Equivalence of pushdown automata

FROM PDA to CFG

Theorem

Let $M = (Q, \Sigma, T, \delta, q_0, Z_0)$ be a PDA. Then there is a content free grammer G such that $L(G) = N(M)$.

Proof

Let M be the PDA and $G = (V, \Sigma, P, S)$ be a content free grammer, where,

i) S is the start symbol

ii) V is the set of objects of the form $[q, A, p]$ where q and p in Q.

A in T

P is the set of production which are define as follows :-

1) $S \rightarrow [q_0, Z_0, q]$ for each q in Q.

2) $[q, A, q_{M-1}] \rightarrow a [q_1, B_1, q_1] [q_2, B_2, q_3] \ldots [q_m, B_M, q_{m-1}]$ for each a in $\varepsilon \vee \Sigma$

$q_1, q_2, \ldots q_{M-1}$ in Q.

$A, B_1, B_2, \ldots B_M$ in Γ.

Such that

$$\delta(q, a, A) = (q_1, B_1, B_2, \ldots B_M)$$

if $M=0$. i.e, $\delta(q, a, A) = (q_1, \varepsilon)$ then,

$$[q, A, q_1] \rightarrow a$$

The variables and productions of a Q have been defined in such a way that a leftmost derivation in G of a sentence, v is a situation of the PDA M when feel the input x.

The variable that appear in any step of a leftmost derivation in G correspond to the symbol on the stack of M.

The intention is that $[q, A, P]$ derive x, it and only if x causes M to erase an A from its stack by some sequence of moves beginning in state q and ending in state p.

To show that $L(G) = N(M)$

By induction on the no. of steps in a derivation that

$$[q, A, p] = x$$

if and only if $(q, x, A) \vdash_M (p, \varepsilon, \varepsilon)$

if part.

Basis :-

if $i=1$ then,

$\delta(q, x, A) = (p, \varepsilon)$ where $x = \varepsilon$ or single input symbol, which produces $[q, A, p] \rightarrow x$ is a production of G.

Induction :-

if $i>1$ and $x = ay$, then

$$(q, ay, A) \vdash (q_1, y_1, B_1, B_2 \dots B_n)$$
$$\vdash (q_1, y_1, y_2, \dots y_n, B_1, B_2 \dots B_n)$$
$$\vdash (q_2, y_2 \dots y_n, B_1 B_2 \dots B_n)$$
$$\vdash$$
$$\vdots$$
$$\vdash (p, \varepsilon, \varepsilon)$$

Here $y = y_1, y_2, \dots y_n$, $A \rightarrow B_1 B_2 \dots B_n$

The input symbol y_i has the effect of popping B_j from the stack after a long sequence of moves, in general remains on the stack unchanged while $y_1, y_2 \dots y_{i-1}$ is read.

