MAE 598 HW-2

1)

$f = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

$g = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix}$

$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} = 12 - 16 = -4$

Equating zero gradient

$4x_1 - 4x_2 = 0$

$x_1 = x_2$

$-4x_1 + 3x_2 + 1 = 0$

$-4x_1 + 3x_1 + 1 = 0$

$-x_1 + 1 = 0$

$\therefore x_1 = 1$

$x_2 = 1$

$\therefore$ Saddle pt $(1,1)$.

$\therefore \frac{1}{2}(x - x_0)^T x - 4$

$x(x - x_0) < 0$

($(x - x_0)$ is a scalar value)

To find the direction, the eigen values of H need to be found out.

$\therefore$

$\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$

$\begin{bmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix} = 0$

$(4-\lambda)(3-\lambda) - 16 = 0$

$12 - 4\lambda - 3\lambda + \lambda^2 - 16 = 0$

$\lambda^2 - 7\lambda - 4 = 0$

By solving the above eqn we get the following eigen value.
(7.53, -0.53).

It proves that the saddle point lies in between two directions.

2) $x_1 + 2x_2 + 3x_3 = 1$ in $\mathbb{R}^3$

$\sqrt{(x_1+1)^2 + x_2^2 + (x_3-1)^2}$ (Given point)

$= (x_1+1)^2 + x_2^2 + (x_3-1)^2$

$= x_1^2 + 2x + 1 + x_2^2 + x_3^2 - 2x_3 + 1$

$= x_1^2 + x_2^2 + x_3^2 + 2x_1 - 2x_3 + 2$

$= x_1^2 + x_2^2 + x_3^2 + 2(x_1 - x_3 + 1)$

Sub $x_1 = 1 - 2x_2 - 3x_3$

$= (1-2x_2-3x_3)^2 + x_2^2 + x_3^2 + 2[(1-2x_2-3x_3) - x_3 + 1]$

$= 1 - 2x_2 - 3x_3 - 2x_2 + 4x_2^2 + 6x_2x_3 - 3x_3 + 6x_2x_3 + 9x_3^2$
$+ 2(2 - 2x_2 - 4x_3) + x_2^2 + x_3^2$

$= 1 - 4x_2 - 6x_3 + 4x_2^2 + 9x_3^2 + 12x_2x_3 + 4 - 4x_2 - 4x_2 - 8x_3$

$= 5 - 8x_2 - 14x_3 + 4x_2^2 + 9x_3^2 + 12x_2x_3 + x_2^2 + x_3^2$

$= 5 - 8x_2 - 14x_3 + 5x_2^2 + 10x_3^2 + 12x_2x_3 = 0$

$5(1+x_2^2+2x_3^2) - 2(4x_2+7x_3-6x_2x_3) = 0$

The above eqn is the function to which gradient need to be derived.

$$g = \begin{bmatrix} \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{bmatrix} = \begin{bmatrix} 5(2x_2) - 2(4-6x_3) \\ 5(4x_3) - 2(7-6x_2) \end{bmatrix}$$

$\therefore 10x_2 - 8 + 12x_3 = 0$

$12x_2 - 14 + 20x_3 = 0$

$\Rightarrow 5x_2 + 6x_3 - 4 = 0$ — (1)

$6x_2 + 10x_3 - 7 = 0.$ — (2)

(1) $\times 6$ & (2) $\times 5$

$30x_2 + 36x_3 - 24 = 0$

$-\ 30x_2 + 50x_3 - 35 = 0$

$(-) \quad (+)$

$-14x_3 + 11 = 0$

$14x_3 = 11$

$x_3 = 11/14$

Sub $x_3 = 11/14$ in (1)

$5x_2 + \frac{66}{14} - 4 = 0$

$5x_2 + \frac{10}{14} = 0$

$5x_2 = \frac{-10}{14} = \frac{-5}{7}$

$x_2 = -1/7$

Sub x_2 & x_3 in the given eqn.

$x_1 - \frac{2}{7} + \frac{33}{14} = 1$

$x_1 - \frac{29}{14} = 1$

$\therefore x_1 = \frac{-15}{14}$

$\therefore$ point $= \left(\frac{-15}{14}, \frac{-1}{7}, \frac{11}{14}\right)$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 12 & 20 \end{bmatrix}$$

It is a positive Hessian, so it is a convex problem.

3) $f(x) = x_1^2 + 2x_2^2$ Let us consider $f(x)$ & $g(x)$ as two convex functions

$g(x) = 3x_1^2 + x_2^2$

For $f(x)$, $H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

For $g(x)$, $H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$

As both Hessians of $f(x)$ & $g(x)$ are positive, both are convex functions.

Let a & b be scalar values such as $a=3$ & $b=2$

$a f(x) = 3x_1^2 + 6x_2^2$ — ①
$b g(x) = 6x_1^2 + 2x^2$ — ②

① + ② $= 9x_1^2 + 8x_2^2$

$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ 0 & 16 \end{bmatrix}$ It is a positive Hessian so it is also a convex function.

For $f[g(x)]$ to be a function, the corresponding Hessian needs to be a positive semi-definite one.

4) $f(x) = x_1^2 + 2x_2^2$ $\qquad g = [2x_1 \quad 4x_2]$

$f(x_1) = 1 + 2 = 3$ $\qquad g_0 = [0 \quad 0]$

$f(x_0) = 0$

$f(x) = 3$

$f(x_0) + g_0^T(x_1 - x_0) = 0 + 0(0)$
$= 0$

$f(x) \geq f(x_0) g_0^T(x_1 - x_0)$

Hence Proved

5) $f = \sum_k (a_k^T P - I)^2$

If $d^T H d \geq 0 \ \forall d \neq 0$
then H is PSD

$g = 2(a_k^T P - I) \cdot a_k$
$= 2a_k^T P \cdot a_k - 2I \cdot a_k$
$= 2a_k^T a_k \cdot P - 2I \cdot a_k$

$d^T H d = \sum_k 2 \cdot d^T a_k a_k^T d$

$d^T a_k = V_k$

$= \sum_k 2 \cdot V_k^2$

$H = 2a_k \cdot a_k^T$
$= 2a_k$

It is a convex problem.

For all values of k
$H = \sum_k 2a_k a_k^T$