Problem - 1

$$\nabla g = \begin{bmatrix} 2(x_1+1) & -\mu_3 & +\mu_1 \\ 2(x_2-2) & +y_2(4) & -\mu_4 \end{bmatrix} = 0$$

$$\boxed{\begin{bmatrix} 2-\mu_3 \\ -2+\mu_2 \end{bmatrix} = 0 \qquad \therefore \mu_2 = 3, \mu_3 = 0}$$

$$\therefore \partial h = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\therefore$ positive definite as $\|H\| \neq 0$

Hence it is global minimum.

Problem - 2

Taking random points,

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \mu_1 \begin{bmatrix} -3(1-x_1^2) \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1+\mu_1 -3(1-x_1^2) \\ \mu_1 - \mu_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= -1 + \mu_1 - 3 + 3x_1^2 = 0$$

$$\mu_1 = \mu_2$$

Problem-3

$$\max_{x_1,x_2,x_3} \quad f = x_1x_2 + x_2x_3 + x_1x_3$$
$$\text{st} = h = x_1 + x_2 + x_3 - 3 = 0$$

$$\min \quad f = -x_1x_2 - x_2x_3 - x_1x_3$$
$$h = x_1 + x_2 + x_3 - 3 = 0$$

$$L = f + \lambda h = -x_1x_2 - x_2x_3 - x_1x_3 + \lambda(x_1 + x_2 + x_3 - 3)$$

$$\frac{\partial L}{\partial x} = -x_2 - x_3 + \lambda = 0$$
$$-x_1 - x_3 + \lambda = 0$$
$$-x_2 - x_1 + \lambda = 0$$

$$x_3 = 3 - x_1 - x_2$$

$$\therefore \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = x_2 = x_3 = \cancel{2}\ 1$$
$$\lambda = 2$$

Reduced gradient:-

$d = x_1, x_2 \qquad h = x_1 + x_2 + x_3 - 3$

$s = x_3 = 3 - x_1 - x_2$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial d}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d} = 0$$

$$\frac{\partial f}{\partial d} = \begin{bmatrix} -x_2 - x_3 \\ -x_1 - x_3 \end{bmatrix} - [-x_2 - x_1](1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -x_2 - x_3 \\ -x_1 - x_3 \end{bmatrix} + \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 = x_2 = x_3 = 1$

Problem-4

$\max_{x_1, x_2} f = 2x_1 + bx_2$

$\min_{x_1, x_2} f = -2x_1 - bx_2$

$g_1 = -x_1^2 - x_2^2 + 5 \geq 0$

$g_2 = \cancel{-x_1} x_2 - x_1 + 2 \geq 0$.

$x_1 = d$

$x_2 = s$.

$n = 2$

$m = 2$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial d} - \left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial h}{\partial s}\right)^{-1} \frac{\partial h}{\partial d}$$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial x_1} = -2$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x_2} = -b$$

$$\frac{\partial h}{\partial s} = \frac{\partial h}{\partial x_2} = \cancel{4} - 2x_2$$

$$\frac{\partial h}{\partial d} = \cancel{4} - 2x_1$$

$$\therefore \frac{\partial f}{\partial d} = -2 - (-b) \times \frac{1}{2x_2} \times (\overset{2x_1}{\cancel{b \cdot 4}}) = 0$$

$= \cancel{-2 - b = 0}$ $= -2 + b\left(\frac{x_1}{x_2}\right) = 0$

$\cancel{\therefore b = 2}$

$-2 + b\left(\frac{1}{2}\right) = 0$

$\therefore b = 4$.

the above condition is contradicting as LHS ≠ RHS at the point (0,0). Hence cannot solve using KKT condition.

Problem-5

$$\min_{x_1, x_2, x_3} \; f = x_1^2 + x_2^2 + x_3^2$$

$$h_1 = \frac{x_1^2}{4} + \frac{x_2^2}{5} + \frac{x_3^2}{25} - 1 = 0$$

$$h_2 = x_1 + x_2 - x_3 = 0$$

$d = x_1$

$s = x_2, x_3$

$n = 3$

$m = 2$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d}$$

$$\boxed{\frac{\partial f}{\partial d} = 2x_1}, \quad \frac{\partial f}{\partial s} = \frac{\partial^2 f}{\partial x_2 \partial x_3} = \frac{\partial f}{\partial x_3}(2x_2) = 0$$

$$\frac{\partial^2 f}{\partial x_3 \partial x_2} = \frac{\partial f}{\partial x_2}(2x_3) = 0$$

$$\boxed{\therefore \frac{\partial f}{\partial s} = 0}$$

$$\frac{\partial h}{\partial s} = \frac{\partial^2 h}{\partial x_2 \partial x_3} = \frac{\partial h}{\partial x}(1) = 0$$

$$\frac{\partial^2 h}{\partial x_3 \partial x_2} = \frac{\partial h}{\partial x}(-1) = 0$$

$$\boxed{\therefore \frac{\partial h}{\partial s} = 0}$$

$\frac{\partial h}{\partial d} = 1$

$\therefore \frac{\partial f}{\partial d} = 2x_1 - \left(0 \times \frac{1}{0}\right)(1) = 0$

$\therefore 2x_1 = 1$

$x_1 = 1/2$

If $h_2$ is considered

$\frac{\partial f}{\partial d} = 2x_1 - \left(0 \times \frac{1}{0}\right)\left(\frac{1}{2}x_1\right) = 0$

$= 2x_1 - 1/2 x_1 = 0$

$\therefore 3/2 x_1 = 0$

$\therefore x_1 = 2/3$

Line search:-

$f(\alpha) = f\left(d_k - \alpha \frac{\partial f}{\partial d}, s_k + \alpha \left(\left(\frac{\partial h}{\partial s}\right)^{-1}\left(\frac{\partial h}{\partial d}\right)\left(\frac{\partial f}{\partial d}\right)\right)^T\right)$

$\varphi(\alpha) = f(d_k, s_k) - \alpha t \left(\frac{\partial f}{\partial d}\right)\left(\frac{\partial f}{\partial d}\right)^T$

The above condition is contradicting as LHS ≠ RHS at the point (0,0). Hence cannot solve using KKT condition.