Engineering Mathematics
[For Semesters I and II]

Third Edition

About the Author

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Chapter 2

Vector Calculus

2.1 INTRODUCTION

In Vector Algebra we mostly deal with constant vectors, viz. vectors which are constant in magnitude and fixed in direction. In Vector Calculus we deal with variable vectors i.e. vectors which are varying in magnitude or direction or both. Corresponding to each value of scalar variable t, if there exists a value of the vector $\bar{F}$, then $\bar{F}$ is called a *vector function of the scalar variable t* and is denoted as $\bar{F} = \bar{F}(t)$. For example, the position of a particle that moves continuously on a curve in space varies with respect to time t. Hence the position vector $\bar{r}$ of the particle with respect to a fixed point is a function of time t, i.e. $\bar{r} = \bar{r}(t)$.

Also the position vector of the particle varies from point to point. Hence it can also be considered as a function of the point. If the points are specified by their rectangular cartesian co-ordinates (x, y, z), then $\bar{r}$ is a function of the scalar variables x, y, z, i.e. $\bar{r} = \bar{r}(x, y, z)$.

A physical quantity, that is a function of the position of a point in space, is called *a scalar point function* or *a vector point function*, according as the quantity is a scalar or vector. Temperature at any point in space and electric potential are examples of scalar point functions. Velocity of a moving particle and gravitational force are examples of vector point functions.

When a point function is defined at every point of a certain region of space, then that region is called a *field*. The field is called a *scalar field* or a *vector field*, according as the point function is a scalar point function or vector point function. In Vector Calculus, though differentiation and integration of vector time functions and vector point functions are dealt with, we will be concerned with the latter only.

2.2 VECTOR DIFFERENTIAL OPERATOR ∇

We now consider an operator ∇ (to be read as 'del') which is useful in defining three quantities known as the gradient, the divergence and the curl, that are useful in engineering applications.

The operator ∇ is defined as

$$\nabla = \bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}$$

∇ is called the vector differential operator, as it behaves like a vector (though not a vector) with $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ as coefficients of $\bar{i}, \bar{j}, \bar{k}$ respectively. When writing $\nabla = \sum \bar{i}\,\frac{\partial}{\partial x}$, it should be noted that $\bar{i}$ and $\frac{\partial}{\partial x}$ are written as the first and second factors respectively.

2.2.1 Gradient of a Scalar Point Function

Let $\phi(x, y, z)$ be a scalar point function defined in a certain region of space. Then the vector point function given by

$$\nabla\phi = \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}\right)\phi = \bar{i}\,\frac{\partial \phi}{\partial x} + \bar{j}\,\frac{\partial \phi}{\partial y} + \bar{k}\,\frac{\partial \phi}{\partial z}$$

is defined as *the gradient of* ϕ and shortly denoted as *grad* ϕ.

Note ☑

1. $\nabla\,\phi$ should not be written as $\phi\,\nabla$.
2. When ∇ combines with ϕ, neither $\cdot$ nor $\times$ should be put between ∇ and ϕ.
3. If ϕ is a constant, $\nabla\,\phi = 0$.
4. $\nabla\,(c_1\phi_1 \pm c_2\phi_2) = c_1\,\nabla\phi_1 \pm c_2\,\nabla\,\phi_2$ where c_1 and c_2 are constants and ϕ_1, ϕ_2 are scalar point functions.
5. $\nabla\,(\phi_1\,\phi_2) = \phi_1\,\nabla\,\phi_2 + \phi_2\,\nabla\,\phi_1$.
6. $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\,\nabla\phi_1 - \phi_1\,\nabla\phi_2}{\phi_2^2}$, if $\phi_2 \neq 0$.
7. If $v = f(u)$, then $\nabla\,v = f'(u)\,\nabla\,u$.

2.2.2 Directional Derivative of a Scalar Point Function $\phi(x, y, z)$

Let P and Q be two neighbouring points whose position vectors with respect to the origin O be $\bar{r}\,(=\overline{OP})$ and $\bar{r} + \overline{\Delta r}\,(=\overline{OQ})$ respectively, so that $\overline{PQ} = \overline{\Delta r}$ and $PQ = \Delta r$. Let ϕ and $\phi + \Delta\phi$ be the values of a scalar point function ϕ at the points P and Q respectively.

Then $\frac{\mathrm{d}\phi}{\mathrm{d}r} = \lim\limits_{\Delta r \to 0}\left(\frac{\Delta\phi}{\Delta r}\right)$ is called the directional derivative of ϕ in the direction OP.

i.e. $\frac{\mathrm{d}\phi}{\mathrm{d}r}$ gives the rate of change of ϕ with respect to the distance measured in the direction of $\bar{r}$.

In particular, $\dfrac{\partial\phi}{\partial x}, \dfrac{\partial\phi}{\partial y}, \dfrac{\partial\phi}{\partial z}$ are the directional derivatives of ϕ at $P(x, y, z)$ in the directions of the co-ordinate axes.

2.2.3 Gradient as a Directional Derivative

Let $\phi(x, y, z)$ be a scalar point function. Then $\phi(x, y, z) = c$ represents, for various values of c, a family of surfaces, called *the level surfaces of the function ϕ.*

Consider two level surfaces of ϕ, namely $\phi = c_1$ and $\phi = c_2$ passing through two neighbouring points $P(\bar{r})$ and $Q(\bar{r}+\Delta\bar{r})$. [Refer to Fig. 2.1]

Let the normal at P to the level surface $\phi = c_1$ meet $\phi = c_2$ at R.

Clearly the least distance between the two surfaces = $PR = \Delta n$.

If $Q\hat{P}R = \theta$, then, from the figure PQR, which is almost a right angled triangle,

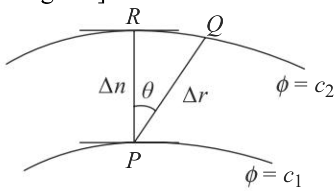

Fig. 2.1

$$\Delta n = \Delta r \cos\theta \tag{1}$$

If $\hat{n}$ is the unit vector along PR, i.e. in the direction of outward drawn normal at P to the surface $\phi = c_1$, then (1) can be written as

$$\Delta n = \hat{n}\cdot\Delta\bar{r}, \qquad \text{where } \Delta\bar{r} = \overline{PQ}.$$

or
$$\mathrm{d}n = \hat{n}\cdot\mathrm{d}\bar{r} \tag{2}$$

$\therefore$
$$\mathrm{d}\phi = \frac{\mathrm{d}\phi}{\mathrm{d}n}\mathrm{d}n$$

$$= \frac{\mathrm{d}\phi}{\mathrm{d}n}\hat{n}\cdot\mathrm{d}\bar{r} \qquad \text{[by using (2)]} \tag{3}$$

Also
$$\mathrm{d}\phi = \frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z \qquad \text{[Refer to Chapter 4 of Part I]}$$

$$= \left(\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}\right)\cdot(\mathrm{d}x\,\bar{i} + \mathrm{d}y\,\bar{j} + \mathrm{d}z\,\bar{k})$$

$$= \nabla\phi\cdot\mathrm{d}\bar{r} \tag{4}$$

From (3) and (4),

$$\nabla\phi\cdot\mathrm{d}\bar{r} = \frac{\mathrm{d}\phi}{\mathrm{d}n}\hat{n}.\mathrm{d}\bar{r}$$

Since $\overline{PQ} = \Delta\bar{r}$ (or d $\bar{r}$) is arbitrary,

$$\nabla\phi = \frac{\mathrm{d}\phi}{\mathrm{d}n}\hat{n} \tag{5}$$

From (1),
$$\frac{\mathrm{d}\phi}{\mathrm{d}n} = \frac{\mathrm{d}\phi}{\mathrm{d}r\,(\cos\theta)}$$

i.e. $$\frac{d\phi}{dr} = \cos\theta \frac{d\phi}{dn}$$

i.e. $$\frac{d\phi}{dr} \le \frac{d\phi}{dn} \quad [\because \cos\theta \le 1]$$

i.e. the maximum directional derivative of ϕ is $\frac{d\phi}{dn}$, that is the directional derivative ϕ in the direction of $\hat{n}$.

Thus, from (5), we get the following interpretation of $\nabla\phi$:

$\nabla\phi$ is a vector whose magnitude is the greatest directional derivative of ϕ and whose direction is that of the outward drawn normal to the level surface $\phi = c$.

WORKED EXAMPLE 2(a)

Example 2.1 Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point P (1, −2, −1), (i) that is maximum, (ii) in the direction of PQ, where Q is (3, −3, −2).

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$= (2xyz + 4z^2)\,\bar{i} + x^2z\,\bar{j} + (x^2y + 8xz)\,\bar{k}$$

$$\therefore \qquad (\nabla\phi)_{(1,-2,-1)} = 8\bar{i} - \bar{j} - 10\bar{k}$$

The magnitude of $(\nabla\phi)_P$ is the greatest directional derivative of ϕ at P.

Thus the maximum directional derivative of ϕ at $(1,-2,-1) = \sqrt{64+1+100} = \sqrt{165}$ units.

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 2\bar{i} - \bar{j} - \bar{k}$$

Directional derivative of ϕ in the direction of $\overline{PQ}$ = Component (or projection) of $\nabla\phi$ along $\overline{PQ}$.

$$= \frac{\nabla\phi\cdot\overline{PQ}}{|\overline{PQ}|}$$

$$= \frac{(8\bar{i} - \bar{j} - 10\bar{k})\cdot(2\bar{i} - \bar{j} - \bar{k})}{\sqrt{4+1+1}}$$

$$= \frac{27}{\sqrt{6}} \text{ units.}$$

Example 2.2 Find the unit normal to the surface $x^3 - xyz + z^3 = 1$ at the point (1, 1, 1).

Note ☑ Unit normal to a surface $\phi = c$ at a point means the unit vector $\hat{n}$ in the direction of the outward drawn normal to the surface at the given point.

The given equation of the surface $x^3 - xyz + z^3 = 1$ is taken as $\phi(x, y, z) = c$.

$\therefore$ $$\phi = x^3 - xyz + z^3$$

$\nabla\phi$ is a vector acting in the direction of the outward drawn normal to the surface $\phi = c$.

Now $$\nabla\phi = (3x^2 - yz)\bar{i} - xz\bar{j} + (3z^2 - xy)\,\bar{k}$$

$$(\nabla\phi)_{(1,1,1)} = 2\bar{i} - \bar{j} + 2\bar{k} = \bar{n} \text{ (= a vector in the direction of the normal)}$$

$\therefore$ $$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$= \frac{1}{3}(2\bar{i} - \bar{j} + 2\bar{k})$$

Example 2.3 Find the directional derivative of the function $\phi = xy^2 + yz^2$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at the point $(-1, 2, 1)$.

The equation of the surface $x \log z - y^2 + 4 = 0$ is identified with $\psi(x, y, z) = c$.

$\therefore$ $$\psi(x, y, z) = x \log z - y^2 \text{ and } c = -4.$$

The direction of the normal to this surface is the same as that of $\nabla\psi$.

Now $$\nabla\psi = (\log z)\bar{i} - 2y\bar{j} + \frac{x}{z}\bar{k}$$

$\therefore$ $$(\nabla\psi)_{(-1,2,1)} = -4\bar{j} - \bar{k} = \bar{b} \text{ (say)}$$

$$\phi = xy^2 + yz^3$$

$$\nabla\phi = y^2\bar{i} + (2xy + z^3)\bar{j} + 3yz^2\bar{k}$$

$$(\nabla\phi)_{(2,-1,1)} = \bar{i} - 3\bar{j} - 3\bar{k}$$

Directional derivative of ϕ in the direction of $\bar{b}$

$$= \frac{\nabla\phi\cdot\bar{b}}{|\bar{b}|}$$

$$= \frac{(\bar{i} - 3\bar{j} - 3\bar{k})\cdot(-4\bar{j} - \bar{k})}{\sqrt{16+1}}$$

$$= \frac{15}{\sqrt{17}} \text{ units.}$$

Example 2.4 Find the angle between the normals to the surface $xy = z^2$ at the points $(-2, -2, 2)$ and $(1, 9, -3)$.

Angle between the two normal lines can be found out as the angle between the vectors acting along the normal lines.

Identifying the equation $xy = z^2$ with $\phi(x, y, z) = c$, we get $\phi = xy - z^2$ and $c = 0$.

$$\nabla\phi = y\bar{i} + x\bar{j} - 2z\bar{k}$$

$$(\nabla\phi)_{(-2,-2,2)} = -2\bar{i} - 2\bar{j} - 4\bar{k} = \bar{n}_1 \text{ (say)}$$

$$(\nabla\phi)_{(1,9,-3)} = 9\bar{i} + \bar{j} + 6\bar{k} = \bar{n}_2 \text{ (say)}$$

$\bar{n}_1$ and $\bar{n}_2$ are vectors acting along the normals to the surface at the given points.

$\therefore$ If θ is the required angle,

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{-44}{\sqrt{24}\cdot\sqrt{118}} = -\frac{11}{\sqrt{177}}$$

$$\therefore \qquad \theta = \cos^{-1}\left\{-\frac{11}{\sqrt{177}}\right\}$$

Example 2.5 Find the angle between the surfaces $x^2 - y^2 - z^2 = 11$ and $xy + yz - zx = 18$ at the point (6, 4, 3).

Angle between two surfaces at a point of intersection is defined as the angle between the respective normals at the point of intersection.

Identifying $x^2 - y^2 - z^2 = 11$ with $\phi = c$,

we have $\quad \phi = x^2 - y^2 - z^2$ and $c = 11$.

$$\therefore \qquad \nabla\phi = 2x\bar{i} - 2y\bar{j} - 2z\bar{k}$$

$$(\nabla\phi)_{(6,4,3)} = 12\bar{i} - 8\bar{j} - 6\bar{k} = \bar{n}_1$$

Identifying $xy + yz - zx = 18$ with $\psi = c'$,

we have $\quad \psi = xy + yz - zx$ and $c' = 18$

$$\nabla\psi = (y-z)\,\bar{i} + (z+x)\bar{j} + (y-x)\bar{k}$$

$$\therefore \qquad \nabla\psi_{\ (6,4,3)} = \bar{i} + 9\bar{j} - 2\bar{k}$$

If θ is the angle between the surfaces at (6, 4, 3), then

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

$$= \frac{-48}{\sqrt{244}\sqrt{86}} = -\frac{24}{\sqrt{61\times 86}}$$

$$\therefore \qquad \theta = \cos^{-1}\left\{\frac{-24}{\sqrt{5246}}\right\}$$

Example 2.6 Find the equation of the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point (1, −1, 2).

Identifying $2xz^2 - 3\,xy - 4x = 7$ with $\phi = c$,

we have $\quad \phi = 2xz^2 - 3xy - 4x$ and $c = 7$.

$$\therefore \qquad \nabla\phi = (2z^2 - 3y - 4)\bar{i} - 3x\bar{j} + 4xz\bar{k}$$

$$(\nabla\phi)_{(1,-1,2)} = 7\bar{i} - 3\bar{j} + 8\bar{k}$$

$(\nabla\phi)_{(1,-1,2)}$ is a vector in the direction of the normal to the surface $\phi = c$.

$\therefore$ D.R.'s of the normal to the surface ($\phi = c$) at the point (1,–1, 2) are (7, –3, 8).

Now the tangent plane is the plane passing through the point (1, –1, 2) and having the line whose D.R.'s are (7, – 3, 8) as a normal.

