Contents

1 Theoretical constructs

1.1 The main principles of classical lattice models

Let Ω be a finite set (the set of *microstates*), let $\mathcal{H} : \Omega \to \mathbb{R}$ be a *hamiltonian*, a specifically chosen random variable. Let $\mathcal{M}(\Omega)$ be the space of probability measures on Ω. In this theory,

the expectation of a random variable $f : \Omega \to \mathbb{R}$ with respect to a measure μ (if not implicitly understood; it's also called the *thermal average*) is denoted as

$$\langle f \rangle_\mu := \mathbb{E}_\mu f = \int_\Omega f d\mu.$$

One of the first questions in statistical mechanics is devoted to the choice of the right measure μ. The choice is governed by Shannon's entropy $S : \mathcal{M}(\Omega) \to \mathbb{R}$, defined as $S(\mu) := -\int_\Omega x \log x d\mu(x)$ (there's a way to understand why S has this form; see [9]). *The maximum entropy principle says*: for a given model of statistical mechanics, choose μ that maximizes S. For example, if there's no other information available about the system, then the measure that maximizes S is the uniform distribution. There are other two typical situations:

(a) If we know that $\langle \mathcal{H} \rangle = U$ for some fixed U (the *internal energy* of the system), then the measure that maximizes S is

$$\mu(\omega) = \frac{e^{-\beta H(\omega)}}{Z}, \quad Z := \sum_{\omega \in \Omega} e^{-\beta \mathcal{H}(\omega)}$$

which is the *Gibbs measure*. As the condition says, we either know the system is isolated, or we know the system exchanges its energy with the external environment but on average the internal energy is preserved. The result can be obtained using Lagrange multipliers; the parameter β, called the *inverse temperature*, is uniquely determined by U (and vice versa: U is uniquely determined by β). In the theory, $\beta = (kT)^{-1}$, where k is the Boltzmann constant and T is the temperature of the system.

(b) If we know additionally that the system exchanges its particles $\mathcal{N}$ with the external environment, and that the expected value of the particles if $\langle \mathcal{N} \rangle = N$, then we obtain a *grand canonical Gibbs distribution* via a similar procedure. It's given by

$$\mu(\omega) := \frac{e^{-\beta(\mathcal{H}(\omega) - \mu N)}}{Z}, \quad Z := \sum_N e^{\beta \mu N} \sum_{\omega \in \Omega} e^{-\beta \mathcal{H}(\omega)}.$$

The parameter μ is identified with the so-called *chemical potential*.

1.2 The thermodynamic limit

1.3 Quantum lattices

1.3.1 A general set-up

I will follow closely the treatment in [12]. Again, we have a lattice $X \subset \mathbb{Z}^n$, but to each site $i \in X$ we attach a copy of a finite-dimensional Hilbert space H_i. To a finite X we attach the tensor product $H_X := \otimes_{i \in X} H_i$.

For infinite lattices, the author of [12] suggests proceeding as follows. Let $\mathfrak{A}_X := \mathrm{End}(H_X)$, and for any two finite subsets $X \subseteq Y \subset \mathbb{Z}^n$, let $\imath : \mathfrak{A}_X \to \mathfrak{A}_Y$ be the inclusion that sends A to $A \otimes 1$ (where 1 is viewed as an endomorphism of $\mathfrak{A}_{Y \setminus X}$). For an infinite $\Lambda \subseteq \mathbb{Z}^n$, the family of all its finite subsets with the inclusions form a direct system. Let $\mathfrak{A}_\Lambda := \varinjlim \mathfrak{A}_X$ be the direct limit taken in the category of C^*-algebras over all finite subsets of Λ. In the literature, this algebra is known as an AF (approximately finite-dimensional) C^*-algebra. The first reference in this theory goes back to Bratteli [4]. See the next subsection for an elaboration on the inductive limit.

Further, for a finite $\Lambda \subset \mathbb{Z}^n$ and a Hamiltonian $\mathcal{H}_\Lambda$, the partition function is defined as

$$Z = \operatorname{tr}_\Lambda e^{-\beta \mathcal{H}_\Lambda}$$

and the expectation of an observable $A \in \mathfrak{A}_\Lambda$ is

$$\langle A \rangle_\Lambda := Z^{-1} \operatorname{tr}_\Lambda (A e^{-\mathcal{H}_\Lambda}).$$

The trace in these formulas is normalized: it's $1/d$ of the usual trace, where d is the dimension of the Hilbert space at one site. An interesting consequence of such normalization is that tr extends than to a norm-one linear functional on the whole $\mathfrak{A} := \mathfrak{A}_{\mathbb{Z}^n}$ (see [12]). The Hamiltonian they choose is given by

$$\mathcal{H}_\Lambda = \sum_{X \subseteq \Lambda} \Phi(X),$$

where Φ is a so-called *interaction*: it's a function from the non-empty finite subsets of $\mathbb{Z}^n$ to self-adjoint operators on them, such that $\Phi(X + i) = \Phi(X)$ for any $i \in \mathbb{Z}^n$ (i.e., it's translational invariant).

The pressure for a finite region Λ in the quantum lattice system is given by

$$P_\Lambda(\Phi) := |\Lambda|^{-1} \ln \operatorname{tr} e^{-H_\Lambda}.$$

One can show that the limit in the sense of van Hove of P_Λ does exist in the quantum setting as well ([12]).

1.3.2 The inductive limit in more detail

For two finite subsets $X \subseteq Y \subset \mathbb{Z}^n$, the inclusion $\imath : \mathfrak{A}_X \to \mathfrak{A}_Y$ that sends A to $A \otimes 1$ is injective; therefore, whenever $X \subseteq Y$, we can view $\mathfrak{A}_X$ as a subalgebra of $\mathfrak{A}_Y$. Hence we can take a union of all such subalgebras coming from finite subsets of $\Lambda \subseteq \mathbb{Z}^n$, and then, to be safe and ensure it's a Banach space, take the closure. So, one can identify (see a proposition below for a rigorous proof)

$$\mathfrak{A}_\Lambda = \varinjlim \mathfrak{A}_X = \operatorname{cl} \left(\bigcup_{X \subset \Lambda,\ |X| < \infty} \mathfrak{A}_X \right) /_{u \sim u \otimes 1}$$

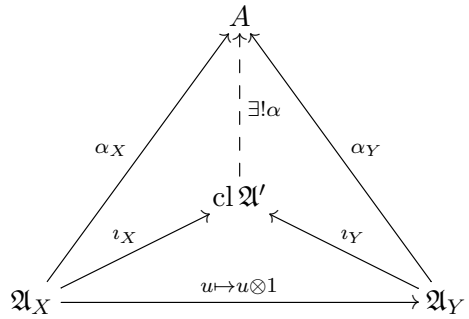
From this point of view, it's easy to understand what the norm is. For A from the dense subspace (the union itself), we just set $\|A\|_\Lambda := \|A\|_X$ if $A \in \mathfrak{A}_X$. The norm extends to the closure by the very process of completeness: for $A \in \mathfrak{A}_\Lambda$, we choose a sequence $A_n \in \mathfrak{A}_{X_n}$ such that $A_n \to A$, and then set $\|A\|_\Lambda := \lim_{n \to \infty} \|A_n\|$.

From Appendix on C^*-algebras, we see that Gelfand-Naymark theorem ensures there is a Hilbert space H such that $\mathfrak{A}_\Lambda \cong \operatorname{End}(H)$.

Further Search 1. I can elaborate on the construction of this Hilbert space. It's more or less constructive and relies on finding pure states. In particular, it would be interesting to see how this H is related to the infinite tensor product $\otimes_{i \in \Lambda} H_i$: what exactly goes wrong?

Proposition 1. In the above set-up, we indeed have $\varinjlim \mathfrak{A}_X = \operatorname{cl} \left(\bigcup_{X \subset \Lambda,\ |X| < \infty} \mathfrak{A}_X \right) /_{u \sim u \otimes 1}$ (isometrically and preserving the $*$-structure).

Proof. Denote $\mathfrak{A}' := \left(\bigcup_{X\subset\Lambda,\ |X|<\infty} \mathfrak{A}_X\right)/_{u\sim u\otimes 1}$. So, we choose morphisms in the category of unital C^*-algebras as bounded unital $*$-homomorphisms with norm less then or equal to one[1]. To prove the statement, all we need to show is that for a unital C^*-algebra A and a bunch of morphisms $\alpha_X : \mathfrak{A}_X \to \mathfrak{A}_Y$ where $X \subseteq Y$ and such that $\alpha_X(u) = \alpha_Y(u \otimes 1)$ (but remember that u is identified with $u \otimes 1$ in the union), there's a unique morphism $\alpha : \mathrm{cl}\,\mathfrak{A}' \to A$. In the language of diagrams, this is saying that



Once α is defined on $\mathfrak{A}'$ with all the mentioned properties, it automatically extends to the closure. So, for $u \in \mathfrak{A}_X$ we set $\alpha(u) := \alpha_X(u)$. This is well defined, for u is identified with $u\otimes 1$ in the union. We get automatically that α is a unital $*$-homorphism since all α_X's are. It's norm is bounded by 1, for $\|\alpha(u)\| \le \|\alpha_X\|\|u\| \le \|u\|$. Thus α is a morphism in the corresponding category. □

Thoughts Aloud 1. So, any element of the union itself can be interpreted as an observable on some finite lattice. But when we take the closure of the union, we get infinite sums of operators. The very procedure of the closure is not attached to the nature of the operators (i.e. than they can be fed with states) but only to the relation between operators themselves. Therefore, we need to define what is $\sum_{n=1}^{\infty} A_n(x)$ for $A_n \in \mathfrak{A}_{X_n}$. An attempt to fantasy is: for $x \in H_\Lambda$, let $\pi_n(x)$ be the projection of x to H_{X_n} (H_{X_n} can be regarded as a subspace of H_Λ). Then we try to set $(\sum A_n)(x) := \sum A_n(\pi_n(x))$. Then $\sum A_n$ is a bounded linear operator, for $\|\sum A_n(\pi_n(x))\| \le (\sum \|A_n\|)\|x\|$. Hmm, why don't we like it? And yet, what's the difference between this and the G-N theorem? Are there fewer operators than in $\mathrm{End}(H_\Lambda)$?

1.4 Relation between classical and quantum lattices

I follow [12] with some minor modifications more appealing to my taste. Let Ω_0 be a finite set of microstates at one site, and let H_0 be a Hilbert space of dimension equal to $|\Omega|$ (which is assigned to one site as well). Let $C(\Omega)$ be the space of observables on Ω. Choose an orthonormal basis e_μ of H_0 labeled my microstates $\mu \in \Omega_0$. Then we have an injection $\imath : C(\Omega_0) \to \mathrm{End}(H_0)$ given by

$$[\imath(f)](e_\mu) := e_{f(\mu)}.$$

In other words, the classical observables are embedded into the quantum observables as diagonal matrices.

1.5 Continuous spins: general principles

I follow closely Section 6.10 of [9]. In case the space of states Ω_0 at a single site is non-compact, the existence of Gibbs measures is no longer guaranteed. For Ω_0 a topological space, one defines the

[1] Otherwise I don't think there's a way to prove that the map induced on the diagram of the injective limit is a bounded operator

following ingredients. Let $\mathcal{B}_0$ be the Borel σ-algebra on Ω_0. For a finite lattice $\Lambda \subset \mathbb{Z}^n$, we supply the space of states with the σ-algebra $\mathcal{B}_\Lambda := \bigotimes_{i\in\Lambda} \mathcal{B}_0$. The natural projections $\pi_\Lambda : \Omega \to \Omega_\Lambda$ allow us to define a σ-algebra on Ω with base in Λ:

$$\sigma_\Lambda := \pi_\Lambda^{-1}(\mathcal{B}_\Lambda).$$

If $S \subseteq \mathbb{Z}^n$ is a possibly infinite lattice, then we supply it with the σ-algebra

$$\sigma_S := \sigma(\bigcup_{\Lambda\subset S,\ \Lambda \text{ finite}} \sigma_\Lambda)$$

(by the last equality I mean the smallest σ-algebra generated by the union).

1.6 Rigged Hilbert spaces

For this subject, I read Antoine's notes [2] (but they are more on lattices of Hilbert spaces, which is a further extension of rigged Hilbert spaces). I also read Section §4 from Chapter 1 of Gelfand's book [8]. It contains a more thorough treatment and results on generalized eigenvectors of self-adjoint operators.

Definition 1. A *rigged Hilbert space* is a triple $(\Phi, \mathcal{H}, \Phi^*)$ such that $\mathcal{H}$ is a Hilbert space, Φ is its dense subspace endowed with a topology stronger than the induced one from $\mathcal{H}$, and Φ^* is the space of continuous complex linear functionals on Φ.

One might also take the space $\Phi^\times$ of continuous complex antilinear functionals (which is done, e.g., in [2]). If we do so, then we can view $\mathcal{H}$ as a linear subspace of $\Phi^\times$.

Why? Here is why: as I understood from our conversations, rigged Hilbert spaces might be helfpul when we deal with unbounded operators. For instance, the Heisenberg principle suggests that either the position q or the momentum operator p is unbounded; if it happens that, say, q is unbounded, it may have empty point spectrum (which means that the model does not contain information about the position of particles). With a certain rigged Hilbert space $(\Phi, \mathcal{H}, \Phi^\times)$, we might extend q to $\Phi^\times$, where it will have a non-empty point spectrum. Also, when the Hamiltonian is unbounded but defined on Φ, then every state of Φ has "finite energy". Usually, Φ is obtained as the common invariant subspace of a family of distinguished observables. The only warning here is that sometimes, when observables are extended to $\Phi^\times$, their spectra might change. In some cases it's desirable, in some cases it's not, one needs to take this into account when employs the RHS formalism.

Question 1. Given any rigged Hilbert space $(\Phi, \mathcal{H}, \Phi^\times)$, is it true that $\mathcal{H}$ is dense in $\Phi^\times$? It is true in examples, but how to show for an abstract space? If $\imath : \Phi \to \mathcal{H}$ is the inclusion map, then we obtain a map $\imath^\times : \mathcal{H}^\times \to \Phi^\times$. Then $\mathcal{H}$ can be viewed as a linear subspace of $\Phi^\times$ via $h \mapsto \imath^\times \psi_h$, where $\psi_h(\varphi) = (h, \varphi)$ and $(,)$ is the Hermitian inner product on $\mathcal{H}$ antilinear in the first slot.

Definition 2. Let $(\Phi, \mathcal{H}, \Phi^*)$ be a rigged Hilbert space and $A : \Phi \to \mathcal{H}$ be a linear operator. An element $f \in \Phi^*$ is called a *generalized eigen-vector* of A if[2] $f(A\varphi) = \lambda f(\varphi)$ for all $\varphi \in \Phi$. In this case, λ is called a *generalized eigen-value* of A.

