 4 & 5 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

By GEM we transform the coefficient matrix A to its row echelon form :

$$U = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1/4 & -1/4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The equations for $Ux = 0$ are

$$\begin{aligned} x_1 + 2x_3 + x_4 &= 0 \\ x_2 - \frac{1}{4}x_3 - \frac{1}{4}x_4 &= 0 \\ x_5 &= 0 \end{aligned}$$

The first, second and the fifth column of A contain the pivots.

So they are linearly independent.

Hence the corresponding columns of A are linearly independent.

Thus $r(A) = 3$, and nullity $(A) = 2$. We find a basis for the nullspace.

Express the **basic variables** (corresponding to the column containing the pivots) x_1, x_2, x_5 in terms of the **free variables** (non basic variable) x_3 and x_4.

$$\begin{aligned} x_1 &= -2x_3 - x_4 \\ x_2 &= \frac{1}{4}x_3 + \frac{1}{4}x_4 \\ x_5 &= 0 \end{aligned}$$

Hence solutions to $Ax = 0$ have the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_3 - x_4 \\ \frac{1}{4}x_3 + \frac{1}{4}x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ \frac{1}{4} \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ \frac{1}{4} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence a basis for $\mathcal{N}(A)$ is

$$\{(-2, \frac{1}{4}, 1, 0, 0)^t,\ (-1, \frac{1}{4}, 0, 1, 0)^t\}.$$

Rank in terms of determinants

Lemma :

Let $C_1, C_2, \ldots, C_n$ be the columns of an $n \times n$ matrix A. Then they are linearly independent iff $\det A$ is nonzero.

Proof :

Suppose that all the columns of A are linearly dependent. Then all the rows of A are also linearly dependent. Hence by the multilinearity property, $\det A$ is zero. Conversely suppose that $\det A = 0$.

$C_1, C_2, \ldots, C_n$ are L.D. $\Leftrightarrow$ one of them can be expressed as a nontrivial linear combination of the others, say $C_1 = \alpha_2 C_2 + \cdots + \alpha_n C_n$. Then

$$\det(C_1, \ldots, C_n) = \det(\alpha_2 C_2 + \cdots + \alpha_n C_n, C_2, \ldots, C_n)$$
$$= \alpha_2 \det(C_2, C_2, \ldots, C_n) + \alpha_3 \det(C_3, C_2, C_3, \ldots, C_n) + \cdots = 0$$

Let U be the row echelon form of A. Then $\det U = 0$.
Hence U has a zero diagonal entry.
Hence there are atmost $n - 1$ pivots in U.
Hence rank $A = \text{rank} U \leq n - 1$.
Hence the columns of A are linearly dependent.

Theorem :

An $m \times n$ matrix A has rank $r \geq 1$ iff $\det M \neq 0$ for some order r minor M of A and $\det N = 0$ for all order $r + 1$ minors N of A.

Proof :

Let U be the row echelon form of A.
Then the first r rows of U are linearly independent if rank $A = r$.
Let M_1 denote the order r minor consisting of the first r rows and the r columns of U containing the pivots.
Then M_1 is an upper triangular matrix whose diagonal entries are the pivots of U.
Hence $\det M_1$ = product of the pivots is nonzero.

Therefore the columns of M_1 are linearly independent.

Let M be the corresponding minor of A.

Then all the columns of M are also linearly independent.

Hence $\det M \neq 0$. Any minor N_1 of U of order $r+1$ must contain a zero row,

hence $\det N_1 = 0$.

Therefore $\det N = 0$ for all order $r+1$ minors N of A.

Conversely suppose that A has an order r minor M with nonzero determinant and all minors of order $r+1$ of A have zero determinant.

Let M_1, be the corresponding minor of U.

Then M_1 has r nonzero rows.

Hence $\operatorname{rank} U \geq r$.

On the other hand if U has $r+1$ nonzero rows then

A will have an order $r+1$ minor with nonzero determinant.

∎

<u>Theorem</u> p vectors $\underset{\sim}{R}_1 = (R_{11}, \cdots, R_{1n}), \cdots, \underset{\sim}{R}_p = (R_{p1}, \cdots, R_{pn})$ are L.I. if and only if the matrix $\underset{\sim}{A}$ with row vectors $\underset{\sim}{R}_1, \cdots, \underset{\sim}{R}_p$: $\underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{R}_1 \\ \vdots \\ \underset{\sim}{R}_p \end{bmatrix}$ has rank p;

they are L.D. (linearly dependent) if and only if

$\text{Rank}(\underset{\sim}{A}) < p$.

If $n < p$, then $\text{rank}(\underset{\sim}{A}) \le \min\{n, p\} < p$.

Hence the p vectors $\underset{\sim}{R}_1, \cdots, \underset{\sim}{R}_p$ are L.D.

<u>Corollary</u> p vectors with $n < p$ components (of $\mathbb{R}^n$) are always L.D.

