Scalar Conservation Laws

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Abstract

This project explores scalar conservation laws in one dimension, beginning with the limitations of the method of characteristics in solving initial value problems (IVPs). We introduce the concept of weak solutions to address these limitations and derive the Rankine-Hugoniot condition to describe shock curves. The investigation reveals that solutions obtained may lack uniqueness, leading to the derivation of the entropy condition to eliminate non-physical solutions. The Riemann problem is examined under different initial states, detailing scenarios where solutions manifest as shock waves or rarefaction waves. Finally, a brief introduction to systems of conservation laws in one dimension is provided, highlighting the complexities and solutions associated with these systems.

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1 INTRODUCTION TO CONSERVATION LAWS

We mainly focus on the following Initial Value Problem,

$$\begin{cases} u_t + q(u)_x = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = g(x), & x \in \mathbb{R}, t = 0 \end{cases} \tag{1}$$

In general, $u = u(x,t)$ represents the *density* or the *concentration* of a physical quantity Q and $q(u)$ is it's *flux function*. If we take a control interval $[x_1, x_2]$, the integral

$$\int_{x_1}^{x_2} u(x,t)\, dx$$

gives the amount of Q between x_1 and x_2. The ***conservation law*** states that the rate of change of Q in the interior is equal to the net flux through the end points of the interval,i.e,

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\, dx = -q(u(x_2,t)) + q(u(x_1,t))$$

assuming that $q > 0$ for a flux along the positive direction of the x-axis. If q and u are smooth functions, then we get

$$\int_{x_1}^{x_2} [u_t(x,t) + q(u(x,t))_x]\, dx = 0$$

Since, the interval $[x_1, x_2]$ is arbitrary, we get the equation

$$u_t + q(u)_x = 0$$

Now, we try to solve the initial value problem defined in (1) using the method of characteristics. Assume that, $x = x(t)$ is the equation of the characteristic based at $(\xi, 0)$ for some $\xi \in \mathbb{R}$, then

$$u(x(t),t) = g(\xi), \quad t > 0$$

Differentiating with respect to t, we get

$$u_x(x(t),t)x'(t) + u_t(x(t),t) = 0, \quad t > 0 \tag{2}$$

Also from (1), along the curve $x = x(t)$;

$$u_t(x(t),t) + q(u(x(t),t))_x = 0$$

$$\Rightarrow u_t(x(t),t) + q'(u(x(t),t)).u_x(x(t),t) = 0$$
$$\Rightarrow u_t(x(t),t) + q'(g(\xi)).u_x(x(t),t) = 0 \tag{3}$$

From (2) and (3);

$$[x'(t) - q'(g(\xi))]\, u_x(x(t), t) = 0$$

Assuming that $u_x(x(t), t) \neq 0$,

$$x'(t) = q'(g(\xi))$$
$$\Rightarrow x(t) = q'(g(\xi))t + \xi \quad (as\ x(0) = \xi)$$

Thus, the characteristics are straight lines with slope $q'(g(\xi))$. Different values of ξ gives different values of the slope.

To compute $u(x,t)$, $t > 0$, we'll go back in time along the characteristic until the base point $(\xi, 0)$. Then $u(x,t) = g(\xi)$. As $x(t) = x$,

$$\xi = x - q'(g(\xi))t$$

Hence,

$$u(x,t) = g\,[x - q'(g(\xi))t]$$

which represents the travelling wave propagating with speed $q'(g(\xi))$.

