High Order Expansion Method for Kuiper's V_n-Statistic — Supplementary Material*

Hong-Yan Zhang*, Zhi-Qiang Feng, Rui-Jia Lin and Yu Zhou

School of Information Science and Technology,
Hainan Normal University, Haikou 571158, China

September 14, 2023

1 Notations and Key Formulas

1.1 $\Phi_n(a,b)$ and $\Phi(a,b)$

Kuiper's $\Phi_n(a,b)$ function is defined by [1,2]

$$\Phi_n(a,b) = \Phi(a,b) + \frac{1}{6\sqrt{n}}\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right)\Phi(a,b) + \mathcal{O}\left(\frac{1}{n}\right). \tag{1}$$

where

$$\Phi(a,b) = \sum_{j=-\infty}^{\infty}\left[\mathrm{e}^{-2j^2(a+b)^2} - \mathrm{e}^{-2(ja+(j-1)b)^2}\right]. \tag{2}$$

1.2 Partial Derivative Operators $\mathcal{D}_n$ and $\mathcal{D}_n^i$

We introduce the first order partial derivative operator $\mathcal{D}_n$ and its powers as follows

$$\mathcal{D}_n = \frac{1}{6\sqrt{n}}\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right), \quad \mathcal{D}_n^i = \frac{1}{6^i n^{i/2}}\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right)^i, \tag{3}$$

then we can reformulate (1) by

$$\Phi_n(a,b) = \Phi(a,b) + \mathcal{D}_n\,\Phi(a,b) + \mathcal{O}\left(\frac{1}{n}\right). \tag{4}$$

1.3 High Order Expansion of $\Phi_n(a,b)$

The generalization of (4) can be written by

$$\begin{aligned}\Phi_n(a,b) &= \exp(\mathcal{D}_n)\Phi(a,b) \\ &= \left[1 + \frac{\mathcal{D}_n}{1!} + \frac{\mathcal{D}_n^2}{2!} + \frac{\mathcal{D}_n^3}{3!} + \cdots + \frac{\mathcal{D}_n^k}{k!}\right]\Phi(a,b) + \mathcal{O}\left(n^{-(k+1)/2}\right).\end{aligned} \tag{5}$$

Consider the i-th term in the k-th order approximation

$$\Phi_n^{(i)}(a,b) = \frac{1}{i!}\,\mathcal{D}_n^i\,\Phi(a,b) = \frac{1}{i!}\frac{1}{(6\sqrt{n})^i}\left[\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right]^i\Phi(a,b) = \frac{1}{n^{i/2}}\cdot\frac{1}{i!6^i}\left[\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right]^i\Phi(a,b) \tag{6}$$

we can obtain

$$\Phi_n(a,b) = \sum_{i=0}^{k}\Phi_n^{(i)}(a,b) + \mathcal{O}\left(\frac{1}{n^{(k+1)/2}}\right) \tag{7}$$

The computation of $\left\{\Phi_n^{(i)}(a,b) : i = 0,1,2,3,\cdots\right\}$ can be done iteratively since we have

$$\Phi_n^{(k+1)}(a,b) = \frac{1}{(k+1)!}\,\mathcal{D}_n^{k+1}\,\Phi(a,b) = \frac{\mathcal{D}_n}{k+1}\left(\frac{1}{k!}\,\mathcal{D}_n^k\,\Phi(a,b)\right) = \frac{1}{k+1}\,\mathcal{D}_n\,\Phi_n^{(k)}(a,b) \tag{8}$$

*Corresponding author: Hong-Yan Zhang; email: hongyan@hainnu.edu.cn; ORCID: 0000-0002-4400-9133

1.4 Functions $Q_i(a,b)$ for Separating Continuous and Discrete Variables

In order to separate the continuous variables a, b and the discrete variable n for the i-th term $\Phi_n^{(i)}(a,b)$, it is convenient for us to introduce the auxiliary functions $Q_i(a,b)$ which are determined by

$$Q_i(a,b) = \frac{1}{6^i i!}\left[\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right]^i \Phi(a,b). \tag{9}$$

1.5 Functions $B_i(c)$ for Calculating the CDF of K_n and V_n

For the purpose of computing the continuous distribution function (CDF) of the statistic $K_n = \sqrt{n}\cdot V_n$, we introduce the family of functions $B_i(c)$ as follows:

$$B_i(c) = \int_{b=0}^{c}\left[\frac{\partial Q_i}{\partial b}\right]_{a=c-b} \mathrm{d}\, b, \quad i = 0, 1, 2, \cdots, k \tag{10}$$

The CDF of K_n can be expressed by

$$\Pr\{K_n \le c\} = \Pr\left\{\sqrt{n}\cdot V_n \le c\right\} = \sum_{i=0}^{k}\frac{B_i(c)}{n^{i/2}} + \mathcal{O}\left(n^{-(k+1)/2}\right), \tag{11}$$

which is a direct generalization of Kuiper's formula [2]. It is a key issue for us to find the expressions for $B_i(c)$. Equivalently, we have

$$\Pr\{V_n \le v\} = \sum_{i=0}^{k}\frac{B_i(v)}{n^{i/2}} + \mathcal{O}\left(n^{-(k+1)/2}\right), \quad v = c/\sqrt{n} \tag{12}$$

For $k = 5$, we have

$$\Pr\{K_n \le c\} = \sum_{i=0}^{5}\frac{B_i(c)}{n^{i/2}} + \mathcal{O}\left(n^{-3}\right), \tag{13}$$

and

$$\Pr\{V_n \le v\} = \sum_{i=0}^{5}\frac{B_i(v)}{n^{i/2}} + \mathcal{O}\left(n^{-3}\right), \quad v = c/\sqrt{n} \tag{14}$$

2 Auxiliary Functions and Partial Derivatives

2.1 Partial Derivatives

For the continuous variable $a, b \in \mathbb{R}$, let

$$u(a,b,j) = 2j^2(a+b)^2, \quad v(a,b,j) = 2[ja + (j-1)b]^2, \quad j \in \mathbb{Z} \tag{15}$$

then

$$\begin{aligned} \frac{\partial u}{\partial a} &= 4j^2(a+b), & \frac{\partial u}{\partial b} &= 4j^2(a+b), \\ \frac{\partial v}{\partial a} &= 4j[ja+(j-1)b], & \frac{\partial v}{\partial b} &= 4(j-1)[ja+(j-1)b] \end{aligned} \tag{16}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial a^2} &= 4j^2, & \frac{\partial^2 u}{\partial b^2} &= 4j^2, & \frac{\partial^2 u}{\partial a \partial b} &= 4j^2 \\ \frac{\partial^2 v}{\partial a^2} &= 4j^2, & \frac{\partial^2 v}{\partial b^2} &= 4(j-1)^2, & \frac{\partial^2 v}{\partial a \partial b} &= 4j(j-1) \end{aligned} \tag{17}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial a}\mathrm{e}^{-u} &= -\frac{\partial u}{\partial a}\mathrm{e}^{-u}, & \frac{\partial}{\partial a}\mathrm{e}^{-v} &= -\frac{\partial v}{\partial a}\mathrm{e}^{-v} \\ \frac{\partial}{\partial b}\mathrm{e}^{-u} &= -\frac{\partial u}{\partial b}\mathrm{e}^{-u}, & \frac{\partial}{\partial b}\mathrm{e}^{-v} &= -\frac{\partial v}{\partial b}\mathrm{e}^{-v} \end{aligned} \tag{18}$$