Determinants

Second-Order Determinants

If $A = [a_{ij}]_{2\times2} \in M_{2\times2}(\mathbb{K})$, Then the determinant of A

written $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is the number $a_{11}a_{22} - a_{12}a_{21}$

Third-Order Determinants

If $A = [a_{ij}]_{3\times3} \in M_{3\times3}(\mathbb{K})$, then its determinant is defined by

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

equivalently

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

It is a fundamental fact, that this defines a unique number.

Here $M_{2\times2}(\mathbb{K})$ is the same as what we denoted by

$\mathbb{K}^{2\times2}$:= collection of 2×2 matrices with entries in

$\mathbb{K}$ = either $\mathbb{R}$ or $\mathbb{C}$.

Determinant of Any Order n

Let $A = [a_{ij}]_{n \times n} \in M_{n \times n}(\mathbb{K})$. The determinant of A written

$$\det A = \begin{vmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{vmatrix}$$

is defined for $n = 1$ by $\det A = a_{11}$, and for $n \geq 2$ by

$$\boxed{\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in} = \sum_{k=1}^{n} a_{ik} C_{ik}}$$

Expansion by i^{th} row

or

$$\boxed{\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots \cdots + a_{nj} C_{nj} = \sum_{k=1}^{n} a_{kj} C_{kj}}$$

Expansion by the j^{th} Column

Let $\underset{\sim}{A} = (a_{ij})$ be an $n \times n$ matrix, we define $\det \underset{\sim}{A}$ inductively. For $n=1$, $\det (a_{11}) = a_{11}$. Let $n > 1$. Suppose we know $\det \underset{\sim}{A}$ where $\underset{\sim}{A}$ is $(n-1) \times (n-1)$.

Let M_{ij} = the determinant of the submatrix $(n-1)\times(n-1)$ size obtained by erasing the ith row and the jth column. M_{ij} is called the minor of the element a_{ij}.

The cofactor of a_{ij} denoted by C_{ij} is defined by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Definition

$$\det \underset{\sim}{A} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots\cdots + a_{in} C_{in}$$

(Expansion by the ith row)

$$= a_{1j} C_{1j} + a_{2j} C_{2j} \cdots\cdots + a_{nj} C_{nj}$$

(Expansion by the jth column)

Here $\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$ $\rightarrow$ called the cofactor of a_{ij}, also written cof a_{ij}.

and M_{ij} called the minor of a_{ij} is the determinant of order $n-1$ obtained by erasing the i^{th} row and the j^{th} column of A

$$\begin{vmatrix} a_{11} \cdots a_{1j} & a_{1n} \\ a_{i1} \cdots a_{ij} \cdots & a_{in} \\ \vdots & \\ a_{n1} \cdots a_{nj} & a_{nn} \end{vmatrix}$$

This definition is unambiguous. It yields the same value for det A no matter which row or column we use in expanding the determinant.

General properties of determinants are the same as the familiar properties of determinants of order 3.

Consider

$$a_{1\ell}C_{1j}+a_{2\ell}C_{2j}+\cdots\cdots+a_{n\ell}C_{nj}=\sum_{k=1}^{n}a_{k\ell}C_{kj}$$

$$\boxed{=\sum_{k=1}^{n}a_{k\ell}\,\mathrm{cof}(a_{kj}) = \delta_{\ell j}\det(A)=\begin{cases}0, & \text{if } \ell\neq j\\ \det A, & \text{if } \ell=j.\end{cases}}$$

In fact, if $\ell \neq j$, then it is the expansion of the determinant in which two columns: ℓ^{th} and j^{th} columns are identical. Hence its value $=0$.
If $\ell = j$, then this is precisely the expansion of $\det A$.
On the same lines

$$\sum_{k=1}^{n}a_{\ell k}C_{ik}=\sum_{k=1}^{n}a_{\ell k}\,\mathrm{cof}(a_{ik})=\delta_{\ell i}\det(A)$$

$$=\begin{cases}0, & \text{if } \ell\neq i\\ \det A, & \text{if } \ell=i.\end{cases}$$

$$\sum_{k=1}^{n}a_{k\ell}C_{kj}=\delta_{\ell j}\det \underset{\sim}{A}=\begin{cases}0, & \ell\neq j\\ \det A, & \ell=j\end{cases}$$

Determinant of transpose of a matrix

Theorem For any $n \times n$ matrix $\underset{\sim}{A}$,

$$\det \underset{\sim}{A} = \det \underset{\sim}{A}^t .$$

Proof. Induct on n. The theorem holds for $n=1$ and $n=2$ trivially. Assume that $\det \underset{\sim}{A} = \det \underset{\sim}{A}^t$ for all $(n-1)\times(n-1)$ matrices. Let $\underset{\sim}{B} = \underset{\sim}{A}^t = [b_{ij}] = [a_{ji}]$. Then expanding by the minors of the first column

$$\det \underset{\sim}{A} = a_{11} \det A_{11} - a_{21} \det A_{21} \cdots + (-1)^{n+1} a_{n1} \det A_{n1}$$

