Exercise of Getting Started

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1 Chapter 1: Foundations

1.1 The Role of Algorithms in Computing

I didn't make the exercise of this section because I didn't find them useful.

1.2 Getting Started

Exercise 1.2-1):
It could be an application like booking. When you search a hotel close to the airport, it gets involved algorithms as searching the hotels close to that airport and it should be searched in a short time period.

Exercise 1.2-2):

$$8n^2 < 64n \cdot \log_2 n \quad \rightarrow \quad n < 8 \cdot \log_2 n$$

Try values until this inequality is false. To $n \lessapprox 43$, insertion sort runs faster than merge sort.

Exercise 1.2-3):

$$100n^2 < 2^n$$

Trying values, for $n \lessapprox 15$, 2^n runs faster than $100n^2$.

Exercise 1.2-4):

View photo of the exercise on the next page.

	1 second	1 minute	1 hour	1 day	1 month	1 year	1 century
$\lg n$	∞	∞	∞	∞	∞	∞	∞
$\sqrt{n}$	10^{12}	$3.6 \cdot 10^{15}$	$1.3 \cdot 10^{19}$	$7.47 \cdot 10^{21}$	$6.91 \cdot 10^{24}$	$9.95 \cdot 10^{26}$	$9.95 \cdot 10^{30}$
n	10^6	$6 \cdot 10^7$	$3.6 \cdot 10^9$	$8.64 \cdot 10^{10}$	$2.63 \cdot 10^{12}$	$3.16 \cdot 10^{13}$	$3.16 \cdot 10^{15}$
$n \lg n$	62746	$2.8 \cdot 10^6$	$1.33 \cdot 10^8$	$2.76 \cdot 10^9$	$7.29 \cdot 10^{10}$	$7.99 \cdot 10^{11}$	$6.87 \cdot 10^{13}$
n^2	1000	7746(aprox)	60000	293939	$1.62 \cdot 10^6$	$5.62 \cdot 10^6$	$5.61 \cdot 10^7$
n^3	100	391 (aprox)	1532	4420	13803	31601	146679
2^n	20(aprox)	26(aprox)	32	36	41	44	51
$n!$	9	11	12	13	15	16	17

Exercise 2.1-1):
Note: Resolved using the logic of C, C++, Java, etc. while iterating over an array on a for loop. Also the number that appears in green, is the number being checked. The number or numbers that appears in red are the numbers being moved.

i	Array
1)	[31, 41, 59, 26, 41, 58]
2)	[31, 41, 59, 26, 41, 58]
3)	[26, 31, 41, 59, 41, 58]
4)	[26, 31, 41, 41, 59, 58]
5)	[26, 31, 41, 41, 58, 59]

Exercise 2.1-2):
Initialization: The loop start getting the first number in the array. In spite of that, it has initialized to 0 the variable sum where the total sum will be stored. Due to that, the invariant holds the first number that will be added to sum.

Maintenance: On each iteration, the loop will hold only the index of the number that will be added, after add it, i will be incremented by 1, holding the next number (i + 1).

Termination: The loop will terminate when the 'n' elements of the array are added. In conclusion, sum it's equivalent of say that $sum = \sum_{i=1}^{n} A[i]$.

Exercise 2.1-3):
click this link to see the resolution → *resolution*

Exercise 2.1-4):

Algorithm 1 Linear Search

```
1: function LINEAR-SEARCH(A, n, x)
2:     for i ← 1 to n do
3:         if A[i] == x then return i
4:     return NIL
```

Initialization: The loop start getting the first element of the array.

Maintenance: On each iteration, the loop takes the next element (i + 1) and compare it with the value being search (x). If it's found return i, else, continue searching that value.

Termination: When all values are read, if x wasn't found in the array, it returns NIL to indicate that no value was found on all the array.

Exercise 2.1-5):

Algorithm 2 ADD-BINARY-INTEGERS

```
1: function ADD-BINARY-INTEGERS(A, B, n)
2:     //Initialize array C with n values
3:     carry ← 0
4:     for i ← 1 to n do
5:         c ← A[i] + B[i]
6:         C[i] ← c  mod 2
7:         carry ← c ÷ 2    //Integer division

8:     C[n] ← carry
9:     return C    //Return the array C with the values
```

Initialization: The loop starts with value of carry to 0, and getting the first bits of A and B.

