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1 Preliminaries

1.1 The main principles of classical lattice models

Let Ω be a finite set (the set of *microstates*), let $\mathcal{H} : \Omega \to \mathbb{R}$ be a *hamiltonian*, a specifically chosen random variable. Let $\mathcal{M}(\Omega)$ be the space of probability measures on Ω. In this theory, the expectation of a random variable $f : \Omega \to \mathbb{R}$ with respect to a measure μ (if not implicitly understood; it's also called the *thermal average*) is denoted as

$$\langle f \rangle_\mu := \mathbb{E}_\mu f = \int_\Omega f d\mu.$$

One of the first questions in statistical mechanics is devoted to the choice of the right measure μ. The choice is governed using Shannon's entropy $S : \mathcal{M}(\Omega) \to \mathbb{R}$, defined as $S(\mu) := -\int_\Omega x \log x d\mu(x)$ (there's a way to understand why S has this form; see [3]). *The maximum entropy principle says*: for a given model of statistical mechanics, choose μ that maximizes S. For example, if there's no other information avaiable about the system, then the measure that maximizes S is the uniform distribution. There are other two typical situations:

(a) If we know that $\langle \mathcal{H} \rangle = U$ for some fixed U (the *internal energy* of the system), then the measure that maximizes S is

$$\mu(\omega) = \frac{e^{-\beta H(\omega)}}{Z}, \quad Z := \sum_{\omega \in \Omega} e^{-\beta \mathcal{H}(\omega)}$$

which is the *Gibbs measure*. This corresponds to the situation when we know the system exchanges its energy with some external thermal reservoir. The result can be obtained using Lagrange multipliers; the parameter β, called the *inverse temperature*, is uniquely determined by U (and vice versa: U is uniquely determined by β). In the theory, $\beta = (kT)^{-1}$, where k is the Boltzmann constant and T is the temperature of the system.

(b) If we know additionally that the system exchanges its particles $\mathcal{N}$ with the external environment, and that the expected value of the particles if $\langle \mathcal{N} \rangle = N$, then we obtain a *grand canonical Gibbs distribution* via a similar procedure. It's given by

$$\mu(\omega) := \frac{e^{-\beta(\mathcal{H}(\omega)-\mu N)}}{Z}, \quad Z := \sum_N e^{\beta\mu N} \sum_{\omega\in\Omega} e^{-\beta\mathcal{H}(\omega)}.$$

The parameter μ is identified with the so-called *chemical potential.*

1.2 The thermodynamic limit

1.3 Quantum lattices

1.3.1 A general set-up

I will follow closely the treatment in [5]. Again, we have a lattice $X \subset \mathbb{Z}^n$, but to each site $i \in X$ we attach a copy of a finite-dimensional Hilbert space H_i. To a finite X we attach the tensor product $H_X := \otimes_{i\in X} H_i$.

For infinite lattices, the author of [5] suggests proceeding as follows. Let $\mathfrak{A}_X := \mathrm{End}(H_X)$, and for any two finite subsets $X \subseteq Y \subset \mathbb{Z}^n$, let $\imath : \mathfrak{A}_X \to \mathfrak{A}_Y$ be the inclusion that sends A to $A \otimes 1$ (where 1 is viewed as an endomorphism of $\mathfrak{A}_{Y\setminus X}$). For an infinite $\Lambda \subseteq \mathbb{Z}^n$, the family of all its finite subsets with the inclusions form a direct system. Let $\mathfrak{A}_\Lambda := \varinjlim \mathfrak{A}_X$ be the direct limit taken in the category of C^*-algebras over all finite subsets of Λ. In the literature, this algebra is known as an AF (approximately finite-dimensional) C^*-algebra. The first reference in this theory goes back to Bratteli [1]. See the next subsection for an elaboration on the inductive limit.

