Contents

1 Preliminaries

1.1 The main principles of classical lattice models

Let Ω be a finite set (the set of *microstates*), let $\mathcal{H} : \Omega \to \mathbb{R}$ be a *hamiltonian*, a specifically chosen random variable. Let $\mathcal{M}(\Omega)$ be the space of probability measures on Ω. In this theory, the expectation of a random variable $f : \Omega \to \mathbb{R}$ with respect to a measure μ (if not implicitly understood; it's also called the *thermal average*) is denoted as

$$\langle f \rangle_\mu := \mathbb{E}_\mu f = \int_\Omega f d\mu.$$

One of the first questions in statistical mechanics is devoted to the choice of the right measure μ. The choice is governed by Shannon's entropy $S : \mathcal{M}(\Omega) \to \mathbb{R}$, defined as $S(\mu) := -\int_\Omega x \log x d\mu(x)$ (there's a way to understand why S has this form; see [5]). *The maximum entropy principle says*: for a given model of statistical mechanics, choose μ that maximizes S. For example, if there's no other information available about the system, then the measure that maximizes S is the uniform distribution. There are other two typical situations:

(a) If we know that $\langle \mathcal{H} \rangle = U$ for some fixed U (the *internal energy* of the system), then the measure that maximizes S is

$$\mu(\omega) = \frac{e^{-\beta H(\omega)}}{Z}, \quad Z := \sum_{\omega \in \Omega} e^{-\beta \mathcal{H}(\omega)}$$

which is the *Gibbs measure*. As the condition says, we either know the system is isolated, or we know the system exchanges its energy with the external environment but on average the internal energy is preserved. The result can be obtained using Lagrange multipliers; the parameter β, called the *inverse temperature*, is uniquely determined by U (and vice versa: U is uniquely determined by β). In the theory, $\beta = (kT)^{-1}$, where k is the Boltzmann constant and T is the temperature of the system.

(b) If we know additionally that the system exchanges its particles $\mathcal{N}$ with the external environment, and that the expected value of the particles if $\langle \mathcal{N} \rangle = N$, then we obtain a *grand canonical Gibbs distribution* via a similar procedure. It's given by

$$\mu(\omega) := \frac{e^{-\beta(\mathcal{H}(\omega) - \mu N)}}{Z}, \quad Z := \sum_N e^{\beta \mu N} \sum_{\omega \in \Omega} e^{-\beta \mathcal{H}(\omega)}.$$

The parameter μ is identified with the so-called *chemical potential*.

1.2 The thermodynamic limit

1.3 Quantum lattices

1.3.1 A general set-up

I will follow closely the treatment in [7]. Again, we have a lattice $X \subset \mathbb{Z}^n$, but to each site $i \in X$ we attach a copy of a finite-dimensional Hilbert space H_i. To a finite X we attach the tensor product $H_X := \otimes_{i \in X} H_i$.

For infinite lattices, the author of [7] suggests proceeding as follows. Let $\mathfrak{A}_X := \mathrm{End}(H_X)$, and for any two finite subsets $X \subseteq Y \subset \mathbb{Z}^n$, let $\imath : \mathfrak{A}_X \to \mathfrak{A}_Y$ be the inclusion that sends A to $A \otimes 1$ (where 1 is viewed as an endomorphism of $\mathfrak{A}_{Y \setminus X}$). For an infinite $\Lambda \subseteq \mathbb{Z}^n$, the family of all its finite subsets with the inclusions form a direct system. Let $\mathfrak{A}_\Lambda := \varinjlim \mathfrak{A}_X$ be the direct limit taken in the category of C^*-algebras over all finite subsets of Λ. In the literature, this algebra is known as an AF (approximately finite-dimensional) C^*-algebra. The first reference in this theory goes back to Bratteli [2]. See the next subsection for an elaboration on the inductive limit.

Further, for a finite $\Lambda \subset \mathbb{Z}^n$ and a Hamiltonian $\mathcal{H}_\Lambda$, the partition function is defined as

$$Z = \mathrm{tr}_\Lambda \, e^{-\beta \mathcal{H}_\Lambda}$$

and the expectation of an observable $A \in \mathfrak{A}_\Lambda$ is

$$\langle A \rangle_\Lambda := Z^{-1} \, \mathrm{tr}_\Lambda (A e^{-\mathcal{H}_\Lambda}).$$

The trace in these formulas is normalized: it's $1/d$ of the usual trace, where d is the dimension of the Hilbert space at one site. An interesting consequence of such normalization is that tr extends than to a norm-one linear functional on the whole $\mathfrak{A} := \mathfrak{A}_{\mathbb{Z}^n}$ (see [7]). The Hamiltonian they choose is given by

$$\mathcal{H}_\Lambda = \sum_{X \subseteq \Lambda} \Phi(X),$$

where Φ is a so-called *interaction*: it's a function from the non-empty finite subsets of $\mathbb{Z}^n$ to self-adjoint operators on them, such that $\Phi(X + i) = \Phi(X)$ for any $i \in \mathbb{Z}^n$ (i.e., it's translational invariant).

The pressure for a finite region Λ in the quantum lattice system is given by

$$P_\Lambda(\Phi) := |\Lambda|^{-1} \ln \mathrm{tr} \, e^{-H_\Lambda}.$$

One can show that the limit in the sense of van Hove of P_Λ does exist in the quantum setting as well ([7]).

1.3.2 The inductive limit in more detail

For two finite subsets $X \subseteq Y \subset \mathbb{Z}^n$, the inclusion $\imath : \mathfrak{A}_X \to \mathfrak{A}_Y$ that sends A to $A \otimes 1$ is injective; therefore, whenever $X \subseteq Y$, we can view $\mathfrak{A}_X$ as a subalgebra of $\mathfrak{A}_Y$. Hence we can take a union of all such subalgebras coming from finite subsets of $\Lambda \subseteq \mathbb{Z}^n$, and then, to be safe and ensure it's a Banach space, take the closure. So, one can identify (see a proposition below for a rigorous proof)

$$\mathfrak{A}_\Lambda = \varinjlim \mathfrak{A}_X = \mathrm{cl} \left(\bigcup_{X \subset \Lambda, \ |X| < \infty} \mathfrak{A}_X \right) /_{u \sim u \otimes 1}$$

From this point of view, it's easy to understand what the norm is. For A from the dense subspace (the union itself), we just set $\|A\|_\Lambda := \|A\|_X$ if $A \in \mathfrak{A}_X$. The norm extends to the closure by the very process of completeness: for $A \in \mathfrak{A}_\Lambda$, we choose a sequence $A_n \in \mathfrak{A}_{X_n}$ such that $A_n \to A$, and then set $\|A\|_\Lambda := \lim_{n\to\infty} \|A_n\|$.

From Appendix on C^*-algebras, we see that Gelfand-Naymark theorem ensures there is a Hilbert space H such that $\mathfrak{A}_\Lambda \cong \mathrm{End}(H)$.

Further Search 1. I can elaborate on the construction of this Hilbert space. It's more or less constructive and relies on finding pure states. In particular, it would be interesting to see how this H is related to the infinite tensor product $\otimes_{i\in\Lambda} H_i$: what exactly goes wrong?

