# CS4522 Advanced Algorithms

Batch 09, L4S1

**Lecture 7: (05 July 2013)**

## Randomized Algorithms & Probabilistic Analysis

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# Announcement

- Mid-semester Quiz after the break!
- You can panic now (again)

# Today's Outline

- Randomized Algorithms
  - Introduction: Hiring Problem
    - Probabilistic analysis
    - Randomized algorithms
  - Randomized Quick Sort
  - Randomized Selection
  - Random Number Generation

# References





- Mainly
  - CLRS, Chapter 5 (pp. 114-128) and others
- Many online resources
- For detailed, in-depth coverage, read
  - "*Randomized Algorithms*" by Motwani and Raghavan

# Intro: Hiring Problem

- **Suppose you need to hire a new assistant**

# Intro: Hiring Problem

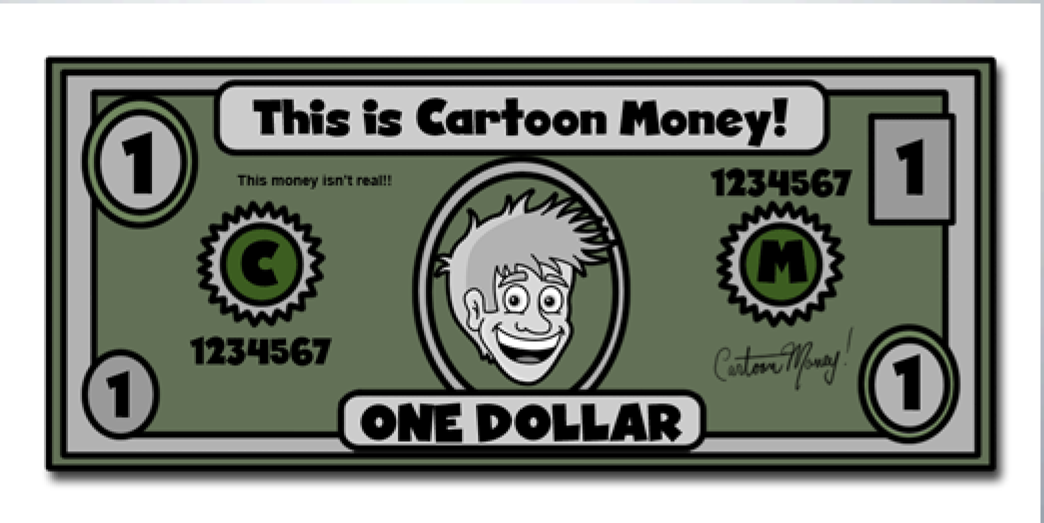




- Suppose you need to hire a new assistant
  - You get a candidate each day from an agency
  - You interview and decide to hire or not

# Intro: Hiring Problem

- Suppose you need to hire a new assistant
  - Need to pay the agency a fee for an interview



# Intro: Hiring Problem

- Suppose you need to hire a new assistant
  - Hiring is more costly
    - Fire the current assistant
    - Pay a large hiring fee to the agency

# Intro: Hiring Problem ...contd

- You always want to have the best person
  - If the interviewed person is better than the current assistant, then hire the new person
  - You are willing to pay the resulting price
- You want to estimate the price of strategy
- Assume
  - Candidates numbered 1,…,*n*
  - After interviewing candidate *i*, determine if *i* is the best seen so far
  - Costs: for interviewing $c_i$ and for hiring $c_h$

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# Intro: Hiring Problem ...contd

**HIRE-ASSISTANT (*n*)**

*best* ← 0 // least qualified, dummy

for *i* ← 1 to *n*

interview candidate *i*

if candidate *i* is better than *best*

*best* ← *i*

hire candidate *i*

# Intro: Hiring Problem ...contd

- If $m$ people hired, total cost $O(n\ c_i + m\ c_h)$
  - If $c_i$ is small, can focus on ($m\ c_h$)
- **Worst-case?**
  - *Hire each person interviewed* (they come in increasing order of quality); total cost = $n\ c_h$
  - But reasonable to expect this will not happen
    - Yet we don't know the order and we don't have a control over that
    - What do we *expect to happen in an average case*?

