Supplementary note for generalized cluster structures on $\mathrm{SL}_n^\dagger$

Dmitriy Voloshyn and Michael Gekhtman

April 8, 2024

Abstract

This is a supplementary note for the main paper *Generalized cluster structures on* $\mathrm{SL}_n^\dagger$ that contains explicit examples of generalized cluster structures compatible with $\pi_{\mathbf{\Gamma}}^\dagger$ in type A_{n-1}, as well as a list of some of the instrinsic problems of the theory. This note will be updated over time.

Contents

1 Summary of the h-convention

In this section, we outline the construction of birational quasi-isomorphism for $\mathcal{GC}_h^\dagger(\mathbf{\Gamma})$, as well as the construction of the initial extended cluster. For all the other information, refer to the main paper [3].

1.1 The maps $\mathcal{F}$, $\mathcal{Q}$ and $\mathcal{G}$

Notation. For a generic element $U \in \mathrm{GL}_n$, the element $U_\oplus \in \mathrm{GL}_n$ is an upper triangular matrix and $U_- \in \mathrm{GL}_n$ is a unipotent lower triangular matrix, such that $U = U_\oplus U_-$.

The map $\mathcal{F}$. Let $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ be a BD triple of type A_{n-1}. Define the sequence $\mathcal{F}_k : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ of rational maps via

$$\mathcal{F}_0(U) := U, \quad \mathcal{F}_k(U) := \tilde{\gamma}^*[\mathcal{F}_{k-1}(U)_-]U, \quad k \geq 1. \tag{1.1}$$

The birational map $\mathcal{F} : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ is defined as the limit

$$\mathcal{F}(U) := \lim_{k\to\infty} \mathcal{F}_k(U). \tag{1.2}$$

Since γ is nilpotent, the sequence $\mathcal{F}_k$ stabilizes at $k = \deg\gamma$, so $\mathcal{F}(U) = \mathcal{F}_{\deg\gamma}(U)$. The inverse of $\mathcal{F}$ is given by

$$\mathcal{F}^{-1}(U) := \tilde{\gamma}^*(U_-)^{-1}U. \tag{1.3}$$

The map $\mathcal{F}$ is neither a Poisson map nor a quasi-isomorphism. However, by means of $\mathcal{F}$ one can construct Poisson birational quasi-isomorphisms. For various invariance properties of $\mathcal{F}$, refer to [3, Section 4.2].

Birational quasi-isomorphisms. Define the birational map $\mathcal{Q} : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ via

$$\mathcal{Q}(U) := \rho(U)^{-1}U\rho(U), \quad \rho(U) := \overrightarrow{\prod_{i=1}}[\tilde{\gamma}^*]^i(U_-). \tag{1.4}$$

The inverse of $\mathcal{Q}$ is given by

$$\mathcal{Q}^{-1}(U) := \mathcal{F}^c(U) := \mathcal{F}(U)\tilde{\gamma}^*(\mathcal{F}(U)_-)^{-1}. \tag{1.5}$$

Let $\pi_{\mathbf{\Gamma}}^\dagger$ and $\pi_{\mathrm{std}}^\dagger$ be the Poisson bivectors associated with an arbitrary BD triple $\mathbf{\Gamma}$ and $\mathbf{\Gamma}_{\mathrm{std}}$ (of type A_{n-1}), respectively. If the r_0 parts of $\pi_{\mathbf{\Gamma}}^\dagger$ and $\pi_{\mathrm{std}}^\dagger$ are the same, then $\mathcal{Q} : (\mathrm{GL}_n, \pi_{\mathrm{std}}^\dagger) \dashrightarrow (\mathrm{GL}_n, \pi_{\mathbf{\Gamma}}^\dagger)$ is a Poisson isomorphism. Moreover, as a map $\mathcal{Q} : (\mathrm{GL}_n, \mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\mathrm{std}})) \dashrightarrow (\mathrm{GL}_n, \mathcal{GC}_h^\dagger(\mathbf{\Gamma}))$, it is a birational quasi-isomorphism, with the marked variables given by

$$\{h_{i+1,i+1} \mid i \in \Gamma_2\}. \tag{1.6}$$

If $\tilde{\mathbf{\Gamma}} \prec \mathbf{\Gamma}$ is another BD triple of type A_{n-1}, then there is a birational quasi-isomorphism $\mathcal{G} : (\mathrm{GL}_n, \mathcal{GC}_h^\dagger(\tilde{\mathbf{\Gamma}})) \dashrightarrow (\mathrm{GL}_n, \mathcal{GC}_h^\dagger(\mathbf{\Gamma}))$. If $\tilde{\mathcal{Q}}$ is defined as the map $\mathcal{Q}$, but with respect to the BD triple $\tilde{\mathbf{\Gamma}}$, then $\mathcal{G} = \mathcal{Q} \circ \tilde{\mathcal{Q}}$. As a map $\mathcal{G} : (\mathrm{GL}_n, \pi_{\tilde{\mathbf{\Gamma}}}^\dagger) \dashrightarrow (\mathrm{GL}_n, \pi_{\mathbf{\Gamma}}^\dagger)$, it is a Poisson isomorphism if the r_0 parts of $\pi_{\tilde{\mathbf{\Gamma}}}^\dagger$ and $\pi_{\mathbf{\Gamma}}^\dagger$ are the same. The marked variables for $\mathcal{G}$ are given by

$$\{h_{i+1,i+1} \mid i \in \Gamma_2 \setminus \tilde{\Gamma}_2\}. \tag{1.7}$$

For more explicit formulas of $\mathcal{G}$, refer to [3, Section 4.4, Section 4.5].

1.2 Initial extended cluster

The initial extended cluster comprises three types of functions: c-functions, φ-functions and h-functions. Only the description of the h-functions depends on the choice of the Belavin-Drinfeld triple.

Description of φ- and c-functions. For an element $U \in \mathrm{GL}_n$, let us set

$$\Phi_{kl}(U) := \left[(U^0)^{[n-k+1,n]} \quad U^{[n-l+1,n]} \quad (U^2)^{\{n\}} \quad \cdots \quad (U^{n-k-l+1})^{\{n\}}\right], \quad k,l \geq 1,\ k+l \leq n; \tag{1.8}$$

$$s_{kl} := \begin{cases} (-1)^{k(l+1)} & n \text{ is even,} \\ (-1)^{(n-1)/2+k(k-1)/2+l(l-1)/2} & n \text{ is odd.} \end{cases} \tag{1.9}$$

Then the φ-functions are given by

$$\varphi_{kl}(U) := s_{kl} \det \Phi_{kl}(U). \tag{1.10}$$

The c-functions are uniquely defined via

$$\det(I + \lambda U) = \sum_{i=0}^{n} \lambda^i s_i c_i(U) \tag{1.11}$$

where $s_i := (-1)^{i(n-1)}$ and I is the identity matrix. Note that $c_0 = I$ and $c_n = \det U$.

Description of the h-functions. Let Π be a set of simple roots of type A_{n-1} and $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ be a BD triple. We identify Π with the interval $[1, n-1]$. For a given $\alpha_0 \in \Pi \setminus \Gamma_2$, set $\alpha_t := \gamma(\alpha_{t-1})$, $t \geq 1$. Recall that the sequence $S^\gamma(\alpha_0) := \{\alpha_t\}_{t\geq 0}$ is the *γ-string* associated to α_0; γ-strings partition Π. For each γ-string $S^\gamma(\alpha_0) = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$, for each $i \in [0, m]$ and $j \in [\alpha_i + 1, n]$, set

$$h_{\alpha_i+1,j}(U) := (-1)^{\varepsilon_{\alpha_i+1,j}} \det[\mathcal{F}(U)]^{[j,n]}_{[\alpha_i+1,n-j+\alpha_i+1]} \prod_{t\geq i+1}^{m} \det[\mathcal{F}(U)]^{[\alpha_t+1,n]}_{[\alpha_t+1,n]} \tag{1.12}$$

where ε_{ij} is defined as

$$\varepsilon_{ij} := (j-i)(n-i), \quad 1 \leq i \leq j \leq n. \tag{1.13}$$

We refer to the functions h_{ij}, $2 \leq i \leq j \leq n$, together with $h_{11}(U) := \det U$ as the *h-functions.*

Frozen variables. In the case of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{GL}_n)$, the frozen variables are given by the set

$$\{c_1, c_2, \ldots, c_{n-1}\} \cup \{h_{i+1,i+1} \mid i \in \Pi \setminus \Gamma_2\} \cup \{h_{11}\}. \tag{1.14}$$

In the case of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{SL}_n)$, $h_{11}(U) = 1$, so this variable is absent. The zero loci of the frozen variables foliate into unions of symplectic leaves of the ambient Poisson variety $(\mathrm{GL}_n, \pi_{\mathbf{\Gamma}}^\dagger)$ or $(\mathrm{SL}_n, \pi_{\mathbf{\Gamma}}^\dagger)$. Moreover, the frozen h-variables do not vanish on $\mathrm{SL}_n^\dagger$.

Initial extended cluster. The initial extended cluster Ψ_0 of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{GL}_n)$ is given by the set

$$\{h_{ij} \mid 2 \leq i \leq j \leq n\} \cup \{\varphi_{kl} \mid k,l \geq 1,\ k+l \leq n\} \cup \{c_1, \ldots, c_{n-1}\} \cup \{h_{11}\}. \tag{1.15}$$

The initial extended cluster of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{SL}_n)$ is obtained from Ψ_0 via removing h_{11}.

A generalized cluster mutation. In the initial extended cluster, only the variable φ_{11} is equipped with a nontrivial *string*, which is given by $(1, c_1, \ldots, c_{n-1}, 1)$. The generalized mutation relation for φ_{11} reads

$$\varphi_{11}\varphi'_{11} = \sum_{r=0}^{n} c_r \varphi_{21}^r \varphi_{12}^{n-r}. \tag{1.16}$$

Other mutations of the initial extended cluster follow the usual pattern from the theory of cluster algebras of geometric type.