Proposition 1. In the above set-up, we indeed have $\varinjlim \mathfrak{A}_X = \mathrm{cl}\left(\bigcup_{X\subset\Lambda,\ |X|<\infty} \mathfrak{A}_X\right)/_{u\sim u\otimes 1}$ (isometrically and preserving the $*$-structure).

Proof. Denote $\mathfrak{A}' := \left(\bigcup_{X\subset\Lambda,\ |X|<\infty} \mathfrak{A}_X\right)/_{u\sim u\otimes 1}$. So, we choose morphisms in the category of unital C^*-algebras as bounded unital $*$-homomorphisms with norm less then or equal to one[1]. To prove the statement, all we need to show is that for a unital C^*-algebra A and a bunch of morphisms $\alpha_X : \mathfrak{A}_X \to \mathfrak{A}_Y$ where $X \subseteq Y$ and such that $\alpha_X(u) = \alpha_Y(u\otimes 1)$ (but remember that u is identified with $u\otimes 1$ in the union), there's a unique morphism $\alpha : \mathrm{cl}\,\mathfrak{A}' \to A$. In the language of diagrams, this is saying that

$$\begin{tikzcd}
& A & \\
& \mathrm{cl}\,\mathfrak{A}' \arrow[u, dashed, "\exists!\alpha"'] & \\
\mathfrak{A}_X \arrow[uur, "\alpha_X"] \arrow[ur, "\imath_X"] \arrow[rr, "u\mapsto u\otimes 1"] & & \mathfrak{A}_Y \arrow[uul, "\alpha_Y"'] \arrow[ul, "\imath_Y"']
\end{tikzcd}$$

Once α is defined on $\mathfrak{A}'$ with all the mentioned properties, it automatically extends to the closure. So, for $u \in \mathfrak{A}_X$ we set $\alpha(u) := \alpha_X(u)$. This is well defined, for u is identified with $u\otimes 1$ in the union. We get automatically that α is a unital $*$-homorphism since all α_X's are. It's norm is bounded by 1, for $\|\alpha(u)\| \le \|\alpha_X\|\|u\| \le \|u\|$. Thus α is a morphism in the corresponding category. $\square$

Thoughts Aloud 1. So, any element of the union itself can be interpreted as an observable on some finite lattice. But when we take the closure of the union, we get infinite sums of operators. The very procedure of the closure is not attached to the nature of the operators (i.e. than they can be fed with states) but only to the relation between operators themselves. Therefore, we need to define what is $\sum_{n=1}^\infty A_n(x)$ for $A_n \in \mathfrak{A}_{X_n}$. An attempt to fantasy is: for $x \in H_\Lambda$, let $\pi_n(x)$ be the projection of x to H_{X_n} (H_{X_n} can be regarded as a subspace of H_Λ). Then we try to set $(\sum A_n)(x) := \sum A_n(\pi_n(x))$. Then $\sum A_n$ is a bounded linear operator, for $\|\sum A_n(\pi_n(x))\| \le (\sum \|A_n\|)\|x\|$. Hmm, why don't we like it? And yet, what's the difference between this and the G-N theorem? Are there fewer operators than in $\mathrm{End}(H_\Lambda)$?

[1] Otherwise I don't think there's a way to prove that the map induced on the diagram of the injective limit is a bounded operator

1.4 Relation between classical and quantum lattices

I follow [7] with some minor modifications more appealing to my taste. Let Ω_0 be a finite set of microstates at one site, and let H_0 be a Hilbert space of dimension equal to $|\Omega|$ (which is assigned to one site as well). Let $C(\Omega)$ be the space of observables on Ω. Choose an orthonormal basis e_μ of H_0 labeled my microstates $\mu \in \Omega_0$. Then we have an injection $\imath : C(\Omega_0) \to \mathrm{End}(H_0)$ given by

$$[\imath(f)](e_\mu) := e_{f(\mu)}.$$

In other words, the classical observables are embedded into the quantum observables as diagonal matrices.

1.5 Continuous spins: general principles

I follow closely Section 6.10 of [5]. In case the space of states Ω_0 at a single site is non-compact, the existence of Gibbs measures is no longer guaranteed. For Ω_0 a topological space, one defines the following ingredients. Let $\mathcal{B}_0$ be the Borel σ-algebra on Ω_0. For a finite lattice $\Lambda \subset \mathbb{Z}^n$, we supply the space of states with the σ-algebra $\mathcal{B}_\Lambda := \bigotimes_{i\in\Lambda} \mathcal{B}_0$. The natural projections $\pi_\Lambda : \Omega \to \Omega_\Lambda$ allow us to define a σ-algebra on Ω with base in Λ:

$$\sigma_\Lambda := \pi_\Lambda^{-1}(\mathcal{B}_\Lambda).$$

If $S \subseteq \mathbb{Z}^n$ is a possibly infinite lattice, then we supply it with the σ-algebra

$$\sigma_S := \sigma\Big(\bigcup_{\Lambda\subset S,\ \Lambda \text{ finite}} \sigma_\Lambda\Big)$$

(by the last equality I mean the smallest σ-algebra generated by the union).

2 The Main Stage

2.1 Statement of the problem and ideas

The current statement, I guess, is the following:

Statement. Consider $\mathbb{Z}^n$ where to each node we attach an infinite-dimensional separable Hilbert space H. Let $\Lambda \subseteq \mathbb{Z}^n$ be an infinite sublattice. Consider the limit

$$\mathfrak{A}_\Lambda := \varinjlim_{X\subset\Lambda,\ X \text{ finite}} \mathfrak{A}_X.$$

If it turns out that $\mathfrak{A}_\Lambda$ is a C^*-algebra, I'd like to do the following: find an ideal $J \triangleleft \mathfrak{A}_\Lambda$ such that the Hilbert space H promised by the Gelfand-Naymark theorem is separable; i.e., $\mathfrak{A}_\Lambda/J \cong \mathrm{End}(H)$ for H separable. It would be also nice to keep embeddings $\mathfrak{A}_X \to \mathfrak{A}_\Lambda$ for finite sublattices $X \subset \Lambda$.

Further Search 2. Given a unital C^*-algebra A, under which conditions on A the Hilbert space given by Gelfand-Naymark theorem is separable?

Idea. To keep everything physically meaningful, I think that J can be tried out as the ideal generated by operators with all but finitely many eigen-states concentrated in a finite sublattice of Λ. Here, I need to refresh my mind with regards to eigen-states.

2.2 Tests on quantum Ising model

In Ising model, to each node of $\mathbb{Z}^2$ we attach a 2-dimensional Hilbert space H with some a priori chosen orthonormal basis e_1, e_2. It corresponds to the states *spin up* and *spin down*.

