Lattice $\mathbb{Z}_n$-QED 1+1

NSF internship under mentorship of Gonzalo Alvarez

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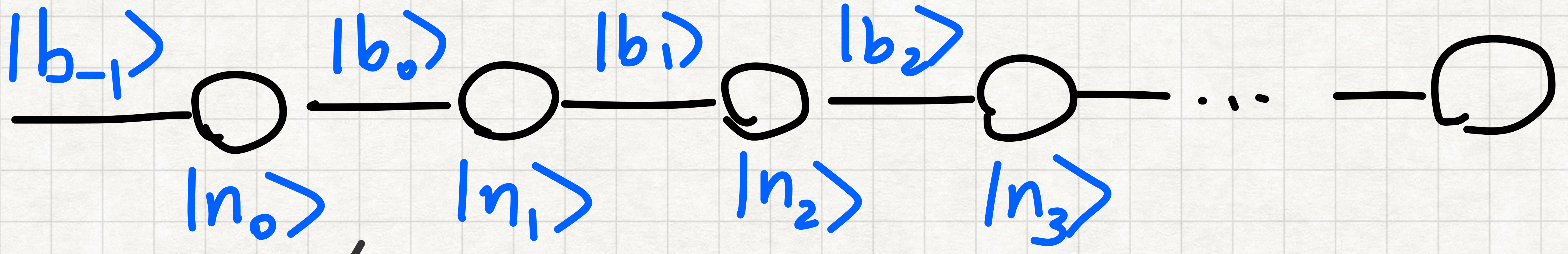
Have a chain of sites:



N sites

Fermions reside on sites;

Bosons reside on links; their state space is n-dim'l

(n from $\mathbb{Z}_n$)

$n = 2s+1$

Operators:

ψ_ℓ – destroys a fermion at site ℓ

$\psi_\ell^\dagger$ – creates — // — ℓ

$U_{\ell\ell+1}$ – increases the state of the boson at link ℓ

For $|\psi\rangle = \cdots \underset{\ell}{\circ}\overset{|k\rangle}{\text{———}}\underset{\ell+1}{\circ}\cdots$, $E_{\ell\ell+1}|\psi\rangle = (k-s)|\psi\rangle$

The Hamiltonian:

$$H = -\frac{n}{2\pi}\sum_{\ell}\left(\psi_\ell^{+} U_{\ell\ell+1}\psi_{\ell+1} + H.c.\right) + 2m\sqrt{\frac{n}{2\pi}}\sum_{\ell}(-1)^{\ell}\psi_\ell^{+}\psi_\ell + \sum_{\ell} E_{\ell\ell+1}^{2}$$