Hence,

$$\Phi(a,b) = \sum_{j=-\infty}^{\infty}[\mathrm{e}^{-u(a,b,j)} - \mathrm{e}^{-v(a,b,j)}]. \tag{19}$$

For the second partial derivative, we have

$$
\begin{aligned}
\frac{\partial^2}{\partial a^2}\mathrm{e}^{-u} &= \frac{\partial}{\partial a}\left[-\frac{\partial u}{\partial a}\mathrm{e}^{-u}\right] = -\frac{\partial^2 u}{\partial a^2}\mathrm{e}^{-u} - \frac{\partial u}{\partial a}\cdot\left(-\frac{\partial u}{\partial a}\right)\mathrm{e}^{-u} = \mathrm{e}^{-u}\left[\left(\frac{\partial u}{\partial a}\right)^2 - \frac{\partial^2 u}{\partial a^2}\right],\\
\frac{\partial^2}{\partial b^2}\mathrm{e}^{-u} &= \frac{\partial}{\partial b}\left[-\frac{\partial u}{\partial b}\mathrm{e}^{-u}\right] = -\frac{\partial^2 u}{\partial b^2}\mathrm{e}^{-u} - \frac{\partial u}{\partial b}\cdot\left(-\frac{\partial u}{\partial b}\right)\mathrm{e}^{-u} = \mathrm{e}^{-u}\left[\left(\frac{\partial u}{\partial b}\right)^2 - \frac{\partial^2 u}{\partial b^2}\right],\\
\frac{\partial^2}{\partial a^2}\mathrm{e}^{-v} &= \frac{\partial}{\partial a}\left[-\frac{\partial v}{\partial a}\mathrm{e}^{-v}\right] = -\frac{\partial^2 v}{\partial a^2}\mathrm{e}^{-v} - \frac{\partial v}{\partial a}\cdot\left(-\frac{\partial v}{\partial a}\right)\mathrm{e}^{-v} = \mathrm{e}^{-v}\left[\left(\frac{\partial v}{\partial a}\right)^2 - \frac{\partial^2 v}{\partial a^2}\right],\\
\frac{\partial^2}{\partial b^2}\mathrm{e}^{-v} &= \frac{\partial}{\partial b}\left[-\frac{\partial v}{\partial b}\mathrm{e}^{-v}\right] = -\frac{\partial^2 v}{\partial b^2}\mathrm{e}^{-v} - \frac{\partial v}{\partial b}\cdot\left(-\frac{\partial v}{\partial b}\right)\mathrm{e}^{-v} = \mathrm{e}^{-v}\left[\left(\frac{\partial v}{\partial b}\right)^2 - \frac{\partial^2 v}{\partial b^2}\right]
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial a\partial b}\mathrm{e}^{-u} &= \frac{\partial}{\partial a}\left[\frac{\partial}{\partial b}\mathrm{e}^{-u}\right] = \frac{\partial}{\partial a}\left[-\frac{\partial u}{\partial b}\mathrm{e}^{-u}\right] = -\frac{\partial^2 u}{\partial a\partial b}\mathrm{e}^{-u} - \frac{\partial u}{\partial b}\frac{\partial}{\partial a}\mathrm{e}^{-u} = \mathrm{e}^{-u}\left[\frac{\partial u}{\partial a}\cdot\frac{\partial u}{\partial b} - \frac{\partial^2 u}{\partial a\partial b}\right]\\
\frac{\partial^2}{\partial a\partial b}\mathrm{e}^{-v} &= \frac{\partial}{\partial a}\left[\frac{\partial}{\partial b}\mathrm{e}^{-v}\right] = \frac{\partial}{\partial a}\left[-\frac{\partial v}{\partial b}\mathrm{e}^{-v}\right] = -\frac{\partial^2 v}{\partial a\partial b}\mathrm{e}^{-v} - \frac{\partial v}{\partial b}\frac{\partial}{\partial a}\mathrm{e}^{-v} = \mathrm{e}^{-v}\left[\frac{\partial v}{\partial a}\cdot\frac{\partial v}{\partial b} - \frac{\partial^2 v}{\partial a\partial b}\right]
\end{aligned}
\tag{21}
$$

Thus

$$
\begin{aligned}
\left[\frac{\partial^2}{\partial a^2} + 2\frac{\partial^2}{\partial a\partial b} + \frac{\partial^2}{\partial b^2}\right]\mathrm{e}^{-u} &= \mathrm{e}^{-u}\left\{\left[\left(\frac{\partial u}{\partial a}\right)^2 + \left(\frac{\partial u}{\partial b}\right)^2\right] - \left[\frac{\partial^2 u}{\partial a^2} + \frac{\partial^2 u}{\partial b^2}\right] + 2\left[\frac{\partial u}{\partial a}\cdot\frac{\partial u}{\partial b} - \frac{\partial^2 u}{\partial a\partial b}\right]\right\}\\
\left[\frac{\partial^2}{\partial a^2} + 2\frac{\partial^2}{\partial a\partial b} + \frac{\partial^2}{\partial b^2}\right]\mathrm{e}^{-v} &= \mathrm{e}^{-v}\left\{\left[\left(\frac{\partial v}{\partial a}\right)^2 + \left(\frac{\partial v}{\partial b}\right)^2\right] - \left[\frac{\partial^2 v}{\partial a^2} + \frac{\partial^2 v}{\partial b^2}\right] + 2\left[\frac{\partial v}{\partial a}\cdot\frac{\partial v}{\partial b} - \frac{\partial^2 v}{\partial a\partial b}\right]\right\}
\end{aligned}
\tag{22}
$$

Substituting (16) and (17) into (22), we can obtain

$$
\begin{aligned}
\left[\frac{\partial^2}{\partial a^2} + 2\frac{\partial^2}{\partial a\partial b} + \frac{\partial^2}{\partial b^2}\right]\mathrm{e}^{-u} &= \mathrm{e}^{-u}\left\{\left[16j^4(a+b)^2 + 16j^4(a+b)^2\right] - \left[4j^2 + 4j^2\right] + 2\left[16j^2(a+b)^2 - 4j^2\right]\right\}\\
&= \mathrm{e}^{-u}\left\{64j^4(a+b)^2 - 16j^2\right\}\\
&= 16j^2\mathrm{e}^{-u}\left[4j^2(a+b)^2 - 1\right] = 16j^2\mathrm{e}^{-u}(2u-1)\\
\left[\frac{\partial^2}{\partial a^2} + 2\frac{\partial^2}{\partial a\partial b} + \frac{\partial^2}{\partial b^2}\right]\mathrm{e}^{-v} &= \mathrm{e}^{-v}\left\{4^2\left[j^2 + (j-1)^2\right]\frac{v}{2} - [4j^2 + 4(j-1)^2] + 2\left[16(j-1)j\frac{v}{2} - 4j(j-1)\right]\right\}\\
&= \mathrm{e}^{-v}\left\{(32j^2 - 32j + 8)v - 16j^2 + 16j - 4\right\}
\end{aligned}
\tag{23}
$$