2 Summary of the g-convention

In this section, we outline the construction of birational quasi-isomorphism for $\mathcal{GC}_g^\dagger(\mathbf{\Gamma})$, as well as the construction of the initial extended cluster. For all the other information, refer to the main paper [3].

2.1 The maps $\mathcal{F}^{\mathrm{op}}$, $\mathcal{Q}^{\mathrm{op}}$ and $\mathcal{G}^{\mathrm{op}}$

Notation. For a generic element $U \in \mathrm{GL}_n$, the element $U_+ \in \mathrm{GL}_n$ is a unipotent upper triangular matrix and $U_\ominus \in \mathrm{GL}_n$ is a lower triangular matrix, such that $U = U_+ U_\ominus$.

The map $\mathcal{F}^{\mathrm{op}}$. Let $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ be a BD triple of type A_{n-1}. Define the sequence $\mathcal{F}_k^{\mathrm{op}} : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ of rational maps via

$$\mathcal{F}_0^{\mathrm{op}}(U) := U, \quad \mathcal{F}_k^{\mathrm{op}}(U) := U\tilde{\gamma}[\mathcal{F}_{k-1}^{\mathrm{op}}(U)_+], \quad k \geq 1. \tag{2.1}$$

The birational map $\mathcal{F}^{\mathrm{op}} : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ is defined as the limit

$$\mathcal{F}^{\mathrm{op}}(U) := \lim_{k\to\infty} \mathcal{F}_k^{\mathrm{op}}(U). \tag{2.2}$$

Since γ is nilpotent, the sequence $\mathcal{F}_k^{\mathrm{op}}$ stabilizes at $k = \deg\gamma$, so $\mathcal{F}^{\mathrm{op}}(U) = \mathcal{F}_{\deg\gamma}^{\mathrm{op}}(U)$. The inverse of $\mathcal{F}^{\mathrm{op}}$ is given by

$$(\mathcal{F}^{\mathrm{op}})^{-1}(U) := U\tilde{\gamma}(U_+)^{-1}. \tag{2.3}$$

The map $\mathcal{F}^{\mathrm{op}}$ is neither a Poisson map nor a quasi-isomorphism. However, by means of $\mathcal{F}^{\mathrm{op}}$ one can construct Poisson birational quasi-isomorphisms in the g-convention. For various invariance properties of $\mathcal{F}^{\mathrm{op}}$, refer to [3, Section 7.1].

Birational quasi-isomorphisms. Define the birational map $\mathcal{Q}^{\mathrm{op}} : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ via

$$\mathcal{Q}^{\mathrm{op}}(U) := \rho^{\mathrm{op}}(U)U(\rho^{\mathrm{op}}(U))^{-1}, \quad \rho^{\mathrm{op}}(U) := \overleftarrow{\prod_{i=1}}[\tilde{\gamma}]^i(U_+). \tag{2.4}$$

The inverse of $\mathcal{Q}^{\mathrm{op}}$ is given by the map

$$(\mathcal{Q}^{\mathrm{op}})^{-1}(U) := \mathcal{F}^{\mathrm{op},c}(U) := \tilde{\gamma}(\mathcal{F}^{\mathrm{op}}(U)_+)^{-1}\mathcal{F}^{\mathrm{op}}(U). \tag{2.5}$$

Let $\pi_{\mathbf{\Gamma}}^\dagger$ and $\pi_{\mathrm{std}}^\dagger$ be the Poisson bivectors associated with an arbitrary BD triple $\mathbf{\Gamma}$ and $\mathbf{\Gamma}_{\mathrm{std}}$ (of type A_{n-1}), respectively. If the r_0 parts of $\pi_{\mathbf{\Gamma}}^\dagger$ and $\pi_{\mathrm{std}}^\dagger$ are the same, then $\mathcal{Q}^{\mathrm{op}} : (\mathrm{GL}_n, \pi_{\mathrm{std}}^\dagger) \dashrightarrow (\mathrm{GL}_n, \pi_{\mathbf{\Gamma}}^\dagger)$

is a Poisson isomorphism. Moreover, as a map $\mathcal{Q}^{\mathrm{op}} : (\mathrm{GL}_n, \mathcal{GC}_g^{\dagger}(\mathbf{\Gamma}_{\mathrm{std}})) \dashrightarrow (\mathrm{GL}_n, \mathcal{GC}_g^{\dagger}(\mathbf{\Gamma}))$, it is a birational quasi-isomorphism, with the marked variables given by

$$\{g_{i+1,i+1} \mid i \in \Gamma_1\}. \tag{2.6}$$

If $\tilde{\mathbf{\Gamma}} \prec \mathbf{\Gamma}$ is another BD triple of type A_{n-1}, then there is a birational quasi-isomorphism $\mathcal{G}^{\mathrm{op}} : (\mathrm{GL}_n, \mathcal{GC}_h^{\dagger}(\tilde{\mathbf{\Gamma}})) \dashrightarrow (\mathrm{GL}_n, \mathcal{GC}_h^{\dagger}(\mathbf{\Gamma}))$. If $\tilde{\mathcal{Q}}^{\mathrm{op}}$ is defined as the map $\mathcal{Q}^{\mathrm{op}}$, but with respect to the BD triple $\tilde{\mathbf{\Gamma}}$, then $\mathcal{G}^{\mathrm{op}} = \mathcal{Q}^{\mathrm{op}} \circ \tilde{\mathcal{Q}}^{\mathrm{op}}$. As a map $\mathcal{G}^{\mathrm{op}} : (\mathrm{GL}_n, \pi^{\dagger}_{\tilde{\mathbf{\Gamma}}}) \dashrightarrow (\mathrm{GL}_n, \pi^{\dagger}_{\mathbf{\Gamma}})$, it is a Poisson isomorphism if the r_0 parts of $\pi^{\dagger}_{\tilde{\mathbf{\Gamma}}}$ and $\pi^{\dagger}_{\mathbf{\Gamma}}$ are the same. The marked variables for $\mathcal{G}^{\mathrm{op}}$ are given by

$$\{g_{i+1,i+1} \mid i \in \Gamma_1 \setminus \tilde{\Gamma}_1\}. \tag{2.7}$$

Explicit formulas for $\mathcal{G}^{\mathrm{op}}$ can be obtained from explicit formulas for $\mathcal{G}$ (refer to [3, Section 4.4, Section 4.5, Section 7.3]).

2.2 Initial extended cluster

The initial extended cluster comprises three types of functions: c-functions, ϕ-functions and g-functions. Only the description of the g-functions depends on the choice of the Belavin-Drinfeld triple.

Description of ϕ- and c-functions. For an element $U \in \mathrm{GL}_n$, let us set

$$\Phi'_{kl}(U) := \begin{bmatrix} (U^0)^{[1,k]} & U^{[1,l]} & (U^2)^{\{1\}} & \cdots & (U^{n-k-l+1})^{\{1\}} \end{bmatrix}, \quad k,l \geq 1, \ k+l \leq n; \tag{2.8}$$

$$s_{kl} := \begin{cases} (-1)^{k(l+1)} & n \text{ is even,} \\ (-1)^{(n-1)/2 + k(k-1)/2 + l(l-1)/2} & n \text{ is odd.} \end{cases} \tag{2.9}$$

Then the ϕ-functions are given by

$$\phi_{kl}(U) := s_{kl} \det \Phi'_{kl}(U) \tag{2.10}$$

The c-functions are uniquely defined via

$$\det(I + \lambda U) = \sum_{i=0}^{n} \lambda^i s_i c_i(U) \tag{2.11}$$

where $s_i := (-1)^{i(n-1)}$ and I is the identity matrix. Note that $c_0 = I$ and $c_n = \det U$ (the c-functions are the same in both g- and h-conventions).

Description of the g-functions. Let Π be a set of simple roots of type A_{n-1} and let $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ be a BD triple of type A_{n-1}. Let $\mathcal{F}^{\mathrm{op}} : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ be the rational map defined by (2.2). We identify Π with the interval $[1, n-1]$. For a given $\alpha_0 \in \Pi \setminus \Gamma_1$, set $\alpha_t := \gamma^*(\alpha_{t-1})$, $t \geq 1$. Recall that the sequence $S^{\gamma^*}(\alpha_0) := \{\alpha_t\}_{t \geq 0}$ is the γ^**-string* associated to α_0; γ^*-strings partition Π. For each $\alpha_0 \in \Pi \setminus \Gamma_1$ and the associated γ^*-string $S^{\gamma^*}(\alpha_0) := \{\alpha_i\}_{i=0}^m$, for every $k \in [0, m]$ and $i \in [\alpha_k + 1, n]$, define

$$g_{i,\alpha_k+1}(U) := \det[\mathcal{F}^{\mathrm{op}}(U)]_{[i,n]}^{[\alpha_k+1, n-i+\alpha_k+1]} \prod_{t \geq k+1}^{m} \det[\mathcal{F}^{\mathrm{op}}(U)]_{[\alpha_t+1,n]}^{[\alpha_t+1,n]}. \tag{2.12}$$

We refer to the functions g_{ij}, $2 \leq j \leq i \leq n$, together with $g_{11}(U) := \det U$ as the *g-functions*.