$\therefore$ Equation of the tangent plane is

$$7(x-1) - 3(y+1) + 8(z-2) = 0$$

i.e. $$7x - 3y + 8z - 26 = 0.$$

Example 2.7 Find the constants a and b, so that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ may cut orthogonally at the point (1,–1, 2).

Two surfaces are said to cut orthogonally at a point of intersection, if the respective normals at that point are perpendicular.

Identifying $5x^2 - 2yz - 9x = 0$ with $\phi_1 = c$,

we have $$\nabla\phi_1 = (10x-9)\bar{i} - 2z\bar{j} - 2y\bar{k}$$

$\therefore$ $$(\nabla\phi_1)_{(1,-1,2)} = \bar{i} - 4\bar{j} + 2\bar{k} = \bar{n}_1 \text{ (say)}$$

Identifying $ax^2y + bz^3 = 4$ with $\phi_2 = c'$.

we have $$(\nabla\phi_2) = 2axy\bar{i} + ax^2\bar{j} + 3bz^2\bar{k}$$

$\therefore$ $$(\nabla\phi_2)_{(1,-1,2)} = -2a\bar{i} + a\bar{j} + 12b\bar{k} = \bar{n}_2 \text{ (say)}$$

Since the surfaces cut orthogonally, $\bar{n}_1 \perp \bar{n}_2$.

i.e. $$\bar{n}_1 \cdot \bar{n}_2 = 0$$

i.e. $$-6a + 24b = 0$$

i.e. $$-a + 4b = 0 \qquad (1)$$

Since (1,–1, 2) is a point of intersection of the two surfaces, it lies on $ax^2y + bz^3 = 4$

$\therefore$ $$-a + 8b = 4 \qquad (2)$$

Solving (1) and (2), we get $a = 4$ and $b = 1$.

Example 2.8 If $\bar{r}$ is the position vector of the point (x, y, z), $\bar{a}$ is a constant vector and $\phi = x^2 + y^2 + z^2$, prove that (i) grad $(\bar{r}\cdot\bar{a}) = \bar{a}$ and (ii) $\bar{r}\cdot$ grad $\phi = 2\phi$.

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

Let $$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

$\therefore$ $$\bar{r}\cdot\bar{a} = a_1x + a_2y + a_3z$$

$\therefore$ $$\text{grad}\,(\bar{r}\cdot\bar{a}) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$$

$$\phi = x^2 + y^2 + z^2$$

$\therefore$ $$\text{grad}\,\phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$\therefore$ $$\bar{r}\cdot\text{grad}\,\phi = 2(x^2 + y^2 + z^2) = 2\phi.$$

Example 2.9 If $\bar{r}$ is the position vector of the point (x, y, z) with respect to the origin, prove that $\nabla(r^n) = nr^{n-2}\bar{r}$.

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$\therefore$ $$r^2 = |\bar{r}^2| = x^2 + y^2 + z^2 \tag{1}$$

$\therefore$ $$\nabla(r^n) = \frac{\partial}{\partial x}(r^n)\bar{i} + \frac{\partial}{\partial y}(r^n)\bar{j} + \frac{\partial}{\partial z}(r^n)\bar{k}$$

$$= nr^{n-1}\left(\frac{\partial r}{\partial x}\bar{i} + \frac{\partial r}{\partial y}\bar{j} + \frac{\partial r}{\partial z}\bar{k}\right) \tag{2}$$

From (l), $$2r\frac{\partial r}{\partial x} = 2x$$

i.e. $$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial r}{\partial z} = \frac{z}{r} \tag{3}$$

Using (3) in (2), we have

$$\nabla(r^n) = nr^{n-1}\left(\frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}\right)$$

$$= nr^{n-2}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= nr^{n-2}\bar{r}.$$

Example 2.10 Find the function ϕ, if grad ϕ

$$= (y^2 - 2xyz^3)\bar{i} + (3 + 2xy - x^2z^3)\bar{j} + (6z^3 - 3x^2yz^2)\bar{k}.$$

$$\nabla\phi = (y^2 - 2xyz^3)\bar{i} + (3 + 2xy - x^2z^3)\bar{j} + (6z^3 - 3x^2yz^2)\bar{k} \tag{1}$$

By definition, $$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k} \tag{2}$$

Comparing (1) and (2), we get

$$\frac{\partial\phi}{\partial x} = y^2 - 2xyz^3 \tag{3}$$

$$\frac{\partial\phi}{\partial y} = 3 + 2xy - x^2z^3 \tag{4}$$

$$\frac{\partial\phi}{\partial z} = 6z^3 - 3x^2yz^2 \tag{5}$$

Integrating both sides of (3) partially with respect to x (i.e. treating y and z as constants),

$$\phi = xy^2 - x^2yz^3 + a \text{ function not containing } x \tag{6}$$

Note ☑ When we integrate both sides of an equation ordinarily with respect to x, we usually add an arbitrary constant in one side. When we integrate partially, we add an arbitrary function of the other variables y and z, i.e. an arbitrary function independent of x.

Integrating (4) partially with respect to y.

$$\phi = 3y + xy^2 - x^2yz^3 + \text{a function not containing } y \tag{7}$$

Integrating (5) partially with respect to z,

$$\phi = \frac{3}{2}z^4 - x^2yz^3 + \text{ a function not containing } z \tag{8}$$

(6), (7) and (8) give only particular forms of ϕ. The general form of ϕ is obtained as follows:

The terms which do not repeat in the R.S's of (6), (7) and (8) should necessarily be included in the value of ϕ.

The terms which repeat should be included only once in the value of ϕ.

The last terms indicate that there is a term of ϕ which is independent of x, y, z, i.e. a constant.

$$\therefore \qquad \phi = 3y + \frac{3}{2}z^4 + xy^2 - x^2yz^3 + c\,.$$

EXERCISE 2(a)

Part A

(Short Answer Questions)

1. Define grad ϕ and give its geometrical meaning.
2. If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla(r) = \frac{1}{r}\bar{r}$.
3. If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla(|\bar{r}|^2) = 2\bar{r}$.
4. If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla f(r) = \frac{1}{r}f'(r)\bar{r}$.
5. Find grad ϕ at the point $(1, -2, -1)$ when $\phi = 3x^2y - y^3z^2$.
6. Find the maximum directional derivative of $\phi = x^3y^2z$ at the point $(1, 1, 1)$.
7. Find the directional derivative of $\phi = xy + yz + zx$ at the point $(1, 2, 3)$ along the x-axis.
8. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum?
9. If the temperature at any point in space is given by $T = xy + yz + zx$, find the direction in which the temperature changes most rapidly with distance from the point $(1, 1, 1)$.
10. The temperature at a point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it fly?

Part B

11. If $\phi = xy + yz + zx$ and $\bar{F} = x^2y\bar{i} + y^2z\bar{j} + z^2x\bar{k}$, find $\bar{F} \cdot$ grad ϕ and $\bar{F} \times$ grad ϕ at the point $(3, -1, 2)$.

12. Find the directional derivative of $\phi = 2xy + z^2$ at the point (1, −1, 3) in the direction of $\bar{i} + 2\bar{j} + 2\bar{k}$.
13. Find the directional derivative of $\phi = xy^2 + yz^3$ at the point P (2, −1, 1) in the direction of PQ where Q is the point (3, 1, 3).
14. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point (2, −2, 3).
15. Find the directional derivative of the scalar function $\phi = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point (3, 1, 3).
16. Find the angle between the normals to the surface $xy^3z^2 = 4$ at the points (−1, −1, 2) and (4, 1, −1).
17. Find the angle between the normals to the surface $x^2 = yz$ at the points (1, 1, 1) and (2, 4, 1).
18. Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point (2, −1, 2).
19. Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, −2, 1).
20. Find the angle between the tangent planes to the surfaces $x \log z - y^2 = -1$ and $x^2y + z = 2$ at the point (1, 1, 1).
21. Find the equation of the tangent plane to the surface $xz^2 + x^2y = z - 1$ at the point (1, −3, 2).
22. Find the values of λ and μ, if the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at the point (1, −1, 2).
23. Find the values of a and b, so that the surfaces $ax^3 - by^2z = (a + 3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at the point (2, −1, −3).
24. Find the scalar point function whose gradient is $(2xy - z^2)\bar{i} + (x^2 + 2yz)\bar{j} + (y^2 - 2zx)\bar{k}$.
25. If $\nabla\phi = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$, find $\phi(x, y, z)$, given that $\phi(1, -2, 2) = 4$.

2.2 THE DIVERGENCE OF A VECTOR

If $\bar{F}(x, y, z)$ is a differentiable vector point function defined at each point (x, y, z) in some region of space, then *the divergence of* $\bar{F}$, denoted as *div* $\bar{F}$, is defined as

$$\begin{aligned}
\text{div}\quad \bar{F} &= \nabla \cdot \bar{F} \\
&= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot \bar{F} \\
&= \bar{i}\cdot\frac{\partial \bar{F}}{\partial x} + \bar{j}\cdot\frac{\partial \bar{F}}{\partial y} + \bar{k}\cdot\frac{\partial \bar{F}}{\partial z}
\end{aligned}$$

Formula for $\nabla\cdot\bar{F}$ ***, when*** $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot(F_1\bar{i} + F_2\bar{j} + F_3\bar{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note ☑ Since $\overline{F}$ is a vector point function , F_1, F_2 and F_3 are scalar point functions and hence $\nabla \cdot \overline{F}$ is also a scalar point function.

2.3.1 Physical meaning of $\nabla \cdot \overline{F}$

(i) If $\overline{V} = V_x \overline{i} + V_y \overline{j} + V_z \overline{k}$ is a vector point function representing the instantaneous velocity of a moving fluid at the point (x, y, z), then $\nabla \cdot \overline{V}$ represents the rate of loss of the fluid per unit volume at that point.

(ii) If the vector point function $\overline{V}$ represents an electric flux, then $\nabla \cdot \overline{V}$ represents the amount of flux that diverges per unit volume in unit time.

(iii) If the vector point function $\overline{V}$ represents heat flux, then $\nabla \cdot \overline{V}$ represents the rate at which heat is issuing from the concerned point per unit volume.

In general, if $\overline{F}$ represents any physical quantity, then $\nabla \cdot \overline{F}$ gives at each point the rate per unit volume at which the physical quantity is issuing from that point. It is due to this physical interpretation of $\nabla \cdot \overline{F}$, it is called the divergence of $\overline{F}$.

2.3.2 Solenoidal Vector

If $\overline{F}$ is a vector such that $\nabla \cdot \overline{F} = 0$ at all points in a given region, then it is said to be a solenoidal vector in that region.

2.3.3 Curl of a Vector

If $\overline{F}$ (x, y, z) is a differentiable vector point function defined at each point (x, y, z) in some region of space, then *the curl of* $\overline{F}$ or *the rotation of* $\overline{F}$, denoted as *curl* $\overline{F}$ or *rot* $\overline{F}$ is defined as

$$\begin{aligned}
\text{Curl } \overline{F} &= \nabla \times \overline{F} \\
&= \left(\overline{i}\,\frac{\partial}{\partial x} + \overline{j}\,\frac{\partial}{\partial y} + \overline{k}\,\frac{\partial}{\partial z} \right) \times \overline{F} \\
&= \overline{i} \times \frac{\partial \overline{F}}{\partial x} + \overline{j} \times \frac{\partial \overline{F}}{\partial y} + \overline{k} \times \frac{\partial \overline{F}}{\partial z}
\end{aligned}$$

Note ☑ Curl $\overline{F}$ is also a vector point function.

Formula for $\boldsymbol{\nabla \times \overline{F}}$, ***when*** $\overline{F} = F_1 \overline{i} + F_2 \overline{j} + F_3 \overline{k}$ (where F_1, F_2 and F_3 are scalar point functions):

$$\begin{aligned}
\nabla \times \overline{F} &= \sum \overline{i} \times \frac{\partial}{\partial x}(F_1 \overline{i} + F_2 \overline{j} + F_3 \overline{k}) \\
&= \sum \frac{\partial F_1}{\partial x}(\overline{i} \times \overline{i}) + \frac{\partial F_2}{\partial x}(\overline{i} \times \overline{j}) + \frac{\partial F_3}{\partial x}(\overline{i} \times \overline{k})
\end{aligned}$$

$$= \sum \left(\frac{\partial F_2}{\partial x} \bar{k} - \frac{\partial F_3}{\partial x} \bar{j} \right)$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \bar{k}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

2.3.4 Physical Meaning of Curl $\bar{F}$

If $\bar{F}$ represents the linear velocity of the point (x, y, z) of a rigid body that rotates about a fixed axis with constant angular velocity $\bar{\omega}$, then curl $\bar{F}$ at that point represents $2\bar{\omega}$.

2.3.5 Irrotational Vector

If $\bar{F}$ is a vector such that $\nabla \times \bar{F} = 0$ at all points in a given region, then it is said to be an irrotational vector in that region.

2.3.6 Scalar Potential of an Irrotational Vector

If $\bar{F}$ is irrotational, then it can be expressed as the gradient of a scalar point function.

Let $\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$

Since $\bar{F}$ is irrotational, Curl $\bar{F} = 0$

i.e.
$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

i.e.
$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \bar{k} = 0$$

$\therefore$
$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}; \ \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}; \ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \qquad (1)$$

Equations (1) are satisfied when

$$F_1 = \frac{\partial \phi}{\partial x}, F_2 = \frac{\partial \phi}{\partial y} \quad \text{and} \quad F_3 = \frac{\partial \phi}{\partial z}$$

$$\therefore \qquad \begin{aligned} \bar{F} &= \bar{i}\,\frac{\partial \phi}{\partial x} + \bar{j}\,\frac{\partial \phi}{\partial y} + \bar{k}\,\frac{\partial \phi}{\partial z} \\ &= \nabla \phi \end{aligned}$$

If $\bar{F}$ is irrotational and $\bar{F} = \nabla \phi$, then ϕ is called the scalar potential of $\bar{F}$.

2.3.7 Expansion Formulae Involving Operations by ∇

Expansion formulas involving operations by one or two ∇'s are given below: The proofs of some of them are given and those of the rest are left as exercise to the students.

1. If $\bar{u}$ and $\bar{v}$ are vector point functions, then $\nabla \cdot (\bar{u} \pm \bar{v}) = \nabla \cdot \bar{u} \pm \nabla \cdot \bar{v}$.
2. If $\bar{u}$ and $\bar{v}$ are vector point functions, then $\nabla \times (\bar{u} \pm \bar{v}) = \nabla \times \bar{u} \pm \nabla \times \bar{v}$.
3. If ϕ is a scalar point function and $\bar{F}$ is a vector point function, then

$$\nabla \cdot (\phi \bar{F}) = \phi(\nabla \cdot \bar{F}) + (\nabla \phi) \cdot \bar{F}$$

Proof:

$$\begin{aligned} \nabla \cdot (\phi \bar{F}) &= \sum \bar{i} \cdot \frac{\partial}{\partial x}(\phi \bar{F}) \\ &= \sum \bar{i} \cdot \left(\phi \frac{\partial \bar{F}}{\partial x} + \frac{\partial \phi}{\partial x} \bar{F} \right) \\ &= \phi \sum \bar{i} \cdot \frac{\partial \bar{F}}{\partial x} + \left(\sum \bar{i}\, \frac{\partial \phi}{\partial x} \right) \cdot \bar{F} \\ &= \phi \left(\nabla \cdot \bar{F} \right) + \left(\nabla \phi \right) \cdot \bar{F} \end{aligned}$$

4. If ϕ is a scalar point function and $\bar{F}$ is a vector point function, then

$$\nabla \times (\phi \bar{F}) = \phi(\nabla \times \bar{F}) + (\nabla \phi) \times \bar{F}$$

5. If $\bar{u}$ and $\bar{v}$ are vector point functions, then $\nabla \cdot (\bar{u} \times \bar{v}) = \bar{v} \cdot \text{curl}\, \bar{u} - \bar{u} \cdot \text{curl}\, \bar{v}$.

