

Supplementary note for generalized cluster structures on SL_n^\dagger

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Abstract

This is a supplementary note for the main paper *Generalized cluster structures on SL_n^\dagger* that contains explicit examples of generalized cluster structures compatible with π_Γ^\dagger in type A_{n-1} , as well as a list of some of the intrinsic problems of the theory. This note will be updated over time.

Contents

1	Summary of the h-convention	2
1.1	The maps \mathcal{F} , \mathcal{Q} and \mathcal{G}	2
1.2	Initial extended cluster	3
2	Summary of the g-convention	4
2.1	The maps $\mathcal{F}^{\mathrm{op}}$, $\mathcal{Q}^{\mathrm{op}}$ and $\mathcal{G}^{\mathrm{op}}$	4
2.2	Initial extended cluster	5
3	Relation between the h- and g-conventions	6
4	Intrinsic problems	7
4.1	The Poisson structure $\mathcal{F}_*(\pi_\Gamma^\dagger)$	7
4.2	Are there cluster structures for \mathcal{F}_m 's?	7
4.3	Are the g - and h -conventions equivalent?	8
4.4	How is $\mathcal{GC}_h^\dagger(\Gamma, \mathrm{SL}_n^\dagger)$ related to $\mathcal{GC}(\Gamma, D(\mathrm{SL}_n))$?	9
5	Examples in $n = 3$ in the h-convention	12
5.1	The standard BD triple	12
5.2	$\Gamma_1 = \{2\}$, $\Gamma_2 = \{1\}$	13
5.3	$\Gamma_1 = \{1\}$, $\Gamma_2 = \{2\}$	14
6	Examples in $n = 4$ in the h-convention	15
6.1	The standard BD triple	15
6.2	Cremmer-Gervais $i \mapsto i + 1$	17
6.3	Cremmer-Gervais $i \mapsto i - 1$	18
7	Examples in $n = 4$ in the g-convention	19
7.1	The standard BD triple	19
7.2	Cremmer-Gervais $i \mapsto i - 1$	20

1 Summary of the h -convention

In this section, we outline the construction of birational quasi-isomorphism for $\mathcal{GC}_h^\dagger(\Gamma)$, as well as the construction of the initial extended cluster. For all the other information, refer to the main paper [3].

1.1 The maps \mathcal{F} , \mathcal{Q} and \mathcal{G}

Notation. For a generic element $U \in \mathrm{GL}_n$, the element $U_\oplus \in \mathrm{GL}_n$ is an upper triangular matrix and $U_- \in \mathrm{GL}_n$ is a unipotent lower triangular matrix, such that $U = U_\oplus U_-$.

The map \mathcal{F} . Let $\Gamma := (\Gamma_1, \Gamma_2, \gamma)$ be a BD triple of type A_{n-1} . Define the sequence $\mathcal{F}_k : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ of rational maps via

$$\mathcal{F}_0(U) := U, \quad \mathcal{F}_k(U) := \tilde{\gamma}^*[\mathcal{F}_{k-1}(U)_-]U, \quad k \geq 1. \quad (1.1)$$

The birational map $\mathcal{F} : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ is defined as the limit

$$\mathcal{F}(U) := \lim_{k \rightarrow \infty} \mathcal{F}_k(U). \quad (1.2)$$

Since γ is nilpotent, the sequence \mathcal{F}_k stabilizes at $k = \deg \gamma$, so $\mathcal{F}(U) = \mathcal{F}_{\deg \gamma}(U)$. The inverse of \mathcal{F} is given by

$$\mathcal{F}^{-1}(U) := \tilde{\gamma}^*(U_-)^{-1}U. \quad (1.3)$$

The map \mathcal{F} is neither a Poisson map nor a quasi-isomorphism. However, by means of \mathcal{F} one can construct Poisson birational quasi-isomorphisms. For various invariance properties of \mathcal{F} , refer to [3, Section 4.2].

Birational quasi-isomorphisms. Define the birational map $\mathcal{Q} : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ via

$$\mathcal{Q}(U) := \rho(U)^{-1}U\rho(U), \quad \rho(U) := \prod_{i=1}^{\rightarrow} [\tilde{\gamma}^*]^i(U_-). \quad (1.4)$$

The inverse of \mathcal{Q} is given by

$$\mathcal{Q}^{-1}(U) := \mathcal{F}^c(U) := \mathcal{F}(U)\tilde{\gamma}^*(\mathcal{F}(U)_-)^{-1}. \quad (1.5)$$

Let π_Γ^\dagger and $\pi_{\mathrm{std}}^\dagger$ be the Poisson bivectors associated with an arbitrary BD triple Γ and Γ_{std} (of type A_{n-1}), respectively. If the r_0 parts of π_Γ^\dagger and $\pi_{\mathrm{std}}^\dagger$ are the same, then $\mathcal{Q} : (\mathrm{GL}_n, \pi_{\mathrm{std}}^\dagger) \dashrightarrow (\mathrm{GL}_n, \pi_\Gamma^\dagger)$ is a Poisson isomorphism. Moreover, as a map $\mathcal{Q} : (\mathrm{GL}_n, \mathcal{GC}_h^\dagger(\Gamma_{\mathrm{std}})) \dashrightarrow (\mathrm{GL}_n, \mathcal{GC}_h^\dagger(\Gamma))$, it is a birational quasi-isomorphism, with the marked variables given by

$$\{h_{i+1, i+1} \mid i \in \Gamma_2\}. \quad (1.6)$$

If $\tilde{\Gamma} \prec \Gamma$ is another BD triple of type A_{n-1} , then there is a birational quasi-isomorphism $\mathcal{G} : (\mathrm{GL}_n, \mathcal{GC}_h^\dagger(\tilde{\Gamma})) \dashrightarrow (\mathrm{GL}_n, \mathcal{GC}_h^\dagger(\Gamma))$. If $\tilde{\mathcal{Q}}$ is defined as the map \mathcal{Q} , but with respect to the BD triple $\tilde{\Gamma}$, then $\mathcal{G} = \mathcal{Q} \circ \tilde{\mathcal{Q}}$. As a map $\mathcal{G} : (\mathrm{GL}_n, \pi_{\tilde{\Gamma}}^\dagger) \dashrightarrow (\mathrm{GL}_n, \pi_\Gamma^\dagger)$, it is a Poisson isomorphism if the r_0 parts of $\pi_{\tilde{\Gamma}}^\dagger$ and π_Γ^\dagger are the same. The marked variables for \mathcal{G} are given by

$$\{h_{i+1, i+1} \mid i \in \Gamma_2 \setminus \tilde{\Gamma}_2\}. \quad (1.7)$$

For more explicit formulas of \mathcal{G} , refer to [3, Section 4.4, Section 4.5].

1.2 Initial extended cluster

The initial extended cluster comprises three types of functions: c -functions, φ -functions and h -functions. Only the description of the h -functions depends on the choice of the Belavin-Drinfeld triple.

Description of φ - and c -functions. For an element $U \in \mathrm{GL}_n$, let us set

$$\Phi_{kl}(U) := [(U^0)^{[n-k+1, n]} \quad U^{[n-l+1, n]} \quad (U^2)^{\{n\}} \quad \dots \quad (U^{n-k-l+1})^{\{n\}}], \quad k, l \geq 1, \quad k + l \leq n; \quad (1.8)$$

$$s_{kl} := \begin{cases} (-1)^{k(l+1)} & n \text{ is even,} \\ (-1)^{(n-1)/2+k(k-1)/2+l(l-1)/2} & n \text{ is odd.} \end{cases} \quad (1.9)$$

Then the φ -functions are given by

$$\varphi_{kl}(U) := s_{kl} \det \Phi_{kl}(U). \quad (1.10)$$

The c -functions are uniquely defined via

$$\det(I + \lambda U) = \sum_{i=0}^n \lambda^i s_i c_i(U) \quad (1.11)$$

where $s_i := (-1)^{i(n-1)}$ and I is the identity matrix. Note that $c_0 = I$ and $c_n = \det U$.

Description of the h -functions. Let Π be a set of simple roots of type A_{n-1} and $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ be a BD triple. We identify Π with the interval $[1, n-1]$. For a given $\alpha_0 \in \Pi \setminus \Gamma_2$, set $\alpha_t := \gamma(\alpha_{t-1})$, $t \geq 1$. Recall that the sequence $S^\gamma(\alpha_0) := \{\alpha_t\}_{t \geq 0}$ is the γ -string associated to α_0 ; γ -strings partition Π . For each γ -string $S^\gamma(\alpha_0) = \{\alpha_0, \alpha_1, \dots, \alpha_m\}$, for each $i \in [0, m]$ and $j \in [\alpha_i + 1, n]$, set

$$h_{\alpha_i+1, j}(U) := (-1)^{\varepsilon_{\alpha_i+1, j}} \det[\mathcal{F}(U)]_{[\alpha_i+1, n-j+\alpha_i+1]}^{[j, n]} \prod_{t \geq i+1}^m \det[\mathcal{F}(U)]_{[\alpha_t+1, n]}^{[\alpha_t+1, n]} \quad (1.12)$$

where ε_{ij} is defined as

$$\varepsilon_{ij} := (j - i)(n - i), \quad 1 \leq i \leq j \leq n. \quad (1.13)$$

We refer to the functions h_{ij} , $2 \leq i \leq j \leq n$, together with $h_{11}(U) := \det U$ as the h -functions.

Frozen variables. In the case of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{GL}_n)$, the frozen variables are given by the set

$$\{c_1, c_2, \dots, c_{n-1}\} \cup \{h_{i+1, i+1} \mid i \in \Pi \setminus \Gamma_2\} \cup \{h_{11}\}. \quad (1.14)$$

In the case of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{SL}_n)$, $h_{11}(U) = 1$, so this variable is absent. The zero loci of the frozen variables foliate into unions of symplectic leaves of the ambient Poisson variety $(\mathrm{GL}_n, \pi_\mathbf{\Gamma}^\dagger)$ or $(\mathrm{SL}_n, \pi_\mathbf{\Gamma}^\dagger)$. Moreover, the frozen h -variables do not vanish on SL_n^\dagger .