2.2.1 The issue with the infinite tensor product

Let's consider first the algebraic tensor product H_∞ of all Hilbert spaces attached to all sites. It's spanned by simple tensors of the form

$$e_{\lambda(1)} \otimes e_{\lambda(2)} \otimes e_{\lambda(3)} \otimes \cdots$$

where $\lambda : \mathbb{N} \to \{e_1, e_2\}$ is a function. There are as my such simple tensors as functions λ; the cardinal number is equal to $|2^{\mathbb{N}}| = |\mathbb{R}|$, i.e., there are uncountably many of them[2]. It's natural to declare such simple tensors an orthonormal basis of H_∞. But then, it's a result of metric spaces theory that if there are uncountably many points such that the distance between any of two is bounded by a positive constant (that doesn't depend on the points), then the space is not separable. This is the case with H_∞. Since it's not separable, its completion $\operatorname{cl} H_\infty$ can't be separable as well. By the way, the same idea is used when one proves l_∞ is not separable.

Just out of curiosity, the same cardinal number occurs when all Hilbert spaces are infinite-dimensional but separable. In this case, we deal with all functions $\lambda : \mathbb{N} \to \mathbb{N}$; the cardinal number of them is again $|\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}|$.

2.2.2 The limit and G-F theorem

For an infinite sublattice $\Lambda \subseteq \mathbb{Z}^n$, the injective limit $\varinjlim_{X \subset \Lambda,\ X \text{ finite}} \mathfrak{A}_X$ can be thought of as the completion of the union of those subalgebras.

The following example explains how the union works.

Example 1. The algebra $\mathfrak{A}_1$ attached to a single site can be identified with the algebra of 2×2 matrices over $\mathbb{C}$. For two nodes, the algebra $\mathfrak{A}_2$ can be identified with matrices of dimension 4×4. The embedding $\mathfrak{A}_1 \to \mathfrak{A}_2$ then does the following:

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto A \otimes 1 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Therefore, for an inifnite sublattice Λ, the algebra $\mathfrak{A}_\Lambda$ can be identified with the space of matrices of infinite size such that only finitely many entries of each of them are non-zero. *Note* the difference with the attempt to take an infinite tensor product of Hilbert spaces: these infinite matrices naturally act upon the space

$$T := \bigoplus_{n=1}^{\infty} \bigotimes_{i \in A \subset \Lambda,\ |A|=n} H_i$$

It's easy to see that this space has an infinite countable basis. This implies that, whatever norm we put on T, the space will not be complete (that's a standard result from functional analysis: in a Banach space, a vector space basis is at least uncountable).

[2] More details on why it's uncountable. Any number $a \in \mathbb{R}$ can be represented as a power series $a = \sum_{i=-\infty}^{\infty} c_i 2^i$, where $c_i \in \{0, 1\}$ and only finitely many c_i for $i > 0$ might be non-zero. Restrict ourselves to $a = \sum_{i=1}^{\infty} c_i 2^{-i}$. Then we have a 1-1 correspondence between functions $\lambda : \mathbb{N} \to \{0, 1\}$ and such numbers.

Way 1 (just a fantasy). Let $\mathfrak{A}'_\Lambda$ be the union of all $\mathfrak{A}_X$ for $X \subset \Lambda$ and X finite. We can substitute the norm on $\mathfrak{A}'_\Lambda$ with the Hilbert-Schmidt norm (see appendix), and then complete $\mathfrak{A}'_\Lambda$ with respect to it. The elements A of the resulting space can be represented as infinite matrices $(a_{nk})_{k,n=1}^{\infty}$ such that $\sum_{n,k} |a_{nk}|^2 < \infty$. We can act with these on a completion of T.

3 Classical lattice models

3.1 Ising model

3.1.1 A general description of the IRF version

There are two versions of the Ising model: the IRF (interaction-round-a-face) model and the vertex model. In the first one, the energy is assigned to vertices; in the second one, the energy is assigned to the bonds between the sites.

Let $\Lambda \subseteq \mathbb{Z}^n$ be a subset of the integer lattice of dimension n. We associate with the lattice the space of microstates $\Omega_\Lambda := \{-1, +1\}^\Lambda$. Therefore, to each node $i \in \Lambda$ there corresponds a *spin* $\omega_i = \pm 1$. For a finite Λ, the hamiltonian of the model is given by

$$\mathcal{H} = \sum_{i,j \in \Lambda,\ i \sim j} \omega_i \omega_j - h \sum_{i \in \Lambda} \omega_i,$$

where $h \in \mathbb{R}$ is some real number that corresponds to the external magnetic field, and $i \sim j$ means the nodes i and j are neighbors on the lattice. We also supply the model with the Gibbs measure defined previously.

3.1.2 Transfer matrices in IRF model (not finished)

To describe the transfer matrices, I restrict myself to a finite cubic lattice $\Lambda \subset \mathbb{Z}^2$ with periodic boundary conditions. Then we can assign energy to each face of the lattice:

$$\epsilon(\text{face}, \omega) := \sum_{i,j \in \text{face},\ i \sim j} \omega_i \omega_j - h \sum_{i \in \text{face}} \omega_i.$$

So the Hamiltonian breaks up into the sum of energies over all faces in Λ:

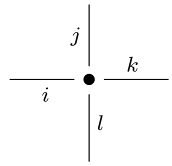
$$H(\omega) = \sum_{F \in \{\text{faces of } \Lambda\}} \epsilon(F, \omega).$$

A *Boltzmann weight* is the quantity $R(F, \omega) := \exp(-\beta \epsilon(F, \omega))$ assigned to a face F. The partition function can be rewritten as

$$Z = \sum_{\omega \in \Omega} \prod_{F \in \text{faces}} R(F, \omega).$$

3.1.3 The vertex model and its transfer matrix

I follow closely [3]. Let Λ be an $n \times m$ cubic lattice in $\mathbb{Z}^2$ with periodic boundary conditions. The states are assigned to the bonds between vertices rather than to the vertices themselves in this model. Let $\Omega_0 = \{1, \dots, n\}$ be the set of possible states of a single bond. For a picture of kind



let ε_{ij}^{kl} denote the energy assigned to the site in this setting. We assume that it doesn't depend on the position of the site but only on the states of the bonds around the sit. The Hamiltonian $\mathcal{H}$ of this model for a particular choice of the state of the lattice is then the sum of ε_{ij}^{kl} over all vertices. The partition function is given by $Z = \sum_{\omega \in \Omega} \exp(-\beta H(\omega))$. A *Boltzmann weight* is the quantity

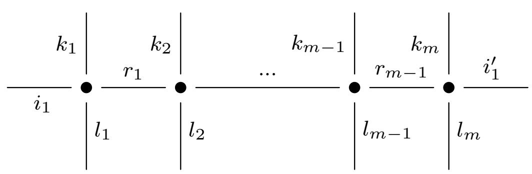
$$R_{ij}^{kl} := \exp(-\beta \varepsilon_{ij}^{kl}).$$

Proposition 2. Let V be an m-dimensional vector space. There exists an endomorphism $T \in \mathrm{End}(V \otimes V^m)$, which is called a *transfer matrix*, such that the partition function of the model is given by

$$Z = \mathrm{tr}_{V^{\otimes m}} (\mathrm{tr}_V \, T)^n$$

where the trace is the usual one (the sum of diagonal elements).

