Notes on Twistor Theory

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Abstract: These are introductory lecture notes on the theory of twistors presented at the GraSP school in 2025.

Contents

1 Spinors and twistors

1.1 Spinors in Minkowski spacetime

Recall that the (orthochronous proper) Lorentz group $SO(1,3)$ can be viewed as the group of matrices acting on $\mathbb{R}^4$ which leave the bilinear form

$$\langle x, \eta y \rangle \doteq x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 = \eta_{ab} x^a y^b, \quad \forall x, y \in \mathbb{R}^4, \tag{1.1}$$

invariant, where η is the Minkowski metric written as

$$\eta = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.2}$$

This means that we are interested in matrices Λ such that

$$\langle \Lambda x, \Lambda \eta y \rangle = \langle x, \eta y \rangle \iff \Lambda^T \eta \Lambda = \eta. \tag{1.3}$$

We usually call $\mathbb{R}^4$ equipped with the metric η *Minkowski spacetime* and the Lorentz transformations Λ can be seen as automorphisms of this space.

Now, given some vector $v \in \mathbb{R}^4$, one can construct a 2×2 Hermitian matrix by

$$v^{\alpha\dot{\alpha}} \doteq \frac{(\sigma_a)^{\alpha\dot{\alpha}}}{\sqrt{2}} v^a = \frac{1}{\sqrt{2}} \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 + v^3 \end{pmatrix}, \tag{1.4}$$

where the σ_a are the Pauli matrices. The indices α and $\dot{\alpha}$ run from $\{0, 1\}$ and $\{\dot{0}, \dot{1}\}$ respectively.

If we compute the determinant of this matrix, we find that

$$2 \det\left(v^{\alpha\dot{\alpha}}\right) = \eta_{ab} v^a v^b. \tag{1.5}$$

Hence, the Lorentz transformations defined by the relations in equation (1.3) can be recast in this notation as the 2×2 matrices which preserve the determinant of the matrix in (1.4):

$$v^{\alpha\dot{\alpha}} \to \tilde{v}^{\alpha\dot{\alpha}} = t^{\alpha}{}_{\beta} v^{\beta\dot{\beta}} \bar{t}^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{1.6}$$

where $\bar{t}^{\dot{\alpha}}{}_{\dot{\beta}} = (t^{\dagger})^{\alpha}{}_{\beta} = (t^{-1})^{\alpha}{}_{\beta}$. Hence, the matrices t are part of the $SL(2,\mathbb{C})$ group of 2×2 complex unitary matrices. The indices α and $\dot{\alpha}$ are then said to live in a $(\frac{1}{2}, 0)$ and a $(0, \frac{1}{2})$ representation of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ respectively.

The transformation in (1.6) defines a linear transformation on the vector v^a preserving its length. We are then led to a group homomorphism $SL(2,\mathbb{C}) \to SO(1,3)$ which is onto. The kernel of this homomorphism consists of the identities $\pm I_{SL(2,\mathbb{C})}$, so, by the fundamental theorem of group homomorphisms [1] we get an *isomorphism*

$$SO(1,3) \simeq SL(2,\mathbb{C})/\{+I, -I\} \doteq PSL(2,\mathbb{C}), \tag{1.7}$$

where $PSL(2,\mathbb{C})$ is sometimes called the *special projective group.* The $SL(2,\mathbb{C})$ is called the *universal cover* of the Lorentz group.

Now, we are led to consider what exactly are the objects on which the elements of $SL(2,\mathbb{C})$ acts on. That is, what exactly is the relationship between the two-component complex vectors which live in the representation space of $SL(2,\mathbb{C})$ and the vectors which live in Minkowski spactime?

We can get a clue of this answer by considering null or light-like vectors. Recall that a vector v^a is called null if $\eta_{ab} v^a v^b = 0$, that is, if its norm vanishes. By equation (1.5), this means that the determinant of $v^{\alpha\dot{\alpha}}$ vanishes. This implies then the the rank of the matrix is 1 and that we can write

$$v^{\alpha\dot{\alpha}}_{\text{null}} = a^{\alpha} \tilde{a}^{\dot{\alpha}}, \tag{1.8}$$

that is, the outer product of two two-component *spinors* a and $\tilde{a}$. The converse is also true: any matrix of the form $a^{\alpha}\tilde{a}^{\dot{\alpha}}$ has rank one (exercise). Because each one of these spinors live in a different representation of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, we say they have *opposite chiralities.* The $\dot{\alpha}$ is said to have *positive chirality* and the α is said to have *negative chirality.*

So now we are working with two-component complex vectors on which we act with elements of $SL(2,\mathbb{C})$. This means that, at the level of a vector space, we are in fact working in $\mathbb{C}^2$. In the next section, we will see how to recover $\mathbb{R}^4$. To upgrade $\mathbb{C}^2$ to a "spacetime", we should equip it with a metric tensor analogous to the Minkowski metric.