Maintenance: On each iteration, the loop takes the next bits values of A and B. Add these values and calculate the value to insert into C and the carry that could exists.

Termination: All values were added and store, now C[0:n - 1] with the result of the sum. To reach the n-th value, adds the last carry value on the position n.

Exercise 2.2-1):
Like the book says, Θ notation is like saying "roughly proportional to n^2 (for example), when n is large." In this case, we remove constants, so the remaining expression is $n^3 + n^2 + n + 3$. The term with the highest exponent is n^3, so at any moment: $n^3 \gg n^2 \gg n$.
Solution: $\Theta(n^3)$.

Exercise 2.2-2):

Algorithm 3 SELECTION-SORT

```
1: function SELECTION-SORT(A, n)
2:     for i ← 1 to n − 1 do
3:         ind_small_elm ← i
4:         for j ← i + 1 to n do
5:             if A[ind_small_elm] > A[j] then
6:                 ind_small_elm ← j
7:         SWAP(A[i], A[ind_small_elm])
```

The invairant is that on each iteration of extern for, it only takes 1 by 1 element. In the inner for, also take all elemnts from i to n, and compare the value of the outter for against the inner for to take the smaller elemnt.

When the algorithm arrives to the last element, all swaps ocurred and the last element will be in the correct place.

The worst case happens when it must iterate on the outher for and also with al the elements from i to n in the inner for. So it's: $\frac{n*(n-1)}{2}$. Thats mean that avoiding all constants values, the solution is: $\Theta(n^2)$.

The best case is not better because you have to check all values in the if, the only instruction that is avoided is the instruction inside the if because the if won't be evaluated to true. But that instruction is insignificant if n it's too big.

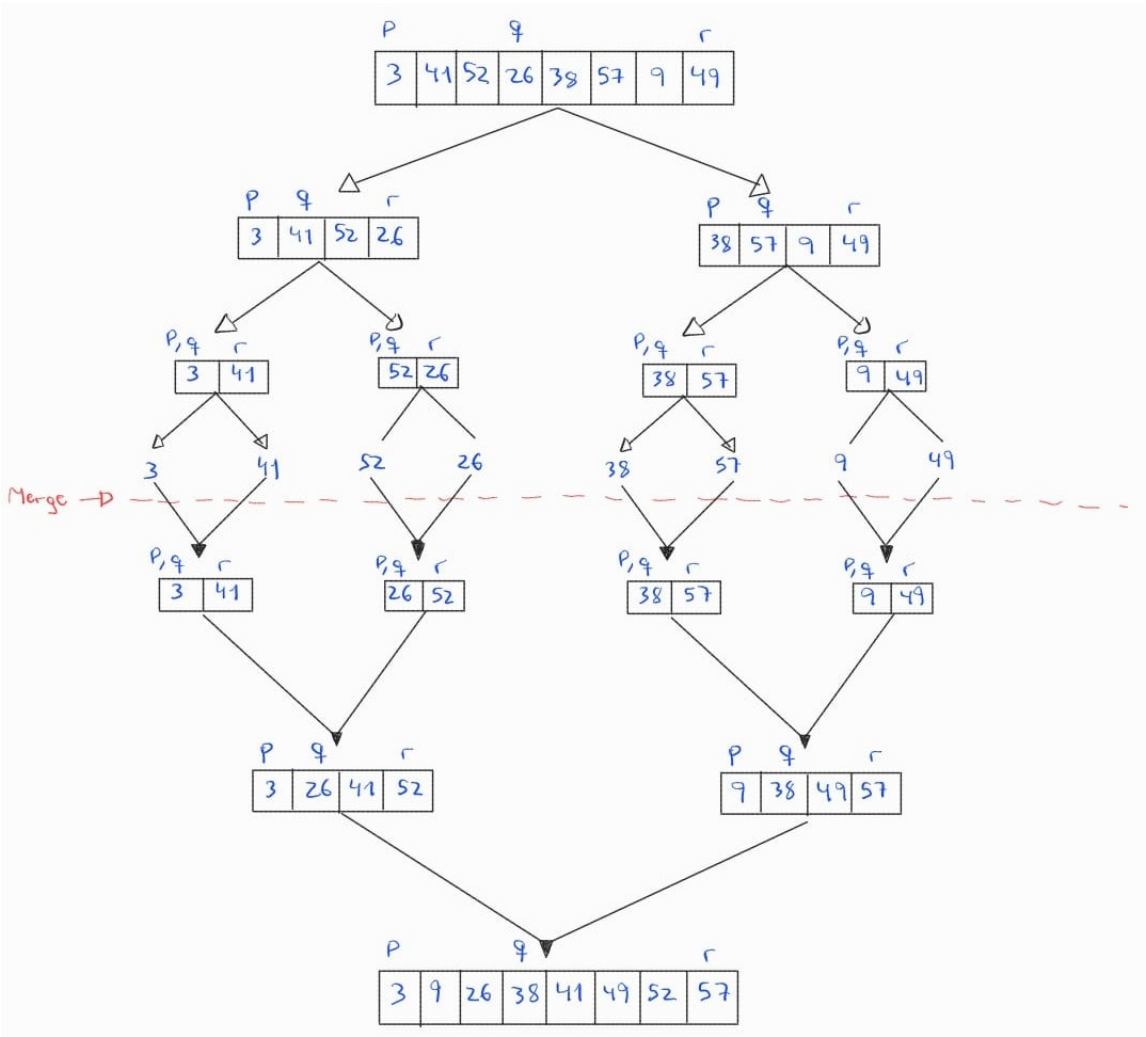
Exercise 2.2-3):
Depends on the value where is storage, if the x value is storage at the first position it will take a constant value to search it. However, if the value is in the last element (worst case) it will spend $constant * n$ time to find that value.

