

Proof of Kruskal's Algorithm

(Proof adapted from Goodaire & Parmenter's *Discrete Mathematics with Graph Theory*.)

Theorem. After running Kruskal's algorithm on a connected weighted graph G , its output T is a minimum weight spanning tree.

Proof. First, T is a spanning tree. This is because:

- **T is a forest.** No cycles are ever created.
- **T is spanning.** Suppose that there is a vertex v that is not incident with the edges of T . Then the incident edges of v must have been considered in the algorithm at some step. The first edge (in edge order) would have been included because it could not have created a cycle, which contradicts the definition of T .
- **T is connected.** Suppose that T is not connected. Then T has two or more connected components. Since G is connected, then these components must be connected by some edges in G , not in T . The first of these edges (in edge order) would have been included in T because it could not have created a cycle, which contradicts the definition of T .

Second, T is a spanning tree of minimum weight. We will prove this using induction. Let T^* be a minimum-weight spanning tree. If $T = T^*$, then T is a minimum weight spanning tree. If $T \neq T^*$, then there exists an edge $e \in T^*$ **of minimum weight** that is not in T . Further, $T \cup e$ contains a cycle C such that:

- a. Every edge in C has weight less than $\text{wt}(e)$. (This follows from how the algorithm constructed T .)
- b. There is some edge f in C that is not in T^* . (Because T^* does not contain the cycle C .)

Consider the tree $T_2 = T \setminus \{e\} \cup \{f\}$:

- a. T_2 is a spanning tree.
- b. T_2 has more edges in common with T^* than T did.
- c. And $\text{wt}(T_2) \leq \text{wt}(T)$. (We exchanged an edge for one that is no more expensive.)

We can redo the same process with T_2 to find a spanning tree T_3 with more edges in common with T^* . By induction, we can continue this process until we reach T^* , from which we see

$$\text{wt}(T) \leq \text{wt}(T_2) \leq \text{wt}(T_3) \leq \cdots \leq \text{wt}(T^*).$$

Since T^* is a minimum weight spanning tree, then these inequalities must be equalities and we conclude that T is a minimum weight spanning tree.