Exercise of Getting Started

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1 Foundations

1.1 The Role of Algorithms in Computing

I didn't make the exercise of this section because I didn't find them useful.

1.2 Getting Started

Exercise 1.2-1):
It could be an application like booking. When you search a hotel close to the airport, it gets involved algorithms as searching the hotels close to that airport and it should be searched in a short time period.

Exercise 1.2-2):

$$8n^2 < 64n \cdot \log_2 n \quad \rightarrow \quad n < 8 \cdot \log_2 n$$

Try values until this inequality is false. To $n \lessapprox 43$, insertion sort runs faster than merge sort.

Exercise 1.2-3):

$$100n^2 < 2^n$$

Trying values, for $n \lessapprox 15$, 2^n runs faster than $100n^2$.

Exercise 1.2-4):

View photo of the exercise on the next page.

	1 second	1 minute	1 hour	1 day	1 month	1 year	1 century
$\lg n$	∞	∞	∞	∞	∞	∞	∞
$\sqrt{n}$	10^{12}	$3.6 \cdot 10^{15}$	$1.3 \cdot 10^{19}$	$7.47 \cdot 10^{21}$	$6.91 \cdot 10^{24}$	$9.95 \cdot 10^{26}$	$9.95 \cdot 10^{30}$
n	10^6	$6 \cdot 10^7$	$3.6 \cdot 10^9$	$8.64 \cdot 10^{10}$	$2.63 \cdot 10^{12}$	$3.16 \cdot 10^{13}$	$3.16 \cdot 10^{15}$
$n \lg n$	62746	$2.8 \cdot 10^6$	$1.33 \cdot 10^8$	$2.76 \cdot 10^9$	$7.29 \cdot 10^{10}$	$7.99 \cdot 10^{11}$	$6.87 \cdot 10^{13}$
n^2	1000	7746(aprox)	60000	293939	$1.62 \cdot 10^6$	$5.62 \cdot 10^6$	$5.61 \cdot 10^7$
n^3	100	391 (aprox)	1532	4420	13803	31601	146679
2^n	20(aprox)	26(aprox)	32	36	41	44	51
$n!$	9	11	12	13	15	16	17

Exercise 2.1-1):
Note: Resolved using the logic of C, C++, Java, etc. while iterating over an array on a for loop. Also the number that appears in green, is the number being checked. The number or numbers that appears in red are the numbers being moved.

i	Array
1)	[31, 41, 59, 26, 41, 58]
2)	[31, 41, 59, 26, 41, 58]
3)	[26, 31, 41, 59, 41, 58]
4)	[26, 31, 41, 41, 59, 58]
5)	[26, 31, 41, 41, 58, 59]

Exercise 2.1-2):
Initialization: The loop start getting the first number in the array. In spite of that, it has initialized to 0 the variable sum where the total sum will be stored. Due to that, the invariant holds the first number that will be added to sum.

Maintenance: On each iteration, the loop will hold only the index of the number that will be added, after add it, i will be incremented by 1, holding the next number (i + 1).

Termination: The loop will terminate when the 'n' elements of the array are added. In conclusion, sum it's equivalent of say that $sum = \sum_{i=1}^{n} A[i]$.

Exercise 2.1-3):
click this link to see the resolution → *resolution*

Exercise 2.1-4):

Algorithm 1 Linear Search

```
1: function LINEAR-SEARCH(A, n, x)
2:     for i ← 1 to n do
3:         if A[i] == x then return i
4:     return NIL
```

Initialization: The loop start getting the first element of the array.

Maintenance: On each iteration, the loop takes the next element (i + 1) and compare it with the value being search (x). If it's found return i, else, continue searching that value.

Termination: When all values are read, if x wasn't found in the array, it returns NIL to indicate that no value was found on all the array.

Exercise 2.1-5):

Algorithm 2 ADD-BINARY-INTEGERS

```
1: function ADD-BINARY-INTEGERS(A, B, n)
2:     //Initialize array C with n values
3:     carry ← 0
4:     for i ← 1 to n do
5:         c ← A[i] + B[i]
6:         C[i] ← c  mod 2
7:         carry ← c ÷ 2    //Integer division

8:     C[n] ← carry
9:     return C    //Return the array C with the values
```

Initialization: The loop starts with value of carry to 0, and getting the first bits of A and B.

Maintenance: On each iteration, the loop takes the next bits values of A and B. Add these values and calculate the value to insert into C and the carry that could exists.

Termination: All values were added and store, now C[0:n - 1] with the result of the sum. To reach the n-th value, adds the last carry value on the position n.

Exercise 2.2-1):
Like the book says, Θ notation is like saying "roughly proportional to n^2 (for example), when n is large." In this case, we remove constants, so the remaining expression is $n^3 + n^2 + n + 3$. The term with the highest exponent is n^3, so at any moment: $n^3 \gg n^2 \gg n$.
Solution: $\Theta(n^3)$.

Exercise 2.2-2):

Algorithm 3 SELECTION-SORT

```
1: function SELECTION-SORT(A, n)
2:     for i ← 1 to n − 1 do
3:         ind_small_elm ← i
4:         for j ← i + 1 to n do
5:             if A[ind_small_elm] > A[j] then
6:                 ind_small_elm ← j
7:         SWAP(A[i], A[ind_small_elm])
```

The invairant is that on each iteration of extern for, it only takes 1 by 1 element. In the inner for, also take all elemnts from i to n, and compare the value of the outter for against the inner for to take the smaller elemnt.

When the algorithm arrives to the last element, all swaps ocurred and the last element will be in the correct place.

The worst case happens when it must iterate on the outher for and also with al the elements from i to n in the inner for. So it's: $\frac{n*(n-1)}{2}$. Thats mean that avoiding all constants values, the solution is: $\Theta(n^2)$.

The best case is not better because you have to check all values in the if, the only instruction that is avoided is the instruction inside the if because the if won't be evaluated to true. But that instruction is insignificant if n it's too big.

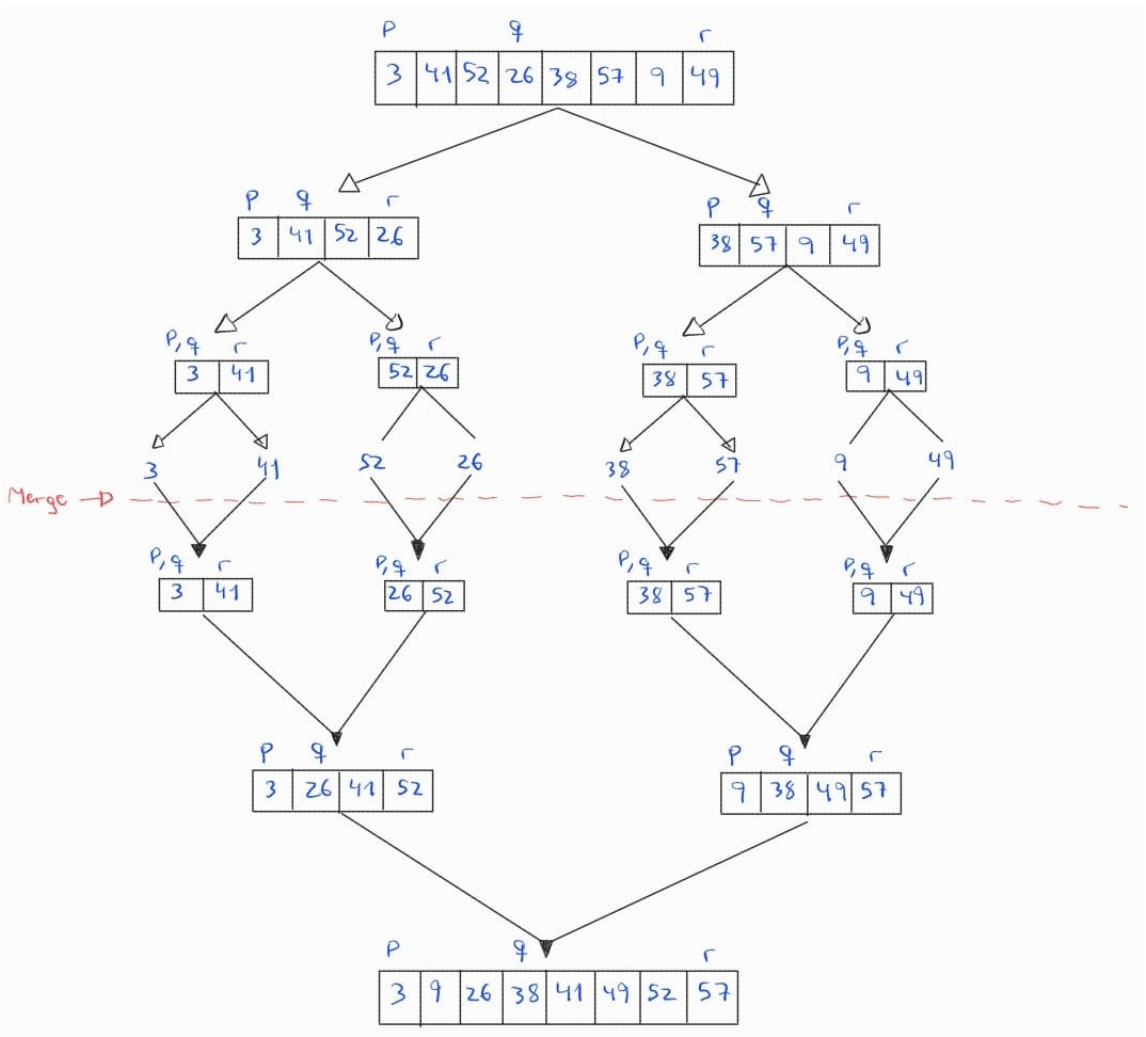
Exercise 2.2-3):
Depends on the value where is storage, if the x value is storage at the first position it will take a constant value to search it. However, if the value is in the last element (worst case) it will spend $constant * n$ time to find that value.