This is done by selecting the Levi-Civita symbols in two dimensions:

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{\dot{\alpha}\dot{\beta}}. \tag{1.9}$$

It is easy to see that this is an element of $SL(2,\mathbb{C})$ and that it is invariant under the action of that group (exercise). The inverses are defined by

$$\epsilon^{\alpha\beta}\epsilon_{\gamma\beta} = \delta^{\alpha}{}_{\beta}, \quad \epsilon^{\alpha\beta}\epsilon_{\alpha\beta} = 2. \tag{1.10}$$

We will then use this tensor to raise and lower indices. Now, unlike the Minkowski metric (1.2), the Levi-Civita symbol in equation (1.9) is anti-symmetric (or skew-symmetric), meaning $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$. Hence, we must fix our convention on how exactly we raise and lower such indices. Our convention will be "lowering to the right and raising to the left", i.e.:

$$a_{\alpha} \doteq a^{\beta}\epsilon_{\beta\alpha}, \quad b^{\alpha} \doteq \epsilon^{\alpha\beta} b_{\beta}. \tag{1.11}$$

The same convention holds for dotted indices. We can use these conventions to define the dual vector of $v^{\alpha\dot{\alpha}}$ defined in equation (1.4), which is written as $v_{\alpha\dot{\alpha}}$ and yields $v^{\alpha\dot{\alpha}}v_{\alpha\dot{\alpha}} = 2\det\left(v^{\alpha\dot{\alpha}}\right)$ (exercise). With these, we can write an line element in $(\mathbb{C}^2, \epsilon_{\alpha\beta})$ as

$$ds^2 = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}dx^{\alpha\dot{\alpha}}dx^{\beta\dot{\beta}}. \tag{1.12}$$

To conclude this section, we introduce the so-called $SL(2,\mathbb{C})$-invariant inner products

$$\langle\kappa\omega\rangle \doteq \kappa^{\alpha}\omega_{\alpha} = \kappa^{\alpha}\omega^{\beta}\epsilon_{\beta\alpha}, \quad [\tilde{\kappa}\tilde{\omega}] \doteq \tilde{\kappa}^{\dot{\alpha}}\tilde{\omega}_{\dot{\alpha}} = \tilde{\kappa}^{\dot{\alpha}}\tilde{\omega}^{\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}}. \tag{1.13}$$

Notice that, unlike the usual notion of an inner-product, the anti-symmetric nature of the Levi-Civita tensor forces these inner-products to be themselves anti-symmetric. Moreover, if one takes either of the inner-products of a spinor κ with itself, one gets a vanishing result (exercise). These inner-products can be related to the usual inner-product of null vectors in Minkowski spacetime: as we saw in equation (1.8), a null vector is written as the product of two spinors of opposite chirality. Hence, one can easily see (exercise) that for two null vectors $v^{\alpha\dot{\alpha}}_{\text{null}} = \kappa^{\alpha}\tilde{\kappa}^{\dot{\alpha}}$ and $w^{\alpha\dot{\alpha}}_{\text{null}} = \omega^{\alpha}\omega^{\dot{\alpha}}$ we have

$$v_{\text{null}} \cdot w_{\text{null}} = \langle\kappa\omega\rangle[\tilde{\kappa}\tilde{\omega}] \tag{1.14}$$

where the dot product is understood to be the usual inner-product in Minkowski spacetime.

The use of twistor theory in the context of string theory and scattering amplitudes has origin in the so-called Parke and Taylor scattering formula of n massless gluons. Since they are massless, the momentum vector can be written as a product of two spinors

$$p_i^{\mu} = \pi_i^{\alpha}\tilde{\pi}_i^{\dot{\alpha}}. \tag{1.15}$$

The scattering amplitude at tree-level is then given in terms of the inner products defined above:

$$\mathcal{A}_n = \frac{\langle\pi_i\pi_j\rangle^4\delta^{(4)}(\sum_k p_k)}{\langle\pi_1\pi_2\rangle\langle\pi_2\pi_3\rangle...\langle\pi_{n-1}\pi_n\rangle\langle\pi_n\pi_1\rangle}. \tag{1.16}$$

The extension of this formula to $\mathcal{N}=4$ super Yang-Mills by Nair led Witten to formulate the latter as a string theory in a specific twistor space.