Averege case is suposing that it's in the middle of the array. The averegage is $\frac{n}{2} = n$ if n it's too big.

Worst case as mentioned before is $\Theta(n)$.

Exercise 2.2-3):
The only thing you could do is a preprocessing step to check if it's alredy sorted or nearly to be sorted and then apply the algorithm who best fits when the best case was achieved.

Exercise 2.3-1):



Exercise 2.3-2):
The "***if*** $p \neq r$" is not useful because $p = 1$ and $q > 1$, so the if will be evaluated to true, and the return (the termination of recursion) will execute without doing any recursion step.

Exercise 2.3-3):
Initialization: The loop starts obtaning the first element to insert it in the sorted array.

Maintenance: On each iteration, the loop insert 1 element of the left or right array, then increment k by 1, to insert on the next iteration the next value 1 by 1. The values that will be inserted with the first loop in the best case is n - *1* where n is the lenght of the subarray/array being sorted.

Termination: In one of the last 2 whiles, it will be inserted all remaining values (could be values in the left or right array) until reach the nth values in the sorted array.

Exercise 2.3-4):
$n = 2$

$$T(2) = 2$$

The solution given says that:

$$T(2) = 2 * \log_2 2 = 2 \cdot 1 = 2$$

Induction hypotesis:

$$T(k) = k \cdot \log_2 k$$

Where k is a power of 2. We want to demostrate that $n = 2k$. Use recurrence to calculate $T(2k)$:

$$T(2k) = 2T(k) + 2k \quad \rightarrow \quad T(2k) = 2 \cdot (k * \log_2 k) + 2k \quad \rightarrow \quad T(2k) = 2k \cdot (\log_2 k + 1) \quad \rightarrow \quad T(2k) = 2k \cdot \log_2 2k$$

As we mentiones before, replacing $n = 2k$, we conclude that the induction hypotesis is correct $n \cdot \log_2 n$

Exercise 2.3-5):

Algorithm 4 INSERTION-SORT-RECURSIVE

```
1: function INSERTION-SORT-RECURSIVE(A, n)
2:     if n == 0 then return
3:     INSERTION-SORT-RECURSIVE(A, n − 1)
4:     key ← A[n]
5:     i ← n − 1
6:     while i ≥ 0 and A[i] > key do
7:         A[i + 1] ← A[i]
8:         i ← i − 1
9:     A[i + 1] ← key
```