Averege case is suposing that it's in the middle of the array. The averegage is $\frac{n}{2} = n$ if n it's too big.

Worst case as mentioned before is $\Theta(n)$.

Exercise 2.2-3):
The only thing you could do is a preprocessing step to check if it's alredy sorted or nearly to be sorted and then apply the algorithm who best fits when the best case was achieved.

Exercise 2.3-1):



Exercise 2.3-2):
The "***if*** $p \neq r$" is not useful because $p = 1$ and $q > 1$, so the if will be evaluated to true, and the return (the termination of recursion) will execute without doing any recursion step.

Exercise 2.3-3):
Initialization: The loop starts obtaning the first element to insert it in the sorted array.

Maintenance: On each iteration, the loop insert 1 element of the left or right array, then increment k by 1, to insert on the next iteration the next value 1 by 1. The values that will be inserted with the first loop in the best case is n - *1* where n is the lenght of the subarray/array being sorted.

Termination: In one of the last 2 whiles, it will be inserted all remaining values (could be values in the left or right array) until reach the nth values in the sorted array.

Exercise 2.3-4):
$n = 2$

$$T(2) = 2$$

The solution given says that:

$$T(2) = 2 * \log_2 2 = 2 \cdot 1 = 2$$

Induction hypotesis:

$$T(k) = k \cdot \log_2 k$$

Where k is a power of 2. We want to demostrate that $n = 2k$. Use recurrence to calculate $T(2k)$:

$$T(2k) = 2T(k) + 2k \quad \rightarrow \quad T(2k) = 2 \cdot (k * \log_2 k) + 2k \quad \rightarrow \quad T(2k) = 2k \cdot (\log_2 k + 1) \quad \rightarrow \quad T(2k) = 2k \cdot \log_2 2k$$

As we mentiones before, replacing $n = 2k$, we conclude that the induction hypotesis is correct $n \cdot \log_2 n$

Exercise 2.3-5):

Algorithm 4 INSERTION-SORT-RECURSIVE

```
1: function INSERTION-SORT-RECURSIVE(A, n)
2:     if n == 0 then return
3:     INSERTION-SORT-RECURSIVE(A, n − 1)
4:     key ← A[n]
5:     i ← n − 1
6:     while i ≥ 0 and A[i] > key do
7:         A[i + 1] ← A[i]
8:         i ← i − 1
9:     A[i + 1] ← key
```

Exercise 2.3-6):

Algorithm 5 BINARY-SEARCH

```
1: function BinarySearch(A, x, l, n)
2:     if l > n then return
3:     mid ← l + n/2
4:     if A[mid] > x then
5:         BinarySearch(A, x, l, mid − 1)
6:     else if A[mid] < x then
7:         BinarySearch(A, x, mid + 1, r)
8:     else   return mid
```

Exercise 2.3-7):
You can't use binary search because while you are sorting most part of the time, it won't be that value in the sorted array. Maybe you should make a variation of binary search to find out in which direction has less difference betwen the number being search and then you should move all values 1 position to right and insert in the new position.

On worst case, imagine that the worst value is at the least position, you should move n items to the right + n / 2 that cost to search all the. So it's $n \cdot \frac{n}{2} = \Theta(n^2)$

Exercise 2.3-8):

Algorithm 6 FIND-SUM-TWO-ELEMENTS

```
1: function FindSumTwoElements(S, n, x)
2:     MergeSort(S, 0, n - 1)    //Θ(n ∗ logn)
3:     i ← 0
4:     j ← n − 1
5:     while i < j do    //Θ(n)
6:         if S[i] + S[j] = x then
7:             return i, j
8:         else if S[i] + S[j] < x then
9:             i ← i + 1
10:        else
11:            j ← j − 1
12:    return No pair found
```

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Exercise 2-1):
a) The number of arrays to be sort are $\frac{n}{k}$, where n is the number of elements in the original array and k the number of elements to be sorted on *insertion sort*. Due to that, on the worst case it should be sorted $\frac{n}{k} \cdot k^2 \quad \rightarrow n \cdot k = \Theta(n \cdot k)$.

b) The recursion will be called until reach $\frac{n}{k}$ elements on a subarray. Hence it won't be called recursively until $\log_2 n$ (when only 1 element is left). Due to that the number of recursions are delimited by $\log_2 \frac{n}{k}$. Finally in the worst case when only is left the first 2 subarrays, it will take n iterations to sort it. Due to that, we explain why it's true the expresion $\Theta(n \cdot \log_2 \frac{n}{k})$.

c)

$$\Theta\left(n \cdot k + n \cdot \log_2 \frac{n}{k}\right) = \Theta(n \cdot \log_2 n)$$

$$\Theta(n \cdot k) = \Theta(n \cdot \log_2 n) \quad \rightarrow \quad k = \Theta(\log_2 n)$$

$$\Theta\left(n \cdot \log_2 \frac{n}{k}\right) = \Theta(n \cdot \log_2 n) \quad \text{//Substituting } k = \Theta(\log_2 n)$$

$$\Theta\left(n \cdot \log_2 \frac{n}{\log_2 n}\right) = \Theta(n \cdot \log_2 n) - \Theta(n \cdot \log_2 \log_2 n) \quad \text{//Dominant term is } \Theta(n \cdot \log_2 n)$$

In conclusion, the biggest posible value which both variants of Merge Sort have the same result is for $k = \Theta(\log_2 n)$.

d) You should see in which value it's better use one algorithm or another to sort and combine the 2 algorithms. As we saw in the section c, this k value is calculated applying that formula.

Exercise 2-2):
a) Do you need to prove that on each iteration, of the inner for 1 element is being moved and it remains on all iterations. In addition to that, you must prove that on the extern for loop it's being incremented by 1.

b) Initialization: The loop starts taking the most right element on the array. So the invariant is that take 1 element at a time.

Maintenance: Loop takes on each iteration an element, compare with the left hand elemnt and if it's smaller than the left hand element, swap both elements. The loop invariant remains as Initialization because on each iteration only 1 element is being checked with another one. After that check, on the next iteration is taken the next left element.

Termination: When this for ends, there are a number of elements equals to i (extern for) that are sorted in the lowest positions of the array.

c) As I said before, on each time that the inner for loop ends, 1 value is sorted. Hence when the extern for loop ends, all values will be sorted on the right position and it's proved the inequality.

d) Both have the same worst-case running time ($\Theta(n^2)$). But Insertion Sort is usually better on the averegage cases because it does less swaps and usually spends $\Theta(n)$ on sort an array. On the other hand, Bubble Sort always spend $Theta(n^2)$ to sort the array.

Exercise 2-3):
a) $\Theta(n)$ because it has to take all values of the array A and add it to $x \cdot p$.

b)

```
NAIVE-HORNER (A, x)
1  y = 0
2  for i = 1 to A.length
3       m = 1
4       for j = 1 to i − 1
5            m = m · x
6       y = y + A[i] · m
```

Figure 1: Resolution extracted from https://atekihcan.github.io/CLRS/02/P02-03/ I was lazy of do that hehehe.

c) For Maintenance:

$$y = a_i + x \sum_{k=0}^{n-(i+1)} a_{k+i+1}x^k$$

$$= a_i x^0 + \sum_{k=0}^{n-i-1} a_{k+i+1}x^{k+1}$$

$$= a_i x^0 + \sum_{k=1}^{n-i} a_{k+i}x^k$$

$$= \sum_{k=0}^{n-i} a_{k+i}x^k.$$

The loop terminates at i = -1:

$$= a_i x^0 + \sum_{k=0}^{n-i-1} a_{k+i+1}x^{k+1} = \sum_{k=0}^{n-i} a_{k+i}x^k.$$

Exercise 2-4):
a) Inversions are: (2, 1), (3, 1), (8, 6), (8, 1), (6, 1)

b) if the set is sor in ascending order, it won't be 0 inversions in the set.

c) Both of them are $\Theta(n^2)$ because you have to take 1 value from the most right position and compare if it's less than the i value (left one) on the array.

d)

Algorithm 7 INVERSIONS

```
1: function INVERSIONS(A, p, r)
2:     //On the else of the first while of Merge, add: Print(L[i], R[j])
```

1.3 Caharcerizing Running Times

Exercise 3.1-1):
We could do the same with saying that k is a multiple of 2. The left subarray has the biggests numbers, and the right one have the lowest numbers. If we want to move the biggests numbers to the right subarray, we should move *n / 2* numbers and then another *n / 2* to the left to move the lowest one at the begginig. Finally we have that we should move $\frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4} = \Omega(n^2)$

Exercise 3.1-2):
The extern for loop, executes in the worst case *n - 1* times. The inner loop, executes *i + 1* to *n* times. As mentioned before, the worst case is for i = 0, then it will be executed *n - 1* times. In conclusion, the wors case is $(n-1) \cdot (n-1) = O(n^2)$.