Since $u(x,t) \equiv g(\xi)$ along the characteristic based at $(\xi, 0)$. We can define u explicitly by the equation

$$G(x,t,u) = u - g\,[x - q'(u)t] = 0 \tag{4}$$

If g and q' are smooth, the *Implicit Function Theorem* implies that equation (4) defines u as a function of (x,t), as long as

$$G_u(x,t,u) = 1 + tq''(u)g'[x - q'(u)t] \neq 0$$

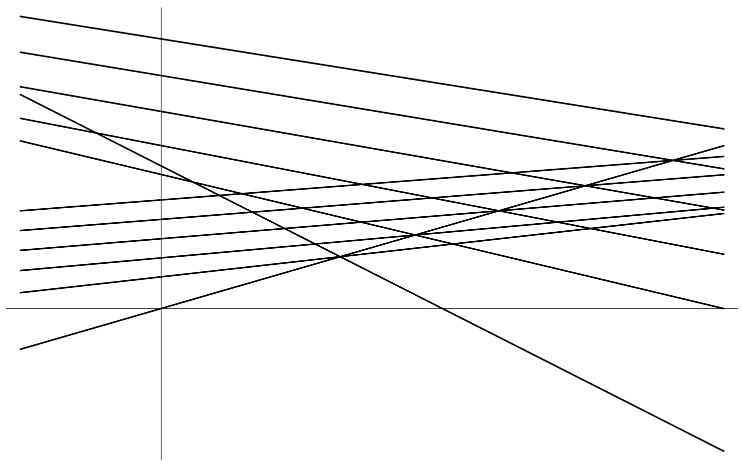
Consequently, if g' and q'' have the same sign then the solution given by the method of characteristics is defined and smooth for all times $t \geq 0$.

Proposition 1.1. *Suppose that $q \in C^2(\mathbb{R}), g \in C^1(\mathbb{R})$ and $g'q'' > 0$ in $\mathbb{R}$. Then formula (4) defines the unique solution u of problem (1) in the half plane $t \geq 0$. Moreover, $u(x,t) \in C^1(\mathbb{R} \times [0, \infty))$.*

Clearly, the characteristics doesn't intersect if g' and q'' have same sign and the solution exists for the given domain. We want to study what happens when g' and q'' have different signs in an interval $[a, b]$. The solution still exists for small times, as $G_u \sim 1$ if $t \sim 0$, but as time goes on we expect the formation of a shock. Let's try to see this with an example.

Example 1.2. Consider the IVP,

$$u_t + (1 - 2u)u_x = 0$$
$$u(x, 0) = \arctan x$$



Here $q(u) = u - u^2,\ q'(u) = 1 - 2u,\ q''(u) = -2$ and $g(\xi) = \arctan\xi,\ g'(\xi) = 1/(1+\xi^2)$. Clearly, sign of g' and q'' are opposite. The equation of characteristics is

$$x'(t) = 1 - 2\arctan\xi$$
$$\Rightarrow x(t) = (1 - 2\arctan\xi)t + \xi \quad (as\ x(0) = \xi)$$

Clearly, the characteristics intersect as we go forward in time from the above image. After the time where the characteristics first intersect, it is impossible to define the solution smoothly. To deal with this we need a new notion of solution.

2 A NEW NOTION OF SOLUTION

We have seen above that the method of characteristics demonstrates that there does not always exist a smooth solution for (1), existing for all times $t > 0$. We, therefore, look for some sort of weak or generalized solution.

2.1 Weak Solution, Rankine-Hugoniot condition

As we cannot find a smooth solution, we must devise some way to interpret a less regular function u that *solves* our initial value problem. However, for that we must *temporarily* assume u to be smooth because the PDE doesn't even makes sense otherwise. Then we multiply the PDE by a smooth function and try to transfer the derivatives so that the resulting expression doesn't involve the derivatives of u directly. Assume,

$$v : \mathbb{R} \times [0, \infty) \to \mathbb{R} \text{ is smooth, with compact support} \tag{5}$$

We call v a *test function*. Multiplying the PDE by v and integrating by parts:

$$\begin{aligned}0 &= \int_0^\infty \int_{-\infty}^\infty (u_t + q(u)_x) v \, dx dt \\ &= -\int_0^\infty \int_{-\infty}^\infty u v_t \, dx dt - \int_{-\infty}^\infty u(x,0) v(x,0) \, dx - \int_0^\infty \int_{-\infty}^\infty q(u) v_x \, dx dt\end{aligned}$$