For the constants λ and ξ, we have

$$
\frac{\partial}{\partial a}\left[\mathrm{e}^{-v}(\lambda v - \xi)\right] = \mathrm{e}^{-v}[\lambda + \xi - \lambda v]\frac{\partial v}{\partial a}, \quad \frac{\partial}{\partial b}\left[\mathrm{e}^{-v}(\lambda v - \xi)\right] = \mathrm{e}^{-v}[\lambda + \xi - \lambda v]\frac{\partial v}{\partial b}. \tag{24}
$$

Therefore,

$$
\left[\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right]\mathrm{e}^{-v}(\lambda v - \xi) = \mathrm{e}^{-v}[\lambda + \xi - \lambda v]\left(\frac{\partial v}{\partial a} + \frac{\partial v}{\partial b}\right) \tag{25}
$$

2.2 Infinite Series

For the convergent series

$$
\sum_{j=-\infty}^{\infty} j^r \mathrm{e}^{-j^2\gamma c^2} \tag{26}
$$

where $c > 0$, $\gamma > 0$ and $r \in \mathbb{N}$, we have

$$
\sum_{j=-\infty}^{\infty} j^r \mathrm{e}^{-j^2\gamma c^2} = \sum_{j=-\infty}^{\infty} (-j)^r \mathrm{e}^{-j^2\gamma c^2} = (-1)^k \sum_{j=-\infty}^{\infty} j^r \mathrm{e}^{-j^2\gamma c^2} \tag{27}
$$

Thus, for the odd $r = 2m+1$, we have

$$
\sum_{j=-\infty}^{\infty} j^{2m+1} \mathrm{e}^{-j^2\gamma c^2} = 0, \quad m = 0, 1, 2, \cdots. \tag{28}
$$

For the even $r = 2m$, we have

$$\sum_{j=-\infty}^{\infty} j^{2k} \mathrm{e}^{-j^2\gamma c^2} = 1 + 2\sum_{j=1}^{\infty} j^{2m} \mathrm{e}^{-j^2\gamma c^2}, \quad m = 0, 1, 2, \cdots . \tag{29}$$

For the convergent series

$$\sum_{j=-\infty}^{\infty} g(j) \mathrm{e}^{-j^2\gamma c^2}, \tag{30}$$

where $g(\cdot)$ is a function, we have

$$\sum_{j=-\infty}^{\infty} g(j) \mathrm{e}^{-j^2\gamma c^2} = \sum_{j=-\infty}^{\infty} g(j+m) \mathrm{e}^{-(j+m)^2\gamma c^2}, \quad m \in \mathbb{Z} \tag{31}$$

since we always have $m + \mathbb{Z} = \mathbb{Z}$ for any $m \in \mathbb{Z}$.

3 High Order Expansion of $\Phi_n(a, b)$

Note that the formulae listed in the subsection 2.1 will be encountered and adopted frequently in this section.

3.1 Expression for $\Phi_n^{(0)}(a, b)$

It is trivial that we have

$$\Phi_n^{(0)}(a, b) = \mathcal{D}_n^0 \, \Phi(a, b) = \Phi(a, b) = \sum_{j=-\infty}^{\infty} \left[\mathrm{e}^{-2j^2(a+b)^2} - \mathrm{e}^{-2(ja+(j-1)b)^2} \right] \tag{32}$$

3.2 Expression for $\Phi_n^{(1)}(a, b)$

It is easy to find that

$$\begin{aligned} \Phi_n^{(1)}(a, b) = \mathcal{D}_n \, \Phi(a, b) &= \frac{1}{6\sqrt{n}} \left[\frac{\partial}{\partial a} + \frac{\partial}{\partial b} \right] \Phi(a, b) \\ &= \frac{1}{6\sqrt{n}} \sum_{j=-\infty}^{\infty} \left[-\frac{\partial u}{\partial a} \mathrm{e}^{-u} + \frac{\partial v}{\partial a} \mathrm{e}^{-v} - \frac{\partial u}{\partial b} \mathrm{e}^{-u} + \frac{\partial v}{\partial b} \mathrm{e}^{-u} \right] \\ &= \frac{1}{6\sqrt{n}} \sum_{j=-\infty}^{\infty} \left\{ -8j^2(a+b)\mathrm{e}^{-u} + 4(2j-1)[ja + (j-1)b]\mathrm{e}^{-v} \right\}. \end{aligned} \tag{33}$$

3.3 Expression for $\Phi_n^{(2)}(a, b)$

With the help of the formulae listed in the subsection 2.1, we have

$$\begin{aligned} \Phi_n^{(2)}(a, b) &= \frac{1}{2!} \cdot \mathcal{D}_n^2 \, \Phi(a, b) = \frac{1/2!}{(6\sqrt{n})^2} \left[\frac{\partial}{\partial a} + \frac{\partial}{\partial b} \right]^2 \Phi(a, b) \\ &= \frac{1/2!}{36n} \sum_{j=-\infty}^{\infty} \left[\frac{\partial^2}{\partial a^2} + 2\frac{\partial^2}{\partial a \partial b} + \frac{\partial^2}{\partial b^2} \right] \left(\mathrm{e}^{-u} - \mathrm{e}^{-v} \right) \\ &= \frac{1/2!}{36n} \sum_{j=-\infty}^{\infty} \left\{ 16j^2\mathrm{e}^{-u}(2u-1) - \mathrm{e}^{-v} \left[(32j^2 - 32j + 8)v - 16j^2 + 16j - 4 \right] \right\} \\ &= \frac{1}{18n} \sum_{j=-\infty}^{\infty} \left\{ 4j^2(2u-1)\mathrm{e}^{-u} - (2j-1)^2(2v-1)\mathrm{e}^{-v} \right\}. \end{aligned} \tag{34}$$

3.4 Expression for $\Phi_n^{(3)}(a,b)$

According to the iterative formula (8), we can calculate the $\Phi_n^{(3)}(a,b)$ from the $\Phi_n^{(2)}(a,b)$.

$$
\begin{aligned}
\Phi_n^{(3)}(a,b) &= \frac{1}{3}\,\mathcal{D}_n\,\Phi_n^{(2)}(a,b) \\
&= \frac{1/3}{6\sqrt{n}}\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right]\frac{1}{18n}\sum_{j=-\infty}^{\infty}\left\{4j^2\mathrm{e}^{-u}(2u-1)-\mathrm{e}^{-v}\left[(8j^2-8j+2)v-4j^2+4j-1\right]\right\} \\
&= \frac{1/3}{108n^{3/2}}\sum_{j=-\infty}^{\infty}\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right]\left\{4j^2\mathrm{e}^{-u}(2u-1)-(2j-1)^2\mathrm{e}^{-v}(2v-1)\right\}.
\end{aligned}
\tag{35}
$$

With the help of (25), we have

$$
\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right]4j^2\mathrm{e}^{-u}(2u-1) = 4j^2\mathrm{e}^{-u}(3-2u)\left(\frac{\partial u}{\partial a}+\frac{\partial u}{\partial b}\right) = 32j^4(a+b)\mathrm{e}^{-u}(3-2u). \tag{36}
$$

and

$$
\begin{aligned}
\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right](2j-1)^2\mathrm{e}^{-v}(2v-1) &= (2j-1)^2\mathrm{e}^{-v}[3-2v]\left(\frac{\partial v}{\partial a}+\frac{\partial v}{\partial b}\right) \\
&= 4(2j-1)^3[ja+(j-1)b]\mathrm{e}^{-v}(3-2v).
\end{aligned}
\tag{37}
$$