On the other hand, supose that the array is divided in 2 subarrays, the left for the higer values and the right for the lower ones. Althought, the extern for must iterate over all elements, so the better case is the same as the

worst (*n - 1*). Then the inner loop is the same as the worst case. The only thing that executes n/2 times is the instruction inside the if and the swap. These operations are constant operations. In conclusion, the lower bound of the asymptotic behaviour is $(n-1)\cdot(n-1) = \Omega(n^2)$

$O(n^2) = \Omega(n^2) = \Theta(n^2)$

Exercise 3.1-3):
Consideramos un array A de tamaño n, dividido en tres partes:

- Primera parte: Las primeras αn posiciones contienen los αn valores más grandes.
- Segunda parte: Las siguientes $(1-2\alpha)n$ posiciones son la parte media.
- Tercera parte: Las últimas αn posiciones contienen los αn valores más grandes después de ordenar.

El número total de movimientos es:

$$\text{Movimientos totales} = \alpha(1-2\alpha)n^2.$$

Para que el argumento tenga sentido, necesitamos que $0 < \alpha < \frac{1}{2}$.

Maximizamos la función:

$$f'(\alpha) = 1 - 4\alpha = 0 \quad \Rightarrow \quad \alpha = \frac{1}{4}.$$

El número total de movimientos es:

$$\frac{1}{4}\cdot\frac{1}{2}n^2 = \frac{1}{8}n^2.$$

Por lo tanto, el tiempo de ejecución en el peor caso es $\Theta(n^2)$.

Exercise 3.2-1):

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n) = \Theta$$

Prove that $\max\{f(n), g(n)\} = O(f(n)+g(n)) \quad \Rightarrow \quad \max\{f(n), g(n)\} \le c_2 \cdot (f(n)+g(n))$.

If the max value is *f(n)*, then we have $f(n) \le c_2 \cdot (f(n)+g(n))$. Also if the max value is *f(n)*, then we have $g(n) \le c_2 \cdot (f(n)+g(n))$ If $c_2 = 1$, and $n \ge n_0$, the result is $\max\{f(n), g(n)\} = O(f(n)+g(n))$.

Prove that $\max\{f(n), g(n)\} = \Omega(f(n)+g(n)) \quad \Rightarrow \quad \max\{f(n), g(n)\} \ge c_1 \cdot (f(n)+g(n))$.

If the max value is *f(n)*, then we have $f(n) \ge c_1 \cdot (f(n)+g(n))$. Also if the max value is *f(n)*, then we have $g(n) \ge c_1 \cdot (f(n)+g(n))$ If $c_1 = \frac{1}{2}$, and $n \ge n_0$, the result is $\max\{f(n), g(n)\} = \Omega(f(n)+g(n))$.

In conclusion: $\max\{f(n), g(n)\} = O(f(n)+g(n)) = \Omega(f(n)+g(n)) = \Theta(f(n)+g(n))$ for $c_1 = \frac{1}{2}$ and $c_2 = 1$.

Exercise 3.2-2):
It's meaningless because de O notation defines asymptoticaly, the upper bound of f(n). This not mean *"Is at least* $O(n^2)$*"* because it's only in the worst possible scenario. Althought, we need to define the ower bound, for example the asymptotic lower bound is $\Omega(n)$ that it's less than $O(n^2)$. Hence we can't say *"At least"* due to it could spend some value betwen $O(n^2)$ on the worst case or $\Omega(n)$ in the best cases.

Exercise 3.2-3):
With the properties of the powers, we have:

$$2^{n+1} = 2^n \cdot 2^1 = O(\max\{2^n, 2^1\}) = O(2^n)$$

The second one is wrong, because with the properties we can't remove the n there, so it should be $O(2^{2n})$.

Exercise 3.2-5):

$$0 \leq c_1 g(n) \leq g(n) \leq c_2 g(n) = \Theta(g(n))$$

$$\Omega(g(n)) = c_1 g(n) \leq g(n) \quad c_1 = \frac{1}{2} \text{ and } n \geq n_0$$

$$O(g(n)) = c_2 g(n) \geq g(n) \quad c_2 = 2 \text{ and } n \geq n_0$$

Exercise 3.2-6):
All the values of the time spent by the algorithm, will be on $\Omega(g(n)) = g(n) = O(g(n))$ if $\Theta(g(n))$. Also we know that $o(g(n)) > O(g(n))$ and $\omega(g(n)) < \Omega(g(n))$, so they won't have values in common.

Exercise 3.2-7):
$\Omega(g(n,m)) = \{f(n,m) :$ if there exists a constant c, n_0 and m_0 such that $cg(n,m) \leq f(g,m)$ for all $n \geq n_0$ or $m \geq m_0$ and $c > 0\}$.

$\Theta(g(n,m)) = \{f(n,m)$ if there exists a constant c_1, c_2, n_0 and m_0 such that $0 \leq < c_1 g(n,m) \leq g(n,m) \leq c_2 g(g,m)$ for all $n \geq n_0$ or $m \geq 0$ and $c_1 > 0$, $c_2 > 0\}$.

Exercise 3.3-1):
f(n) is monotically increasing because there is an $f(m) \leq f(n)$. For *g(n)* it's the same. Then we have:

$$f(m_1) \leq f(n_1) + g(m_2) \leq g(n_2) \quad \Rightarrow f(m_1) + g(m_2) \leq f(n_1) + g(n_2)$$

Demostrate $f(g(n))$:

$$f(g(m)) \leq f(g(n)) \quad m \leq n$$

Demostrate $f(n) \cdot g(n)$ are non negative for an $n_1 > 0, n_2 > 0, m_1 < n_1 and m_2 < n_2$

$$f(m_1) * g(m_2) \leq f(n_1) * g(n_2)$$

Exercise 3.3-2):

$$\lfloor \alpha n \rfloor = \alpha n \qquad \lceil (1-\alpha)n \rceil = (1-\alpha)n$$

$$\alpha n + (1-\alpha)n = n(\alpha + 1 - \alpha) = n$$

Exercise 3.3-3):
Applying the properties of exponentials, we have

$$(n + o(n))^k = n^k + o(n^k) \quad \Rightarrow \quad \Theta(\max\{n^k, n^k\}) = \Theta(n^k)$$

With the property of ceil and floor, we know $\lceil n \rceil^k \leq n^k \leq \lfloor n \rfloor^k$, the problem says $\Theta(n^k) = n^k$

Exercise 3.3-4):

a) Equation $a^{\log_b c} = c^{\log_b a}$

$$\log_b c = x \qquad \log_b a = y$$

$$a^x = (b^y)^x \quad \Rightarrow \quad a^{\log_b c} = b^{yx}$$

Use the relation $c = b^x$

$$c^y = (b^x)^y \quad \Rightarrow \quad c^{\log_b a} = b^{xy}$$

$$a^{\log_b c} = b^{yx} = c^{\log_b a}$$

b) Demostrate equations 3.26, 2.27 y 3.28.

3.26)

$$n! = \sqrt{2\pi n} * \left(\frac{n}{e}\right)^n + 1 \quad \Rightarrow \quad (2\pi n)^{\frac{1}{2}} * \left(\frac{n}{e}\right)^n + 1$$

Removing the constant values that are $2, \pi, e, 1$, the reamining values are: $n^{\frac{1}{2}} \cdot n^n = o(n^n)$.

3.27)

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Prove $n! = \omega(2^n)$, that's mean:

$$\frac{n!}{2^n} \to \infty.$$

Dividiendo por 2^n,

$$\frac{n!}{2^n} \approx \sqrt{2\pi n}\left(\frac{n}{2e}\right)^n.$$

For a bigger value of n , $\frac{n}{2e} > 1$, the term tends to infinity. We can conclude that

$$n! = \omega(2^n).$$

c) $\log_2(\Theta(n)) = \Theta(\log_2 n)$

$$\log_2(c_1 \cdot n) \leq \log_2(f(n)) \leq \log_2(c_2 \cdot n) \quad \Rightarrow \quad \log_2 c_1 + \log_2 n \leq \log_2(f(n)) \leq \log_2 c_2 + \log_2 n.$$

$\log_2 c_1$ and $\log_2 c_2$ are constants. Hence we have: $\log_2(f(n)) = \Theta(\log_2 n) \quad \Rightarrow \quad \log_2(\Theta(n)) = \Theta(\log_2 n)$

Exercise 3.3-5):
As we saw, $\lceil \log_2 n \rceil! = \log_2 n!$ applying the formula 3.28, we have that $\log_2 n = \Theta(n \log_2 n)$. In conclusión, it's polinomially bounded.

$\lceil \log_2 \log_2 n \rceil! = \log_2 \log_2 n = \log_2 \Theta n * \log_2 n$ with that we can show that is **not** polinomally bounded.

Exercise 3.3-6):
for the definition, we know $\log_2^* n < 5$ (rarely will be upper to 5). If we supose that takes the value of $n = 2^{65536} = 10^{\frac{65536}{\log_2 10}} \approx 10^{19.728}$ The result of $\log_2^* 10^1 9.728 = 5$ for example, then we have $\log_2 5 \approx 2.322$.