Frozen variables. In the case of $\mathcal{GC}_g^\dagger(\mathbf{\Gamma}, \mathrm{GL}_n)$, the frozen variables are given by the set

$$\{c_1, c_2, \dots, c_{n-1}\} \cup \{g_{i+1,i+1} \mid i \in \Pi \setminus \Gamma_1\} \cup \{g_{11}\}. \tag{2.13}$$

In the case of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{SL}_n)$, $g_{11}(U) = 1$, so this variable is absent. The zero loci of the frozen variables foliate into unions of symplectic leaves of the ambient Poisson variety $(\mathrm{GL}_n, \pi_{\mathbf{\Gamma}}^\dagger)$ or $(\mathrm{SL}_n, \pi_{\mathbf{\Gamma}}^\dagger)$. Moreover, the frozen h-variables do not vanish on $\mathrm{SL}_n^\dagger$.

Initial extended cluster. The initial extended cluster Ψ_0 of $\mathcal{GC}_g^\dagger(\mathbf{\Gamma}, \mathrm{GL}_n)$ is given by the set

$$\{g_{ij} \mid 2 \leq j \leq i \leq n\} \cup \{\phi_{kl} \mid k, l \geq 1,\ k + l \leq n\} \cup \{c_1, \dots, c_{n-1}\} \cup \{g_{11}\}. \tag{2.14}$$

The initial extended cluster of $\mathcal{GC}_g^\dagger(\mathbf{\Gamma}, \mathrm{SL}_n)$ is obtained from Ψ_0 via removing h_{11}.

A generalized cluster mutation. In the initial extended cluster, only the variable ϕ_{11} is equipped with a nontrivial *string*, which is given by $(1, c_1, \dots, c_{n-1}, 1)$. The generalized mutation relation for ϕ_{11} reads

$$\phi_{11}\phi_{11}' = \sum_{r=0}^{n} c_r \phi_{21}^r \phi_{12}^{n-r}. \tag{2.15}$$

Other mutations of the initial extended cluster follow the usual pattern from the theory of cluster algebras of geometric type.

3 Relation between the h- and g-conventions

In this section, we briefly mention the relation between the g- and the h-conventions. Let $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ be an arbitrary BD triple of type A_{n-1}.

Variables. The c-variables in both the h- and the g-conventions are the same. For the other variables in the initial extended clusters, the connection is as follows.

1) For ϕ- and φ-functions, $\phi_{kl}(W_0^{-1}UW_0) = \varphi_{kl}(U)$ where $W_0 := \sum_{i=1}^{n-1}(-1)^{i+1}e_{n-i+1,i}$.

2) For g_{ij} and h_{ji} from the initial extended clusters of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma})$ and $\mathcal{GC}_g^\dagger(\mathbf{\Gamma}^{\mathrm{op}})$, $g_{ij}(U) = (-1)^{\varepsilon_{ji}} h_{ji}(U^T)$ where $\varepsilon_{ji} := (n-j)(i-j)$.

Quivers. The initial quiver $Q_g(\mathbf{\Gamma})$ for the g-convention can be obtained from the initial quiver $Q_h(\mathbf{\Gamma}^{\mathrm{op}})$ for the h-convention via the following steps:

- Replace each vertex φ_{kl} with ϕ_{kl}, $2 \leq k + l \leq n$, $k, l \geq 1$ and each h_{ji} with g_{ij}, $2 \leq j \leq i \leq n$;
- For each g_{ij}, $2 \leq j \leq i \leq n$, reverse the orientation of the arrows in its neighborhood;
- For the vertices ϕ_{kl} with $k + l = n$ and $k \geq 2$, add an arrow $\phi_{kl} \to \phi_{k-1,l+1}$;
- Remove the arrow $\phi_{1,n-1} \to g_{11}$.

Mutation equivalence. In $n = 3$, the initial extended cluster of $\mathcal{GC}_g^\dagger(\mathbf{\Gamma}, \mathrm{GL}_3)$ can be obtained from the initial extended cluster of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{GL}_3)$ (for any $\mathbf{\Gamma}$) via a sequence of mutations (see Section 4.3). We conjecture that there is no such sequence in $n \geq 4$.

Birational quasi-isomorphisms. Define $\mathcal{F}$, $\mathcal{Q}$ and $\mathcal{G}$ relative the BD triple $\mathbf{\Gamma}$, and define $\mathcal{F}^{\text{op}}$, $\mathcal{Q}^{\text{op}}$ and $\mathcal{G}^{\text{op}}$ relative the opposite BD triple $\mathbf{\Gamma}^{\text{op}}$. Then $\mathcal{F}(U^T) = \mathcal{F}^{\text{op}}(U)^T$, $\mathcal{Q}(U^T) = \mathcal{Q}^{\text{op}}(U)^T$, $\mathcal{G}(U^T) = \mathcal{G}^{\text{op}}(U)^T$.

4 Intrinsic problems

4.1 The Poisson structure $\mathcal{F}_*(\pi^\dagger_{\mathbf{\Gamma}})$

Let $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ be a BD triple of type A_{n-1}. Define a rational map $\mathcal{C} : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ via

$$\mathcal{C}(U) := U \cdot \rho(U) = U \prod_{k=1}^{\rightarrow} \tilde{\gamma}^*(U_-), \quad U \in \mathrm{GL}_n\,. \tag{4.1}$$

The map $\mathcal{C}$ is in fact birational, with the inverse given by

$$\mathcal{C}^{-1}(U) = U \cdot \tilde{\gamma}^*(U_-)^{-1}, \quad U \in \mathrm{GL}_n\,. \tag{4.2}$$

Set $\pi_{\mathcal{F}} := \mathcal{F}_*(\pi^\dagger_{\mathbf{\Gamma}})$. Since $\mathcal{F}^c(U) = \mathcal{F}(U)\tilde{\gamma}^*(\mathcal{F}(U)_-)^{-1}$, the following diagram is commutative:

$$\begin{tikzcd}
(\mathrm{GL}_n, \pi^\dagger_{\mathbf{\Gamma}}) \arrow[r, dashed, "\mathcal{F}^c"] \arrow[d, dashed, "\mathcal{F}"'] & (\mathrm{GL}_n, \pi^\dagger_{\text{std}}) \arrow[dl, dashed, "\mathcal{C}"] \\
(\mathrm{GL}_n, \pi_{\mathcal{F}}) &
\end{tikzcd} \tag{4.3}$$

Moreover, all the arrows are birational Poisson isomorphisms (provided the r_0-parts are the same for all Poisson bivectors). The Poisson bracket $\{\cdot,\cdot\}_{\mathcal{F}}$ that corresponds to $\pi_{\mathcal{F}}$ is given by

$$\begin{aligned}
\{f,g\}_{\mathcal{F}} =& \langle R_0\pi_0[U,\nabla_U f], [U,\nabla_U g]\rangle + \langle \pi_0[U,\nabla_U f], \nabla^L_U g\rangle + \\
&+\langle \pi_> \nabla^L_U f, \nabla^L_U g\rangle - \langle \pi_> \nabla^R_U f, \nabla^R_U g\rangle + \\
&+\langle \frac{1}{1-\gamma}\pi_> \nabla^R_U f, \nabla^R_U g\rangle - \langle \nabla^R_U f, \frac{1}{1-\gamma}\pi_> \nabla^R_U g\rangle + \\
&+\langle \pi_\leq \nabla^L_U f, \mathrm{Ad}_{U\tilde{\gamma}^*(U_-)^{-1}} \frac{1}{1-\gamma}\pi_> \nabla^R_U g\rangle - \langle \mathrm{Ad}_{U\tilde{\gamma}^*(U_-)^{-1}} \frac{1}{1-\gamma}\pi_> \nabla^R_U f, \pi_\leq \nabla^L_U g\rangle.
\end{aligned} \tag{4.4}$$

Recall that $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}(U) = \tilde{\gamma}^*(U_-)^{-1} \cdot U, \quad U \in \mathrm{GL}_n\,. \tag{4.5}$$

We find it very intriguing that the maps $\mathcal{C}^{-1}$ and $\mathcal{F}^{-1}$ have very similar formulas. In a sense, $\pi_{\mathcal{F}}$ sits in between $\pi^\dagger_{\text{std}}$ and $\pi^\dagger_{\mathbf{\Gamma}}$, and it can be twisted into either of the Poisson structures via an application of $(\mathcal{F}^{-1})_*$ or $(\mathcal{C}^{-1})_*$. Is there anything interesting that one can say about $\pi_{\mathcal{F}}$, as well as about the induced compatible generalized cluster structure on GL_n?

4.2 Are there cluster structures for $\mathcal{F}_m$'s?

Let us fix a BD triple $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ of type A_{n-1} and set

$$\begin{aligned}
\{f,g\}_+(U) :=& \langle \pi_> \nabla^R_U f, \nabla^R_U g\rangle - \langle \pi_> \nabla^L_U f, \nabla^L_U g\rangle + \\
&+\langle R_0\pi_0[\nabla_U f, U], [\nabla_U g, U]\rangle - \langle \pi_0[\nabla_U f, U], \nabla^L_U g\rangle, \quad U \in \mathrm{GL}_n,
\end{aligned} \tag{4.6}$$

where $\nabla_U^R f = U \cdot \nabla_U f$ and $\nabla_U^L f = \nabla_U f \cdot U$. Let $\hat{h}_{ij}(U) := \det U_{[i,n-j+i]}^{[j,n]}$. During a numerical experimentation[1], we noticed that

$$\{\log \hat{h}_{ij}, \log \hat{h}_{ks}\}_{\text{std}}^{\dagger} = \{\log \mathcal{F}_m^*(\hat{h}_{ij}), \log \mathcal{F}_m^*(\hat{h}_{ks})\}_+ = \{\log \mathcal{F}^*(\hat{h}_{ij}), \log \mathcal{F}^*(\hat{h}_{ks})\}_{\mathbf{\Gamma}}^{\dagger}$$

for all $m \in [0, \deg \gamma]$ (r_0 elements are assumed to be the same). A natural question arises: does there exist a sequence of Poisson varieties[2] (V_m, π_m) such that π_m reduces to $\{\cdot,\cdot\}_+$ for the flag minors of $\mathcal{F}_m$, and such that there is a generalized cluster structure $\mathcal{GC}_m$ on V_m compatible with π_m?