Further, for a finite $\Lambda \subset \mathbb{Z}^n$ and a Hamiltonian $\mathcal{H}_\Lambda$, the partition function is defined as

$$Z = \mathrm{tr}_\Lambda\, e^{-\beta\mathcal{H}_\Lambda}$$

and the expectation of an observable $A \in \mathfrak{A}_\Lambda$ is

$$\langle A \rangle_\Lambda := Z^{-1}\,\mathrm{tr}_\Lambda(Ae^{-\mathcal{H}_\Lambda}).$$

The trace in these formulas is normalized: it's $1/d$ of the usual trace, where d is the dimension of the Hilbert space at one site. An interesting consequence of such normalization is that tr extends than to a norm-one linear functional on the whole $\mathfrak{A} := \mathfrak{A}_{\mathbb{Z}^n}$ (see [5]). The Hamiltonian they choose is given by

$$\mathcal{H}_\Lambda = \sum_{X\subseteq\Lambda} \Phi(X),$$

where Φ is a so-called *interaction*: it's a function from the non-empty finite subsets of $\mathbb{Z}^n$ to self-adjoint operators on them, such that $\Phi(X + i) = \Phi(X)$ for any $i \in \mathbb{Z}^n$ (i.e., it's translational invariant).

The pressure for a finite region Λ in the quantum lattice system is given by

$$P_\Lambda(\Phi) := |\Lambda|^{-1} \ln \operatorname{tr} e^{-H_\Lambda}.$$

One can show that the limit in the sense of van Hove of P_Λ does exist in the quantum setting as well ([5]).

1.3.2 The inductive limit in more detail

For two finite subsets $X \subseteq Y \subset \mathbb{Z}^n$, the inclusion $\imath : \mathfrak{A}_X \to \mathfrak{A}_Y$ that sends A to $A \otimes 1$ is injective; therefore, whenever $X \subseteq Y$, we can view $\mathfrak{A}_X$ as a subalgebra of $\mathfrak{A}_Y$. Hence we can take a union of all such subalgebras coming from finite subsets of $\Lambda \subseteq \mathbb{Z}^n$, and then, to be safe and ensure it's a Banach space, take the closure. So, one can identify (see a proposition below for a rigorous proof)

$$\mathfrak{A}_\Lambda = \varinjlim \mathfrak{A}_X = \mathrm{cl}\left(\bigcup_{X \subset \Lambda,\ |X| < \infty} \mathfrak{A}_X \right) /_{u \sim u \otimes 1}$$

From this point of view, it's easy to understand what the norm is. For A from the dense subspace (the union itself), we just set $\|A\|_\Lambda := \|A\|_X$ if $A \in \mathfrak{A}_X$. The norm extends to the closure by the very process of completeness: for $A \in \mathfrak{A}_\Lambda$, we choose a sequence $A_n \in \mathfrak{A}_{X_n}$ and then set $\|A\|_\Lambda := \lim_{n\to\infty} \|A_n\|$.

From Appendix on C^* algebras, we see that Gelfand-Naymark theorem ensures there is a Hilbert space H such that $\mathfrak{A}_\Lambda \cong \mathrm{End}(H)$.

Further Search 1. I can elaborate on the construction of this Hilbert space. It's more or less constructive and relies on finding pure states. In particular, it would be interesting to see how this H is related to the infinite tensor product $\otimes_{i\in\Lambda} H_i$: what exactly goes wrong?

Proposition 1. In the above set-up, we indeed have $\varinjlim \mathfrak{A}_X = \mathrm{cl}\left(\bigcup_{X\subset\Lambda,\ |X|<\infty} \mathfrak{A}_X\right) /_{u\sim u\otimes 1}$ (isometrically and preserving the $*$-structure).