On the other hand, for the same n value, we have $\log_2 10^1 9.728 \approx 65.535$ then looking the table, Applying the logarithm iteratively, we can see:

1. $\log_2 65.535 \approx 6.02$.
2. $\log_2 6.02 \approx 1.37$
3. $\log_2 1.37 \approx 0.45$

Because we could apply 3 times the iteration, $\log_2^*(\log_2 n) = 3$. **In conclusion**, $\log_2^*(\log_2 n) > \log_2(\log_2^* n)$

Exercise 3.3-7):
Substitute $\phi = \frac{1+\sqrt{5}}{2}$ in $x^2 = x + 1$.

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+\sqrt{5}}{2} + 1 \quad \Rightarrow \quad \frac{3+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}$$

With the $\widehat{\phi}$ is do the same procedure.

Exercise 3.3-9):
supose $\log_2 k = \log_2 n$, then operating we have:

$$\frac{k * \log_2 n}{\log_2 n} = \Theta\left(\frac{n}{\log_2 n}\right) \quad \Rightarrow \quad k = \Theta\left(\frac{n}{\log_2 n}\right)$$

1.4 Divide and Conquer

Exercise 4.1-1):
$T(n) = 8T(\lfloor \frac{n}{2} \rfloor + \Theta(1))$. As mentioned before, the ceil and floor doesn't matter on anylizing algorithms when n is too big. Due to that, the result is the same $T(n) = \Theta(n^3)$.

Exercise 4.1-2):
The length of the matrix that you pass as a fourth parameter, now it should be $k * n$, for bigger values of n and k, is the same as says, $T(n) = 8T(\frac{k*n}{2}) + \Theta(1)$. Now the result is $T(n) = \Theta(k * n^3)$.

The second option is the same as the first one. Inconclusion any of them is asymptoticaly faster than the other one, both of them have the same speed.

Exercise 4.1-3):
Now is $\Theta(n^2)$ the driving function because you need to combine the solutions. Then $T(n) = 8T(\frac{n}{2}) * \Theta(n^2)$. Applying the master theorem, $n^{\log_2 8} = 3$, case 1 applies again. $f(n) = O(n^{3-\epsilon})$ for any positive $\epsilon \leq 1$

Exercise 4.1-4):
Exercise resolved in cpp on the file MatrixAddRecursive.cpp (click on that to go to the file).

The cost of that algorithm is $T(n) = 4T(\frac{n}{2}) + \Theta(1)$. The whatershed function is: $n^{\log_2 4} = n^2$. We have that $f(n) = O(n^{2-\epsilon})$ for a positive $\epsilon <= 2$. So we are on the case 1, the solution is: $T(n) = \Theta(n^2)$.

The case we have: $T(n) = 4T(\frac{n}{2}) + \Theta(n^2)$. The whatershed function is: $n^{\log_2 4} = n^2 = f(n) = \Theta(n^2)$. Applying case 2, the solution is: $T(n) = \Theta(n^2 \log_2 n)$.

Exercise 4.2-1):

$$
\begin{aligned}
S_1 &= 8 - 2 = 6 \\
S_2 &= 1 + 3 = 4 \\
S_3 &= 7 + 5 = 12 \\
S_4 &= 4 - 6 = -2 \\
S_5 &= 1 + 5 = 6 \\
S_6 &= 6 + 2 = 8 \\
S_7 &= 3 - 5 = -2 \\
S_8 &= 4 + 2 = 6 \\
S_9 &= 1 - 7 = -6 \\
S_{10} &= 6 + 8 = 14
\end{aligned}
$$

The P values are:

$$
\begin{aligned}
P_1 &= 1 * 6 = 6 \\
P_2 &= 4 * 2 = 8 \\
P_3 &= 12 * 6 = 72 \\
P_4 &= 5 * -2 = -10 \\
P_5 &= 6 * 8 = 48 \\
P_6 &= -2 * 6 = -12 \\
P_7 &= -6 * 14 = -84
\end{aligned}
$$

The result of the matrix is:

$$C_{11} = 48 - 10 - 8 - 12 = 18$$
$$C_{12} = 6 + 8 = 12$$
$$C_{21} = 72 - 10 = 62$$
$$C_{22} = 48 + 6 - 72 + 84 = 66$$

$$C = \begin{bmatrix} 18 & 12 \\ 62 & 66 \end{bmatrix}$$

Exercise 4.2-2):
You can find the pseudocode on that website (it's not mine, it's from another one): https://atekihcan.github.io/CLRS/04/E04.02-02/.

I implemented it in C++ using the code of exercise 4.1-4 (adding the function to substract matrices). You can find the result of that code clicking there: StrassensAlgorithm.cpp.

Exercise 4.2-3):
The largest k should be 7 because it's the number os time that the algorithm is called recursively. Also we can prove that applying the case 1 of the master theorem to $T(n) = aT\left(\frac{n}{2}\right) + \Theta(1)$ where $a = k$ for the reasons explained at the beggining.

Exercise 4.2-4):
Strassen's alrogithm is better that the navie way of multyplying matrices.

Then we know that the Strassen's algorithm spends $\Theta(n^{\log_2 7}) = \Theta(n^{2.8})$. Then applying the *68 x 68* where $n = 68$, applying this method the amount of time required is $68^{2.8} = 135215$. For $n = 70$, the result is $70^{2.8} = 146647$. The last one is $n = 72$ and the result is $72^{2.8} = 158683$. The Strassen's alrogithm is a little bit inneficienter than the Pan ways to multiply this particular matrices.

Exercise 4.2-6):
It will be done using block matrices (join 2 differents matrices in 1 matrice to operate). Then the solution will be $M^2 \quad \Rightarrow \quad O((2n)^\alpha) = O(n^\alpha)$.

Exercise 4.7-2):
To check that is polynominal-growt we need to proof $\frac{f(n)}{n^c}$ for $f(n) = n^2$. We need to check if there exists such a constat c that: $\frac{n^2}{n^c} = n^{2-c}$. In that case for c = 3 we have: $\frac{n^2}{n^3} = n^-1 = \frac{1}{n}$. Since $\frac{1}{n}$ is bounded for any $n \geq 1$, $f(n) = n^2$ satisfies the polynominal growth condition.

On the other hand we will apply the same argumentation. we need to check if there exists any c for $\frac{2^n}{n^c}$. We supose the case of $\lim_{n\to\infty} \frac{2^n}{n^c}$. In that case 2^n growth exponentialy to the ∞ while n^c growth polynomially. Hence the exponentialy is not bounded and 2^n does not satisfies the polynominal-growth.

Exercise 4.7-3):
If n tends to ∞, the smallest possible value of $f(n)$ will be 0, assuming it has a negative exponent. On the other hand, if $f(n)$ is a polynomial function like n^2, all values will be well-defined because it satisfies the polynomial-growth condition.

Exercise 4.7-4):
As explained before 2^n doesn't satisfies the polinominal-growth condition.

To proof $f(\Theta(n)) = \Theta(f(n))$ we need a constant c that scales in the same asymptotic way. In that case: $f(nc) = 2^{nc} = (2^n)^c = \Theta(f(n))$. So this is a valid example.

Exercise 4.7-5):

a) $a_1 = a_2 = a_3 = 1$, $b_1 = 2$, $b_2 = 3$, $b_4 = 6$. We need to find a p value that $\left(\frac{1}{2}\right)^p + \left(\frac{1}{3}\right)^p + \left(\frac{1}{6}\right)^p = 1$. For $p = 0$, the result is 3. for $p = 1$, the result is: $p = 0.9999....$ Then we know that $0 < p < 1$. $f(x) = n \log_2 n$

$$\begin{aligned} T(n) &= \Theta\left(n^p \left(1 + \int_1^n \frac{f(x)}{x^{p+1}}\,\mathrm{d}x\right)\right) \\ &= \Theta\left(n^p \left(1 + \int_1^n x^{-p} \log_2 x\,\mathrm{d}x\right)\right) \\ &= 1 + \frac{n^{1-p}\log_2 n}{1-p} - \frac{n^{1-p}}{(1-p)^2} \end{aligned}$$

For $p = 1$ we see the biggest order is $T(n) = \Theta(n \log_2 n)$.

1.5 Probabilistic Analysis and Randomized Alrogithms

Note: Most exercises of this part weren't done because it wasn't my scope of understanding that book. Anyway, I just suck at learning it and I dont have too much free time to roll up my sleves and learn that perfectly. Maybe in the future I will try this again.

Exercise 5.1-1):
You know who is better because best starts on 0, that is the lees rank that u can achieve. The biggest rank is bound at the upper with a maximum score of n. Betwen on that interval of scores you can index eassly each worker by the score he got.

Exercise 5.1-2):
An imlementation could be multiply a for the value that was got by *Random(0, 1)* and then make another math operation with b for example to introduce aleatority and give as a result the number between a and b. That operations could be: $a + \lfloor (Random(0, 1) \cdot (b - a + 1)) \rfloor$ That would give us a number between [0, b - a + 1] and then adding a, we take a value between [a, b].

The expected running time is $O(1)$.

Exercise 5.1-3):
We see the 2 posible values that exists with p and 1 - p are:

$$\begin{aligned} P(00) &= (1-p)(1-p), \\ P(01) &= (1-p)p, \\ P(10) &= p(1-p), \\ P(11) &= pp. \end{aligned}$$

01 and 10 happends with the same probability. If we have 01 we return 0. On the other hand if we have 10 return 1. If neither of that values, keep trying until get 1 of these values.