Initial extended cluster. The initial extended cluster Ψ_0 of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{GL}_n)$ is given by the set

$$\{h_{ij} \mid 2 \leq i \leq j \leq n\} \cup \{\varphi_{kl} \mid k, l \geq 1, \quad k + l \leq n\} \cup \{c_1, \dots, c_{n-1}\} \cup \{h_{11}\}. \quad (1.15)$$

The initial extended cluster of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{SL}_n)$ is obtained from Ψ_0 via removing h_{11} .

A generalized cluster mutation. In the initial extended cluster, only the variable φ_{11} is equipped with a nontrivial *string*, which is given by $(1, c_1, \dots, c_{n-1}, 1)$. The generalized mutation relation for φ_{11} reads

$$\varphi_{11}\varphi'_{11} = \sum_{r=0}^n c_r \varphi_{21}^r \varphi_{12}^{n-r}. \quad (1.16)$$

Other mutations of the initial extended cluster follow the usual pattern from the theory of cluster algebras of geometric type.

2 Summary of the g -convention

In this section, we outline the construction of birational quasi-isomorphism for $\mathcal{GC}_g^\dagger(\mathbf{\Gamma})$, as well as the construction of the initial extended cluster. For all the other information, refer to the main paper [3].

2.1 The maps \mathcal{F}^{op} , \mathcal{Q}^{op} and \mathcal{G}^{op}

Notation. For a generic element $U \in \text{GL}_n$, the element $U_+ \in \text{GL}_n$ is a unipotent upper triangular matrix and $U_- \in \text{GL}_n$ is a lower triangular matrix, such that $U = U_+ U_-$.

The map \mathcal{F}^{op} . Let $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ be a BD triple of type A_{n-1} . Define the sequence $\mathcal{F}_k^{\text{op}} : \text{GL}_n \dashrightarrow \text{GL}_n$ of rational maps via

$$\mathcal{F}_0^{\text{op}}(U) := U, \quad \mathcal{F}_k^{\text{op}}(U) := U \tilde{\gamma}[\mathcal{F}_{k-1}^{\text{op}}(U)_+], \quad k \geq 1. \quad (2.1)$$

The birational map $\mathcal{F}^{\text{op}} : \text{GL}_n \dashrightarrow \text{GL}_n$ is defined as the limit

$$\mathcal{F}^{\text{op}}(U) := \lim_{k \rightarrow \infty} \mathcal{F}_k^{\text{op}}(U). \quad (2.2)$$

Since γ is nilpotent, the sequence $\mathcal{F}_k^{\text{op}}$ stabilizes at $k = \deg \gamma$, so $\mathcal{F}^{\text{op}}(U) = \mathcal{F}_{\deg \gamma}^{\text{op}}(U)$. The inverse of \mathcal{F}^{op} is given by

$$(\mathcal{F}^{\text{op}})^{-1}(U) := U \tilde{\gamma}(U_+)^{-1}. \quad (2.3)$$

The map \mathcal{F}^{op} is neither a Poisson map nor a quasi-isomorphism. However, by means of \mathcal{F}^{op} one can construct Poisson birational quasi-isomorphisms in the g -convention. For various invariance properties of \mathcal{F}^{op} , refer to [3, Section 7.1].

Birational quasi-isomorphisms. Define the birational map $\mathcal{Q}^{\text{op}} : \text{GL}_n \dashrightarrow \text{GL}_n$ via

$$\mathcal{Q}^{\text{op}}(U) := \rho^{\text{op}}(U) U (\rho^{\text{op}}(U))^{-1}, \quad \rho^{\text{op}}(U) := \prod_{i=1}^{\leftarrow} [\tilde{\gamma}]^i(U_+). \quad (2.4)$$

The inverse of \mathcal{Q}^{op} is given by the map

$$(\mathcal{Q}^{\text{op}})^{-1}(U) := \mathcal{F}^{\text{op},c}(U) := \tilde{\gamma}(\mathcal{F}^{\text{op}}(U)_+)^{-1} \mathcal{F}^{\text{op}}(U). \quad (2.5)$$

Let $\pi_{\mathbf{\Gamma}}^\dagger$ and π_{std}^\dagger be the Poisson bivectors associated with an arbitrary BD triple $\mathbf{\Gamma}$ and $\mathbf{\Gamma}_{\text{std}}$ (of type A_{n-1}), respectively. If the r_0 parts of $\pi_{\mathbf{\Gamma}}^\dagger$ and π_{std}^\dagger are the same, then $\mathcal{Q}^{\text{op}} : (\text{GL}_n, \pi_{\text{std}}^\dagger) \dashrightarrow (\text{GL}_n, \pi_{\mathbf{\Gamma}}^\dagger)$

is a Poisson isomorphism. Moreover, as a map $\mathcal{Q}^{\text{op}} : (\text{GL}_n, \mathcal{GC}_g^\dagger(\Gamma_{\text{std}})) \dashrightarrow (\text{GL}_n, \mathcal{GC}_g^\dagger(\Gamma))$, it is a birational quasi-isomorphism, with the marked variables given by

$$\{g_{i+1,i+1} \mid i \in \Gamma_1\}. \quad (2.6)$$

If $\tilde{\Gamma} \prec \Gamma$ is another BD triple of type A_{n-1} , then there is a birational quasi-isomorphism $\mathcal{G}^{\text{op}} : (\text{GL}_n, \mathcal{GC}_h^\dagger(\tilde{\Gamma})) \dashrightarrow (\text{GL}_n, \mathcal{GC}_h^\dagger(\Gamma))$. If $\tilde{\mathcal{Q}}^{\text{op}}$ is defined as the map \mathcal{Q}^{op} , but with respect to the BD triple $\tilde{\Gamma}$, then $\mathcal{G}^{\text{op}} = \mathcal{Q}^{\text{op}} \circ \tilde{\mathcal{Q}}^{\text{op}}$. As a map $\mathcal{G}^{\text{op}} : (\text{GL}_n, \pi_{\tilde{\Gamma}}^\dagger) \dashrightarrow (\text{GL}_n, \pi_\Gamma^\dagger)$, it is a Poisson isomorphism if the r_0 parts of $\pi_{\tilde{\Gamma}}^\dagger$ and π_Γ^\dagger are the same. The marked variables for \mathcal{G}^{op} are given by

$$\{g_{i+1,i+1} \mid i \in \Gamma_1 \setminus \tilde{\Gamma}_1\}. \quad (2.7)$$

Explicit formulas for \mathcal{G}^{op} can be obtained from explicit formulas for \mathcal{G} (refer to [3, Section 4.4, Section 4.5, Section 7.3]).

2.2 Initial extended cluster

The initial extended cluster comprises three types of functions: c -functions, ϕ -functions and g -functions. Only the description of the g -functions depends on the choice of the Belavin-Drinfeld triple.

Description of ϕ - and c -functions. For an element $U \in \text{GL}_n$, let us set

$$\Phi'_{kl}(U) := [(U^0)^{[1,k]} \quad U^{[1,l]} \quad (U^2)^{\{1\}} \quad \dots \quad (U^{n-k-l+1})^{\{1\}}], \quad k, l \geq 1, \quad k + l \leq n; \quad (2.8)$$

$$s_{kl} := \begin{cases} (-1)^{k(l+1)} & n \text{ is even,} \\ (-1)^{(n-1)/2+k(k-1)/2+l(l-1)/2} & n \text{ is odd.} \end{cases} \quad (2.9)$$

Then the ϕ -functions are given by

$$\phi_{kl}(U) := s_{kl} \det \Phi'_{kl}(U) \quad (2.10)$$

The c -functions are uniquely defined via

$$\det(I + \lambda U) = \sum_{i=0}^n \lambda^i s_i c_i(U) \quad (2.11)$$

where $s_i := (-1)^{i(n-1)}$ and I is the identity matrix. Note that $c_0 = I$ and $c_n = \det U$ (the c -functions are the same in both g - and h -conventions).

Description of the g -functions. Let Π be a set of simple roots of type A_{n-1} and let $\Gamma := (\Gamma_1, \Gamma_2, \gamma)$ be a BD triple of type A_{n-1} . Let $\mathcal{F}^{\text{op}} : \text{GL}_n \dashrightarrow \text{GL}_n$ be the rational map defined by (2.2). We identify Π with the interval $[1, n-1]$. For a given $\alpha_0 \in \Pi \setminus \Gamma_1$, set $\alpha_t := \gamma^*(\alpha_{t-1})$, $t \geq 1$. Recall that the sequence $S^{\gamma^*}(\alpha_0) := \{\alpha_t\}_{t \geq 0}$ is the γ^* -string associated to α_0 ; γ^* -strings partition Π . For each $\alpha_0 \in \Pi \setminus \Gamma_1$ and the associated γ^* -string $S^{\gamma^*}(\alpha_0) := \{\alpha_i\}_{i=0}^m$, for every $k \in [0, m]$ and $i \in [\alpha_k + 1, n]$, define

$$g_{i, \alpha_k+1}(U) := \det[\mathcal{F}^{\text{op}}(U)]_{[i, n]}^{[\alpha_k+1, n-i+\alpha_k+1]} \prod_{t \geq k+1}^m \det[\mathcal{F}^{\text{op}}(U)]_{[\alpha_t+1, n]}^{[\alpha_t+1, n]}. \quad (2.12)$$

We refer to the functions g_{ij} , $2 \leq j \leq i \leq n$, together with $g_{11}(U) := \det U$ as the g -functions.