Hops a fermion at $\ell+1$ to the left; increases the state of the boson

Hops to the right; decreases bosonic state

Staggered fermions energy

energy from links

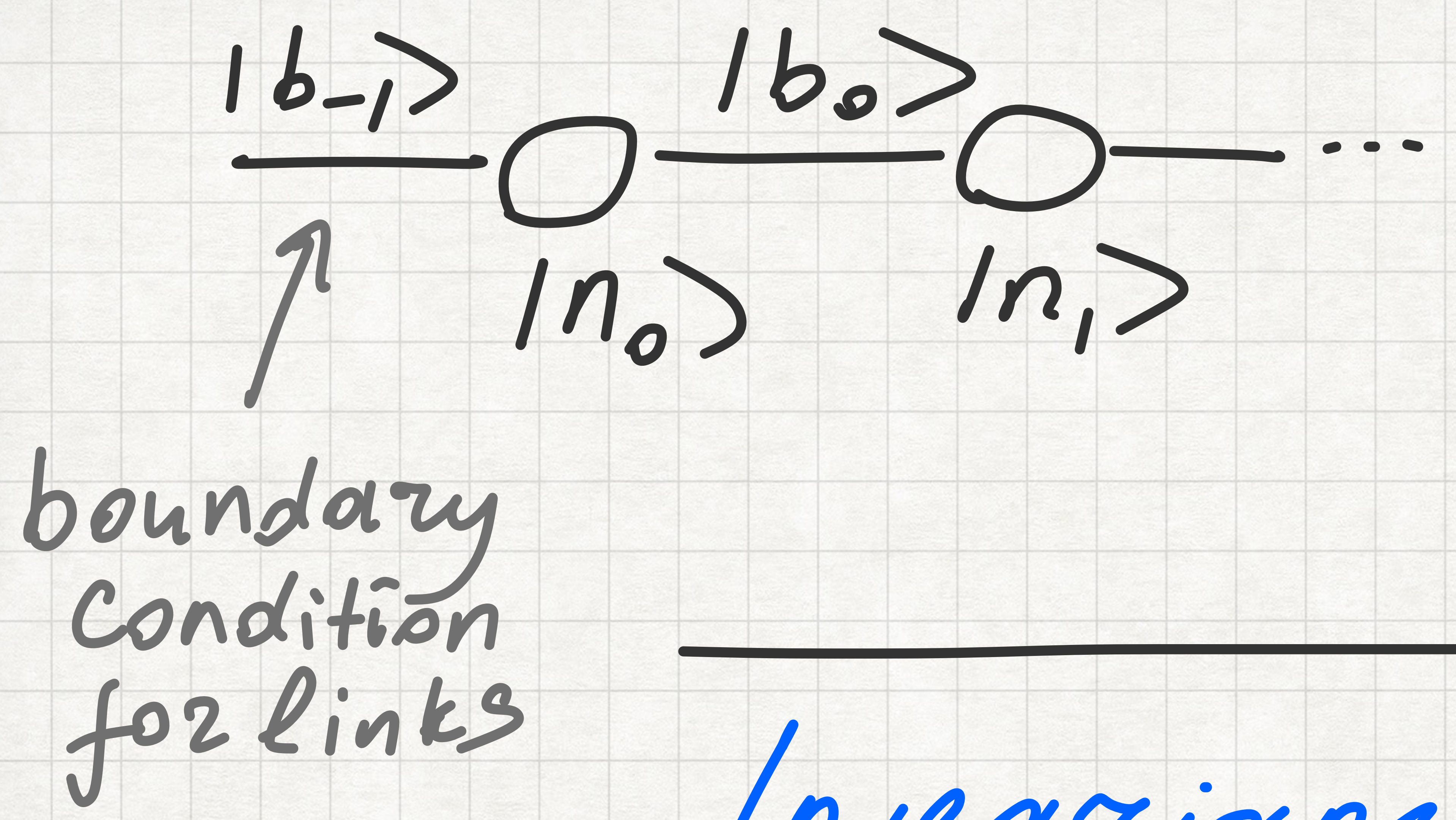
The Gauss Law:

$$G_\ell := \psi_\ell^{+}\psi_\ell + \frac{1}{2}\left((-1)^{\ell} - 1\right) - \left(E_{\ell\ell+1} - E_{\ell-1\ell}\right)$$

The physical space:

$|\psi\rangle$ s.t. $G_\ell|\psi\rangle = 0$ for all ℓ

Its dim'n is $n\cdot 2^{N}$



Gauss law says:

$$b_\ell = b_{\ell-1} + n_\ell + \frac{1}{2}\left((-1)^\ell - 1\right)$$

Invariance properties of H:

1) H preserves subspaces with fixed $|b_{-1}\rangle$
2) H preserves the number of fermions

Corollary: basic kets in $|gs\rangle$ have the same num of fermions and b_{-1}

$\lambda_1 :=$ the lowest e-value of H.

Proposition $\lambda_1(m)$ is a concave C^1-function of m.