Algorithm 8 Generación de bit imparcial a partir de BIASED-RANDOM

```
procedure UNBIASED-RANDOM
    while true do
        first ← BIASED-RANDOM()
        second ← BIASED-RANDOM()
        if first = 0 and second = 1 then
            return 0
        else if first = 1 and second = 0 then
            return 1
```

Running time is $\frac{2}{2p(1-p)} = \frac{1}{p(1-p)} = O\left(\frac{1}{p(1-p)}\right)$

Exercise 5.2-1):
First of all we will use the same indicator random variable as mentioned in the book few pharagraphs before. The probability of being hired is $\frac{1}{n}$ *from 1 to n - 1*, That's mean because it's presented in a random order, it could be someone before you that have better scores and he will be hired instead of you.

To hire exactly n times, its $\frac{1}{n!}$ because there is n permutations where you hire everyone but only one of them is correct.

Exercise 5.2-2):
$\frac{(n-1)(n-2)!}{n!}$ There is n - 1 positions to place the best candidate and also you have n - 2 possibilities of place 2 person to hire them at the same time.

Exercise 5.2-3):
Let $X = \sum_{i=1}^{n} X_i$, where X_i is the value of the i-th die. We use indicator random variables $I_i(k)$, which take the value 1 if die i shows the value k, and 0 otherwise. Then we can write:

$$X_i = \sum_{k=1}^{6} k \cdot I_i(k)$$

The expected value of X_i is:

$$E[X_i] = E\left[\sum_{k=1}^{6} k \cdot I_i(k)\right] = \sum_{k=1}^{6} k \cdot E[I_i(k)]$$

Since $E[I_i(k)] = \frac{1}{6}$, we get:

$$E[X_i] = \sum_{k=1}^{6} k \cdot \frac{1}{6} = \frac{1}{6}(1+2+3+4+5+6) = 3.5$$

Now, the expected value of X is:

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = n \cdot 3.5$$

Exercise 5.2-5):
$X_i = Icustomergetshisownhat$. Probability of customer i recieves his own hat is $E[X_i] = PrI = \frac{1}{n}$.

$$E[x_i] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{n} = \frac{n}{n} = 1$$

The number of costumers that receives his own hat is 1.

Exercise 5.2-6):
$X_ij = Ianinversionwasfound.$ Probability of the inversion ij to be found is $E[X_ij] = PrI = \frac{1}{2}$. Number of inversions is:

$$E[X_{ij}] = E\left[\sum_{1\leq i\leq j\leq n}^{n} X_{ij}\right] = \sum_{1\leq i\leq j\leq n}^{n} E[X_{ij}] = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}.$$

2 Sorting and Order Statics

2.1 Heapsort

Exercise 6.1-1):
The minimum number of elements is: 2^h and the maximum is $2^{h+1} - 1$

Exercise 6.1-2):
With the tree of the figure 6.1 for example the element of position 4 (whose value is 8) it's on height $\lfloor \log_2 4 \rfloor = 2$. And as the book explained before, That's the second level.

Exercise 6.1-3):
You have to keep in mind the property that for max-heap, $A[parent] \geq A[child]$. Following that, you achieve the tree like in figure 6.1.

Exercise 6.1-4):
In the most right position on the leafs.

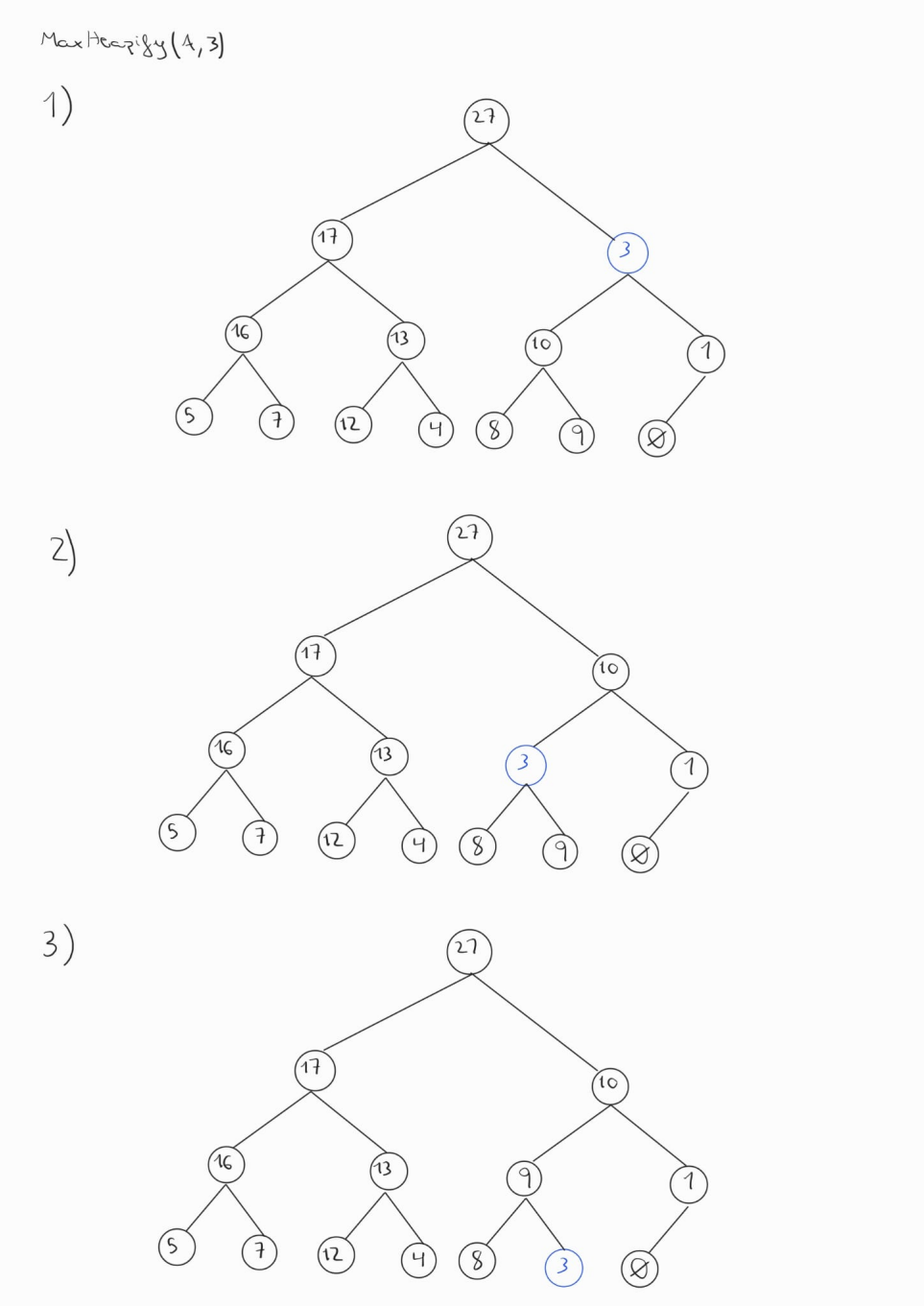
Exercise 6.1-5):
Between the root or level 1 or 2.

Exercise 6.1-6):
If it's sorted in ascending order, yes, it's a min-heap.

Exercise 6.1-7):
No, The parent with value 15 has a child with value 16 and the condition of max-heap isn't satisfy.

Exercise 6.1-8):
We are going to supose the same tree as figure 6.1. That tree has 10 elements. We know the leafs are 9, 3, 2, 4, 1 in this case, that's 5 leafs. In that case the number 9 are at the index 6, which is the same as say $\frac{10}{2} + 1$ as the statement said. With the other numbers is the same as that.

Exercise 6.2-1):
Done on the next page



Exercise 6.2-2):
Supose A_1 and A_2 are the childrens of the root. They are also the roots of another subtrees. We know they are balanced so one subtree can't have more nodes that the other subtree. If one tree has more than $\frac{2n}{3}$ nodes, the other one must have $\frac{n}{3}$ and that would violate the properties of the binary trees.

The smallest possible value is $\alpha = \frac{2}{3}$.

Exercise 6.2-3):
The pseudocode used as reference is the one that appears on page 165. It would be the same but only changing the conditions of the first 2 if. Also we can change the variable name *largest* with *lowest* The changes would be $if l \leq A.heap_size and A[l] < A[i] then lowest = l$, on the other if would be $if r \leq A.heap_size and A[r] < A[lowest] then lowest = r$. The reamining lines of the pseudocode would be the same.