Frozen variables. In the case of $\mathcal{GC}_g^\dagger(\Gamma, \mathrm{GL}_n)$, the frozen variables are given by the set

$$\{c_1, c_2, \dots, c_{n-1}\} \cup \{g_{i+1, i+1} \mid i \in \Pi \setminus \Gamma_1\} \cup \{g_{11}\}. \quad (2.13)$$

In the case of $\mathcal{GC}_h^\dagger(\Gamma, \mathrm{SL}_n)$, $g_{11}(U) = 1$, so this variable is absent. The zero loci of the frozen variables foliate into unions of symplectic leaves of the ambient Poisson variety $(\mathrm{GL}_n, \pi_\Gamma^\dagger)$ or $(\mathrm{SL}_n, \pi_\Gamma^\dagger)$. Moreover, the frozen h -variables do not vanish on SL_n^\dagger .

Initial extended cluster. The initial extended cluster Ψ_0 of $\mathcal{GC}_g^\dagger(\Gamma, \mathrm{GL}_n)$ is given by the set

$$\{g_{ij} \mid 2 \leq j \leq i \leq n\} \cup \{\phi_{kl} \mid k, l \geq 1, k + l \leq n\} \cup \{c_1, \dots, c_{n-1}\} \cup \{g_{11}\}. \quad (2.14)$$

The initial extended cluster of $\mathcal{GC}_g^\dagger(\Gamma, \mathrm{SL}_n)$ is obtained from Ψ_0 via removing h_{11} .

A generalized cluster mutation. In the initial extended cluster, only the variable ϕ_{11} is equipped with a nontrivial *string*, which is given by $(1, c_1, \dots, c_{n-1}, 1)$. The generalized mutation relation for ϕ_{11} reads

$$\phi_{11}\phi'_{11} = \sum_{r=0}^n c_r \phi_{21}^r \phi_{12}^{n-r}. \quad (2.15)$$

Other mutations of the initial extended cluster follow the usual pattern from the theory of cluster algebras of geometric type.

3 Relation between the h - and g -conventions

In this section, we briefly mention the relation between the g - and the h -conventions. Let $\Gamma := (\Gamma_1, \Gamma_2, \gamma)$ be an arbitrary BD triple of type A_{n-1} .

Variables. The c -variables in both the h - and the g -conventions are the same. For the other variables in the initial extended clusters, the connection is as follows.

- 1) For ϕ - and φ -functions, $\phi_{kl}(W_0^{-1}UW_0) = \varphi_{kl}(U)$ where $W_0 := \sum_{i=1}^{n-1} (-1)^{i+1} e_{n-i+1, i}$.
- 2) For g_{ij} and h_{ji} from the initial extended clusters of $\mathcal{GC}_h^\dagger(\Gamma)$ and $\mathcal{GC}_g^\dagger(\Gamma^{\mathrm{op}})$, $g_{ij}(U) = (-1)^{\varepsilon_{ji}} h_{ji}(U^T)$ where $\varepsilon_{ji} := (n-j)(i-j)$.

Quivers. The initial quiver $Q_g(\Gamma)$ for the g -convention can be obtained from the initial quiver $Q_h(\Gamma^{\mathrm{op}})$ for the h -convention via the following steps:

- Replace each vertex φ_{kl} with ϕ_{kl} , $2 \leq k+l \leq n$, $k, l \geq 1$ and each h_{ji} with g_{ij} , $2 \leq j \leq i \leq n$;
- For each g_{ij} , $2 \leq j \leq i \leq n$, reverse the orientation of the arrows in its neighborhood;
- For the vertices ϕ_{kl} with $k+l = n$ and $k \geq 2$, add an arrow $\phi_{kl} \rightarrow \phi_{k-1, l+1}$;
- Remove the arrow $\phi_{1, n-1} \rightarrow g_{11}$.

Mutation equivalence. In $n = 3$, the initial extended cluster of $\mathcal{GC}_g^\dagger(\Gamma, \mathrm{GL}_3)$ can be obtained from the initial extended cluster of $\mathcal{GC}_h^\dagger(\Gamma, \mathrm{GL}_3)$ (for any Γ) via a sequence of mutations (see Section 4.3). We conjecture that there is no such sequence in $n \geq 4$.

Birational quasi-isomorphisms. Define \mathcal{F} , \mathcal{Q} and \mathcal{G} relative the BD triple $\mathbf{\Gamma}$, and define \mathcal{F}^{op} , \mathcal{Q}^{op} and \mathcal{G}^{op} relative the opposite BD triple $\mathbf{\Gamma}^{\text{op}}$. Then $\mathcal{F}(U^T) = \mathcal{F}^{\text{op}}(U)^T$, $\mathcal{Q}(U^T) = \mathcal{Q}^{\text{op}}(U)^T$, $\mathcal{G}(U^T) = \mathcal{G}^{\text{op}}(U)^T$.

4 Intrinsic problems

4.1 The Poisson structure $\mathcal{F}_*(\pi_{\mathbf{\Gamma}}^\dagger)$

Let $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ be a BD triple of type A_{n-1} . Define a rational map $\mathcal{C} : \text{GL}_n \dashrightarrow \text{GL}_n$ via

$$\mathcal{C}(U) := U \cdot \rho(U) = U \prod_{k=1}^{\rightarrow} \tilde{\gamma}^*(U_-), \quad U \in \text{GL}_n. \quad (4.1)$$

The map \mathcal{C} is in fact birational, with the inverse given by

$$\mathcal{C}^{-1}(U) = U \cdot \tilde{\gamma}^*(U_-)^{-1}, \quad U \in \text{GL}_n. \quad (4.2)$$

Set $\pi_{\mathcal{F}} := \mathcal{F}_*(\pi_{\mathbf{\Gamma}}^\dagger)$. Since $\mathcal{F}^c(U) = \mathcal{F}(U) \tilde{\gamma}^*(\mathcal{F}(U)_-)^{-1}$, the following diagram is commutative:

$$\begin{array}{ccc} (\text{GL}_n, \pi_{\mathbf{\Gamma}}^\dagger) & \xrightarrow{\mathcal{F}^c} & (\text{GL}_n, \pi_{\text{std}}^\dagger) \\ \mathcal{F} \downarrow & \swarrow \mathcal{C} & \\ (\text{GL}_n, \pi_{\mathcal{F}}) & & \end{array} \quad (4.3)$$

Moreover, all the arrows are birational Poisson isomorphisms (provided the r_0 -parts are the same for all Poisson bivectors). The Poisson bracket $\{\cdot, \cdot\}_{\mathcal{F}}$ that corresponds to $\pi_{\mathcal{F}}$ is given by

$$\begin{aligned} \{f, g\}_{\mathcal{F}} = & \langle R_0 \pi_0 [U, \nabla_U f], [U, \nabla_U g] \rangle + \langle \pi_0 [U, \nabla_U f], \nabla_U^L g \rangle + \\ & + \langle \pi_{>} \nabla_U^L f, \nabla_U^L g \rangle - \langle \pi_{>} \nabla_U^R f, \nabla_U^R g \rangle + \\ & + \langle \frac{1}{1-\gamma} \pi_{>} \nabla_U^R f, \nabla_U^R g \rangle - \langle \nabla_U^R f, \frac{1}{1-\gamma} \pi_{>} \nabla_U^R g \rangle + \\ & + \langle \pi_{\leq} \nabla_U^L f, \text{Ad}_{U \tilde{\gamma}^*(U_-)^{-1}} \frac{1}{1-\gamma} \pi_{>} \nabla_U^R g \rangle - \langle \text{Ad}_{U \tilde{\gamma}^*(U_-)^{-1}} \frac{1}{1-\gamma} \pi_{>} \nabla_U^R f, \pi_{\leq} \nabla_U^L g \rangle. \end{aligned} \quad (4.4)$$

Recall that \mathcal{F}^{-1} is given by

$$\mathcal{F}^{-1}(U) = \tilde{\gamma}^*(U_-)^{-1} \cdot U, \quad U \in \text{GL}_n. \quad (4.5)$$

We find it very intriguing that the maps \mathcal{C}^{-1} and \mathcal{F}^{-1} have very similar formulas. In a sense, $\pi_{\mathcal{F}}$ sits in between π_{std}^\dagger and $\pi_{\mathbf{\Gamma}}^\dagger$, and it can be twisted into either of the Poisson structures via an application of $(\mathcal{F}^{-1})_*$ or $(\mathcal{C}^{-1})_*$. Is there anything interesting that one can say about $\pi_{\mathcal{F}}$, as well as about the induced compatible generalized cluster structure on GL_n ?

4.2 Are there cluster structures for \mathcal{F}_m 's?

Let us fix a BD triple $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ of type A_{n-1} and set

$$\begin{aligned} \{f, g\}_+(U) := & \langle \pi_{>} \nabla_U^R f, \nabla_U^R g \rangle - \langle \pi_{>} \nabla_U^L f, \nabla_U^L g \rangle + \\ & + \langle R_0 \pi_0 [\nabla_U f, U], [\nabla_U g, U] \rangle - \langle \pi_0 [\nabla_U f, U], \nabla_U^L g \rangle, \quad U \in \text{GL}_n, \end{aligned} \quad (4.6)$$

where $\nabla_U^R f = U \cdot \nabla_U f$ and $\nabla_U^L f = \nabla_U f \cdot U$. Let $\hat{h}_{ij}(U) := \det U_{[i, n-j+i]}^{[j, n]}$. During a numerical experimentation¹, we noticed that

$$\{\log \hat{h}_{ij}, \log \hat{h}_{ks}\}_{\text{std}}^\dagger = \{\log \mathcal{F}_m^*(\hat{h}_{ij}), \log \mathcal{F}_m^*(\hat{h}_{ks})\}_+ = \{\log \mathcal{F}^*(\hat{h}_{ij}), \log \mathcal{F}^*(\hat{h}_{ks})\}_\Gamma^\dagger$$

for all $m \in [0, \deg \gamma]$ (r_0 elements are assumed to be the same). A natural question arises: does there exist a sequence of Poisson varieties² (V_m, π_m) such that π_m reduces to $\{\cdot, \cdot\}_+$ for the flag minors of \mathcal{F}_m , and such that there is a generalized cluster structure \mathcal{GC}_m on V_m compatible with π_m ?

