SRM Institute of Science and Technology
Ramapuram Campus
Department of Mathematics

18MAB101T - Calculus And Linear Algebra

Year/Sem: I/I

Branch: Common to ALL B.Tech. except B.Tech. (Business Systems)

Unit – I

MATRICES

Part – C

1. Find the eigen values of $A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$.

Solution:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements = 2 + 1 - 3 = 0$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements$

$$= \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -7 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = -5 + (-6) + (-2) = -5 - 6 - 2 = -13$$

$S_3 = Determinant\ of\ A = |A|$ = 2 (-5)-2 (-6) $-7(2)$ = -10 + 12 – 14 = -12

Therefore, the characteristic equation of A is $\lambda^3 - 13\lambda + 12 = 0$

$$\begin{array}{c|cccc} 3 & 1 & 0 & -13 & 12 \\ & 0 & 3 & 9 & -12 \\ \hline & 1 & 3 & -4 & 0 \end{array}$$

$(\lambda - 3)(\lambda^2 + 3\lambda - 4) = 0$

$$\Rightarrow \lambda = 3\ , \lambda = \frac{-3 \pm \sqrt{3^2 - 4(1)(-4)}}{2(1)} = \frac{-3\ \pm \sqrt{25}}{2} = \frac{-3 \pm 5}{2} = \frac{-3+5}{2}, \frac{-3-5}{2} = 1, -4$$

Therefore, the eigen values are $\lambda = 3$, 1 and -4.

2. The product of two eigen values of the matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **is 16. Find the third eigen value.**

Solution: Let the eigen values of the matrix be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1 \lambda_2 = 16$

We know that $\lambda_1 \lambda_2 \lambda_3 = |A|$ (Since product of the eigen values is equal to the determinant of the matrix)

$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$ = 6(9-1)+2(-6+2) +2(2-6) = 48-8-8 = 32

$Therefore, \lambda_1 \lambda_2 \lambda_3 = 32 \Rightarrow 16\lambda_3 = 32 \Rightarrow \lambda_3 = 2$

3. Show that the matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ **satisfies its own characteristic equation.**

Solution: Let A $= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 = 2,$

$S_2 = |A| = 1 - (-4) = 5$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$

$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Therefore, the given matrix satisfies its own characteristic equation.

4. If A = $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ $\mathbf{write\ A^2\ interms\ of\ A\ and\ I, using\ Cayley - Hamilton\ theorem.}$

Solution: Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 6$, $S_2 = |A| = 5$

Therefore, the characteristic equation is $\lambda^2 - 6\lambda + 5 = 0$

By Cayley-Hamilton theorem, $A^2 - 6A + 5I = 0$

$$\text{(i.e.) } A^2 = 6A - 5I$$

5. Determine A^4 If A = $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, using Cayley – Hamilton theorem.

<u>Solution</u>: Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 0$

$S_2 = |A| = -5$

Therefore, the characteristic equation is $\lambda^2 - 5 = 0$

By Cayley-Hamilton theorem, $A^2 - 5I = 0$ (i.e.) $A^2 = 5I$

$$A^2 = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\text{Therefore } A^4 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

6. Given A = $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, Find A^{-1} using Cayley – Hamilton theorem.

<u>Solution:</u> The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$,

Here, $S_1 = 4$ and $S_2 = -5 \Rightarrow \lambda^2 - 4\lambda - 5 = 0$.

By Cayley – Hamilton theorem $A^2 - 4A - 5I = 0$.

Multiply by A^{-1}, we get $A - 4I - 5A^{-1} = 0$ $\therefore A^{-1} = \frac{1}{5}[A - 4I] = \begin{bmatrix} \frac{-3}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{-1}{5} \end{bmatrix}$

7. Determine the nature of the following quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2$.

<u>Solution</u>: The matrix of the quadratic form is Q = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The eigen values of the matrix are 1, 2, 0

Therefore, the quadratic form is Positive Semi-definite.

8. Discuss the nature of the quadratic form $2x^2+3y^2+2z^2+2xy$ without reducing it to canonical form.

Solution: The matrix of the quadratic form is Q = $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

$D_1 = 2(+ve)$

$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5(+ve)$

$D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(6-0) - 1(2-0) + 0 = 12 - 2 = 10(+ve)$

Therefore, the quadratic form is positive definite.

9. Find the quadratic form of the matrix A= $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 2 \\ -1 & 2 & 5 \end{bmatrix}$**.**

Solution: Let A = $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 2 \\ -1 & 2 & 5 \end{bmatrix}$

Quadratic form is $X^T A X$, where $X^T = (x, y, z)$

Therefore, Q.F= $(x \ y \ z)\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 2 \\ -1 & 2 & 5 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x^2 + 3y^2 + 5z^2 - 2zx + 4yz$

10. If the eigen vectors of a 2 × 2 matrix A are $X_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $X_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then verify that they are mutually orthogonal. Also find normalized matrix N.

Solution: X_1 and X_2 are said to be mutually orthogonal if $X_1^T X_2 = 0$.

$$(1 \ \ -2)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$$

Modal matrix M=$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Normalized matrix N=$\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

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