(uses standard results from perturbation theory; it's important that $H = H^\dagger$)

Proved estimates:

$$\lim_{m\to\infty} \frac{\lambda_1}{m} = -2\left[\frac{N}{2}\right]\sqrt{\frac{n}{2\pi}}$$

$$\lambda_1 < -2m\left[\frac{N}{2}\right]\sqrt{\frac{n}{2\pi}}$$

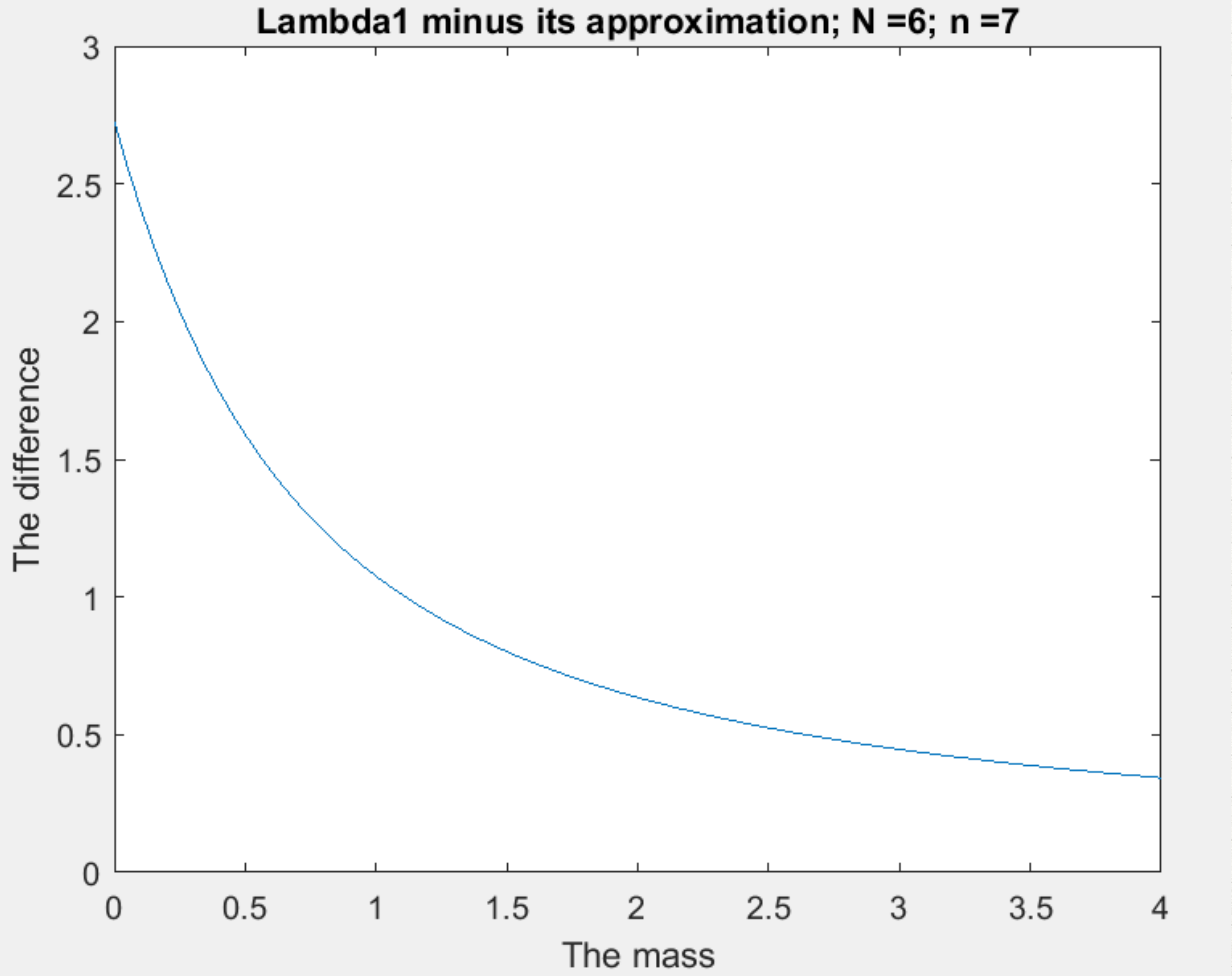
So, $\lambda_1 \approx -Nm\sqrt{\frac{n}{2\pi}}$, $m>0$

$$\lim_{m\to-\infty} \frac{\lambda_1}{m} = 2\left[\frac{N+1}{2}\right]\sqrt{\frac{n}{2\pi}}$$

$$\lambda_1 < 2m\left[\frac{N+1}{2}\right]\sqrt{\frac{n}{2\pi}}$$

So, $\lambda_1 \approx Nm\sqrt{\frac{n}{2\pi}}$, $m<0$

How good are the estimates?