Putting the initial condition $u = g$ on $\mathbb{R} \times \{t = 0\}$, we obtain

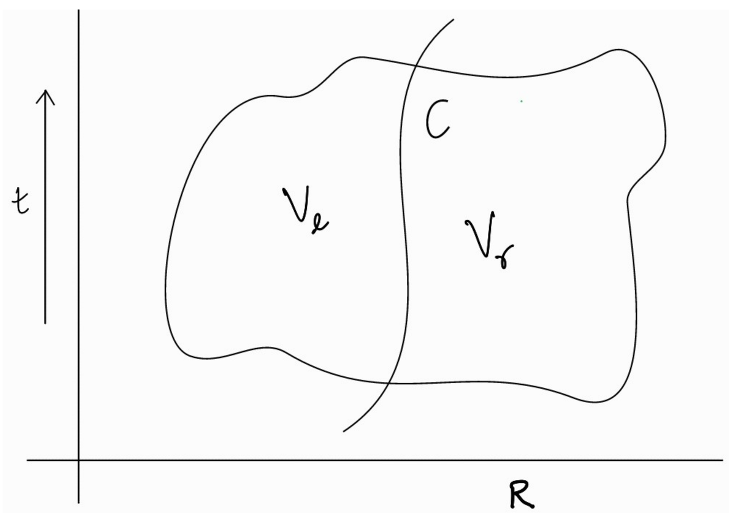
$$\int_0^\infty \int_{-\infty}^\infty [u v_t + q(u) v_x] \, dx dt + \int_{-\infty}^\infty g v(x,0) \, dx = 0 \tag{6}$$

This equality does not involve any derivatives of u and is valid even if u is only bounded.

Definition 2.1. *We say that $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is a weak solution of (1), provided equality (6) holds for each test function v satisfying (5).*

Now suppose that we have a weak solution of (1), then what can we deduce about this solution from the equality (6)? Let's try to see this for u with a particularly simple structure. Suppose that in some open region $V \subset \mathbb{R} \times (0, \infty)$, u is smooth on either side of a smooth curve C.

Let V_l be the part on the left and V_r on the right of the curve. Assuming u is a weak solution of (1) and C^1 in V_l and V_r.



As we get done with our assumptions, choose a test function v with compact support in V_l. Then (6) becomes

$$0 = \int_0^\infty \int_{-\infty}^\infty [u v_t + q(u) v_x] \, dx dt = -\int_0^\infty \int_{-\infty}^\infty [u_t + q(u)_x] v \, dx dt \tag{7}$$

This equality holds for all test functions v with compact support in V_l, so

$$u_t + q(u)_x = 0 \text{ in } V_l \tag{8}$$

Similarly,

$$u_t + q(u)_x = 0 \text{ in } V_r \tag{9}$$

Now, choose a test function v with compact support in V such that it doesn't necessarily vanish along the curve C. Then again from (6),

$$\begin{aligned}
0 &= \int_0^\infty \int_{-\infty}^\infty [uv_t + q(u)v_x]\, dxdt \\
&= \int\int_{V_l} [uv_t + q(u)v_x]\, dxdt + \int\int_{V_r} [uv_t + q(u)v_x]\, dxdt
\end{aligned}$$

Now, taking the integral on the left part of V, using Gauss Theorem, we get

$$\begin{aligned}
\int\int_{V_l} [uv_t + q(u)v_x]\, dxdt &= -\int\int_{V_l} [u_t + q(u)_x] v\, dxdt + \int_C (u_l \nu^2 + q(u_l)\nu^1) v\, dl \\
&= \int_C (u_l \nu^2 + q(u_l)\nu^1) v\, dl
\end{aligned}$$