4.3 Are the g- and h-conventions equivalent?

By the equivalence we mean that the initial extended clusters of $\mathcal{GC}_h^{\dagger}(\mathbf{\Gamma})$ and $\mathcal{GC}_g^{\dagger}(\mathbf{\Gamma})$ can be obtained from one another via a sequence of mutations (and the variables are equal as elements of $\mathcal{O}(\mathrm{GL}_n)$). In [3] we verified that the frozen variables in $\mathcal{GC}_g^{\dagger}(\mathbf{\Gamma}, \mathrm{GL}_n)$ coincide with the frozen variables in $\mathcal{GC}_h^{\dagger}(\mathbf{\Gamma}, \mathrm{GL}_n)$ for any BD triple $\mathbf{\Gamma}$. As for the equivalence, we were able to confirm for $n = 3$ and all BD triples $\mathbf{\Gamma}$ that $\mathcal{GC}_g^{\dagger}(\mathbf{\Gamma}, \mathrm{GL}_3) = \mathcal{GC}_h^{\dagger}(\mathbf{\Gamma}, \mathrm{GL}_3)$. We conjecture that they are not equivalent for $n \geq 4$. Below we provide examples of mutation sequences that transform the initial cluster of $\mathcal{GC}_h^{\dagger}(\mathbf{\Gamma}, \mathrm{GL}_3)$ into the initial cluster of $\mathcal{GC}_g^{\dagger}(\mathbf{\Gamma}, \mathrm{GL}_3)$. In each case, we know all such sequences of minimal length (available upon request). Let us denote by φ'_{kl} and h'_{ij} the variables in the resulting extended cluster in $\mathcal{GC}_h^{\dagger}(\mathbf{\Gamma}, \mathrm{GL}_3)$.

Case $\Gamma_1 = \Gamma_2 = \emptyset$. The minimal length is 10, the number of distinct sequences of minimal length is 8. An example of such a sequence:

$$\varphi_{12} \to \varphi_{21} \to \varphi_{11} \to h_{23} \to \varphi_{12} \to h_{23} \to \varphi_{11} \to \varphi_{21} \to h_{23} \to \varphi_{21}. \tag{4.7}$$

The correspondence between the variables is given by $\varphi'_{kl}(U) = \phi_{kl}(U)$ and $h'_{ij}(U) = g_{ji}(U)$.

Case $\Gamma_1 = \{2\}$, $\Gamma_2 = \{1\}$. The minimal length is 11 and the number of sequences is 6. An example of such a sequence:

$$\varphi_{12} \to \varphi_{21} \to \varphi_{11} \to h_{22} \to h_{23} \to \varphi_{12} \to h_{23} \to \varphi_{11} \to \varphi_{21} \to h_{23} \to \varphi_{21}. \tag{4.8}$$

The correspondence between the variables is given by $\varphi'_{kl}(U) = \phi_{kl}(U)$, $h'_{23}(U) = g'_{32}(U)$, $h'_{22}(U) = g_{33}(U)$, $h_{33}(U) = g_{22}(U)$.

Case $\Gamma_1 = \{1\}$, $\Gamma_2 = \{2\}$. The minimal length is 13 and the number of sequences is 30. An example of such a sequence:

$$\varphi_{12} \to h_{23} \to \varphi_{12} \to \varphi_{11} \to h_{23} \to \varphi_{21} \to \varphi_{11} \to h_{23} \to h_{33} \to \varphi_{12} \to \varphi_{11} \to \varphi_{21} \to \varphi_{11}. \tag{4.9}$$

The correspondence between the variables is given by $\varphi'_{kl}(U) = \phi_{kl}(U)$, $h'_{23}(U) = g'_{32}(U)$, $h'_{33}(U) = g_{22}(U)$, $h_{22}(U) = g_{33}(U)$.

[1] We have verified this identity in $n = 3$, $n = 4$ and $n = 5$ for all BD triples.

[2] Of course, one can set V_m to be the spectrum of the ring generated by the flags of $\mathcal{F}_m$. We are interested in the largest possible variety $V_m \subseteq \mathrm{SL}_n$ with the mentioned properties.

4.4 How is $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{SL}_n^\dagger)$ related to $\mathcal{GC}(\mathbf{\Gamma}, D(\mathrm{SL}_n))$?

In the work [1], the initial extended cluster of the generalized cluster structure $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\mathrm{std}}, \mathrm{SL}_n^\dagger)$ was obtained from the initial extended cluster of $\mathcal{GC}(\mathbf{\Gamma}_{\mathrm{std}}, D(\mathrm{SL}_n))$ via a sequence of mutations denoted as $\mathcal{S}$. A natural question arises: if $\mathbf{\Gamma}$ is any aperiodic oriented BD triple of type A_{n-1}, can the initial extended cluster of $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{SL}_n^\dagger)$ be obtained from the initial extended cluster of $\mathcal{GC}(\mathbf{\Gamma}, D(\mathrm{SL}_n))$ that was described in [2]? We found such mutation sequences[3] in $n = 3$ and $n = 4$ for all BD triples. We conjecture that the same holds for $n \geq 5$; however, we do not see a relatively simple way of proving it for an arbitrary n (as one can see below, the mutation sequences become rather long and unpredictable).

Let us recall that the initial extended cluster of $\mathcal{GC}(\mathbf{\Gamma}, D(\mathrm{SL}_n))$ comprises 5 types of functions: the g-functions, the h-functions, the φ-functions, the f-functions and the c-functions. To resolve the conflict of notation, we will mark the g- and h-functions in $\mathcal{GC}(\mathbf{\Gamma}, D(\mathrm{SL}_n))$ with a bar. The $\mathcal{S}$ sequence in $n = 3$ is given by

$$\mathcal{S} := \bar{g}_{32} \to \bar{g}_{22} \to \bar{g}_{33} \to f_{11} \to \bar{g}_{32}, \tag{4.10}$$

and in $n = 4$,

$$\begin{aligned}\mathcal{S} :=& \bar{g}_{42} \to \bar{g}_{32} \to \bar{g}_{43} \to \bar{g}_{22} \to \bar{g}_{33} \to \bar{g}_{44} \to f_{21} \to f_{11} \to f_{12} \to \\ &\to \bar{g}_{42} \to \bar{g}_{32} \to \bar{g}_{43} \to \bar{g}_{33} \to \bar{g}_{42}.\end{aligned} \tag{4.11}$$

Below we list the mutation sequences for $n = 3$ and $n = 4$, as well as the correspondence between the variables. The variables in the resulting extended cluster of $\mathcal{GC}(\mathbf{\Gamma}, D(\mathrm{SL}_n))$ will be denoted as $\bar{g}'$, $\bar{h}'$ and f'. The c- and φ-variables for $\mathcal{GC}(\mathbf{\Gamma}, D(\mathrm{SL}_n))$ and $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{SL}_n^\dagger)$ are the same. The correspondence between the coordinates (X, Y) in $D(\mathrm{SL}_n)$ and U in SL_n is given by

$$D(\mathrm{SL}_n) \ni (X, Y) \mapsto U := X^{-1}Y \in \mathrm{SL}_n \,.$$

Note that in the case of $D(\mathrm{GL}_n)$, the below correspondence between the variables is up to an additional factor of $(\det X)^\ell$ for some ℓ that depends on the given variable.

Case $\Gamma_1 = \Gamma_2 = \emptyset$, $n = 3$. The mutation sequence is given by $\mathcal{S}$. The correspondence is given by $\bar{g}'_{32}(X, Y) = h_{33}(U)$, $f'_{11} = h_{22}(U)$, $\bar{g}'_{22}(X, Y) = h_{23}(U)$.

Case $\Gamma_1 = \{2\}$, $\Gamma_2 = \{1\}$, $n = 3$. The mutation sequence is given by

$$\mathcal{S} \to \bar{h}_{12} \to \bar{h}_{22}. \tag{4.12}$$

The correspondence is given by $\bar{h}'_{22}(X, Y) = h_{33}(U)$, $f'_{11}(X, Y) = h_{22}(U)$, $\bar{g}'_{22}(X, Y) = h_{23}(U)$.

Case $\Gamma_1 = \{1\}$, $\Gamma_2 = \{2\}$, $n = 3$. The mutation sequence is given by

$$\mathcal{S} \to \bar{h}_{13} \to \bar{h}_{23} \to \bar{h}_{33} \to \bar{g}_{33} \to \bar{g}_{22} \to \bar{h}_{13} \to \bar{h}_{23} \to \bar{h}_{33}. \tag{4.13}$$

The correspondence is given by $\bar{g}'_{33}(X, Y) = h_{23}(U)$, $\bar{h}'_{33}(X, Y) = h_{22}(U)$, $\bar{g}'_{32}(X, Y) = h_{33}(U)$.

[3] However, we didn't verify whether the sequences are of minimal possible length.

Case $\Gamma_1 = \Gamma_2 = \emptyset$, $n = 4$. The mutation sequence is given by $\mathcal{S}$. The correspondence is given by $\bar{g}'_{42}(X,Y) = h_{44}(U)$, $\bar{g}'_{32}(X,Y) = h_{34}(U)$, $\bar{g}'_{22}(X,Y) = h_{24}(U)$, $\bar{g}'_{33}(X,Y) = h_{33}(U)$, $f'_{21}(X,Y) = h_{23}(U)$, $f'_{12}(X,Y) = h_{22}(U)$.