Finally, we have

$$
\Phi_n^{(3)}(a,b) = \frac{1/3}{108n^{3/2}}\sum_{j=-\infty}^{\infty}\left\{32j^4(a+b)(3-2u)\mathrm{e}^{-u}-4(2j-1)^3[ja+(j-1)b](3-2v)\mathrm{e}^{-v}\right\}. \tag{38}
$$

3.5 Expression for $\Phi_n^{(4)}(a,b)$

The $\Phi_n^{(4)}(a,b)$ can be computed from $\Phi_n^{(3)}(a,b)$ via the iterative formula (8).

$$
\begin{aligned}
\Phi_n^{(4)}(a,b) &= \frac{1}{4}\,\mathcal{D}_n\,\Phi_n^{(3)}(a,b) \\
&= \frac{1/4}{6\sqrt{n}}\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right]\frac{1}{324n^{3/2}}\sum_{j=-\infty}^{\infty}\left\{32j^4(a+b)(3-2u)\mathrm{e}^{-u}-4(2j-1)^3[ja+(j-1)b](3-2v)\mathrm{e}^{-v}\right\} \\
&= \frac{1/4}{1944n^2}\sum_{j=-\infty}^{\infty}\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right]\left\{32j^4(a+b)(3-2u)\mathrm{e}^{-u}-4(2j-1)^3[ja+(j-1)b](3-2v)\mathrm{e}^{-v}\right\}.
\end{aligned}
\tag{39}
$$

By

$$
\begin{aligned}
&\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right]32j^4(a+b)(3-2u)\mathrm{e}^{-u} \\
&= 64j^4(a+b)(3-2u)\mathrm{e}^{-u}+32j^4(a+b)(2u-5)\mathrm{e}^{-u}\left(\frac{\partial u}{\partial a}+\frac{\partial u}{\partial b}\right) \\
&= 64j^4\mathrm{e}^{-u}(4u^2-12u+3)
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
&\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right]4(2j-1)^3[ja+(j-1)b](3-2v)\mathrm{e}^{-v} \\
=&4(2j-1)^4(3-2v)\mathrm{e}^{-v}+4(2j-1)^3[ja+(j-1)b](2v-5)\mathrm{e}^{-v}\left(\frac{\partial v}{\partial a}+\frac{\partial v}{\partial b}\right) \\
=&4(2j-1)^4\mathrm{e}^{-v}(4v^2-12v+3)
\end{aligned}
\tag{41}
$$

we immediately obtain

$$
\Phi_n^{(4)}(a,b) = \frac{1/4}{1944n^2}\sum_{j=-\infty}^{\infty}\left\{64j^4\mathrm{e}^{-u}(4u^2-12u+3)-4(2j-1)^4\mathrm{e}^{-v}(4v^2-12v+3)\right\} \tag{42}
$$

3.6 Expression for $\Phi_n^{(5)}(a,b)$

The $\Phi_n^{(5)}(a,b)$ can be computed from $\Phi_n^{(4)}(a,b)$ via the iterative formula (8). Actually, we have

$$\begin{aligned}\Phi_n^{(5)}(a,b) &= \frac{1}{5}\,\mathcal{D}_n\,\Phi_n^{(4)}(a,b)\\ &= \frac{1/5}{6\sqrt{n}}\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right]\frac{1}{1944n^2}\sum_{j=-\infty}^{\infty}\left\{16j^4\mathrm{e}^{-u}(4u^2-12u+3)-(2j-1)^4\mathrm{e}^{-v}(4v^2-12v+3)\right\}\\ &= \frac{1/5}{11664n^{5/2}}\sum_{j=-\infty}^{\infty}\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right]\left\{16j^4\mathrm{e}^{-u}(4u^2-12u+3)-(2j-1)^4\mathrm{e}^{-v}(4v^2-12v+3)\right\}.\end{aligned} \tag{43}$$

With the help of

$$\begin{aligned}\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right]16j^4\mathrm{e}^{-u}(4u^2-12u+3) &= 16j^4(-4u^2+20u-15)\mathrm{e}^{-u}\left(\frac{\partial u}{\partial a}+\frac{\partial u}{\partial b}\right)\\ &= 128j^6(a+b)\mathrm{e}^{-u}(-4u^2+20u-15)\end{aligned} \tag{44}$$

and

$$\begin{aligned}\left[\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right](2j-1)^4\mathrm{e}^{-v}(4v^2-12v+3) &= (2j-1)^4(-4v^2+20v-15)\mathrm{e}^{-v}\left(\frac{\partial v}{\partial a}+\frac{\partial v}{\partial b}\right)\\ &= 4(2j-1)^5(-4v^2+20v-15)\mathrm{e}^{-v}[aj+b(j-1)],\end{aligned} \tag{45}$$

we have

$$\begin{aligned}&\Phi_n^{(5)}(a,b)\\ =&\frac{1/5}{11664n^{5/2}}\sum_{j=-\infty}^{\infty}\left\{128j^6(a+b)\mathrm{e}^{-u}(-4u^2+20u-15)+4(2j-1)^5[aj+b(j-1)](4v^2-20v+15)\mathrm{e}^{-v}\right\}\end{aligned} \tag{46}$$

3.7 Expressions for $Q_i(a,b)$

When the expressions for $\Phi_n^{(i)}(a,b)$ are available, we can find the expressions of $Q_i(a,b)$ easily. For the order parameter $k=5$, we have

$$\begin{aligned}\Phi_n(a,b) &= \sum_{i=0}^{k}\Phi_n^{(i)}(a,b)+\mathcal{O}\left(n^{-(k+1)/2}\right)=\sum_{i=0}^{5}\frac{Q_i(a,b)}{n^{i/2}}+\mathcal{O}\left(n^{-(k+1)/2}\right)\\ &= \sum_{j=-\infty}^{\infty}\left[\mathrm{e}^{-u}-\mathrm{e}^{-v}\right]\\ &\quad-\frac{1}{6\sqrt{n}}\sum_{j=-\infty}^{\infty}\left[8j^2(a+b)\mathrm{e}^{-u}-4(2j-1)[ja+(j-1)b]\mathrm{e}^{-v}\right]\\ &\quad+\frac{1/2}{36n}\sum_{j=-\infty}^{\infty}\left\{16j^2\mathrm{e}^{-u}(2u-1)-4(2j-1)^2\mathrm{e}^{-v}(2v-1)\right\}\\ &\quad-\frac{1/3}{108n^{3/2}}\sum_{j=-\infty}^{\infty}\left\{32j^4(a+b)\mathrm{e}^{-u}(2u-3)-4(2j-1)^3[ja+(j-1)b]\mathrm{e}^{-v}(2v-3)\right\}\\ &\quad+\frac{1/4}{1944n^2}\sum_{j=-\infty}^{\infty}\left\{64j^4\mathrm{e}^{-u}(4u^2-12u+3)-4(2j-1)^4\mathrm{e}^{-v}(4v^2-12v+3)\right\}\\ &\quad-\frac{1/5}{11664n^{5/2}}\sum_{j=-\infty}^{\infty}\left\{128j^6(a+b)\mathrm{e}^{-u}(4u^2-20u+15)\right.\\ &\qquad\qquad\left.-4(2j-1)^5[aj+b(j-1)](4v^2-20v+15)\mathrm{e}^{-v}\right\}\\ &\quad+\mathcal{O}\left(n^{-3}\right).\end{aligned} \tag{47}$$