Here, $\nu = (\nu^1, \nu^2)$ is the unit normal to the curve C, pointing from V_l to V_r, and u_l is the limit from the left. Similarly,

$$\int\int_{V_r} [uv_t + q(u)v_x]\, dxdt = -\int_C (u_r \nu^2 + q(u_r)\nu^1) v\, dl$$

where, u_r is the limit from the right of C. Substituting the values of these integrals, we get

$$\int_C [(q(u_l) - q(u_r))\nu^1 + (u_l - u_r)\nu^2] v\, dl = 0$$

This equality holds for all test functions v, so

$$(q(u_l) - q(u_r))\nu^1 + (u_l - u_r)\nu^2 = 0 \tag{10}$$

Suppose, C is defined parametrically as $\{(x,t) | x = s(t)\}$ for some smooth function $s(.) : [0, \infty) \to \mathbb{R}$. Then,

$$\boldsymbol{\nu} = (\nu^1, \nu^2) = (1 + s'^2)^{-1/2}(1, -s')$$

Then (10) implies,

$$(q(u_l) - q(u_r)) = s'(u_l - u_r) \tag{11}$$

in V along C. This is the ***Rankine-Hugoniot condition***. This condition tells us that even though along the curve these quantities may vary but the expression $(q(u_l) - q(u_r))$ and $s'(u_l - u_r)$ must always balance. It gives us a choice of the curve C for which the solution can be defined in case of a shock wave. But we shall see that the weak solution so obtained might not be unique, as we see in the following example, and we need other condition to ensure the uniqueness of the solution.

Example 2.2. Let's take the *Burger's Equation* with the following initial condition,

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = g(x), & x \in \mathbb{R}, t = 0 \end{cases}$$

where

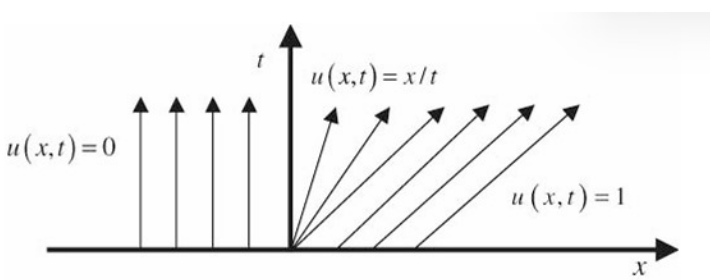
$$g(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

The characteristics are the straight line,

$$x(t) = g(x_0)t + x_0$$

. Therefore, $u = 0$ if $x < 0$ and $u = 1$ if $x > t$. The region $0 < x < t$ is not covered by the characteristics, so we connect the states 0 and 1 continuously by *rarefaction wave*. We get the following weak solution

$$u(x,t) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{t} & 0 < x < t \\ 1 & x \geq t \end{cases}$$



But, unfortunately this is not the only weak solution. There also exists a shock wave solution. As,

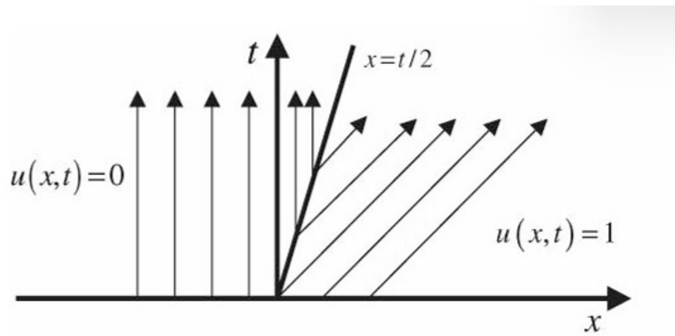
$$u_l = 0,\ u_r = 1,\ q(u_l) = 0,\ q(u_r) = \frac{1}{2}$$

the Rankine-Hugoniot condition yields

$$s'(t) = \frac{q(u_l) - q(u_r)}{u_l - u_r} = \frac{1}{2}$$

As the discontinuity arises at $x = 0$, the shock curve starts at $s(0) = 0$, so we get the following weak solution

$$w(x,t) = \begin{cases} 0 & x < \dfrac{t}{2} \\ 1 & x > \dfrac{t}{2} \end{cases}$$



This actually is a *non physical solution* as we shall see in the next section.