Case $\Gamma_1 = \{3\}$, $\Gamma_2 = \{1\}$, $n = 4$. The mutation sequence is given by

$$\mathcal{S} \to \bar{h}_{12} \to \bar{h}_{22}. \tag{4.14}$$

The correspondence is given by $\bar{h}'_{22}(X,Y) = h_{44}(U)$, $\bar{g}'_{32}(X,Y) = h_{34}(U)$, $\bar{g}'_{22}(X,Y) = h_{24}(U)$, $\bar{g}'_{33}(X,Y) = h_{33}(U)$, $f'_{21}(X,Y) = h_{23}(U)$, $f'_{12}(X,Y) = h_{22}(U)$.

Case $\Gamma_1 = \{3\}$, $\Gamma_2 = \{2\}$, $n = 4$. The mutation sequence is given by

$$\mathcal{S} \to \bar{h}_{13} \to \bar{h}_{23} \to \bar{h}_{33} \to f_{11}. \tag{4.15}$$

The correspondence is given by $f'_{11}(X,Y) = h_{44}(U)$, $\bar{g}'_{32}(X,Y) = h_{34}(U)$, $\bar{g}'_{22}(X,Y) = h_{24}(U)$, $\bar{g}'_{33}(X,Y) = h_{33}(U)$, $f'_{21}(X,Y) = h_{23}(U)$, $f'_{12}(X,Y) = h_{22}(U)$.

Case $\Gamma_1 = \{1\}$, $\Gamma_2 = \{3\}$, $n = 4$. The mutation sequence is given by

$$\begin{aligned} S \to &\bar{h}_{14} \to \bar{h}_{24} \to \bar{h}_{34} \to \bar{h}_{44} \to \bar{g}_{44} \to \bar{g}_{43} \to \bar{g}_{22} \to \\ \to &\bar{h}_{14} \to \bar{h}_{24} \to \bar{h}_{34} \to \bar{h}_{44} \to \bar{g}_{44} \to \bar{g}_{22} \to f_{21} \to \\ \to &\bar{h}_{14} \to \bar{h}_{24} \to \bar{h}_{34} \to \bar{h}_{44}. \end{aligned} \tag{4.16}$$

The correspondence is given by $\bar{g}'_{42}(X,Y) = h_{44}(U)$, $\bar{g}'_{32}(X,Y) = h_{34}(U)$, $\bar{g}'_{43}(X,Y) = h_{24}(U)$, $\bar{g}'_{33}(X,Y) = h_{33}(U)$, $g'_{44}(X,Y) = h_{23}(U)$, $h'_{44}(X,Y) = h_{22}(U)$.

Case $\Gamma_1 = \{1\}$, $\Gamma_2 = \{2\}$, $n = 4$. The mutation sequence is given by

$$\begin{aligned} S \to &\bar{h}_{13} \to \bar{h}_{23} \to \bar{h}_{33} \to f_{11} \to \bar{g}_{22} \to \\ \to &\bar{h}_{13} \to \bar{h}_{23} \to \bar{h}_{33} \to \bar{g}_{22} \to f_{21} \to \\ \to &\bar{h}_{13} \to \bar{h}_{23}. \end{aligned} \tag{4.17}$$

The correspondence is given by $\bar{g}'_{42}(X,Y) = h_{44}(U)$, $\bar{g}'_{32}(X,Y) = h_{34}(U)$, $f'_{11}(X,Y) = h_{24}(U)$, $\bar{g}'_{33}(X,Y) = h_{33}(U)$, $\bar{h}'_{33}(X,Y) = h_{23}(U)$, $\bar{h}'_{23}(X,Y) = h_{22}(U)$.

Case $\Gamma_1 = \{2\}$, $\Gamma_2 = \{3\}$, $n = 4$. The mutation sequence is given by

$$S \to \bar{h}_{14} \to \bar{h}_{24} \to \bar{h}_{34} \to \bar{h}_{44} \to \bar{g}_{44} \to \bar{g}_{43} \to \bar{g}_{32} \to \bar{h}_{14} \to \bar{h}_{24} \to \bar{h}_{34} \to \bar{h}_{44} \to \bar{g}_{44}. \tag{4.18}$$

The correspondence is given by $\bar{g}'_{42}(X,Y) = h_{44}(U)$, $\bar{g}'_{43}(X,Y) = h_{34}(U)$, $\bar{g}'_{22}(X,Y) = h_{24}(U)$, $\bar{g}'_{44}(X,Y) = h_{33}(U)$, $f'_{21}(X,Y) = h_{23}(U)$, $f'_{12}(X,Y) = h_{22}(U)$.

Case $\Gamma_1 = \{2\}$, $\Gamma_2 = \{1\}$, $n = 4$. The mutation sequence is given by

$$\mathcal{S} \to \bar{h}_{12} \to \bar{h}_{22} \to \bar{g}_{32} \to \bar{h}_{12}. \tag{4.19}$$

The correspondence is given by $\bar{g}'_{42}(X,Y) = h_{44}(U)$, $\bar{h}'_{22}(X,Y) = h_{34}(U)$, $\bar{g}'_{22}(X,Y) = h_{24}(U)$, $\bar{h}'_{12}(X,Y) = h_{33}(U)$, $f'_{21}(X,Y) = h_{23}(U)$, $f'_{12}(X,Y) = h_{22}(U)$.

Case of Cremmer-Gervais, $\Gamma_1 = \{2,3\}$, $\Gamma_2 = \{1,2\}$, $\gamma(i) = i-1$, $i \in \Gamma_1$. The mutation sequence is given by

$$S \to \bar{h}_{12} \to \bar{h}_{22} \to \bar{h}_{13} \to \bar{h}_{23} \to \bar{h}_{33} \to \bar{g}_{32} \to \bar{h}_{12} \to \bar{g}_{32} \to \bar{g}_{33} \to f_{11}. \tag{4.20}$$

The correspondence is given by $f'_{11}(X,Y) = h_{44}(U)$, $\bar{h}'_{22}(X,Y) = h_{34}(U)$, $\bar{g}'_{22}(X,Y) = h_{24}(U)$, $\bar{h}'_{12}(X,Y) = h_{33}(U)$, $f'_{21}(X,Y) = h_{23}(U)$, $f'_{12}(X,Y) = h_{22}(U)$.

Case of Cremmer-Gervais, $\Gamma_1 = \{1,2\}$, $\Gamma_2 = \{2,3\}$, $\gamma(i) = i+1$, $i \in \Gamma_1$. The mutation sequence is given by

$$\begin{aligned} S \to &\bar{h}_{13} \to \bar{h}_{23} \to \bar{h}_{33} \to \bar{h}_{14} \to \bar{h}_{24} \to \bar{h}_{34} \to \bar{h}_{44} \to f_{11} \to \bar{g}_{22} \to \bar{g}_{44} \to \bar{g}_{43} \to \bar{g}_{32} \to \\ \to &\bar{h}_{13} \to \bar{h}_{23} \to \bar{h}_{33} \to \bar{h}_{14} \to \bar{h}_{24} \to \bar{h}_{34} \to \bar{h}_{44} \to \bar{g}_{44} \to f_{21} \to g_{32} \to g_{22} \to h_{14} \to \\ \to &h_{24} \to h_{13} \to h_{34} \to h_{44} \to h_{23} \to g_{33} \to h_{44} \to f_{11} \to h_{33}. \end{aligned} \tag{4.21}$$

The correspondence is given by $\bar{g}'_{42}(X,Y) = h_{44}(U)$, $\bar{g}'_{43}(X,Y) = h_{34}(U)$, $\bar{g}'_{33}(X,Y) = h_{24}(U)$, $\bar{g}'_{44}(X,Y) = h_{33}(U)$, $f'_{11}(X,Y) = h_{23}(U)$, $\bar{h}'_{33}(X,Y) = h_{22}(U)$.

5 Examples in $n = 3$ in the h-convention

5.1 The standard BD triple

The initial quiver is illustrated in Figure 1.

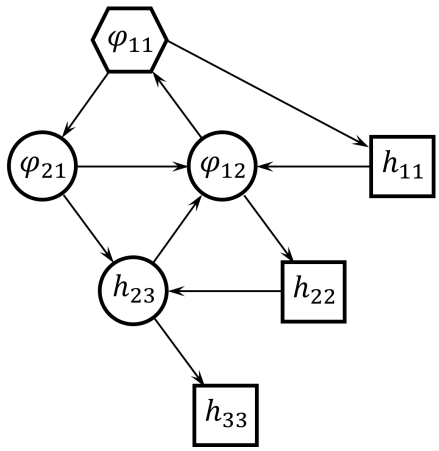

Figure 1. *The initial quiver for* $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{std}, \mathrm{GL}_3)$.

The initial variables. The variables in the initial extended cluster are given as follows:

$$c_1(U) = \mathrm{tr}(U), \quad c_2(U) = \frac{1}{2!}(\mathrm{tr}(U)^2 - \mathrm{tr}(U^2)); \tag{5.1}$$

$$\varphi_{21}(U) = u_{13}, \quad \varphi_{12}(U) = \det U_{[1,2]}^{[2,3]} \tag{5.2}$$

$$\varphi_{11}(U) = -\det \begin{bmatrix} u_{13} & (U^2)_{13} \\ u_{23} & (U^2)_{23} \end{bmatrix} = u_{23} \det U_{[1,2]}^{[2,3]} + u_{13} \det U_{[1,2]}^{\{1,3\}}; \tag{5.3}$$

$$h_{23}(U) = -u_{23}u_{33} - u_{13}u_{32}, \quad h_{22}(U) = u_{33} \det U_{[2,3]}^{[2,3]} + u_{32} \det U_{\{1,3\}}^{[2,3]}; \tag{5.4}$$

$$h_{11}(U) = \det U, \quad h_{33}(U) = u_{33}. \tag{5.5}$$

Some 1-step mutations.

$$\varphi_{11}'(U) = \det \begin{bmatrix} u_{12} & u_{13} \\ (U^2)_{12} & (U^2)_{13} \end{bmatrix} = u_{12} \det U_{[1,2]}^{[2,3]} + u_{13} \det U_{\{1,3\}}^{[2,3]}. \tag{5.6}$$

5.2 $\Gamma_1 = \{2\}$, $\Gamma_2 = \{1\}$

The initial quiver is illustrated in Figure 2.