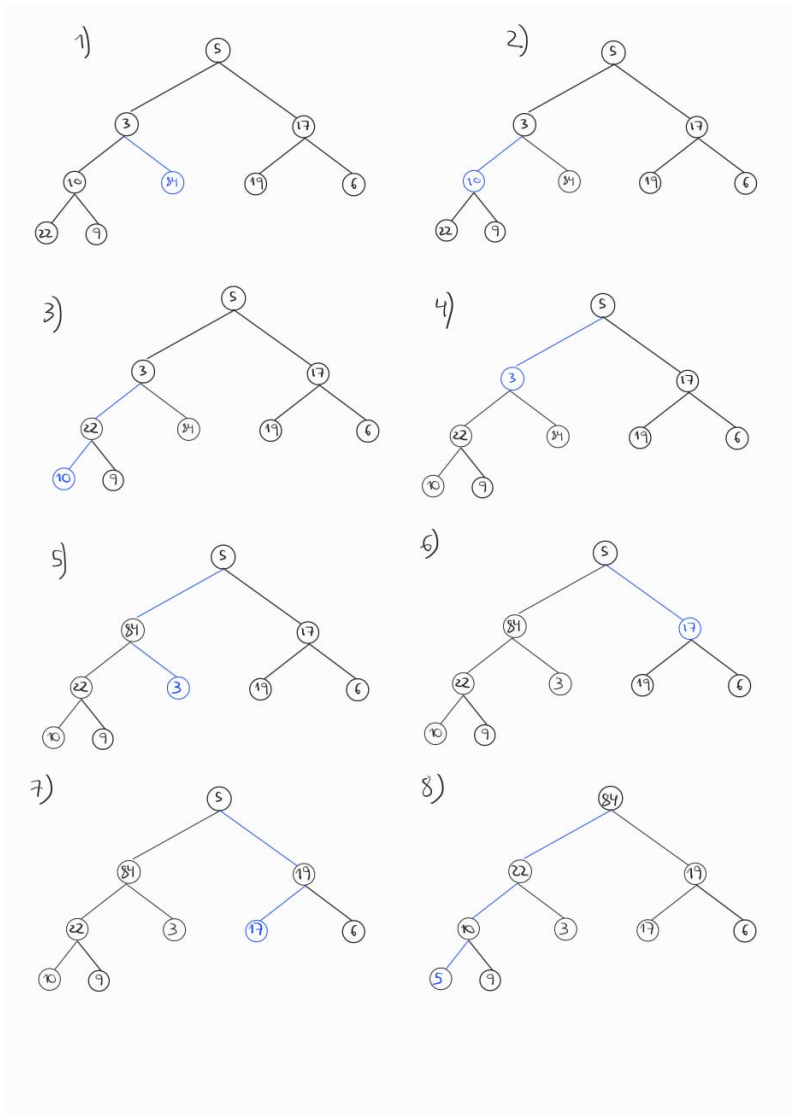
Exercise 6.2-4):
it won't continue making recursive calls, because that mean the subtree on the root has the biggest value posible following the definition of max-heap.

Exercise 6.2-5):
Anything because from A.heap-size / 2 it's located all the leafs of the main tree. Due to that, any comparasions will be done.

Exercise 6.2-6):
Resolved on the file: 6.2-6.cpp

Exercise 6.2-7):
Supose that you start from the root. Also supose that from the inmediate children of the root, it will be called recursively. As we are going down until reach a leaf, also we know a tree has $\log_2 n$ levels in that case it's the same as the amount of time the recursive call will be done. Due to that, the worst case is $O(\log_2 n)$ as the sentence says.

Exercise 6.3-1):



Exercise 6.3-2):

$$\frac{n}{2^h} \geq \frac{1}{2} \quad \Rightarrow \quad n \geq \frac{1}{2}2^h \quad \Rightarrow \quad \log_2 n \geq h$$

Because h is the height of the tree, and the height starts from 0, the result is:

$$0 \leq h \leq \lfloor \log_2 n \rfloor$$

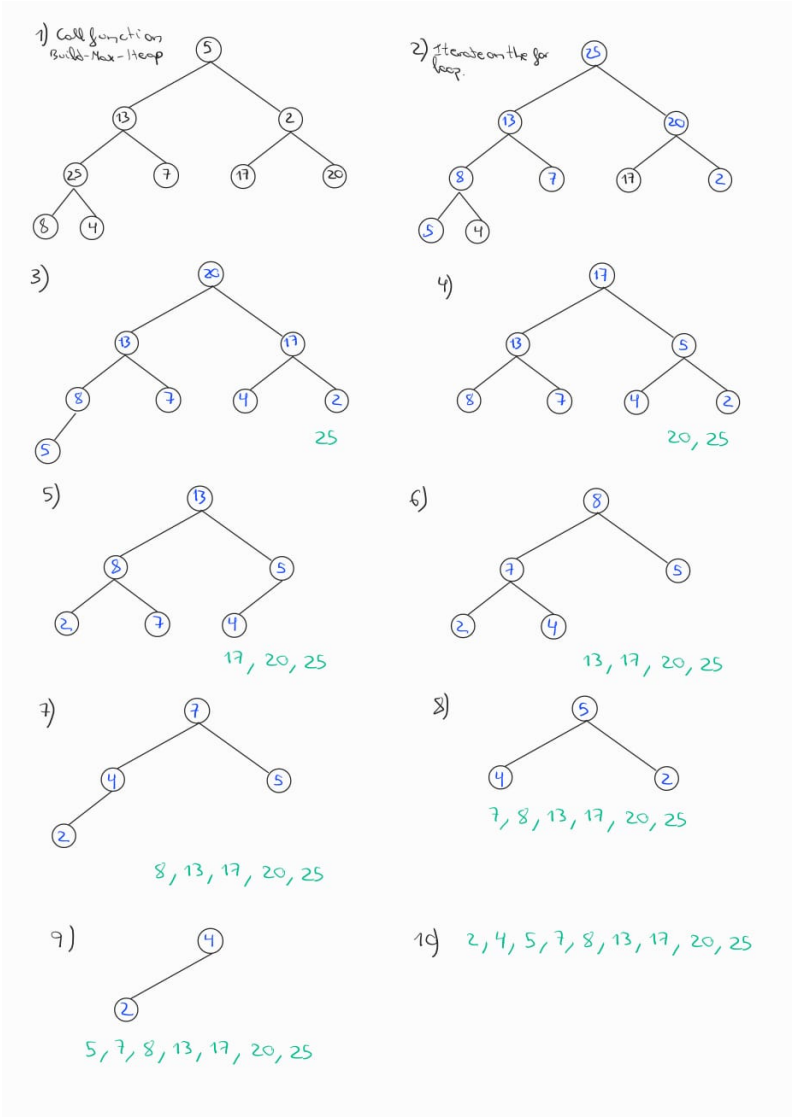
Exercise 6.3-3):

Doing from bootom-up you have guarantees that the lees posible subtree is ordered. Hence on the fathers subtrees, when you order them you have guarantees that the child subtree is ordered. While on the other way we cant guarantee that.

Exercise 6.3-4):

We know that on each height on the tree, the numbers of elements got doubled. Also, the number of elements in the heap, is at least n and the ttal number of elements are $2^h + 1$. Due to that the result is what the sentence is asking about proof.

Exercise 6.4-1):



Exercise 6.4-2):
Initialization: At the start of the loop (before the first iteration), the subarray $A[1 : n]$ is not sorted, but in the first iteration of the loop, the `MaxHeapify` function is executed over the entire array, ensuring that the array becomes a **max-heap**. Then, the largest element (at the root of the max-heap) is placed in its final position, that is, at the last position of the array, ensuring that the subarray $A[1 : n]$ contains the smallest elements, while $A[n]$ contains the largest element.

Maintenance: In each iteration of the loop, the following steps occur:

1. The subarray $A[1 : i]$ is still a **max-heap** containing the i smallest elements of $A[1 : n]$.
2. Then, the element at the root (the largest one) is moved to position i. At this point, the subarray $A[1 : i - 1]$ contains the $i - 1$ smallest elements sorted (i.e., the largest ones).
3. The subarray $A[i : n]$ now contains largest elements sorted.
4. The size of the heap is reduced by one (since the largest value has been placed in its correct position), and the `MaxHeapify` function is applied again to restore the heap property in the subarray $A[1 : i - 1]$.
5. This process repeats until the array is completely sorted.

Termination: The loop terminates when i reaches 1, that is, when the array is completely sorted. At the end of all iterations, the subarray $A[1 : n]$ will be sorted in increasing order, and the max-heap property will have been maintained at all times.

Exercise 6.4-3):
It would be $\Omega(n \log_2 n)$ because on the inner loop it will always remove the largest value, replace the root value with the less one, and then execute the function, it makes all time expend $\Theta(n \log_2 n)$.

Exercise 6.4-4):
Explained on the before exercise. Same logic as this.

Exercise 6.4-5):
It's the same explaination as the 2 before exercise. In this case the first function outside the loop could be in the average time, but the inner loop will be always $\Theta(n \log_2 n)$.

Exercise 6.5-1):
It just extract the element at the first position (because it's the biggest one). Then do the swap as HeapSort with the last element, to move at the first position the next biggest element.

Exercise 6.5-2):
Is added on the index with the value of textitheap_size, then when calling the function *Max_heap_increase_key*, it will compare the value of the key with its parent keys, and will change the position if the parent key is less than the new key to be inserted. When that condition is not satisfied (or the new key reaches the root), it will end the procedure of insert because it will be inserted in the correct position.

Exercise 6.5-3):
the functions *Min-Heap-Minimum* and *Min-Heap-Exctract-Min* are the same as the max version because the min-heap will have the least value possible on the root. On the *Min-Heap-Decrease-Key*, the only change needed is at the while condition. The max version has $whilei > 0$ *and* $A[Parent(i)].key < A[i].key$, while the min version would be $whilei > 0$ *and* $A[Parent(i)].key > A[i].key$, changing the less operator by the greater operator. The change needed on *Min-Heap-Insert* is $-\infty$ by ∞.