Proof. Denote $\mathfrak{A}' := \left(\bigcup_{X\subset\Lambda,\ |X|<\infty} \mathfrak{A}_X\right) /_{u\sim u\otimes 1}$. So, we choose morphisms in the category of unital C^*-algebras as bounded unital $*$-homorphisms with norm less then or equal to one[1]. To prove the statement, all we need to show is that for a unital C^*-algebra A and a bunch of morphisms $\alpha_X : \mathfrak{A}_X \to \mathfrak{A}_Y$ where $X \subseteq Y$ and such that $\alpha_X(u) = \alpha_Y(u \otimes 1)$ (but remember that u is identified with $u \otimes 1$ in the union), there's a unique morphism $\alpha : \mathrm{cl}\,\mathfrak{A}' \to A$. In the language of diagrams, this is saying that

$$\begin{tikzcd}
 & A & \\
 & \mathrm{cl}\,\mathfrak{A}' \arrow[u, dashed, "\exists!\alpha"] & \\
\mathfrak{A}_X \arrow[uur, "\alpha_X"] \arrow[ur, "\imath_X"] \arrow[rr, "u\mapsto u\otimes 1"] & & \mathfrak{A}_Y \arrow[uul, "\alpha_Y"'] \arrow[ul, "\imath_Y"']
\end{tikzcd}$$

[1]Otherwise I don't think there's a way to prove that the map induced on the diagram of the injective limit is a bounded operator

Once α is defined on $\mathfrak{A}'$ with all the mentioned properties, it automatically extends to the closure. So, for $u \in \mathfrak{A}_X$ we set $\alpha(u) := \alpha_X(u)$. This is well defined, for u is identified with $u \otimes 1$ in the union. We get automatically that α is a unital $*$-homorphism since all α_X's are. It's norm is bounded by 1, for $\|\alpha(u)\| \leq \|\alpha_X\| \|u\| \leq \|u\|$. Thus α is a morphism in the corresponding category. $\square$

1.4 Relation between classical and quantum lattices

I follow [5] with some minor modifications more appealing to my taste. Let Ω_0 be a finite set of microstates at one site, and let H_0 be a Hilbert space of dimension equal to $|\Omega|$ (which is assigned to one site as well). Let $C(\Omega)$ be the space of observables on Ω. Choose an orthonormal basis e_μ of H_0 labeled my microstates $\mu \in \Omega_0$. Then we have an injection $\imath : C(\Omega_0) \to \mathrm{End}(H_0)$ given by

$$[\imath(f)](e_\mu) := e_{f(\mu)}.$$

In other words, the classical observables are embedded into the quantum observables as diagonal matrices.

1.5 Continuous spins: general principles

I follow closely Section 6.10 of [3]. In case the space of states Ω_0 at a single site is non-compact, the existence of Gibbs measures is no longer guaranteed. For Ω_0 a topological space, one defines the following ingredients. Let $\mathcal{B}_0$ be the Borel σ-algebra on Ω_0. For a finite lattice $\Lambda \subset \mathbb{Z}^n$, we supply the space of states with the σ-algebra $\mathcal{B}_\Lambda := \bigotimes_{i\in\Lambda} \mathcal{B}_0$. The natural projections $\pi_\Lambda : \Omega \to \Omega_\Lambda$ allow us to define a σ-algebra on Ω with base in Λ:

$$\sigma_\Lambda := \pi_\Lambda^{-1}(\mathcal{B}_\Lambda).$$

If $S \subseteq \mathbb{Z}^n$ is a possibly infinite lattice, then we supply it with the σ-algebra

$$\sigma_S := \sigma\Big(\bigcup_{\Lambda \subset S,\ \Lambda \text{ finite}} \sigma_\Lambda \Big)$$

(by the last equality I mean the smallest σ-algebra generated by the union).

2 Classical lattice models

2.1 Ising model

2.1.1 A general description of the IRF version

There are two versions of the Ising model: the IRF (interaction-round-a-face) model and the vertex model. In the first one, the energy is assigned to vertices; in the second one, the energy is assigned to the bonds between the sites.

Let $\Lambda \subseteq \mathbb{Z}^n$ be a subset of the integer lattice of dimension n. We associate with the lattice the space of microstates $\Omega_\Lambda := \{-1, +1\}^\Lambda$. Therefore, to each node $i \in \Lambda$ there corresponds a *spin* $\omega_i = \pm 1$. For a finite Λ, the hamiltonian of the model is given by

$$\mathcal{H} = \sum_{i,j \in \Lambda,\ i \sim j} \omega_i \omega_j - h \sum_{i \in \Lambda} \omega_i,$$

where $h \in \mathbb{R}$ is some real number that corresponds to the external magnetic field, and $i \sim j$ means the nodes i and j are neighbors on the lattice. We also supply the model with the Gibbs measure defined previously.