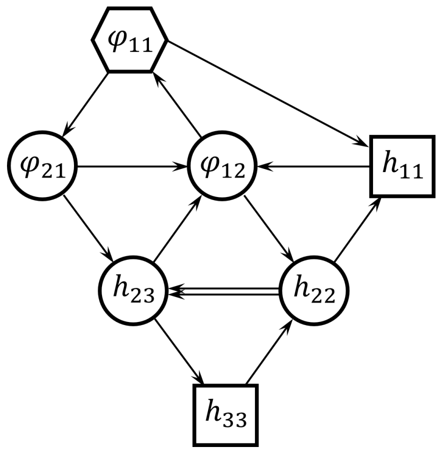

Figure 2. *The initial quiver for* $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{GL}_3)$ *with* $\Gamma_1 = \{2\}$, $\Gamma_2 = \{1\}$.

The initial variables. All the variables in the initial extended cluster are as in $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\mathrm{std}}, \mathrm{GL}_3)$ except the variable h_{33}, which is given by

$$h_{33}(U) = u_{33} \det U_{[2,3]}^{[2,3]} + u_{23} \det U_{[2,3]}^{\{1,3\}}. \tag{5.7}$$

Birational quasi-isomorphisms. The birational quasi-isomorphism

$$\mathcal{Q} : (\mathrm{GL}_3, \mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\mathrm{std}})) \dashrightarrow (\mathrm{GL}_3, \mathcal{GC}_h^\dagger(\mathbf{\Gamma}))$$

is given by

$$\mathcal{Q}(U) = (I - \alpha(U) e_{32}) U (I + \alpha(U) e_{32}), \quad \alpha(U) := \frac{\det U_{[2,3]}^{\{1,3\}}}{\det U_{[2,3]}^{[2,3]}}. \tag{5.8}$$

5.3 $\Gamma_1 = \{1\}$, $\Gamma_2 = \{2\}$

The initial quiver is illustrated in Figure 3.

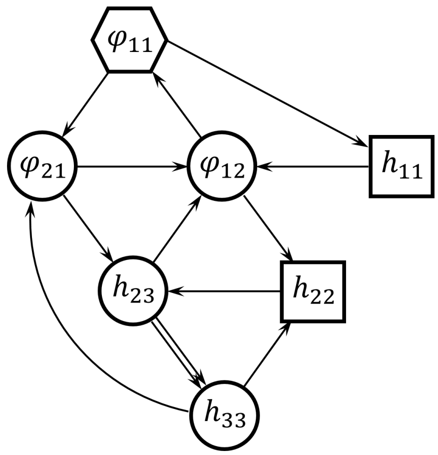

Figure 3. *The initial quiver for* $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{GL}_3)$ *with* $\Gamma_1 = \{1\}$, $\Gamma_2 = \{2\}$.

The initial variables. All the variables in the initial extended cluster are as in $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\mathrm{std}}, \mathrm{GL}_3)$ except the variables h_{23} and h_{22}; these are given by

$$h_{23}(U) = -u_{23}u_{33} - u_{13}u_{32}, \quad h_{22}(U) = u_{33} \det U_{[2,3]}^{[2,3]} + u_{32} \det U_{[2,3]}^{\{1,3\}}. \tag{5.9}$$

Birational quasi-isomorphisms. The birational quasi-isomorphism

$$\mathcal{Q} : (\mathrm{GL}_3, \mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\mathrm{std}})) \dashrightarrow (\mathrm{GL}_3, \mathcal{GC}_h^\dagger(\mathbf{\Gamma}))$$

is given by

$$\mathcal{Q}(U) = (I - \alpha(U)e_{21})U(I + \alpha(U)e_{21}), \quad \alpha(U) := \frac{u_{32}}{u_{33}}. \tag{5.10}$$

6 Examples in $n = 4$ in the h-convention

6.1 The standard BD triple

The initial quiver for $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\rm std}, \mathrm{GL}_4)$ is illustrated in Figure 4.

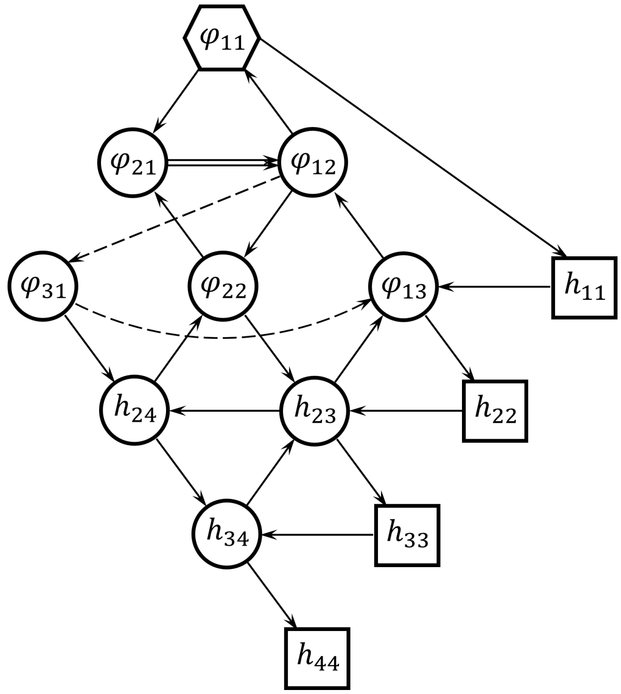

Figure 4. *The initial quiver for* $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{std}, \mathrm{GL}_4)$.

The initial variables. The φ-variables are given by

$$\varphi_{11}(U) = -\det \begin{bmatrix} u_{14} & (U^2)_{14} & (U^3)_{14} \\ u_{24} & (U^2)_{24} & (U^3)_{24} \\ u_{34} & (U^2)_{34} & (U^3)_{34} \end{bmatrix}; \tag{6.1}$$

$$\varphi_{12}(U) = \det \begin{bmatrix} u_{13} & u_{14} & (U^2)_{14} \\ u_{23} & u_{24} & (U^2)_{24} \\ u_{33} & u_{34} & (U^2)_{34} \end{bmatrix}, \quad \varphi_{21}(U) = \det \begin{bmatrix} u_{14} & (U^2)_{14} \\ u_{24} & (U^2)_{24} \end{bmatrix}; \tag{6.2}$$

$$\varphi_{31}(U) = -u_{14}, \quad \varphi_{22}(U) = \det U_{[1,2]}^{[3,4]}, \quad \varphi_{13}(U) = -\det U_{[1,3]}^{[2,4]}. \tag{6.3}$$

The h-variables are given by

$$h_{24}(U) = u_{24}, \quad h_{23}(U) = \det U_{[2,3]}^{[3,4]}, \quad h_{22}(U) = \det U_{[2,4]}^{[2,4]}; \tag{6.4}$$

$$h_{34}(U) = -u_{34}, \quad h_{33}(U) = \det U_{[3,4]}^{[3,4]}, \quad h_{44}(U) = u_{44}. \tag{6.5}$$

The c-variables are given by

$$c_1(U) = -\operatorname{tr} U, \quad c_2(U) = \frac{1}{2!}\left(\operatorname{tr}(U)^2 - \operatorname{tr}(U^2)\right), \tag{6.6}$$

$$c_3(U) = -\frac{1}{3!}\left(\operatorname{tr}(U)^3 - 3\operatorname{tr}(U)\operatorname{tr}(U^2) + 2\operatorname{tr}(U^3)\right). \tag{6.7}$$

List of 1-step mutations. Here's what one obtains after a 1-step mutation of the initial cluster in each direction:

$$\varphi_{12}'(U) = \det U_{[1,2]}^{[3,4]} \det \begin{bmatrix} u_{14} & (U^2)_{14} \\ u_{34} & (U^2)_{34} \end{bmatrix} + \begin{bmatrix} u_{14} & (U^2)_{14} \\ u_{24} & (U^2)_{24} \end{bmatrix} \det U_{[1,2]}^{\{2,4\}}; \tag{6.8}$$

$$\varphi_{21}'(U) = -\det \begin{bmatrix} u_{14} & (U^2)_{13} & (U^2)_{14} \\ u_{24} & (U^2)_{23} & (U^2)_{24} \\ u_{34} & (U^2)_{33} & (U^2)_{34} \end{bmatrix}; \tag{6.9}$$

$$\varphi_{31}'(U) = -\det U_{[1,3]}^{\{1\}\cup[3,4]}, \quad \varphi_{31}'(U) = -u_{24} \det U_{[2,4]}^{[2,4]} - u_{14} \det U_{[2,4]}^{\{1\}\cup[3,4]}; \tag{6.10}$$

$$\varphi_{22}'(U) = -u_{14} \det U_{[2,3]}^{\{1,4\}} - u_{24} \det U_{[2,3]}^{\{2,4\}} - u_{34} \det U_{[2,3]}^{[3,4]}. \tag{6.11}$$

$$h_{24}'(U) = -\det U_{\{1,3\}}^{[3,4]}, \quad h_{34}'(U) = -\det U_{\{2,4\}}^{[3,4]}; \tag{6.12}$$

$$h_{23}'(U) = u_{14} \det U_{[2,4]}^{[2,4]} - u_{24} \det U_{\{1\}\cup[3,4]}^{[2,4]}. \tag{6.13}$$

6.2 Cremmer-Gervais $i \mapsto i+1$

The initial quiver for $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{GL}_4)$ is illustrated in Figure 5.