Staggered configurations:

$$|st_2(b_{-1})\rangle = \overset{b_{-1}}{\text{——}} \bigcirc \overset{b_{-1}}{\text{——}} \otimes \overset{b_{-1}}{\text{——}} \bigcirc \overset{b_{-1}}{\text{——}} \otimes \text{——} \cdots \overset{b_{-1}}{\text{——}} \textcircled{?}$$

$$|st_1(b_{-1})\rangle = \overset{b_{-1}}{\text{——}} \otimes \overset{b_{-1}+1}{\text{——}} \bigcirc \overset{b_{-1}}{\text{——}} \otimes \overset{b_{-1}+1}{\text{——}} \bigcirc \text{——} \cdots \overset{?}{\text{——}} \textcircled{?}$$

Will be important later...

Proved:

$$\lim_{m\to+\infty} |gs\rangle = |st_2(s)\rangle \qquad (n = 2s+1)$$

$$\lim_{m\to-\infty} |gs\rangle = |st_1(s)\rangle$$

(a technical subtlety: assume $\arg\langle gs|st_i(s)\rangle = 0$)

Consequence:

For $m \gg 0$, $\left[\frac{N}{2}\right]$ num of fermions in $|gs\rangle$;

For $m \ll 0$, $\left[\frac{N+1}{2}\right]$ —//—

A ground state favors $b_{-1} = s$.

Non-degeneracy of $|gs\rangle$

Proposition For $m \ll 0$ and $m \gg 0$, the ground state is non-degenerate

Some tools for other m?

Perron-Frobenius thm: if it were that the entries of H are ≤ 0, the $|gs\rangle$ would be non-degenerate.

Gershgorin thm: E-values of a matrix reside in certain discs on a $\mathbb{C}$-plane; if k-discs are disjoint from $n-k$, then exactly k e-values are there.

(continuation of the previous slide)

With Gershgorin's theorem, can derive a sufficient condition for non-degeneracy:

$$\frac{n}{2\hbar}(2N-3) < 4m\sqrt{\frac{n}{2\hbar}}+1$$

Roughly, $\dfrac{N\sqrt{n}}{m} < \sqrt{8\hbar}$ — useless

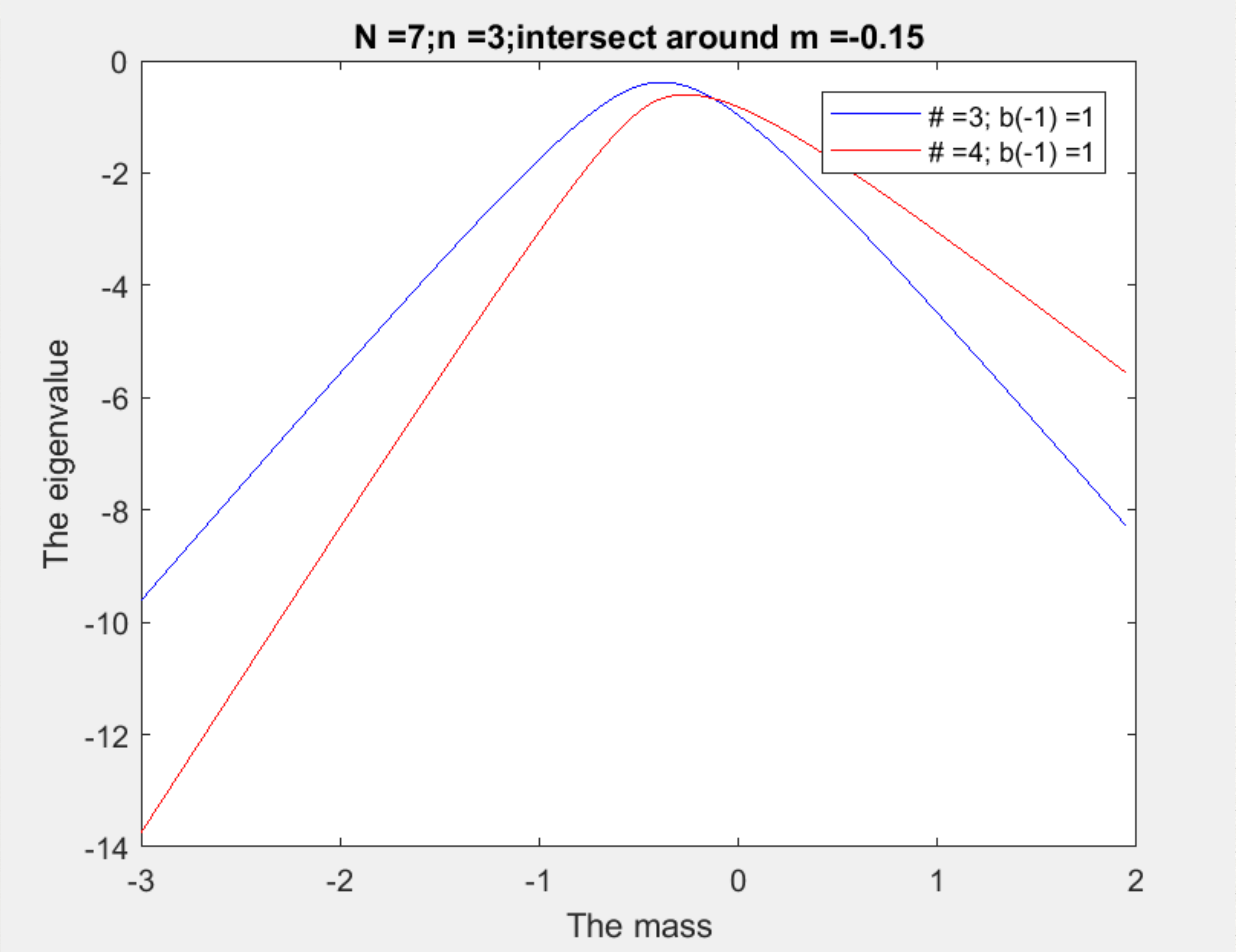
Actually, numerics show $|gs\rangle$ is degenerate for N odd

The reason: for $m \gg 0$, $|gs\rangle$ has $\left[\frac{N}{2}\right]$ fermions

for $m \ll 0$, $|gs\rangle$ has $\left[\frac{N+1}{2}\right]$ fermions

At some m, there has to be a transition.

(numerics – next slide...)