Proof:

$$\begin{aligned} \nabla \cdot (\bar{u} \times \bar{v}) &= \sum \bar{i} \cdot \frac{\partial}{\partial x}(\bar{u} \times \bar{v}) \\ &= \sum \bar{i} \cdot \left(\frac{\partial \bar{u}}{\partial x} \times \bar{v} + \bar{u} \times \frac{\partial \bar{v}}{\partial x} \right) \\ &= \sum \bar{i} \cdot \left(\frac{\partial \bar{u}}{\partial x} \times \bar{v} \right) - \sum \bar{i} \cdot \left(\frac{\partial \bar{v}}{\partial x} \times \bar{u} \right) \\ &= \sum \left(\bar{i} \times \frac{\partial \bar{u}}{\partial x} \right) \cdot \bar{v} - \sum \left(\bar{i} \times \frac{\partial \bar{v}}{\partial x} \right) \cdot \bar{u} \end{aligned}$$

[∵ the value of a scalar triple product is unaltered, when dot and cross are interchanged]

$$= \left(\sum \bar{i} \times \frac{\partial \bar{u}}{\partial x}\right) \cdot \bar{v} - \left(\sum \bar{i} \times \frac{\partial \bar{v}}{\partial x}\right) \cdot \bar{u}$$

$$= \bar{v} \cdot \text{Curl}\,\bar{u} - \bar{u} \cdot \text{Curl}\,\bar{v}$$

6. If $\bar{u}$ and $\bar{v}$ are vector point functions, then

$$\nabla \times (\bar{u} \times \bar{v}) = (\nabla \cdot \bar{v})\bar{u} + (\bar{v} \cdot \nabla)\bar{u} - (\nabla \cdot \bar{u})\bar{v} - (\bar{u} \cdot \nabla)\bar{v}$$

Note ☑ In this formula, $\nabla \cdot \bar{v}$ and $\bar{v} \cdot \nabla$ are not the same, $\nabla \cdot \bar{v}$ means div $\bar{v}$, but $\bar{v} \cdot \nabla$ represents the operator $v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$, if $\bar{v} = v_x \bar{i} + v_y \bar{j} + v_z \bar{k}$. Thus $(\bar{v} \cdot \nabla)\bar{u}$ represents $v_x \frac{\partial \bar{u}}{\partial x} + v_y \frac{\partial \bar{u}}{\partial y} + v_z \frac{\partial \bar{u}}{\partial z}$.

7. If $\bar{u}$ and $\bar{v}$ are vector point functions, then $\nabla(\bar{u} \cdot \bar{v}) = \bar{v} \times \text{curl}\,\bar{u} + \bar{u} \times \text{curl}\,\bar{v}$ $+ (\bar{v} \cdot \nabla)\bar{u} + (\bar{u} \cdot \nabla)\bar{v}$.

8. If ϕ is a scalar point function, then div (grad ϕ) = $\nabla \cdot (\nabla\phi) = \nabla^2\phi$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the *Laplacian operator* and $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$ is called *the Laplacian of* ϕ. $\nabla^2\phi = 0$ is called the Laplace equation.

Note ☑ ∇^2 can also operate on a vector point function $\bar{F}$ resulting in $\nabla^2\bar{F} = \frac{\partial^2\bar{F}}{\partial x^2} + \frac{\partial^2\bar{F}}{\partial y^2} + \frac{\partial^2\bar{F}}{\partial z^2}$.

9. If ϕ is a scalar point function, then curl (grad ϕ) = $\nabla \times (\nabla\phi) = 0$.

Proof grad $\phi = \left(\frac{\partial\phi}{\partial x}\right)\bar{i} + \left(\frac{\partial\phi}{\partial y}\right)\bar{j} + \left(\frac{\partial\phi}{\partial z}\right)\bar{k}$

$$\therefore \text{curl (grad } \phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right)\bar{i} + \left(\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z}\right)\bar{j} + \left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right)\bar{k}$$

$$= 0.$$

Note ☑ This result means that (grad ϕ) is always an irrotational vector.

10. If $\bar{F}$ is a vector point function, then div $(\text{curl}\,\bar{F}) = \nabla\cdot(\nabla\times\bar{F}) = 0$.

Proof: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\therefore \quad \text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

$$\therefore \text{ div}\left(\text{curl}\,\bar{F}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z} + \frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x} + \frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y}$$

$$= 0.$$

Note ☑ This result means that (curl $\bar{F}$) is always a solenoidal vector.

11. If $\bar{F}$ is a vector point function, then

$$\text{curl }(\text{curl}\,\bar{F}) = \nabla\times(\nabla\times\bar{F}) = \nabla(\nabla\cdot\bar{F}) - \nabla^2\bar{F}.$$

Proof: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

Then $\quad \text{curl}\,\bar{F} = \left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right)\bar{i} + \left(\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}\right)\bar{j} + \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right)\bar{k}$

$\therefore$ curl (curl $\bar{F}$)

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right) & \left(\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}\right) & \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right) \end{vmatrix}$$

$$= \sum\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right]\bar{i}$$

$$= \sum\left[\left(\frac{\partial^2 F_2}{\partial y\,\partial x} + \frac{\partial^2 F_3}{\partial z\,\partial x}\right) - \left(\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]\bar{i}$$

$$= \sum\left[\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x\,\partial y} + \frac{\partial^2 F_3}{\partial x\,\partial z}\right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]\bar{i}$$

$$=\sum\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F_1\right]\bar{i}$$

$$=\sum\left[\frac{\partial}{\partial x}\left(\nabla\cdot\bar{F}\right)-\nabla^2 F_1\right]\bar{i}$$

$$=\left[\bar{i}\,\frac{\partial}{\partial x}\left(\nabla\cdot\bar{F}\right)+\bar{j}\,\frac{\partial}{\partial y}\left(\nabla\cdot\bar{F}\right)+\bar{k}\,\frac{\partial}{\partial z}\left(\nabla\cdot\bar{F}\right)\right]-\nabla^2\left[F_1\bar{i}+F_2\bar{j}+F_3\bar{k}\right]$$

$$=\nabla\left(\nabla\cdot\bar{F}\right)-\nabla^2\bar{F}.$$

12. If $\bar{F}$ is a vector point function, then

$$\text{grad (div}\,\bar{F})=\nabla(\nabla\cdot\bar{F})=\nabla\times(\nabla\times\bar{F})+\nabla^2\bar{F}\,.$$

Note ☑ Rewriting the formula (11), this result is obtained.

WORKED EXAMPLE 2(b)

Example 2.1 When $\phi = x^3 + y^3 + z^3 - 3xyz$, find $\nabla\phi$, $\nabla\cdot\nabla\phi$ and $\nabla\times\nabla\phi$ at the point (1, 2, 3).

$$\phi = x^3 + y^3 + z^3 - 3x\,yz$$

$$\nabla\phi=\sum\left(\frac{\partial\phi}{\partial x}\right)\bar{i}$$

$$=3(x^2-yz)\bar{i}+3(y^2-zx)\bar{j}+3(z^2-xy)\bar{k}$$

$$\nabla\cdot\nabla\phi=\frac{\partial}{\partial x}[3(x^2-yz)]+\frac{\partial}{\partial y}[3(y^2\ \ zx)]+\frac{\partial}{\partial z}[3(z^2-xy)]$$

$$=6(x+y+z)$$

$$\nabla\times\nabla\phi=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3(x^2-yz) & 3(y^2-zx) & 3(z^2-xy)\end{vmatrix}$$

$$=(-3x+3x)\bar{i}-(-3y+3y)\bar{j}+(-3z+3z)\bar{k}$$

Note ☑ $\nabla\times\nabla\phi = 0$, for any ϕ, as per the expansion formula (9).

$$(\nabla\phi)_{(1,2,3)}=-15\bar{i}+3\bar{j}+21\bar{k}$$

$$(\nabla\cdot\nabla\phi)_{(1,2,3)} = 36 \quad\text{and}\quad (\nabla\times\nabla\phi)_{(1,2,3)} = 0.$$

Example 2.2 If $\bar{F}=(x^2-y^2+2xz)\bar{i}+(xz-xy+yz)\bar{j}+(z^2+x^2)\bar{k}$, find $\nabla\cdot\bar{F}, \nabla(\nabla\cdot\bar{F}), \nabla\times\bar{F}, \nabla\cdot(\nabla\times\bar{F})$ and $\nabla\times(\nabla\times\bar{F})$ at the point (1, 1, 1).

$$\overline{F}=(x^2-y^2+2xz)\overline{i}+(xz-xy+yz)\overline{j}+(z^2+x^2)\overline{k}$$

$$\begin{aligned}\nabla\cdot\overline{F}&=\frac{\partial}{\partial x}(x^2-y^2+2xz)+\frac{\partial}{\partial y}(xz-xy+yz)+\frac{\partial}{\partial z}(z^2+x^2)\\&=(2x+2z)+(-x+z)+2z\\&=x+5z\end{aligned}$$

$$\begin{aligned}\nabla(\nabla\cdot\overline{F})&=\frac{\partial}{\partial x}(x+5z)\overline{i}+\frac{\partial}{\partial y}(x+5z)\overline{j}+\frac{\partial}{\partial z}(x+5z)\overline{k}\\&=\overline{i}+5\overline{k}\end{aligned}$$

$$\begin{aligned}\nabla\times\overline{F}&=\begin{vmatrix}\overline{i}&\overline{j}&\overline{k}\\\dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\x^2-y^2+2xz&xz-xy+yz&z^2+x^2\end{vmatrix}\\&=-(x+y)\overline{i}-(2x-2x)\overline{j}+(y+z)\overline{k}\\&=-(x+y)\overline{i}+(y+z)\overline{k}\end{aligned}$$

$$\begin{aligned}\nabla\cdot(\nabla\times\overline{F})&=\frac{\partial}{\partial x}[-(x+y)]+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}(y+z)\\&=-1+0+1=0\end{aligned}$$

Note ☑ $\nabla\cdot(\nabla\times\overline{F})=0$, for any $\overline{F}$, as per the expansion formula

$$\begin{aligned}\nabla\times(\nabla\times\overline{F})&=\begin{vmatrix}\overline{i}&\overline{j}&\overline{k}\\\dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\-(x+y)&0&y+z\end{vmatrix}\\&=\overline{i}+\overline{k}\end{aligned}$$

$\therefore$
$$(\nabla\cdot\overline{F})_{(1,1,1)}=6;\quad[\nabla(\nabla\cdot\overline{F})]_{(1,1,1)}=\overline{i}+5\overline{k};$$
$$(\nabla\times\overline{F})_{(1,1,1)}=-2\overline{i}+2\overline{k};\quad[\nabla\cdot(\nabla\times\overline{F})]_{(1,1,1)}=0;$$
$$[\nabla\times(\nabla\times\overline{F})]_{(1,1,1)}=\overline{i}+\overline{k}$$

Example 2.3 If $\bar{a}$ is a constant vector and $\overline{r}$ is the position vector of the point (x, y, z,) with respect to the origin, prove that (i) grad $(\overline{a}\cdot\overline{r})=\overline{a}$, (ii) div $(\overline{a}\times\overline{r})=0$ and (iii) curl $(\overline{a}\times\overline{r})=2\overline{a}$.

$$\overline{r}=x\overline{i}+y\overline{j}+z\overline{k}$$

Let $\overline{a}=a_1\overline{i}+a_2\overline{j}+a_3\overline{k}$, where a_1, a_2, a_3 are constants.

$\therefore$
$$\overline{a}\cdot\overline{r}=a_1x+a_2y+a_3z$$

$$\therefore \quad \text{grad } (\bar{a}\cdot\bar{r}) = \sum \frac{\partial}{\partial x}(a_1x + a_2y + a_3z)\bar{i}$$
$$= a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$
$$= \bar{a}$$

$$\bar{a}\times\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$
$$= (a_2z - a_3y)\bar{i} + (a_3x - a_1z)\bar{j} + (a_1y - a_2x)\bar{k}$$

$$\therefore \quad \text{div}(\bar{a}\times\bar{r}) = \frac{\partial}{\partial x}(a_2z - a_3y) + \frac{\partial}{\partial y}(a_3x - a_1z) + \frac{\partial}{\partial z}(a_1y - a_2x)$$
$$= 0$$

$$\text{curl}(\bar{a}\times\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2z - a_3y & a_3x - a_1z & a_1y - a_2x \end{vmatrix}$$
$$= (a_1 + a_1)\bar{i} - (-a_2 - a_2)\bar{j} + (a_3 + a_3)\bar{k}$$
$$= 2(a_1\bar{i} + a_2\bar{j} + a_3\bar{k})$$
$$= 2\bar{a}$$

Example 2.4 Show that $\bar{u} = (2x^2 + 8xy^2z)\bar{i} + (3x^3y - 3xy)\bar{j} - (4y^2z^2 + 2x^3z)\bar{k}$ is not solenoidal, but $\bar{v} = xyz^2\bar{u}$ is solenoidal.

$$\nabla\cdot\bar{u} = \frac{\partial}{\partial x}(2x^2 + 8xy^2z) + \frac{\partial}{\partial y}(3x^3y - 3xy) + \frac{\partial}{\partial z}\{-(4y^2z^2 + 2x^3z)\}$$
$$= (4x + 8y^2z) + (3x^3 - 3x) - (8y^2z + 2x^3)$$
$$= x^3 + x$$
$$\neq 0, \text{ for all points } (x, y, z)$$

$\therefore$ $\bar{u}$ is not solenoidal.

$$\bar{v} = xyz^2\bar{u}$$
$$= (2x^3yz^2 + 8x^2y^3z^3)\bar{i} + (3x^4y^2z^2 - 3x^2y^2z^2)\bar{j} - (4xy^3z^4 + 2x^4yz^3)\bar{k}$$

$$\therefore \quad \nabla\cdot\bar{v} = (6x^2yz^2 + 16xy^3z^3) + (6x^4yz^2 - 6x^2yz^2) - (16xy^3z^3 + 6x^4yz^2)$$
$$= 0, \text{ for all points } (x, y, z)$$

$\therefore$ $\bar{v}$ is solenoidal.

Example 2.5 Show that $\bar{F} = (y^2 - z^2 + 3yz - 2x)\bar{i} + (3xz + 2xy)\bar{j} + (3xy - 2xz + 2z)\bar{k}$ is both solenoidal and irrotational.

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x - 2x + 2$$

$$= 0, \text{ for all points } (x, y, z)$$

$\therefore\ \bar{F}$ is a solenoidal vector.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$$

$$= (3x - 3x)\bar{i} - (3y - 2z + 2z - 3y)\bar{j} + (3z + 2y - 2y - 3z)\bar{k}$$

$$= 0, \text{ for all points } (x, y, z)$$

$\therefore\ \bar{F}$ is an irrotational vector.

Example 2.6 Show that $\bar{F} = (y^2 + 2xz^2)\bar{i} + (2xy - z)\bar{j} + (2x^2z - y + 2z)\bar{k}$ is irrotational and hence find its scalar potential.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (y^2 + 2xz^2) & (2xy - z) & (2x^2z - y + 2z) \end{vmatrix}$$

$$= (-1 + 1)\bar{i} - (4xz - 4xz)\bar{j} + (2y - 2y)\bar{k}$$

$$= 0, \text{ for all points } (x, y, z)$$

$\therefore\ \bar{F}$ is irrotational.