2.1.2 Transfer matrices in IRF model (not finished)

To describe the transfer matrices, I restrict myself to a finite cubic lattice $\Lambda \subset \mathbb{Z}^2$ with periodic boundary conditions. Then we can assign energy to each face of the lattice:

$$\epsilon(\text{face}, \omega) := \sum_{i,j \in \text{face},\ i \sim j} \omega_i \omega_j - h \sum_{i \in \text{face}} \omega_i.$$

So the Hamiltonian breaks up into the sum of energies over all faces in Λ:

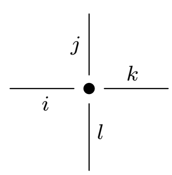
$$H(\omega) = \sum_{F \in \{\text{faces of } \Lambda\}} \epsilon(F, \omega).$$

A *Boltzmann weight* is the quantity $R(F, \omega) := \exp(-\beta \epsilon(F, \omega))$ assigned to a face F. The partition function can be rewritten as

$$Z = \sum_{\omega \in \Omega} \prod_{F \in \text{faces}} R(F, \omega).$$

2.1.3 The vertex model and its transfer matrix

I follow closely [2]. Let Λ be an $n \times m$ cubic lattice in $\mathbb{Z}^2$ with periodic boundary conditions. The states are assigned to the bonds between vertices rather than to the vertices themselves in this model. Let $\Omega_0 = \{1, \ldots, n\}$ be the set of possible states of a single bond. For a picture of kind



let ε_{ij}^{kl} denote the energy assigned to the site in this setting. We assume that it doesn't depend on the position of the site but only on the states of the bonds around the sit. The Hamiltonian $\mathcal{H}$ of this model for a particular choice of the state of the lattice is then the sum of ε_{ij}^{kl} over all vertices. The partition function is given by $Z = \sum_{\omega \in \Omega} \exp(-\beta H(\omega))$. A *Boltzmann weight* is the quantity

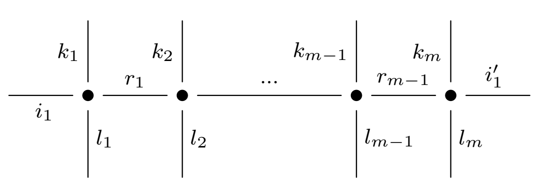
$$R_{ij}^{kl} := \exp(-\beta \varepsilon_{ij}^{kl}).$$

Proposition 2. Let V be an m-dimensional vector space. There exists an endomorphism $T \in \mathrm{End}(V \otimes V^m)$, which is called a *transfer matrix*, such that the partition function of the model is given by

$$Z = \mathrm{tr}_{V^{\otimes m}} (\mathrm{tr}_V\, T)^n$$

where the trace is the usual one (the sum of diagonal elements).

Proof. Consider a row in the cubic lattice, for a moment assuming that the boundary conditions on the ends (the states i_1 and i_1') may not be the same



Let us fix the end states i_1, i'_1, $k_1, \dots, k_m$ and $l_1, \dots, l_m$. The contribution to Z when only r_i's are running over Ω_0 is given by

$$T^{i'_1 l_1 \cdots l_m}_{i_1 k_1 \cdots k_m} := \sum_{r_1, \dots, r_{m-1}} R^{r_1 l_1}_{i_1 k_1} \cdots R^{i'_1 l_m}_{r_{m-1} k_m}.$$

Let V be an m-dimensional vector space spanned by some $e_1, \dots, e_m$. Define an endomorphism $T \in \mathrm{End}(V \otimes V^{\otimes m})$ by setting on the basis elements

$$T(e_{i_1} \otimes e_{k_1} \otimes \cdots \otimes e_{k_m}) = \sum_{i'_1, l_1, \dots, l_m} T^{i'_1 l_1 \cdots l_m}_{i_1 k_1 \cdots k_m} e_{i'_1} \otimes e_{l_1} \otimes \cdots \otimes e_{l_m}.$$