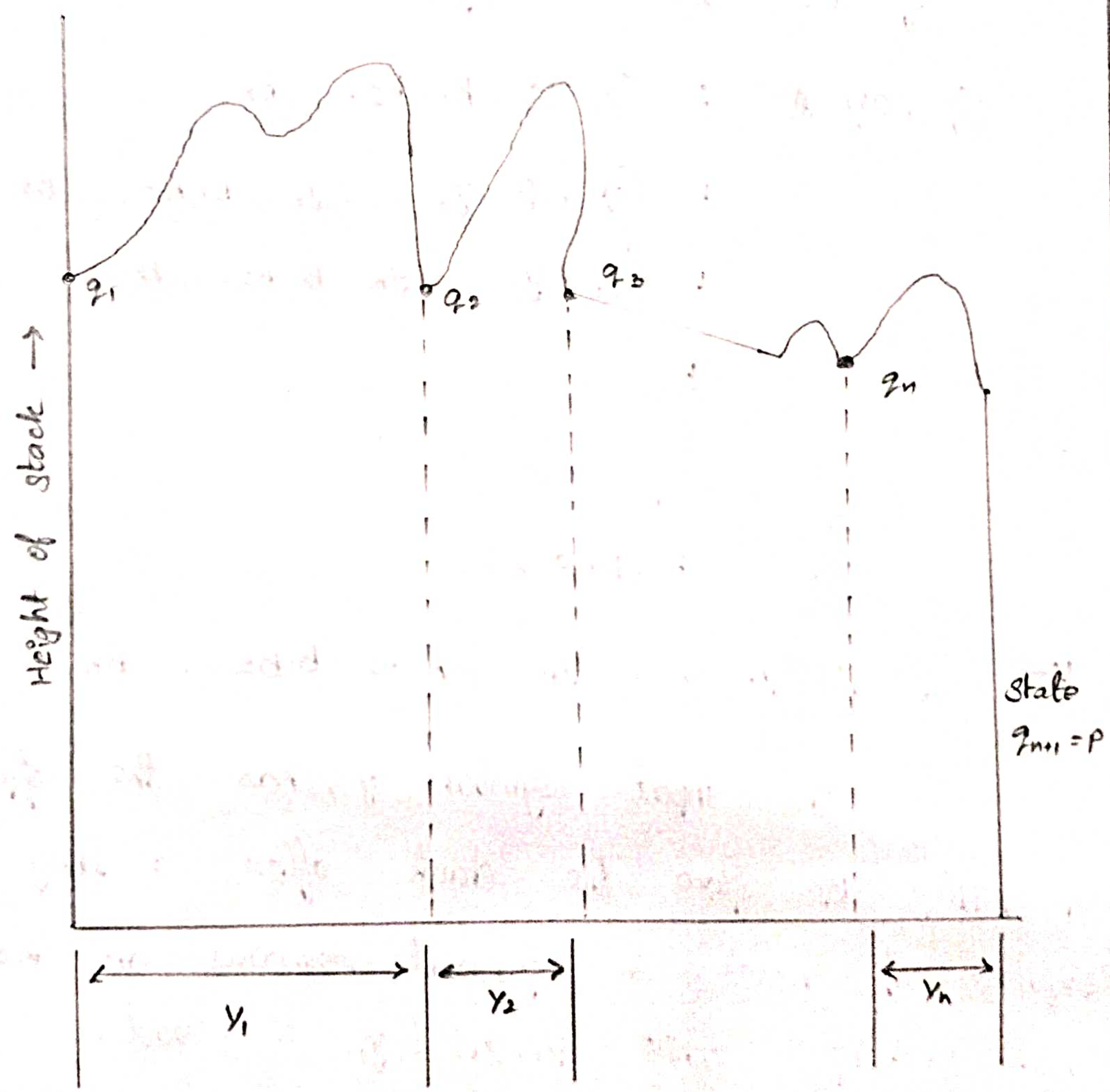
And also $Q = \{q_1, q_2 \ldots q_{n+1}\}$ where $q_{n+1} = p$

$\therefore (q_j, y_j, B_j) \vdash (q_{j+1}, \varepsilon, \varepsilon)$

Thus it produces the production

$(q_j, B_j, q_{j+1}) \Rightarrow y_j$ for $1 \leq j \leq n$

The popping of B_j is shown.



if $(q, ay, A) \vdash (q_1, y, B_1 B_2 \ldots B_n)$ then

$[q, A, P]$ a $[q_1, B, q_2]$ $[q_2, B_2, q_3]$ ......

$[q_m, B_n, q_{n+1}]$

$[q, A, P] \Rightarrow a\ y_1 y_2 \ldots y_n = x$

Only it part

suppose $[q, A, P]^i \Rightarrow x$

This is proved by induction on i, the no. of steps.

Basis :

if $i = 1$, then

$[q, x, A] \vdash^* [p, \varepsilon, \varepsilon]$

Because $[q, A, P] \rightarrow x$ must be a production of G

$\delta(q, x, A) = (p, \varepsilon)$

INDUCTION :-

If $i > n$, then $(q_j, x_j, B_j) \vdash^* (q_{j+1}, \varepsilon, \varepsilon)$ for $i \le j \le n$ if we insert $B_{j+1} \ldots B_n$ at the bottom of each stack in the above sequence of ID's

$(q_j, x_j, B_j B_{j+1} \ldots B_n) \vdash^* (q_{j+1}, \varepsilon, B_{j+1} \ldots B_n)$

From (q, x, A) derivation, we known that

$(q, x, A) \vdash^* (q_1, x_1, x_2 \ldots x_n, B_1 B_2 \ldots B_n)$ is a legal move of M.