Let the scalar potential of $\bar{F}$ be ϕ.

$$\therefore \qquad \bar{F} = \nabla\phi$$

$$= \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\therefore \qquad \frac{\partial\phi}{\partial x} = y^2 + 2xz^2$$

Integrating partially w.r.t. x;

$$\phi = xy^2 + x^2z^2 + \text{a function independent of } x \tag{1}$$

$$\frac{\partial\phi}{\partial y} = 2xy - z$$

Integrating partially w.r.t. y;

$$\phi = xy^2 - yz + \text{a function independent of } y \tag{2}$$

$$\frac{\partial\phi}{\partial z} = 2x^2z - y + 2z$$

Integrating partially w.r.t. z;

$$\phi = x^2z^2 - yz + z^2 + \text{a function independent of } z \qquad (3)$$

From (1), (2), (3), we get $\phi = xy^2 + x^2z^2 - yz + z^2 + c$.

Example 2.7 Find the values of the constants a, b, c, so that $\bar{F} = (axy + bz^3)\bar{i} + (3x^2 - cz)\bar{j} + (3xz^2 - y)\bar{k}$ may be irrotational. For these values of a, b, c, find also the scalar potential of $\bar{F}$.

$\bar{F}$ is irrotational.

$$\therefore \quad \nabla \times \bar{F} = 0$$

i.e.
$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (axy + bz^3) & (3x^2 - cz) & (3xz^2 - y) \end{vmatrix} = 0$$

i.e.
$$(-1 + c)\bar{i} - (3z^2 - 3bz^2)\bar{j} + (6x - ax)\bar{k} = 0$$

$$\therefore \quad c - 1 = 0, \quad 3z^2(1 - b) = 0, \quad x(6 - a) = 0$$

$$\therefore \quad a = 6, \quad b = 1, \quad c = 1.$$

Using these values of a, b, c,

$$\bar{F} = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$$

Let ϕ be the scalar potential of $\bar{F}$.

$$\therefore \quad \bar{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\frac{\partial\phi}{\partial x} = 6xy + z^3, \quad \frac{\partial\phi}{\partial y} = 3x^2 - z, \quad \frac{\partial\phi}{\partial z} = 3xz^2 - y$$

Integrating partially w.r.t. the concerned variables,

$$\phi = 3x^2y + xz^3 + \text{a function independent of } x \qquad (1)$$

$$\phi = 3x^2y - yz + \text{a function independent of } y \qquad (2)$$

$$\phi = xz^3 - yz + \text{a function independent of } z \qquad (3)$$

From(l), (2) and (3), we get

$$\phi = 3x^2y + xz^3 - yz + c$$

Example 2.8 If ϕ and ψ are scalar point functions, prove that (i) $\phi\,\nabla\phi$ is irrotational and (ii) $\nabla\phi \times \nabla\psi$ is solenoidal.

$$\begin{aligned} \nabla \times \phi\nabla\phi &= \phi(\nabla \times \nabla\phi) + \nabla\phi \times \nabla\phi, \text{ by expansion formula} \\ &= \phi(0) + 0, \text{ by expansion formula} \\ &= 0 \end{aligned}$$

$\therefore$ $\phi\nabla\phi$ is irrotational

$$\nabla\cdot(\bar{u}\times\bar{v})=\bar{v}\cdot\text{curl}\,\bar{u}-\bar{u}\cdot\text{curl}\,\bar{v}, \text{ by expansion formula}$$

$\therefore$ $$\nabla\cdot(\nabla\phi\times\nabla\psi)=\nabla\psi\cdot\text{curl}\,(\nabla\phi)-\nabla\phi\cdot\text{curl}\,(\nabla\psi)$$
$$=\nabla\psi\cdot 0-\nabla\phi\cdot 0=0.$$

$\therefore$ $(\nabla\phi\times\nabla\psi)$ is solenoidal.

Example 2.9 If $r=|\bar{r}|$, where $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla^2(r^n)=n(n+1)\,r^{n-2}$ and hence deduce that $\frac{1}{r}$ satisfies Laplace equation.

We have already proved, in worked example (9) of the previous section, that $\nabla(r^n)=nr^{n-2}\bar{r}$.

Now $$\nabla^2(r^n)=\nabla\cdot(\nabla r^n)$$
$$=\nabla\cdot(nr^{n-2}\bar{r})$$
$$=n\left[\nabla(r^{n-2})\cdot\bar{r}+r^{n-2}(\nabla\cdot\bar{r})\right], \text{ by expansion formula}$$
$$=n\left[(n-2)\,r^{n-4}\,\bar{r}\cdot\bar{r}+3r^{n-2}\right]$$

$$\left[\text{since } \nabla\cdot\bar{r}=\nabla\cdot(x\bar{i}+y\bar{j}+z\bar{k})\right.$$
$$=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)$$
$$\left.=3\right]$$

$\therefore$ $$\nabla^2(r^n)=n\,[(n-2)\,r^{n-4}\,r^2+3r^{n-2}]$$
$$=n\,(n+1)\,r^{n-2}$$

Taking $n=-1$ in the above result,

$$\nabla^2\left(\frac{1}{r}\right)=(-1)(0)\,r^{-3}=0$$

i.e. $\frac{1}{r}$ satisfies Laplace equation.

Example 2.10 If u and v are scalar point functions, prove that

$$\nabla\cdot(u\nabla v-v\nabla u)=u\nabla^2 v-v\nabla^2 u.$$

$$\nabla\cdot(u\nabla v-v\nabla u)=\nabla\cdot(u\nabla v)-\nabla\cdot(v\nabla u)$$
$$=(\nabla u\cdot\nabla v+u\nabla^2 v)-(\nabla v\cdot\nabla u+v\nabla^2 u)$$

[by the expansion formula]

$$=u\nabla^2 v-v\nabla^2 u.$$

Example 2.11 If u and v are scalar point functions and $\bar{F}$ is a vector point function such that $u\bar{F}=\nabla v$, prove that $\bar{F}\cdot$ curl $\bar{F}=0$.

Given $$u\bar{F}=\nabla v$$

$\therefore$ $$\bar{F}=\frac{1}{u}\nabla v$$

$\therefore$ $$\text{Curl}\,\bar{F} = \nabla\times\left(\frac{1}{u}\nabla v\right)$$

$$= \frac{1}{u}(\nabla\times\nabla v) + \nabla\left(\frac{1}{u}\right)\times\nabla v, \text{ by expansion formula}$$

$$= \nabla\left(\frac{1}{u}\right)\times\nabla v, \text{ since } \nabla\times\nabla v = 0.$$

Now $$\bar{F}\cdot\text{curl}\,\bar{F} = \frac{1}{u}\left\{\nabla v\cdot\left(\nabla\frac{1}{u}\times\nabla v\right)\right\}$$

$$= \frac{1}{u}(0),$$ [$\because$ Two factors are equal in the scalar triple product]

$$= 0.$$

Example 2.12 If $r = |\bar{r}|$, where $\bar{r}$ is the position vector of the point (x, y, z) with respect to the origin, prove that (i) $\nabla f(r) = \frac{f'(r)}{r}\bar{r}$ and

(ii) $$\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r).$$

$$r^2 = x^2 + y^2 + z^2$$

$\therefore$ $$2r\frac{\partial r}{\partial x} = 2x$$

$\therefore$ $\frac{\partial r}{\partial x} = \frac{x}{r}$. Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Now $$\nabla f(r) = \sum \bar{i}\,\frac{\partial}{\partial x}f(r)$$

$$= f'(r)\sum\frac{\partial r}{\partial x}\bar{i}$$

$$= f'(r)\sum\frac{x}{r}\bar{i}$$

$$= \frac{f'(r)}{r}\bar{r}$$

$$\nabla^2 f(r) = \nabla\cdot\nabla f(r)$$

$$= \nabla\cdot\left\{\frac{f'(r)}{r}\bar{r}\right\}$$

$$= \nabla\left\{\frac{f'(r)}{r}\right\}\cdot\bar{r} + \frac{f'(r)}{r}\nabla\cdot\bar{r}$$

$$= \left\{\frac{r f''(r) - f'(r)}{r^2}\right\}\nabla(r)\cdot\bar{r} + 3\frac{f'(r)}{r} \quad [\because \nabla\cdot\bar{r} = 3]$$

$$= \left\{ \frac{r f''(r) - f'(r)}{r^2} \right\} \frac{1}{r} \bar{r} \cdot \bar{r} + \frac{3 f'(r)}{r}$$

$$\left[\because \nabla (r) = \sum \frac{\partial r}{\partial x} \bar{i} = \sum \frac{x}{r} i = \frac{1}{r} \bar{r} \right]$$

$$= \left\{ \frac{r f''(r) - f'(r)}{r^2} \right\} \frac{1}{r} (r^2) + \frac{3 f'(r)}{r}$$

$$= f''(r) + \frac{2}{r} f'(r)$$

Example 2.13 Find $f(r)$ if the vector $f(r)\ \bar{r}$ is both solenoidal and irrotational.

$f(r)\ \bar{r}$ is solenoidal

$\therefore \quad \nabla \cdot \{ f(r) \bar{r} \} = 0$

i.e. $\quad \nabla f(r) \cdot \bar{r} + f(r) \nabla \cdot \bar{r} = 0$ [by expansion formula]

i.e. $\quad \dfrac{f'(r)}{r} \bar{r} \cdot \bar{r} + 3 f(r) = 0$ [Refer to the previous problem]

i.e. $\quad r f'(r) + 3 f(r) = 0$

i.e. $\quad \dfrac{f'(r)}{f(r)} + \dfrac{3}{r} = 0$

Integrating both sides w.r.t. r,

$$\log f(r) + 3 \log r = \log c$$

i.e. $$\log r^3 f(r) = \log c$$

$\therefore$ $$f(r) = \frac{c}{r^3} \qquad (1)$$

$f(r)\ \bar{r}$ is also irrotational

$\therefore \quad \nabla \times \{ f(r) \bar{r} \} = 0$

i.e. $$\nabla f(r) \times \bar{r} + f(r) \nabla \times \bar{r} = 0 \text{ [by expansion formula]}$$

i.e. $$\frac{f'(r)}{r} (\bar{r} \times \bar{r}) + 0 = 0 \left[\because \nabla \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 \right]$$

i.e. $$\frac{f'(r)}{r} (0) + 0 = 0$$

This is true for all values of $f(r)$ (2)

From (1) and (2), we get that $f(r)\ \bar{r}$ is both solenoidal and irrotational if $f(r) = \dfrac{c}{r^3}$

Example 2.14 If ϕ is a scalar point function, prove that $\nabla \phi$ is both solenoidal and irrotational, provided ϕ is a solution of Laplace equation.

$\nabla \cdot \nabla \phi = 0$, only when $\nabla^2 \phi = 0$.

i.e, $\nabla \phi$ is solenoidal, only when $\nabla^2 \phi = 0$ (1)

$\nabla \times \nabla \phi = 0$, always [by expansion formula]

i.e. $\nabla \phi$ is irrotational always (2)

From (1) and (2),

$\nabla \phi$ is both solenoidal and irrotational, when $\nabla^2 \phi = 0$, i.e. when ϕ is a solution of Laplace equation.

Example 2.15 If $\bar{F}$ is solenoidal, prove that curl curl curl curl $\bar{F} = \nabla^4 \bar{F}$.

Since $\bar{F}$ is solenoidal, $\nabla \cdot \bar{F} = 0$ (1)

By the expansion formula,

$$\text{Curl curl } \bar{F} = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F} \quad (2)$$

$$= -\nabla^2 \bar{F} \quad \text{[by (1)]}$$

$\therefore$ Curl curl curl curl $\bar{F}$ = curl curl $(-\nabla^2 \bar{F})$

$$= -\text{ curl curl } (\nabla^2 \bar{F})$$

$$= -[\nabla\{\nabla \cdot \nabla^2 \bar{F}\} - \nabla^2 (\nabla^2 \bar{F})], \text{ by using (2)}$$

$$= -[\nabla\{\nabla^2 (\nabla \cdot \bar{F})\} - \nabla^4 \bar{F}], \quad \text{by interchanging the operations } \nabla \cdot \text{ and } \nabla^2$$

$$= \nabla^4 \bar{F} \quad \{\text{by using (1)}\}$$

EXERCISE 2(b)

Part A

(Short Answer Questions)

1. Define divergence and curl of a vector point function.
2. Give the physical meaning of $\nabla \cdot \bar{F}$
3. Give the physical meaning of $\nabla \times \bar{F}$
4. When is a vector said to be (i) solenoidal, (ii) irrotational ?
5. Prove that the curl of any vector point function is solenoidal.
6. Prove that the gradient of any scalar point function is irrotational.
7. If $\bar{r}$ is the position vector of the point (x, y, z) w.r.t. the origin, find div $\bar{r}$ and curl $\bar{r}$.
8. If $\bar{F} = 3xyz^2\,\bar{i} + 2xy^3\,\bar{j} - x^2yz\,\bar{k}$, find $\nabla \cdot \bar{F}$ at the point (1,–1, 1).
9. If $\bar{F} = (x^2 + yz)\,\bar{i} + (y^2 + 2zx)\,\bar{j} + (z^2 + 3xy)\,\bar{k}$, find $\nabla \times \bar{F}$ at the point (2,–1, 2).
10. If $\bar{F} = (x + y + 1)\,\bar{i} + \bar{j} - (x + y)\,\bar{k}$, show that $\bar{F}$ is perpendicular to curl $\bar{F}$.
11. If $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, prove that curl curl $\bar{F} = 0$.
12. Show that $\bar{F} = (x + 2y)\,\bar{i} + (y + 3z)\,\bar{j} + (x - 2z)\bar{k}$ is solenoidal.
13. Show that $\bar{F} = (\sin y + z)\,\bar{i} + (x \cos y - z)\,\bar{j} + (x - y)\bar{k}$ is irrotational.
14. Find the value of λ, so that $\bar{F} = \lambda y^4 z^2 \bar{i} + 4x^3 z^2 \bar{j} + 5x^2 y^2 \bar{k}$ may be solenoidal.
15. Find the value of λ, if $\bar{F} = (2x - 5y)\bar{i} + (x + \lambda y)\bar{j} + (3x - z)\bar{k}$ is solenoidal.

16. Find the value of a, if $\overline{F} = (axy - z^3)\overline{i} + (a-2)x^2\overline{j} + (1-a)xz^2\,\overline{k}$ is irrotational.
17. Find the values of a, b, c, so that the vector $\overline{F} = (x+y+az)\overline{i} + (bx+2y-z)\overline{j} + (-x+cy+2z)\overline{k}$ may be irrotational.
18. If $\bar{u}$ and $\bar{v}$ are irrotational, prove that $(\bar{u}\times\bar{v})$ is solenoidal.
19. If ϕ_1 and ϕ_2 are scalar point functions, prove that $\nabla \times (\phi_1\,\nabla\phi_2) = -\nabla\times(\phi_2\nabla\phi_1)$.
20. If $\nabla\phi$ is a solenoidal vector, prove that ϕ is a solution of Laplace equation.