Exercise 2.3-6):

Algorithm 5 BINARY-SEARCH

```
1: function BinarySearch(A, x, l, n)
2:     if l > n then return
3:     mid ← l + n/2
4:     if A[mid] > x then
5:         BinarySearch(A, x, l, mid − 1)
6:     else if A[mid] < x then
7:         BinarySearch(A, x, mid + 1, r)
8:     else   return mid
```

Exercise 2.3-7):
You can't use binary search because while you are sorting most part of the time, it won't be that value in the sorted array. Maybe you should make a variation of binary search to find out in which direction has less difference betwen the number being search and then you should move all values 1 position to right and insert in the new position.

On worst case, imagine that the worst value is at the least position, you should move n items to the right + n / 2 that cost to search all the. So it's $n \cdot \frac{n}{2} = \Theta(n^2)$

Exercise 2.3-8):

Algorithm 6 FIND-SUM-TWO-ELEMENTS

```
1: function FindSumTwoElements(S, n, x)
2:     MergeSort(S, 0, n - 1)    //Θ(n ∗ logn)
3:     i ← 0
4:     j ← n − 1
5:     while i < j do    //Θ(n)
6:         if S[i] + S[j] = x then
7:             return i, j
8:         else if S[i] + S[j] < x then
9:             i ← i + 1
10:        else
11:            j ← j − 1
12:    return No pair found
```

Problems of page 45

Exercise 2-1):
a) The number of arrays to be sort are $\frac{n}{k}$, where n is the number of elements in the original array and k the number of elements to be sorted on *insertion sort*. Due to that, on the worst case it should be sorted $\frac{n}{k} \cdot k^2 \quad \rightarrow n \cdot k = \Theta(n \cdot k)$.

b) The recursion will be called until reach $\frac{n}{k}$ elements on a subarray. Hence it won't be called recursively until $\log_2 n$ (when only 1 element is left). Due to that the number of recursions are delimited by $\log_2 \frac{n}{k}$. Finally in the worst case when only is left the first 2 subarrays, it will take n iterations to sort it. Due to that, we explain why it's true the expresion $\Theta(n \cdot \log_2 \frac{n}{k})$.

c)

$$\Theta\left(n \cdot k + n \cdot \log_2 \frac{n}{k}\right) = \Theta(n \cdot \log_2 n)$$

$$\Theta(n \cdot k) = \Theta(n \cdot \log_2 n) \quad \rightarrow \quad k = \Theta(\log_2 n)$$

$$\Theta\left(n \cdot \log_2 \frac{n}{k}\right) = \Theta(n \cdot \log_2 n) \quad \text{//Substituting } k = \Theta(\log_2 n)$$

$$\Theta\left(n \cdot \log_2 \frac{n}{\log_2 n}\right) = \Theta(n \cdot \log_2 n) - \Theta(n \cdot \log_2 \log_2 n) \quad \text{//Dominant term is } \Theta(n \cdot \log_2 n)$$

In conclusion, the biggest posible value which both variants of Merge Sort have the same result is for $k = \Theta(\log_2 n)$.

d) You should see in which value it's better use one algorithm or another to sort and combine the 2 algorithms. As we saw in the section c, this k value is calculated applying that formula.

Exercise 2-2):
a) Do you need to prove that on each iteration, of the inner for 1 element is being moved and it remains on all iterations. In addition to that, you must prove that on the extern for loop it's being incremented by 1.

b) Initialization: The loop starts taking the most right element on the array. So the invariant is that take 1 element at a time.

Maintenance: Loop takes on each iteration an element, compare with the left hand elemnt and if it's smaller than the left hand element, swap both elements. The loop invariant remains as Initialization because on each iteration only 1 element is being checked with another one. After that check, on the next iteration is taken the next left element.