$(q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$ for $j = 1, 2, \ldots n$

The proof concludes with the observation that,

$[q, A, P] \overset{*}{\underset{G}{\Rightarrow}} x$ with $q = q_0$ and $A = Z$

$[q_0, Z_0, P] \overset{*}{\Rightarrow} x$ iff and only if

$(q_0, x, Z_0) \vdash^* (p, \varepsilon, \varepsilon)$

$\therefore S^* \Rightarrow x$ if and only if

$(q_0, X, Z_0) \vdash^* (p, \varepsilon, \varepsilon)$ for some state in p

That is x is in $L(G)$ if and only if x is in $N(M)$.

Example :-

Convert the PDA $P = \{(p, q), \{0, 1\} = \{x, Z_0\}, \delta, q, Z_0\}$

$\delta(q, 1, Z_0) = \{(q, xZ_0)\}$

$\delta(q, 1, x) = \{(q, xx)\}$

$\delta(q, 0, x) = \{(p, x)\}$

$\delta(q, \varepsilon, Z_0) = \{(q, \varepsilon)\}$

$$\delta(p, 1, x) = \{(p, \epsilon)\}$$
$$\delta(p, 0, Z_0) = \{(q, Z_0)\}$$

Solution :-

The variables involved in this PDA are,

$$V = \{S, [q, x, q], [q, x, p], [p, x, q], [p, x, p], [q, Z_0, q], [q, Z_0, p], [p, Z_0, q], [p, Z_0, p]\}$$

The production are,

$$S \to [q, Z_0, q]$$
$$S \to [q, Z_0, p]$$

i) $\delta(q, 1, Z_0) = \{(q, xZ_0)\}$

$$[q, Z_0, q] \to 1\,[q, x, q]\,[q, Z_0, q]$$
$$\quad\quad 1\,[q, x, p]\,[q, Z_0, q]$$

$$[q, Z_0, p] \to 1\,[q, x, q]\,[q, Z_0, p]$$
$$\quad\quad 1\,[q, x, p]\,[q, Z_0, p]$$

ii) $\delta(q, 1, x) = \{(q, xx)\}$

$$[q, x, q] \to 1\,[q, x, q]\,[q, x, q]$$
$$\quad\quad 1\,[q, x, p]\,[p, x, q]$$

iii) $\delta(q, 0, x) = \{(p, x)\}$

$[q, x, q] \to 0[p, x, q]$

$[q, x, p] \to 0[p, x, p]$

iv) $\delta(q, \varepsilon, Z_0) = \{(q, \varepsilon)\}$

$[q, Z_0, q] \to \varepsilon$

v) $\delta(p, 1, x) = \{(p, \varepsilon)\}$

$[p, x, p] \to 1$

vi) $\delta(p, 0, Z_0) = \{(q, Z_0)\}$

$[p, Z_0, q] \to 0[q, Z_0, q]$

$[p, Z_0, p] \to 0[q, Z_0, p]$

After Eliminating the unwanted variables

The CFG is given by.

$S \to [q, Z_0, q]$

$[q, Z_0, q] \to \{1[q, x, p][p, Z_0, q], \varepsilon\}$

$[q, x, p] \to \{1[q, x, p][p, x, p], 0[p, x, p]\}$

$[p, x, p] \to 1$

$[p, Z_0, q] \to 0[q, Z_0, q]$

$S \to [q, Z_0, q]$

$\ldots, Z_0, q] \rightarrow 1\ [q, x, p], [p, Z_0, q] \mid \varepsilon$

$[q, x, p] \rightarrow 1\ [q, x, p][p, x, p] \mid 0\ [p, x, p]\}$

$[p, Z_0, q] \rightarrow 0\ [q, Z_0, q]$

$[p, x, p] \rightarrow 1$