Part B

21. If $u = x^2yz$ and $v = xy - 3z^2$, find (i) $\nabla\cdot(\nabla u\times\nabla v)$ and $\nabla\times(\nabla u\times\nabla v)$ at the point (1, 1, 0).
22. Find the directional drivative of $\nabla\cdot(\nabla\phi)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2 z = 3x + z^2$, where $\phi = x^2y^2z^2$
23. If $\overline{F} = 3x^2\,\overline{i} + 5xy^2\,\overline{j} + xyz^3\,\overline{k}$, find $\nabla\cdot\overline{F}, \nabla(\nabla\cdot\overline{F}), \nabla\times\overline{F}, \nabla\cdot(\nabla\times\overline{F})$ and $\nabla\times(\nabla\times\overline{F})$ at the point (1, 2, 3)
24. If $\bar{a}$ is a constant vector and $\bar{r}$ is the position vector of (x, y, z) w.r.t. the origin, prove that $\nabla\times[(\bar{a}\cdot\bar{r})\bar{r}] = \bar{a}\times\bar{r}$.
25. Prove that $\overline{F} = 3yz\overline{i} + 2zx\,\overline{j} + 4xy\,\overline{k}$ is not irrotational, but $(x^2yz^3)\,\overline{F}$ is irrotational. Find also its scalar potential.
26. Show that $\overline{F} = (z^2 + 2x + 3y)\,\overline{i} + (3x + 2y + z)\,\overline{j} + (y + 2zx)\,\overline{k}$ is irrotational, but not solenoidal. Find also its scalar potential.
27. Find the constants a, b, c, so that $\overline{F} = (x + 2y + az)\,\overline{i} + (bx - 3y - z)\,\overline{j} + (4x + cy + 2z)\,\overline{k}$ may be irrotational. For these values of a, b, c, find its scalar potential also.
28. Find the smallest positive integral values of a, b, c, if $\overline{F} = axyz^3\,\overline{i} + bx^2z^3\,\overline{j} + cx^2yz^2\,\overline{k}$ is irrotational. For these values of a, b, c, find its scalar potential also.
29. If $\bar{r}$ is the position vector of the point (x, y, z) w.r.t. the origin, prove that

    (i) $\nabla\cdot\left(\frac{1}{r}\bar{r}\right) = \frac{2}{r}$ and (ii) $\nabla\left[\nabla\cdot\left(\frac{1}{r}\bar{r}\right)\right] = -\frac{2}{r^3}$
30. Find the value of n, if $r^n\,\bar{r}$ is both solenoidal and irrotational, when $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

2.4 LINE INTEGRAL OF VECTOR POINT FUNCTIONS

Let $\overline{F}\,(x, y, z)$ be a vector point function defined at all points in some region of space and let C be a curve in that region (Fig. 2.2).

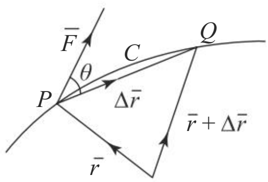

Fig. 2.2

Let the position vectors of two neighbouring points P and Q on C be $\bar{r}$ and $\bar{r} + \Delta\bar{r}$. Then $\overline{PQ} = \Delta\bar{r}$ If $\overline{F}$ acts at P in a direction that makes an angle θ with $\overline{PQ}$, then $\overline{F} \cdot \Delta\overline{r} = F(\Delta r)\cos\theta$

In the limit, $\overline{F} \cdot \mathrm{d}\overline{r} = F\, \mathrm{d}r \cos\theta$.

Note ☑ Physically $\overline{F} \cdot \mathrm{d}\overline{r}$ means the elemental work done by the force $\overline{F}$ through the displacement $\mathrm{d}\overline{r}$.

Now the integral $\int_C \overline{F} \cdot \mathrm{d}\overline{r}$ is defined as *the line integral of* $\overline{F}$ *along the curve C*.

Since $\int_C \overline{F} \cdot \mathrm{d}\overline{r} = \int_C F \cos\theta\, \mathrm{d}r$, it is also called the line integral of the tangential component of $\overline{F}$ along C.

Note ☑ (1) $\int_{A\,(C)}^{B} \overline{F} \cdot \mathrm{d}\overline{r}$ depends not only on the curve C but also on the terminal points A and B.

(2) Physically $\int_{A\,(C)}^{B} \overline{F} \cdot \mathrm{d}\overline{r}$ denotes the total work done by the force $\overline{F}$ in displacing a particle form A to B along the curve C.

(3) If the value of $\int_{A}^{B} \overline{F} \cdot \mathrm{d}\overline{r}$ does not depend on the curve C, but only on the terminal points A and B, $\overline{F}$ is called a *Conservative vector*. Similarly, if the work done by a force $\overline{F}$ in displacing a particle from A to B does not depend on the curve along which the particle gets displaced but only on A and B, the force $\overline{F}$ is called a *Conservative force*.

(4) If the path of integration C is a closed curve, the line integral is denoted as $\oint_C \overline{F} \cdot \mathrm{d}\overline{r}$.

(5) When $\overline{F} = F_1\,\overline{i} + F_2\,\overline{j} + F_3\,\overline{k}$,

$$\int_C \overline{F} \cdot \mathrm{d}\overline{r} = \int_C (F_1\,\overline{i} + F_2\,\overline{j} + F_3\,\overline{k}) \cdot (\mathrm{d}x\,\overline{i} + \mathrm{d}y\,\overline{j} + \mathrm{d}z\,\overline{k})$$

$$(\because \overline{r} = x\overline{i} + y\overline{j} + z\overline{k})$$

$$= \int_C (F_1 \mathrm{d}x + F_2 \mathrm{d}y + F_3 \mathrm{d}z)$$, which is evaluated as in the problems in Chapter 5 of Part I.

(6) $\int_C \phi\,\mathrm{d}r$, where ϕ is a scalar point function and $\int_C \overline{F} \times \mathrm{d}\overline{r}$ are also line integrals.

2.4.1 Condition for $\overline{F}$ to be Conservative

If $\overline{F}$ is an irrotational vector, it is conservative.

Proof: Since $\bar{F}$ is irrotational, it can be expressed as $\nabla\phi$. i.e., $\bar{F} = \nabla\phi$

$$\int_A^B \bar{F}\cdot d\bar{r} = \int_A^B \nabla\phi\cdot d\bar{r}$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}\right)\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \int_A^B d\phi$$

$= [\phi]_A^B$, whatever be the path of integration

$= \phi(B) - \phi(A)$

$\therefore$ $\bar{F}$ is conservative.

Note ☑ If $\bar{F}$ is irrotational (and hence conservative) and C is a closed curve, then $\oint_C \bar{F}\cdot d\bar{r} = 0$.

[$\because$ $\phi(A) = \phi(B)$, as A and B coincide]

2.4.2 Surface Integral of Vector Point Function

Let S be a two sided surface, one side of which is considered arbitrarily as the positive side.

Let $\bar{F}$ be a vector point function defined at all points of S.

Let dS be the typical elemental surface area in S surrounding the point (x, y, z).

Let $\hat{n}$ be the unit vector normal to the surface S at (x, y, z) drawn in the positive side (or outward direction)

Let θ be the angle between $\bar{F}$ and $\hat{n}$.

$\therefore$ The normal component of $\bar{F} = \bar{F}\cdot\hat{n} = F\cos\theta$

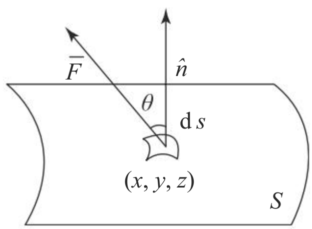

Fig. 2.3

The integral of this normal component through the elemental surface area dS over the surface S is called *the surface integral of* $\bar{F}$ *over* S and denoted as $\int_S F\cos\theta\, dS$ or $\int_S \bar{F}\cdot\hat{n}\, dS$.

If $d\bar{S}$ is a vector whose magnitude is dS and whose direction is that of $\hat{n}$, then $d\bar{S} = \hat{n}\, dS$.

$\therefore$ $\int_S \bar{F}\cdot\hat{n}\, dS$ can also be written as $\int_S \bar{F}\cdot d\bar{S}$.

Note ☑ (1) If S is a closed surface, the outer surface is usually chosen as the positive side.

(2) $\int_S \phi \, d\bar{S}$ and $\int_S \bar{F} \times d\bar{S}$, where ϕ is a scalar point function are also surface integrals.

(3) When evaluating $\int_S \bar{F} \cdot \hat{n} \, d\bar{S}$, the surface integral is first expressed in the scalar form and then evaluated as in problems in Chapter 5 of part I.

To evaluate a surface integral in the scalar form, we convert it into a double integral and then evaluate. Hence the surface integral $\int_S \bar{F} \cdot d\bar{S}$ is also denoted as $\iint_S \bar{F} \cdot d\bar{S}$.

WORKED EXAMPLE 2(c)

Example 2.1 Evaluate $\int_C \phi \, d\bar{r}$, where C is the curve $x = t$, $y = t^2$, $z = (1 - t)$ and $\phi = x^2 y (1 + z)$ from $t = 0$ to $t = 1$.

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\therefore \quad d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

Hence the given line integral $I = \int_C x^2 y(1+z)\,(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$

$$= \bar{i} \int_C x^2 y(1+z)\,dx + \bar{j} \int_C x^2 y(1+z)\,dy + \bar{k} \int_C x^2 y\,(1+z)\,dz$$

$$= \bar{i} \int_0^1 t^4\,(2-t)\,dt + \bar{j} \int_0^1 t^4\,(2-t)\,2t\,dt + \bar{k} \int_0^1 t^4\,(2-t)\,(-dt)$$

$$= \bar{i}\left[2\frac{t^5}{5} - \frac{t^6}{6}\right]_0^1 + \bar{j}\left[4\frac{t^6}{6} - 2\frac{t^7}{7}\right]_0^1 + \bar{k}\left[-2\frac{t^5}{5} + \frac{t^6}{6}\right]_0^1$$

$$= \frac{7}{30}\bar{i} + \frac{8}{21}\bar{j} - \frac{7}{30}\bar{k}.$$

Example 2.2 If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$, evaluate $\int_C \bar{F} \times d\bar{r}$, where C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from (0, 0, 0) to (1, 2, 1).

$$\bar{F} \times d\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ xy & -z & x^2 \\ dx & dy & dz \end{vmatrix}$$

$$= -(z\,dz + x^2\,dy)\bar{i} - (xy\,dz - x^2\,dx)\,\bar{j} + (xy\,dy + z d\,x)\,\bar{k}$$

$\therefore$ The given line integral

$$= \int_C [-(z\mathrm{d}z + x^2\ \mathrm{d}y)\bar{i} - (xy\ \mathrm{d}z - x^2\ \mathrm{d}x)\ \bar{j} + (xy\,\mathrm{d}y + z\mathrm{d}x)\bar{k}]$$

$$= \int_0^1 [-(t^3 \cdot 3t^2 \cdot + t^4 \cdot 2)\,\mathrm{d}t\ \bar{i} - (2t^3 \cdot 3t^2 - t^4 \cdot 2t)\,\mathrm{d}t\ \bar{j} + (2t^3 \cdot 2 + t^3\ 2\ t)\,\mathrm{d}t\ \bar{k}]$$

[$\because$ (0, 0, 0,) corresponds to $t = 0$ and (1, 2, 1) corresponds to $t = 1$]

$$= -\bar{i}\int_0^1 (3t^5 + 2t^4)\,\mathrm{d}t - \bar{j}\int_0^1 (6t^5 - 2t^5)\,\mathrm{d}t + \bar{k}\int_0^1 (4t^3 + 2t^4)\,\mathrm{d}t$$

$$= -\bar{i}\left(3\frac{t^6}{6} + 2\frac{t^5}{5}\right)_0^1 - \bar{j}\left(4\frac{t^6}{6}\right)_0^1 + \bar{k}\left(t^4 + 2\frac{t^5}{5}\right)_0^1$$

$$= -\frac{9}{10}\bar{i} - \frac{2}{3}\bar{j} + \frac{7}{5}\bar{k}$$

Example 2.3 Find the work done when a force $\bar{F} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}$ displaces a particle in the xy-plane from (0, 0) to (1, 1) along the curve (i) $y = x$, (ii) $y^2 = x$. Comment on the answer.

$$W = \text{Work done by } \bar{F} = \int_C \bar{F}\cdot \mathrm{d}\bar{r}$$

$$= \int_C [(x^2 - y^2 + x)\bar{i} - (2xy + y)\,\bar{j}]\cdot(\mathrm{d}x\,\bar{i} + \mathrm{d}y\,\bar{j} + \mathrm{d}z\,\bar{k})$$

$$= \int_C [(x^2 - y^2 + x)\,\mathrm{d}x - (2xy + y)\,\mathrm{d}y]$$

Case (i) C is the line $y = x$.

$$\therefore \quad W_1 = \int_{\substack{y = x \\ (\mathrm{d}y = \mathrm{d}x)}} [(x^2 - y^2 + x)\,\mathrm{d}x - (2xy + y)\,\mathrm{d}y]$$

$$= \int_0^1 (-2x^2)\,\mathrm{d}x$$

$$= -\frac{2}{3}$$

Case (ii) C is the curve $y^2 = x$.

$$\therefore \quad W_2 = \int_{\substack{x = y^2 \\ (\mathrm{d}x = 2y\,\mathrm{d}y)}} [(x^2 - y^2 + x)\,\mathrm{d}x - (2xy + y)\,\mathrm{d}y]$$

$$= \int_0^1 (2y^5 - 2y^3 - y)\,\mathrm{d}y$$

$$= -\frac{2}{3}$$

Comment As the works done by the force, when it moves the particle along two different paths from (0, 0) to (1, 1), are equal, the force may be a conservative force.

In fact, $\bar{F}$ is a conservative force, as $\bar{F}$ is irrotational.

It can be verified that the work done by $\bar{F}$ when it moves the particle from (0, 0) to (1, 1) along any other path (such as $x^2 = y$) is also equal to $-\dfrac{2}{3}$.

Example 2.4 Find the work done by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r} = \cos t\bar{i} + \sin t\,\bar{j} + t\bar{k}$ from $t = 0$ to $t = 2\pi$.

From the vector equation of the curve C, we get the parametric equations of the curve as $x = \cos t$, $y = \sin t$, $z = t$.

Work done by
$$\bar{F} = \int_C \bar{F}\cdot \mathrm{d}\bar{r}$$
$$= \int_C (z\bar{i} + x\bar{j} + y\bar{k})\cdot(\mathrm{d}x\,\bar{i} + \mathrm{d}y\,\bar{j} + \mathrm{d}z\,\bar{k})$$
$$= \int_C (z\,\mathrm{d}x + x\,\mathrm{d}y + y\,\mathrm{d}z)$$
$$= \int_0^{2\pi} \left[t(-\sin t) + \cos^2 t + \sin t\right]\mathrm{d}t$$
$$= \left[t\cos t - \sin t + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) - \cos t\right]_0^{2\pi}$$
$$= (2\pi + \pi - 1) - (-1)$$
$$= 3\pi$$

Example 2.5 Evaluate $\oint_C \bar{F}\cdot \mathrm{d}\bar{r}$, where $\bar{F} = (\sin y)\bar{i} + x(1+\cos y)\bar{j} + z\bar{k}$ and C is the circle $x^2 + y^2 = a^2$ in the xy-plane.