2.2 Entropy condition

To discard the non-physical solutions we need another condition that can also work as a uniqueness criteria. We know from above discussion that the general form of conservation law (1) takes the constant value $u(\xi, 0) = g(\xi)$ along the projected characteristic

$$x(t) = g'(\xi)t + \xi, \quad \xi \in \mathbb{R},\ t \geq 0 \tag{12}$$

Now we typically encounter the crossing of characteristics and discontinuities in the solution if we move *forward* in time. So, let us consider the class of piecewise-smooth weak solutions of (1) with the property that if we move backwards in t along any characteristic, we will not encounter any lines of discontinuity for u.

So, suppose at some point on the curve C that u has distinct left and right limits, u_l and u_r. Then from (12), we get

$$q'(u_l) > s'(t) > q'(u_r) \tag{13}$$

These inequalities are called the ***entropy condition***. Geometrically, this means that the slope of a shock curve is less than the slope of left characteristics and more than the slope of right characteristics.

Further, assuming that q is strictly convex, $q'' \geq C > 0$ for some $C \in \mathbb{R}$, then along any shock curve

$$u_l > u_r$$

.

Lemma 2.3. *A one sided jump estimate Assume $q : \mathbb{R} \to \mathbb{R}$ is smooth and strictly convex and g is strictly increasing. Then there exists a constant C such that the solution u is satisfies the inequality*

$$u(x+z,t) - u(x,t) \leq \frac{C}{t} z \tag{14}$$

for all $t > 0$ and $x, z \in \mathbb{R}$, $z > 0$.

Proof. We know the equation,

$$G(x,t,u) \equiv u - g[x - q'(u)t] = 0$$

defines the unique classical solution of (1), at least for small times.
Then by Implicit function theorem,

$$u_x = -\frac{G_x}{G_u} = \frac{g'}{1 + tg'q''}$$

From our assumptions, let $q'' \geq c > 0$ and $g' > 0$ we get

$$u_x \leq \frac{C}{t}$$

where $C = 1/c$. Using the mean value theorem on the space variable, we get

$$u(x+z,t) - u(x,t) = u_x(x+z*)z,$$
$$\Rightarrow u(x+z,t) - u(x,t) \leq \frac{C}{t} z$$

for some $z > 0$ and $z* \in (x, x+z)$. □

*We call inequality (14) the **entropy condition**.*

Definition 2.4. We say that a function $u \in \mathbb{L}^\infty(\mathbb{R} \times (0,\infty))$ is an ***entropy solution*** of (1) provided

$$\int_0^\infty \int_{-\infty}^\infty \{uv_t + q(u)v_x\}\, dxdt + \int_{-\infty}^\infty gv(x,0)\, dx = 0$$

for all test functions $v \in \mathbf{C}_c^\infty(\mathbb{R} \times [0,\infty))$ and

$$u(x+z,t) - u(x,t) \leq C(1 + \frac{1}{t})z$$

for some constant $C \geq 0$ and $x, z \in \mathbb{R}$, $t > 0$, with $z > 0$.