Exercise 6.5-4):

Algorithm 9 MAX-HEAP-DECREASE-KEY(A, i, k)

1: **Input:** A max-heap A, index i, new key value k with $k \leq A[i]$.key
2: **Output:** Updated max-heap with $A[i]$.key $= k$
3: **if** $k > A[i]$.key **then**
4: **error** "new key is larger than current key"
5: $A[i]$.key $\leftarrow k$
6: **call** MAX-HEAPIFY(A, i)

Exercise 6.5-5):
It could contain any value. Also it could take a value larger than the key and it will cause an error, so adding that value, we make sure that no error will occur.

Exercise 6.5-6):
Max-Heapify "sinks" the element correcting the errors at the directoin to their childs. The violation that can happens is that the new key value is larger than it's father, o fix that it wuld need to buuble up. So with the function *Max-Heapify* this can't be done, because it asume that all subtrees are max-heaps. Therefore, it couldn't be that case and it could be positioned in a bad position.

Exercise 6.5-7):
Initialization: At the beggining, we have a new element in a leaf, the a and b conditions are true but the c condition can occur with the new element inserted.

Maintenance: On each iteration it will interchange the child with the father if the father has lees value key than the child. Therefore the condition a and b will be satisfied on each iteration if that change is done. But the biggest subtrees couldnt be satisfied leading to the option c. Therefore it will be changing all time with their fathers until satisfies the condition that $A[1 : A.heap - size]$ satisfies the max-heap property.

Termination: When the while loops end. The condition a and b will be true for all possible subtrees, but the condition c wont' be satisfied because the start invariant which says $A[1 : A.heap - size]$ satisfies the max-heap property, it will be true to every subtree. Hence it's satisfies the condition when the first call is done to that function.

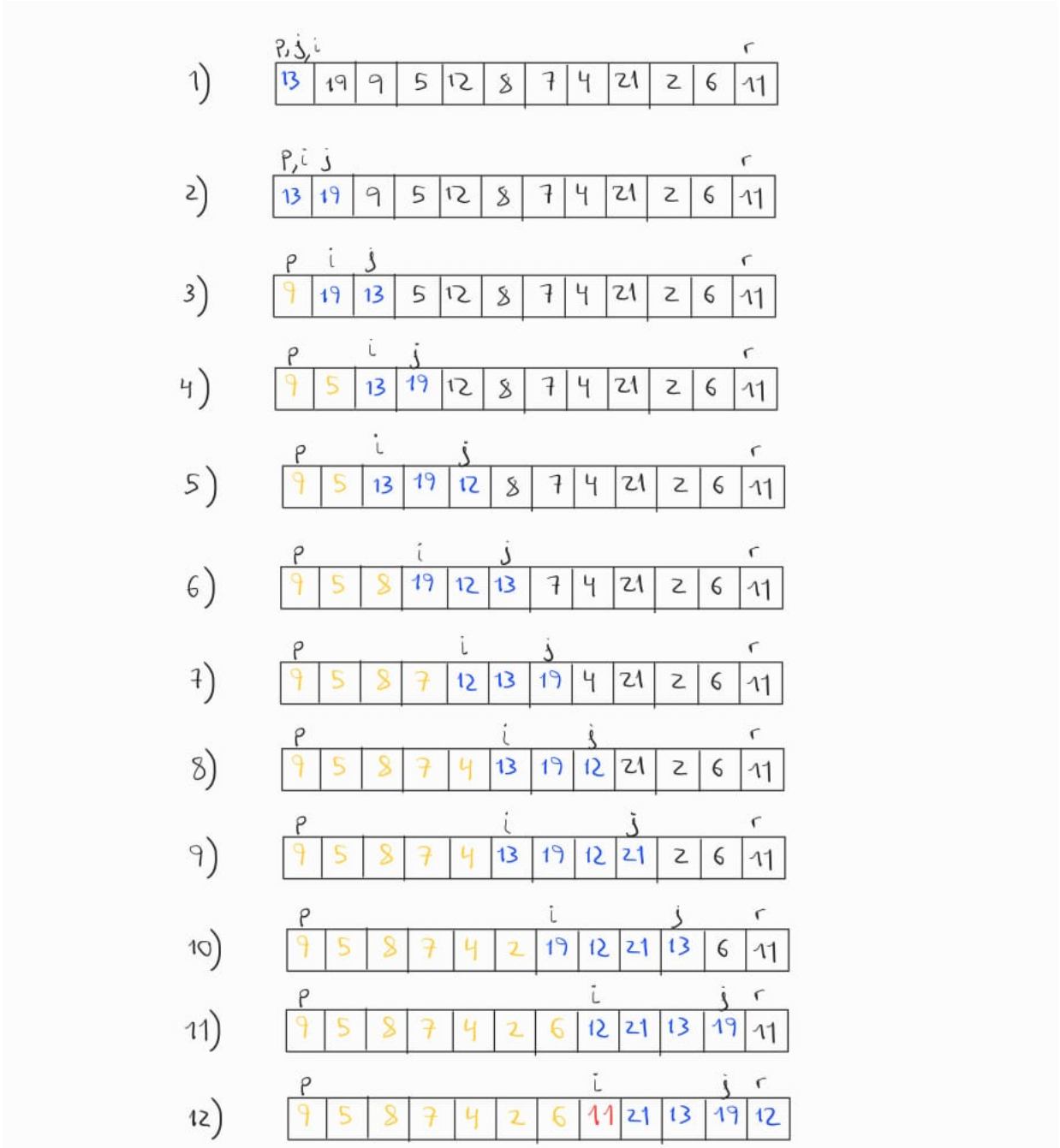
Exercise 6.5-8):
The idea is to move to the bottom just the key values, it will make only 1 assigment. Then when you need to retrieve the objects with that values, you only need to retrieve the key value and find which object is associated with that key value.

Exercise 6.5-9):
To each element that is inserted, you start associating a key with the biggest possible value. Doing that you make sure that the first value inserted it will be the first value to be extracted when the function *Max-Heap-Extract-Max* is called (is the analogous) function for the pop function of a queue.

Exercise 6.5-10):
Replace the node to be deleted witht he last node of the heap. Update the size of the heap (incrementing it by 1), and then call *Max-Heapify* to sort again the elements. This function called has a runtime of $O(\log_2 n)$ as we explained at exercise done before. Due to that this function has a complexity of $O(\log_2 n)$ too.

2.2 QuickSort

Exercise 7.1-1):



Exercise 7.1-2):
It returns the last value of the array, in other words, q = r. To fix that we could introduce a boolean value initialized to false. If any swap happends, set that boolean to true. At the end, if the boolean is false, return $\lfloor (p+r)/2 \rfloor$.

Exercise 7.1-3):
At the beggining, $p = 1$ and $r = n$ where n it's the size of the array. Due to that, the for loop iterates n elements of the array.

Exercise 7.1-4):
Change the $if A[j] \leq x$ by $if A[j] > x$ To sort in decrease order.

Exercise 7.2-1):
Induction hypotesis used is $T(m) = cm^2$. $T(n) \leq c(n-1)^2 + kn \leq cn^2$. Due to that, the solution is $T(n) = cn^2$.

Exercise 7.2-2):
We are on the case a of figure 7.5. Then $T(n) = T(n-1) + \Theta(n) = \Theta(n^2)$.

Exercise 7.2-3):
Because all elements are less than the pivot, we are on the partition case of the exercise 7.2-1 because 1 subarray will be empty and the other one will have $n-1$ values. Hence the running time is $\Theta(n^2)$.

Exercise 7.2-4):
We have to remaind that Insertion Sort is good when all elements are almost sorted. In that case insertion sort is $O(n+d)$ where d is too small. Therefore the running time is $O(n)$. While QuickSort, have to select the pivot, then compare the elements, move them depends of the pivot, etc. Due to that, Insertion Sort is $O(n \log_2 n)$ which is slower than $O(n)$.

Exercise 7.2-5):
This exercise is the same as the Figure 7.4 but with constant values. One array will be partitioned by the contstant α and the other one by β. Because β is bigger, this partition will end later than α. However each one will have a depth $\log_{1/\alpha} n$ and the other one $\log_{1/\beta} n$.

Exercise 7.2-6):
Assume we have an array of n distinct elements where all permutations are equally likely. For the partition to be at least as balanced as $(1-\alpha) : \alpha$ (with $0 < \alpha \leq \frac{1}{2}$), the pivot must be chosen among the elements ranked between

$$\lceil \alpha n \rceil \quad \text{and} \quad \lfloor (1-\alpha)n \rfloor.$$

The number of such "good" pivot positions is approximately

$$\lfloor (1-\alpha)n \rfloor - \lceil \alpha n \rceil + 1 \approx (1-2\alpha)n.$$

Since each element is equally likely to be chosen as the pivot, the probability of a balanced split is

$$\frac{(1-2\alpha)n}{n} = 1 - 2\alpha.$$

Exercise 7.3-1):
Because for the same inputs on each execution, the result it isn't the same because there is a random factor on the algorithm.

Exercise 7.3-2):
The best one is: $T(n) = 2T(\frac{n}{2}) + O(1) = O(n)$.

The worst is : $T(n) = T(n-1) + O(1) = O(n)$.

2.3 Sorting in Linear Time

Exercise 8.1-1):
Following the theorem 8.1, the possible depth is $\log_2 h$.

