Problem-1

$$\Delta g = \begin{bmatrix} 2(x_1+1) & -\mu_3 & +\mu_1 \\ 2(x_2-2) & +y_2(4) & -\mu_4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2-\mu_3 \\ -2+\mu_2 \end{bmatrix} = 0$$

$\therefore \mu_2 = 3, \mu_3 = 0$

$$\therefore \partial h = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\therefore$ positive definite as $\|H\| \neq 0$

Hence it is global minimum.

Problem-2

Taking random points,

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \mu_1 \begin{bmatrix} -3(1-x_1)^2 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1+\mu_1 - 3(1-x_1 s)^2 \\ \mu_1 - \mu_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$= -1 + \mu_1 - 3 + 3x_1^2 = 0$

$\mu_1 = \mu_2$

:- Problem

$$\max_{x_1,x_2,x_3} = f = x_1x_2 + x_2x_3 + x_1x_3$$
$$h = x_1 + x_2 + x_3 - 3 = 0$$

$3 - d_1 - s = 0$

$s d = x_1$

$d s = x_2 + x_3$

$$\therefore \min_{x_1,x_2,x_3} f = -x_1x_2 - x_2x_3 - x_1x_1$$
$$h = 3 - x_1 - x_2 - x_3 = 0$$

$n = 3$

$m = 1$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial d} - \left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d}$$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial x_1} = x_2 + x_3$$

$$\frac{\partial f}{\partial s} = \frac{\partial^2 f}{\partial x_2 \partial x_3} = \frac{\partial f}{\partial x_3}(x_1 + x_3) = \cancel{\frac{\partial f}{}}\ \cancel{x_3} = 1$$

$$\frac{\partial^2 f}{\partial x_3 \partial x_2} = \frac{\partial f}{\partial x_2}(x_2 + x_1) = 1$$

$$\therefore \frac{\partial f}{\partial s} = 1$$

$$\frac{\partial h}{\partial s} = \frac{\partial^2 h}{\partial x_2 \partial x_3} = \frac{\partial h}{\partial x_3}(-1) = 0$$

$$\frac{\partial^2 h}{\partial x_3 \partial x_2} = \frac{\partial h}{\partial x_2}(-1) = 0$$

$$\cancel{\frac{\partial h}{\partial d} =} -1$$

$$\therefore \frac{\partial f}{\partial d} = (x_2 + x_3) - 1 \times \frac{1}{0}(-1) = 0$$

$$\therefore x_2 + x_3 = 0$$

$$\therefore x_2 = -x_3$$ (Only possible (x = 0)].

$$\therefore 3 \cancel{-x_1} + \cancel{x_3 - x_2} = 0$$

$$\therefore 3 - x_1 = 0$$

$$\therefore x_1 = 3$$

$$\therefore x_2, x_3 = 0$$

$$\lambda^T = -\left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial h}{\partial s}\right)^{-1}$$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial d} + \lambda^T \frac{\partial h}{\partial d}$$

$$0 = \frac{\partial f}{\partial s} + \lambda^T \frac{\partial h}{\partial s}.$$

$$\therefore L(x_1, x_2, x_3) = (-x_1x_2 - x_2x_3 - x_1x_3) + \lambda(3 - x_1 - x_2 - x_3)$$

$$\therefore \frac{\partial L}{\partial x} = \begin{bmatrix} -x_2 - x_3 + \lambda(-1) \\ -x_1 - x_3 + \lambda(-1) \\ -x_2 - x_1 + \lambda(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\cancel{-x}\ x_2 + x_3 + \lambda = 0$$
$$x_1 + x_3 + \lambda = 0$$
$$x_2 + x_1 + \lambda = 0$$

$$\cancel{x_1 +} \quad 3 - x_1 - x_2 - x_3 = 0 \left[\frac{\partial L}{\partial \lambda} = 0 \right]$$
$$3 - x_1 - x_2 + x_3$$

$$\therefore 3 - x_1 + \lambda = 0$$

Problem-4

$\max_{x_1, x_2} f = 2x_1 + bx_2$

$\min_{x_1 x_2} f = -2x_1 - bx_2$

$g_1 = -x_1^2 - x_2^2 + 5 \geq 0$

$g_2 = x_2 - x_1 + 2 \geq 0$

$\boxed{\begin{array}{l} x_1 = d \\ x_2 = s \\ n = 2 \\ m = 2 \end{array}}$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial d} - \left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial h}{\partial s}\right)^{-1} \frac{\partial h}{\partial d}$$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial x_1} = -2$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x_2} = -b$$

$$\frac{\partial h}{\partial s} = \frac{\partial h}{\partial x_2} = 1$$

$$\frac{\partial h}{\partial d} = -1$$

$$\therefore \frac{\partial f}{\partial d} = -2 - (-b) \times \frac{1}{1} \times (-1) = 0$$

$$= -2 - b = 0$$

$$\therefore b = -2$$

The above condition is contradicting as LHS ≠ RHS at the points (0,0). Hence cannot solve using KKT condition.

Problem-5

$$\min_{x_1, x_2, x_3} f = x_1^2 + x_2^2 + x_3^2$$

$$h_1 = \frac{x_1^2}{4} + \frac{x_2^2}{5} + \frac{x_3^2}{25} - 1 = 0$$

$$h_2 = x_1 + x_2 - x_3 = 0$$

$d = x_1$

$s = x_2, x_3$

$n = 3$

$m = 2$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d}$$

$$\boxed{\frac{\partial f}{\partial d} = 2x_1}, \quad \frac{\partial f}{\partial s} = \frac{\partial^2 f}{\partial x_2 \partial x_3} = \frac{\partial f}{\partial x_3}(2x_2) = 0$$

$$\frac{\partial^2 f}{\partial x_3 \partial x_2} = \frac{\partial f}{\partial x_2}(2x_3) = 0$$

$$\boxed{\therefore \frac{\partial f}{\partial s} = 0}$$

$$\frac{\partial h}{\partial s} = \frac{\partial^2 h}{\partial x_2 \partial x_3} = \frac{\partial h}{\partial x}(1) = 0$$

$$\frac{\partial^2 h}{\partial x_3 \partial x_2} = \frac{\partial h}{\partial x}(-1) = 0$$

$$\boxed{\therefore \frac{\partial h}{\partial s} = 0}$$

$\frac{\partial h}{\partial d} = 1$

$\therefore \frac{\partial f}{\partial d} = 2x_1 - (0 \times \frac{1}{0})(1) = 0$

$\therefore 2x_1 = 1$

$x_1 = 1/2$

If h_2 is considered

$\frac{\partial f}{\partial d} = 2x_1 - (0 \times \frac{1}{0})(\frac{1}{2} x_1) = 0$

$= 2x_1 - 1/2 x_1 = 0$

$\therefore 3/2 x_1 = 0$

$\therefore x_1 = 2/3$

Line search:-

$f(\alpha) = f(d_k - \alpha \frac{\partial f}{\partial d}, s_k + \alpha\left(\left(\frac{\partial h}{\partial s}\right)^{-1}\left(\frac{\partial h}{\partial d}\right)\right)\left(\frac{\partial f}{\partial d}\right)^{T})$

$\varphi(\alpha) = f(d_k, s_k) - \alpha t \left(\frac{\partial f}{\partial d}\right)\left(\frac{\partial f}{\partial d}\right)^{T}$

The above condition is contradicting as LHS ≠ RHS at the point (0,0). Hence cannot solve using KKT condition.