Termination: When this for ends, there are a number of elements equals to i (extern for) that are sorted in the lowest positions of the array.

c) As I said before, on each time that the inner for loop ends, 1 value is sorted. Hence when the extern for loop ends, all values will be sorted on the right position and it's proved the inequality.

d) Both have the same worst-case running time ($\Theta(n^2)$). But Insertion Sort is usually better on the averegage cases because it does less swaps and usually spends $\Theta(n)$ on sort an array. On the other hand, Bubble Sort always spend $Theta(n^2)$ to sort the array.

Exercise 2-3):
a) $\Theta(n)$ because it has to take all values of the array A and add it to $x \cdot p$.

b)

```
NAIVE-HORNER (A, x)
1  y = 0
2  for i = 1 to A.length
3       m = 1
4       for j = 1 to i − 1
5            m = m · x
6       y = y + A[i] · m
```

Figure 1: Resolution extracted from https://atekihcan.github.io/CLRS/02/P02-03/ I was lazy of do that hehehe.

c) For Maintenance:

$$y = a_i + x \sum_{k=0}^{n-(i+1)} a_{k+i+1}x^k$$

$$= a_i x^0 + \sum_{k=0}^{n-i-1} a_{k+i+1}x^{k+1}$$

$$= a_i x^0 + \sum_{k=1}^{n-i} a_{k+i}x^k$$

$$= \sum_{k=0}^{n-i} a_{k+i}x^k.$$

The loop terminates at i = -1:

$$= a_i x^0 + \sum_{k=0}^{n-i-1} a_{k+i+1}x^{k+1} = \sum_{k=0}^{n-i} a_{k+i}x^k.$$

Exercise 2-4):
a) Inversions are: (2, 1), (3, 1), (8, 6), (8, 1), (6, 1)

b) if the set is sor in ascending order, it won't be 0 inversions in the set.

c) Both of them are $\Theta(n^2)$ because you have to take 1 value from the most right position and compare if it's less than the i value (left one) on the array.

d)

Algorithm 7 INVERSIONS

```
1: function INVERSIONS(A, p, r)
2:     //On the else of the first while of Merge, add: Print(L[i], R[j])
```

1.3 Caharcerizing Running Times

Exercise 3.1-1):
We could do the same with saying that k is a multiple of 2. The left subarray has the biggests numbers, and the right one have the lowest numbers. If we want to move the biggests numbers to the right subarray, we should move *n / 2* numbers and then another *n / 2* to the left to move the lowest one at the begginig. Finally we have that we should move $\frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4} = \Omega(n^2)$

Exercise 3.1-2):
The extern for loop, executes in the worst case *n - 1* times. The inner loop, executes *i + 1* to *n* times. As mentioned before, the worst case is for i = 0, then it will be executed *n - 1* times. In conclusion, the wors case is $(n-1) \cdot (n-1) = O(n^2)$.

On the other hand, supose that the array is divided in 2 subarrays, the left for the higer values and the right for the lower ones. Althought, the extern for must iterate over all elements, so the better case is the same as the

worst (*n - 1*). Then the inner loop is the same as the worst case. The only thing that executes n/2 times is the instruction inside the if and the swap. These operations are constant operations. In conclusion, the lower bound of the asymptotic behaviour is $(n-1) \cdot (n-1) = \Omega(n^2)$

$O(n^2) = \Omega(n^2) = \Theta(n^2)$

Exercise 3.1-3):
Consideramos un array A de tamaño n, dividido en tres partes:

- Primera parte: Las primeras αn posiciones contienen los αn valores más grandes.
- Segunda parte: Las siguientes $(1-2\alpha)n$ posiciones son la parte media.
- Tercera parte: Las últimas αn posiciones contienen los αn valores más grandes después de ordenar.

El número total de movimientos es:

$$\text{Movimientos totales} = \alpha(1-2\alpha)n^2.$$

Para que el argumento tenga sentido, necesitamos que $0 < \alpha < \frac{1}{2}$.

Maximizamos la función:

$$f'(\alpha) = 1 - 4\alpha = 0 \quad \Rightarrow \quad \alpha = \frac{1}{4}.$$

El número total de movimientos es:

$$\frac{1}{4} \cdot \frac{1}{2}n^2 = \frac{1}{8}n^2.$$

Por lo tanto, el tiempo de ejecución en el peor caso es $\Theta(n^2)$.