# Probabilistic Analysis

- Probabilistic analysis means
  - *Analyzing problems using probability*
- In such analysis
  - Use knowledge of or make assumptions about distribution of inputs
    - Without this, cannot use probabilistic analysis
  - Compute an *expected* cost or running time
    - Expectation taken over all possible inputs
    - ➔ averaging the cost/running time over that space

# Probabilistic Analysis

…contd

- **Example: Hiring problem/algorithm**
  - *Assume candidates come in random order*
    - Can compare any two and decide who is better
    - There is a *total order* on the candidates
    - ➔ can rank each candidate with a unique number between 1 and $n$; *rank*($i$) denotes rank of $i$
    - Convention: higher rank means better qualified
    - The ordered list <*rank*(1), *rank*(2),…, *rank*($i$)> is a permutation of the list <1, 2, …, $n$>
    - List of ranks equally likely to be any one of $n$! permutations of numbers 1 through $n$

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# Randomized Algorithms

- **As we saw, probabilistic analysis requires we know about the distribution on inputs**
  - **But this may not be so in many cases**
- **In our current HIRE-ASSISTANT algorithm**
  - **Candidates may seem to come randomly**
    - **But cannot know if this is correct or not**
  - **To have a *randomized algorithm* for this**
    - **Need to have greater control on interviewing order**
    - **What can we do?**

# Randomized Algorithms ...contd

- Randomized algorithm for hiring problem
  - Agency has *n* candidates; list sent in advance
  - Each day ***we randomly choose whom to interview***
  - A significant change!
    - Instead of assuming, we enforce random order
- ***Randomized algorithm***
  - **Behavior is determined by input and by values produced by a *random-number generator***

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# Randomized Hiring Algorithm

**RANDOMIZED-HIRE-ASSISTANT (*n*)**

- randomly permute the candidate list
- *best* ← 0 // least qualified, dummy
- for *i* ← 1 to *n*
  - interview candidate *i*
  - if candidate *i* is better than *best*
    - *best* ← *i*
    - hire candidate *i*

# Randomized Algorithms



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- A randomized algorithm
  - An algorithm that makes random choices during execution
  - Random numbers used for making decisions
  - Behavior determined by a random-number generator (in addition to the input)

# Basics

- Random decision making is introduced to reduce the chance of a worst-case scenario
- Randomized algorithms have no worst-case behavior due a particular input
  - i.e., no bad inputs
  - But only bad random numbers !!

# Basics ...contd

- We compute *expected running-time* (the average case), considering probabilities when necessary

- Randomized strategy useful when
  - There are many ways we can proceed
  - But, difficult to determine a way guaranteed to be good

# Basics ...contd

- If many alternatives are good, we can simply choose one randomly
- If we have to make many choices, a random selection of good and bad choices can be a good strategy if,…
  - Benefits of good choices outweigh the costs of bad choices

# Another Example

- Suppose a teacher wants to give a quiz in the class on the day a homework is due to make sure students did their own work
- But giving a quiz for every homework will consume time from limited class-time
- Practical solution might be to do this for 50% of homework
- How to decide when to give quizzes?

# Another Example ...contd





- Announcing in advance is not effective
- Giving un-announced quizzes on alternate homeworks → students will figure out
- Giving quizzes on “important” topics?