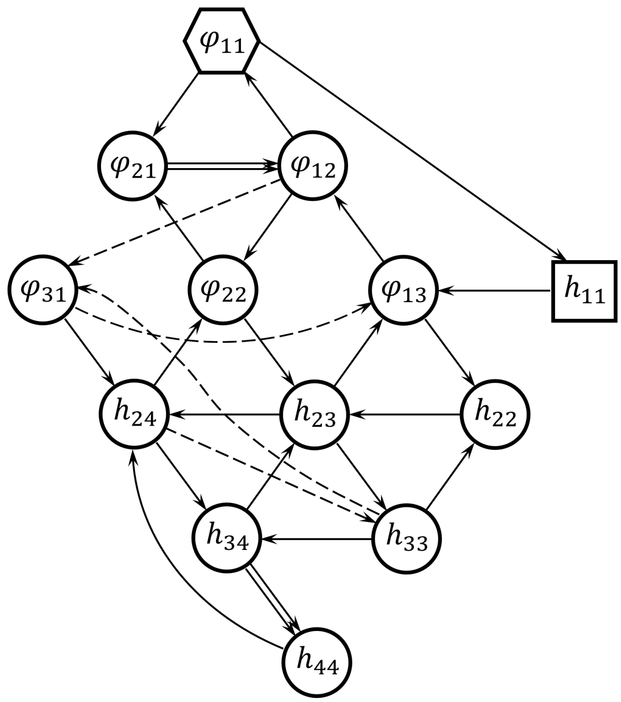

Figure 5. *The initial quiver for* $\mathcal{GC}_g^\dagger(\mathbf{\Gamma}, \mathrm{GL}_4)$ *for* $\Gamma_1 = \{1,2\}$, $\Gamma_2 = \{2,3\}$, $\gamma : i \mapsto i+1$.

The initial variables. In the initial extended cluster, all cluster and frozen variables are given as in $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\mathrm{std}}, \mathrm{GL}_4)$ except for the variables h_{22}, h_{23}, h_{24}, h_{33}, h_{34}. Let us set

$$\ell_1(U) := \det U_{[3,4]}^{[3,4]} u_{44} + \det U_{\{2,4\}}^{[3,4]} u_{43} + \det U_{\{1,4\}}^{[3,4]} u_{42}; \tag{6.14}$$

$$\ell_2(U) := \det U_{[3,4]}^{\{2,4\}} u_{44} + \det U_{\{2,4\}}^{\{2,4\}} u_{43} + \det U_{\{1,4\}}^{\{2,4\}} u_{42}; \tag{6.15}$$

$$\ell_3(U) := \det U_{[3,4]}^{[2,3]} u_{44} + \det U_{\{2,4\}}^{[2,3]} u_{43} + \det U_{\{1,4\}}^{[2,3]} u_{42}. \tag{6.16}$$

Then the h-variables are given by:

$$h_{24}(U) = u_{24} \cdot \ell_1(U) + u_{14}\ell_2(U), \quad h_{34}(U) = -u_{34}u_{44} - u_{24}u_{43} - u_{14}u_{42}, \quad h_{44}(U) = u_{44}; \tag{6.17}$$

$$h_{23}(U) = \det U_{[2,3]}^{[3,4]} \ell_1(U) + \det U_{\{1,3\}}^{[3,4]} \ell_2(U) + \det U_{[1,2]}^{[3,4]} \ell_3(U), \quad h_{33}(U) = \ell_1(U); \tag{6.18}$$

$$h_{22}(U) = \det U_{[2,4]}^{[2,4]} \ell_1(U) + \det U_{\{1\}\cup[3,4]}^{[2,4]} \ell_2(U) + \det U_{[1,2]\cup\{4\}}^{[2,4]} \ell_3(U). \tag{6.19}$$

Birational quasi-isomorphisms. TBD

6.3 Cremmer-Gervais $i \mapsto i-1$

The initial quiver for $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}, \mathrm{GL}_4)$ is illustrated in Figure 6.

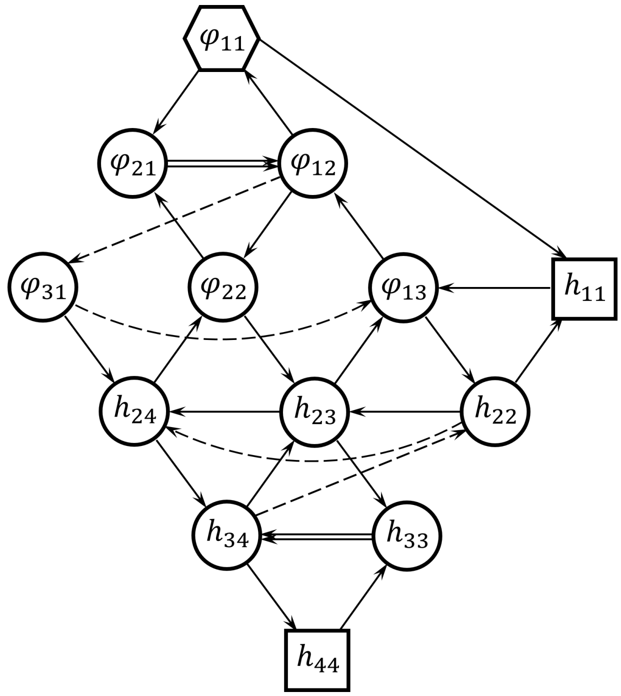

Figure 6. *The initial quiver for* $\mathcal{GC}_g^\dagger(\mathbf{\Gamma}, \mathrm{GL}_4)$ *for* $\Gamma_1 = \{2,3\}$, $\Gamma_2 = \{1,2\}$, $\gamma : i \mapsto i-1$.

The initial variables. In the initial extended cluster, all cluster and frozen variables are given as in $\mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\mathrm{std}}, \mathrm{GL}_4)$ except for the variables h_{34}, h_{33}, h_{44}. These are given by:

$$h_{34}(U) = -u_{34} \det U_{[2,4]}^{[2,4]} - u_{24} \det U_{[2,4]}^{\{1\}\cup[3,4]}; \tag{6.20}$$

$$h_{33}(U) = \det U_{[3,4]}^{[3,4]} \det U_{[2,4]}^{[2,4]} + \det U_{\{2,4\}}^{[3,4]} \det U_{[2,4]}^{\{1\}\cup[3,4]} + \det U_{[2,3]}^{[3,4]} \det U_{[2,4]}^{[1,2]\cup\{4\}}; \tag{6.21}$$

$$\begin{aligned} h_{44}(U) =& u_{44} \left(\det U_{[3,4]}^{[3,4]} \det U_{[2,4]}^{[2,4]} + \det U_{\{2,4\}}^{[3,4]} \det U_{[2,4]}^{\{1\}\cup[3,4]} + \det U_{[2,3]}^{[3,4]} \det U_{[2,4]}^{[1,2]\cup\{4\}}\right) + \\ &+ u_{34} \left(\det U_{[3,4]}^{\{2,4\}} \det U_{[2,4]}^{[2,4]} + \det U_{\{2,4\}}^{\{2,4\}} \det U_{[2,4]}^{\{1\}\cup[3,4]} + \det U_{[2,3]}^{\{2,4\}} \det U_{[2,4]}^{[1,2]\cup\{4\}}\right) + \\ &+ u_{24} \left(\det U_{[3,4]}^{\{1,4\}} \det U_{[2,4]}^{[2,4]} + \det U_{\{2,4\}}^{\{1,4\}} \det U_{[2,4]}^{\{1\}\cup[3,4]} + \det U_{[2,3]}^{\{1,4\}} \det U_{[2,4]}^{[1,2]\cup\{4\}}\right). \end{aligned} \tag{6.22}$$

Birational quasi-isomorphisms. TBD

7 Examples in $n = 4$ in the g-convention

7.1 The standard BD triple

The initial quiver for $\mathcal{GC}_g^\dagger(\mathbf{\Gamma}_{\rm std}, \mathrm{GL}_4)$ is illustrated in Figure 7.

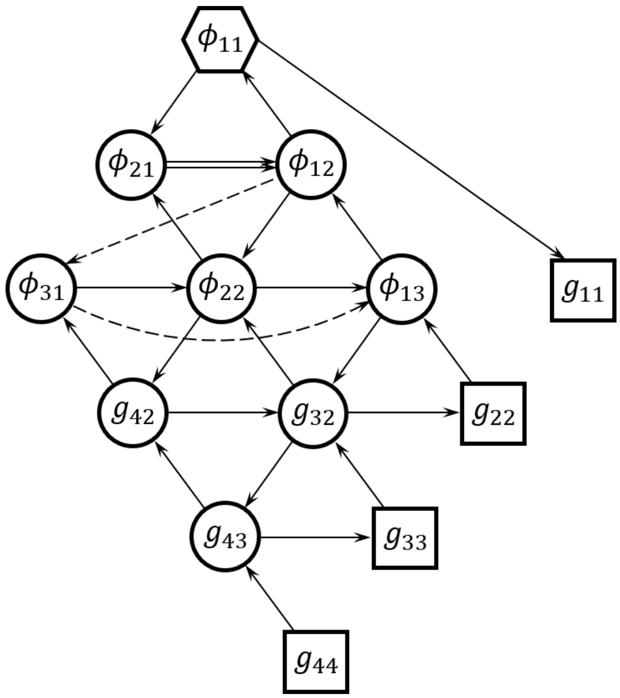

Figure 7. *The initial quiver for* $\mathcal{GC}_g^\dagger(\mathbf{\Gamma}_{std}, \mathrm{GL}_4)$.