Separating the continuous variables a, b and the discrete variable n, we immediately have

$$\begin{cases} Q_0(a,b) = \sum_{j=-\infty}^{\infty} \left[\mathrm{e}^{-2j^2(a+b)^2} - \mathrm{e}^{-2[ja+(j-1)b]^2} \right] \\ Q_1(a,b) = -\frac{1}{6} \sum_{j=-\infty}^{\infty} \left\{ 8j^2(a+b)\mathrm{e}^{-2j^2(a+b)^2} - 4(2j-1)[ja+(j-1)b]\mathrm{e}^{-v} \right\} \\ Q_2(a,b) = \frac{1}{72} \sum_{j=-\infty}^{\infty} \left\{ 16j^2\mathrm{e}^{-u}(2u-1) - 4(2j-1)^2\mathrm{e}^{-v}(2v-1) \right\} \\ Q_3(a,b) = -\frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{ 32j^4(a+b)\mathrm{e}^{-u}(2u-3) - 4(2j-1)^3[ja+(j-1)b]\mathrm{e}^{-v}(2v-3) \right\} \\ Q_4(a,b) = \frac{1}{7776} \sum_{j=-\infty}^{\infty} \left\{ 64j^4\mathrm{e}^{-u}(4u^2-12u+3) - 4(2j-1)^4\mathrm{e}^{-v}(4v^2-12v+3) \right\} \\ Q_5(a,b) = -\frac{1/5}{11664} \sum_{j=-\infty}^{\infty} \left\{ 128j^6(a+b)\mathrm{e}^{-u}(-4u^2+20u-15) + 4(2j-1)^5[aj+b(j-1)](4v^2-20v+15)\mathrm{e}^{-v} \right\} \end{cases} \tag{48}$$

4 Expressions for $B_i(c)$ in High Order Expansion Formula

4.1 Expression for $B_0(c)$

Since

$$\begin{aligned} \frac{\partial Q_0}{\partial b} &= \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial b} \left[\mathrm{e}^{-u} - \mathrm{e}^{-v} \right] = \sum_{j=-\infty}^{\infty} \left[-\mathrm{e}^{-u} \cdot \frac{\partial u}{\partial b} + \mathrm{e}^{-v} \cdot \frac{\partial v}{\partial b} \right] \\ &= \sum_{j=-\infty}^{\infty} \left[-4j^2(a+b)\mathrm{e}^{-2j^2(a+b)^2} + 4(j-1)[aj+b(j-1)]\mathrm{e}^{-2[aj+b(j-1)]^2} \right] \end{aligned} \tag{49}$$

and

$$\left[\frac{\partial Q_0}{\partial b} \right]_{a=c-b} = \sum_{j=-\infty}^{\infty} \left\{ -4j^2c\mathrm{e}^{-2j^2c^2} + 4(j-1)[cj-b]\mathrm{e}^{-2(cj-b)^2} \right\}, \tag{50}$$

we have

$$\begin{aligned} B_0(c) &= \int_{b=0}^{c} \left[\frac{\partial Q_0}{\partial b} \right]_{a=c-b} \mathrm{d}\, b \\ &= \sum_{j=-\infty}^{\infty} \int_{b=0}^{c} \left\{ -4j^2c\mathrm{e}^{-2j^2c^2} + 4(j-1)[cj-b]\mathrm{e}^{-2(cj-b)^2} \right\} \mathrm{d}\, b \\ &= \sum_{j=-\infty}^{\infty} \left\{ -4bj^2c\mathrm{e}^{-2j^2c^2} + (j-1)\mathrm{e}^{-2(cj-b)^2} \right\} \Bigg|_{b=0}^{c} \\ &= \sum_{j=-\infty}^{\infty} \left\{ -4j^2c^2\mathrm{e}^{-2j^2c^2} + (j-1)\mathrm{e}^{-2c^2(j-1)^2} - (j-1)\mathrm{e}^{-2c^2j^2} \right\} \\ &= \sum_{j=-\infty}^{\infty} \left\{ -4j^2c^2\mathrm{e}^{-2j^2c^2} + j\mathrm{e}^{-2c^2j^2} - (j-1)\mathrm{e}^{-2c^2j^2} \right\} \\ &= \sum_{j=-\infty}^{\infty} (1-4j^2c^2)\mathrm{e}^{-2j^2c^2} \\ &= 1 - 2\sum_{j=1}^{\infty} (4j^2c^2-1)\mathrm{e}^{-2j^2c^2} \end{aligned} \tag{51}$$

Obviously, the $B_0(c)$ is the same as the $A(c)$ obtained by Kuiper [2].

4.2 Expression for $B_1(c)$

Since

$$\begin{aligned}
-\frac{\partial Q_1}{\partial b} =& \frac{1}{6}\sum_{j=-\infty}^{\infty}\frac{\partial}{\partial b}\left[8j^2(a+b)\mathrm{e}^{-u} - 4(2j-1)[ja+(j-1)b]\mathrm{e}^{-v}\right] \\
=& \frac{1}{6}\sum_{j=-\infty}^{\infty}\left[8j^2\mathrm{e}^{-u} - 8j^2(a+b)\mathrm{e}^{-u}\cdot\frac{\partial u}{\partial b} - 4(j-1)(2j-1)\mathrm{e}^{-v} + 4(2j-1)[aj+b(j-1)]\mathrm{e}^{-v}\cdot\frac{\partial v}{\partial b}\right] \\
=& \frac{1}{6}\sum_{j=-\infty}^{\infty}\left\{8j^2\mathrm{e}^{-2j^2(a+b)^2}[1-4j^2(a+b)^2] - 4(j-1)(2j-1)\mathrm{e}^{-2[aj+b(j-1)]^2}[1-4(aj+b(j-1))^2]\right\}
\end{aligned} \tag{52}$$

and

$$-\left[\frac{\partial Q_1}{\partial b}\right]_{a=c-b} = \frac{1}{6}\sum_{j=-\infty}^{\infty}\left\{8j^2\mathrm{e}^{-2j^2c^2}[1-4j^2c^2] - 4(j-1)(2j-1)\mathrm{e}^{-2(cj-b)^2}[1-4(cj-b)^2]\right\}, \tag{53}$$

we have

$$\begin{aligned}
-B_1(c) =& \int_{b=0}^{c}\left[\frac{\partial Q_1}{\partial b}\right]_{a=c-b}\mathrm{d}\, b \\
=& \frac{1}{6}\sum_{j=-\infty}^{\infty}\int_{b=0}^{c}\left\{8j^2\mathrm{e}^{-2j^2c^2}[1-4j^2c^2] - 4(j-1)(2j-1)\mathrm{e}^{-2(cj-b)^2}[1-4(cj-b)^2]\right\}\mathrm{d}\, b \\
=& \frac{1}{6}\sum_{j=-\infty}^{\infty}\left\{8bj^2\mathrm{e}^{-2j^2c^2}[1-4j^2c^2] + 4(j-1)(2j-1)\mathrm{e}^{-2(cj-b)^2}(cj-b)\right\}\Bigg|_{b=0}^{c} \\
=& \frac{1}{6}\sum_{j=-\infty}^{\infty}\left\{8cj^2\mathrm{e}^{-2j^2c^2}[1-4j^2c^2] + 4(j-1)(2j-1)\left[\mathrm{e}^{-2c^2(j-1)^2}c(j-1) - \mathrm{e}^{-2c^2j^2}cj\right]\right\} \\
=& \frac{1}{6}\sum_{j=-\infty}^{\infty}\mathrm{e}^{-2j^2c^2}\left\{8cj^2[1-4j^2c^2] + 4j(2j+1)cj - 4(j-1)(2j-1)cj\right\} \\
=& \frac{8}{3}\sum_{j=1}^{\infty}cj^2(3-4c^2j^2)\mathrm{e}^{-2c^2j^2}
\end{aligned} \tag{54}$$

Thus

$$B_1(c) = \frac{8}{3}\sum_{j=1}^{\infty}cj^2(4c^2j^2-3)\mathrm{e}^{-2c^2j^2}, \tag{55}$$

which is the same as the $B(c)$ obtained by Kuiper [2].