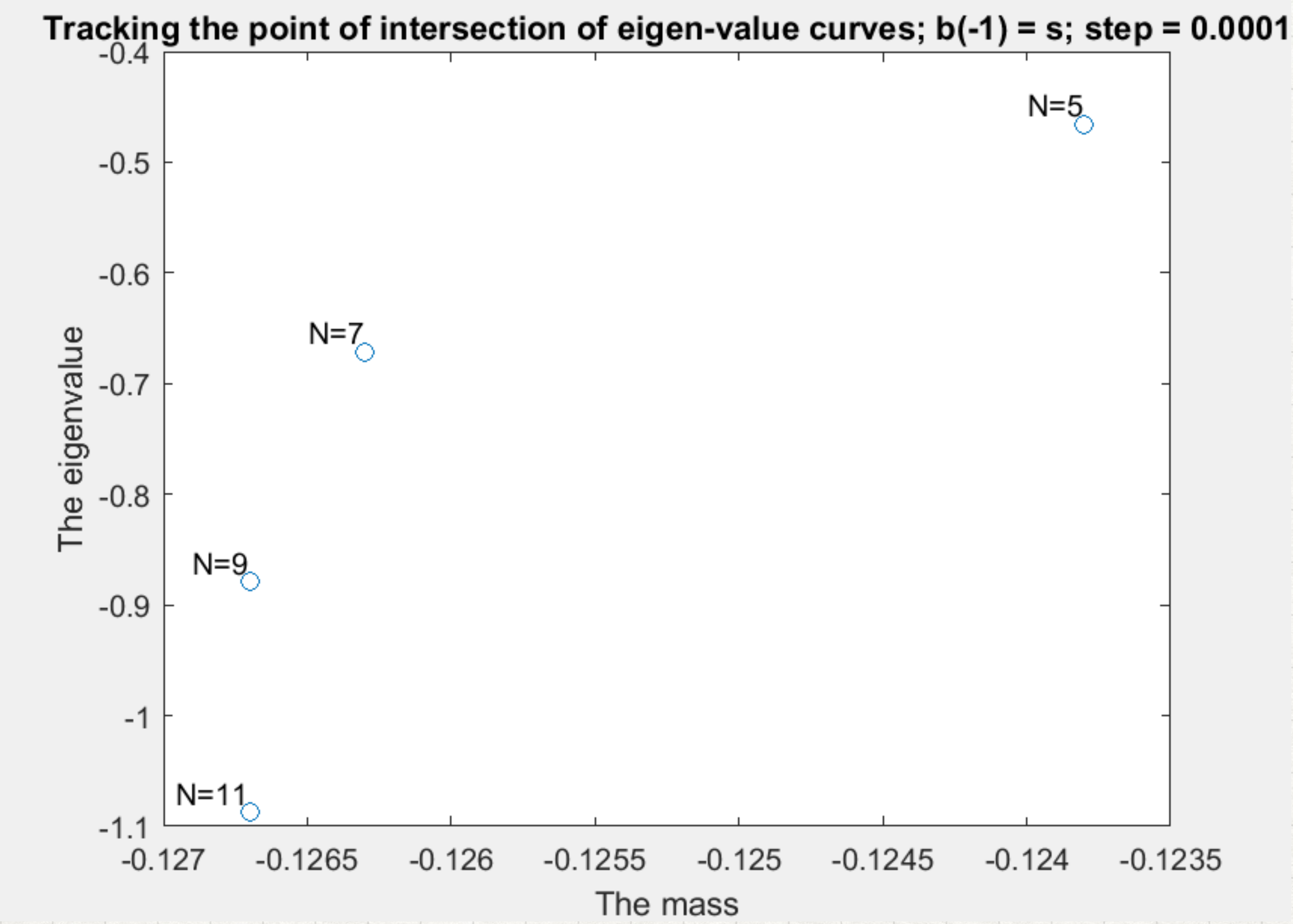
The lowest e-values from two subspaces are compared: #fermions=3; #fermions=4