The initial variables. The ϕ- and c-variables, as elements of $\mathcal{O}(\mathrm{GL}_4)$, are given by the following formulas:

$$\phi_{11}(U) = \det \begin{bmatrix} u_{21} & (U^2)_{21} & (U^3)_{21} \\ u_{31} & (U^2)_{31} & (U^3)_{31} \\ u_{41} & (U^2)_{41} & (U^3)_{41} \end{bmatrix}, \quad \phi_{12}(U) = -\det \begin{bmatrix} u_{21} & u_{22} & (U^2)_{21} \\ u_{31} & u_{32} & (U^2)_{31} \\ u_{41} & u_{42} & (U^2)_{41} \end{bmatrix}; \tag{7.1}$$

$$\phi_{21}(U) = \det \begin{bmatrix} u_{31} & (U^2)_{31} \\ u_{41} & (U^2)_{41} \end{bmatrix}, \quad \phi_{31}(U) = u_{41}, \quad \phi_{22}(U) = \det U^{[1,2]}_{[3,4]}, \quad \phi_{13}(U) = \det U^{[1,3]}_{[2,4]}; \tag{7.2}$$

$$c_1(U) = -\operatorname{tr} U, \quad c_2(U) = \frac{1}{2!}\left(\operatorname{tr}(U)^2 - \operatorname{tr}(U^2)\right), \tag{7.3}$$

$$c_3(U) = -\frac{1}{3!}\left(\operatorname{tr}(U)^3 - 3\operatorname{tr}(U)\operatorname{tr}(U^2) + 2\operatorname{tr}(U^3)\right). \tag{7.4}$$

The g-variables are given by

$$g_{11}(U) := \det U, \quad g_{ij}(U) = \det U^{[j,n-i+j]}_{[i,n]}, \quad 2 \le j \le i \le n. \tag{7.5}$$

7.2 Cremmer-Gervais $i \mapsto i-1$

The initial quiver for $\mathcal{GC}_g^{\dagger}(\mathbf{\Gamma}, \mathrm{GL}_4)$ is illustrated in Figure 8.

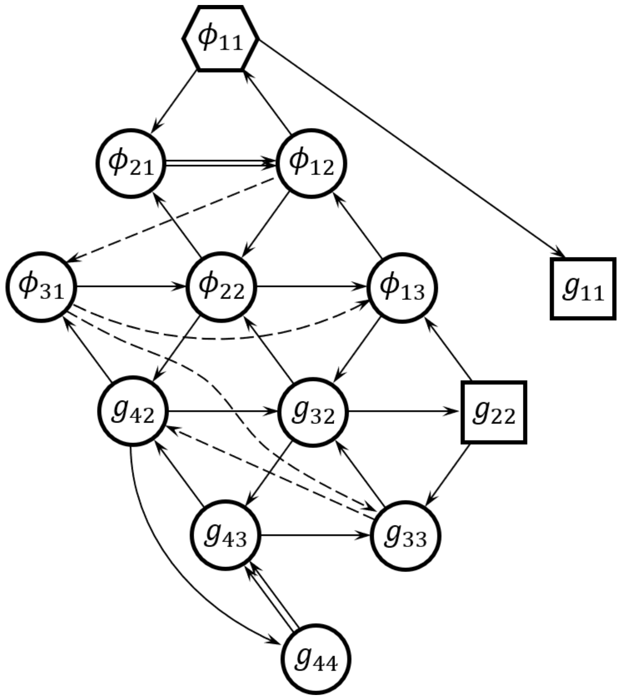

Figure 8. *The initial quiver for* $\mathcal{GC}_g^{\dagger}(\mathbf{\Gamma}, \mathrm{GL}_4)$ *for* $\Gamma_1 = \{2,3\}$, $\Gamma_2 = \{1,2\}$, $\gamma : i \mapsto i-1$.

The initial variables. Let us set

$$\ell_1(U) := \det U_{[3,4]}^{[3,4]} u_{44} + \det U_{[3,4]}^{\{2,4\}} u_{34} + \det U_{[3,4]}^{\{1,4\}} u_{24}; \tag{7.6}$$

$$\ell_2(U) := \det U_{\{2,4\}}^{[3,4]} u_{44} + \det U_{\{2,4\}}^{\{2,4\}} u_{34} + \det U_{\{2,4\}}^{\{1,4\}} u_{24}; \tag{7.7}$$

$$\ell_3(U) := \det U_{[2,3]}^{[3,4]} u_{44} + \det U_{[2,3]}^{\{2,4\}} u_{34} + \det U_{[2,3]}^{\{1,4\}} u_{24}. \tag{7.8}$$

The g-variables are given by the following formulas:

$$g_{42}(U) = u_{42} \cdot \ell_1(U) + u_{41}\ell_2(U), \quad g_{43}(U) = u_{43}u_{44} + u_{42}u_{34} + u_{41}u_{24}, \quad g_{44}(U) = u_{44}; \tag{7.9}$$

$$g_{32}(U) = \det U_{[3,4]}^{[2,3]} \ell_1(U) + \det U_{[3,4]}^{\{1,3\}} \ell_2(U) + \det U_{[3,4]}^{[1,2]} \ell_3(U), \quad g_{33}(U) = \ell_1(U); \tag{7.10}$$

$$g_{22}(U) = \det U_{[2,4]}^{[2,4]} \ell_1(U) + \det U_{[2,4]}^{\{1\}\cup[3,4]} \ell_2(U) + \det U_{[2,4]}^{[1,2]\cup\{4\}} \ell_3(U). \tag{7.11}$$

Birational quasi-isomorphisms. There is a birational quasi-isomorphism

$$\mathcal{Q}^{\mathrm{op}} : (\mathrm{GL}_4, \mathcal{GC}_g^{\dagger}(\mathbf{\Gamma}_{\mathrm{std}})) \dashrightarrow (\mathrm{GL}_4, \mathcal{GC}_g^{\dagger}(\mathbf{\Gamma})), \quad \mathcal{Q}^{\mathrm{op}}(U) := \rho^{\mathrm{op}}(U) U (\rho^{\mathrm{op}}(U))^{-1} \tag{7.12}$$

where the rational map $\rho^{\mathrm{op}} : \mathrm{GL}_n \dashrightarrow \mathrm{GL}_n$ is given by

$$\rho^{\mathrm{op}}(U) = \left(I + \frac{u_{34}}{u_{44}} e_{12} \right) \cdot \left(I + \frac{\det U_{\{2,4\}}^{[3,4]}}{\det U_{[3,4]}^{[3,4]}} e_{12} + \frac{u_{24}}{u_{44}} e_{13} + \frac{u_{34}}{u_{44}} e_{23} \right). \tag{7.13}$$

The marked variables for $\mathcal{Q}^{\mathrm{op}}$ are g_{33} and g_{44}. Define the BD triples $\tilde{\mathbf{\Gamma}} := (\{2\},\{1\},2\mapsto 1)$ and $\hat{\mathbf{\Gamma}} := (\{3\},\{2\},3\mapsto 2)$. There is a pair of complementary birational quasi-isomorphisms

$$\mathcal{G} : (\mathrm{GL}_4, \mathcal{GC}_g^{\dagger}(\tilde{\mathbf{\Gamma}}) \dashrightarrow (\mathrm{GL}_4, \mathcal{GC}_g^{\dagger}(\mathbf{\Gamma})), \quad \mathcal{G}' : (\mathrm{GL}_4, \mathcal{GC}_g^{\dagger}(\hat{\mathbf{\Gamma}})) \dashrightarrow (\mathrm{GL}_4, \mathcal{GC}_g^{\dagger}(\mathbf{\Gamma})). \tag{7.14}$$

They are given by

$$\mathcal{G}^{\mathrm{op}}(U) = G^{\mathrm{op}}(U)\cdot U \cdot G^{\mathrm{op}}(U)^{-1}, \quad G^{\mathrm{op}}(U) := \left(I + \frac{u_{34}}{u_{44}}e_{12}\right)\cdot\left(I + \frac{u_{24}}{u_{44}}e_{13} + \frac{u_{34}}{u_{44}}e_{23}\right); \tag{7.15}$$

$$(\mathcal{G}^{\mathrm{op}})'(U) = G'(U)\cdot U\cdot G'(U)^{-1}, \quad G'(U) := (I + \alpha_1(U)e_{12} + \alpha_2(U)e_{13}), \tag{7.16}$$

$$\alpha_1(U) = \frac{\det U^{[3,4]}_{\{2,4\}}u_{44} + \det U^{\{2,4\}}_{\{2,4\}}u_{34}}{\det U^{[3,4]}_{[3,4]}u_{44} + \det U^{\{2,4\}}_{[3,4]}u_{34}}, \quad \alpha_2(U) = -\frac{\det U^{[3,4]}_{[2,3]}u_{44} + \det U^{\{2,4\}}_{[2,3]}u_{34}}{\det U^{[3,4]}_{[3,4]}u_{44} + \det U^{\{2,4\}}_{[3,4]}u_{34}}. \tag{7.17}$$

The marked variable for $\mathcal{G}$ is g_{44}, and the marked variable for $\mathcal{G}'$ is g_{33}.

References

[1] M. Gekhtman, M. Shapiro and A. Vainshtein, 'Drinfeld double of GL_n and generalized cluster structures', *Proc. Lond. Math. Soc.* **116**(3) (2018), 429–484. doi:10.1112/plms.12086

[2] D. Voloshyn, 'Multiple generalized cluster structures on $D(\mathrm{GL}_n)$', *Forum of Mathematics, Sigma* **11**(46) (2023), 1–78. doi:10.1017/fms.2023.44

[3] D. Voloshyn and M. Gekhtman, 'Generalized cluster structures on $\mathrm{SL}_n^\dagger$', Preprint, 2023. arXiv:2312.04859