[2] I do not use A^*, for it goes from $\mathcal{H}^*$ to Φ^* and thus its domain may not include every functional from Φ^*. It's probably a subtle moment. If we demanded $A(\Phi) \subseteq \Phi$, then we would be able to switch to A^*, but this just doesn't cover all examples we want.

Example 1. One example of a rigged Hilbert space is the triple $(S, L^2(\mathbb{R}), S^*)$, where S is the Schwartz space and S^* is the space of distributions correspondingly. For a fixed $h \neq 0$, consider a unitary operator $U_h(f)(x) := f(x-h)$. Then U_h does not have eigen-vectors[3] in $L^2(\mathbb{R})$. However, if we view U_h as an operator on the space of distributions S^*, where S is the Schwartz space, then the distributions $x \mapsto e^{i\lambda x}$ are generalized eigen-vectors of U_h, for

$$(e^{i\lambda x}, U_h\varphi) = \int_{\mathbb{R}} e^{i\lambda x}\varphi(x-h)dx = e^{i\lambda h}\int_{\mathbb{R}} e^{i\lambda x}\varphi(x)dx = e^{i\lambda h}(e^{i\lambda x}, \varphi),$$

where $\varphi \in S$, and the pairing is the usual one between S and S^*.

Definition 3. (From Gelfand's book [8]) Let $A : \Phi \to \mathcal{H}$ be a linear operator and let Φ'_λ be the set of generalized eigen-vectors of A that correspond to a value λ. With an element $\varphi \in \Phi$ and the value λ, we associate a functional $\tilde{\varphi}_\lambda : \Phi'_\lambda \to \mathbb{C}$ such that $\tilde{\varphi}_\lambda(F_\lambda) = F_\lambda(\varphi)$. The map $\varphi \mapsto \tilde{\varphi}_\lambda$ is called the *spectral decomposition of φ with respect to the operator A*. We say that the set of generalized eigen-vectors of A is *complete* if the condition $\tilde{\varphi}_\lambda \equiv 0$ implies $\varphi = 0$.

Example 2. The set of generalized eigen-vectors[4] of U_h is complete, for the expression $F_\lambda(\varphi)$ is, up to a constant, the Fourie transform of φ. This is the consequence of Plancherel's theorem (or the fact that the Fourie transfrom is an isometry when viewed as an operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

A theorem of interest for us, drawn from [8]:

Theorem 1. *Any (possibly unbounded) self-adjoint operator has a complete set of generalized eigen-vectors with real eigen-values.*

2 The Main Stage

2.1 Statement of the problem and ideas

The current statement, I guess, is the following:

Statement. Consider $\mathbb{Z}^n$ where to each node we attach an infinite-dimensional separable Hilbert space H. Let $\Lambda \subseteq \mathbb{Z}^n$ be an infinite sublattice. Consider the limit

$$\mathfrak{A}_\Lambda := \varinjlim_{X \subset \Lambda,\ X \text{ finite}} \mathfrak{A}_X.$$

If it turns out that $\mathfrak{A}_\Lambda$ is a C^*-algebra, I'd like to do the following: find an ideal $J \triangleleft \mathfrak{A}_\Lambda$ such that the Hilbert space H promised by the Gelfand-Naymark theorem is separable; i.e., $\mathfrak{A}_\Lambda / J \cong \mathrm{End}(H)$ for H separable. It would be also nice to keep embeddings $\mathfrak{A}_X \to \mathfrak{A}_\Lambda$ for finite sublattices $X \subset \Lambda$.

Further Search 2. Given a unital C^*-algebra A, under which conditions on A the Hilbert space given by Gelfand-Naymark theorem is separable?

Idea. To keep everything physically meaningful, I think that J can be tried out as the ideal generated by operators with all but finitely many eigen-states concentrated in a finite sublattice of Λ. Here, I need to refresh my mind with regards to eigen-states.

[3] I do not understand Gelfand's argument. It relies on applying the classical Fourie transform, but there's no formula for it in $L^2(\mathbb{R})$, only in $L^1(\mathbb{R})$, so there still might be an eigenvector in $L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$. So, if we assume that $f \in L^1(\mathbb{R}) \subset L^2(\mathbb{R})$ is an eigen-vector of U_h, then $f(x-h) = \lambda f(x)$ for every $x \in \mathbb{R}$. But the Fourie transform yields $e^{-ihx}F(x) = \lambda F(x)$, so $F(x) \equiv 0$ (since it's for any h) and hence $f(x) \equiv 0$.

[4] I don't know why the functions $x \mapsto e^{i\lambda x}$ exhaust the genearlized eigen-vectors. Might think about it later.

2.2 Tests on quantum Ising model

In Ising model, to each node of $\mathbb{Z}^2$ we attach a 2-dimensional Hilbert space H with some a priori chosen orthonormal basis e_1, e_2. It corresponds to the states *spin up* and *spin down*.

2.2.1 The issue with the infinite tensor product

Let's consider first the algebraic tensor product H_∞ of all Hilbert spaces attached to all sites. It's spanned by simple tensors of the form

$$e_{\lambda(1)} \otimes e_{\lambda(2)} \otimes e_{\lambda(3)} \otimes \cdots$$

where $\lambda : \mathbb{N} \to \{e_1, e_2\}$ is a function. There are as my such simple tensors as functions λ; the cardinal number is equal to $|2^{\mathbb{N}}| = |\mathbb{R}|$, i.e., there are uncountably many of them[5]. It's natural to declare such simple tensors an orthonormal basis of H_∞. But then, it's a result of metric spaces theory that if there are uncountably many points such that the distance between any of two is bounded by a positive constant (that doesn't depend on the points), then the space is not separable. This is the case with H_∞. Since it's not separable, its completion $\operatorname{cl} H_\infty$ can't be separable as well. By the way, the same idea is used when one proves l_∞ is not separable.

Just out of curiosity, the same cardinal number occurs when all Hilbert spaces are infinite-dimensional but separable. In this case, we deal with all functions $\lambda : \mathbb{N} \to \mathbb{N}$; the cardinal number of them is again $|\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}|$.

2.2.2 The limit and G-F theorem

For an infinite sublattice $\Lambda \subseteq \mathbb{Z}^n$, the injective limit $\varinjlim_{X \subset \Lambda,\ X \text{ finite}} \mathfrak{A}_X$ can be thought of as the completion of the union of those subalgebras.

The following example explains how the union works.

Example 3. The algebra $\mathfrak{A}_1$ attached to a single site can be identified with the algebra of 2×2 matrices over $\mathbb{C}$. For two nodes, the algebra $\mathfrak{A}_2$ can be identified with matrices of dimension 4×4. The embedding $\mathfrak{A}_1 \to \mathfrak{A}_2$ then does the following:

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto A \otimes 1 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Therefore, for an inifnite sublattice Λ, the algebra $\mathfrak{A}_\Lambda$ can be identified with the space of matrices of infinite size such that only finitely many entries of each of them are non-zero. *Note* the difference with the attempt to take an infinite tensor product of Hilbert spaces: these infinite matrices naturally act upon the space

$$T := \bigoplus_{n=1}^{\infty} \bigotimes_{i \in A \subset \Lambda,\ |A|=n} H_i$$

It's easy to see that this space has an infinite countable basis. This implies that, whatever norm we put on T, the space will not be complete (that's a standard result from functional analysis: in a Banach space, a vector space basis is at least uncountable).

[5] More details on why it's uncountable. Any number $a \in \mathbb{R}$ can be represented as a power series $a = \sum_{i=-\infty}^{\infty} c_i 2^i$, where $c_i \in \{0, 1\}$ and only finitely many c_i for $i > 0$ might be non-zero. Restrict ourselves to $a = \sum_{i=1}^{\infty} c_i 2^{-i}$. Then we have a 1-1 correspondence between functions $\lambda : \mathbb{N} \to \{0, 1\}$ and such numbers.

Way 1 (just a fantasy). Let $\mathfrak{A}'_\Lambda$ be the union of all $\mathfrak{A}_X$ for $X \subset \Lambda$ and X finite. We can substitute the norm on $\mathfrak{A}'_\Lambda$ with the Hilbert-Schmidt norm (see appendix), and then complete $\mathfrak{A}'_\Lambda$ with respect to it. The elements A of the resulting space can be represented as infinite matrices $(a_{nk})_{k,n=1}^{\infty}$ such that $\sum_{n,k} |a_{nk}|^2 < \infty$. We can act with these on a completion of T. However, it's not a C^*-algebra; it is a Banach algebra though. There might be something in this approach.

Way 2. We can complete with respect to the operator norm. I can't prove this, but the evidence is that we obtain the space of compact operators on $\mathrm{cl}\, T$. That's very good. If our Hamiltonian is normalized in such a way that in the limit it gives a bounded operator, then we can employ the Hilbert-Schmidt theorem and find an orthonormal basis in $\mathrm{cl}\, T$ of eigen-values of the limiting Hamiltonian.

2.3 Thoughts on C^*-algebras approach

So the idea was the following: since it's not sometimes clear what a limiting Hilbert space should look like, we can take the limit of the corresponding algebras of observables and then, by Gelfand-Naymark theorem, find an underlying Hilbert space, hopefully a separable one. But the dream will not come true:

Statement. Even in quantum Ising model, the Gelfand-Naymark representation from the proof of the theorem (see Appendix) yields a non-separable Hilbert space when corresponds to an infinite lattice.

Evidence. Let's have a more careful look at how the representation is constructed. Let A be a unital C^*-algebra, let's say. If A is taken as the C^*-algebra corresponding to an infinite lattice in quantum Ising model, then A is an AF-algebra (approximately finite). In particular, it is separable and infinite-dimensional. Now, to construct the representation, for every non-zero $z \in A$ we pick a representation π_z such that $\|\pi_z(z)\xi_z\| = \|z\|$, where ξ_z is the cyclic vector of π_z, and then we take the direct sum of those. Clearly, the sum is uncountable, for as a set the algebra A is uncountable, so a basis of the resulting Hilbert space cannot be countable. □

So, the proof of Gelfand-Naymark theorem, even though more or less constructive, does not yield a way to construct a separable Hilbert space. The algorithm might be polished, I guess. Which $z \in A$ we might restrict to? Which are sufficient? The problem is that the choice of z's is *set-theoretic*, it's not *functionally-analytic*.

The C^*-algebras approach also has a downside that we lose unbounded observables. For example, in free fermions on infinite chain, if we don't normalize the Hamiltonian, the limit results in an unbounded operator, so the limiting C^*-algebra doesn't capture this. We've tried to use different normalizations, but this yielded either the zero operator or the identity, something trivial. One might tweak the eigen-values so that their absolute values are less than 1; but then, what's the meaning of the limiting observable?

To sum up, the downside: the limiting C^*-algebra loses both separability of the Hilbert space and does not contain unbounded observables like the total energy of the lattice.

I'd like to mention Segal's article [15]. From what I understood from his article, it's not necessary to find a faithful representation of the whole limiting C^*-algebra. There's the following result, *the upside of the C^*-algebras approach*, which is a corollary of a more general statement that can be found in the article:

Statement. Let A be a C^*-algebra and $u \in A$ be self-adjoint.Then for any α from the spectrum of u, there exists an irreducible representation ϕ of A and a non-zero element x of the space on which A is represented such that $\phi(u)x = \alpha x$.

The result is great in a sense that even the continuous spectrum of a self-adjoint element can be realized (at least partly) as a point spectrum. This reminds of the rigged Hilbert space approach that also gives a way to realize the continuum spectrum as a point one (through generalized eigenvalues).

Further Search 3. Segal in [15] mentions that it's actually not an issue that unbounded operators don't land in the limiting C^*-algebras, for they can be treated in terms of bounded operators. What did he mean?

3 Classical lattice models

3.1 Ising model

3.1.1 A general description of the IRF version

There are two versions of the Ising model: the IRF (interaction-round-a-face) model and the vertex model. In the first one, the energy is assigned to vertices; in the second one, the energy is assigned to the bonds between the sites.

Let $\Lambda \subseteq \mathbb{Z}^n$ be a subset of the integer lattice of dimension n. We associate with the lattice the space of microstates $\Omega_\Lambda := \{-1, +1\}^\Lambda$. Therefore, to each node $i \in \Lambda$ there corresponds a *spin* $\omega_i = \pm 1$. For a finite Λ, the hamiltonian of the model is given by

$$\mathcal{H} = \sum_{i,j \in \Lambda,\ i \sim j} \omega_i \omega_j - h \sum_{i \in \Lambda} \omega_i,$$

where $h \in \mathbb{R}$ is some real number that corresponds to the external magnetic field, and $i \sim j$ means the nodes i and j are neighbors on the lattice. We also supply the model with the Gibbs measure defined previously.

3.1.2 Transfer matrices in IRF model (not finished)

To describe the transfer matrices, I restrict myself to a finite cubic lattice $\Lambda \subset \mathbb{Z}^2$ with periodic boundary conditions. Then we can assign energy to each face of the lattice:

$$\epsilon(\text{face}, \omega) := \sum_{i,j \in \text{face},\ i \sim j} \omega_i \omega_j - h \sum_{i \in \text{face}} \omega_i.$$

So the Hamiltonian breaks up into the sum of energies over all faces in Λ:

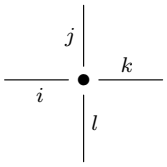
$$H(\omega) = \sum_{F \in \{\text{faces of } \Lambda\}} \epsilon(F, \omega).$$

A *Boltzmann weight* is the quantity $R(F, \omega) := \exp(-\beta \epsilon(F, \omega))$ assigned to a face F. The partition function can be rewritten as

$$Z = \sum_{\omega \in \Omega} \prod_{F \in \text{faces}} R(F, \omega).$$

3.1.3 The vertex model and its transfer matrix

I follow closely [5]. Let Λ be an $n \times m$ cubic lattice in $\mathbb{Z}^2$ with periodic boundary conditions. The states are assigned to the bonds between vertices rather than to the vertices themselves in this model. Let $\Omega_0 = \{1, \dots, n\}$ be the set of possible states of a single bond. For a picture of kind



let ε_{ij}^{kl} denote the energy assigned to the site in this setting. We assume that it doesn't depend on the position of the site but only on the states of the bonds around the sit. The Hamiltonian $\mathcal{H}$ of this model for a particular choice of the state of the lattice is then the sum of ε_{ij}^{kl} over all vertices. The partition function is given by $Z = \sum_{\omega \in \Omega} \exp(-\beta H(\omega))$. A *Boltzmann weight* is the quantity

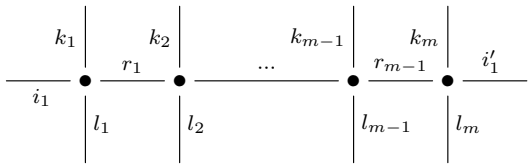
$$R_{ij}^{kl} := \exp(-\beta \varepsilon_{ij}^{kl}).$$

Proposition 2. Let V be an m-dimensional vector space. There exists an endomorphism $T \in \mathrm{End}(V \otimes V^m)$, which is called a *transfer matrix*, such that the partition function of the model is given by

$$Z = \mathrm{tr}_{V^{\otimes m}}(\mathrm{tr}_V T)^n$$

where the trace is the usual one (the sum of diagonal elements).

