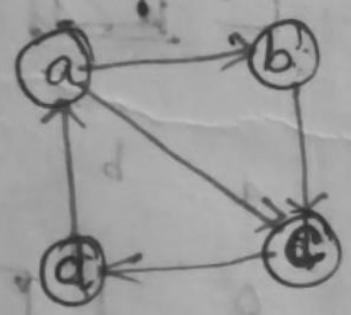


Analysis:

Time complexity is $O(n^3)$. This operation is located within 3 Nested for loops.

→ Eg:-

Obtain the transitive closure for the following digraph



using Warshall's Algorithm.

Note:-
 $R^{(0)} \Rightarrow$ initial nodes
io) Adj matrix
 $R^{(1)} \Rightarrow$ To reach, 'a'
is the intermediate
 $R^{(2)} \Rightarrow a+b, R^{(3)} \Rightarrow a,b,c$
 $R^{(4)} \Rightarrow a,b,c,d$

① Adjacency Matrix

ie) $R^{(0)}$

	a	b	c	d
a	0	1	1	0
b	0	0	1	0
c	0	0	0	1
d	1	0	0	0

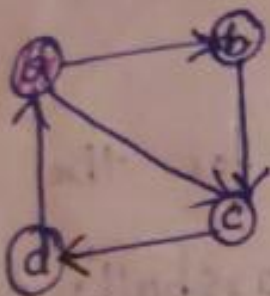
\Rightarrow Edge is present w else 0.

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$$R^{(0)} =$$

0	1	1	0
0	0	1	0
0	0	0	1
1	0	0	0

* If there is a direct connection use 1 else 0.
Ans is same as transitive closure

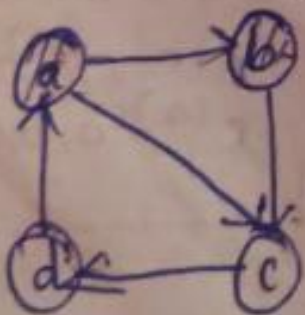


$$R^{(1)}$$

	a	b	c	d
a	0	1	1	0
b	0	0	1	0
c	0	0	0	1
d	1	1	0	0

In $R^{(1)} \Rightarrow$ Source - dest (a-b)
intermediate node 'a'.

- 2 ways, 1) directly d-b.
2) d-a-b.
io) $R_{(ij)}^{(k)} \leftarrow R_{(ij)}^{(k-1)}$



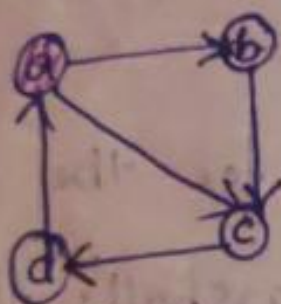
$$R^{(2)}$$

	a	b	c	d
a	0	1	1	0
b	0	0	1	0
c	0	0	0	1
d	1	1	1	0

$R^{(0)}$

$$R^{(0)} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

* if there is a direct connection use 1 else 0.
Final Ans is same as transitive closure.


 (1)
 $R^{(1)}$

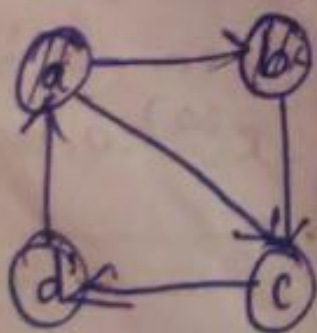
$$R^{(1)} = \begin{bmatrix} a & b & c & d \\ a & 0 & 1 & 1 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 1 \\ d & 1 & 1 & 0 & 0 \end{bmatrix}$$

In $R^{(1)}$ \Rightarrow Source-dest (a-b)
intermediate node 'a'.

2 ways, 1) directly d-b.

2) d-a-b.

$$R_{(i,j)}^{(k)} \leftarrow R_{(i,j)}^{(k-1)}$$

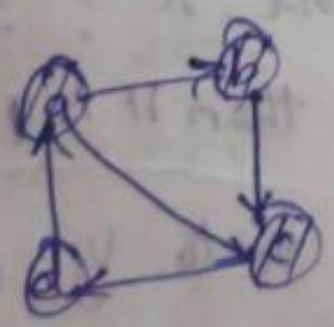

 (2)
 $R^{(2)}$

$$R^{(2)} = \begin{bmatrix} a & b & c & d \\ a & 0 & 1 & 1 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{bmatrix}$$

Here intermediate Vertices are a and b.
 we have to find a path length of upto
 3 edges going through intermediate
 Vertices a and b. We will keep adjacency
 matrix of $R^{(1)}$ as it is and add more
 1's for path length 3 with interme-
 diate Vertices a and b.

for instance: We get $d-a-b-c$.

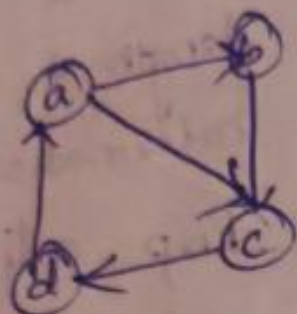
Hence matrix $[d][c] = 1$.



$R^{(3)} =$

	a	b	c	d
a	0	1	1	1
b	0	0	1	1
c	0	0	0	1
d	1	1	1	1

Here intermediate Vertices a, b,
 Matrix $[a][d] = 1$.



$R^{(4)} =$

	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
d	1	1	1	1

We can compute all the elements of matrix $R^{(k)}$ from previous $R^{(k-1)}$.

Let $r_{ij}^{(k)}$ is the element in row i j^{th} column of matrix $R^{(k)}$, if there is a path from v_i to v_j then it is 1 else 0.

The path from v_i to v_j can be computed in 2 cases.

* List of vertices that does not contain k^{th} vertex is noted. Then

paths from v_i to v_j with intermediate vertices numbered not higher than $k-1$.

Then set $r_{ij}^{k-1} = 1$.

* The path not containing k^{th} vertex v_k in intermediate vertices.

This indicates path from v_i to v_k with each intermediate vertex numbered not higher than $(k-1)$. $\therefore r_{ik}^{(k-1)} = 1$.

This indicates path from v_k to v_j with each intermediate vertex numbered not higher than $(k-1)$. $\therefore r_{kj}^{(k-1)} = 1$.

We can generalize r^k from r^{k-1} as.

$$r_{ij}^{(k)} = \boxed{r_{ij}^{(k-1)} \text{ or } r_{ik}^{(k-1)} \text{ and } r_{kj}^{(k-1)}}$$

Assume,

$$i = 1 \text{ to } 4$$

$$j = 1 \text{ to } 4$$

$$k = 1$$

$$r_{ij} = r_{ij}^{(k-1)} \text{ or } r_{ik}^{(k-1)} \text{ and } r_{kj}^{(k-1)} \leftarrow r_{ij}^{(k-1)}, j=1,2$$

$$r_{ii} = r_{ii}^{(k-1)} \text{ or } r_{ii}^{(k-1)} \text{ and } r_{ii}^{(k-1)}$$

$$= 0 \text{ or } 0 \text{ or } 0$$

$$= 0$$

$$r_{12} = r_{12}^{(k-1)} \text{ or } r_{11}^{(k-1)} \text{ and } r_{12}^{(k-1)}$$

$$= 1 \text{ or } 0 \text{ and } 1$$

$$= 1$$

$$r_{42} = r_{42}^{(k-1)} \text{ or } r_{41}^{(k-1)} \text{ and } r_{12}^{(k-1)}$$

$$= 0 \text{ or } 1 \text{ and } 1$$

$$= 0 \text{ or } 1 = 1$$