4.3 Expression for $B_2(c)$

Since

$$\begin{aligned}
\frac{\partial Q_2}{\partial b} =& \frac{1}{72}\sum_{j=-\infty}^{\infty}\frac{\partial}{\partial b}\left\{16j^2\mathrm{e}^{-u}(2u-1) - 4(2j-1)^2\mathrm{e}^{-v}(2v-1)\right\} \\
=& \frac{1}{72}\sum_{j=-\infty}^{\infty}\left\{16j^2\mathrm{e}^{-u}(3-2u)\cdot\frac{\partial u}{\partial b} - 4(2j-1)^2\mathrm{e}^{-v}(3-2v)\cdot\frac{\partial v}{\partial b}\right\} \\
=& \frac{1}{72}\sum_{j=-\infty}^{\infty}\Big\{64j^4(a+b)\mathrm{e}^{-2j^2(a+b)^2}(3-4j^2(a+b)^2) \\
& -16(j-1)(2j-1)^2[aj+b(j-1)]\mathrm{e}^{-2[aj+b(j-1)]^2}[3-4(aj+b(j-1))^2]\Big\}
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
&\left[\frac{\partial Q_2}{\partial b}\right]_{a=c-b} \\
&= \frac{1}{72}\sum_{j=-\infty}^{\infty}\left\{64cj^4\mathrm{e}^{-2j^2c^2}(3-4j^2c^2) - 16(j-1)(2j-1)^2(cj-b)\mathrm{e}^{-2(cj-b)^2}[3-4(cj-b)^2]\right\},
\end{aligned} \tag{57}$$

we have

$$
\begin{aligned}
B_2(c) &= \int_{b=0}^{c} \left[\frac{\partial Q_2}{\partial b}\right]_{a=c-b} \mathrm{d}\, b \\
&= \frac{1}{72} \sum_{j=-\infty}^{\infty} \int_{b=0}^{c} \left\{64cj^4 \mathrm{e}^{-2j^2c^2}(3-4j^2c^2) - 16(j-1)(2j-1)^2(cj-b)\mathrm{e}^{-2(cj-b)^2}[3-4(cj-b)^2]\right\} \mathrm{d}\, b \\
&= \frac{1}{72} \sum_{j=-\infty}^{\infty} \left\{64bcj^4 \mathrm{e}^{-2j^2c^2}(3-4j^2c^2) + 4(j-1)(2j-1)^2\mathrm{e}^{-2(cj-b)^2}[4(cj-b)^2-1]\right\} \Big|_{b=0}^{c} \\
&= \frac{1}{72} \sum_{j=-\infty}^{\infty} \left\{64c^2j^4 \mathrm{e}^{-2j^2c^2}(3-4j^2c^2) + 4(j-1)(2j-1)^2\mathrm{e}^{-2c^2(j-1)^2}[4c^2(j-1)^2-1]\right\} \\
&\quad - \frac{1}{72} \sum_{j=-\infty}^{\infty} \left\{4(j-1)(2j-1)^2\mathrm{e}^{-2c^2j^2}[4c^2j^2-1]\right\} \\
&= \frac{1}{72} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-2j^2c^2} \left[64c^2j^4(3-4j^2c^2) + 4j(2j+1)^2[4c^2j^2-1] - 4(j-1)(2j-1)^2(4c^2j^2-1)\right] \\
&= \frac{1}{72} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-2j^2c^2} \left[64c^2j^4(3-4j^2c^2) + 4(4c^2j^2-1)(12j^2-4j+1)\right] \\
&= \frac{1}{18} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-2j^2c^2} \left[16c^2j^4(3-4j^2c^2) + (4c^2j^2-1)(12j^2+1)\right] \\
&= -\frac{1}{18} + \frac{1}{9} \sum_{j=1}^{\infty} \left\{\mathrm{e}^{-2j^2c^2} \left[4c^2j^2(-16c^2j^4+24j^2+1) - 12j^2 - 1\right]\right\}
\end{aligned}
\tag{58}
$$

4.4 Expression for $B_3(c)$

By

$$
\begin{aligned}
-\frac{\partial Q_3}{\partial b} &= \frac{1}{324} \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial b} \left\{32j^4(a+b)\mathrm{e}^{-u}(2u-3) - 4(2j-1)^3[ja+(j-1)b]\mathrm{e}^{-v}(2v-3)\right\} \\
&= \frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{32j^4\mathrm{e}^{-u}(2u-3) + 32j^4(a+b)\mathrm{e}^{-u}(5-2u)\cdot\frac{\partial u}{\partial b}\right\} \\
&\quad - \frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{4(j-1)(2j-1)^3\mathrm{e}^{-v}(2v-3) + 4(2j-1)^3[ja+(j-1)b]\mathrm{e}^{-v}(5-2v)\cdot\frac{\partial v}{\partial b}\right\} \\
&= \frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{32j^4\mathrm{e}^{-2j^2(a+b)^2}[4j^2(a+b)^2(6-4j^2(a+b)^2)-3]\right\} \\
&\quad - \frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{4(j-1)(2j-1)^3\mathrm{e}^{-2(aj+b(j-1))^2}[4(aj+b(j-1))^2(6-4(aj+b(j-1))^2)-3]\right\}
\end{aligned}
\tag{59}
$$

and

$$
\begin{aligned}
-\left[\frac{\partial Q_3}{\partial b}\right]_{a=c-b} &= \frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{32j^4\mathrm{e}^{-2j^2c^2}[4j^2c^2(6-4j^2c^2)-3]\right\} \\
&\quad - \frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{4(j-1)(2j-1)^3\mathrm{e}^{-2(cj-b)^2}[4(cj-b)^2(6-4(cj-b)^2)-3]\right\},
\end{aligned}
\tag{60}
$$