4.3 Are the g - and h -conventions equivalent?

By the equivalence we mean that the initial extended clusters of $\mathcal{GC}_h^\dagger(\Gamma)$ and $\mathcal{GC}_g^\dagger(\Gamma)$ can be obtained from one another via a sequence of mutations (and the variables are equal as elements of $\mathcal{O}(\text{GL}_n)$). In [3] we verified that the frozen variables in $\mathcal{GC}_g^\dagger(\Gamma, \text{GL}_n)$ coincide with the frozen variables in $\mathcal{GC}_h^\dagger(\Gamma, \text{GL}_n)$ for any BD triple Γ . As for the equivalence, we were able to confirm for $n = 3$ and all BD triples Γ that $\mathcal{GC}_g^\dagger(\Gamma, \text{GL}_3) = \mathcal{GC}_h^\dagger(\Gamma, \text{GL}_3)$. We conjecture that they are not equivalent for $n \geq 4$. Below we provide examples of mutation sequences that transform the initial cluster of $\mathcal{GC}_h^\dagger(\Gamma, \text{GL}_3)$ into the initial cluster of $\mathcal{GC}_g^\dagger(\Gamma, \text{GL}_3)$. In each case, we know all such sequences of minimal length (available upon request). Let us denote by φ'_{kl} and h'_{ij} the variables in the resulting extended cluster in $\mathcal{GC}_h^\dagger(\Gamma, \text{GL}_3)$.

Case $\Gamma_1 = \Gamma_2 = \emptyset$. The minimal length is 10, the number of distinct sequences of minimal length is 8. An example of such a sequence:

$$\varphi_{12} \rightarrow \varphi_{21} \rightarrow \varphi_{11} \rightarrow h_{23} \rightarrow \varphi_{12} \rightarrow h_{23} \rightarrow \varphi_{11} \rightarrow \varphi_{21} \rightarrow h_{23} \rightarrow \varphi_{21}. \quad (4.7)$$

The correspondence between the variables is given by $\varphi'_{kl}(U) = \phi_{kl}(U)$ and $h'_{ij}(U) = g_{ji}(U)$.

Case $\Gamma_1 = \{2\}, \Gamma_2 = \{1\}$. The minimal length is 11 and the number of sequences is 6. An example of such a sequence:

$$\varphi_{12} \rightarrow \varphi_{21} \rightarrow \varphi_{11} \rightarrow h_{22} \rightarrow h_{23} \rightarrow \varphi_{12} \rightarrow h_{23} \rightarrow \varphi_{11} \rightarrow \varphi_{21} \rightarrow h_{23} \rightarrow \varphi_{21}. \quad (4.8)$$

The correspondence between the variables is given by $\varphi'_{kl}(U) = \phi_{kl}(U)$, $h'_{23}(U) = g'_{32}(U)$, $h'_{22}(U) = g_{33}(U)$, $h_{33}(U) = g_{22}(U)$.

Case $\Gamma_1 = \{1\}, \Gamma_2 = \{2\}$. The minimal length is 13 and the number of sequences is 30. An example of such a sequence:

$$\varphi_{12} \rightarrow h_{23} \rightarrow \varphi_{12} \rightarrow \varphi_{11} \rightarrow h_{23} \rightarrow \varphi_{21} \rightarrow \varphi_{11} \rightarrow h_{23} \rightarrow h_{33} \rightarrow \varphi_{12} \rightarrow \varphi_{11} \rightarrow \varphi_{21} \rightarrow \varphi_{11}. \quad (4.9)$$

The correspondence between the variables is given by $\varphi'_{kl}(U) = \phi_{kl}(U)$, $h'_{23}(U) = g'_{32}(U)$, $h'_{33}(U) = g_{22}(U)$, $h_{22}(U) = g_{33}(U)$.

¹We have verified this identity in $n = 3$, $n = 4$ and $n = 5$ for all BD triples.

²Of course, one can set V_m to be the spectrum of the ring generated by the flags of \mathcal{F}_m . We are interested in the largest possible variety $V_m \subseteq \text{SL}_n$ with the mentioned properties.

4.4 How is $\mathcal{GC}_h^\dagger(\Gamma, \text{SL}_n^\dagger)$ related to $\mathcal{GC}(\Gamma, D(\text{SL}_n))$?

In the work [1], the initial extended cluster of the generalized cluster structure $\mathcal{GC}_h^\dagger(\Gamma_{\text{std}}, \text{SL}_n^\dagger)$ was obtained from the initial extended cluster of $\mathcal{GC}(\Gamma_{\text{std}}, D(\text{SL}_n))$ via a sequence of mutations denoted as \mathcal{S} . A natural question arises: if Γ is any aperiodic oriented BD triple of type A_{n-1} , can the initial extended cluster of $\mathcal{GC}_h^\dagger(\Gamma, \text{SL}_n^\dagger)$ be obtained from the initial extended cluster of $\mathcal{GC}(\Gamma, D(\text{SL}_n))$ that was described in [2]? We found such mutation sequences³ in $n = 3$ and $n = 4$ for all BD triples. We conjecture that the same holds for $n \geq 5$; however, we do not see a relatively simple way of proving it for an arbitrary n (as one can see below, the mutation sequences become rather long and unpredictable).

Let us recall that the initial extended cluster of $\mathcal{GC}(\Gamma, D(\text{SL}_n))$ comprises 5 types of functions: the g -functions, the h -functions, the φ -functions, the f -functions and the c -functions. To resolve the conflict of notation, we will mark the g - and h -functions in $\mathcal{GC}(\Gamma, D(\text{SL}_n))$ with a bar. The \mathcal{S} sequence in $n = 3$ is given by

$$\mathcal{S} := \bar{g}_{32} \rightarrow \bar{g}_{22} \rightarrow \bar{g}_{33} \rightarrow f_{11} \rightarrow \bar{g}_{32}, \quad (4.10)$$

and in $n = 4$,

$$\begin{aligned} \mathcal{S} := & \bar{g}_{42} \rightarrow \bar{g}_{32} \rightarrow \bar{g}_{43} \rightarrow \bar{g}_{22} \rightarrow \bar{g}_{33} \rightarrow \bar{g}_{44} \rightarrow f_{21} \rightarrow f_{11} \rightarrow f_{12} \rightarrow \\ & \rightarrow \bar{g}_{42} \rightarrow \bar{g}_{32} \rightarrow \bar{g}_{43} \rightarrow \bar{g}_{33} \rightarrow \bar{g}_{42}. \end{aligned} \quad (4.11)$$

Below we list the mutation sequences for $n = 3$ and $n = 4$, as well as the correspondence between the variables. The variables in the resulting extended cluster of $\mathcal{GC}(\Gamma, D(\text{SL}_n))$ will be denoted as \bar{g}' , \bar{h}' and f' . The c - and φ -variables for $\mathcal{GC}(\Gamma, D(\text{SL}_n))$ and $\mathcal{GC}_h^\dagger(\Gamma, \text{SL}_n^\dagger)$ are the same. The correspondence between the coordinates (X, Y) in $D(\text{SL}_n)$ and U in SL_n is given by

$$D(\text{SL}_n) \ni (X, Y) \mapsto U := X^{-1}Y \in \text{SL}_n.$$

Note that in the case of $D(\text{GL}_n)$, the below correspondence between the variables is up to an additional factor of $(\det X)^\ell$ for some ℓ that depends on the given variable.

Case $\Gamma_1 = \Gamma_2 = \emptyset$, $n = 3$. The mutation sequence is given by \mathcal{S} . The correspondence is given by $\bar{g}'_{32}(X, Y) = h_{33}(U)$, $f'_{11} = h_{22}(U)$, $\bar{g}'_{22}(X, Y) = h_{23}(U)$.

Case $\Gamma_1 = \{2\}$, $\Gamma_2 = \{1\}$, $n = 3$. The mutation sequence is given by

$$\mathcal{S} \rightarrow \bar{h}_{12} \rightarrow \bar{h}_{22}. \quad (4.12)$$

The correspondence is given by $\bar{h}'_{22}(X, Y) = h_{33}(U)$, $f'_{11}(X, Y) = h_{22}(U)$, $\bar{g}'_{22}(X, Y) = h_{23}(U)$.

Case $\Gamma_1 = \{1\}$, $\Gamma_2 = \{2\}$, $n = 3$. The mutation sequence is given by

$$\mathcal{S} \rightarrow \bar{h}_{13} \rightarrow \bar{h}_{23} \rightarrow \bar{h}_{33} \rightarrow \bar{g}_{33} \rightarrow \bar{g}_{22} \rightarrow \bar{h}_{13} \rightarrow \bar{h}_{23} \rightarrow \bar{h}_{33}. \quad (4.13)$$

The correspondence is given by $\bar{g}'_{33}(X, Y) = h_{23}(U)$, $\bar{h}'_{33}(X, Y) = h_{22}(U)$, $\bar{g}'_{32}(X, Y) = h_{33}(U)$.

³However, we didn't verify whether the sequences are of minimal possible length.

Case $\Gamma_1 = \Gamma_2 = \emptyset$, $n = 4$. The mutation sequence is given by \mathcal{S} . The correspondence is given by $\bar{g}'_{42}(X, Y) = h_{44}(U)$, $\bar{g}'_{32}(X, Y) = h_{34}(U)$, $\bar{g}'_{22}(X, Y) = h_{24}(U)$, $\bar{g}'_{33}(X, Y) = h_{33}(U)$, $f'_{21}(X, Y) = h_{23}(U)$, $f'_{12}(X, Y) = h_{22}(U)$.