1.2 Complexifying Minkowski

We saw that we have a natural correspondence between $(\mathbb{R}^4, \eta_{ab})$ and $(\mathbb{C}^2, \epsilon_{\alpha\beta})$ if we look at them as metric spaces with a non-positive definite metric. In general, a generic spacetime $(\mathcal{M}, g)$ can be fully described if we consider the line element

$$ds^2 = g_{ab}(x)dx^a dx^b, \tag{1.17}$$

for a generic metric g. We can then define the *complexification* of $(\mathcal{M}, g)$, denoted by $(\mathcal{M}_{\mathbb{C}}, g)$, as allowing the coordinates x^a to take complex values and making $g(x)$ a holomorphic function of our coordinates (meaning there is no dependence on the complex conjugate $\bar{x}^a$). We will focus on the complexification of Minkowski spacetime throughout these lectures.

This means that we will be effectively working in $\mathbb{C}^4$. The isometry group is $SO(4,\mathbb{C})$ which is locally isomorphic to $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. This means that any vector on $\mathcal{M}_{\mathbb{C}}$ can be represented by a pair of $SL(2,\mathbb{C})$ indices.

Even though the line element looks the same after complexification, i.e.,

$$ds^2 = \eta_{ab}dx^a dx^b = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \tag{1.18}$$

we now allow the coordinates to take complex values, which makes the notion of a "signature" irrelevant. In fact, one can obtain any version of "Minkowski" spacetime by selecting different real "slices" of $\mathcal{M}_{\mathbb{C}}$.

How do we do this? The answer is by adopting different complex conjugations. Notice that we can coordinatize the complex space by $x^{\alpha\dot{\alpha}}$ defined in (1.4). If we define the complex conjugate of $x^{\alpha\dot{\alpha}}$ by

$$(x^{\alpha\dot{\alpha}})^\dagger \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{x}^0 + \bar{x}^3 & \bar{x}^1 - i\bar{x}^2 \\ \bar{x}^1 - i\bar{x}^2 & \bar{x}^0 - \bar{x}^3 \end{pmatrix}, \tag{1.19}$$

that is, simply taking the conjugate transpose of $x^{\alpha\dot{\alpha}}$, then we can set $x^{\alpha\dot{\alpha}} = (x^{\alpha\dot{\alpha}})^\dagger$ to recover the real part of the coordinates and hence *Lorentzian signature*. We notice that this conjugation is carried over to the spinors:

$$\bar{\kappa}^{\dot{\alpha}} = (\bar{a}, \bar{b}), \quad \bar{\tilde{\omega}}^{\alpha} = (\bar{c}, \bar{d}). \tag{1.20}$$

This conjugation allows us to write the null vector correspondence in (1.8) for any real null vector in terms of κ^α and $\bar{\kappa}^{\dot{\alpha}}$.

This seems simple, but we can also recover Euclidean signature if we define a slightly different complex conjugation:

$$\hat{x}^{\alpha\dot{\alpha}} \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{x}^0 - \bar{x}^3 & -\bar{x}^1 + i\bar{x}^2 \\ -\bar{x}^1 - i\bar{x}^2 & \bar{x}^0 + \bar{x}^3 \end{pmatrix}. \tag{1.21}$$

We now demand that $x^{\alpha\dot{\alpha}} = \hat{x}^{\alpha\dot{\alpha}}$, which forces (exercise)

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + iy^3 & iy^1 + y^2 \\ iy^1 - y^2 & x^0 - iy^3 \end{pmatrix}. \tag{1.22}$$

with $x^0, y^1, y^2, y^3 \in \mathbb{R}$ (just take $x^j = z^j + iy^j$). Taking the determinant now gives (exercise)

$$2\det\big(x^{\alpha\dot{\alpha}}\big) = (x^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2, \tag{1.23}$$

which is just the Euclidean metric on $\mathbb{R}^4$. As in the Lorentzian case, this induces a complex conjugation on the spinors which goes as

$$\hat{\kappa}^{\alpha} = (-\bar{b}, \bar{a}), \quad \hat{\tilde{\omega}}^{\dot{\alpha}} = (-\bar{d}, \bar{c}). \tag{1.24}$$

One can check (exercise) that this operation does *not* square to the identity and in fact one needs to apply this conjugation four times to go back to the original spinor. This has a nice consequence: there are no non-trivial combinations $\kappa^\alpha \tilde{\omega}^{\dot{\alpha}}$ which is preserved under the hat-conjugation, which means there are no real null vectors in Euclidean space.