Proof. Consider a row in the cubic lattice, for a moment assuming that the boundary conditions on the ends (the states i_1 and i_1') may not be the same



Let us fix the end states i_1, i_1', $k_1, \dots, k_m$ and $l_1, \dots, l_m$. The contribution to Z when only r_i's are running over Ω_0 is given by

$$T_{i_1 k_1 \cdots k_m}^{i_1' l_1 \cdots l_m} := \sum_{r_1, \dots, r_{m-1}} R_{i_1 k_1}^{r_1 l_1} \cdots R_{r_{m-1} k_m}^{i_1' l_m}.$$

Let V be an m-dimensional vector space spanned by some $e_1, \dots, e_m$. Define an endomorphism $T \in \mathrm{End}(V \otimes V^{\otimes m})$ by setting on the basis elements

$$T(e_{i_1} \otimes e_{k_1} \otimes \cdots \otimes e_{k_m}) = \sum_{i_1', l_1, \dots, l_m} T_{i_1 k_1 \cdots k_m}^{i_1' l_1 \cdots l_m} e_{i_1'} \otimes e_{l_1} \otimes \cdots \otimes e_{l_m}.$$

If wee unfreeze the endpoints with states i_1 and i_1' and let them run over Ω_0, then we see that the contribution to Z of the whole row (with still fixed states on the vertical bonds and now $i_1 = i_1'$) is

given by $\mathrm{tr}_V(T)^{l_1\dots l_m}_{k_1\dots k_m}$. Now, if the row was the first one and we consider the next one to it, and let $l_1,\dots,l_m$ run over Ω_0, then the contribution to Z is

$$\sum_{l_1,\dots,l_m} \mathrm{tr}_V(T)^{l_1\dots l_m}_{k_1\dots k_m}\,\mathrm{tr}_V(T)^{j_1\dots j_m}_{l_1\dots l_m} = [(\mathrm{tr}_V(T))^2]^{j_1\dots j_m}_{k_1\dots k_m}$$

(the last equality was not obvious to me due to a mess with indices, but it can be checked easily). Continuing in this fashion, the contribution to Z with fixed states of the vertical bonds on the ends is given by $[(\mathrm{tr}_V(T))^n]^{l_1\dots l_m}_{k_1\dots k_m}$. Now, applying the periodic condition $k_j = l_j$ and summing over all possible states of the ends, we finally find that $Z = \mathrm{tr}_{V^{\otimes m}}[\mathrm{tr}_V(T)]^n$. □

I think I can say that a transfer matrix is just a batch of all possible microstates of a row ingeniously packed into a linear endomorphism.

3.2 Gaussian free field model

I follow closely Chapter 8 from [9]. In Gaussian free field model, the space of states at a single site is chosen to be $\Omega_0 := \mathbb{R}$. Accordingly, the space of states on a region $\Lambda \subseteq \mathbb{Z}^n$ is given by $\Omega_\Lambda := \mathbb{R}^\Lambda$. The Hamiltonian of the model is on the lattice Λ is chosen to be

$$\mathcal{H} := \frac{\beta}{4n} \sum_{i\sim j,\ \{i,j\}\cap\Lambda\neq\emptyset} (\omega_i - \omega_j)^2 + \frac{m^2}{2}\sum_{i\in\mathbb{Z}^n}\omega_i^2,$$

where β is the inverse temperature, $\omega_i \in \Omega_0$ is the assigned spin at site $i \in \mathbb{Z}^n$, and m is the mass.

A couple of comments on the choice of the Hamiltonian:

1) The factor $(\omega_i - \omega_j)^2$ tells us that the interaction favors the agreement of neighboring spins;

2) Since the space of states at single site is non-compact, we penalize large values of spin by adding the factor $m^2/2 \cdot \omega_i^2$ for each one;

3) Notice the condition under the first summation. It tells us that we also take into the account the boundary of Λ (there might be different boundary conditions though).

Fix a finite lattice $\Lambda \subset \mathbb{Z}^n$ and a state $\eta \in \Omega$ (it serves as a boundary condition for Λ). For a state $\omega_\Lambda \in \Omega_\Lambda$, by $\mathcal{H}(\omega_\Lambda)$ we mean that we plug into the Hamiltionian the state that equals ω_Λ on Λ and η on the complement of Λ.

In Subsection 1.5 we specified a way of choosing σ-algebras on the spaces of states. Let $\sigma_{\mathbb{Z}^n}$ be such σ-algebra on the whole $\mathbb{Z}^n$. For $A \in \sigma_{\mathbb{Z}^n}$, the Gibbs measure in this model is defined as

$$\mu(A) := \int_A \frac{e^{-\mathcal{H}(\omega_\Lambda)}}{Z}\prod_{i\in\Lambda} d\omega_i,$$

where $d\omega_i$ is the Lebesgue measure on $\mathbb{R}$ assigned to the site $i \in \mathbb{Z}^n$ and Z is the obviously chosen partition function.

There's a way to define Gibbs measures for infinite Λ as well (explained in [9], I postpone its description here for a moment). The case of massless GFF is drastically different from the case of massive GFF. For instance, Theorem 8.19 in [9] says that there are no infinite-volume Gibbs measures in $n = 1$ and $n = 2$ cases. Nevertheless, Theorem 8.21 in the same reference tells us that there are infintely many infinite-volume Gibbs measures when $n \geq 3$. In the massive case, the GFF model has infinitely many infinite-volume Gibbs measures for any n (see Theorem 8.28 in [9]).

3.3 O(N)-symmetric model

I follow Chapter 9 from [9]. In O(N)-model, we take $\Omega_0 := S^{N-1}$, so the spins might have an arbitrary direction. For a finite lattice $\Lambda \subseteq \mathbb{Z}^n$, the Hamiltonian (in the absence of a magnetic field) is usually written as

$$\mathcal{H} = -\beta \sum_{i\sim j,\ \{i,j\}\cap\Lambda\neq\emptyset} \langle \omega_i, \omega_j \rangle ,$$

where $\omega_i \in \Omega_0$ is a spin at site i, and the brackets denote the standard inner product in $\mathbb{R}^N$. For different N's, we obtain some familiar models: for $N = 1$ we have the Ising model; for $N = 2$ we get the XY-model; and for $N = 3$ we obtain the Heisenberg model.

The definition of finite-volume Gibbs measures is similar to the case of GFF model. At each site i, we have Lebesgue measure $d\omega_i$ on S^{N-1}. We fix a boundary condition, which is the choice of a state $\eta \in \Omega$, and then for measurable sets A we set

$$\mu(A) := \int_A \frac{e^{-\mathcal{H}(\omega_\Lambda)}}{Z} \prod_{i\in\Lambda} d\omega_i,$$

where Z is the obvious partition function and $\omega_\Lambda \in \Omega_\Lambda$; by $\mathcal{H}(\omega_\Lambda)$ I mean that we plug in a state equal to ω_Λ on Λ and η outside of Λ.

One might be interested in the following questions with regards to O(N)-models:

1) Is there an orientational long-range order? In my understanding, the mathematical formalism of this question is whether the correlations $\mathbb{E}_\mu \langle \omega_i, \omega_j \rangle$ converge to zero as $\|i - j\| \to \infty$;

2) Is there a spontaneous magnetization? The formalism in my understanding is: for any infinite-volume Gibbs measure μ, is it true that $\lim_{n\to\infty} \left\langle \|m_{B(n)}\| \right\rangle_\mu \neq 0$? Here $B(n)$ is a cube of size n and $m_{B(n)} := \frac{1}{|B(n)|} \sum_{i\in B(n)} \omega_i$ is the *magnetization density*.

The answers to both questions are negative for $N \geq 2$ and $n = 1, 2$. This is due to the following theorem, which can be also stated for a more general Hamiltonian:

Theorem 2. *(Mermin-Wagner) For $N \geq 2$ and $n = 1, 2$, all infinite-volume Gibbs measures are invariant under the action of the rotation group.*

Maybe, I will write why the answers are negative a bit later.

4 Free Bosons

4.1 Truncated Free Bosons

We have here one site with Hilbert space $\mathcal{H}_M$ of dimension M, and basis $\{|0\rangle, |1\rangle, \cdots, |M-1\rangle\}$. We should here say $|0\rangle_M$, $|1\rangle_M$, etc, but will omit the extra subindex for clarity. We define creation $a^\dagger$ and destruction operators[6] a, (they should really be a_M and $a_M^\dagger$ but will omit the subindex) such that $a|0\rangle = 0$ (the zero of the Hilbert space), $a|n\rangle = \sqrt{n}|n-1\rangle$ for all $0 < n < M$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ for all $0 \leq n < M - 1$, $a^\dagger|M-1\rangle = 0$. Let $H := H_M := M^{-1}a^\dagger a$ be the "Hamiltonian" operator for free bosons. Then H is self-adjoint, every element of the basis is an eigenvector of H, and with eigenvalues[7] $\{0, 1/M, 2/M, \cdots, (M-1)/M\}$.

[6] Have checked that they are indeed adjoint to each other even in the truncated case.

[7] Indeed, $H|0\rangle = 0$ and for $M - 1 \geq m > 0$ we have $H|m\rangle = \frac{\sqrt{m}\sqrt{m}}{M}$.

Consider the the algebra $\mathcal{U}_M$ generated by a and $a^\dagger$ and the identity I. H belongs to this algebra. It's a C^*-algebra (even a von Neumann algebra): we included in the list of generators all conjugates and we supplied the algebra with the operator norm, which always satisfies the C^*-identity. It's a von Neumann algebra because any finite-dimensional C^*-algebra is a von Neumann algebra.

Regarding the dimension, it's clear to me that the algebra is finite-dimensional and that the dimension can be bounded from below by $3M-2$. I haven't figured out yet the exact dimension.

4.2 Infinite Free Bosons

We have here one site with separable Hilbert space $\mathcal{H}_\infty$ of infinite dimension, and basis $\{|0\rangle, |1\rangle, \cdots\}$; we define creation $a^\dagger$ and destruction operators a, such that $a|0\rangle = 0$, $a|n\rangle = \sqrt{n}|n-1\rangle$ for all $0 < n$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ for all $0 \leq n$. Notice that both a and $a^\dagger$ are unbounded but their domains are dense in $\mathcal{H}_\infty$.

Statement. The approach with energy density does not work. Precisely, the following is true: for every $M \geq 1$, extend $H_M = \frac{1}{M} a_M^\dagger a_M$ to act on $\mathcal{H}_\infty$ by setting it equal to zero on the orthogonal complement of $\mathcal{H}_M \subset \mathcal{H}_\infty$. Then there is a well defined limit in the strong topology: $H_\infty|\psi\rangle = \lim_{M\to\infty} H_M|\psi\rangle$. However, $H_\infty = 0$. It might make sense if we divide not by the dimension but by the size of the lattice.

Proof. Indeed, for any basis ket $|k\rangle$, we have $\lim_{M\to} H_M|k\rangle = \lim_{k\to\infty}(k/M)|k\rangle = 0$, so the sequence H_M converges strongly to zero. □

Statement. Redefine H_M as $H_M := a_M^\dagger a_M$ and extend by zero onto the orthogonal complement of $\mathcal{H}_M$, so $H_M|k\rangle = k|k\rangle$ for $k \leq M-1$. Then H_M converges strongly to some unbounded self-adjoint (hence closed) operator H_∞ on a dense subspace of $\mathcal{H}_\infty$.

The fact that H_∞ is unbounded means that it's not regarded as an observable in the language of C^*-algebras. This point of view was adapted by Segal and is also mentioned in the C^*-algebras approach in [10].

Proof. The domain of H_∞ would be the dense subspace

$$\operatorname{dom} H_\infty = \{|\psi\rangle = \sum_{k=0}^{\infty} \psi_k|k\rangle \in \mathcal{H}_\infty \mid \sum_{k=0}^{\infty} k|\psi_k| < \infty\}.$$

So, for any $|\psi\rangle$ such that $\sum_{k=0}^{\infty} k|\psi_k| < \infty$, we can safely define[8]

$$H_\infty|\psi\rangle := \lim_{M\to\infty} H_M|\psi\rangle = \sum_{k=0}^{\infty} k\psi_k|k\rangle.$$

The operator is clearly unbounded, for $\|H_\infty|k\rangle\| \to \infty$.

Now let's find the domain of $H_\infty^\dagger$. By definition of the adjoint of an unbounded operator, $|\phi\rangle \in \operatorname{dom}(H_\infty^\dagger)$ if and only if there exists $|\theta\rangle \in \mathcal{H}_\infty$ such that for every $|\psi\rangle \in \operatorname{dom}(H_\infty)$ we have $\langle\phi|H_\infty|\psi\rangle = \langle\theta|\psi\rangle$. In components, this equality means that $\phi_k^*\psi_k k = \theta_k^*\psi_k$, hence $|\phi\rangle$ must reside in $\operatorname{dom}(H_\infty)$. Since H_∞ is obviously symmetric and $\operatorname{dom}(H_\infty) = \operatorname{dom}(H_\infty^\dagger)$, it is self-adjoint. From the general theory of unbounded operators we know that the adjoint is always closed, hence any $H_\infty = H_\infty^\dagger$ is closed. □

[8] It's a general fact from the theory of Banach spaces that a series $\sum_{k=1}^{\infty} x_k$ converges iff the series $\sum_{k=1}^{\infty} \|x_k\|$ converges.

The limit (in operator norm, weak, strong?) of the sequence of operators H_M exits (?), is unique, and let's call it H_∞. Then H_∞ is self adjoint (is it really?) and its spectrum is $\sigma(H_\infty) = [0, 1)$ (does it include 1?). Moreover, every $1/n$ for $n >= 1$ is an eigenvalue of H_∞, and the corresponding eigenvectors form a basis of $\mathcal{H}_\infty$.

For $P \geq 1$, let v_P, w_P be both vectors in $\mathcal{H}_P$, and consider their extension to any $M \geq P$ by the same name. then $lim_{M\to\infty}\langle v_P|H_M|w_P\rangle$ exists, and is equal to $\langle v_P|H_\infty|w_P\rangle$, where v_P, w_P are considered vectors of $\mathcal{H}_\infty$.

Consider the algebra $\mathcal{U}_\infty$ generated by a and $a^\dagger$ and the identity I. Does H_∞ belong to this algebra? What's the dimension of this algebra? Is this a C* algebra? Is it a von Neumann algebra? Etc. Is this algebra a limit in some sense from the sequence of algebras $\mathcal{U}_M$.