If wee unfreeze the endpoints with states i_1 and i'_1 and let them run over Ω_0, then we see that the contribution to Z of the whole row (with still fixed states on the vertical bonds and now $i_1 = i'_1$) is given by $\mathrm{tr}_V(T)^{l_1 \dots l_m}_{k_1 \dots k_m}$. Now, if the row was the first one and we consider the next one to it, and let $l_1, \dots, l_m$ run over Ω_0, then the contribution to Z is

$$\sum_{l_1, \dots, l_m} \mathrm{tr}_V(T)^{l_1 \dots l_m}_{k_1 \dots k_m} \, \mathrm{tr}_V(T)^{j_1 \dots j_m}_{l_1 \dots l_m} = [(\mathrm{tr}_V(T))^2]^{j_1 \dots j_m}_{k_1 \dots k_m}$$

(the last equality was not obvious to me due to a mess with indices, but it can be checked easily). Continuing in this fashion, the contribution to Z with fixed states of the vertical bonds on the ends is given by $[(\mathrm{tr}_V(T))^n]^{l_1 \dots l_m}_{k_1 \dots k_m}$. Now, applying the periodic condition $k_j = l_j$ and summing over all possible states of the ends, we finally find that $Z = \mathrm{tr}_{V^{\otimes m}}[\mathrm{tr}_V(T)]^n$. $\square$

I think I can say that a transfer matrix is just a batch of all possible microstates of a row ingeniously packed into a linear endomorphism.

2.2 Gaussian free field model

I follow closely Chapter 8 from [3]. In Gaussian free field model, the space of states at a single site is chosen to be $\Omega_0 := \mathbb{R}$. Accordingly, the space of states on a region $\Lambda \subseteq \mathbb{Z}^n$ is given by $\Omega_\Lambda := \mathbb{R}^\Lambda$. The Hamiltonian of the model is on the lattice Λ is chosen to be

$$\mathcal{H} := \frac{\beta}{4n} \sum_{i \sim j, \ \{i,j\} \cap \Lambda \neq \emptyset} (\omega_i - \omega_j)^2 + \frac{m^2}{2} \sum_{i \in \mathbb{Z}^n} \omega_i^2,$$

where β is the inverse temperature, $\omega_i \in \Omega_0$ is the assigned spin at site $i \in \mathbb{Z}^n$, and m is the mass.

A couple of comments on the choice of the Hamiltonian:

1) The factor $(\omega_i - \omega_j)^2$ tells us that the interaction favors the agreement of neighboring spins;

2) Since the space of states at single site is non-compact, we penalize large values of spin by adding the factor $m^2/2 \cdot \omega_i^2$ for each one;

3) Notice the condition under the first summation. It tells us that we also take into the account the boundary of Λ (there might be different boundary conditions though).

Fix a finite lattice $\Lambda \subset \mathbb{Z}^n$ and a state $\eta \in \Omega$ (it serves as a boundary condition for Λ). For a state $\omega_\Lambda \in \Omega_\Lambda$, by $\mathcal{H}(\omega_\Lambda)$ we mean that we plug into the Hamiltonian the state that equals ω_Λ on Λ and η on the complement of Λ.

In Subsection 1.5 we specified a way of choosing σ-algebras on the spaces of states. Let $\sigma_{\mathbb{Z}^n}$ be such σ-algebra on the whole $\mathbb{Z}^n$. For $A \in \sigma_{\mathbb{Z}^n}$, the Gibbs measure in this model is defined as

$$\mu(A) := \int_A \frac{e^{-\mathcal{H}(\omega_\Lambda)}}{Z} \prod_{i\in\Lambda} d\omega_i,$$

where $d\omega_i$ is the Lebesgue measure on $\mathbb{R}$ assigned to the site $i \in \mathbb{Z}^n$ and Z is the obviously chosen partition function.