Proof. Consider a row in the cubic lattice, for a moment assuming that the boundary conditions on the ends (the states i_1 and i_1') may not be the same



Let us fix the end states i_1, i_1', $k_1, \dots, k_m$ and $l_1, \dots, l_m$. The contribution to Z when only r_i's are running over Ω_0 is given by

$$T_{i_1 k_1 \cdots k_m}^{i_1' l_1 \cdots l_m} := \sum_{r_1, \dots, r_{m-1}} R_{i_1 k_1}^{r_1 l_1} \cdots R_{r_{m-1} k_m}^{i_1' l_m}.$$

Let V be an m-dimensional vector space spanned by some $e_1, \dots, e_m$. Define an endomorphism $T \in \mathrm{End}(V \otimes V^{\otimes m})$ by setting on the basis elements

$$T(e_{i_1} \otimes e_{k_1} \otimes \cdots \otimes e_{k_m}) = \sum_{i_1', l_1, \dots, l_m} T_{i_1 k_1 \cdots k_m}^{i_1' l_1 \cdots l_m} e_{i_1'} \otimes e_{l_1} \otimes \cdots \otimes e_{l_m}.$$

If wee unfreeze the endpoints with states i_1 and i_1' and let them run over Ω_0, then we see that the contribution to Z of the whole row (with still fixed states on the vertical bonds and now $i_1 = i_1'$) is given by $\mathrm{tr}_V(T)_{k_1 \dots k_m}^{l_1 \dots l_m}$. Now, if the row was the first one and we consider the next one to it, and let $l_1, \dots, l_m$ run over Ω_0, then the contribution to Z is

$$\sum_{l_1, \dots, l_m} \mathrm{tr}_V(T)_{k_1 \dots k_m}^{l_1 \dots l_m} \, \mathrm{tr}_V(T)_{l_1 \dots l_m}^{j_1 \dots j_m} = [(\mathrm{tr}_V(T))^2]_{k_1 \dots k_m}^{j_1 \dots j_m}$$

(the last equality was not obvious to me due to a mess with indices, but it can be checked easily). Continuing in this fashion, the contribution to Z with fixed states of the vertical bonds on the ends is given by $[(\mathrm{tr}_V(T))^n]_{k_1\dots k_m}^{l_1\dots l_m}$. Now, applying the periodic condition $k_j = l_j$ and summing over all possible states of the ends, we finally find that $Z = \mathrm{tr}_{V^{\otimes m}}[\mathrm{tr}_V(T)]^n$. $\square$

I think I can say that a transfer matrix is just a batch of all possible microstates of a row ingeniously packed into a linear endomorphism.

3.2 Gaussian free field model

I follow closely Chapter 8 from [5]. In Gaussian free field model, the space of states at a single site is chosen to be $\Omega_0 := \mathbb{R}$. Accordingly, the space of states on a region $\Lambda \subseteq \mathbb{Z}^n$ is given by $\Omega_\Lambda := \mathbb{R}^\Lambda$. The Hamiltonian of the model is on the lattice Λ is chosen to be

$$\mathcal{H} := \frac{\beta}{4n} \sum_{i\sim j,\ \{i,j\}\cap\Lambda\neq\emptyset} (\omega_i - \omega_j)^2 + \frac{m^2}{2}\sum_{i\in\mathbb{Z}^n} \omega_i^2,$$

where β is the inverse temperature, $\omega_i \in \Omega_0$ is the assigned spin at site $i \in \mathbb{Z}^n$, and m is the mass.

A couple of comments on the choice of the Hamiltonian:

1) The factor $(\omega_i - \omega_j)^2$ tells us that the interaction favors the agreement of neighboring spins;

2) Since the space of states at single site is non-compact, we penalize large values of spin by adding the factor $m^2/2 \cdot \omega_i^2$ for each one;

3) Notice the condition under the first summation. It tells us that we also take into the account the boundary of Λ (there might be different boundary conditions though).

Fix a finite lattice $\Lambda \subset \mathbb{Z}^n$ and a state $\eta \in \Omega$ (it serves as a boundary condition for Λ). For a state $\omega_\Lambda \in \Omega_\Lambda$, by $\mathcal{H}(\omega_\Lambda)$ we mean that we plug into the Hamiltonian the state that equals ω_Λ on Λ and η on the complement of Λ.

In Subsection 1.5 we specified a way of choosing σ-algebras on the spaces of states. Let $\sigma_{\mathbb{Z}^n}$ be such σ-algebra on the whole $\mathbb{Z}^n$. For $A \in \sigma_{\mathbb{Z}^n}$, the Gibbs measure in this model is defined as

$$\mu(A) := \int_A \frac{e^{-\mathcal{H}(\omega_\Lambda)}}{Z} \prod_{i\in\Lambda} d\omega_i,$$

where $d\omega_i$ is the Lebesgue measure on $\mathbb{R}$ assigned to the site $i \in \mathbb{Z}^n$ and Z is the obviously chosen partition function.

There's a way to define Gibbs measures for infinite Λ as well (explained in [5], I postpone its description here for a moment). The case of massless GFF is drastically different from the case of massive GFF. For instance, Theorem 8.19 in [5] says that there are no infinite-volume Gibbs measures in $n = 1$ and $n = 2$ cases. Nevertheless, Theorem 8.21 in the same reference tells us that there are infintely many infinite-volume Gibbs measures when $n \geq 3$. In the massive case, the GFF model has infinitely many infinite-volume Gibbs measures for any n (see Theorem 8.28 in [5]).

3.3 O(N)-symmetric model

I follow Chapter 9 from [5]. In O(N)-model, we take $\Omega_0 := S^{N-1}$, so the spins might have an arbitrary direction. For a finite lattice $\Lambda \subseteq \mathbb{Z}^n$, the Hamiltonian (in the absence of a magnetic field) is usually written as

$$\mathcal{H} = -\beta \sum_{i \sim j, \quad \{i,j\} \cap \Lambda \neq \emptyset} \langle \omega_i, \omega_j \rangle ,$$

where $\omega_i \in \Omega_0$ is a spin at site i, and the brackets denote the standard inner product in $\mathbb{R}^N$. For different N's, we obtain some familiar models: for $N = 1$ we have the Ising model; for $N = 2$ we get the XY-model; and for $N = 3$ we obtain the Heisenberg model.

The definition of finite-volume Gibbs measures is similar to the case of GFF model. At each site i, we have Lebesgue measure $d\omega_i$ on S^{N-1}. We fix a boundary condition, which is the choice of a state $\eta \in \Omega$, and then for measurable sets A we set

$$\mu(A) := \int_A \frac{e^{-\mathcal{H}(\omega_\Lambda)}}{Z} \prod_{i \in \Lambda} d\omega_i,$$

where Z is the obvious partition function and $\omega_\Lambda \in \Omega_\Lambda$; by $\mathcal{H}(\omega_\Lambda)$ I mean that we plug in a state equal to ω_Λ on Λ and η outside of Λ.

