AML5103 | Applied Probability and Statistics | Problem Set-1

1. A student taking a test has to select 7 out of 10 questions. How many different choices does she have if:
   (a) there are no other restrictions?
   (b) she has to answer exactly 2 of the last 4?
   (c) she has to answer exactly 2 of the first 6?
   (d) she has to answer at least 3 of the first 5?

2. A bone marrow transplant can be made only between two people with all six human leukocyte antigens (HLA) being compatible. Each person's halotype consists of 2 HLA-A antigens, 2 HLA-B antigens, and 2 HLA-DR antigens. There are a total of 18, 40, and 14 antigens for types A, B, and DR, respectively. The order of the antigens is irrelevant and the types can be repeated. How many possible antigen strings are there?

3. If eight identical blackboards are to be divided among 4 schools, how many divisions are possible? What if each school must receive at least one blackboard?

4. 9 computers are brought in for servicing (and machines are serviced one at a time). Of the 9 computers, 3 are PCs, 4 are Macs, and 2 are Linux machines. Assume that all computers of the same type are indistinguishable (i.e., all the PCs are indistinguishable, all the Macs are indistinguishable, etc.).
   (a) In how many distinguishable ways can the computers be ordered for servicing?
   (b) In how many distinguishable ways can the computers be ordered if the first 5 machines serviced must include all 4 Macs?
   (c) In how many distinguishable ways can the computers be ordered if 2 PCs must be in the first three and 1 PC must be in the last three computers serviced?

5. 100 units of stabilizing weights are to be placed into 5 vehicles. Because of different vehicle characteristics, vehicle 1 needs at least 10 units, vehicles 2 and 3 at least 12 each, vehicles 6 and 7 travel in a convoy and they need at least 4 combined. How many distributions of these weight units are feasible?

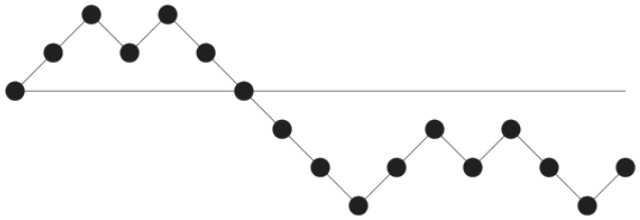
6. Eleven soccer players are to be divided into 4 functional groups: 3 forwards, 3 midfields, 4 defenses, and 1 goalie. There are only 2 people who can play goalie. Both of these two players can play any other position. Of the remaining 9, 4 can play only forward or midfield; the other 5 can play only defense or midfield. We want to calculate the number

of possible ways to divide the team into the 4 functional groups. Follow the hint below and get to the answer:

$$\text{Select 1 goalie out of 2 in } \binom{?}{?} \text{ ways } \textbf{AND} \begin{cases} \text{Remaining goalie plays defense} \\ \textbf{OR} \\ \text{Remaining goalie plays midfield} \\ \textbf{OR} \\ \text{Remaining goalie plays forward} \end{cases}$$

7. In how many ways can r identical server requests be distributed among n servers so that the ith server receives *at least* m_i requests, for each $i = 1, 2, \ldots, n$? You can assume that $r \geq (m_1 + m_2 + \cdots + m_n)$.

8. Suppose a particle starting from the origin can move *only* up or down; the *binomial option pricing model* addresses stock price movements using such an idea.



Show that the number of ways the particle can move from the origin to position k in n steps is $\binom{n}{\frac{n+k}{2}}$. Assume that $n + k$ is even.

9. Imagine a criminal appeals court consisting of five judges; let's call them A, B, C, D, and E. The judges meet regularly to vote (independently, of course) on the fate of prisoners who have petitioned for a review of their convictions. The result of each of the court's deliberations is determined by a simple majority; for a petitioner to be granted or denied a new trial requires three or more votes. Based on long-term record keeping, it is known that A votes correctly 95% of the time; i.e., when A votes to either uphold or to reverse the original conviction, he is wrong only 5% of the time. Similarly, B, C, D, and E vote correctly $95\%, 90\%, 90\%$, and 80% of the time. (There are, of course, two different ways a judge can make a mistake. The judge may uphold a conviction, with new evidence later showing that the petitioner was in fact innocent. Or the judge may vote to reverse a conviction when in fact the petitioner is actually guilty, as determined by the result of a second conviction at the new trial.) Suppose we want to calculate the probability that the court, as an entity, makes an incorrect decision.

   (a) Write the sample space showing any two outcomes in it clearly. Explain what the outcomes mean.

   Hint: consider $\{0, 1\}$.

   (b) How many outcomes n are there in the sample space?

(c) How many outcomes $n(E)$ are there in the event of interest?
Hint: consider selecting at least 3 slots from 5.

(d) Explain briefly why or why not the probability of the event of interest can be calculated as $n(E)/n$.

10. Mr. Brown needs to take 1 tablet of type A and 1 tablet of type B together on a regular basis. One tablet of type A corresponds to a 1 mg dosage, and so does 1 tablet of type B. He keeps these two types of tablets in two separately labeled bottles as they cannot be differentiated easily. One day, on a buisness trip, Mr. Brown brought 10 tablets of type A and 10 tablets of type B.. Unfortunately, he drops the bottles and breaks them. He does not have the time to go to a pharmacy to buy a new set of tablets but he needs to take his required dosage of both tablets A and B. The safe dosage that he needs for both tablets A and B is given by

$$0.9\,\text{mg} \leq \text{safe dosage} \leq 1.1\,\text{mg}.$$

Taking either an excess or a shortage of the required intake will result in serious health issues.

(a) Suppose that after investigating the broken bottles, Mr. Brown finds 2 tablets that are still intact in the bottle for tablet A. The other 18 tablets are found to be mixed in a pile. Is it better for him to take one known tablet from the bottle and one from the pile, or take two tablets from the pile? Answer this by calculating the respective probabilities that he will not have any serious health issues for both options.

(b) Suppose that after investigating the broken bottles, Mr. Brown finds that the tablets are all mixed up. What is the probability that he will not have any serious health issues if he randomly picks 2 tablets?

11. A total of 28% of American males smoke cigarettes, 7% smoke cigars, and 5% smoke both cigars and cigarettes. Let A and B represent the events that a randomly chosen person is a cigarette smoker and a cigar smoker, respectively. Explain in plain English what the following compound events represent and calculate their probabilities: (1) $(A \cup B)^c$ (2) $B \cap A^c$.

12. What is more likely? Provide quantitative support.

(a) Obtaining at least one 6 in 4 rolls of a single die.

(b) Obtaining at least one 12 in 24 rolls of a pair of dice.

13. Data was collected from the residents of a town and displayed as follows:

		Income		
		<$25k	$25k – $70k	> 70k
Age (years)	< 25	952	1,050	53
	25 – 45	456	2,055	1,570
	> 45	54	952	1,008

Answer the following:

(a) What fraction of people are less than 25 years old?

(b) What is the probability that a randomly chosen person is more than 25 years old?

(c) What fraction of people earn less than $70,000?

(d) What is the probability that a randomly chosen person is less than 25 years old and earns more than $70,000?

(e) What fraction of people among those who earn less than $25,000 are between 25-45 years old?

(f) If the next random person you see happens to be more than 45 years old, what is the probability that the person earns less than $70,000?