Given integral
$$= \int_C \left[(\sin y)\bar{i} + x(1+\cos y)\bar{j} + z\bar{k}\right]\cdot(\mathrm{d}x\,\bar{i} + \mathrm{d}y\,\bar{j} + \mathrm{d}z\,\bar{k})$$
$$= \int_{\left(\substack{x^2+y^2=a^2\\ z=0}\right)} \left[\sin y\,\mathrm{d}x + x(1+\cos y)\,\mathrm{d}y + z\,\mathrm{d}z\right]$$
$$= \int_{x^2+y^2=a^2} \left[\sin y\,\mathrm{d}x + x(1+\cos y)\,\mathrm{d}y\right]$$

Since C is a closed curve, we use the parametric equations of C, namely $x = a\cos\theta$, $y = a\sin\theta$ and the parameter θ as the variable of integration. To move around the circle C once completely, θ varies from 0 to 2π.

Now, given integral
$$= \int \left[(\sin y\,\mathrm{d}x + x\cos y\,\mathrm{d}y) + x\,\mathrm{d}y\right]$$
$$= \int \left[\mathrm{d}(x\sin y)\; x\,\mathrm{d}y\right]$$
$$= \int_0^{2\pi} \left\{\mathrm{d}\left[a\cos\theta\cdot\sin(a\sin\theta)\right] + a^2\cos\theta\,\mathrm{d}\theta\right\}$$

$$= \left[a\cos\theta \cdot \sin(a\sin\theta) + \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{2\pi}$$

$$= \pi a^2$$

Example 2.6 Find the work done by the force $\bar{F} = (2xy + z^3)\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k}$, when it moves a particle from (1, −2, 1) to (3, 1, 4) along any path.

To evaluate the work done by a force, the equation of the path and the terminal points must be given. As the equation of the path is not given in this problem, we guess that the given force $\bar{F}$ is conservative. Let us verify whether $\bar{F}$ is conservative, i.e. irrotational.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= (0-0)\bar{i} - (3z^2 - 3z^2)\bar{j} + (2x - 2x)\bar{k}$$

$$= 0$$

$\therefore$ $\bar{F}$ is irrotational and hence conservative.

$\therefore$ Work done by $\bar{F}$ depends only on the terminal points.

Since $\bar{F}$ is irrotational, let $\bar{F} = \nabla\phi$.

It is easily found that $\phi = x^2y + z^3x + c$.

Work done by $\displaystyle \bar{F} = \int_{(1,-2,1)}^{(3,1,4)} \bar{F}\cdot d\bar{r}$

$$= \int_{(1,-2,1)}^{(3,1,4)} \nabla\phi \cdot d\bar{r}$$

$$= \int_{(1,-2,1)}^{(3,1,4)} d\phi$$

$$= [\phi(x,y,z)]_{(1,-2,1)}^{(3,1,4)}$$

$$= \left[x^2y + z^2x + c\right]_{(1,-2,1)}^{(3,1,4)}$$

$$= (201 + c) - (-1 + c)$$

$$= 202.$$

Example 2.7 Find the work done by the force $\bar{F} = y(3x^2y - z^2)\bar{i} + x(2x^2y - z^2)$ $\bar{j} - 2xyz\bar{k}$, when it moves a particle around a closed curve C.

To evaluate the work done by a force, the equation of the path C and the terminal points must be given.

Since C is a closed curve and the particle moves around this curve once completely, any point (x_0, y_0, z_0) can be taken as the initial as well as the final point.

But the equation of C is not given. Hence we guess that the given force $\bar{F}$ is conservative, i.e. irrotational. Actually it is so, as verified below.

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y^2x^2 - yz^2 & 2x^3y - z^2x & -2xyz \end{vmatrix}$$

$$= (-2xz + 2xz)\bar{i} - (-2yz + 2yz)\bar{j} + (6x^2y - z^2 - 6x^2y + z^2)\bar{k}$$

$$= 0$$

Since $\bar{F}$ is irrotational, let $\bar{F} = \nabla\phi$.

$$\therefore \text{ Work done by } \bar{F} = \oint_C \bar{F}\cdot \mathrm{d}\bar{r}$$

$$= \oint_C \nabla\phi\cdot \mathrm{d}\bar{r}$$

$$= \int_{(x_0, y_0, z_0)}^{(x_0, y_0, z_0)} \mathrm{d}\phi$$

$$= \phi(x_0, y_0, z_0) - \phi(x_0, y_0, z_0)$$

$$= 0$$

Example 2.8 Evaluate $\iint_S \bar{A}\cdot \mathrm{d}\bar{S}$, where $\bar{A} = 12x^2y\bar{i} - 3yz\bar{j} + 2z\bar{k}$ and S is the portion of the plane $x + y + z = 1$ included in the first octant (Fig. 2.4).

Given integral $\mathrm{I} = \iint_S \bar{A}\cdot\hat{n}\,\mathrm{d}\bar{S}$, where $\hat{n}$ is the unit normal to the surface S given by

$\phi = c$,

i.e. $x + y + z = 1$

$$\therefore \quad \phi = x + y + z$$

$$\nabla\phi = \bar{i} + \bar{j} + \bar{k}$$

$$\hat{n} = \frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})$$

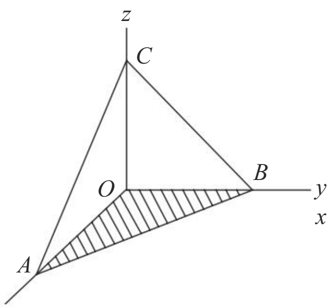

Fig. 2.4

$$\therefore \quad \mathrm{I} = \iint_S (12x^2y\bar{i} - 3yz\bar{j} + 2z\bar{k})\cdot\frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})\,\mathrm{d}S$$

$$= \frac{1}{\sqrt{3}}\iint_S (12x^2y - 3yz + 2z)\,\mathrm{d}S$$

To convert the surface integral as a double integral, we project the surface S on the xoy - plane. Then dS cos γ= dA, where γ is the angle between the surface S and the xoy - plane, i.e. the angle between $\hat{n}$ and $\bar{k}$. $\therefore \cos \gamma = \hat{n} \cdot \bar{k}$

$$\therefore \qquad dS = \frac{dA}{\hat{n} \cdot k} = \frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

$$\therefore \qquad I = \frac{1}{\sqrt{3}} \iint_{\Delta OAB} (12x^2 y - 3yz + 2z) \frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

[$\because$ the projection of S on the xoy plane is ΔOAB]

$$= \iint_{\Delta OAB} \left\{12x^2 y - 3y(1 - x - y) + 2(1 - x - y)\right\} dx\,dy$$

Note ☑ To express the integrand as a function of x and y only, z is expressed as a function of x and y from the equation of S.

$$I = \int_0^1 \int_0^{1-y} (12x^2 y + 3xy + 3y^2 - 5y - 2x + 2)\, dx\,dy$$

$$= \int_0^1 \left[4y(1 - y)^3 + \frac{3y}{2}(1 - y)^2 + 3y^2(1 - y) - 5y(1 - y) - (1 - y)^2 + 2(1 - y)\right] dy$$

$$= \frac{49}{120}$$

Example 2.9 Evaluate $\iint_S \bar{F} \cdot d\bar{S}$, where $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant.

Given integral $I = \iint_S \bar{F} \cdot \hat{n}\, dS$, where $\hat{n}$ is the unit normal to the surface S given by

$$\phi = c \text{ i.e. } \quad x^2 + y^2 + z^2 = 1.$$

$$\phi = x^2 + y^2 + z^2$$

$$\therefore \qquad \nabla\phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\therefore \qquad \hat{n} = \frac{2x\bar{i} + 2y\bar{j} + 2z\bar{k}}{\sqrt{4(x^2 + y^2 + z^2)}}$$

$$= \frac{2(x\bar{i} + y\bar{j} + z\bar{k})}{\sqrt{4 \times 1}} \qquad [\because \text{the point } (x, y, z) \text{ lies on } S]$$

$$= x\bar{i} + y\bar{j} + z\bar{k}.$$

$$\therefore \qquad I = \iint_S (yz\bar{i} + zx\bar{j} + xy\bar{k}) \cdot (x\bar{i} + y\bar{j} + z\bar{k})\, dS$$

$$= \iint_S 3xyz\, dS$$

$$= \iint_R 3xyz \frac{dx\, dy}{\hat{n} \cdot \bar{k}},$$

where R is the region in the xy-plane bounded by the circle $x^2 + y^2 = 1$ and lying in the first quadrant.

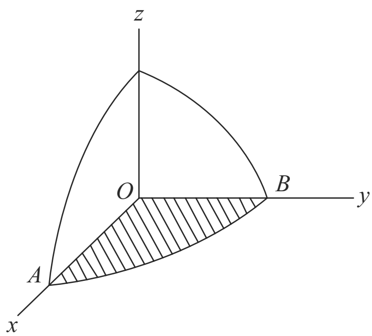

Fig. 2.5

$$I = \iint_R 3xyz \frac{dx\, dy}{z}$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} 3xy\, dx\, dy$$

$$= \int_0^1 3y \left(\frac{x^2}{2}\right)_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2} \int_0^1 y\,(1 - y^2)\, dy$$

$$= \frac{3}{2} \left(\frac{y^2}{2} - \frac{y^4}{4}\right)_0^1$$

$$= \frac{3}{8}$$

Example 2.10 Evaluate $\iint_S \bar{F} \cdot dS$ if, $\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$ and S is the surface of the cylinder $x^2 + y^2 = 9$ contained in the first octant between the planes $z = 0$ and $z = 2$.

Given integral $I = \iint_S \bar{F} \cdot \hat{n}\, dS$, where $\hat{n}$ is the unit normal to the surface S given by $\phi = c$, i.e. $x^2 + y^2 = 9$

$\therefore \quad \phi = x^2 + y^2$

$$\nabla\phi = 2x\bar{i} + 2y\bar{j}$$

$$\therefore \quad \hat{n} = \frac{2xi + 2y\bar{j}}{\sqrt{4(x^2 + y^2)}}$$

$$= \frac{2(x\bar{i} + y\bar{j})}{\sqrt{4 \times 9}} \quad [\because \text{ the point } (x, y, z) \text{ lies on } S]$$

$$= \frac{1}{3}(x\bar{i} + y\bar{j})$$

$$\therefore\ \mathrm{I} = \iint_s (yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k})\cdot\frac{1}{3}(x\bar{i} + y\bar{j})\,\mathrm{d}S$$

$$= \frac{1}{3}\iint_s (xyz + 2y^3)\,\mathrm{d}S$$

$$= \frac{1}{3}\iint_R (xyz + 2y^3)\frac{\mathrm{d}x\,\mathrm{d}z}{\hat{n}\cdot\bar{j}}$$

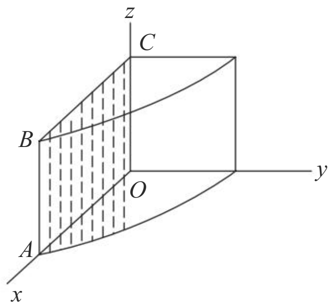

Fig. 2.6

where R is the rectangular region OABC in the xoz-plane, got by projecting the cylindrical surface S on the xoz-plane (Fig. 2.6).

$$\mathrm{I} = \frac{1}{3}\iint_R (xyz + 2y^3)\frac{\mathrm{d}x\,\mathrm{d}z}{\frac{y}{3}}$$

$$= \iint_R (xz + 2y^2)\,\mathrm{d}x\,\mathrm{d}z$$

$$= \int_0^2\int_0^3 [xz + 2(9 - x^2)]\,\mathrm{d}x\,\mathrm{d}z$$

$$= \int_0^2\left(\frac{9}{2}z + 18\times 3 - 2\times 9\right)\mathrm{d}z$$

$$= \left(\frac{9}{2}\cdot\frac{z^2}{2} + 36\cdot z\right)_0^2 = 81$$

EXERCISE 2(c)

Part A

(Short Answer Questions)

1. Define line integral of a vector point function.
2. When is a force said to be conservative? State also the condition to be satisfied by a conservative force.
3. If $\bar{F}$ is irrotational, prove that it is conservative.
4. Define surface integral of a vector point function.
5. Explain how $\iint_S \bar{F}\cdot\mathrm{d}\bar{S}$ is evaluated.
6. Evaluate $\int_C \bar{r}\cdot\mathrm{d}\bar{r}$, where C is the line $y = x$ in the xy-plane from (1, 1) to (2, 2).
7. Find the work done by the force $\bar{F} = x\bar{i} + 2y\bar{j}$ when it moves a particle on the curve $2y = x^2$ from (0, 0) to (2, 2).
8. Prove that the force $\bar{F} = (2x + yz)\bar{i} + (xz - 3)\bar{j} + xy\bar{k}$ is conservative.

9. Evaluate $\iint\limits_S (yz\bar{i} + zx\bar{j} + xy\bar{k}) \cdot d\bar{S}$, where S is the region bounded by $x = 0$, $x = a$, $y = 0$, $y = b$ and lying in the xoy-plane.
10. Find the work done by a conservative force when it moves a particle around a closed curve.

Part B

11. Evaluate $\int\limits_C \phi \, d\bar{r}$, where $\phi = 2xyz^2$ and C is the curve given by $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$.
12. Evaluate $\int\limits_C \bar{F} \times d\bar{r}$ along the curve $x = \cos t$, $y = 2 \sin t$, $z = \cos t$ from $t = 0$ to $t = \dfrac{\pi}{2}$, given that $\bar{F} = 2x\bar{i} + y\bar{j} + z\bar{k}$.
13. Evaluate $\int\limits_C \bar{F} \times d\bar{r}$ along the curve $x = \cos\theta$, $y = \sin\theta$, $z = 2\cos\theta$ from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$, given that $\bar{F} = 2y\bar{i} - z\bar{j} + x\bar{k}$
14. Evaluate $\int\limits_C \bar{F} \cdot d\bar{r}$ along the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, given that $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$.
15. Find the work done by the force $\bar{F} = 3xy\bar{i} - y^2\bar{j}$, when it moves a particle along the curve $y = 2x^2$ in the xy-plane from (0, 0) to (1, 2).
16. Find the work done by the force $\bar{F} = (y + 3z)\bar{i} + (2z + x)\bar{j} + (3x + 2y)\bar{k}$, when it moves a particle along the curve $x = a \cos t$, $y = a \sin t$, $z = \dfrac{2at}{\pi}$ between the points $(a, 0, 0)$ and $(0, a, a)$.
17. Find the work done by the force $\bar{F} = (2y + 3)\bar{i} + xz\bar{j} + (yz - x)\bar{k}$ when it moves a particle along the line segment joining the origin and the point (2, 1, 1).
18. Evaluate $\oint\limits_C \bar{F} \cdot d\bar{r}$, where C is the circle $x^2 + y^2 = 4$ in the xy-plane, if $\bar{F} = (2x - y - z)\bar{i} + (x + y - z^2)\bar{j} + (3x - 2y + 4z)\bar{k}$.
19. Find the work done by the force $\bar{F} = (x^2 + y^2)\bar{i} + (x^2 + z^2)\bar{j} + y\bar{k}$, when it moves a particle along the upper half of the circle $x^2 + y^2 = 1$ from the point $(-1, 0)$ to the point $(1, 0)$.
20. Find the work done by the force $\bar{F} = (e^x z - 2xy)\bar{i} + (1 - x^2)\bar{j} + (e^x + z)\bar{k}$, when it moves a particle from $(0, 1, -1)$ to $(2, 3, 0)$ along any path.
21. Find the work done by the force $\bar{F} = (y^2 \cos x + z^3)\bar{i} + (2y \sin x - 4)\bar{j} + (3xz^2 + 2)\bar{k}$, when it moves a particle from $(0, 1, -1)$ to $\left(\dfrac{\pi}{2}, -1, 2\right)$ along any path.