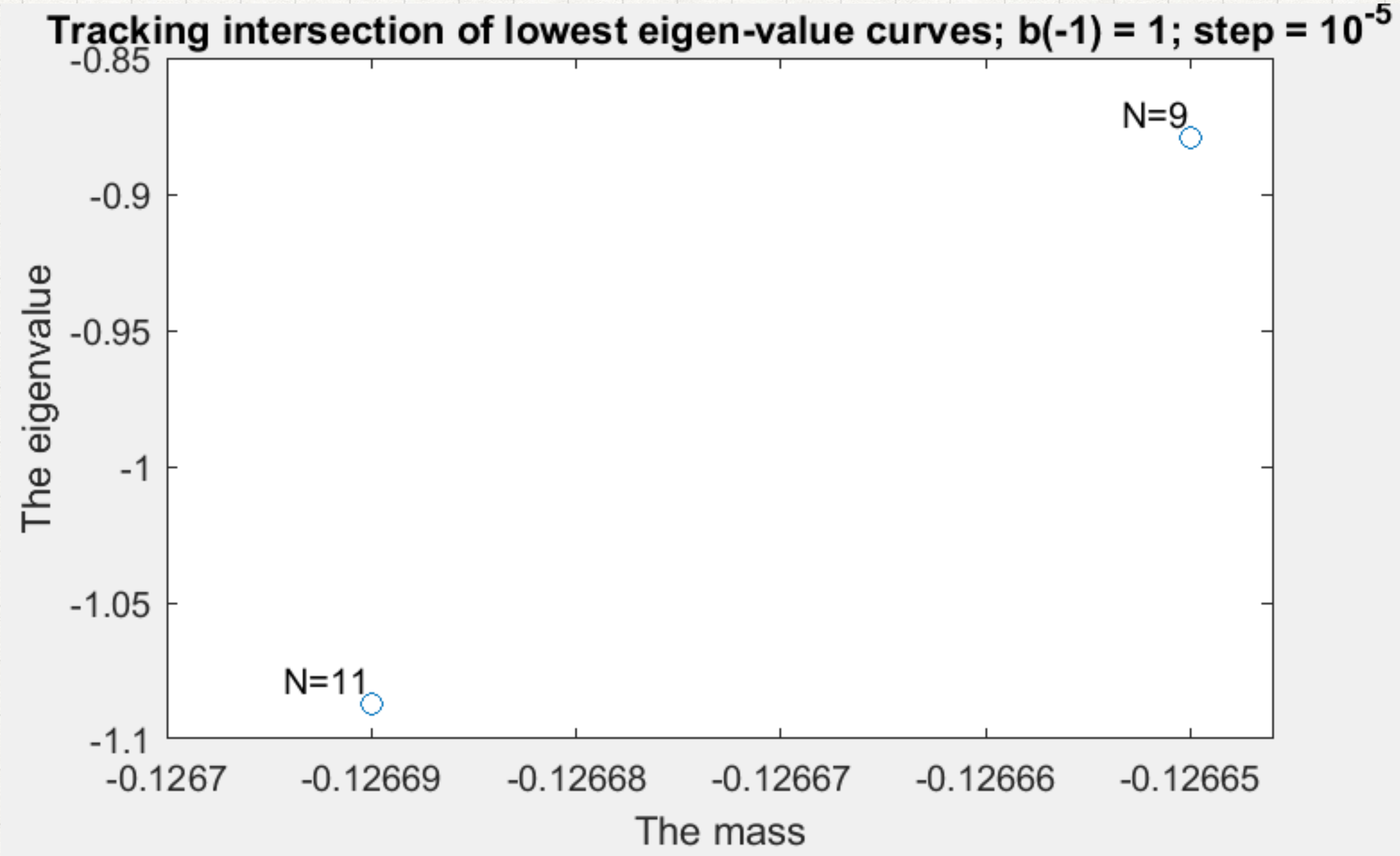
A natural conjecture:

For every N odd, let $\hat{m}(N)$ be the "mass of intersection" of the two ℓ-values curves. Let $\hat{m} := \lim_{N \to \infty} \hat{m}(N)$.

Then there is a phase transition at $m = \hat{m}$.

Numerics for the conjecture:





N=13 is already not manageable: H on the physical space has 603, 979, 776 entries ...

Wilson loop

$$\Sigma := \frac{1}{N} \sum_{e} \langle E_{e e+1} \rangle .$$

Proved: $\lim\limits_{m \to \infty} \Sigma = 0$; $\lim\limits_{m \to -\infty} \Sigma = \frac{1}{N}\left[\frac{N}{2}\right] \approx 0.5^{*}$

* not in agreement with Ercolessi's paper: her numerics predict ≈ 0.7, but our rigorous treatment — ≈ 0.5.

The main way to go:

$$\Sigma_\infty := \lim_{N\to\infty} \Sigma.$$

Find a mass m s.t. left & right derivatives of Σ_∞ exist at m but are not equal.
Or at least prove it exists.