# Another Example ...contd





- **Easiest is to *flip a coin to decide***

# Another Example ...contd

- Easiest is to *flip a coin to decide*
  - 50% probability for a quiz on a homework
  - Expected number of quizzes = 1/2 of the number of homeworks
  - It is possible, but unlikely, that there is no quiz for the whole semester (or the opposite) – unless the coin is biased
  - → *a randomized algorithm*

# Basics ...contd

- Randomized strategy particularly useful when faced with a malicious "adversary"
  - Will deliberately try to feed a bad input to the algorithm
  - → Randomness commonly used/applied in cryptography
  - Issue: *pseudo*-random number generator

# Basics ...contd

- Two types of randomized algorithms
  1. Always gives the correct answer; may require some time/resources to execute → *Las Vegas algorithms*
  2. Can complete quickly (bounded resource usage) but the answer may not be 100% accurate → *Monte Carlo algorithms*

- Special complexity classes, analysis

# Randomized Quick Sort



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- Recall Quick Sort from Lecture 2
- (See also pp. 170-185, Ch. 7 of CLRS)
- *Randomized Quick Sort* Version 1
  - Before sorting the array, randomly permute the elements
    - Enforces the property that every permutation is equally likely
    - Makes the running time independent of the original input ordering

# Randomized Quick Sort



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- *Randomized Quick Sort* Version 2
  - Modifying the original PARTITION procedure, perform a randomized-PARTITION
  - Slight changes to original procedures
  - At each step, before partitioning, exchange $A[p]$ with another element chosen randomly from $A[p \ldots r]$

# Random Numbers?

- Assume: we have a ***random-number generator*** of the form **RANDOM(*a*,*b*)**
  - Returns a random integer between *a* and *b*, inclusive, with each being equally likely
- In practice, true randomness cannot be achieved with a computer
- Most programming environments provide ***pseudo-random number generators***

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# Randomized-QuickSort, V2

**Input: Unsorted sub-array $A[p..r]$**

**Output: Sorted sub-array $A[p..r]$**

**RAND-QUICKSORT ($A, p, r$)**

**if $p < r$**

**then $q \leftarrow$ RAND-PARTITION($A, p, r$)**

**RAND-QUICKSORT (A, $p$, $q$-1)**

**RAND-QUICKSORT (A, $q$+1, $r$)**

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# Randomized-Partition Algorithm

**Input: same as for PARTITION( )**

**Output: same as for PARTITION( )**

**RAND-PARTITION** $(A, p, r)$

$i \leftarrow$ **RANDOM**$(p, r)$

**Exchange** $A[p] \leftrightarrow A[i]$

**return PARTITION**$(A, p, r)$

# Complexity Analysis

- **Worst-case: discussed earlier**
  - **Running time $\Theta(n^2)$**
- **Average-case (expected) running time of randomized Quick Sort is $\Theta(n \lg n)$**
  - **Details: pp. 180-184 in CLRS**

# Average-case Analysis



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- **Intuitively, average-case running time of randomized Quick Sort is $\Theta(n \lg n)$**
  - **partitioning splits the array such that a fraction of elements are on one side**
  - **recursion tree has depth $\Theta(\lg n)$**
  - **$\Theta(n)$ work is performed at these $\Theta(\lg n)$ levels**

# Average-case Analysis ...contd





- Observations for precise analysis
  - Value *q* returned by PARTITION depends only on the rank of x = *A*[*p*] among *A*[*p*…*r*]
    - The rank of x, *rank*(x), in a set is the number of elements less than or equal to x in the set
  - Due to swapping with a random element first, *rank*(x)=*i* for *i*=1,2,…,*n* with probability 1/*n*
    - Assumptions: *n*=*r*-*p*+1 (# elements in *A*[*p*…*r*]), elements are distinct

# Average-case Analysis …contd

- **Observations for precise analysis …contd**
  - **If *rank*(x)=1: $q=j=p$ is returned, low side of partition contains 1 element $A[p]$**
    - **This event occurs with probability $1/n$**
  - **If *rank*(x)=2: the smallest element will go to $A[p]$; $q$, low side, probability same as above**
  - **If *rank*(x) > 2: low side of partition has $i$ elements; probability=$1/n$ for each $i$=2,…,$n$-1**