Exercise 8.1-3):
Supose for the sake of contradiction, that there exists a comparison sort that runs in linear time, say $O(n)$, on at least half of the inputs. That is, there exists a constant $c > 0$ such that for at least $\frac{n!}{2}$ inputs,

$$T(n) \leq c\,n.$$

Let the running time on the remaining inputs be at most $O(n \log n)$ (since no algorithm can beat the $\Omega(n \log n)$ worst-case lower bound). Then the overall average running time would be bounded by

$$\frac{1}{n!}\left(\frac{n!}{2} \cdot c\,n + \frac{n!}{2} \cdot O(n \log n)\right) = \frac{1}{2} \cdot c\,n + \frac{1}{2} \cdot O(n \log n) = O(n \log n).$$

This suggests that the average running time is only a constant factor times $n \log n$. However, the decision tree lower bound implies that almost all inputs (i.e., a non-negligible fraction) must incur $\Omega(n \log n)$ comparisons to reach the lower bound. In other words, if half the inputs were as fast as $O(n)$, the average could not be as high as $\Omega(n \log n)$.

Thus, no comparison sort can run in $O(n)$ time on at least half of the $n!$ inputs.

What about a fraction $1/n$ or $1/2^n$?
If an algorithm were to run in linear time on a fraction $1/n$ (or $1/2^n$) of the inputs, the contribution of those inputs to the overall average would be

$$\frac{1}{n!}\left(\frac{n!}{n} \cdot O(n)\right) = O(1) \quad \text{(or even smaller for } 1/2^n\text{)},$$

while the remaining inputs (which form almost all of the $n!$ inputs) must still incur $\Omega(n \log n)$ comparisons. Therefore, the overall average remains $\Omega(n \log n)$.

Exercise 8.1-4):
We show that any comparison sort must take $\Omega(n \log n)$ comparisons even for inputs that satisfy the following promise: For every index i with $i \bmod 4 = 0$, the element that is initially at position i will, after sorting, appear in one of the positions $i-1$, i, or $i+1$. For the remaining positions (those with $i \bmod 4 \neq 0$), no constraint is given.

Consider the set S of all input orders consistent with this promise. Note that the elements in positions i with $i \bmod 4 \neq 0$ may appear in any order. Thus, there are at least $(3n/4)!$ possible orderings for these elements. By Stirling's approximation, we have

$$\log_2\big((3n/4)!\big) = \Omega(n \log n).$$

Since any comparison sort must distinguish among all these distinct inputs, its decision tree must have at least $\Omega(2^{n \log n})$ leaves. This implies that in the worst case the sort must perform at least $\Omega(n \log n)$ comparisons.

Exercise 8.2-1):

Exercise 8.2-2):
It's stable because u start order them on B from the final, so the last values added on the vector A, is suposed to be on the last position. If vector A has two equals values, the first one that appear starting from the final of the array, is placed at the position k, while the next element with the same value, will be placed at $k-1$, making that algorithm stable.

Exercise 8.2-3):
In this case the first value inserted on the array will be on a position k on the output array. The second time that appear an element with the same value, will be situated on $k-1$ changing the order when the elements were

inserted. In that case the **stable** property is not guarantee.

Exercise 8.2-4):
Initialization:
Before the loop starts, the array C has been computed such that $C[i]$ stores the position where the last element of value i should be placed in B. Since no elements have been copied yet, the invariant holds.

Maintenance:
At the start of each iteration, the last unplaced element in A of value i is correctly positioned in $B[C[i]]$. After placing it, $C[i]$ is decremented, ensuring the next occurrence of i is placed correctly in the next iteration. Thus, the invariant holds throughout the loop.

Termination:
When the loop ends, all elements from A have been placed in B in a stable order. Since every element was placed in its correct position using C, and $C[i]$ was decremented correctly, B is now a sorted version of A.

Thus, the loop invariant holds, proving the correctness of COUNTING-SORT.

Exercise 8.2-5):

Algorithm 10 Counting Sort In-Place

Require: Array A of size n, maximum value k
Ensure: Sorted array A

```
Initialize array C of size k with zeros
for j = 0 to n − 1 do
    C[A[j]] ← C[A[j]] + 1
index ← 0
for i = 0 to k − 1 do
    while C[i] > 0 do
        A[index] ← i
        index ← index + 1
        C[i] ← C[i] − 1
```

Exercise 8.2-6):
It's just consist of making the first 2 for loops of counting sort. Then when it gives a and b, just substract and return $C[b] - C[a-1]$ to give the answer to that question.

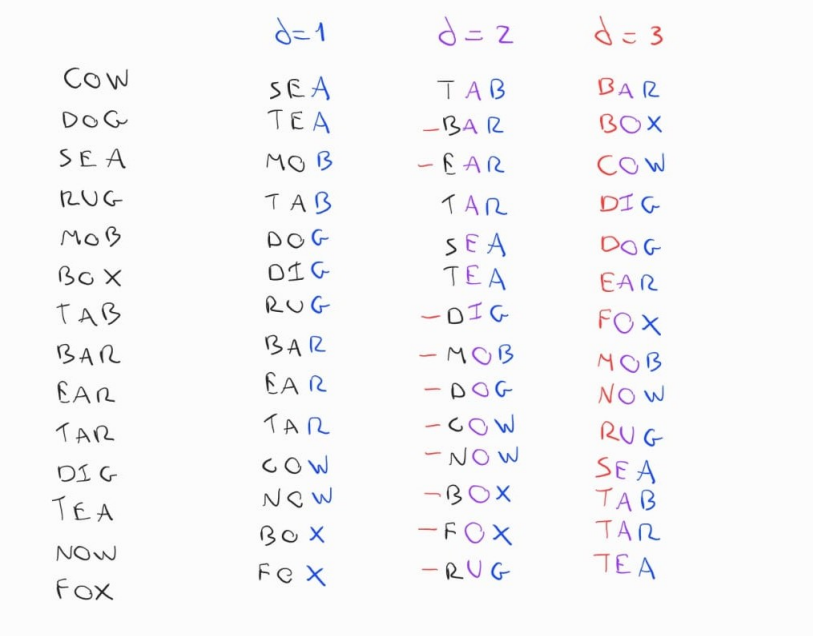
Exercise 8.2-7):

Algorithm 11 Counting Sort for Numbers with Fractional Parts

Require: Array A of n numbers in range $[0, k]$ with at most d decimal digits
Ensure: Sorted array A in $O(n + 10^d k)$ time

```
Multiply each A[i] by 10^d to convert to an integer
Find max value k' ← 10^d k
Initialize count array C of size k' + 1 with zeros
for j ← 0 to n − 1 do
    C[A[j]] ← C[A[j]] + 1
for i ← 1 to k' do
    C[i] ← C[i] + C[i − 1]
for j ← n − 1 to 0 do
    Place A[j] in sorted position using C
    C[A[j]] ← C[A[j]] − 1
Divide each A[i] by 10^d to restore original scale
```

Exercise 8.3-1):



Exercise 8.3-2):
Insertion sort, merge sort are stables. On the other hand, heapsort and QuickSort aren't stables.

Exercise 8.3-3):
Base Case: For $d = 1$, radix sort uses a stable sort on one digit. Thus, the array is correctly sorted by that digit.

Inductive Hypothesis: Assume that after sorting by $d - 1$ digits, the array is correctly sorted with respect to the $d - 1$ least significant digits.

Inductive Step: Now, sort by the dth digit using a stable sort. Because the sort is stable, elements with equal dth digits retain their relative order from the previous pass (sorted by the lower $d - 1$ digits). Thus, the entire array becomes sorted by all d digits.

Conclusion: By induction, radix sort sorts the array correctly. The proof requires the stability assumption in the inductive step, to preserve the order established by the lower digits when sorting by a higher digit.

Exercise 8.3-5):
It could be make using radix sort because they are positive numbers. We need to know if the n^3 are dozens, hundreds, to set the value d for the radix sort algorithm. Then apply radix sort to sort it in $O(d(n))$. Keep in mind that values that have less length that n^3, will be evaluated to 0 when their length is exceded, making them be at the first position.

Exercise 8.3-6):
It will require d passes. In the worst case there will be 10 piles of card at the same time.

2.4 Medians and Order Statics

Exercise 9.1-1):
Solution on cpp: Exercise9.1_1.cpp.

Exercise 9.1-2):
As explained before it's necesary $3\lfloor \frac{n}{2} \rfloor$ if it's odd, or $3(n-2)/2$ if it's even.

Exercise 9.1-3):
To determine the fastest horse need $\frac{25}{n} + 1$ where n it's the amount of horses that race at a time. It will need also the same, to take the 3 fastest one. Instead of only store the fastest one on the sixth race, it will need to store the 3 best horses on the sixth race.

Exercise 9.1-4):
Each pair requires 1 comparison to find both the local maximum and minimum. After pairing, you have $\frac{n}{2}$ maximum candidates and $\frac{n}{2}$ minimums. For odd n, one extra comparison is needed for the unpaired element. Next, you need $\frac{n}{2} - 1$ comparisons to find the overall maximum. Similarly, $\frac{n}{2} - 1$ comparisons are required to find the minimum. Thus, the total number of comparisons is $\lceil \frac{3n}{2} \rceil - 2$. This is the lower bound for the number of comparisons in the worst case.