One might be interested in the following questions with regards to O(N)-models:

1) Is there an orientational long-range order? In my understanding, the mathematical formalism of this question is whether the correlations $\mathbb{E}_\mu \langle \omega_i, \omega_j \rangle$ converge to zero as $\|i - j\| \to \infty$;

2) Is there a spontaneous magnetization? The formalism in my understanding is: for any infinite-volume Gibbs measure μ, is it true that $\lim_{n \to \infty} \left\langle \|m_{B(n)}\| \right\rangle_\mu \neq 0$? Here $B(n)$ is a cube of size n and $m_{B(n)} := \frac{1}{|B(n)|} \sum_{i \in B(n)} \omega_i$ is the *magnetization density.*

The answers to both questions are negative for $N \geq 2$ and $n = 1, 2$. This is due to the following theorem, which can be also stated for a more general Hamiltonian:

Theorem 1. *(Mermin-Wagner) For $N \geq 2$ and $n = 1, 2$, all infinite-volume Gibbs measures are invariant under the action of the rotation group.*

Maybe, I will write why the answers are negative a bit later.

4 Free Bosons

4.1 Truncated Free Bosons

We have here one site with Hilbert space $\mathcal{H}_M$ of dimension M, and basis $\{|0\rangle, |1\rangle, \cdots, |M-1\rangle\}$. We should here say $|0\rangle_M$, $|1\rangle_M$, etc, but will omit the extra subindex for clarity. We define creation $a^\dagger$ and destruction operators[3] a, (they should really be a_M and $a_M^\dagger$ but will omit the subindex) such that $a|0\rangle = 0$ (the zero of the Hilbert space), $a|n\rangle = \sqrt{n}|n-1\rangle$ for all $0 < n < M$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ for all $0 \leq n < M - 1$, $a^\dagger|M-1\rangle = 0$. Let $H := H_M := M^{-1}a^\dagger a$ be the "Hamiltonian" operator for free bosons. Then H is self-adjoint, every element of the basis is an eigenvector of H, and with eigenvalues[4] $\{0, 1/M, 2/M, \cdots, (M-1)/M\}$.

[3] Have checked that they are indeed adjoint to each other even in the truncated case.

[4] Indeed, $H|0\rangle = 0$ and for $M - 1 \geq m > 0$ we have $H|m\rangle = \frac{\sqrt{m}\sqrt{m}}{M}$.

Consider the the algebra $\mathcal{U}_M$ generated by a and $a^\dagger$ and the identity I. H belongs to this algebra. It's a C^*-algebra (even a von Neumann algebra): we included in the list of generators all conjugates and we supplied the algebra with the operator norm, which always satisfies the C^*-identity. It's a von Neumann algebra because any finite-dimensional C^*-algebra is a von Neumann algebra.

Regarding the dimension, it's clear to me that the algebra is finite-dimensional and that the dimension can be bounded from below by $3M-2$. I haven't figured out yet the exact dimension.

4.2 Infinite Free Bosons

We have here one site with separable Hilbert space $\mathcal{H}_\infty$ of infinite dimension, and basis $\{|0\rangle, |1\rangle, \cdots\}$; we define creation $a^\dagger$ and destruction operators a, such that $a|0\rangle = 0$, $a|n\rangle = \sqrt{n}|n-1\rangle$ for all $0 < n$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ for all $0 \leq n$. Notice that both a and $a^\dagger$ are unbounded but their domains are dense in $\mathcal{H}_\infty$.

Statement. The approach with energy density does not work. Precisely, the following is true: for every $M \geq 1$, extend $H_M = \frac{1}{M} a_M^\dagger a_M$ to act on $\mathcal{H}_\infty$ by setting it equal to zero on the orthogonal complement of $\mathcal{H}_M \subset \mathcal{H}_\infty$. Then there is a well defined limit in the strong topology: $H_\infty|\psi\rangle = \lim_{M\to\infty} H_M|\psi\rangle$. However, $H_\infty = 0$.

Proof. Indeed, for any basis ket $|k\rangle$, we have $\lim_{M\to} H_M|k\rangle = \lim_{k\to\infty}(k/M)|k\rangle = 0$, so the sequence H_M converges strongly to zero. □

Statement. Redefine H_M as $H_M := a_M^\dagger a_M$ and extend by zero onto the orthogonal complement of $\mathcal{H}_M$, so $H_M|k\rangle = k|k\rangle$ for $k \leq M-1$. Then H_M converges strongly to some unbounded self-adjoint (hence closed) operator H_∞ on a dense subspace of $\mathcal{H}_\infty$.

Proof. The domain of H_∞ would be the dense subspace

$$\operatorname{dom} H_\infty = \{|\psi\rangle = \sum_{k=0}^{\infty} \psi_k |k\rangle \in \mathcal{H}_\infty \mid \sum_{k=0}^{\infty} k|\psi_k| < \infty\}.$$

So, for any $|\psi\rangle$ such that $\sum_{k=0}^{\infty} k|\psi_k| < \infty$, we can safely define[5]

$$H_\infty|\psi\rangle := \lim_{M\to\infty} H_M|\psi\rangle = \sum_{k=0}^{\infty} k\psi_k|k\rangle.$$

The operator is clearly unbounded, for $\|H_\infty|k\rangle\| \to \infty$.

Now let's find the domain of $H_\infty^\dagger$. By definition of the adjoint of an unbounded operator, $|\phi\rangle \in \operatorname{dom}(H_\infty^\dagger)$ if and only if there exists $|\theta\rangle \in \mathcal{H}_\infty$ such that for every $|\psi\rangle \in \operatorname{dom}(H_\infty)$ we have $\langle\phi|H_\infty|\psi\rangle = \langle\theta|\psi\rangle$. In components, this equality means that $\phi_k^*\psi_k k = \theta_k^*\psi_k$, hence $|\phi\rangle$ must reside in $\operatorname{dom}(H_\infty)$. Since H_∞ is obviously symmetric and $\operatorname{dom}(H_\infty) = \operatorname{dom}(H_\infty^\dagger)$, it is self-adjoint. From the general theory of unbounded operators we know that the adjoint is always closed, hence any $H_\infty = H_\infty^\dagger$ is closed. □

The limit (in operator norm, weak, strong?) of the sequence of operators H_M exits (?), is unique, and let's call it H_∞. Then H_∞ is self adjoint (is it really?) and its spectrum is $\sigma(H_\infty) = [0, 1)$

[5] It's a general fact from the theory of Banach spaces that a series $\sum_{k=1}^{\infty} x_k$ converges iff the series $\sum_{k=1}^{\infty} \|x_k\|$ converges.

(does it include 1?). Moreover, every $1/n$ for $n >= 1$ is an eigenvalue of H_∞, and the corresponding eigenvectors form a basis of $\mathcal{H}_\infty$.

For $P \geq 1$, let v_P, w_P be both vectors in $\mathcal{H}_P$, and consider their extension to any $M \geq P$ by the same name. then $lim_{M\to\infty}\langle v_P|H_M|w_P\rangle$ exists, and is equal to $\langle v_P|H_\infty|w_P\rangle$, where v_P, w_P are considered vectors of $\mathcal{H}_\infty$.

Consider the algebra $\mathcal{U}_\infty$ generated by a and $a^\dagger$ and the identity I. Does H_∞ belong to this algebra? What's the dimension of this algebra? Is this a C* algebra? Is it a von Neumann algebra? Etc. Is this algebra a limit in some sense from the sequence of algebras $\mathcal{U}_M$.