Case $\Gamma_1 = \{3\}$, $\Gamma_2 = \{1\}$, $n = 4$. The mutation sequence is given by

$$\mathcal{S} \rightarrow \bar{h}_{12} \rightarrow \bar{h}_{22}. \quad (4.14)$$

The correspondence is given by $\bar{h}'_{22}(X, Y) = h_{44}(U)$, $\bar{g}'_{32}(X, Y) = h_{34}(U)$, $\bar{g}'_{22}(X, Y) = h_{24}(U)$, $\bar{g}'_{33}(X, Y) = h_{33}(U)$, $f'_{21}(X, Y) = h_{23}(U)$, $f'_{12}(X, Y) = h_{22}(U)$.

Case $\Gamma_1 = \{3\}$, $\Gamma_2 = \{2\}$, $n = 4$. The mutation sequence is given by

$$\mathcal{S} \rightarrow \bar{h}_{13} \rightarrow \bar{h}_{23} \rightarrow \bar{h}_{33} \rightarrow f_{11}. \quad (4.15)$$

The correspondence is given by $f'_{11}(X, Y) = h_{44}(U)$, $\bar{g}'_{32}(X, Y) = h_{34}(U)$, $\bar{g}'_{22}(X, Y) = h_{24}(U)$, $\bar{g}'_{33}(X, Y) = h_{33}(U)$, $f'_{21}(X, Y) = h_{23}(U)$, $f'_{12}(X, Y) = h_{22}(U)$.

Case $\Gamma_1 = \{1\}$, $\Gamma_2 = \{3\}$, $n = 4$. The mutation sequence is given by

$$\begin{aligned} \mathcal{S} &\rightarrow \bar{h}_{14} \rightarrow \bar{h}_{24} \rightarrow \bar{h}_{34} \rightarrow \bar{h}_{44} \rightarrow \bar{g}_{44} \rightarrow \bar{g}_{43} \rightarrow \bar{g}_{22} \rightarrow \\ &\rightarrow \bar{h}_{14} \rightarrow \bar{h}_{24} \rightarrow \bar{h}_{34} \rightarrow \bar{h}_{44} \rightarrow \bar{g}_{44} \rightarrow \bar{g}_{22} \rightarrow f_{21} \rightarrow \\ &\rightarrow \bar{h}_{14} \rightarrow \bar{h}_{24} \rightarrow \bar{h}_{34} \rightarrow \bar{h}_{44}. \end{aligned} \quad (4.16)$$

The correspondence is given by $\bar{g}'_{42}(X, Y) = h_{44}(U)$, $\bar{g}'_{32}(X, Y) = h_{34}(U)$, $\bar{g}'_{43}(X, Y) = h_{24}(U)$, $\bar{g}'_{33}(X, Y) = h_{33}(U)$, $g'_{44}(X, Y) = h_{23}(U)$, $h'_{44}(X, Y) = h_{22}(U)$.

Case $\Gamma_1 = \{1\}$, $\Gamma_2 = \{2\}$, $n = 4$. The mutation sequence is given by

$$\begin{aligned} \mathcal{S} &\rightarrow \bar{h}_{13} \rightarrow \bar{h}_{23} \rightarrow \bar{h}_{33} \rightarrow f_{11} \rightarrow \bar{g}_{22} \rightarrow \\ &\rightarrow \bar{h}_{13} \rightarrow \bar{h}_{23} \rightarrow \bar{h}_{33} \rightarrow \bar{g}_{22} \rightarrow f_{21} \rightarrow \\ &\rightarrow \bar{h}_{13} \rightarrow \bar{h}_{23}. \end{aligned} \quad (4.17)$$

The correspondence is given by $\bar{g}'_{42}(X, Y) = h_{44}(U)$, $\bar{g}'_{32}(X, Y) = h_{34}(U)$, $f'_{11}(X, Y) = h_{24}(U)$, $\bar{g}'_{33}(X, Y) = h_{33}(U)$, $\bar{h}'_{33}(X, Y) = h_{23}(U)$, $\bar{h}'_{23}(X, Y) = h_{22}(U)$.

Case $\Gamma_1 = \{2\}$, $\Gamma_2 = \{3\}$, $n = 4$. The mutation sequence is given by

$$\mathcal{S} \rightarrow \bar{h}_{14} \rightarrow \bar{h}_{24} \rightarrow \bar{h}_{34} \rightarrow \bar{h}_{44} \rightarrow \bar{g}_{44} \rightarrow \bar{g}_{43} \rightarrow \bar{g}_{32} \rightarrow \bar{h}_{14} \rightarrow \bar{h}_{24} \rightarrow \bar{h}_{34} \rightarrow \bar{h}_{44} \rightarrow \bar{g}_{44}. \quad (4.18)$$

The correspondence is given by $\bar{g}'_{42}(X, Y) = h_{44}(U)$, $\bar{g}'_{43}(X, Y) = h_{34}(U)$, $\bar{g}'_{22}(X, Y) = h_{24}(U)$, $\bar{g}'_{44}(X, Y) = h_{33}(U)$, $f'_{21}(X, Y) = h_{23}(U)$, $f'_{12}(X, Y) = h_{22}(U)$.

Case $\Gamma_1 = \{2\}$, $\Gamma_2 = \{1\}$, $n = 4$. The mutation sequence is given by

$$\mathcal{S} \rightarrow \bar{h}_{12} \rightarrow \bar{h}_{22} \rightarrow \bar{g}_{32} \rightarrow \bar{h}_{12}. \quad (4.19)$$

The correspondence is given by $\bar{g}'_{42}(X, Y) = h_{44}(U)$, $\bar{h}'_{22}(X, Y) = h_{34}(U)$, $\bar{g}'_{22}(X, Y) = h_{24}(U)$, $\bar{h}'_{12}(X, Y) = h_{33}(U)$, $f'_{21}(X, Y) = h_{23}(U)$, $f'_{12}(X, Y) = h_{22}(U)$.

Case of Cremmer-Gervais, $\Gamma_1 = \{2, 3\}$, $\Gamma_2 = \{1, 2\}$, $\gamma(i) = i - 1$, $i \in \Gamma_1$. The mutation sequence is given by

$$S \rightarrow \bar{h}_{12} \rightarrow \bar{h}_{22} \rightarrow \bar{h}_{13} \rightarrow \bar{h}_{23} \rightarrow \bar{h}_{33} \rightarrow \bar{g}_{32} \rightarrow \bar{h}_{12} \rightarrow \bar{g}_{32} \rightarrow \bar{g}_{33} \rightarrow f_{11}. \quad (4.20)$$

The correspondence is given by $f'_{11}(X, Y) = h_{44}(U)$, $\bar{h}'_{22}(X, Y) = h_{34}(U)$, $\bar{g}'_{22}(X, Y) = h_{24}(U)$, $\bar{h}'_{12}(X, Y) = h_{33}(U)$, $f'_{21}(X, Y) = h_{23}(U)$, $f'_{12}(X, Y) = h_{22}(U)$.

Case of Cremmer-Gervais, $\Gamma_1 = \{1, 2\}$, $\Gamma_2 = \{2, 3\}$, $\gamma(i) = i + 1$, $i \in \Gamma_1$. The mutation sequence is given by

$$\begin{aligned} S \rightarrow \bar{h}_{13} \rightarrow \bar{h}_{23} \rightarrow \bar{h}_{33} \rightarrow \bar{h}_{14} \rightarrow \bar{h}_{24} \rightarrow \bar{h}_{34} \rightarrow \bar{h}_{44} \rightarrow f_{11} \rightarrow \bar{g}_{22} \rightarrow \bar{g}_{44} \rightarrow \bar{g}_{43} \rightarrow \bar{g}_{32} \rightarrow \\ \rightarrow \bar{h}_{13} \rightarrow \bar{h}_{23} \rightarrow \bar{h}_{33} \rightarrow \bar{h}_{14} \rightarrow \bar{h}_{24} \rightarrow \bar{h}_{34} \rightarrow \bar{h}_{44} \rightarrow \bar{g}_{44} \rightarrow f_{21} \rightarrow g_{32} \rightarrow g_{22} \rightarrow h_{14} \rightarrow \\ \rightarrow h_{24} \rightarrow h_{13} \rightarrow h_{34} \rightarrow h_{44} \rightarrow h_{23} \rightarrow g_{33} \rightarrow h_{44} \rightarrow f_{11} \rightarrow h_{33}. \end{aligned} \quad (4.21)$$

The correspondence is given by $\bar{g}'_{42}(X, Y) = h_{44}(U)$, $\bar{g}'_{43}(X, Y) = h_{34}(U)$, $\bar{g}'_{33}(X, Y) = h_{24}(U)$, $\bar{g}'_{44}(X, Y) = h_{33}(U)$, $f'_{11}(X, Y) = h_{23}(U)$, $\bar{h}'_{33}(X, Y) = h_{22}(U)$.

5 Examples in $n = 3$ in the h -convention

5.1 The standard BD triple

The initial quiver is illustrated in Figure 1.

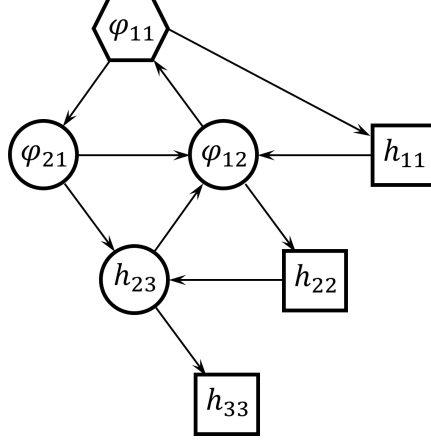


Figure 1. The initial quiver for $\mathcal{GC}_h^\dagger(\Gamma_{std}, \mathrm{GL}_3)$.