There's a way to define Gibbs measures for infinite Λ as well (explained in [3], I postpone its description here for a moment). The case of massless GFF is drastically different from the case of massive GFF. For instance, Theorem 8.19 in [3] says that there are no infinite-volume Gibbs measures in $n = 1$ and $n = 2$ cases. Nevertheless, Theorem 8.21 in the same reference tells us that there are infintely many infinite-volume Gibbs measures when $n \geq 3$. In the massive case, the GFF model has infinitely many infinite-volume Gibbs measures for any n (see Theorem 8.28 in [3]).

2.3 O(N)-symmetric model

I follow Chapter 9 from [3]. In O(N)-model, we take $\Omega_0 := S^{N-1}$, so the spins might have an arbitrary direction. For a finite lattice $\Lambda \subseteq \mathbb{Z}^n$, the Hamiltonian (in the absence of a magnetic field) is usually written as

$$\mathcal{H} = -\beta \sum_{i\sim j,\ \ \{i,j\}\cap\Lambda\neq\emptyset} \langle \omega_i, \omega_j \rangle ,$$

where $\omega_i \in \Omega_0$ is a spin at site i, and the brackets denote the standard inner product in $\mathbb{R}^N$. For different N's, we obtain some familiar models: for $N = 1$ we have the Ising model; for $N = 2$ we get the XY-model; and for $N = 3$ we obtain the Heisenberg model.

The definition of finite-volume Gibbs measures is similar to the case of GFF model. At each site i, we have Lebesgue measure $d\omega_i$ on S^{N-1}. We fix a boundary condition, which is the choice of a state $\eta \in \Omega$, and then for measurable sets A we set

$$\mu(A) := \int_A \frac{e^{-\mathcal{H}(\omega_\Lambda)}}{Z} \prod_{i\in\Lambda} d\omega_i,$$

where Z is the obvious partition function and $\omega_\Lambda \in \Omega_\Lambda$; by $\mathcal{H}(\omega_\Lambda)$ I mean that we plug in a state equal to ω_Λ on Λ and η outside of Λ.

One might be interested in the following questions with regards to O(N)-models:

1) Is there an orientational long-range order? In my understanding, the mathematical formalism of this question is whether the correlations $\mathbb{E}_\mu \langle \omega_i, \omega_j \rangle$ converge to zero as $\|i - j\| \to \infty$;

2) Is there a spontaneous magnetization? The formalism in my understanding is: for any infinite-volume Gibbs measure μ, is it true that $\lim_{n\to\infty} \left\langle \|m_{B(n)}\| \right\rangle_\mu \neq 0$? Here $B(n)$ is a cube of size n and $m_{B(n)} := \frac{1}{|B(n)|} \sum_{i\in B(n)} \omega_i$ is the *magnetization density.*

The answers to both questions are negative for $N \geq 2$ and $n = 1, 2$. This is due to the following theorem, which can be also stated for a more general Hamiltonian:

Theorem 1. *(Mermin-Wagner) For $N \geq 2$ and $n = 1, 2$, all infinite-volume Gibbs measures are invariant under the action of the rotation group.*

Maybe, I will write why the answers are negative a bit later.

3 Questions

3.1 For a discussion

3.2 Just for myself

Question 1. I do not see any issue with regards to defining the Gelfand spectrum for non-commutative Banach algebras. Why is it defined only for commutative ones? Only because it appears in the set-up of G-N theorem?

4 Appendix

4.1 C^*-algebras

I think a few results from this theory are worth mentioning in the notes. Recall the definition of a C^*-algebra itself:

Definition 1. A (unital) C^*-algebra A is a Banach space over $\mathbb{C}$ that is also a (unital) algebra such that the multiplication is a bounded bilinear map of norm 1. It's also supplied with an involution $* : A \to A$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$.

A straightforward[2] consequence from the axioms is that $\|a\| = \|a^*\|$ and in the unital case $\|1\| = 1$.