5 Free Fermions

5.1 Finite Chain

Consider one site with Hilbert space of dimension two, and basis $\{|0\rangle, |1\rangle\}$; physically we say that the state $|0\rangle$ is empty, and the state $|1\rangle$ occupied; we define creation $c^\dagger$ and destruction operators c, such that $c|0\rangle = 0$, $c|1\rangle = |0\rangle$, $c^\dagger|0\rangle = |1\rangle$, $c^\dagger|1\rangle = 0$. For N sites, we extend the definitions by making N tensor products, so that the total dimension is 2^N. (Even though we use the word *site* or *sites*, we use here momentum space for simplicity, so that the H below will be already in "diagonal form.") Given a $N-$tensor product state $|v\rangle \equiv |v_0\rangle \otimes |v_1\rangle \otimes \cdots \otimes |v_{N-1}\rangle$, the v_m are either 0 or 1. Let the parity up to m operator P_m be the diagonal operator such that $P_m|v\rangle = p(|v\rangle, m)|v\rangle$, where $p(|v\rangle, m)$ is the number 1 or -1, and is given by

$$p(|v\rangle, m) \equiv (-1)^{\sum_{0\leq m'<m} v_{m'}}. \tag{5.1}$$

In other words, $P_m|v\rangle$ is $|v\rangle$ if the number of $|1\rangle$ strictly preceding mth position is even; $-|v\rangle$ if odd.

We write d_m to mean $I \otimes I \otimes \cdots \otimes c \otimes I \otimes \cdots \otimes I$, where c is at location m, and I is the one site identity. We write $c_m(N) \equiv d_m P_m$, where P_m is the parity operator up to m, as defined before. Note that $c_m(N)$ acts on the full space $\{|0\rangle, |1\rangle\}^{\otimes N}$. *I drop now the* (N) *from the* c_m.

Example 4. Let $|w\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |0\rangle \cdots$, and remember that we count locations from 0. Then $c_2|w\rangle = -|0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots$, with a -1 because the sum of 1s before location 2 is odd. On the other hand $c_3^\dagger|w\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle \otimes \cdots$, with a $+1$ because the sum of 1s before location 3 is even. Obviously $c_2^\dagger|w\rangle = c_3|w\rangle = 0$ as we can't destroy if there's no particle, and we can't create if there's already a particle, because these are fermions and accept only one particle per location.

We write $c_m^\dagger c_m$ to mean the tensor product operator with $c^\dagger$ and c at location m, identities elsewhere, *and with appropriate parities,* so it's $d_m^\dagger d_m$ (the parity operators commute with d_m's and cancel each other out). Let

$$H_N = N^{-1} \sum_{m=0}^{m=N-1} e_m c_m^\dagger c_m \tag{5.2}$$

be the "Hamiltonian" operator for free fermions, where $e_m = -2\cos(2\pi m/N)$ for $0 \leq m < N$ integers; this is called the *dispersion* relation for free periodic fermions. Then H_N is a self-adjoint (symmetric) matrix of rank 2^N. Note that $c^\dagger c|0\rangle = 0$ and $c^\dagger c|1\rangle = |1\rangle$.

5.1.1 Eigenvalues and Eigenvectors of the Hamiltonian

All this is just matrix diagonalization, but because of the form of the 2^N matrix H, we can find its eigenvalues and eigenvectors in a compact way, that we know how to describe.

Let $S = \{f : \{0, 1, \cdots, N-1\} \to \{0, 1\}\}$, that can be thought of as binary numbers with N digits. There are 2^N functions f in S and for each, the number $N^{-1} \sum_{m=0}^{m<N} f(m) e_m$ is an eigenvalue of H_N with eigenvector[9]

$$\bigotimes_{m \in I(f)} c_m^\dagger |0\rangle,$$

where $I(f) = \{m \in \{0, 1, \cdots, N-1\}; f(m) = 1\}$, and $|0\rangle = |0\rangle \otimes |0\rangle \cdots |0\rangle$ the fully "empty" state. In particular, the lowest eigenvector of H_N is

$$\bigotimes_{m \in I_{min}} c_m^\dagger |0\rangle,$$

with eigenvalue $N^{-1} \sum_{m \in I_{min}} e_m$, where $I_{min} = \{m; 0 \le m \le N/4\} \cup \{m; 3N/4 \le m < N\}$.

Exercise 1. Express the largest eigenvector and eigenvalue to make sure you understand the construction.

Proof. The largest eigenvalue is determined by the set $I_{max} := \{m;\ \frac{N}{4} \le m \le \frac{3N}{4}\}$. I have understood the construction and worked out the details. □

5.1.2 Fermionic Density

Let $D_N = N^{-1} \sum_{m=0}^{m<N} c_m^\dagger c_m$ be the density operator. Then D_N is self-adjoint, diagonal, and with eigenvalues[10] $\{0, 1/N, 2/N, \cdots, 1\}$. Moreover, D_N and H_N commute (physically, H_N "conserves" the particle density or the number of particles), and the eigenvector of H_N characterized by $f \in S$, is also an eigenvector of D_N with eigenvalue equal to the number of elements of $I(f)$ divided by N.

Consider the algebra $\mathcal{U}_M$ generated by c_m (with parity and in the 2^N space) and $c_m^\dagger$ and the identity I. The operators H_N and D_N belong to this algebra. Since the involution preserves the list of generators, it's also a finite-dimensional C^*-algebra. Because of its dimension, it's also a von Neumann algebra. What is its dimension?

5.2 Infinite Chain

(A possible way, a guess of mine) Define a set

$$\mathcal{B}_\infty := \{|\psi_f\rangle = |f(0)\rangle \otimes |f(1)\rangle \otimes \cdots \mid f : \mathbb{N} \to \{0, 1\},\ f(i) = 0 \text{ for all but finitely many } i\}.$$

For the infinite chain, define the space of states via

$$\mathcal{H}_\infty := \mathrm{cl}\,\mathrm{span}\,\mathcal{B}_\infty$$

where by cl I mean the completion. This space is obviously separable. The elements from $\mathcal{B}_\infty$ are declared to constitute an orthonormal basis. The basic kets, therefore, describe a state when some finite number of fermions occupies some sites. Define also subspaces

$$\mathcal{H}_M := \mathrm{span}\{|\psi_f\rangle \in \mathcal{B}_\infty \mid f(k + M) = 0 \text{ for all } k \ge 0\}.$$

[9] Checked on a sheet of paper.

[10] The eigenvectors are given by all possible functions $f : \{0, \ldots, N-1\} \to \{|0\rangle, |1\rangle\}$; i.e., all elements $f(0) \otimes f(1) \otimes \cdots \otimes f(N-1)$ are eigenvectors.

Note that

$$\mathcal{H}_M^{\perp} = \text{span}\{|\psi_f\rangle \in \mathcal{B}_\infty \mid f(k+M) = 1 \text{ for some } k \geq 0\}.$$

For every $M \geq 1$ we extend H_M to act on $\mathcal{H}_\infty$ by setting it equal to zero on $\mathcal{H}_M^{\perp}$.

Statement. When H_M are defined with $1/N$ factor, they converge strongly to zero.

A comment on the statement. It's interesting that we get zero no matter whether we divide by the dimension of the space of states (the case of bosons) or by the size of the chain. Here, the reason is that I prohibited infinitely many fermions to occupy the chain. I think, first, it's not physically relevant, and second, this makes $\mathcal{H}_\infty$ non-separable. But if we allow infinitely many fermions, the limiting operator $\mathcal{H}_\infty$ is non-trivial.

Proof. Indeed, choose a basic ket $|\psi_f\rangle \in \mathcal{B}_\infty$ and let $N := \max\{i \mid f(i) = 1\}$. According to the finite chain case, we have

$$\lim_{M\to\infty} H_M|\psi_f\rangle = \lim_{M\to\infty,\ M\geq N} \frac{1}{M} \sum_{m=0}^{m<N} f(m)(-2\cos(\frac{2\pi m}{M})) = 0.$$

Thus $H_\infty = 0$. Note that the sum goes up to N, but inside of cosines we have $1/M$. □

Statement. Define H_M without the factor of $1/M$. Then H_M converge strongly to an unbounded self-adjoint operator H_∞ defined via

$$H_\infty|\psi_f\rangle := -|I(f)||\psi_f\rangle,$$

where by $|I(f)|$ I mean the number of elements in $I(f)$.

Proof. We see that

$$\lim_{M\to\infty} H_M|\psi_f\rangle = \lim_{M\to\infty,\ M\geq N} \sum_{m=0}^{m<N} f(m)(-2\cos\frac{2\pi m}{M})|\psi_f\rangle = -|I(f)||\psi_f\rangle,$$

so $H_M|\psi_f\rangle$ is diagonal, though unbounded. We've already encountered an unbounded diagonal operator in the case of infinite free bosons, and we've shown it's self-adjoint[11] □

The last statement actually suggests what a factor we might pick for H_M's to obtain an observable in the limit. We see that we can take $1/|I(f)|$:

Statement. Now try H_M's via the formula

$$H_M|\psi_f\rangle = -\frac{1}{|I(f)|}\left(\sum_{m=0}^{m<M} f(m)e_m\right)|\psi_f\rangle.$$

Then they converge strongly to the identity operator on $\mathcal{H}_\infty$.

Statement. The density operators D_N converge strongly to zero. If we redefine the density operators as $D_N := \sum_{m=0}^{m<N} c_m^\dagger c_m$, then D_N converge strongly to the operator defined as $D_\infty|\psi_f\rangle = |I_f||\psi_f\rangle$ (which is unbounded, as shown before). Therefore, if we define D_N's as $D_N|\psi_f\rangle := \frac{1}{|I_f|}\sum_{m=0}^{m<N} c_m^\dagger c_m$, then they converge to the identity operator.

[11] I mention it carefully since in the theory of unbounded operators not everything that seems symmetric (swings freely in the slots of the inner product) is self-adjoint. The latter condition is stronger.

Statement. Consider the algebras $\mathcal{U}_M$ spanned by $c_0, \ldots, c_{M-1}$. Their direct limit $\mathcal{U}_\infty = \lim_{M\to\infty} \mathcal{U}_M$ is the closure of their union in the operator norm. Now, the question was: do H_∞ and D_∞ belong to $\mathcal{U}_\infty$? The unbounded versions of H_∞ and D_∞ certainly do not. But if we choose a proper normalization (for instance, divide by $|I_f|$), then they do belong, for they are equal to the identity.

Statement. Let H_N be defined with a factor of $1/N$. Let $\lambda_{\min}(N)$ and $\lambda_{\max}(N)$ the respectively the lowest and the highest eigen-values of H_N. Then

$$\lim_{N\to\infty} \lambda_{\min}(N) = \frac{-2}{\pi}, \quad \lim_{N\to\infty} \lambda_{\max}(N) = \frac{2}{\pi}.$$

Proof. Indeed, recall that $I_{\min} = \{m; 0 \le m \le N/4\} \cup \{m; 3N/4 \le m < N\}$ and $I_{\max}$ is its complement. For $\lambda_{\min}$, we see that

$$\begin{aligned}
\lim_{N\to\infty} \lambda_{\min}(N) &= -2 \lim_{N\to\infty} \left(\sum_{m=0}^{m\le N/4} \cos(2\pi m/N)\frac{1}{N} + \sum_{m\ge 3N/4}^{m<N-1} \cos(2\pi m/N)\frac{1}{N} \right) = \\
&= -4 \lim_{N\to\infty} \left(\sum_{m=0}^{m\le N/4} \cos(2\pi m/N)\frac{1}{N} \right) = \frac{-4}{2\pi} \lim_{N\to\infty} \sum_{m=0}^{m\le N/4} \cos(x_m)\Delta x_m = \\
&= \frac{-2}{\pi} \int_0^{\pi/2} \cos(x)\, dx = \frac{-2}{\pi},
\end{aligned}$$

where $x_m = (2\pi m)/N$. A similar argument applies for $\lambda_{\max}$. □

The gap in this model is clearly $O(N^{-1})$, for we need to add or exclude just one fermion.

Idea. (A wrong way but which might be a starting point). For every H_N with a factor of $1/N$, let's order its eigenvalues so that

$$\lambda_{\min}(N) = \lambda_1(N) \le \lambda_2(N) \le \cdots \le \lambda_{2^N}(N) = \lambda_{\max}(N).$$

Define

$$\lambda_i(\infty) := \lim_{N\to\infty} \lambda_i(N).$$

It seems that these values could constitute a countable collection of eigen-values for a dreamed H_∞. However, the gap is zero:

$$\lambda_{i+1}(\infty) - \lambda_i(\infty) = \lim_{N\to\infty} (\lambda_{i+1}(N) - \lambda_i(N)) = \lim_{N\to\infty} \frac{-2}{N} \cos(\frac{2\pi m_{i,N}}{N}) = 0,$$

for two neighboring eigenvalues would differ by no more than one fermion.

Idea. Since the previous idea doesn't work, let's do the following. Let

$$\mathcal{F} := \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ is non-decreasing}\}.$$

For every $f \in \mathcal{F}$, define

$$\lambda_f := \lim_{N\to\infty} \lambda_{f(N)}(N).$$

How many of such λ_f's do we actually have? We've seen that a lot of them might coincide because the order of gaps is $O(N^{-1})$. Being devoid of such knowledge, we proceed further:

$$\mathcal{F}_\infty := \frac{\mathcal{F}}{f \sim g \text{ iff } \lambda_f = \lambda_g}.$$

Let $\mathcal{H}_\infty$ be a vector space spanned freely by elements of $\mathcal{F}_\infty$. Define a hermitian inner product on $\mathcal{H}_\infty$ by declaring all elements of $\mathcal{F}_\infty$ to be orthonormal. Now, complete $\mathcal{H}_\infty$. We are ready to define the limiting Hamiltonian:

$$H_\infty[f] := \lambda_f[f], \quad [f] \in \mathcal{H}_\infty.$$

The operator H_∞ is self-adjoint, for it's diagonal with real eigen-values. It's also bounded, for the eigen-values are less than 1. Indeed, if the series $\sum_{[f]\in\mathcal{F}_\infty} a_f[f]$ converges (as a net), then the series $\sum_{[f]\in\mathcal{F}_\infty} \lambda_f a_f[f]$ converges as well, for

$$\sum_{[f]\in\mathcal{F}_\infty} |\lambda_f a_f| \leq \sum_{[f]\in\mathcal{F}_\infty} |a_f|.$$

The questions to answer rigorously are the following:

1. Is $\mathcal{H}_\infty$ separable? I expect that the answer is no;
2. What's the best way to represent elements of the basis? The elements of the basis do not determine a particular arrangement of fermions but rather a collection of possible arrangements with equal energy density. Is there a way to embed the spaces $\mathcal{H}_N$ into $\mathcal{H}_\infty$?

(What was written before) The limit (in operator norm, weak, strong?) of the sequence of operators H_M exits (?), is unique, and let's call it H_∞. Then H_∞ is self adjoint (is it really?) and its spectrum is $\sigma(H_\infty) = [-2/\pi, 2/\pi]$.

What are the eigenvalues (if any) and eigenvectors of H_∞?