22. Find the work done by the force $\overline{F} = y^2\overline{i} + 2(xy+z)\overline{j} + 2y\overline{k}$, when it moves a particle around a closed curve C.
23. Evaluate $\iint_S \overline{F} \cdot \mathrm{d}\overline{S}$, where $\overline{F} = xy\overline{i} - x^2\overline{j} + (x+z)\overline{k}$ and S is the part of the plane $2x + 2y + z = 6$ included in the first octant.
24. Evaluate $\iint_S \overline{F} \cdot \mathrm{d}\overline{S}$, where $\overline{F} = y\overline{i} - x\overline{j} + 4\overline{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant.
25. Evaluate $\iint_S \overline{F} \cdot \mathrm{d}\overline{S}$, where $\overline{F} = z\overline{i} + x\overline{j} - 3y^2z\overline{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between the planes $z = 0$ and $z = 5$.

2.5 INTEGRAL THEOREMS

The following three theorems in Vector Calculus are of importance from theoretical and practical considerations:

1. Green's theorem in a plane
2. Stoke's theorem
3. Gauss Divergence theorem

Green's theorem in a plane provides a relationship between a line integral and a double integral.

Stoke's theorem, which is a generalisation of Green's theorem, provides a relationship between a line integral and a surface integral. In fact, Green's theorem can be deduced as a particular case of Stoke's theorem. Gauss Divergence theorem provides a relationship between a surface integral and a volume integral.

We shall give the statements of the above theorems (without proof) below and apply them to solve problems:

2.5.1 Green's Theorem in a Plane

If C is a simple closed curve enclosing a region R in the xy-plane and $P(x, y)$, $Q(x, y)$ and its first order partial derivatives are continuous in R, then

$$\oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$

where C is described in the anticlockwise direction.

2.5.2 Stoke's Theorem

If S is an open two sided surface bounded by a simple closed curve C and if $\overline{F}$ is a vector point function with continuous first order partial derivatives on S, then

$$\oint_C \overline{F} \cdot \mathrm{d}\overline{r} = \iint_S \operatorname{curl} \overline{F} \cdot \mathrm{d}\overline{S},$$

where C is described in the anticlockwise direction as seen from the positive tip of the outward drawn normal at any point of the surface S.

2.5.3 Gauss Divergence Theorem

If S is a closed surface enclosing a region of space with volume V and if $\bar{F}$ is a vector point function with continuous first order partial derivatives in V, then

$$\iint_S \bar{F} \cdot \mathrm{d}\bar{S} = \iiint_V (\operatorname{div} \bar{F})\, \mathrm{d}v\,.$$

2.5.4 Deduction of Green's Theorem from Stoke's Theorem

Stoke's theorem is $\quad \oint_C \bar{F} \cdot \mathrm{d}\bar{r} = \iint_S \operatorname{curl} \bar{F} \cdot \mathrm{d}\bar{S} \qquad (1)$

Take S to be a plane surface (region) R in the xy-plane bounded by a simple closed curve C.

Also take $\bar{F} = P(x, y)\bar{i} + Q(x, y)\bar{j}$

Then curl $\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$

$= \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\bar{k} \qquad$ [$\because$ P and Q are functions of x and y]

Inserting all these in (1), we get

$$\oint_C (P\bar{i} + Q\bar{j}) \cdot (\mathrm{d}x\bar{i} + \mathrm{d}y\bar{j} + \mathrm{d}z\bar{k}) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \bar{k} \cdot \mathrm{d}\bar{S}$$

i.e. $\quad \oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_R (Q_x - P_y)\, \bar{k} \cdot \hat{n}\, \mathrm{d}S \qquad (2)$

Now $\hat{n}$ is the unit vector in the outward drawn normal direction to the surface R. Normal at any point of R, that lies in the xy-plane, is parallel to the z-axis.

Taking the positive direction of the z-axis as the positive (outward) direction of $\hat{n}$, we get $\hat{n} = \bar{k}$.

Using this in (2), we get

$$\oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_R (Q_x - P_y)\, \mathrm{d}x\, \mathrm{d}y$$

[$\because$ $\mathrm{d}S$ = elemental plane surface area in the xy-plane
= $\mathrm{d}x\,\mathrm{d}y$]

Note ☑ If the surface S is a plane surface, problems in Stoke's theorem will reduce to problems in Green's theorem in a plane.

2.5.5 Scalar form of Stoke's Theorem

Take $\bar{F} = P\bar{i} + Q\bar{j} + R\bar{k}$ in Stoke′s theorem, where P, Q, R are functions of x, y, z.

Then $$\bar{F}\cdot \mathrm{d}\bar{r} = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z \qquad (1)$$

$$\text{Curl } \bar{F} = (R_y - Q_z)\bar{i} + (P_z - R_x)\bar{j} + (Q_x - P_y)\bar{k}$$

$$\begin{aligned}\therefore \quad \text{Curl } \bar{F}\cdot \mathrm{d}\bar{S} &= \text{curl } \bar{F}\cdot \hat{n}\,\mathrm{d}S \\ &= (R_y - Q_z)(\hat{n}\cdot\bar{i})\mathrm{d}S + (P_z - R_x)(\hat{n}\cdot\bar{j})\mathrm{d}S + (Q_x - P_y)(\hat{n}\cdot\bar{k})\mathrm{d}S \\ &= (R_y - Q_z)\,\mathrm{d}y\,\mathrm{d}z + (P_z - R_x)\,\mathrm{d}z\,\mathrm{d}x + (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y \qquad (2)\end{aligned}$$

$[\because \mathrm{d}S\cos\gamma = \mathrm{d}x\,\mathrm{d}y$ i.e. $\mathrm{d}S\ (\hat{n}\cdot\bar{k}) = \mathrm{d}x\,\mathrm{d}y]$

Inserting (1) and (2) in Stoke's theorem, it reduces to the scalar form

$$\int_C (P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z) = \iint_S [(R_y - Q_z)\,\mathrm{d}y\,\mathrm{d}z + (P_z - R_x)\,\mathrm{d}z\,\mathrm{d}x + (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y] \qquad (3)$$

Note ☑ If we take P and Q as functions of x and y only and $R = 0$ in (3), we get Green's theorem in a plane.

2.5.6 Scalar form of Gauss Divergence Theorem

Take $\bar{F} = P\bar{i} + Q\bar{j} + R\bar{k}$ in Divergence theorem, where P, Q, R are functions of x, y, z.

Then div $$\bar{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \qquad (1)$$

$$\begin{aligned}\bar{F}\cdot \mathrm{d}\bar{S} &= \bar{F}\cdot\hat{n}\,\mathrm{d}S \\ &= P(\hat{n}\cdot\bar{i})\,\mathrm{d}S + Q\,(\hat{n}\cdot\bar{j})\,\mathrm{d}S + R\,(\hat{n}\cdot\bar{k})\,\mathrm{d}S \\ &= P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}z\,\mathrm{d}x + R\,\mathrm{d}x\,\mathrm{d}y \qquad (2)\end{aligned}$$

Inserting (1) and (2) in Divergence theorem, we get

$$\iint_S (P\mathrm{d}y\,\mathrm{d}z + Q\mathrm{d}z\,\mathrm{d}x + R\mathrm{d}x\,\mathrm{d}y) = \iiint_V (P_x + Q_y + R_z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (3)$$

which is the scalar form of Divergence theorem.

Note ☑ (3) is also called *Green's theorem in space*.

2.5.7 Green's Identities

In Divergence theorem, take $\bar{F} = \phi\nabla\psi$, where ϕ and ψ are scalar point functions.
Then div $(\phi\,\nabla\,\psi) = \nabla\cdot(\phi\,\nabla\,\psi)$

$$= \phi\,\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

$\therefore$ Divergence theorem becomes

$$\iint_S \phi\nabla\psi\cdot \mathrm{d}\bar{S} = \iiint_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,\mathrm{d}V \qquad (1)$$

Interchanging ϕ and ψ,

$$\iint_S \psi\nabla\phi\cdot d\bar{S} = \iiint_V (\psi\nabla^2\phi + \nabla\phi\cdot\nabla\psi)dV \qquad (2)$$

(1) – (2) gives,

$$\iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\bar{S} = \iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV \qquad (3)$$

Note ☑ (1) or (2) is called *Green's first identity*. and (3) is called *Green's second identity*.

WORKED EXAMPLE 2(d)

Example 2.1 Verify Green's theorem in a plane with respect to $\int_C (x^2 - y^2)\, dx + 2xy\, dy]$, where C is the boundary of the rectangle in the xoy-plane bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$.

(OR)

Verify Stoke's theorem for a vector field defined by $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$ in the rectangular region in the xoy-plane bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$.

Stoke's theorem is $\int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot d\bar{S}$

Now
$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$
$$= (2y + 2y)\,\bar{k}$$

∴ We have to verify that

$$\int_C [(x^2 - y^2)\bar{i} + 2xy\,\bar{j}]\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k}) = \iint_S 4y\bar{k}\cdot\hat{n}\,dS$$

i.e.
$$\int_C [(x^2 - y^2)\,dx + 2xy\,dy] = \iint_R 4y\,dx\,dy \text{[Fig. 2. 7]} \qquad (1)$$

$$\left[\because \hat{n} = \bar{k}\right]$$

(1) is also the result for the given function as per Green´s theorem.

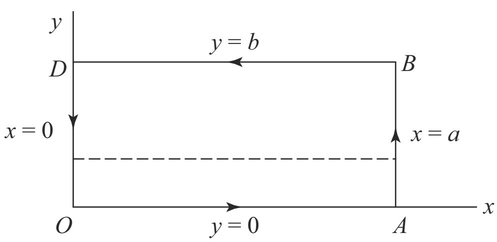

Fig. 2.7

$$\text{L.S. of (1)} = \underset{\substack{OA \\ \begin{pmatrix} y=0 \\ dy=0 \end{pmatrix}}}{\int} + \underset{\substack{AB \\ \begin{pmatrix} x=a \\ dx=0 \end{pmatrix}}}{\int} + \underset{\substack{BD \\ \begin{pmatrix} y=b \\ dy=0 \end{pmatrix}}}{\int} + \underset{\substack{DO \\ \begin{pmatrix} x=0 \\ dx=0 \end{pmatrix}}}{\int} [(x^2 - y^2)\,dx + 2xy\,dy]$$

[∵ the boundary C consists of 4 lines]

$$= \int_0^a x^2 dx + \int_0^b 2ay\,dy + \int_a^0 (x^2 - b^2)\,dx + 0$$

Note ☑ To simplify the line integral along each line, we make use of the equation of the line and the corresponding value of dx or dy

$$= \left(\frac{x^3}{3}\right)_0^a + a(y^2)_0^b - \left(\frac{x^3}{3} - b^2 x\right)_0^a$$

$$= 2\,ab^2$$

$$\text{R.S. of (1)} = \int_0^b \int_0^a 4y\,dx\,dy$$

$$= \int_0^b 4y\,(x)_0^a\,dy = 2a(y^2)_0^b$$

$$= 2\,a\,b^2.$$

Since L.S. of (1) = R.S. of (1), Stoke's theorem (Green′s theorem) is verified.

Example 2.2 Verify Green's theorem in a plane for $\int_C [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy]$, where C is the boundary of the region defined by the lines $x = 0, y = 0$ and $x + y = 1$.

$$\text{Green's theorem is } \int_C (P\,dx + Q\,dy) = \iint_R (Q_x - P_y)\,dx\,dy$$

∴ For the given integral,

$$\int_C [3x^2 - 8y^2]\,dx + (4y - 6xy)\,dy] = \iint_R 10y\,dx\,dy \text{ [Fig. 2.8]} \qquad (1)$$

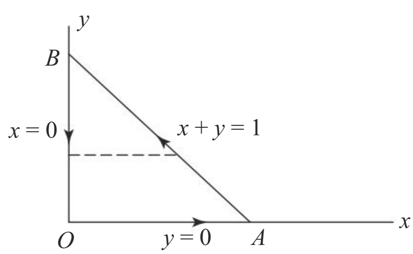

Fig. 2.8

$$\text{L.S. of (1)} = \underset{\substack{OA \\ \begin{pmatrix} y=0 \\ dy=0 \end{pmatrix}}}{\int} + \underset{\substack{AB \\ \begin{pmatrix} x+y=1 \\ x=1-y \\ dx=-dy \end{pmatrix}}}{\int} + \underset{\substack{BO \\ \begin{pmatrix} x=0 \\ dx=0 \end{pmatrix}}}{\int} [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$$

$$= \int_0^1 3x^2 \mathrm{d}x + \int_0^1 [\{3(1-y)^2 - 8y^2\}(-\mathrm{d}y) + \{4y - 6y\,(1-y)\}\,\mathrm{d}y] + \int_1^0 4y\,\mathrm{d}y$$

$$= \int_0^1 3x^2\,\mathrm{d}x + \int_0^1 (11y^2 + 4y - 3)\,\mathrm{d}y - \int_0^1 4y\,\mathrm{d}y = 1 + \left(\frac{11}{3} + 2 - 3\right) - 2$$

$$= \frac{5}{3}$$

R.S. of (1) $= \int_0^1 \int_0^{1-y} 10y\,\mathrm{d}x\,\mathrm{d}y$

$$= \int_0^1 10y\,(1-\mathrm{y})\,\mathrm{d}y$$

$$= \left(5y^2 - 10\frac{y^3}{3}\right)_0^1 = \frac{5}{3}$$

Since L.S. of (1) = R.S. of (1), Green's theorem is verified.

Example 2.3 Use Stoke's theorem to evaluate $\int_C \overline{F} \cdot \mathrm{d}\overline{r}$, where $\overline{F} = (\sin x - y)$ $\overline{i} - \cos x\overline{j}$ and C is the boundary of the triangle whose vertices are (0, 0), $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

Note ☑ Evaluating $\int_C \overline{F} \cdot \mathrm{d}\overline{r}$ by using Stoke's theorem means expressing the line integral in terms of its equivalent surface integral and then evaluating the surface integral.

By Stoke's theorem, $\int_C \overline{F} \cdot \mathrm{d}\overline{r} = \iint_S \mathrm{curl}\,\overline{F} \cdot \mathrm{d}\overline{S}$, where S is any open two-sided surface bounded by C. [Fig. 2.9]

To simplify the work, we shall choose S as the plane surface R in the xoy-plane bounded by C.

$\therefore \int_C \overline{F} \cdot \mathrm{d}\overline{r} = \iint_R \mathrm{curl}\,\overline{F} \cdot \overline{k}\,\mathrm{d}x\,\mathrm{d}y$ [$\because$ For the xoy-plane, $\hat{n} = \overline{k}$ and $\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y$]

For this problem,

$$\mathrm{curl}\,\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\sin x - y) & -\cos x & 0 \end{vmatrix}$$

$$= (\sin x + 1)\overline{k}$$

$$\therefore \text{The given line integral} = \iint_R (1+\sin x)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (1+\sin x)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \left(x - \cos x\right)_{\frac{\pi y}{2}}^{\frac{\pi}{2}} \,\mathrm{d}y$$

$$= \int_0^1 \left(\frac{\pi}{2} - \frac{\pi y}{2} + \cos\frac{\pi y}{2}\right)\mathrm{d}y$$

$$= \left(\frac{\pi}{2}y - \frac{\pi y^2}{4} + \frac{2}{\pi}\sin\frac{\pi y}{2}\right)_0^1$$

$$= \frac{\pi}{4} + \frac{2}{\pi}.$$

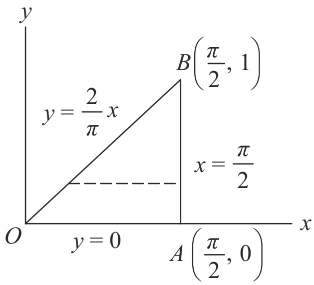

Fig. 2.9

Example 2.4 Use Green's theorem in a plane to evaluate $\oint_C [(2x - y)\,\mathrm{d}x + (x + y)\,\mathrm{d}y]$, where C is the boundary of the circle $x^2 + y^2 = a^2$ in the xoy-plane.