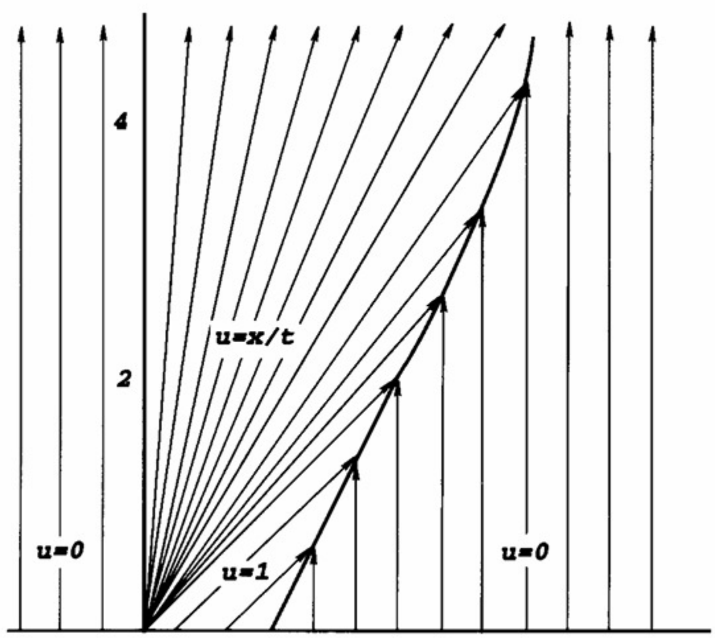
Theorem 2.5. *(Uniqueness of entropy solution) Assume q is convex and smooth. Then there exists-up to a set of measure zero-at most on entropy solution of (1).*

Example 2.6. Let's again take the Burger's equation with a different initial condition

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

For $0 \leq t \leq 2$, we can find the define the solution as follows with the help of Example 2.2

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{t} & 0 < x < t \\ 1 & t < x < 1 + \dfrac{t}{2} \\ 0 & x > 1 + \dfrac{t}{2} \end{cases} \quad (0 \leq t \leq 2)$$



Checking the Entropy condition on the shock curve $x(t) = 1 + \frac{t}{2}$, $u_l = 1 > u_r = 0$. For the rarefaction wave in the region $0 < x < t$, we shall take the case where $0 < x < x + z < t$, for $z > 0$,then

$$u(x+z) - u(x,t) = \frac{x+z}{t} - \frac{x}{t} \leq \frac{C}{t} z$$

for $C = 1$.

For times $t \geq 2$ we find the shock wave parameterized as $x = s(t)$, with $u = x/t$ to the left of $s(t)$ and $u = 0$ to the right. By Rankine-Hugoniot condition,

$$s'(t) = \frac{1}{2}\left(\frac{s(t)}{t}\right)^2 \Big/ \frac{s(t)}{t}$$
$$\Rightarrow s'(t) = \frac{s(t)}{2t} \qquad (t \geq 2)$$

Additionally, $s(2) = 2$ and solving this ODE we get $s(t) = (2t)^{1/2}(t \geq 2)$. Hence, we get the solution

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 < x < (2t)^{1/2} \\ 0, & x > (2t)^{1/2} \end{cases} \qquad (t \geq 2)$$

Checking the Entropy condition for the rarefaction wave in the region $0 < x < (2t)^{1/2}$, we shall take the case where $0 < x < x + z < (2t)^{1/2}$, for $z > 0$,then

$$u(x+z) - u(x,t) = \frac{x+z}{t} - \frac{x}{t} \leq \frac{C}{t}z$$

for $C = 1$.

Since, the entropy condition is satisfied for all times $t \geq 0$, the solution so obtained is the unique solution for our initial value problem.

3 RIEMANN's PROBLEM

The initial-value problem (1) with the piecewise-constant initial function

$$g(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases} \tag{15}$$

is called the *Riemann's problem* for the scalar conservation law (1). Here, $u_l, u_r \in \mathbb{R}$ are the left and right initial states with $u_l \neq u_r$.

Theorem 3.1. *(Solution of Riemann's Problem)*
Assuming q is uniformly convex, C^2 and $G = (q')^{-1}$.