Let D_∞ be the corresponding limit of the D_N operators. Show it's self-adjoint with spectrum $\sigma(D_\infty) = [0, 1]$. What are the eigenvalues (if any) and eigenvectors of D_∞?

Consider the algebra $\mathcal{U}_\infty$ generated by c_i and $c_i^\dagger$ and the identity I with some completion. How does one "complete" or "close" this algebra? Does H_∞ belong to this algebra? What about D_∞? What's the dimension of this algebra? Is this a C* algebra? Is it a von Neumann algebra? Etc. Is this algebra a limit in some sense from the sequence of algebras $\mathcal{U}_N$.

6 Pure Gauge in 1D

TBW

7 Pure Gauge in 2D

TBW

8 Gauge and Matter in 1D

TBW

9 Antiferromagnetic spin-$\frac{1}{2}$ chain

I follow [1]. He considers the following Hamiltonian[12]:

$$H = \sum_{i=1}^{L} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z,$$

[12]Being an antiferromagnetic material means that the spins are aligned in opposite directions so that there's no resulting global magnetization.

where, I guess, $S_i = (S_i^x, S_i^y, S_i^z)^t$ forms the 1/2-spin operator. According to [13], $S = \frac{1}{2}\sigma$, where σ is a vector with matrix components, called *Pauli matrices*, which are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Also, $S_{L+1} = S_1$ and L is even. Just for myself, so the Pauli matrices live in U(2) (not SU(2)) and their determinants are equal to -1.

We know the following:

1. If $\Delta = 1$, then there's a unique ground state with a gap of $O(L^{-1})$. The author presents a rigorous proof based on Perron-Frobenius theorem, which I, of course, do not understand;
2. For $-1 < \Delta \leq 1$ there's a unique ground state and no gap (but this had been known non-rigorously). The author proves rigorously that for $\Delta > -1$, there is either no gap or no unique ground state (but he can't distinguish between these two cases).
3. If $\Delta \leq -1$, the system is ferromagnetic.

There's also a description of expected behavior at higher spins.

Further Search 4. As I understand, different spins are realized as different representations of SU(2). The trivial representation corresponds to the zero spin; the 1/2-spin corresponds to the identity representation. I've been looking at Woit's book [16] and at unpublished notes by Noah Miller [14]. The spin itself (as a number) is realized as a certain eigenvalue of some special matrix.

10 The meaning of Gauge

The Quantum Ising model has a *global* symmetry $\tilde{S}_i = -S_i$, that leaves the form of the Hamiltonian invariant in the new variables, and so it's the same system. By *global* I mean independent of i. The Quantum Ising Gauge model (see Hunter L.'s write up) has a *local* symmetry.

11 Gauge and Matter in 1D

11.1 Definitions of the full space

We consider again a 1D chain, put bosons of dimension n on the links, with group $Z(n)$ and define E, A, V and U as before. We won't use A and V in the Hamiltonian. This time we add fermions on the sites, where at site l, the operator $\psi_\ell^\dagger$ creates a fermion, and the operator ψ_ℓ destroys a fermion at that site. The fermion operators *anti-commute* on *different* sites. Then let

$$H = -\frac{n}{2\pi}\sum_\ell (\psi_\ell^\dagger U_{\ell,\ell+1}\psi_{\ell+1} + H.c.) + 2m\sqrt{\frac{n}{2\pi}}\sum_\ell (-1)^\ell \psi_\ell^\dagger \psi_\ell + \sum_\ell E_{\ell,\ell+1}^2, \tag{11.1}$$ {eq:1dQEDLat

where n is an odd non-negative integer, and m is a real number. We use the following version of the chain with a free end on the left that will determine the states of all the other links under the Gauss law:

$$\overset{-1}{\text{———}}\bigcirc^0 \overset{0}{\text{———}}\bigcirc^1 \overset{1}{\text{———}} \cdots \overset{N-2}{\text{———}}\bigcirc^{N-1} \tag{11.2}$$ {eq:1chain}

The full space on N sites has dimension $(2n)^N$, the 2^N comes from the fermions (electrons and positrons), and the n^N from the bosons (phonons). The actual physical space we work with is specified by the kernel of the Gauss law (see below).

A basis element at site l is denoted as $|n_l, b_l\rangle$, where $n_l \in \{0,1\}$ indicates the presence of a fermion at site l and $b_l \in \{0,\dots,n-1\}$ indicates the state of the boson to the right of the fermion. Such basis is called *computational*.

Recall how the operators act. Let $|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle$ be an arbitrary basic state. Then

(a)

$$\begin{aligned}&\psi_\ell^\dagger|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle = \\ &= (-1)^{n_0+n_1+\dots+n_{\ell-1}}(1-n_\ell)|n_0\, n_1\, \dots\, n_\ell = 1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle;\end{aligned}$$

(b)

$$\begin{aligned}&\psi_\ell|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle = \\ &= (-1)^{n_0+n_1+\dots+n_{\ell-1}}n_\ell|n_0\, n_1\, \dots\, n_\ell = 0\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle;\end{aligned}$$

(c)

$$\psi_\ell^\dagger\psi_\ell|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle = n_\ell|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle;$$

(d) For $n = 2s+1$ odd,

$$E_{\ell,\ell+1}|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle = (b_\ell - s)|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle;$$

$$E_{\ell,\ell+1}^2|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle = (b_\ell - s)^2|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle;$$

(e)

$$\begin{aligned}&U_{\ell,\ell+1}|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle = \\ &\quad =|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_\ell + 1\, \dots\, b_{N-2}\rangle\end{aligned}$$

(f)

$$\begin{aligned}&\psi_\ell^\dagger U_{\ell,+1}\psi_{\ell+1}|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle = \\ &= n_{\ell+1}(1-n_\ell)|n_0\, n_1\, \dots\, n_\ell = 1\, n_{\ell+1} = 0\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_\ell + 1\, \cdots\, b_{N-2}\rangle.\end{aligned} \tag{11.3}$$ {eq:ham_fir_

So, when we have a fermion at site $l+1$ and no fermion at site l, the operator $\psi_l^\dagger U_{l,l+1}\psi_{l+1}$ shifts the fermion to site l and increments the state of the boson on the corresponding lift. Similarly, its Hermitian conjugate acts as

$$\begin{aligned}&\psi_{\ell+1}^\dagger U_{\ell,\ell+1}^\dagger\psi_\ell|n_0\, n_1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_{N-2}\rangle = \\ &= n_\ell(1-n_{\ell+1})|n_0\, n_1\, \dots\, n_\ell = 0\, n_{\ell+1} = 1\, \dots\, n_{N-1}\, b_{-1}\, b_0\, \dots\, b_\ell - 1\, \cdots\, b_{N-2}\rangle.\end{aligned}$$

From the letter: in QED fermions (matter) occupy even sites, and anti-fermions (anti-matter) occupy odd sites. The same for QCD, except that fermions in QED are electrons, and fermions in QCD are quarks.

11.2 Gauss Law and Physical Space

Let the Gauss law be

$$G_\ell := \psi_\ell^\dagger \psi_\ell + \frac{1}{2}((-1)^\ell - 1) - (E_{\ell,\ell+1} - E_{\ell-1,\ell}). \tag{11.4}$$ {eq:1dQED_Ga

We restrict the full space to the physical space of states $|\psi\rangle$ that satisfy $G_\ell|\psi\rangle = 0 \pmod n$. The physical space has dimension $n2^N$; see Hunter L's explanation for why this is the case.

I wanted to expand a bit on the Gauss law. So, all pieces that comprise G_ℓ are diagonal, hence G_ℓ is diagonal itself and its kernel is spanned by some of the basic kets. If $|n_0\, n_1\, \ldots\, n_{N-1}\, b_{-1}\, b_0\, \ldots\, b_{N-2}\rangle$ is such, then the action of G_ℓ upon it is given by

$$G_\ell|n_0\, n_1\, \ldots\, n_{N-1}\, b_{-1}\, b_0\, \ldots\, b_{N-2}\rangle =$$
$$= (n_\ell + \frac{1}{2}((-1)^\ell - 1) + b_{\ell-1} - b_\ell)|n_0\, n_1\, \ldots\, n_{N-1}\, b_{-1}\, b_0\, \ldots\, b_{N-2}\rangle,$$

We see that b_ℓ is determined by $b_{\ell-1}$ via

$$b_\ell = b_{\ell-1} + n_\ell + \frac{1}{2}((-1)^\ell - 1).$$

So the physical space is

$$\mathrm{span}(|n_0\, n_1\, \ldots\, n_{N-1}\, b_{-1}\, b_0\, \ldots\, b_{N-2}\rangle \mid b_\ell = b_{\ell-1} + n_\ell + \frac{1}{2}((-1)^\ell - 1) \text{ for every } \ell).$$

Recursively solving the equation for b_ℓ, we find that

$$b_\ell = b_{-1} + n_0 + n_1 + \cdots + n_\ell - \left[\frac{\ell+1}{2}\right],$$

where [] denotes the integer part.

Proposition 3. Each term in the Hamiltonian defined as in (11.1) on the chain (11.2) with a free left link leaves the physical space invariant.

Proof. The first term that shifts a fermion to the left acts according to the formula 11.3. If $n_\ell = 0$ and $n_{\ell+1} = 1$, then the term acts non-trivially and the state on ℓth link is updated according to

$$b_\ell + 1 = b_{-1} + n_0 + n_1 + \cdots + (n_\ell + 1) - \left[\frac{\ell+1}{2}\right],$$

so we simply add 1 to both sides. For $k \geq 1$, the equation of states on the other links are also preserved:

$$b_{\ell+k} = b_{-1} + n_0 + \cdots + (n_\ell - 1) + (n_{\ell+1} + 1) + \cdots + n_{\ell+k} - \left[\frac{\ell+1}{2}\right].$$

Similarly for Hermitian conjugate term (now subtract 1 from both sides in the equation for b_ℓ). The other terms in the Hamiltonian are diagonal, so they obviously preserve the physical space. Thus the Hamiltonian itself does preserves the physical space. □

11.3 Limits

Ercolessi et al [7] numerically take both limits *first* the $N \to \infty$ limit and then the $n \to \infty$ limit and find a "phase transition" at some value of $\tilde{m}$. Order of the limits here FIXME TODO. For now, perhaps is enough to take $N \to \infty$ for n finite, and prove that the limiting (topological, C*, Von Neuman??) algebra exists, and the limiting H exits. To define a phase transition I need an "order parameter" which will be something like a magnetization in a spin model. Let $W = N^{-1} \sum_l E_{l,l+1}$ be the "Wilson loop" operator. Let $|gs\rangle$ be the ground state of H. Let $w_1(\tilde{m}, n, N) = \langle gs|W|gs\rangle$, and let $w(\tilde{m}, n) = \lim_{N\to\infty} w_1(\tilde{m}, n, N)$. This quantity w as a function of $\tilde{m}$ will have a first derivative everywhere, except for a value of $\tilde{m}$, where the left and right derivatives both exist but are different. We say that the model goes through a phase transition at that value of $\tilde{m}$. Obviously, that value will be dependent on n, and we are interested in $n \to \infty$.

11.3.1 Staggered configurations

We define two *staggered* states: $|\,\mathrm{st}_1(b_{-1})\rangle$ and $|\,\mathrm{st}_2(b_{-1})\rangle$ in the following way: for the first one, all even sites are occupied and odd are empty; for the second one, all even sites are empty and all odd sites are occupied. The states on the links are determined by b_{-1}, which we include in the definition.

Statement. For $|\,\mathrm{st}_2(b_{-1})\rangle$, $b_\ell = b_{-1}$; for $|\,\mathrm{st}_1(b_{-1})\rangle$, $b_\ell = \begin{cases} b_{-1}+1, & \ell = 2k; \\ b_{-1}, & \ell = 2k+1. \end{cases}$. Pictorially,

$$|\,\mathrm{st}_2(b_{-1})\rangle = \quad \overset{b_{-1}}{\text{———}} \bigcirc \overset{b_{-1}}{\text{———}} \bigotimes \overset{b_{-1}}{\text{———}} \cdots \overset{b_{-1}}{\text{———}} ?$$

$$|\,\mathrm{st}_1(b_{-1})\rangle = \quad \overset{b_{-1}}{\text{———}} \bigotimes \overset{b_{-1}+1}{\text{———}} \bigcirc \overset{b_{-1}}{\text{———}} \cdots \overset{b_{-1}+?}{\text{———}} ?$$

where $\otimes$ means that the site is occupied. This is a simple computation using aforementioned formulas derived from the Gauss law.

11.3.2 Wilson loop for $m \gg 0$

Split the Hamiltonian as $H = A + 2m\sqrt{n/2\pi}D$, where A is the sum of the first term and the third terms of H and $D = \sum_l (-1)^l \psi_l^\dagger \psi_l$.

Proposition 4. No matter what b_{-1} is, we have

$$\lim_{m\to\infty} m^{-1} \langle \mathrm{gs}\,|H|\,\mathrm{gs}\rangle = \lim_{m\to\infty} m^{-1} \langle \mathrm{st}_2\,|H|\,\mathrm{st}_2\rangle = 2\sqrt{\frac{n}{2\pi}} \langle \mathrm{st}_2\,|D|\,\mathrm{st}_2\rangle = -2\left[\frac{N}{2}\right]\sqrt{\frac{n}{2\pi}}.$$

As a consequence, if λ_1 is the eigen-value corresponding to $|\,\mathrm{gs}\rangle$, then $\lambda_1 = O(m)$. More precisely,

$$\lim_{m\to\infty} \frac{\lambda_1}{m} = -2\left[\frac{N}{2}\right]\sqrt{\frac{n}{2\pi}}.$$

This implies that for $m \gg 0$ and N even (or large), the lowest energy level can be approximated as

$$\lambda_1(m) \approx -Nm\sqrt{\frac{n}{2\pi}}.$$

Proof. Notice that the eigen-vectors of H and $m^{-1}H$ are the same (and their natural order is preserved). From the definition of gs_2, we see that

$$m^{-1}\langle \mathrm{st}_2 |H| \mathrm{st}_2\rangle \geq m^{-1}\langle \mathrm{gs} |H| \mathrm{gs}\rangle = \min_{\|\psi\|=1} \langle\psi|H|\psi\rangle \geq m^{-1} \min_{\|\psi\|=1} A\psi + 2\sqrt{\frac{n}{2\pi}}\langle \mathrm{st}_2 |D| \mathrm{st}_2\rangle$$

Note that

$$\lim_{m\to\infty} m^{-1}\langle \mathrm{st}_2 |H| \mathrm{st}_2\rangle = 2\sqrt{\frac{n}{2\pi}}\langle \mathrm{st}_2 |D| \mathrm{st}_2\rangle$$

Hence, taking the limit in the above inequality, we obtain the statement. With regards to λ_1, we write $\langle \mathrm{gs} |H| \mathrm{gs}\rangle = \lambda_1\langle \mathrm{gs} | \mathrm{gs}\rangle = \lambda_1$. □

Lemma 1. Assume that $\lambda_1 \neq -\langle \mathrm{st}_2(s)| \mathrm{gs}\rangle^{-2}$. Then any ground state admits a decomposition

$$|gs\rangle = \alpha| \mathrm{st}_2(s)\rangle + |v\rangle$$

where $|v\rangle$ is orthogonal to $| \mathrm{st}_2(b_{-1})\rangle$ for any $b_{-1} \in \mathbb{Z}_n$ and s is such that $n = 2s + 1$. In other words, the only staggered configuration (of second type) that might be present in $| \mathrm{gs}\rangle$ corresponds to $b_{-1} = s$.