The initial variables. The variables in the initial extended cluster are given as follows:

$$c_1(U) = \mathrm{tr}(U), \quad c_2(U) = \frac{1}{2!}(\mathrm{tr}(U)^2 - \mathrm{tr}(U^2)); \quad (5.1)$$

$$\varphi_{21}(U) = u_{13}, \quad \varphi_{12}(U) = \det U_{[1,2]}^{[2,3]} \quad (5.2)$$

$$\varphi_{11}(U) = -\det \begin{bmatrix} u_{13} & (U^2)_{13} \\ u_{23} & (U^2)_{23} \end{bmatrix} = u_{23} \det U_{[1,2]}^{[2,3]} + u_{13} \det U_{[1,2]}^{\{1,3\}}; \quad (5.3)$$

$$h_{23}(U) = -u_{23}u_{33} - u_{13}u_{32}, \quad h_{22}(U) = u_{33} \det U_{[2,3]}^{[2,3]} + u_{32} \det U_{\{1,3\}}^{[2,3]}; \quad (5.4)$$

$$h_{11}(U) = \det U, \quad h_{33}(U) = u_{33}. \quad (5.5)$$

Some 1-step mutations.

$$\varphi'_{11}(U) = \det \begin{bmatrix} u_{12} & u_{13} \\ (U^2)_{12} & (U^2)_{13} \end{bmatrix} = u_{12} \det U_{[1,2]}^{[2,3]} + u_{13} \det U_{\{1,3\}}^{[2,3]}. \quad (5.6)$$

5.2 $\Gamma_1 = \{2\}, \Gamma_2 = \{1\}$

The initial quiver is illustrated in Figure 2.

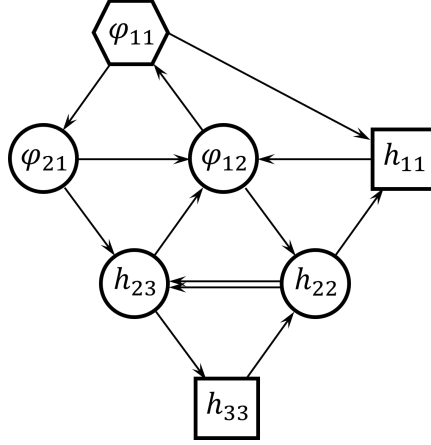


Figure 2. The initial quiver for $\mathcal{GC}_h^\dagger(\Gamma, \mathrm{GL}_3)$ with $\Gamma_1 = \{2\}, \Gamma_2 = \{1\}$.

The initial variables. All the variables in the initial extended cluster are as in $\mathcal{GC}_h^\dagger(\Gamma_{\mathrm{std}}, \mathrm{GL}_3)$ except the variable h_{33} , which is given by

$$h_{33}(U) = u_{33} \det U_{[2,3]}^{[2,3]} + u_{23} \det U_{[2,3]}^{\{1,3\}}. \quad (5.7)$$

Birational quasi-isomorphisms. The birational quasi-isomorphism

$$\mathcal{Q} : (\mathrm{GL}_3, \mathcal{GC}_h^\dagger(\Gamma_{\mathrm{std}})) \dashrightarrow (\mathrm{GL}_3, \mathcal{GC}_h^\dagger(\Gamma))$$

is given by

$$\mathcal{Q}(U) = (I - \alpha(U)e_{32})U(I + \alpha(U)e_{32}), \quad \alpha(U) := \frac{\det U_{[2,3]}^{\{1,3\}}}{\det U_{[2,3]}^{[2,3]}}. \quad (5.8)$$

5.3 $\Gamma_1 = \{1\}, \Gamma_2 = \{2\}$

The initial quiver is illustrated in Figure 3.

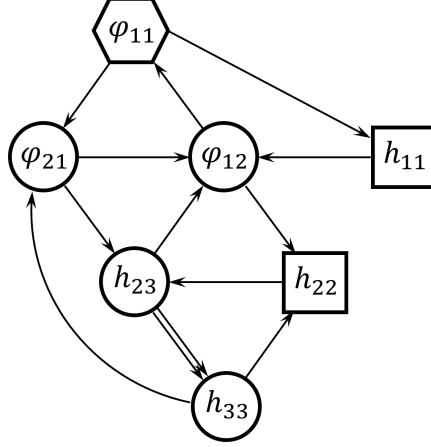


Figure 3. The initial quiver for $\mathcal{GC}_h^\dagger(\Gamma, \mathrm{GL}_3)$ with $\Gamma_1 = \{1\}, \Gamma_2 = \{2\}$.

The initial variables. All the variables in the initial extended cluster are as in $\mathcal{GC}_h^\dagger(\Gamma_{\mathrm{std}}, \mathrm{GL}_3)$ except the variables h_{23} and h_{22} ; these are given by

$$h_{23}(U) = -u_{23}u_{33} - u_{13}u_{32}, \quad h_{22}(U) = u_{33} \det U_{[2,3]}^{[2,3]} + u_{32} \det U_{[2,3]}^{\{1,3\}}. \quad (5.9)$$

Birational quasi-isomorphisms. The birational quasi-isomorphism

$$\mathcal{Q} : (\mathrm{GL}_3, \mathcal{GC}_h^\dagger(\Gamma_{\mathrm{std}})) \dashrightarrow (\mathrm{GL}_3, \mathcal{GC}_h^\dagger(\Gamma))$$

is given by

$$\mathcal{Q}(U) = (I - \alpha(U)e_{21})U(I + \alpha(U)e_{21}), \quad \alpha(U) := \frac{u_{32}}{u_{33}}. \quad (5.10)$$

6 Examples in $n = 4$ in the h -convention

6.1 The standard BD triple

The initial quiver for $\mathcal{GC}_h^\dagger(\Gamma_{\text{std}}, \text{GL}_4)$ is illustrated in Figure 4.

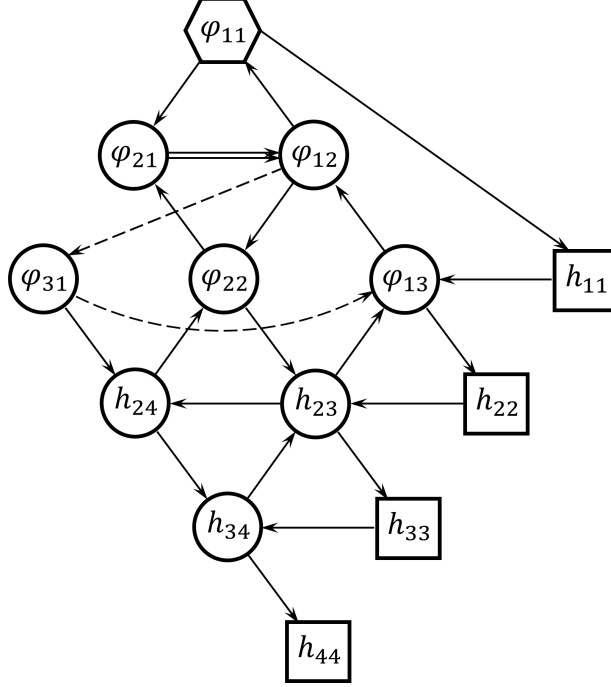


Figure 4. The initial quiver for $\mathcal{GC}_h^\dagger(\Gamma_{\text{std}}, \text{GL}_4)$.

The initial variables. The φ -variables are given by

$$\varphi_{11}(U) = -\det \begin{bmatrix} u_{14} & (U^2)_{14} & (U^3)_{14} \\ u_{24} & (U^2)_{24} & (U^3)_{24} \\ u_{34} & (U^2)_{34} & (U^3)_{34} \end{bmatrix}; \quad (6.1)$$

$$\varphi_{12}(U) = \det \begin{bmatrix} u_{13} & u_{14} & (U^2)_{14} \\ u_{23} & u_{24} & (U^2)_{24} \\ u_{33} & u_{34} & (U^2)_{34} \end{bmatrix}, \quad \varphi_{21}(U) = \det \begin{bmatrix} u_{14} & (U^2)_{14} \\ u_{24} & (U^2)_{24} \end{bmatrix}; \quad (6.2)$$

$$\varphi_{31}(U) = -u_{14}, \quad \varphi_{22}(U) = \det U_{[1,2]}^{[3,4]}, \quad \varphi_{13}(U) = -\det U_{[1,3]}^{[2,4]}. \quad (6.3)$$

The h -variables are given by

$$h_{24}(U) = u_{24}, \quad h_{23}(U) = \det U_{[2,3]}^{[3,4]}, \quad h_{22}(U) = \det U_{[2,4]}^{[2,4]}; \quad (6.4)$$

$$h_{34}(U) = -u_{34}, \quad h_{33}(U) = \det U_{[3,4]}^{[3,4]}, \quad h_{44}(U) = u_{44}. \quad (6.5)$$

The c -variables are given by

$$c_1(U) = -\text{tr } U, \quad c_2(U) = \frac{1}{2!} (\text{tr}(U)^2 - \text{tr}(U^2)), \quad (6.6)$$

$$c_3(U) = -\frac{1}{3!} (\text{tr}(U)^3 - 3 \text{tr}(U) \text{tr}(U^2) + 2 \text{tr}(U^3)). \quad (6.7)$$