(i) If $u_l > u_r$, the entropy solution of the Riemann problem is x

$$u(x,t) := \begin{cases} u_l, & if \ \frac{x}{t} < s'(t) \\ u_r, & if \ \frac{x}{t} > s'(t) \end{cases} \qquad (x \in \mathbb{R},\ t > 0)$$

where

$$s'(t) := \frac{q(u_l) - q(u_r)}{u_l - u_r}$$

(ii) If $u_l < u_r$, the entropy solution of the Riemann problem is

$$u(x,t) := \begin{cases} u_l, & if\ \ \dfrac{x}{t} < q'(u_l) \\ G(\dfrac{x}{t}), & if\ \ q'(u_l) < \dfrac{x}{t} < q'(u_r) \\ u_r, & if\ \ \dfrac{x}{t} > q'(u_r) \end{cases} \quad (x \in \mathbb{R},\ t > 0)$$

In the first case, the states are separated by a *shock wave* and in the second case, the states are separated by a *rarefaction wave*.

Proof. (i) As in this case the shock wave satisfies the Rankine-Hugoniot condition hence it is clearly a weak solution. Also, $u_l > u_r$ the entropy condition holds as well. Therefore, it is a unique solution of (1).

(ii) For this case we have to check if u solves the conservation law in the region $q'(u_l) < \frac{x}{t} < q'(u_r)$. Consider

$$u(x,t) = v(\frac{x}{t})$$

we try to find what should v be to solve our PDE.

$$\begin{aligned} u_t + q(u)_x &= u_t + q'(u)u_x \\ &= -v'(\frac{x}{t})\frac{x}{t^2} + q'(v)v'(\frac{x}{t}) \\ &= v'(\frac{x}{t})\frac{1}{t}[q'(v) - \frac{x}{t}] \end{aligned}$$

Assuming, v' never vanishes, we get $q'(v) = \frac{x}{t}$. Hence,

$$u(x,t) = G(\frac{x}{t})$$

solves the conservation law in this region. Also, $G(q'(u_l)) = u_l$ and $G(q'(u_r)) = u_r$. Therefore, u is continuous and hence a weak solution.

Now, moving on to check the entropy condition. We shall take the case, for $z > 0$, when $q'(u_l)t < x + z < q'(u_r)t$, then

$$u(x+z,t) - u(x,t) = G(\frac{x+z}{t}) - G(\frac{x}{t}) \leq \frac{C}{t}z$$

for any $C > 0$. Hence, a unique solution.

$\square$

4 SYSTEM OF CONSERVATION LAWS

Now, we give a brief introduction to what happens when we have a *system of conservation laws*.

Generally, we investigate the vector function

$$\mathbf{u} = \mathbf{u}(x,t) = (u^1(x,t), u^2(x,t), ..., u^m(x,t)), \quad x \in \mathbb{R}^n, t \geq 0$$

where the components are the densities of various conserved quantities in some physical system. Then in any smooth, bounded region $U \subset \mathbb{R}^n$, the integral

$$\int_U \mathbf{u}(x,t)\, dx$$

represents the total amount of of these quantities in U at time t. As conservation law says that the rate of change within U is equal to the net flux through ∂U, that is

$$\frac{d}{dt}\int_U \mathbf{u}\, dx = -\int_{\partial U} \mathbf{F}(\mathbf{u})\nu\, dS = -\int_U div\mathbf{F}(\mathbf{u})\, dx$$

where $\mathbf{F} : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ and ν is the unit outward normal along U. As the region U is arbitrary, we get the initial value problem for the *system of conservation laws*:

$$\begin{cases} \mathbf{u}_t + div\mathbf{F}(\mathbf{u}) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \mathbf{u} = \mathbf{g} & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

where $\mathbf{g} = (g^1, g^2, ..., g^m)$ denote the initial distribution of $\mathbf{u} = (u^1, u^2, ..., u^m)$.