5 Free Fermions

5.1 Finite Chain

Consider one site with Hilbert space of dimension two, and basis $\{|0\rangle, |1\rangle\}$; physically we say that the state $|0\rangle$ is empty, and the state $|1\rangle$ occupied; we define creation $c^\dagger$ and destruction operators c, such that $c|0\rangle = 0$, $c|1\rangle = |0\rangle$, $c^\dagger|0\rangle = |1\rangle$, $c^\dagger|1\rangle = 0$. For N sites, we extend the definitions by making N tensor products, so that the total dimension is 2^N. (Even though we use the word *site* or *sites*, we use here momentum space for simplicity, so that the H below will be already in "diagonal form.") Given a $N-$tensor product state $|v\rangle \equiv |v_0\rangle \otimes |v_1\rangle \otimes \cdots \otimes |v_{N-1}\rangle$, the v_m are either 0 or 1. Let the parity up to m operator P_m be the diagonal operator such that $P_m|v\rangle = p(|v\rangle, m)|v\rangle$, where $p(|v\rangle, m)$ is the number 1 or -1, and is given by

$$p(|v\rangle, m) \equiv (-1)^{\sum_{0\leq m'<m} v_{m'}}. \tag{5.1}$$

In other words, $P_m|v\rangle$ is $|v\rangle$ if the number of $|1\rangle$ strictly preceding mth position is even; $-|v\rangle$ if odd.

We write d_m to mean $I \otimes I \otimes \cdots \otimes c \otimes I \otimes \cdots \otimes I$, where c is at location m, and I is the one site identity. We write $c_m(N) \equiv d_m P_m$, where P_m is the parity operator up to m, as defined before. Note that $c_m(N)$ acts on the full space $\{|0\rangle, |1\rangle\}^{\otimes N}$. *I drop now the* (N) *from the* c_m.

Example 2. Let $|w\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |0\rangle \cdots$, and remember that we count locations from 0. Then $c_2|w\rangle = -|0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots$, with a -1 because the sum of 1s before location 2 is odd. On the other hand $c_3^\dagger|w\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle \otimes \cdots$, with a $+1$ because the sum of 1s before location 3 is even. Obviously $c_2^\dagger|w\rangle = c_3|w\rangle = 0$ as we can't destroy if there's no particle, and we can't create if there's already a particle, because these are fermions and accept only one particle per location.

We write $c_m^\dagger c_m$ to mean the tensor product operator with $c^\dagger$ and c at location m, identities elsewhere, *and with appropriate parities,* so it's $P_m d_m^\dagger d_m P_m$. (Careful that P_m and d_m or $d_m^\dagger$ do not commute.) Let

$$H_N = N^{-1} \sum_{m=0}^{m=N-1} e_m c_m^\dagger c_m \tag{5.2}$$

be the "Hamiltonian" operator for free fermions, where $e_m = -2\cos(2\pi m/N)$ for $0 \leq m < N$ integers; this is called the *dispersion* relation for free periodic fermions. Then H_N is a self-adjoint (symmetric) matrix of rank 2^N.

5.1.1 Eigenvalues and Eigenvectors of the Hamiltonian

All this is just matrix diagonalization, but because of the form of the 2^N matrix H, we can find its eigenvalues and eigenvectors in a compact way, that we know how to describe.

Let $S = \{f : \{0, 1, \cdots, N-1\} \to \{0, 1\}\}$, that can be thought of as binary numbers with N digits. There are 2^N functions f in S and for each, the number $N^{-1} \sum_{m=0}^{m<N} f(m) e_m$ is an eigenvalue of H_N with eigenvector

$$\bigotimes_{m \in I(f)} c_m^\dagger |0\rangle,$$

where $I(f) = \{m \in \{0, 1, \cdots, N-1\}; f(m) = 1\}$, and $|0\rangle = |0\rangle \otimes |0\rangle \cdots |0\rangle$ the fully "empty" state. In particular, the lowest eigenvector of H_N is

$$\bigotimes_{m \in I_{min}} c_m^\dagger |0\rangle,$$

with eigenvalue $N^{-1} \sum_{m \in I_{min}} e_m$, where $I_{min} = \{m; 0 \leq m \leq N/(8\pi)\} \cup \{m; 3N/(8\pi) \leq m < N\}$.

Exercise: Express the largest eigenvector and eigenvalue to make sure you understand the construction.

5.1.2 Fermionic Density

Let $D_N = N^{-1} \sum_{m=0}^{m<N} c_m^\dagger c_m$ be the density operator. Then D_N is self-adjoint, diagonal, and with eigenvalues $\{0, 1/N, 2/N, \cdots, 1\}$. Moreover, D_N and H_N commute (physically, H_N "conserves" the particle density or the number of particles), and the eigenvector of H_N characterized by $f \in S$, is also an eigenvector of D_N with eigenvalue equal to the number of elements of $I(f)$ divided by N.

Consider the algebra $\mathcal{U}_M$ generated by c_m (with parity and in the 2^N space) and $c_m^\dagger$ and the identity I. H_N and D_N belong to this algebra. What's the dimension of this algebra? Is this a C* algebra? Is it a von Neumann algebra? Etc.

5.2 Infinite Chain

We have here a separable Hilbert space $\mathcal{H}_\infty$ of infinite dimension, and basis $\{|0\rangle, |1\rangle, \cdots\}$. For every $M \geq 1$ we extend H_M to act on $\mathcal{H}_\infty$ (does this extension make sense?). The limit (in operator norm, weak, strong?) of the sequence of operators H_M exits (?), is unique, and let's call it H_∞. Then H_∞ is self adjoint (is it really?) and its spectrum is $\sigma(H_\infty) = [-2/\pi, 2/\pi]$.

What are the eigenvalues (if any) and eigenvectors of H_∞?

Let D_∞ be the corresponding limit of the D_N operators. Show it's self-adjoint with spectrum $\sigma(D_\infty) = [0, 1]$. What are the eigenvalues (if any) and eigenvectors of D_∞?

Consider the algebra $\mathcal{U}_\infty$ generated by c_i and $c_i^\dagger$ and the identity I with some completion. How does one "complete" or "close" this algebra? Does H_∞ belong to this algebra? What about D_∞? What's the dimension of this algebra? Is this a C* algebra? Is it a von Neumann algebra? Etc. Is this algebra a limit in some sense from the sequence of algebras $\mathcal{U}_N$.

6 Pure Gauge in 1D

TBW

7 Pure Gauge in 2D

TBW

8 Gauge and Matter in 1D

TBW

9 Questions

9.1 For a discussion

Question 1. What's a momentum space? And what's the meaning of diagonilized Hamiltonian?

Question 2. It's been confusing to me that c_m is regarded as an operator on a site and as an operator with parity on the tensor product of sites. I changed the notation so that c means the operator on a site and c_m is the operator on the tensor product with parity.

Question 3. Is there a typo in the definition of $p(|v\rangle, m)$? I feel that m has to be included in the summation, for otherwise P_m commutes with d_m.

9.2 Just for myself

Question 4. Derive the grand canonical Gibbs distribution to see why the sum is over N. It bothers me, I just can't stand this formula before I see its derivation.