List of 1-step mutations. Here's what one obtains after a 1-step mutation of the initial cluster in each direction:

$$\varphi'_{12}(U) = \det U_{[1,2]}^{[3,4]} \det \begin{bmatrix} u_{14} & (U^2)_{14} \\ u_{34} & (U^2)_{34} \end{bmatrix} + \begin{bmatrix} u_{14} & (U^2)_{14} \\ u_{24} & (U^2)_{24} \end{bmatrix} \det U_{[1,2]}^{\{2,4\}}; \quad (6.8)$$

$$\varphi'_{21}(U) = -\det \begin{bmatrix} u_{14} & (U^2)_{13} & (U^2)_{14} \\ u_{24} & (U^2)_{23} & (U^2)_{24} \\ u_{34} & (U^2)_{33} & (U^2)_{34} \end{bmatrix}; \quad (6.9)$$

$$\varphi'_{31}(U) = -\det U_{[1,3]}^{\{1\} \cup [3,4]}, \quad \varphi'_{31}(U) = -u_{24} \det U_{[2,4]}^{[2,4]} - u_{14} \det U_{[2,4]}^{\{1\} \cup [3,4]}; \quad (6.10)$$

$$\varphi'_{22}(U) = -u_{14} \det U_{[2,3]}^{\{1,4\}} - u_{24} \det U_{[2,3]}^{\{2,4\}} - u_{34} \det U_{[2,3]}^{[3,4]}. \quad (6.11)$$

$$h'_{24}(U) = -\det U_{\{1,3\}}^{[3,4]}, \quad h'_{34}(U) = -\det U_{\{2,4\}}^{[3,4]}; \quad (6.12)$$

$$h'_{23}(U) = u_{14} \det U_{[2,4]}^{[2,4]} - u_{24} \det U_{\{1\} \cup [3,4]}^{[2,4]}. \quad (6.13)$$

6.2 Cremmer-Gervais $i \mapsto i + 1$

The initial quiver for $\mathcal{GC}_h^\dagger(\Gamma, \text{GL}_4)$ is illustrated in Figure 5.

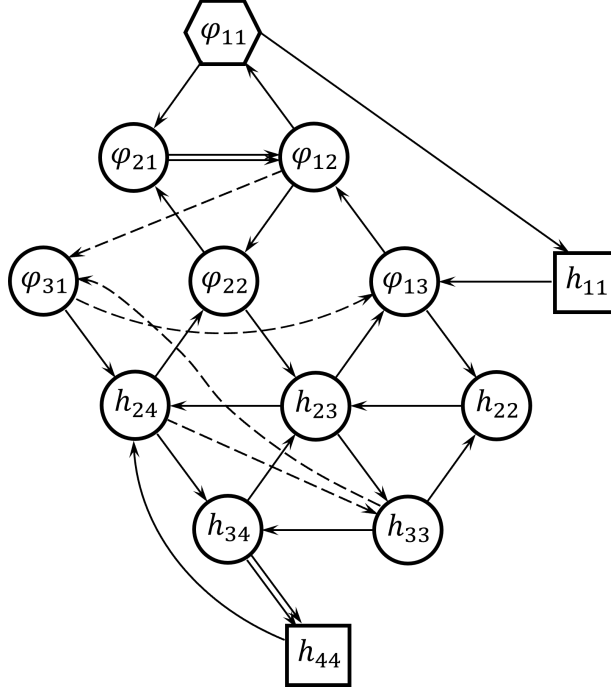


Figure 5. The initial quiver for $\mathcal{GC}_g^\dagger(\Gamma, \text{GL}_4)$ for $\Gamma_1 = \{1, 2\}$, $\Gamma_2 = \{2, 3\}$, $\gamma : i \mapsto i + 1$.

The initial variables. In the initial extended cluster, all cluster and frozen variables are given as in $\mathcal{GC}_h^\dagger(\Gamma_{\text{std}}, \text{GL}_4)$ except for the variables h_{22} , h_{23} , h_{24} , h_{33} , h_{34} . Let us set

$$\ell_1(U) := \det U_{[3,4]}^{[3,4]} u_{44} + \det U_{\{2,4\}}^{[3,4]} u_{43} + \det U_{\{1,4\}}^{[3,4]} u_{42}; \quad (6.14)$$

$$\ell_2(U) := \det U_{[3,4]}^{\{2,4\}} u_{44} + \det U_{\{2,4\}}^{\{2,4\}} u_{43} + \det U_{\{1,4\}}^{\{2,4\}} u_{42}; \quad (6.15)$$

$$\ell_3(U) := \det U_{[3,4]}^{[2,3]} u_{44} + \det U_{\{2,4\}}^{[2,3]} u_{43} + \det U_{\{1,4\}}^{[2,3]} u_{42}. \quad (6.16)$$

Then the h -variables are given by:

$$h_{24}(U) = u_{24} \cdot \ell_1(U) + u_{14} \ell_2(U), \quad h_{34}(U) = -u_{34} u_{44} - u_{24} u_{43} - u_{14} u_{42}, \quad h_{44}(U) = u_{44}; \quad (6.17)$$

$$h_{23}(U) = \det U_{[2,3]}^{[3,4]} \ell_1(U) + \det U_{\{1,3\}}^{[3,4]} \ell_2(U) + \det U_{[1,2]}^{[3,4]} \ell_3(U), \quad h_{33}(U) = \ell_1(U); \quad (6.18)$$

$$h_{22}(U) = \det U_{[2,4]}^{[2,4]} \ell_1(U) + \det U_{\{1\} \cup [3,4]}^{[2,4]} \ell_2(U) + \det U_{[1,2] \cup \{4\}}^{[2,4]} \ell_3(U). \quad (6.19)$$

Birational quasi-isomorphisms. TBD

6.3 Cremmer-Gervais $i \mapsto i - 1$

The initial quiver for $\mathcal{GC}_h^\dagger(\Gamma, \text{GL}_4)$ is illustrated in Figure 6.

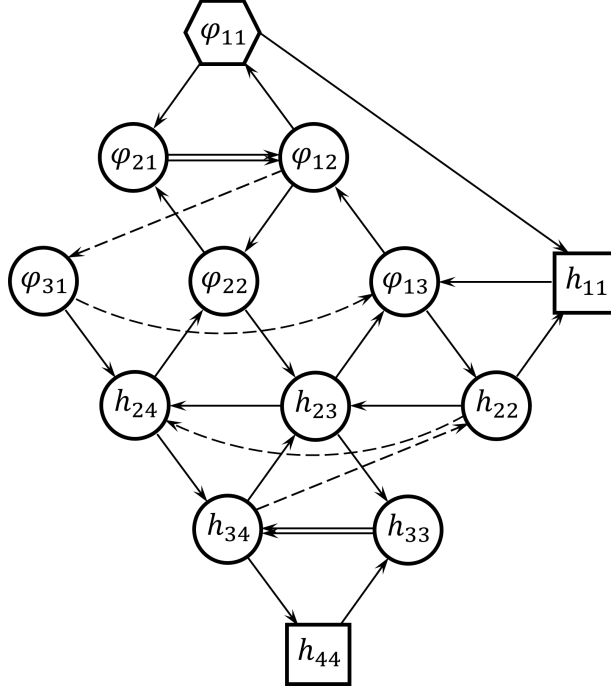


Figure 6. The initial quiver for $\mathcal{GC}_g^\dagger(\Gamma, \text{GL}_4)$ for $\Gamma_1 = \{2, 3\}$, $\Gamma_2 = \{1, 2\}$, $\gamma : i \mapsto i - 1$.

The initial variables. In the initial extended cluster, all cluster and frozen variables are given as in $\mathcal{GC}_h^\dagger(\Gamma_{\text{std}}, \text{GL}_4)$ except for the variables h_{34} , h_{33} , h_{44} . These are given by:

$$h_{34}(U) = -u_{34} \det U_{[2,4]}^{[2,4]} - u_{24} \det U_{[2,4]}^{\{1\} \cup [3,4]}; \quad (6.20)$$

$$h_{33}(U) = \det U_{[3,4]}^{[3,4]} \det U_{[2,4]}^{[2,4]} + \det U_{\{2,4\}}^{[3,4]} \det U_{[2,4]}^{\{1\} \cup [3,4]} + \det U_{[2,3]}^{[3,4]} \det U_{[2,4]}^{[1,2] \cup \{4\}}; \quad (6.21)$$

$$\begin{aligned} h_{44}(U) = & u_{44} \left(\det U_{[3,4]}^{[3,4]} \det U_{[2,4]}^{[2,4]} + \det U_{\{2,4\}}^{[3,4]} \det U_{[2,4]}^{\{1\} \cup [3,4]} + \det U_{[2,3]}^{[3,4]} \det U_{[2,4]}^{[1,2] \cup \{4\}} \right) + \\ & + u_{34} \left(\det U_{[3,4]}^{\{2,4\}} \det U_{[2,4]}^{[2,4]} + \det U_{\{2,4\}}^{\{2,4\}} \det U_{[2,4]}^{\{1\} \cup [3,4]} + \det U_{[2,3]}^{\{2,4\}} \det U_{[2,4]}^{[1,2] \cup \{4\}} \right) + \\ & + u_{24} \left(\det U_{[3,4]}^{\{1,4\}} \det U_{[2,4]}^{[2,4]} + \det U_{\{2,4\}}^{\{1,4\}} \det U_{[2,4]}^{\{1\} \cup [3,4]} + \det U_{[2,3]}^{\{1,4\}} \det U_{[2,4]}^{[1,2] \cup \{4\}} \right). \end{aligned} \quad (6.22)$$

Birational quasi-isomorphisms. TBD

7 Examples in $n = 4$ in the g -convention

7.1 The standard BD triple

The initial quiver for $\mathcal{GC}_g^\dagger(\Gamma_{\text{std}}, \text{GL}_4)$ is illustrated in Figure 7.