we have

$$
\begin{aligned}
-B_3(c) =& \int_{b=0}^{c} \left[\frac{\partial Q_3}{\partial b}\right]_{a=c-b} \mathrm{d}\, b \\
=& \frac{1}{324} \sum_{j=-\infty}^{\infty} \int_{b=0}^{c} \left\{32 j^4 \mathrm{e}^{-2j^2c^2} [4j^2c^2(6-4j^2c^2)-3]\right\} \mathrm{d}\, b \\
& - \frac{1}{324} \sum_{j=-\infty}^{\infty} \int_{b=0}^{c} \left\{4(j-1)(2j-1)^3 \mathrm{e}^{-2(cj-b)^2} [4(cj-b)^2(6-4(cj-b)^2)-3]\right\} \mathrm{d}\, b \\
=& \frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{32 c j^4 \mathrm{e}^{-2j^2c^2} [4j^2c^2(6-4j^2c^2)-3]\right\} \\
& + \frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{4(j-1)(2j-1)^3 (cj-b) \mathrm{e}^{-2(cj-b)^2} [4(cj-b)^2-3]\right\} \Big|_{b=0}^{c} \\
=& \frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{32 c j^4 \mathrm{e}^{-2j^2c^2} [4j^2c^2(6-4j^2c^2)-3]\right\} \\
& + \frac{1}{324} \sum_{j=-\infty}^{\infty} \left\{4(j-1)(2j-1)^3 c(j-1) \mathrm{e}^{-2c^2(j-1)^2} [4c^2(j-1)^2-3]\right. \\
& \qquad\qquad \left. -4cj(j-1)(2j-1)^3 \mathrm{e}^{-2c^2j^2} [4c^2j^2-3]\right\} \\
=& \frac{1}{324} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-2j^2c^2} \left\{32 c j^4 [4j^2c^2(6-4j^2c^2)-3] + (4c^2j^2-3) 4cj [j(2j+1)^3 - (j-1)(2j-1)^3]\right\} \\
=& \frac{1}{324} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-2j^2c^2} \left\{32 c j^4 [4j^2c^2(6-4j^2c^2)-3] + (4c^2j^2-3) 4cj [-1+8j-12j^2+32j^3]\right\} \\
=& \frac{1}{324} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-2j^2c^2} \left\{32 c j^4 [4j^2c^2(6-4j^2c^2)-3] + (4c^2j^2-3) 4c [8j^2+32j^4]\right\} \\
=& \frac{16}{81} \sum_{j=1}^{\infty} c j^2 \mathrm{e}^{-2j^2c^2} \left\{-16c^4j^6 + 40c^2j^4 + 4c^2j^2 - 15j^2 - 3\right\}
\end{aligned}
\tag{61}
$$

Hence

$$
B_3(c) = \frac{16}{81} \sum_{j=1}^{\infty} c j^2 \mathrm{e}^{-2j^2c^2} \left\{16c^4j^6 - 40c^2j^4 - 4c^2j^2 + 15j^2 + 3\right\} \tag{62}
$$

4.5 Expression for $B_4(c)$

According to

$$
\begin{aligned}
\frac{\partial Q_4}{\partial b} =& \frac{1}{7776} \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial b} \left\{64 j^4 \mathrm{e}^{-u}(4u^2-12u+3) - 4(2j-1)^4 \mathrm{e}^{-v}(4v^2-12v+3)\right\} \\
=& \frac{1}{1944} \sum_{j=-\infty}^{\infty} \left\{16 j^4 \mathrm{e}^{-u}(-4u^2+20u-15) \cdot \frac{\partial u}{\partial b} + (2j-1)^4 \mathrm{e}^{-v}(4v^2-20v+15) \cdot \frac{\partial v}{\partial b}\right\} \\
=& \frac{1}{1944} \sum_{j=-\infty}^{\infty} \left\{64 j^6 (a+b) \mathrm{e}^{-2j^2(a+b)^2} (-16j^4(a+b)^4 + 40j^2(a+b)^2 - 15)\right\} \\
& + \frac{1}{1944} \sum_{j=-\infty}^{\infty} \left\{4(j-1)(2j-1)^4 (aj+b(j-1)) \mathrm{e}^{-2(aj+b(j-1))^2} [16(aj+b(j-1))^4 \right. \\
& \qquad\qquad \left. -40(aj+b(j-1))^2 + 15]\right\}
\end{aligned}
\tag{63}
$$

and

$$\left[\frac{\partial Q_4}{\partial b}\right]_{a=c-b} = \frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{64j^6 c\mathrm{e}^{-2j^2c^2}(-16j^4c^4+40j^2c^2-15)\Big\} + \frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{4(j-1)(2j-1)^4(cj-b)\mathrm{e}^{-2(cj-b)^2}[16(cj-b)^4-40(cj-b)^2+15]\Big\} \tag{64}$$

we can find that

$$\begin{aligned}
B_4(c) = &\int_{b=0}^{c}\left[\frac{\partial Q_4}{\partial b}\right]_{a=c-b}\mathrm{d}\,b\\
=&\frac{1}{1944}\sum_{j=-\infty}^{\infty}\int_{b=0}^{c}\Big\{64j^6 c\mathrm{e}^{-2j^2c^2}(-16j^4c^4+40j^2c^2-15)\Big\}\,\mathrm{d}\,b\\
&+\frac{1}{1944}\sum_{j=-\infty}^{\infty}\int_{b=0}^{c}\Big\{4(j-1)(2j-1)^4(cj-b)\mathrm{e}^{-2(cj-b)^2}[16(cj-b)^4-40(cj-b)^2+15]\Big\}\,\mathrm{d}\,b\\
=&\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{64j^6 c^2\mathrm{e}^{-2j^2c^2}(-16j^4c^4+40j^2c^2-15)\Big\}\\
&+\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{(j-1)(2j-1)^4\mathrm{e}^{-2(cj-b)^2}[16(cj-b)^4-24(cj-b)^2+3]\Big\}\Big|_{b=0}^{c}\\
=&\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{64j^6 c^2\mathrm{e}^{-2j^2c^2}(-16j^4c^4+40j^2c^2-15)\Big\}\\
&+\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{(j-1)(2j-1)^4\mathrm{e}^{-2c^2(j-1)^2}[16c^4(j-1)^4-24c^2(j-1)^2+3]\Big\}\\
&-\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{(j-1)(2j-1)^4\mathrm{e}^{-2c^2j^2}[16c^4j^4-24c^2j^2+3]\Big\}\\
=&\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{64j^6 c^2\mathrm{e}^{-2j^2c^2}(-16j^4c^4+40j^2c^2-15)\Big\}\\
&+\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{[j(2j+1)^4-(j-1)(2j-1)^4]\mathrm{e}^{-2c^2j^2}[16c^4j^4-24c^2j^2+3]\Big\}\\
=&\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{64j^6 c^2\mathrm{e}^{-2j^2c^2}(-16j^4c^4+40j^2c^2-15)\Big\}\\
&+\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{(80j^4-32j^3+40j^2-8j+1)\mathrm{e}^{-2c^2j^2}[16c^4j^4-24c^2j^2+3]\Big\}\\
=&\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{\mathrm{e}^{-2j^2c^2}[(80j^4-32j^3+40j^2-8j+1)(16c^4j^4-24c^2j^2+3)\\
&\qquad -64c^2j^6(16c^4j^4-40c^2j^2+15)]\Big\}\\
=&\frac{1}{1944}\sum_{j=-\infty}^{\infty}\Big\{\mathrm{e}^{-2j^2c^2}[16c^4j^4(-64c^2j^6+240j^4+40j^2+1)\\
&\qquad -24c^2j^2(120j^4+40j^2+1)+120j^2(2j^2+1)+3]\Big\}\\
=&\frac{1}{648}+\frac{1}{972}\sum_{j=1}^{\infty}\Big\{\mathrm{e}^{-2j^2c^2}[16c^4j^4(-64c^2j^6+240j^4+40j^2+1)\\
&\qquad -24c^2j^2(120j^4+40j^2+1)+120j^2(2j^2+1)+3]\Big\}
\end{aligned} \tag{65}$$