***Exercise*:** Define the complex conjugate

$$\overline{x^{\alpha\dot{\alpha}}} \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{x}^0 + \bar{x}^3 & \bar{x}^1 + i\bar{x}^2 \\ \bar{x}^1 - i\bar{x}^2 & \bar{x}^0 - \bar{x}^3 \end{pmatrix}. \tag{1.25}$$

Now, demand that $x^{\alpha\dot{\alpha}} = \overline{x^{\alpha\dot{\alpha}}}$. What kind of signature do we get? What is the action of this conjugation on the spinors? What kind of spinors are they?

1.3 Twistor space

We are now ready to define the twistor space. This will be a subset of the 3-dimensional complex projective space, $\mathbb{CP}^3$. We can take many subsets of this space and in the following lectures we will explore the different possibilities.

The $\mathbb{CP}^3$ space is obtained from $\mathbb{C}^4$ with homogeneous coordinates $Z^A = (Z^1, Z^2, Z^3, Z^4)$ excluding the origin and with the equivalence relation

$$rZ^A \sim Z^A, \forall r \in \mathbb{C}\backslash\{0\} \equiv \mathbb{C}^\times. \tag{1.26}$$

The *twistor space* of $\mathcal{M}_\mathbb{C}$, denoted as $\mathbb{PT}$, is obtained by first dividing the homogenous coordinates Z^A into two Weyl spinors of opposite chirality

$$Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha), \tag{1.27}$$

subjected to the constraint

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha. \tag{1.28}$$

Equations (1.28) are referred to as *incidence relations* and they select a complex plane $\mathbb{C}^2 \subset \mathbb{C}^4$, much like a linear equation of the form $y = ax$ selects a line in the real plane with coordinates (x, y). By considering the scaling relation in (1.26), we define a $\mathbb{CP}^1 \subset \mathbb{PT}$.

Now, the relationship of $\mathbb{PT}$ with spacetime is rather intriguing if we consider that the linear coefficient $x^{\alpha\dot{\alpha}}$ in (1.28) in fact corresponds to a point in spacetime. The *twistor correspondence* then tells us that a point in Minkowski spacetime corresponds to a linearly and holomorphically embedded Riemann sphere in twistor space.

In fact, *any* holomorphic linear embedding of a Riemann sphere can be put in the form of the incidence relations (1.28). This is done by considering $\sigma_a = (\sigma_0, \sigma_1)$ as homogeneous coordinates on $\mathbb{CP}^1$ and defining the maps

$$\mu^{\dot{\alpha}} = b^{\dot{\alpha}a}\sigma_a, \quad \lambda_a = c^a_\alpha\sigma_a, \tag{1.29}$$

where $(b^{\dot{\alpha}a}, c^a_\alpha)$ are 8 complex parameters to be determined. We can use the 3 automorphisms of $\mathbb{CP}^3$ together with the projective reescaling (1.26) to reduce these to 4 complex degrees of freedom, giving

$$\mu^{\dot{\alpha}} = b^{\dot{\alpha}a}\sigma_a, \quad \lambda_a = \delta^a_\alpha\sigma_a. \tag{1.30}$$

So, a point x in spacetime corresponds to a linearly embedded Riemann sphere (sometimes called a "line") $\mathbb{CP}^1 \equiv X \subset \mathbb{PT}$, making this relation highly non-local.

We can ask the converse question: what does a *point* in twistor space correspond to in spacetime? If we consider a point $Z \in \mathbb{PT}$ as the intersection of two lines X and Y, then

$$X \cap Y = \{Z \in \mathbb{PT}\} \iff \mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha, \quad \mu^{\dot{\alpha}} = y^{\alpha\dot{\alpha}}\lambda_\alpha. \tag{1.31}$$

This relationship yields

$$(x-y)^{\alpha\dot{\alpha}}\lambda_\alpha = \epsilon^{\alpha\beta}(x-y)_\beta{}^{\dot{\alpha}}\lambda_\alpha = 0, \tag{1.32}$$

which is only non-trivial in two-dimensions if $(x-y)^{\alpha\dot{\alpha}} \propto \lambda^\alpha$. This is just a consequence of the anti-symmetry of the ϵ tensor. We can use the free index to write

$$(x-y)^{\alpha\dot{\alpha}} = \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}, \tag{1.33}$$

for some $\tilde{\lambda}^{\dot{\alpha}}$. But we saw that this means that $(x-y)^{\alpha\dot{\alpha}}$ is a null vector and thus the points x and y are null separated! Furthermore, the point Z in twistor space is obtained by varying the choice of $\tilde{\lambda}^{\dot{\alpha}}$. The result is a 2-plane where every tangent vector has the form $\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$, which is called an α-plane.

In summary, a point in Minkowski spacetime corresponds to a line in twistor space while a point in twistor space corresponds to a null vector in Minkowski spacetime.

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