Likewise, expanding $\det \underset{\sim}{B}$ by the minors of the first row,

$$\det \underset{\sim}{B} = b_{11} \det B_{11} - b_{12} \det B_{12} + \cdots + (-1)^{n+1} b_{1n} B_{1n}$$

Since $B_{1i} = (A_{i1})^t$ for all i, by induction hypothesis $\det B_{1i} = \det A_{i1}$. Hence $\det \underset{\sim}{A} = \det \underset{\sim}{B}$. ∎

Cramer's Rule

Theorem If the system of equations $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$, where $A \in M_{n\times n}(\mathbb{K})$, $\underset{\sim}{x} = [x_1, \cdots, x_n]^t$ and $\underset{\sim}{b} = [b_1, \cdots b_n]^t$ has a nonzero coefficient determinant $D = \det \underset{\sim}{A}$, then the system has precisely one solution. This sol.^{n} is given by

$$\boxed{x_j = \frac{D_j}{D}, \quad j = 1, 2, \ldots n.}$$

Here D_j is the determinant obtained from D by replacing in D the j^{th} column by the column $[b_1, \cdots, b_n]^t$.

Proof. Multiply the first eq.^{n} by C_{1j}, the second by C_{2j} ...., the n^{th} by C_{nj} and add:

$$C_{1j}(a_{11}x_1 + \cdots + a_{1n}x_n) + \cdots\cdots + C_{nj}(a_{n1}x_1 + \cdots\cdots + a_{nn}x_n)$$
$$= b_1 C_{1j} + \cdots\cdots + b_n C_{nj}.$$

$M_{n\times n}(\mathbb{K}) \equiv \mathbb{K}^{n\times n}$ matrices of size $n\times n$ with entries in $\mathbb{K}$.

Hence

$$(a_{11}C_{1j} \cdots + a_{n1}C_{nj})x_1 + \cdots\cdots + (a_{1n}C_{1j} + \cdots a_{nn}C_{nj})x_n$$
$$= b_1 C_{1j} + \cdots\cdots + b_n C_{nj}$$

Since

$$\sum_{l} a_{kl} C_{kj} = \delta_{lj} \det \underset{\sim}{A} ,$$

We obtain

$$(\det \underset{\sim}{A})\, x_j = D_j ,$$

Hence, if $\det \underset{\sim}{A} \neq 0$, $\quad x_j = \dfrac{D_j}{\det(\underset{\sim}{A})} = \dfrac{D_j}{D}$,

$j = 1, 2, \ldots\ldots, n$.

This also proves that if $\det A \neq 0$, then the system has a unique solution.

∎

Lemma If $\underset{\sim}{A} \in M_{n\times n}(\mathbb{K})$ is an upper diagonal or a lower diagonal matrix, Then

$\det \underset{\sim}{A}$ = The product of the diagonal elements.

Proof. Let $\underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ & & & & \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$

be an upper diagonal matrix. Expand $\det \underset{\sim}{A}$ by the minors of the first column, to obtain

$$\det \underset{\sim}{A} = a_{11} \det \begin{bmatrix} a_{22} & \cdots & a_{2n} \\ 0 & & \\ \vdots & & \\ 0 & & a_{nn} \end{bmatrix} \xrightarrow[\text{the process}]{\text{Continue}}$$

$$= a_{11}\, a_{22} \cdots\cdots a_{nn}.$$

∎

Properties:

$$\boxed{\begin{array}{ll}(i) & \det \underset{\sim}{A} = \det \underset{\sim}{A}^t \\ (ii) & \det(\underset{\sim}{A}\underset{\sim}{B}) = \det(\underset{\sim}{A}) \cdot \det(\underset{\sim}{B})\end{array}}$$

(iii) If $\underset{\sim}{A}$ is invertible, then $\det \underset{\sim}{A} \neq 0$ and $\det \underset{\sim}{A}^{-1} = \frac{1}{\det \underset{\sim}{A}}$.

Rank and Determinants

Theorem. Let $\underset{\sim}{A} = (a_{ij})$ be an $n \times n$ matrix. Then

$$\boxed{\operatorname{rank} \underset{\sim}{A} = \max\left\{ r : \det \underset{\sim}{B} \neq 0, \text{ where } \underset{\sim}{B} \text{ is some } r \times r \text{ square submatrix of } \underset{\sim}{A} \right\}}$$

In particular, for a square matrix $\underset{\sim}{A}_{n \times n}$, ~~det A ≠ 0~~

$\det \underset{\sim}{A} \neq 0 \iff \operatorname{rank} \underset{\sim}{A} = n \iff \underset{\sim}{A}$ is invertible

Examples

(1) The matrix

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \\ 1 & 2 & \cdots & n \end{bmatrix}_{(n+1)\times n}$$

has rank $= n$. In fact $\det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} = 1 \neq 0$

(2) The matrix

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 5 & 4 & 3 \\ 4 & 7 & 7 & 3 \\ 2 & 3 & 1 & 3 \end{bmatrix}$$

has rank 2.