There is no need to think about alleviating the mild condition $\lambda_1 \neq -\langle \mathrm{st}_2(s)| \mathrm{gs}\rangle^{-2}$, for this is vacuous when the mass is large, and the lemma itself is needed only to prove that $\lim_{m\to\infty} | \mathrm{gs}\rangle = | \mathrm{st}_2(s)\rangle$.

Proof. Only for this proof, write H as $H = U+D+E$, where U, D and E correspond to respectively the first, the second and the third terms of the Hamiltonian (so, D is as before except that I absorbed the constant into it). Pick a ground state and decompose it as

$$| \mathrm{gs}\rangle = \sum_{i\in\mathbb{Z}_n} \alpha_i| \mathrm{st}_2(i)\rangle + |v\rangle,$$

where $|v\rangle \perp | \mathrm{st}_2(i)\rangle$ for all $i \in \mathbb{Z}_n$. Note that H is not only Hermitian but also symmetric, so we can assume α_i's are real[13]. Define additionally a state $|\phi\rangle$ such that

$$|\phi\rangle = \sum_{i\in\mathbb{Z}_n} \beta_i| \mathrm{st}_2(i)\rangle + |v\rangle,\ \beta_i \in \mathbb{R},\ \sum_{i\in\mathbb{Z}_n} \beta_i^2 = \sum_{i\in\mathbb{Z}_n} \alpha_i^2 =: \alpha^2.$$

So, $\langle\phi|\phi\rangle = 1$. The idea is to find a nice objective function to formulate a minimization problem for β_i's on the $(n-1)$-dimensional sphere of radius $|\alpha|$, whose solution has to be the ground state $| \mathrm{gs}\rangle$. Notice that U sends $| \mathrm{st}_2(j)\rangle$ to a vector that is ortogonal to all $| \mathrm{st}_2(i)\rangle$. This implies that

$$\langle \mathrm{st}_2(j)|H| \mathrm{st}_2(i)\rangle = \langle \mathrm{st}_2(j)|D + E| \mathrm{st}_2(i)\rangle = \delta_{ij}(-[N/2] + (j-s)^2(N-1)) =: \delta_{ij}c_j \qquad (11.5)$$ {eq:sthst}

where δ_{ij} is the Kronecker symbol. Next, since $| \mathrm{gs}\rangle$ is an eigen-vector,

$$H| \mathrm{gs}\rangle = \lambda_1| \mathrm{gs}\rangle = \sum_{i\in\mathbb{Z}_n} \alpha_i H| \mathrm{st}_2(i)\rangle + H|v\rangle$$

[13] In more detail, the ground state can be decomposed into its real and imaginary parts: $| \mathrm{gs}\rangle = \Re| \mathrm{gs}\rangle + i\Im| \mathrm{gs}\rangle$. Then, with an appropriate scaling, both parts are also ground states, but now with real components. Proving the result for the real parts, we prove the result for $| \mathrm{gs}\rangle$ as well (but then α might be complex).

hence

$$H|v\rangle = \sum_{i\in\mathbb{Z}_n} (\lambda_1\alpha_i|\,\mathrm{st}_2(i)\rangle - \alpha_i H|\,\mathrm{st}_2(i)\rangle) + \lambda_1|v\rangle;$$

Combining this with (11.5), we see that

$$\langle \mathrm{st}_2(j)|H|v\rangle = \alpha_j(\lambda_1 - (j-s)^2(N-1) + [N/2]) = \alpha_j(\lambda_1 - c_j).$$

Consider $\langle \mathrm{st}_2(j)|H|\phi\rangle$.

$$\begin{aligned}\langle \mathrm{st}_2(j)|H|\phi\rangle &= \sum_{i\in\mathbb{Z}_n} \beta_i\langle \mathrm{st}_2(j)|H|\,\mathrm{st}_2(i)\rangle + \langle \mathrm{st}_2(j)|H|v\rangle = \\ &= \beta_j\langle \mathrm{st}_2(j)|D|\,\mathrm{st}_2(j)\rangle + \beta_j\langle \mathrm{st}_2(j)|E|\,\mathrm{st}_2(j)\rangle + \langle \mathrm{st}_2(j)|H|v\rangle = \\ &= \beta_j(-[N/2] + (j-s)^2(N-1)) + \alpha_j(\lambda_1 - (j-s)^2(N-1) + [N/2]) = \\ &= \beta_j c_j + \alpha_j(\lambda_1 - c_j),\end{aligned}$$

For $\langle v|H|\phi\rangle$ we have

$$\begin{aligned}\langle v|H|\phi\rangle &= \sum_{i\in\mathbb{Z}_n} \beta_i\langle v|H|\,\mathrm{st}_2(i)\rangle + \langle v|H|v\rangle = \\ &= \beta_j\alpha_j(\lambda_1 - c_j) + \langle v|H|v\rangle.\end{aligned}$$

Now we can evaluate $\langle\phi|H|\phi\rangle$:

$$\langle\phi|H|\phi\rangle = \sum_{j\in\mathbb{Z}_n} \big(\beta_j^2 c_j + 2\beta_j\alpha_j(\lambda_1 - c_j)\big) + n\langle v|H|v\rangle$$

We are ready to define an objective function $f : S^{n-1} \to \mathbb{R}$,

$$f(\vec{\beta}) := f(\beta_0, \ldots, \beta_{n-1}) := \sum_{j\in\mathbb{Z}_n} (\beta_j^2 c_j + 2\beta_j\alpha_j(\lambda_1 - c_j)), \quad \sum_{j\in\mathbb{Z}_n} \beta_j^2 = \alpha^2.$$

Let's look at the chart given by $\beta_s := \sqrt{\alpha^2 - \sum_{j\neq s}\beta_j^2}$. Notice that $c_s = (s-s)^2 = 0$. In this chart, the coordinate representation of f is

$$f(\vec{\beta}) = \sum_{j\neq s}(\beta_j^2 c_j + 2\beta_j\alpha_j(\lambda_1 - c_j)) + 2\frac{\alpha_s}{\sqrt{\alpha^2 - \sum_{j\neq s}\beta_j^2}}$$

The partial derivative with respect to β_j for $j \neq s$ is

$$\frac{\partial f}{\partial \beta_j} = 2\beta_j c_j + 2\alpha_j(\lambda_1 - c_j) + 2\frac{\beta_j\alpha_s}{\beta_s^3};$$

Now the trick. By the very definition of $|\,\mathrm{gs}\rangle$, the collection $\beta_j := \alpha_j$ has to minimize f. Therefore, the partial derivatives of f with respect to β_j's (for $j \neq s$) in this chart at $\beta_j = \alpha_j$ are zero:

$$2\alpha_j c_j + 2\alpha_j(\lambda_1 - c_j) + 2\frac{\alpha_j\alpha_s}{\alpha_s^3} = 0.$$

One of the solutions of the above equation is $\alpha_j = 0$. If $\lambda_1 = -\alpha_s^{-2}$, there might be something else, but we assumed $\lambda_1 \neq -\alpha_s^{-2}$ from the beginning. So, since $\alpha_j = 0$ $(j \neq s)$ is the unique solution, it's the one that corresponds to the decomposition of $|\,\mathrm{gs}\rangle$. Thus $|\,\mathrm{gs}\rangle = \alpha_s|\,\mathrm{st}_2(s)\rangle + |v\rangle$. □

Caution. Without saying a few words beforehand, we can't talk about $\lim_{m\to\infty} |\,\mathrm{gs}\rangle = |\,\mathrm{st}_2(s)\rangle$, for this limit depends on the way we choose $|\,\mathrm{gs}\rangle$ for every value of mass. We can be whimsical and choose $|\,\mathrm{gs}\rangle = \alpha|\,\mathrm{st}_2\rangle + |v\rangle$ with $\arg\alpha$ (the argument of the complex number) running around $[0, 2\pi)$. Then the limiting vector is not unique. A limiting value might be $|\,\mathrm{st}_2(s)\rangle$, but it also might be something like $e^{i\pi/3}|\,\mathrm{st}_2(s)\rangle$. One natural way to avoid this is to assume $\langle \mathrm{gs}\,|\,\mathrm{st}_2(s)\rangle > 0$, i.e. we normalize the argument of the component of gs corresponding to the configuration $|\,\mathrm{st}_2(s)\rangle$.

Proposition 5. Assume that the choice of the ground state for every mass is made in such a way that $\langle \mathrm{gs}\,|\,\mathrm{st}_2(s)\rangle > 0$. Then the limit exists and is given by $\lim_{m\to\infty} |\,\mathrm{gs}\rangle = |\,\mathrm{st}_2(s)\rangle$.

Proof. First, let's show that for any basic ket $|\psi\rangle$ that is not equal to $|\,\mathrm{st}_2(b_{-1})\rangle$ (for any b_{-1}), $\lim_{m\to\infty}\langle\psi|\,\mathrm{gs}\rangle = 0$. Indeed, we see that

$$\langle\psi|\,\mathrm{gs}\rangle = \frac{1}{\lambda_1}\langle\psi|H|\,\mathrm{gs}\rangle = \frac{1}{\lambda_1}\langle\psi|A|\,\mathrm{gs}\rangle + \frac{2m}{\lambda_1}\sqrt{\frac{n}{2\pi}}\langle\psi|D|\,\mathrm{gs}\rangle = \frac{1}{\lambda_1}\langle\psi|A|\,\mathrm{gs}\rangle + \frac{2m}{\lambda_1}\sqrt{\frac{n}{2\pi}}\cdot a\langle\psi|\,\mathrm{gs}\rangle$$

where a is an eigen-value of D (which is not equal to $-[N/2]$ since $\psi \neq |\,\mathrm{st}_2(b_{-1})\rangle$). The above can be rewritten as

$$\langle\psi|\,\mathrm{gs}\rangle = \frac{1}{\lambda_1}\frac{1}{1-\frac{2ma}{\lambda_1}\sqrt{\frac{n}{2\pi}}}\langle\psi|A|gs\rangle.$$

We see that

$$1 - \frac{2ma}{\lambda_1}\sqrt{\frac{n}{2\pi}} \to 1 + a[N/2] \neq 0$$

therefore

$$\lim_{m\to\infty}\langle\psi|\,\mathrm{gs}\rangle = 0.$$

By Lemma 1, $|\,\mathrm{gs}\rangle$ admits a decomposition $|\,\mathrm{gs}\rangle = \alpha|\,\mathrm{st}_2(s)\rangle + |v\rangle$ (α might be complex in general). Clearly, for $\psi = |\,\mathrm{st}_2(b_{-1})\rangle$ with b_{-1}, the value $\langle\psi|\,\mathrm{gs}_2\rangle$ tends to zero as well. Therefore,

$$|\alpha|^2 = |\langle \mathrm{st}_2(s)|\,\mathrm{gs}\rangle|^2 \to |\langle \mathrm{gs}\,|\,\mathrm{gs}\rangle|^2 = 1.$$

Now the assumptiong $\alpha > 0$ comes into play. The only possible value is then $\alpha = 1$. Thus the limit exists and is equal to $|\,\mathrm{st}_2(s)\rangle$. □

Corollary 5.1. Let $\Sigma = N^{-1}\sum\langle E_{l,l+1}\rangle$ be the electric field operator. Then[14] $\lim_{m\to\infty}\Sigma = 0$.

Proof. Indeed, we can assume that $\lim_{m\to\infty}|\,\mathrm{gs}\rangle = |\,\mathrm{st}_2(s)\rangle$. Then $\langle \mathrm{gs}\,|E_{l,l+1}|\,\mathrm{gs}\rangle \to \langle \mathrm{st}_2(s)|E_{l,l+1}|\,\mathrm{st}_2(s)\rangle = 0$. □

Corollary 5.2. Let Σ be again an electric field operator. If the mass m is large, then Σ is bounded for *any* N and n. In particular, for large m there's a sequence $N_k \to \infty$ and $n_k \to \infty$ such that Σ converges to a number.

Proof. This follows right from the definition of the limit. For $\varepsilon = 1$, there is m_0 such that for all $m \geq m_0$ we have $|\Sigma| < 1$. So, Σ is bounded by 1 for all N, n and $m \geq m_0$. A compactness argument then gives us subsequences $N_k \to \infty$ and $n_k \to \infty$ such that $\lim_{k\to\infty}\Sigma$ is a number. □

We do not specify the order of the limits in the previous proposition. Whatever order we choose (or whatever fancy subsequences we choose, may be N depending on n) – there is still a subsequence for which Σ does converge.

[14] No matter how we choose the ground states for every value of mass since we're under the Hermitian inner product.

11.4 Idea: an equivalent model without links

Consider QED 1+1 with the Hamiltonian specified in Equation 11.1, which reads

$$H = -\frac{n}{2\pi}\sum_{\ell}(\psi_\ell^\dagger U_{\ell,\ell+1}\psi_{\ell+1} + H.c.) + 2m\sqrt{\frac{n}{2\pi}}\sum_{\ell}(-1)^\ell\psi_\ell^\dagger\psi_\ell + \sum_{\ell}E_{\ell,\ell+1}^2$$

Specifying the state of a boson on one chosen link assigns the states of bosons to all the other links. This suggests that we can consider a family of Hamiltonians parametrized by the state of the marked boson. View the chain as before:

$$\overset{-1}{\text{———}}\bigcirc^0\overset{0}{\text{———}}\bigcirc^1\overset{1}{\text{———}}\cdots\overset{N-2}{\text{———}}\bigcirc^{N-1}$$

Let $q \in \mathbb{Z}_n$ be a state on the leftmost link. Recall that the physical space (determined by the Gauss law) is given by

$$P = \text{span}(|n_0\, n_1\, \ldots\, n_{N-1}\, b_{-1}\, b_0\, \ldots\, b_{N-2}\rangle \mid b_\ell = b_{\ell-1} + n_\ell + \frac{1}{2}((-1)^\ell - 1) \text{ for every } \ell).$$

We can split P into the direct sum according to what value of b_{-1} is:

$$P = P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1},$$

where

$$P_j = \text{span}(|n_0\, n_1\, \ldots\, n_{N-1}\, b_{-1}\, b_0\, \ldots\, b_{N-2}\rangle \in P \mid b_{-1} = j).$$

Now, let's observe what is the restriction of H to each subspace P_j. Denote $H_j := H|_{P_j}$.