We take the case when space dimension is one, that is the system of conservation laws in one space dimension,

$$\begin{cases} \mathbf{u}_t + \mathbf{F}(\mathbf{u})_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \mathbf{u} = \mathbf{g} & \text{on } \mathbb{R} \times \{t=0\} \end{cases} \tag{16}$$

Definition 4.1. *(Weak Solution)* We say that $\mathbf{u} \in L^\infty(\mathbb{R} \times (0,\infty); \mathbf{R}^m)$ is a weak solution of the initial value problem (16) provided the equality

$$\int_0^\infty \int_{-\infty}^\infty [\mathbf{u}.\mathbf{v}_t + \mathbf{F}(\mathbf{u}).\mathbf{v}_x]\, dxdt + \int_{-\infty}^\infty \mathbf{g}\mathbf{v}|_{t=0}\, dx = 0$$

holds for all test functions

$$\mathbf{v} : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m \text{ is smooth, with compact support.}$$

Definition 4.2. *(Rankine-Hugoniot condition)* Suppose that a weak solution $\mathbf{u}$ exists for the IVP (16) which is smooth on either side of a curve C, represented parametrically as $\{(x,t)|x = s(t)\}$, along which $\mathbf{u}$ has jump discontinuities. Then parallel to the case of single conservation law, we get

$$\mathbf{F}(\mathbf{u}_l) - \mathbf{F}(\mathbf{u}_r) = s'(\mathbf{u}_l - \mathbf{u}_r)$$

where $\mathbf{u}_l$ and $\mathbf{u}_r$ are the left and right limits, respectively.

Definition 4.3. Two smooth functions $\Phi, \Psi : \mathbb{R}^m \to \mathbb{R}$ comprise an ***entropy/entropy-flux pair*** for the conservation law $\mathbf{u}_t + \mathbf{F}(\mathbf{u})_x = 0$ provided Φ is convex and

$$D\Phi(z)D\mathbf{F}(z) = D\Psi(z) \quad (z \in \mathbb{R}^m).$$

Definition 4.4. We call $\mathbf{u}$ an ***entropy solution*** of (16) provided $\mathbf{u}$ is a weak solution and satisfies the inequalities

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \Phi(\mathbf{u})v_t + \Psi(\mathbf{u})v_x \, dxdt \geq 0 \\ \text{for each } v \in C_c^\infty(\mathbb{R} \times (0,\infty)), v \geq 0 \end{cases}$$

for each entropy/entropy-flux pair (Φ, Ψ)

Definition 4.5. Two smooth functions $\Phi, \Psi : \mathbb{R} \to \mathbb{R}$ comprise an ***entropy/entropy-flux pair*** for the *scalar conservation law* $u_t + F(u)_x = 0$ provided Φ is convex and

$$\Phi'(z)F'(z) = \Psi'(z) \quad (z \in \mathbb{R}).$$

Definition 4.6. We call $u \in C([0,\infty), L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times (0,\infty))$ an entropy solution of (1) provided u satisfies the inequalities

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \Phi(u)v_t + \Psi(u)v_x \, dxdt \geq 0 \\ \text{for each } v \in C_c^\infty(\mathbb{R} \times (0,\infty)), v \geq 0 \end{cases}$$

for each entropy/entropy-flux pair (Φ, Ψ), and $u(.,t) \to g$ in L^1 as $t \to 0$.

The process of ensuring the uniqueness of a system of more than one conservation law is much more challenging and less straightforward as the presence of multiple interacting waves and the complexity of the interactions make the analysis more difficult.

Theorem 4.7. *(Uniqueness of entropy solutions for a single conservation law) There exists-up to a set of measure zero-at most one entropy solution of (1).*

5 Summary

We defined a new notion of solution, the weak solution, by trying to reduce the number of restrictions on the classical solution. Then we derived the Rankine-Hugoniot condition to determine the weak solution in the case of a shock curve. We saw that this solution may not be unique, so we derived the entropy condition that discards the non-physical solutions and ensures the uniqueness of the solution. We then investigated the conservation law with a particular case of piecewise smooth initial condition, the Riemann problem. Then we moved to systems of conservation laws in one dimension, where the definition of weak solution and the Rankine-Hugoniot condition were just extensions of the case of a single conservation law. However, for the entropy solution, we introduced the entropy/entropy-flux pairs. Also, we concluded that the process of ensuring the uniqueness of a system of more than one conservation law is much more complex and not as straightforward as in the case of a single conservation law.

References

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