4.6 Expression for $B_5(c)$

Simple but cumbersome calculations show that

$$
\begin{aligned}
-\frac{\partial Q_5}{\partial b} =& \frac{1/5}{11664} \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial b} \big\{128j^6(a+b)\mathrm{e}^{-u}(4u^2-20u+15) \\
& \qquad\qquad -4(2j-1)^5[aj+b(j-1)](4v^2-20v+15)\mathrm{e}^{-v}\big\} \\
=& \frac{1/5}{11664} \sum_{j=-\infty}^{\infty} \left\{128j^6\mathrm{e}^{-u}(4u^2-20u+15)+128j^6(a+b)\mathrm{e}^{-u}(-4u^2+28u-35)\cdot\frac{\partial u}{\partial b}\right\} \\
& - \frac{1/5}{11664} \sum_{j=-\infty}^{\infty} \big\{4(j-1)(2j-1)^5\mathrm{e}^{-v}(4v^2-20v+15)\big\} \\
& - \frac{1/5}{11664} \sum_{j=-\infty}^{\infty} \left\{4(2j-1)^5(aj+b(j-1))\mathrm{e}^{-v}(-4v^2+28v-35)\cdot\frac{\partial v}{\partial b}\right\} \\
=& \frac{1/5}{11664} \sum_{j=-\infty}^{\infty} \big\{128j^6\mathrm{e}^{-u}(-8u^3+60u^2-90u+15) \\
& \qquad -4(j-1)(2j-1)^5\mathrm{e}^{-v}(-8v^3+60v^2-90v+15)\big\} \\
=& \frac{1/5}{11664} \sum_{j=-\infty}^{\infty} \left\{128j^6\mathrm{e}^{-2(a+b)^2j^2}(-64(a+b)^6j^6+240(a+b)^4j^4-180(a+b)^2j^2+15)\right\} \\
& - \frac{1/5}{11664} \sum_{j=-\infty}^{\infty} \Big\{4(j-1)(2j-1)^5\mathrm{e}^{-2(aj+b(j-1))^2}(-64(aj+b(j-1))^6 \\
& \qquad\qquad +240(aj+b(j-1))^4-180(aj+b(j-1))^2+15)\Big\}
\end{aligned}
\tag{66}
$$

thus we have

$$
\begin{aligned}
& -\left[\frac{\partial Q_5}{\partial b}\right]_{a=c-b} \\
=& \frac{1/5}{11664} \sum_{j=-\infty}^{\infty} \left\{128j^6\mathrm{e}^{-2c^2j^2}(-64c^6j^6+240c^4j^4-180c^2j^2+15)\right\} \\
& - \frac{1/5}{11664} \sum_{j=-\infty}^{\infty} \left\{4(j-1)(2j-1)^5\mathrm{e}^{-2(cj-b)^2}[-64(cj-b)^6+240(cj-b)^4-180(cj-b)^2+15]\right\}
\end{aligned}
\tag{67}
$$

Hence, the $B_5(c)$ can be written by

$$
\begin{aligned}
&-B_5(c)\\
=&\int_{b=0}^{c}\left[\frac{\partial Q_5}{\partial b}\right]_{a=c-b}\mathrm{d}\,b\\
=&\frac{1/5}{11664}\sum_{j=-\infty}^{\infty}\int_{b=0}^{c}\left\{128j^6\mathrm{e}^{-2c^2j^2}(-64c^6j^6+240c^4j^4-180c^2j^2+15)\right\}\mathrm{d}\,b\\
&-\frac{1/5}{11664}\sum_{j=-\infty}^{\infty}\int_{b=0}^{c}\left\{4(j-1)(2j-1)^5\mathrm{e}^{-2(cj-b)^2}(-64(cj-b)^6+240(cj-b)^4-180(cj-b)^2+15)\right\}\mathrm{d}\,b\\
=&\frac{1/5}{11664}\sum_{j=-\infty}^{\infty}\left\{128cj^6\mathrm{e}^{-2c^2j^2}(-64c^6j^6+240c^4j^4-180c^2j^2+15)\right\}\\
&+\frac{1/5}{11664}\sum_{j=-\infty}^{\infty}\left\{4(j-1)(2j-1)^5(cj-b)\mathrm{e}^{-2(cj-b)^2}(16(cj-b)^4-40(cj-b)^2+15)\right\}\Big|_{b=0}^{c}\\
=&\frac{1/5}{11664}\sum_{j=-\infty}^{\infty}\left\{128cj^6\mathrm{e}^{-2c^2j^2}(-64c^6j^6+240c^4j^4-180c^2j^2+15)\right\}\\
&+\frac{1/5}{11664}\sum_{j=-\infty}^{\infty}\left\{4c(j-1)^2(2j-1)^5\mathrm{e}^{-2c^2(j-1)^2}(16c^4(j-1)^4-40c^2(j-1)^2+15)\right\}\\
&-\frac{1/5}{11664}\sum_{j=-\infty}^{\infty}\left\{4cj(j-1)(2j-1)^5\mathrm{e}^{-2c^2j^2}(16c^4j^4-40c^2j^2+15)\right\}\\
=&\frac{1/5}{11664}\sum_{j=-\infty}^{\infty}\left\{128cj^6\mathrm{e}^{-2c^2j^2}(-64c^6j^6+240c^4j^4-180c^2j^2+15)\right\}\\
&+\frac{1/5}{11664}\sum_{j=-\infty}^{\infty}\left\{4cj(192j^5-80j^4+160j^3-40j^2+12j-1)\mathrm{e}^{-2c^2j^2}(16c^4j^4-40c^2j^2+15)\right\}\\
=&\frac{2}{3645}\sum_{j=1}^{\infty}\Big\{16\mathrm{e}^{-2c^2j^2}\left[16c^5j^6(-32c^2j^6+168j^4+40j^2+3)-40c^3j^4(84j^4+40j^2+3)\right.\\
&\qquad\left.+15cj^2(56j^4+40j^2+3)\right]\Big\}
\end{aligned}
\tag{68}
$$

Therefore,

$$
\begin{aligned}
&B_5(c)\\
=&\frac{2}{3645}\sum_{j=1}^{\infty}\Big\{16\mathrm{e}^{-2c^2j^2}\left[16c^5j^6(32c^2j^6-168j^4-40j^2-3)+40c^3j^4(84j^4+40j^2+3)\right.\\
&\qquad\left.-15cj^2(56j^4+40j^2+3)\right]\Big\}
\end{aligned}
\tag{69}
$$

Acknowledgements

This work was supported in part by the Hainan Provincial Natural Science Foundation of China under grant number 720RC616 and in part by the National Natural Science Foundation of China under grant number 62167003.

References

[1] D. A. Darling. The Kolmogorov-Smirnov, Cramer-von Mises Tests. *The Annals of Mathematical Statistics*, 28(4):823–838, 1957. `http://www.jstor.org/stable/2237048`, published by Institute of Mathematical Statistics.

[2] Nicolas H. Kuiper. Tests concerning random points on a circle. In *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen*, volume 63 of *A: Mathemaical Statistics*, pages 38–47, 1960. Available online: `https://core.ac.uk/download/pdf/82631793.pdf`.