Idea. Does there exist a nice basis in each P_j that allows us to write the action of H_j as if there were no links?

11.4.1 Order of basis

I show on an example what the order I choose. For $N = 2$ and $n = 3$, writing a basic ket as $|n_0\, n_1\, b_{-1}\, b_0\rangle$, we have

$$|0\,0\,0\,2\rangle \prec |0\,0\,1\,0\rangle \prec |0\,0\,2\,1\rangle \prec\prec |1\,0\,0\,0\rangle \prec |1\,0\,1\,1\rangle \prec |1\,0\,2\,2\rangle \prec$$

$$|0\,1\,0\,2\rangle \prec |0\,1\,1\,0\rangle \prec |0\,1\,2\,1\rangle \prec |1\,1\,0\,0\rangle \prec |1\,1\,1\,1\rangle \prec |1\,1\,2\,2\rangle.$$

So the rule is: first, split the basis into groups $F(n_0, \ldots, n_{N-1})$, each of which depends only on the states of the sites. We declare that

$$F(n_0, \ldots, n_{N-1}) \prec F(\tilde{n}_0, \ldots, \tilde{n}_{N-1})$$

if and only if, as a binary number,

$$\underline{n_{N-1}\cdots n_0} \leq \underline{\tilde{n}_{N-1}\cdots\tilde{n}_0}.$$

Inside of each group, we order the kets by b_{-1} (the other b_ℓ's are completely determined by the states on the links and b_{-1}).

11.4.2 Some observations

Statement. Let $A_\ell := \psi_\ell^\dagger U_{\ell,\ell+1}\psi_{\ell+1}$. Fix the states $\underline{n}_0, \ldots, \hat{n}_\ell, \hat{n}_{\ell+1}, \ldots, \underline{n}_{N-1}, \underline{b}_{-1}, \underline{b}_0, \ldots, \hat{b}_\ell, \ldots, \underline{b}_{N-2}$ (the hat means that we skip that site or link and I use the underline to mean that the state is fixed).

$$V := V(\underline{n}_0, \ldots, \hat{n}_\ell, \hat{n}_{\ell+1}, \ldots, \underline{n}_{N-1}, \underline{b}_{-1}, \underline{b}_0, \ldots, \hat{b}_\ell, \ldots, \underline{b}_{N-2}) :=$$
$$= \mathrm{span}\{|n_0\, n_1\, \ldots\, n_{N-1}\, b_{-1}\, b_0\, \ldots\, b_{N-2} \in P \mid n_i = \underline{n}_i,\ i \neq \ell, \ell+1;\ b_i = \underline{b}_i,\ i \neq \ell;\ n_\ell \neq n_{\ell+1}\}.$$

Basically, I define a subspace in which everything is fixed except that b_ℓ can vary and n_ℓ with $n_{\ell+1}$ cannot be equal. Notice that the dimension of this subspace is equal to $2 \cdot n$. We might say about the restriction of A_ℓ to V the following:

1) As a matrix, the restriction of A_ℓ to V might be represented as

$$A_\ell|_V = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

where I is the identity $n \times n$ matrix.

2) We have

$$(A_\ell + A_\ell^\dagger)^2|_V = AA^\dagger + A^\dagger A = I,$$

where I is the identity matrix. This is a simple computation.

12 Additional comments

12.1 Commutators and Anti-commutators

Let $[A, B] \equiv AB - BA$. In the bosonic truncated example, prove (well, "prove" is too strong for this straightforward result, I mean, just note) that $[a, a^\dagger] \neq I$. This is way more general: No operators A and B over a *finite* dimensional Hilbert space can satisfy $[A, B] = I$. (Hint: Use the trace and the circular property of the trace.) For the non-truncated (non-truncated above, it's always truncated below) infinite dimensional bosonic example, note that $[a, a^\dagger] = I$. Physics jargon: This commutator is the starting point for the "Heisenberg uncertainty principle" and some models require it out of principle, hence its importance.

Let $\{A, B\} = AB + BA$. This is physics notation and unfortunately clashes with the set $\{\}$ notation. Some use $[A, B]_+$ but I won't here, at least for now. In the fermionic examples, note that $\{c_i^\dagger, c_j\} = \delta_{i,j} I$, and that $\{c_i, c_j\} = 0$ for finite and infinite number of sites N. Remember that for N sites, $c_i^\dagger$ actually acts on a 2^N dimensional space, and *includes* the "fermionic sign," that is the parity operator. (I suppose it'd be too tedious to write $d_i P_i \equiv c_i$.)

12.2 Quantum Field Theory

You wanted me to explain what QFT means. This is going to be opinionated, so feel free to disagree. I'll give you two "definitions": one good that I like and one bad that can be found in some (bad) literature, where people try to "patch" it to make it sensible without success. As I'm biased toward the lattice, I'd prefer to work on the lattice definition. A QFT is the double infinity: that is, a model where we have infinite sites with an already infinite (separable) Hilbert space on each single site. For example, take the (already infinite and separable) bosonic example above with

one site as defined, and add N sites by tensor product, and take N to infinity. You'll already get the desired commutation $[a_i^\dagger, a_j] = \delta_{i,j} I$, and $[a_i, a_j] = 0$. Now for the bad definition: Take the bosonic example above with one site, and instead of adding sites, put a *continuum* variable to each a and $a^\dagger$ and make it $a(x)$, and define an algebra there. You want the following property: $[a(x)^\dagger, a(y)] = \delta(x-y)I$, where you have to define that "Dirac delta δ" in some sort of Schwartz space of tempered distributions. You'll see that there's no way to have operators like that, I mean, that obey that commutation relation in the continuum. People have worked to salvage this definition, but nothing has worked in the continuum if there are interactions, that is, beyond the simplest quadratic Hamiltonian you can write.

You can also "define" QFT in the Lagrangian formalism, with both good and bad definitions: that is, with lattice and continuum definitions. On the lattice, you'll have a countable and infinite number of integrals that you need to make sense of. On the continuum, you'll have a non-countable number of integrals to deal with, and make sense of it. According to A. Jaffe and E. Witten, (the guys that wrote the text for the Yang-Mills millenium problem), Balaban was the one that has had more success with proving existence of YM theories, (existence in lower dimensions than 3+1), and Balaban has worked with the good definition (the lattice one), albeit in the Lagrangian or analysis or partition function formalism, however you want to call it.

12.3 Hamiltonian versus Lagrangian

We'll use only the Hamiltonian approach and thus mostly algebra (plus topology) as opposed to analysis. But here I write this, just in case, we find that some things are better seen with analysis. How are the Hamiltonian (algebraic) and Lagrangian (analysis, partition function) approaches connected? Answer: By the transfer matrix method; see Hunter L.'s write up on this topic. Are the Hamiltonian and Lagrangian formulations of a given model equivalent? Short Answer: Yes. Longer answer: We would have to define "equivalent" something like "yields the same physics"; we would also have to have a definition of the model in question in at least one approach (Hamiltonian or Lagrangian) and convert to the other using the Transfer Matrix. With all that said, the answer is yes.

12.4 Mapping from First Quantization to Second Quantization

We won't work in First Quantization but I can write this mapping for a general H or at least for some examples if you are interested in this info. FIXME. TODO.

13 Pure Gauge in 1D

OK, here I'll borrow from Hunter L.'s 6.2. Lattice $Z(n)$-QED Model in 1-d. I want to do this thing first WITHOUT fermions so just pure gauge; and this may be too trivial given that we're in one dimension. As in the Hamiltonian formalism we don't include physical time this is pure QED in 1+1 (if you read the literature), but it needs to be qualified, actually *doubly* qualified: on the lattice and with a $Z(n)$ group instead of $U(1)$ (we'll go to U(1), soon, though).

But let's start with all finite, so we have two finite integers: n in $Z(n)$ so our one-site Hilbert space has size n, and then we have N: the number of sites. Instead of putting our Hilbert space on the sites, we'll put them on the links. We assume periodic boundary conditions (PBC) (for now I think it's probably better than open boundary conditions), and so we have also N links: there's a

link connecting the last site with the first, and our lattice looks more like a ring than a line. Now, I need to define some operators. I will define operators on each link, and then think of them as acting on the n^N Hilbert space by tensor product. If I look at the Hamiltonian of $Z(n)$ lattice QED in 1D in Eq. (6.2) of Hunter's write up, I see that only the last term $E^2_{i,i+1}$ remains. So I have to define this so-called "electric field" operator. Here the subindex $i, i+1$ refers to the lattice link that connects sites i with $i+1$; Let's rather write that as $E^2_{i,x}$ where i, x says: the link that starts at site i and goes in the forward x direction. But for simplicity and because we're in 1D, I'll drop the x from the subindex.

Scaling the Hamiltonian will be very important here. While in the previous examples we had only one term, and so only an overall factor that would make the spectrum be in $[0, 1]$ as opposed to $[0, \infty]$ with straightforward conversion between the two definitions, in the following examples, we'll have more than one term in H, and therefore, we would get unrelated results if the *relative* scaling of the various Hamiltonian terms is different. I will then choose E_i to be the one with *tilde* in Hunter's writing. Moreover, I restrict this thing to n odd. So here we go: Let $\{|0\rangle, |1\rangle, \cdots, |n-1\rangle\}$ with n odd, be the basis of the Hilbert space on link i, x, we then define the non-negative integer s such that $n = 2s+1$, and the operator $E_i|k\rangle = (k-s)|k\rangle$, which is of course diagonal, with integer eigenvalues $-s$ to s in steps of 1. We extend this definition by tensor product to the space of N sites and dimension n^N. For example if $N = 3$, and $|v\rangle = |k_0\rangle|k_1\rangle|k_2\rangle$, then $E_1|v\rangle = (k_1 - s)|v\rangle$, and so E_i is a square matrix with n^N rows. Note that in this tensor product extension, there isn't a fermionic sign or Parity operator, because the E operators are bosons and commute on different sites.

We of course need a Hamiltonian here; Because we don't have fermions, just pure gauge, I'll define it as

$$H = \sum_{i=0}^{i<N} E_i^2, \tag{13.1}$$

so that it's just another square matrix with n^N rows, and note that I omitted the identities in the tensor product, because I would have to write $I \otimes I \otimes \cdots E_i^2 \otimes I \otimes I$ with E_i^2 at location i, instead of just E_i^2. Before finding eigenvalues and eigenvectors of H, we need to deal with the gauge symmetry.

13.1 Gauge Symmetry

Unlike in non-gauge models, here we need to weed out states that don't follow Gauss Law. So, we'll define the Gauss law operator, and consider its kernel as a (Hilbert) subspace of the n^N dimensional Hilbert space considered before. Because this *is* a subspace, things should be fine, and we should reconsider all operators defined before to be constrained to that space. This Hilbert subspace is going to be called "physical" as opposed to the original space that contains physical and unphysical states. We will then forget forever the original big space; I say big, because obviously the physical space will be smaller than the original. Let G acting on N sites be

$$G = \sum_{i=0}^{i=N-1} E_{i+1} - E_i, \tag{13.2}$$

with $E_N \equiv E_0$. Physical jargon justification: Because we don't have fermions (electric charge in QED) then the divergence of the electric field over a closed loop should give zero. And because on a one dimensional lattice, the only loop seems to be the one going from beginning to end, we get the sum over all sites of the difference.

The operator G so defined is exactly the zero operator, so we don't have to restrict anything. Oops.

13.2 Algebra of Operators

Let me define another operator called $U_i|k\rangle = |k+1 \pmod n\rangle$ and I also define $U_i^\dagger|k\rangle = |k-1 \pmod n\rangle$, so that these operators wrap around. Obviously this operator is extended to N sites by tensor products, and is a bosonic operator, which means it commutes on different sites. On the same site, $U^\dagger U = UU^\dagger = I$ so that it is unitary. But note that if n is finite, then (on the same site) $[E, U] \neq U$ due to the "border" basis states; note that it's not equal but "almost" equal, and by that I mean equal on states of the basis other than the border ones. Now, as for the physical jargon, we need an operator A, such that $[E, A] = iI$ where $i = \sqrt{-1}$ (same site here), so that we need the "counterpart" magnetic field A to the electric field E. We already know (see previous bosonic example, cross ref. here FIXME TODO) that this A doesn't exist if n is finite. But it exists when n is infinite, as we will see. But before that let me define the exponentiation of E as $V = \exp(i\beta E)$ (one site only, and extend by tensor products), and I need to think about the β variable, let's set it to one for now $\beta \equiv 1$.

It may be better to work on just one site with $N = 1$ for this. Prove that the V above exists and is unique for n finite. Prove that if n is finite, then there exists a unique A such that $U = \exp(i\alpha A)$, where I need to adjust the constant α which may depend on n, and let just set it to 1 $\alpha \equiv 1$ for now. This is all on one fixed site. This is just saying that the exponential and logarithm of certain matrices exist. I guess I could put the exercise: Find the generators of the algebra that includes U, V, E, A, and H as defined above, *and* the identity I. What's the dimension of this algebra. Note that H is symmetric, and find its eigenvalues.

Now we take the n to infinity but keep N finite. We can even take $N = 1$ for simplicity. Prove that $[E, A] = iI$ and that equivalently $[E, U] = U$ so that we have the QED "actual" commutations. I think we can say that U is a representation of the infinite group $U(1)$, so we have even that. I guess we'd like to show that computations done for n finite converge to this n infinity case that is separable, and does represent QED in one spacial dimension (and is equivalent to QED 1+1 in Lagrangian formalism, where the last 1 is the time dimension).

What happens if we take both limits n to infinity but also N to infinity? We may want to postpone this question for latter, not sure.

I'm not sure if going to 2D now, which is the smallest dimension that has plaquettes... or I should perhaps introduce fermions to this 1D case. So the order of the below sections may change.

14 Pure Gauge in 2D

TBW

15 Gauge and Matter in 1D

TBW

16 Questions

16.1 For a discussion

16.2 Just for myself

In one article found this: the totality of numbers associated with a given observable is called its *spectrum*. And indeed, if we regard the algebra of continuous functions as a Banach algebra, then the spectrum of anything is its image; in quantum mechanics, we indeed look at the eigenvalues, which from Banach algebras point of view form their spectra.

In Segal's article I saw that a product of two observables may not be an observable, so he proposed an algebraic structure where a multiplication is not assumed but only powers of elements.

Also from one article: More recent studies have indicated that all self-adjoint operators may not be adequate as the model for the algebra of observables of every physical system. The $C*$-algebras are a long step from this special model, but still not into the chaos of abstract structures consistent with the general features of physical systems.

From Kadison: Given a state f and an observable A, the value $f(A)$ is the expectation of the observable A when these two have physical meanings.

16.3 Notes from conversations

16.3.1 07/08

We might want:

1. Prove that the limits for the Wilson loop taken in different orders yield different results;
2. Is there a gap between the ground state and the first excited state in QED 1+1?
3. Look at the papers of people who tried to prove these statements in other models.