# Average-case Analysis ...contd

- Size of low side of partition (*q*-*p*+1) is
  - 1 with probability 2/*n*
  - *i* with probability 1/*n* for *i*=2,3,…,*n*-1
- Recurrence for the average (expected) running time of randomized Quick Sort

$$T(n) = \frac{1}{n}\left( T(1) + T(n-1) + \sum_{q=1}^{n-1}\big(T(q) + T(n-q)\big) \right) + \Theta(n)$$

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# Average-case Analysis ...contd

- **Can simplify and re-write this as**

$$T(n) = \frac{2}{n}\sum_{q=1}^{n-1} T(k) + \Theta(n)$$

- **Solve the recurrence (substitution method)**

$$T(n) \le a\, n \lg n + b$$

- **Average running time is *O*(*n* lg*n*)**

# Randomized Selection



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- Discussion based on CLRS pp. 213-222
  - In Chapter 9, Medians and Order Statistics

- Selection problem
  - Input: a set A of $n$ distinct numbers and a number $i$ such that $1 \leq i \leq n$
  - Output: the element x of A that is larger than exactly $i$-1 other elements of A

# Randomized Selection





- Selection problem can be solved in $O(n \lg n)$ time
  - Sort the numbers first and then index the *i*-th element in the array
- But can be done faster, in $O(n)$ average-time, using a randomized algorithm

# Randomized-Select Algorithm

**RAND-SELECT**$(A, p, r, i)$

if $p = r$

then return $A[p]$

$q \leftarrow$ **RAND-PARTITION**$(A, p, r)$

$k \leftarrow q - p + 1$

if $i = k$ then return $A[q]$ // pivot is the answer

elseif $i < k$

then return **RAND-SELECT** $(A, p, q\text{-}1, i)$

else return **RAND-SELECT** $(A, q\text{+}1, r, i - k)$

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# Random Number Generation

- We consider the generation of *pseudorandom numbers*
  - Satisfy most statistical properties of random numbers and appear to be random
- In many cases, we need a *sequence of random numbers*
  - So use of the system clock may not work
  - Numbers should look independent

# Random Number Generation

- Standard method
  - *Linear congruential generator*
  - First described by Lehmer in 1951
  - Numbers $x_1, x_2, \ldots$ are generated satisfying

    $$x_{i+1} = a\, x_i \bmod m$$
  - $x_0$ is called the *seed* (must be given, ≠ 0)
  - a and m to be selected suitably

# Random Number Generation

- Standard method
  - E.g., if $m=11$, $a=7$ and $x_0=1$, then we get
    7, 5, 2, 3, 10, 4, 6, 9, 8, 1, 7, 5, 2, …
  - Here, after $m-1=10$, the sequence repeats
  - If m is a prime, then there are always choices for a that give a *full period* of m-1

# Random Number Generation

- Standard method
  - For some a, this will not happen
  - e.g., if m=11, a=5 and $x_0$=1, then we get a sequence with a shorter period

    5, 3, 4, 9, 1, 5, 3, 4, …
  - Generally, the 31-bit prime

    $m=2^{31}-1 = 2{,}147{,}483{,}647$ and

    $a = 7^5 = 16{,}807$ are commonly used

    (this $a=7^5$ gives a full period generator)

# Random Number Generation

- When m is a prime, $x_i$ is never 0
- Sometimes we need a random real number in the between 0 and 1
  - This is obtained easily by dividing the above formula by m
  - Normalize to get 0 and 1

# Conclusion





- Randomized algorithms
  - Introduction, Hiring problem
  - Randomized Quick Sort, Selection, Random Number Generation
- Next time: Graph algorithms

# References





- Randomized Algorithms & Probabilistic Analysis [CLRS Chapter 5]
- The lecture slides are based on the slides prepared by Prof. Sanath Jayasena for this class in previous years.