Question 5. I do not see any issue with regards to defining the Gelfand spectrum for non-commutative Banach algebras. Why is it defined only for commutative ones? Only because it appears in the set-up of G-N theorem?

Question 6. Does the 2nd G-N theorem require a unit? I mean, if not, then something is still mapped to the unit of the operator algebra; that doesn't look natural, but might happen.

In one article found this: the totality of numbers associated with a given observable is called its *spectrum*. And indeed, if we regard the algebra of continuous functions as a Banach algebra, then the spectrum of anything is its image; in quantum mechanics, we indeed look at the eigenvalues, which from Banach algebras point of view form their spectra.

In Segal's article I saw that a product of two observables may not be an observable, so he proposed an algebraic structure where a multiplication is not assumed but only powers of elements.

Also from one article: More recent studies have indicated that all self-adjoint operators may not be adequate as the model for the algebra of observables of every physical system. The $C*$-algebras are a long step from this special model, but still not into the chaos of abstract structures consistent with the general features of physical systems.

From Kadison: Given a state f and an observable A, the value $f(A)$ is the expectation of the observable A when these two have physical meanings.

10 Appendix

10.1 C^*-algebras

Recall the definition of a C^*-algebra:

Definition 1. A (unital) C^*-algebra A is a Banach space over $\mathbb{C}$ that is also a (unital) algebra such that the multiplication is a bounded bilinear map of norm 1. It's also supplied with an involution[6] $* : A \to A$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$.

A straightforward[7] consequence from the axioms is that $\|a\| = \|a^*\|$ and in the unital case $\|1\| = 1$.

10.1.1 Gelfand-Naymark: commutative case

I think it worth mentioning:

Theorem 2. *(Gelfand-Naymark, 1st) We have the following:*

(i) Any possibly non-unital commutative C^*-algebra is isomorphic to the space $C_0(\Omega)$ of continuous functions vanishing[8] at ∞ on some locally compact Hausdorff topological space Ω;

(ii) Any unital commutative C^*-algebra is isomorphic to $C(\Omega)$ for Ω a compact Hausdorff space.

There's an accurate description of both the isomorphism and the space Ω: the space Ω is the Gelfand spectrum of A, and the isomorphism is the Gelfand representation.

Here are the definitions. Let A be a Banach algebra[9]. Its *Gelfand spectrum* is the space $\Omega \subset A^*$ of all characters[10] of A that is endowed with ω^*-topology induced from A^*. The isomorphism then is the Gelfand representation $\Gamma_A : A \to C_0(\Omega)$ that sends an element $a \in A$ to a continuous map (vanishing at ∞) $\hat{a}$ such that[11]

Proposition 3. The Gelfand spectrum Ω of a Banach algebra A is a locally compact Hausdorff space; if A is in addition unital, then Ω is Hausdorff and compact.

Proof. Obviously, Ω is Hausdorff since ω^*-topology is Hausdorff. Consider $\Omega' := \Omega \cup \{0\}$. If $\chi_\alpha \to f$ in ω^*-topology, then obviously f preserves multiplication (but it might become zero), so Ω' is a closed subset of the unit ball in A^*. Now it's the consequence of Banach-Alaoglu theorem that Ω' is compact. Being a closed subset of a compact Hausdorff space, we see that Ω is locally compact and Hausdorff. In case A is unital, 0 is an isolated point of Ω', for if there was a net $\chi_\alpha \in \Omega$ such that $\chi_\alpha \to 0$, then $1 = \chi_\alpha(1) \to 0$, which is a contradiction, and thus Ω is compact. $\square$

The beauty of the result and the proof are mesmerizing, so I couldn't resist working out the details. Here I'd like to record a sketch for the unital case (the non-unital has to come from the unital by the process of adjoining an identity). One can find a full proof in [1].

[6] An involution has the same properties as the usual conjugation operators. So, $(a+b)^* = a^* + b^*$, $(\lambda a)^* = \lambda^* a^*$, $1^* = 1$, $(ab)^* = b^* a^*$, $(a^*)^* = a$.

[7] From the boundedness of the multiplication we obtain $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$, hence $\|a\| \leq \|a\|^*$. Switching to $a \mapsto a^*$, get $\|a\| = \|a\|^*$. For the unit, $\|1\| = \|1\|^2$ since $1^* = 1$.

[8] Precisely, a function $\varphi : \Omega \to \mathbb{C}$ is vanishing at infinity if for any $\varepsilon > 0$ there exists a compact subset $\Delta \subseteq \Omega$ such that $|\phi(t)| < \varepsilon$ when $t \notin \Delta$.

[9] I.e. a Banach space that's also an algebra such that the multiplication is a bounded bilinear map with norm 1.

[10] A character is a non-zero linear functional that preserves the multiplication. It turns out that any character $\chi : A \to \mathbb{C}$ has norm $\|\chi\| \leq 1$, so they're automatically continuous, for if $\chi(a) = 1$ for some $a \in A$ with $\|a\| < 1$, then let $b := \sum_{k=1}^{\infty} a^k$. It follows that $a + ab = b$, hence $1 + \chi(b) = \chi(b)$, which is absurd.

[11] The continuity of $\hat{a}$ follows right from the definition of ω^*-topology. It indeed vanishes at ∞, for $\hat{a}$ is a continuous map on the compactification $\Omega' := \Omega \cup \{0\}$ such that $\hat{a}(0) = 0$ (look at the formula defining $\hat{a}$), hence for any $\varepsilon > 0$ there's $U_\varepsilon \ni 0$ such that $|\hat{a}(\chi)| < \varepsilon$ when $t \in U_\varepsilon$. But U_ε^c is compact, as a closed subset of the compact space Ω', so $\hat{a}$ vanishes at ∞.

Sketch of the proof of the 1st G-F theorem. First of all, in case of C^*-algebras it is automatic that every character preserves the $*$-structure. This fact implies that Γ_A is a unital $*$-homomorphism. Next, one shows that the spectrum $\sigma(a)$ of any $a \in A$ coincides with the range of $\hat{a}$. Indeed, $\lambda \in \sigma(a)$ if and only if $a-\lambda$ is not invertbile, if and only if $a-\lambda$ belongs to some maximal ideal; there is a $1-1$ correspondence between maximal ideals and characters in unital commutative Banach algebras, so we find χ such that $\chi(a-\lambda) = \lambda$, i.e., $\hat{a}(\chi) = \lambda$. A consequence of this is that the spectral radius $r(a)$ of any a coincides with the norm of $\hat{a}$. Now, if a is self-adjoint, then $\|a\| = r(a) = \|\hat{a}\|$, which means that Γ_A works as an isometry on self-adjoint elements. If $a \in A$ is an arbitrary one, then we use the C^*-structure and the fact that a^*a is self-adjoint:

$$\|a\|^2 = \|a^*a\| = \|\hat{a}^*\hat{a}\| = \|\hat{a}\|^2.$$

Thus Γ_A is an isometric embedding. Its image is a closed subalgebra and contains the identity of $C(\Omega)$. For surjectivity, one applies the Stone-Weierstrass theorem: the functions $\hat{a}$ separate the points (for two different $\chi_1 \neq \chi_2$, just take any $a \in \operatorname{Ker}\chi_1 \backslash \operatorname{Ker}\chi_2$, and then $0 = \hat{a}(\chi_1) \neq \hat{a}(\chi_2)$). $\square$

10.1.2 Gelfand-Naymark: non-commutative case

In this section, I describe the Gelfand-Naymark-Segal construction that is used to show that any C^* algebra can be isometrically represented in some Hilbert space. The construction relies on the notions of states and pure states. I assume everywhere that there's a fixed unital C^*-algebra A. A good source for the relevant results is [1].