In free fermions, the gap is $O(N^{-1})$. *Spontaneous symmetry breaking* means that the ground state is *degenerate* (i.e., the corresponding eigen-space is of dimension bigger than 1). One can indeed encounter this phenomenon in nature. There are arguments from thermodynamics (not rigorous) about phase transitions (the most challenging problems).

17 Appendix

17.1 C^*-algebras

Recall the definition of a C^*-algebra:

Definition 4. A (unital) C^*-algebra A is a Banach space over $\mathbb{C}$ that is also a (unital) algebra such that the multiplication is a bounded bilinear map of norm 1. It's also supplied with an involution[15] $* : A \to A$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$.

A straightforward[16] consequence from the axioms is that $\|a\| = \|a^*\|$ and in the unital case $\|1\| = 1$.

[15] An involution has the same properties as the usual conjugation operators. So, $(a+b)^* = a^* + b^*$, $(\lambda a)^* = \lambda^* a^*$, $1^* = 1$, $(ab)^* = b^*a^*$, $(a^*)^* = a$.

[16] From the boundedness of the multiplication we obtain $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$, hence $\|a\| \leq \|a\|^*$. Switching to $a \mapsto a^*$, get $\|a\| = \|a\|^*$. For the unit, $\|1\| = \|1\|^2$ since $1^* = 1$.

17.1.1 Gelfand-Naymark: commutative case

I think it worth mentioning:

Theorem 3. *(Gelfand-Naymark, 1st) We have the following:*

(i) Any possibly non-unital commutative C^*-algebra is isomorphic to the space $C_0(\Omega)$ of continuous functions vanishing[17] at ∞ on some locally compact Hausdorff topological space Ω;

(ii) Any unital commutative C^*-algebra is isomorphic to $C(\Omega)$ for Ω a compact Hausdorff space.

There's an accurate description of both the isomorphism and the space Ω: the space Ω is the Gelfand spectrum of A, and the isomorphism is the Gelfand representation.

Here are the definitions. Let A be a Banach algebra[18]. Its *Gelfand spectrum* is the space $\Omega \subset A^*$ of all characters[19] of A that is endowed with ω^*-topology induced from A^*. The isomorphism then is the Gelfand representation $\Gamma_A : A \to C_0(\Omega)$ that sends an element $a \in A$ to a continuous map (vanishing at ∞) $\hat{a}$ such that[20]

Proposition 6. The Gelfand spectrum Ω of a Banach algebra A is a locally compact Hausdorff space; if A is in addition unital, then Ω is Hausdorff and compact.

Proof. Obviously, Ω is Hausdorff since ω^*-topology is Hausdorff. Consider $\Omega' := \Omega \cup \{0\}$. If $\chi_\alpha \to f$ in ω^*-topology, then obviously f preserves multiplication (but it might become zero), so Ω' is a closed subset of the unit ball in A^*. Now it's the consequence of Banach-Alaoglu theorem that Ω' is compact. Being a closed subset of a compact Hausdorff space, we see that Ω is locally compact and Hausdorff. In case A is unital, 0 is an isolated point of Ω', for if there was a net $\chi_\alpha \in \Omega$ such that $\chi_\alpha \to 0$, then $1 = \chi_\alpha(1) \to 0$, which is a contradiction, and thus Ω is compact. □

The beauty of the result and the proof are mesmerizing, so I couldn't resist working out the details. Here I'd like to record a sketch for the unital case (the non-unital has to come from the unital by the process of adjoining an identity). One can find a full proof in [3].

Sketch of the proof of the 1st G-F theorem. First of all, in case of C^*-algebras it is automatic that every character preserves the $*$-structure. This fact implies that Γ_A is a unital $*$-homomorphism. Next, one shows that the spectrum $\sigma(a)$ of any $a \in A$ coincides with the range of $\hat{a}$. Indeed, $\lambda \in \sigma(a)$ if and only if $a - \lambda$ is not invertbile, if and only if $a - \lambda$ belongs to some maximal ideal; there is a $1-1$ correspondence between maximal ideals and characters in unital commutative Banach algebras, so we find χ such that $\chi(a - \lambda) = \lambda$, i.e., $\hat{a}(\chi) = \lambda$. A consequence of this is that the spectral radius $r(a)$ of any a coincides with the norm of $\hat{a}$. Now, if a is self-adjoint, then $\|a\| = r(a) = \|\hat{a}\|$, which

[17] Precisely, a function $\varphi : \Omega \to \mathbb{C}$ is vanishing at infinity if for any $\varepsilon > 0$ there exists a compact subset $\Delta \subseteq \Omega$ such that $|\phi(t)| < \varepsilon$ when $t \notin \Delta$.

[18] I.e. a Banach space that's also an algebra such that the multiplication is a bounded bilinear map with norm 1.

[19] A character is a non-zero linear functional that preserves the multiplication. It turns out that any character $\chi : A \to \mathbb{C}$ has norm $\|\chi\| \leq 1$, so they're automatically continuous, for if $\chi(a) = 1$ for some $a \in A$ with $\|a\| < 1$, then let $b := \sum_{k=1}^{\infty} a^k$. It follows that $a + ab = b$, hence $1 + \chi(b) = \chi(b)$, which is absurd.

[20] The continuity of $\hat{a}$ follows right from the definition of ω^*-topology. It indeed vanishes at ∞, for $\hat{a}$ is a continuous map on the compactification $\Omega' := \Omega \cup \{0\}$ such that $\hat{a}(0) = 0$ (look at the formula defining $\hat{a}$), hence for any $\varepsilon > 0$ there's $U_\varepsilon \ni 0$ such that $|\hat{a}(\chi)| < \varepsilon$ when $t \in U_\varepsilon$. But U_ε^c is compact, as a closed subset of the compact space Ω', so $\hat{a}$ vanishes at ∞.

means that Γ_A works as an isometry on self-adjoint elements. If $a \in A$ is an arbitrary one, then we use the C^*-structure and the fact that a^*a is self-adjoint:

$$\|a\|^2 = \|a^*a\| = \|\hat{a}^*\hat{a}\| = \|\hat{a}\|^2.$$

Thus Γ_A is an isometric embedding. Its image is a closed subalgebra and contains the identity of $C(\Omega)$. For surjectivity, one applies the Stone-Weierstrass theorem: the functions $\hat{a}$ separate the points (for two different $\chi_1 \neq \chi_2$, just take any $a \in \operatorname{Ker}\chi_1 \backslash \operatorname{Ker}\chi_2$, and then $0 = \hat{a}(\chi_1) \neq \hat{a}(\chi_2)$). □

17.1.2 Gelfand-Naymark: non-commutative case

In this section, I describe the Gelfand-Naymark-Segal construction that is used to show that any C^* algebra can be isometrically represented in some Hilbert space. The construction relies on the notions of states and pure states. I assume everywhere that there's a fixed unital C^*-algebra A. A good source for the relevant results is [3].

Definition 5. A state f is a positive linear functional f (i.e., $f(z^*z) \geq 0$ for any z) that is normalized at the unit: $f(1) = 1$.

One can show that states form a convex ω^*-compact set: it coincides with all bounded linear functionals whose norm is 1 and which is achieved at the identity. By Krein-Millman theorem, this set is the closed convex hull of so-called *extreme points*[21]. These extreme points are called *pure states*. For any state f, one can find a representation π in some Hilbert space H, and a vector $\xi \in H$ such that

$$f(x) = (\pi(x)\xi, \xi).$$

Moreover, ξ can be chosen in such a way that the span of $\{\pi(x)\xi \mid x \in A\}$ is dense in H. The construction also tells us that π is irreducible if and only if f is a pure states. The steps to find π are the following:

We define $\pi(x)$ for every $x \in A$ as the multiplication by x on the left: $\pi(x)(y) = xy$. Since the chosen state f is positive, it yields a positive sesquilinear form $(x, y) := f(y^*x)$ on A. The form has a kernel $N := \{(x, x) = 0\}$, so the quotient A/N is a vector space with an inner product. It turns out that N is also a left ideal with respect to the multiplication, hence every $\pi(x)$ lifts to A/N. Now, simply complete A/N and extend each $\pi(x)$ to a bounded linear operator on the obtained Hilbert space. The work to be done is to show that every $\pi(x)$ is bounded. But this turns out to be true. The vector ξ is then $\xi := 1 + N$. If in the end we do not obtain that $\operatorname{span}\{\pi(x)\xi \mid x \in A\}$ is dense, simply restrict the representation to this subspace.

Next, a series of propositions in [3] tells us that for any self-adjoint $x \in A$, we can find a pure state f such that $f(x) = \|x\|$. This makes us think that instead of constructing representations through pure states, we can actually start with elements of A. A corollary of this is that for every element $z \in A$, there's an irreducible representation π and a vector ξ in its Hilbert space such that $\|\pi(z)\xi\| = \|z\|$. This is enough to prove the Gelfand-Naymark theorem:

Theorem 4. *(Gelfand-Naymark) For any unital C^*-algebra A, there's an isometric representation in some Hilbert space.*

Proof. As we discussed above, for every non-zero $z \in A$ we pick a representation π_z such that $\|\pi_z(z)\xi_z\| = \|z\|$. The direct some of all of these representations is then injective. It's also a result of C^*-algebra homomorphisms that injectivity implies that π is isometric. □

[21] I.e., points that are not interior points of intervals lying inside that set.

17.1.3 Properties of AF algebras (not finished)

Definition 6. A (unital) C^*-algebra A is called *approximately finite-dimensional* if A is an inductive limit of a sequence of finite-dimensional (unital) C^*-algebras.

I'd like to record some interesting properties of such algebras. They may not be important for our purposes, though. An example of an AF algebra appearing in these notes is the space of observables attached to an infinite sublattice when a Hilbert space at a single site is finite-dimensional.

Proposition 7. (see [4]) Let A be a unital C^*-algebra. Then A is an AF-algebra if and only if the following two conditions are fullfilled:

(i) A is separable;

(ii) If $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$, then there exist a finite-dimensional C^*-subalgebra $B \subseteq A$ and elements $y_1, \ldots, y_n \in B$ such that $\|x_i - y_i\| < \varepsilon$, $i = 1, \ldots, n$.

Furthermore, if A is AF, and A_1 is a finite-dimensional C^*-subalgebra of A, there exists an increasing sequence $A_2 \subseteq A_3 \subseteq \cdots$ of finite-dimensional C^*-subalgebras such that $A_1 \subseteq A_2$ and $A = \bigcup_i A_i$.

Some aspects of the statement are not clear to me, and if needed, can delve into: so, the closure is not taken in the second bullet, is this right?

A couple of interesting results on pure states:

Proposition 8. (see [4]) Let A be an AF-algebra and let ω_1 and ω_2 be pure states of A such that the associated representations π_1 and π_2 are faithful. Then there exists an automorphism α of A such that $\omega_1 = \omega_2 \circ \alpha$.

The next proposition is basically saying that a state is pure iff its restriction to each finite-dimensional subalgebra is pure.

Proposition 9. (see [4]) Let A be an AF-algebra and let ω be a state of A such that the associated representation is faithful. Then ω is pure if and only if there exists an increasing subsequence A_n of finite-dimensional $*$-subalgebras of A all containing the identity and such that $A = \varinjlim A_n$ and $\omega | A_n$ is pure for all n.

17.2 Hilbert-Schmidt operators

The results on Hilbert-Schmidt operators can be found in a concise form in [6] on page 268.

Let H be an infinite-dimensional Hilbert space and $\mathfrak{A} := \mathrm{End}(H)$ be the space of bounded linear endomorphisms. If $e_1, \ldots, e_n, \ldots$ is an orthonormal basis of H, for $A \in \mathfrak{A}$ set

$$\|A\|_2 := \sqrt{\sum_i \|Ae_i\|^2}.$$

It turns out that $\|A\|_2$ does not depend on the choice of the orthonormal basis. The operator A is called a *Hilbert-Schmidt operator* if $\|A\|_2 < \infty$. Denote the corresponding space as $\mathfrak{A}_2$. These operators satisfy the following properties:

(a) $\|A\| \leq \|A\|_2$;

(b) If $T \in \mathfrak{A}$ and $A \in \mathfrak{A}_2$, then

$$\|TA\|_2 \le \|T\|\|A\|_2, \quad \|AT\|_2 \le \|A\|_2\|T\|;$$

(c) $\|A\|_2 = \|A^*\|_2$;

(d) Algebraically, $\mathfrak{A}_2$ is a two-sided ideal of $\mathfrak{A}$ (non-closed in the operator norm topology in infinite-dimensional case);

(e) The subspace of finite-rank operators is contained in $\mathfrak{A}_2$ and is dense there;

(f) $A \in \mathfrak{A}_2$ iff $|A| := \sqrt{A^*A} \in \mathfrak{A}_2$. In this case, $\|A\|_2 = \||A|\|_2$;

(g) Hilbert-Schmidt operators are compact. Moreover, if $\lambda_1, \dots, \lambda_n, \dots$ are eigenvalues of $|A|$ (each repeated as many times as its multiplicity), then $|A| \in \mathfrak{A}_2$ if and only if $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. In this case, $\|A\|_2 = \sqrt{\sum_{n=1}^{\infty} \lambda_n^2}$.

17.3 Quantum physics

17.3.1 Spin

I read a chapter on spins in Landau's book [13]. Here I record what I understand about spins. So, the *spin* of a particle is something that behaves like a classical angular momentum (at least, it has some properties of that). If $\psi(x, y, z)$ is a wave function that depends on three coordinates, then we'd like to include a variable σ, which corresponds to the spin, and the wave function then depends on four variables: $\psi(x, y, z; \sigma)$. We have a spin operator $S = (S_x, S_y, S_z)$ that we can apply to the wave function. As Landau says, we apply it somehow straight to the variable σ. What's the meaning of $S\psi$? I guess, this means that we conduct a measurement of the spin of a particle that's in the state ψ. There's an heuristic reasoning, which I don't understand, but which yields the following commutation realtions among components of S:

$$\{S_y, S_z\} = iS_x, \quad \{S_z, S_x\} = iS_y, \quad \{S_x, S_y\} = iS_z,$$

where the bracket denotes, I guess, the anti-commutator. They also derive

$$S^2 = s(s+1),$$

where s is the least upper bound of eigenvalues of S_z (then $-s$ is the greatest lower bound of the eigenvalues). From this they derive that $s \in \frac{1}{2}\mathbb{Z}$. As I understand, that relation means

$$S_x^2 + S_y^2 + S_z^2 = s(s+1)I,$$

where I is the identity operator, though it is not clear why we have such an equation. Depending on s, the components are realized as square matrices of different sizes. If $s = 1/2$, then the components are 2×2 matrices (Pauli matrices). For $s = 1$, they are 3×3 matrices. One can find a relation between s and their sizes. Also, I don't think that the relation between S^2 and s can be derived from the commutation relations between components of S.

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