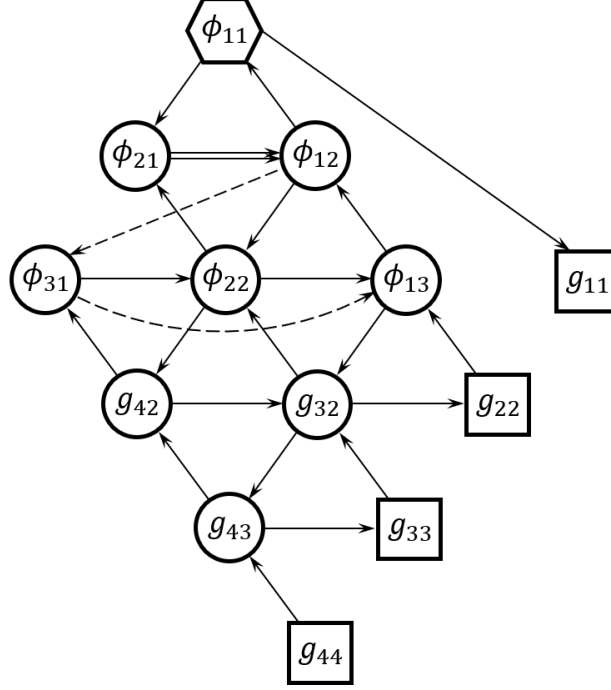


Figure 7. The initial quiver for $\mathcal{GC}_g^\dagger(\Gamma_{\text{std}}, \text{GL}_4)$.

The initial variables. The ϕ - and c -variables, as elements of $\mathcal{O}(\text{GL}_4)$, are given by the following formulas:

$$\phi_{11}(U) = \det \begin{bmatrix} u_{21} & (U^2)_{21} & (U^3)_{21} \\ u_{31} & (U^2)_{31} & (U^3)_{31} \\ u_{41} & (U^2)_{41} & (U^3)_{41} \end{bmatrix}, \quad \phi_{12}(U) = -\det \begin{bmatrix} u_{21} & u_{22} & (U^2)_{21} \\ u_{31} & u_{32} & (U^2)_{31} \\ u_{41} & u_{42} & (U^2)_{41} \end{bmatrix}; \quad (7.1)$$

$$\phi_{21}(U) = \det \begin{bmatrix} u_{31} & (U^2)_{31} \\ u_{41} & (U^2)_{41} \end{bmatrix}, \quad \phi_{31}(U) = u_{41}, \quad \phi_{22}(U) = \det U_{[3,4]}^{[1,2]}, \quad \phi_{13}(U) = \det U_{[2,4]}^{[1,3]}; \quad (7.2)$$

$$c_1(U) = -\text{tr } U, \quad c_2(U) = \frac{1}{2!} (\text{tr}(U)^2 - \text{tr}(U^2)), \quad (7.3)$$

$$c_3(U) = -\frac{1}{3!} (\text{tr}(U)^3 - 3 \text{tr}(U) \text{tr}(U^2) + 2 \text{tr}(U^3)). \quad (7.4)$$

The g -variables are given by

$$g_{11}(U) := \det U, \quad g_{ij}(U) = \det U_{[i,n]}^{[j,n-i+j]}, \quad 2 \leq j \leq i \leq n. \quad (7.5)$$

7.2 Cremmer-Gervais $i \mapsto i - 1$

The initial quiver for $\mathcal{GC}_g^\dagger(\Gamma, \text{GL}_4)$ is illustrated in Figure 8.

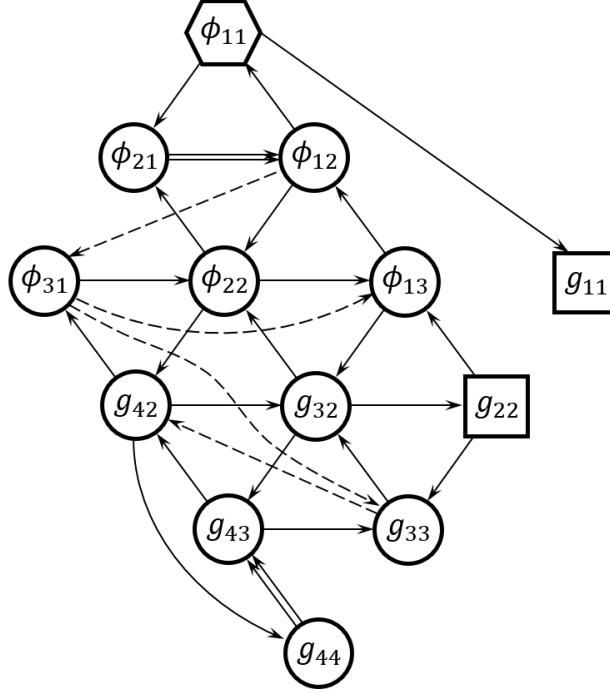


Figure 8. The initial quiver for $\mathcal{GC}_g^\dagger(\Gamma, \text{GL}_4)$ for $\Gamma_1 = \{2, 3\}$, $\Gamma_2 = \{1, 2\}$, $\gamma : i \mapsto i - 1$.

The initial variables. Let us set

$$\ell_1(U) := \det U_{[3,4]}^{[3,4]} u_{44} + \det U_{[3,4]}^{\{2,4\}} u_{34} + \det U_{[3,4]}^{\{1,4\}} u_{24}; \quad (7.6)$$

$$\ell_2(U) := \det U_{\{2,4\}}^{[3,4]} u_{44} + \det U_{\{2,4\}}^{\{2,4\}} u_{34} + \det U_{\{2,4\}}^{\{1,4\}} u_{24}; \quad (7.7)$$

$$\ell_3(U) := \det U_{[2,3]}^{[3,4]} u_{44} + \det U_{[2,3]}^{\{2,4\}} u_{34} + \det U_{[2,3]}^{\{1,4\}} u_{24}. \quad (7.8)$$

The g -variables are given by the following formulas:

$$g_{42}(U) = u_{42} \cdot \ell_1(U) + u_{41} \ell_2(U), \quad g_{43}(U) = u_{43} u_{44} + u_{42} u_{34} + u_{41} u_{24}, \quad g_{44}(U) = u_{44}; \quad (7.9)$$

$$g_{32}(U) = \det U_{[3,4]}^{[2,3]} \ell_1(U) + \det U_{[3,4]}^{\{1,3\}} \ell_2(U) + \det U_{[3,4]}^{[1,2]} \ell_3(U), \quad g_{33}(U) = \ell_1(U); \quad (7.10)$$

$$g_{22}(U) = \det U_{[2,4]}^{[2,4]} \ell_1(U) + \det U_{[2,4]}^{\{1\} \cup [3,4]} \ell_2(U) + \det U_{[2,4]}^{[1,2] \cup \{4\}} \ell_3(U). \quad (7.11)$$

Birational quasi-isomorphisms. There is a birational quasi-isomorphism

$$\mathcal{Q}^{\text{op}} : (\text{GL}_4, \mathcal{GC}_g^\dagger(\Gamma_{\text{std}})) \dashrightarrow (\text{GL}_4, \mathcal{GC}_g^\dagger(\Gamma)), \quad \mathcal{Q}^{\text{op}}(U) := \rho^{\text{op}}(U) U (\rho^{\text{op}}(U))^{-1} \quad (7.12)$$

where the rational map $\rho^{\text{op}} : \text{GL}_n \dashrightarrow \text{GL}_n$ is given by

$$\rho^{\text{op}}(U) = \left(I + \frac{u_{34}}{u_{44}} e_{12} \right) \cdot \left(I + \frac{\det U_{\{2,4\}}^{[3,4]}}{\det U_{[3,4]}^{[3,4]}} e_{12} + \frac{u_{24}}{u_{44}} e_{13} + \frac{u_{34}}{u_{44}} e_{23} \right). \quad (7.13)$$

The marked variables for \mathcal{Q}^{op} are g_{33} and g_{44} . Define the BD triples $\tilde{\Gamma} := (\{2\}, \{1\}, 2 \mapsto 1)$ and $\hat{\Gamma} := (\{3\}, \{2\}, 3 \mapsto 2)$. There is a pair of complementary birational quasi-isomorphisms

$$\mathcal{G} : (\text{GL}_4, \mathcal{GC}_g^\dagger(\tilde{\Gamma})) \dashrightarrow (\text{GL}_4, \mathcal{GC}_g^\dagger(\Gamma)), \quad \mathcal{G}' : (\text{GL}_4, \mathcal{GC}_g^\dagger(\hat{\Gamma})) \dashrightarrow (\text{GL}_4, \mathcal{GC}_g^\dagger(\Gamma)). \quad (7.14)$$

They are given by

$$\mathcal{G}^{\text{op}}(U) = G^{\text{op}}(U) \cdot U \cdot G^{\text{op}}(U)^{-1}, \quad G^{\text{op}}(U) := \left(I + \frac{u_{34}}{u_{44}} e_{12} \right) \cdot \left(I + \frac{u_{24}}{u_{44}} e_{13} + \frac{u_{34}}{u_{44}} e_{23} \right); \quad (7.15)$$

$$(\mathcal{G}^{\text{op}})'(U) = G'(U) \cdot U \cdot G'(U)^{-1}, \quad G'(U) := (I + \alpha_1(U) e_{12} + \alpha_2(U) e_{13}), \quad (7.16)$$

$$\alpha_1(U) = \frac{\det U_{\{2,4\}}^{[3,4]} u_{44} + \det U_{\{2,4\}}^{\{2,4\}} u_{34}}{\det U_{[3,4]}^{[3,4]} u_{44} + \det U_{[3,4]}^{\{2,4\}} u_{34}}, \quad \alpha_2(U) = -\frac{\det U_{[2,3]}^{[3,4]} u_{44} + \det U_{[2,3]}^{\{2,4\}} u_{34}}{\det U_{[3,4]}^{[3,4]} u_{44} + \det U_{[3,4]}^{\{2,4\}} u_{34}}. \quad (7.17)$$

The marked variable for \mathcal{G} is g_{44} , and the marked variable for \mathcal{G}' is g_{33} .

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