Definition 2. A state f is a positive linear functional f (i.e., $f(z^*z) \geq 0$ for any z) that is normalized at the unit: $f(1) = 1$.

One can show that states form a convex ω^*-compact set: it coincides with all bounded linear functionals whose norm is 1 and which is achieved at the identity. By Krein-Millman theorem, this set is the closed convex hull of so-called *extreme points*[12]. These extreme points are called *pure states*. For any state f, one can find a representation π in some Hilbert space H, and a vector $\xi \in H$ such that

$$f(x) = (\pi(x)\xi, \xi).$$

Moreover, ξ can be chosen in such a way that the span of $\{\pi(x)\xi \mid x \in A\}$ is dense in H. The construction also tells us that π is irreducible if and only if f is a pure states. The steps to find π are the following:

We define $\pi(x)$ for every $x \in A$ as the multiplication by x on the left: $\pi(x)(y) = xy$. Since the chosen state f is positive, it yields a positive sesquilinear form $(x, y) := f(y^*x)$ on A. The form has a kernel $N := \{(x, x) = 0\}$, so the quotient A/N is a vector space with an inner product. It turns out that N is also a left ideal with respect to the multiplication, hence every $\pi(x)$ lifts to A/N. Now, simply complete A/N and extend each $\pi(x)$ to a bounded linear operator on the obtained Hilbert space. The work to be done is to show that every $\pi(x)$ is bounded. But this turns out to be true. The vector ξ is then $\xi := 1 + N$. If in the end we do not obtain that $\operatorname{span}\{\pi(x)\xi \mid x \in A\}$ is dense, simply restrict the representation to this subspace.

Next, a series of propositions in [1] tells us that for any self-adjoint $x \in A$, we can find a pure state f such that $f(x) = \|x\|$. This makes us think that instead of constructing representations through pure states, we can actually start with elements of A. A corollary of this is that for every

[12] I.e., points that are not interior points of intervals lying inside that set.

element $z \in A$, there's an irreducible representation π and a vector ξ in its Hilbert space such that $\|\pi(z)\xi\| = \|z\|$. This is enough to prove the Gelfand-Naymark theorem:

Theorem 3. *(Gelfand-Naymark) For any unital C^*-algebra A, there's an isometric representation in some Hilbert space.*

Proof. As we discussed above, for every non-zero $z \in A$ we pick a representation π_z such that $\|\pi_z(z)\xi_z\| = \|z\|$. The direct some of all of these representations is then injective. It's also a result of C^*-algebra homomorphisms that injectivity implies that π is isometric. □

10.1.3 Properties of AF algebras (not finished)

Definition 3. A (unital) C^*-algebra A is called *approximately finite-dimensional* if A is an inductive limit of a sequence of finite-dimensional (unital) C^*-algebras.

I'd like to record some interesting properties of such algebras. They may not be important for our purposes, though. An example of an AF algebra appearing in these notes is the space of observables attached to an infinite sublattice when a Hilbert space at a single site is finite-dimensional.

Proposition 4. (see [2]) Let A be a unital C^*-algebra. Then A is an AF-algebra if and only if the following two conditions are fullfilled:

(i) A is separable;

(ii) If $x_1, \dots, x_n \in A$ and $\varepsilon > 0$, then there exist a finite-dimensional C^*-subalgebra $B \subseteq A$ and elements $y_1, \dots, y_n \in B$ such that $\|x_i - y_i\| < \varepsilon$, $i = 1, \dots, n$.

Furthermore, if A is AF, and A_1 is a finite-dimensional C^*-subalgebra of A, there exists an increasing sequence $A_2 \subseteq A_3 \subseteq \cdots$ of finite-dimensional C^*-subalgebras such that $A_1 \subseteq A_2$ and $A = \bigcup_i A_i$.

Some aspects of the statement are not clear to me, and if needed, can delve into: so, the closure is not taken in the second bullet, is this right?

A couple of interesting results on pure states:

Proposition 5. (see [2]) Let A be an AF-algebra and let ω_1 and ω_2 be pure states of A such that the associated representations π_1 and π_2 are faithful. Then there exists an automorphism α of A such that $\omega_1 = \omega_2 \circ \alpha$.

The next proposition is basically saying that a state is pure iff its restriction to each finite-dimensional subalgebra is pure.

Proposition 6. (see [2]) Let A be an AF-algebra and let ω be a state of A such that the associated representation is faithful. Then ω is pure if and only if there exists an increasing subsequence A_n of finite-dimensional $*$-subalgebras of A all containing the identity and such that $A = \varinjlim A_n$ and $\omega|A_n$ is pure for all n.

10.2 Hilbert-Schmidt operators

The results on Hilbert-Schmidt operators can be found in a concise form in [4] on page 268.

Let H be an infinite-dimensional Hilbert space and $\mathfrak{A} := \mathrm{End}(H)$ be the space of bounded linear endomorphisms. If $e_1, \ldots, e_n, \ldots$ is an orthonormal basis of H, for $A \in \mathfrak{A}$ set

$$\|A\|_2 := \sqrt{\sum_i \|Ae_i\|^2}.$$

It turns out that $\|A\|_2$ does not depend on the choice of the orthonormal basis. The operator A is called a *Hilbert-Schmidt operator* if $\|A\|_2 < \infty$. Denote the corresponding space as $\mathfrak{A}_2$. These operators satisfy the following properties:

(a) $\|A\| \leq \|A\|_2$;

(b) If $T \in \mathfrak{A}$ and $A \in \mathfrak{A}_2$, then

$$\|TA\|_2 \leq \|T\|\|A\|_2, \quad \|AT\|_2 \leq \|A\|_2\|T\|;$$

(c) $\|A\|_2 = \|A^*\|_2$;

(d) Algebraically, $\mathfrak{A}_2$ is a two-sided ideal of $\mathfrak{A}$ (non-closed in the operator norm topology in infinite-dimensional case);

(e) The subspace of finite-rank operators is contained in $\mathfrak{A}_2$ and is dense there;

(f) $A \in \mathfrak{A}_2$ iff $|A| := \sqrt{A^*A} \in \mathfrak{A}_2$. In this case, $\|A\|_2 = \||A|\|_2$;

(g) Hilbert-Schmidt operators are compact. Moreover, if $\lambda_1, \ldots, \lambda_n, \ldots$ are eigenvalues of $|A|$ (each repeated as many times as its multiplicity), then $|A| \in \mathfrak{A}_2$ if and only if $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. In this case, $\|A\|_2 = \sqrt{\sum_{n=1}^{\infty} \lambda_n^2}$.

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