The results of interest, I think, are the following:

Theorem 2. *(Gelfand-Naymark, 1st) We have the following:*

(i) Any possibly non-unital C^*-algebra is isomorphic to the space $C_0(\Omega)$ of continuous functions vanishing[3] at ∞ on some locally compact Hausdorff topological space Ω;

(ii) Any unital C^*-algebra is isomorphic to $C(\Omega)$ for Ω a compact Hausdorff space.

There's an accurate description of both the isomorphism and the space Ω: the space Ω is the Gelfand spectrum of A, and the isomorphism is the Gelfand representation.

Here are the definitions. Let A be a Banach algebra[4]. Its *Gelfand spectrum* is the space $\Omega \subset A^*$ of all characters[5] of A that is endowed with ω^*-topology induced from A^*. The isomorphism then is the Gelfand representation $\Gamma_A : A \to C_0(\Omega)$ that sends an element $a \in A$ to a continuous map (vanishing at ∞) $\hat{a}$ such that[6]

---

[2]From the boundedness of the multiplication we obtain $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$, hence $\|a\| \leq \|a\|^*$. Switching to $a \mapsto a^*$, get $\|a\| = \|a\|^*$. For the unit, $\|1\| = \|1\|^2$ since $1^* = 1$.

[3]Precisely, a function $\varphi : \Omega \to \mathbb{C}$ is vanishing at infinity if for any $\varepsilon > 0$ there exists a compact subset $\Delta \subseteq \Omega$ such that $|\phi(t)| < \varepsilon$ when $t \notin \Delta$.

[4]I.e. a Banach space that's also an algebra such that the multiplication is a bounded bilinear map with norm 1.

[5]A character is a non-zero linear functional that preserves the multiplication. It turns out that any character $\chi : A \to \mathbb{C}$ has norm $\|\chi\| \leq 1$, so they're automatically continuous, for if $\chi(a) = 1$ for some $a \in A$ with $\|a\| < 1$, then let $b := \sum_{k=1}^{\infty} a^k$. It follows that $a + ab = b$, hence $1 + \chi(b) = \chi(b)$, which is absurd.

[6]The continuity of $\hat{a}$ follows right from the definition of ω^*-topology. It indeed vanishes at ∞, for $\hat{a}$ is a continuous map on the compactification $\Omega' := \Omega \cup \{0\}$ such that $\hat{a}(0) = 0$ (look at the formula defining $\hat{a}$), hence for any $\varepsilon > 0$ there's $U_\varepsilon \ni 0$ such that $|\hat{a}(\chi)| < \varepsilon$ when $t \in U_\varepsilon$. But U_ε^c is compact, as a closed subset of the compact space Ω', so $\hat{a}$ vanishes at ∞.

Proposition 3. The Gelfand spectrum Ω of a Banach algebra A is a locally compact Hausdorff space; if A is in addition unital, then Ω is Hausdorff and compact.

Proof. Obviously, Ω is Hausdorff since ω^*-topology is Hausdorff. Consider $\Omega' := \Omega \cup \{0\}$. If $\chi_\alpha \to f$ in ω^*-topology, then obviously f preserves multiplication (but it might become zero), so Ω' is a closed subset of the unit ball in A^*. Now it's the consequence of Banach-Alaoglu theorem that Ω' is compact. Being a subset of a compact Hausdorff space, we see that Ω is locally compact and Hausdorff. In case A is unital, 0 is an isolated point of Ω', for if there was a net $\chi_\alpha \in \Omega$ such that $\chi_\alpha \to 0$, then $1 = \chi_\alpha(1) \to 0$, which is a contradiction, and thus Ω is compact. □

In general, for possibly non-commutative C^*-algebras we have

Theorem 3. *(Gelfand-Naymark, 2nd) Any C^*-algebra is $*$-isometrically isomorphic to* $\mathrm{End}(H)$ *for some Hilbert space H.*

Further Search 2. Might understand the construction of this operator algebra in detail. I've seen that the proof is constructive. That's the matter of finding pure states and defining the corresponding irreducible representations. The direct sum of those will yield the result.

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