By Green's theorem in a plane,

$$\oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_R (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\therefore \oint_C [(2x - y)\,\mathrm{d}x + (x + y)\,\mathrm{d}y] = \iint_R [1-(-1)]\,\mathrm{d}x\,\mathrm{d}y$$

$$= 2\iint_R \mathrm{d}x\,\mathrm{d}y \text{[Fig. 2.10]}$$

$$= 2 \times \text{area of the region } R$$

$$= 2\pi a^2$$

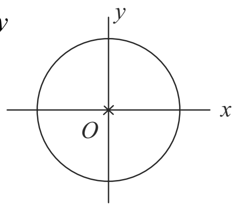

Fig. 2.10

Example 2.5 Verify Stoke's theorem when $\bar{F} = (2xy - x^2)\,\bar{i} - (x^2 - y^2)\,\bar{j}$ and C is the boundary of the region enclosed by the parabolas $y^2 = x$ and $x^2 = y$.

$$\text{Stoke's theorem is } \int_C \bar{F}\cdot \mathrm{d}\bar{r} = \iint_S \text{curl } \bar{F}\cdot\hat{n}\,\mathrm{d}S$$

$$\text{Now curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - x^2 & -x^2 + y^2 & 0 \end{vmatrix}$$

$$= (-2x - 2x)\bar{k}$$

$$= -4x\bar{k}$$

∴ Stoke's theorem becomes

$$\int_C [(2xy - x^2)\,dx - (x^2 - y^2)\,dy] = \iint_R -4x\bar{k}\cdot\bar{k}\,dx\,dy$$

$$= \iint_R -4x\,dx\,dy \text{ [Fig. 2.11]} \tag{1}$$

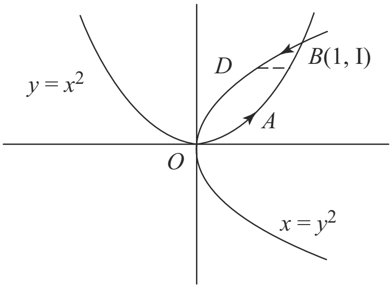

Fig. 2.11

$$\text{L.S. of (1)} = \underset{\substack{OAB \\ \left(\substack{y=x^2 \\ dy=2x\,dx}\right)}}{\int} + \underset{\substack{BDO \\ \left(\substack{x=y^2 \\ dx=2y\,dy}\right)}}{\int} [(2xy - x^2)\,dx - (x^2 - y^2)\,dy]$$

$$= \int_0^1 (2x^5 - x^2)\,dx + \int_1^0 (3y^4 - 2y^5 + y^2)\,dy$$

[∵ the coordinates of B are found as (1, 1) by solving the equations $y = x^2$ and $x = y^2$]

$$= \frac{-3}{5}$$

$$\text{R.S. of (1)} = \int_0^1 \int_{y^2}^{\sqrt{y}} -4x\,dx\,dy$$

$$= -2\int_0^1 (x^2)_{y^2}^{\sqrt{y}}\,dy$$

$$= -2\int_0^1 (y - y^4)\,dy$$

$$= -\frac{3}{5}$$

Since L.S. of (1) = R.S. of (1), Stoke's theorem is verified.

Example 2.6 Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int_C (x\,dy - y\,dx)$ Hence find the area bounded (i) by the parabola $y^2 = 4ax$ and its latus rectum and (ii) by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

By Green's theorem, $\displaystyle\int_C (P\,dx + Q\,dy) = \iint_R (Q_x - P_y)\,dx\,dy$

Taking $P = -\dfrac{y}{2}$ and $Q = \dfrac{x}{2}$, we get

$$\frac{1}{2}\int_C (x\,dy - y\,dx) = \iint_R \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right] dx\,dy$$
$$= \iint_R dx\,dy$$

=Area of the region R enclosed by C.

(i) Area bounded by the parabola and its latus rectum $= \dfrac{1}{2}\displaystyle\int_C (x\,dy - y\,dx)$ [Fig. 2.12]

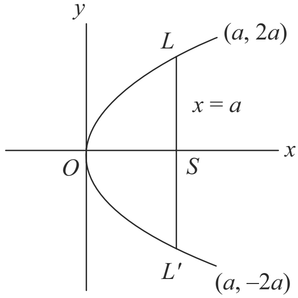

Fig. 2.12

$$= \frac{1}{2}\left[\int_{LOL'} + \int_{L'SL} (x\,dy - y\,dx)\right]$$

$$= \frac{1}{2}\left[\int_{\left(\substack{x = \frac{y^2}{4a} \\ dx = \frac{y}{2a}dy}\right)} (x\,dy - y\,dx) + \int_{\substack{x=a \\ dx=0}} (x\,dy - y\,dx)\right]$$

$$= \frac{1}{2}\left[\int_{2a}^{-2a} -\frac{y^2}{4a}\,dy + \int_{-2a}^{2a} a\,dy\right]$$

$$= \int_0^{2a} \frac{y^2}{4a}\,dy + \int_0^{2a} a\,dy$$

$$= \frac{8}{3}a^2.$$

(ii) Area bounded by the ellipse

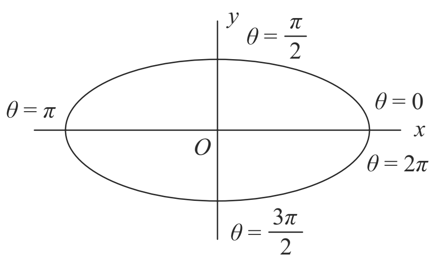

Fig. 2.13

$$= \frac{1}{2}\int_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_{\left(\substack{x = a\cos\theta \\ y = b\sin\theta}\right)} (x\,dy - y\,dx) \quad \text{[Fig. 2.13]}$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \pi ab.$$

Example 2.7 Use Stoke's theorem to prove that (i) curl (grad ϕ) = 0 and (ii) div (curl $\bar{F}$) = 0.

(i) Stoke's theorem is $\iint_S \text{curl}\, \bar{F} \cdot d\bar{S} = \oint_C \bar{F} \cdot d\bar{r}$

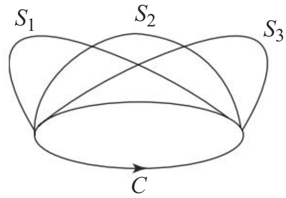

Fig. 2.14

Taking $\bar{F} = \text{grad}\, \phi$, we have

$$\iint_S \text{curl (grad}\, \phi) \cdot d\bar{S} = \oint_C \text{grad}\, \phi \cdot d\bar{r}$$
$$= \oint_C d\phi$$
$$= 0$$

The above result is true for any open two-sided surface S, provided it is bounded by the same simple closed curve C. [Fig. 2.14]

$\therefore$ curl (grad ϕ) $\cdot\, d\bar{S} = 0$, [for any S and hence for any $d\bar{S}$]

$\therefore$ curl (grad ϕ) =0

(ii) Gauss divergence theorem is

$\iiint_V (\text{div}\, \bar{F})\, dv = \iint_S \bar{F} \cdot d\bar{S}$, where S is a closed surface enclosing a volume V.

Replacing $\bar{F}$ by curl $\bar{F}$, we have

$$\iiint_V \text{div (curl}\, \bar{F})\, dv = \iint_S \text{curl}\, \bar{F} \cdot d\bar{S} \tag{1}$$

The surface integral in (1) appears to be the same as that in Stoke's theorem. However Stoke's theorem cannot be used to simplify the R.S. of (1), as S is a closed surface. In order to enable us to use Stoke's theorem, we divide S into two parts S_1 and S_2 by a plane. S_1 and S_2 are open two-sided surfaces, each of which is bounded by the same closed curve C as given in Fig. 2.15.

Now (1) becomes

$$\iiint_V \text{div (curl}\, \bar{F})\, dv = \iint_{S_1} \text{curl}\, \bar{F} \cdot d\bar{S} + \iint_{S_2} \text{curl}\, \bar{F} \cdot d\bar{S}$$
$$= \oint_C \bar{F} \cdot d\bar{r} + \oint_C \bar{F} \cdot d\bar{r}$$

[by Stoke's theorem]

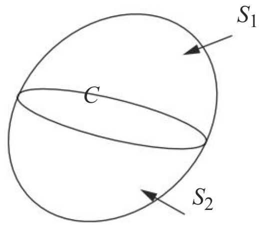

Fig. 2.15

Note ☑ If C is described in the anticlockwise sense as seen from the positive tip of the outer normal to S_1, it will be described in the clockwise sense as seen from the positive tip of the outer normal to S_2.

$$\therefore \quad \iiint_V \text{div (curl}\, \bar{F})\, dv = \oint_C \bar{F} \cdot d\bar{r} - \oint_C \bar{F} \cdot d\bar{r}$$
$$= 0.$$

Since V is arbitrary, div (curl $\bar{F}$) = 0.

Example 2.8 Evaluate $\int_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$, by using Stoke's theorem, where C is the boundary of the rectangle defined by $0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$.

The scalar form of Stoke's theorem is $\int_C (P\,dx + Q\,dy + R\,dz) = \iint_S [(R_y - Q_z)$ $dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy]$

Taking $P = \sin z$, $Q = -\cos x$, $R = \sin y$, we get

$$\int_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$$

$$= \iint_S (\cos y\,dy\,dz + \cos z\,dz\,dx + \sin x\,dx\,dy)$$

$$= \iint_S \sin x\,dx\,dy$$ [$\because$ S is the rectangle in the $z = 3$ plane and hence $dz = 0$]

$$= \int_0^1 \int_0^\pi \sin x\,dx\,dy$$

$$= -(\cos x)_0^\pi$$

$$= 2$$

Example 2.9 Evaluate $\iint_S (x\,dy\,dz + 2y\,dz\,dx + 3z\,dx\,dy)$ where S is the closed surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Scalar from of divergence theorem is

$$\iint_S (P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy) = \iiint_V (P_x + Q_y + R_z)\,dV$$

Taking $P = x$, $Q = 2y$, $R = 3z$,

the given surface integral = $\iiint_V 6\,dv = 6V$

$$= 6 \times \frac{4}{3}\pi a^3$$

$$= 8\pi a^3.$$

Example 2.10 Verify Stoke's theorem for $\bar{F} = xy\bar{i} - 2yz\,\bar{j} - zx\bar{k}$ where S is the open surface of the rectangular parallelopiped formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ above the xoy-plane.

Stoke's theorem is $\int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot d\bar{S}$

Here $$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix}$$

$$= 2y\bar{i} + z\bar{j} + x\bar{k}$$

$\therefore$ Stoke's theorem takes form

$$\int_C (xy\,dx - 2yz\,dy - zx\,dz) = \iint_S (2y\bar{i} + z\bar{j} - x\bar{k}) \cdot d\bar{S} \tag{1}$$

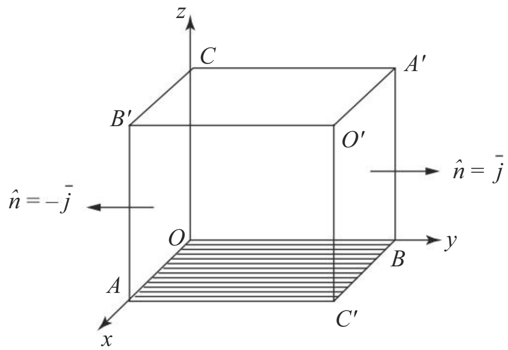

Fig. 2.16

The open cuboid S is made up of the five faces $x = 0, x = 1, y = 0, y = 2$ and $z = 3$ and is bounded by the rectangle $OAC'B$ lying on the xoy-plane. [Fig. 2.16]

$$\text{L.S. of (1)} = \int_{OAC'B} (xy\,dx - 2yz\,dy - zx\,dz)$$

$$= \int_{OAC'B} xy\,dx \text{ [Fig. 2.17]} \qquad (\because \text{ the boundary } C \text{ lies on the plane } z = 0)$$

$$= \int_{\substack{OA \\ \begin{pmatrix} y=0 \\ dy=0 \end{pmatrix}}} + \int_{\substack{AC' \\ \begin{pmatrix} x=1 \\ dx=0 \end{pmatrix}}} + \int_{\substack{C'B \\ \begin{pmatrix} y=2 \\ dy=0 \end{pmatrix}}} + \int_{\substack{BO \\ \begin{pmatrix} x=0 \\ dx=0 \end{pmatrix}}} (xy\,dx)$$

$$= \int_1^0 2x\,dx$$

$$= -1.$$

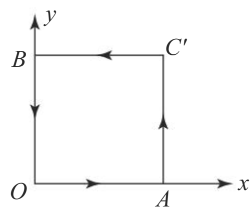

Fig. 2.17

$$\text{R.S. of (1)} = \iint_{\begin{pmatrix} x=0 \\ \hat{n}=-i \end{pmatrix}} + \iint_{\begin{pmatrix} x=1 \\ \hat{n}=i \end{pmatrix}} + \iint_{\begin{pmatrix} y=0 \\ \hat{n}=-\bar{j} \end{pmatrix}} + \iint_{\begin{pmatrix} y=2 \\ \hat{n}=\bar{j} \end{pmatrix}} + \iint_{\begin{pmatrix} z=3 \\ \hat{n}=\bar{k} \end{pmatrix}} (2y\bar{i} + z\bar{j} - x\bar{k}) \cdot \hat{n}\,ds$$

Note ☑ $\hat{n}$ is the unit normal at any point of the concerned surface. For example, at any point of the plane surface $y = 0$, the outward drawn normal is parallel to the y-axis, but opposite in direction. $\therefore$ $\hat{n}$ at any point of $y = 0$ is equal to $-\bar{j}$.

Similarly $\hat{n}$ for $y = 2$ is equal to $\bar{j}$ and so on.
Using the relevant value of $\hat{n}$ and simplifying the integrand, we have

$$\text{R. S. of (1)} = -\iint_{x=0} 2y\,dS + \iint_{x=1} 2y\,dS - \iint_{y=0} z\,dS + \iint_{y=2} z\,dS - \iint_{z=3} x\,dS$$

$$= -\int_0^3\int_0^2 2y\,dy\,dz + \int_0^3\int_0^2 2y\,dy\,dz - \int_0^1\int_0^3 z\,dz\,dx + \int_0^1\int_0^3 z\,dz\,dx - \int_0^2\int_0^1 x\,dx\,dy$$

[$\because$ Elemental plane (surface) area dS on $x = 0$ and $x = 1$ are equal, each equal to dy dz etc.]

$$= -\int_0^2\int_0^1 x\,dx\,dy \qquad (\because \text{ the other integrals cancel themselves})$$

$$= -\int_0^2 \left(\frac{x^2}{2}\right)_0^1 dy$$

$$= -1.$$

Thus Stoke's theorem is verified.

Example 2.11 Verify Stoke′s theorem for $\bar{F} = y^2 z\bar{i} + z^2 x\bar{j} + x^2 y\bar{k}$ where S is the open surface of the cube formed by the planes $x = \pm a$, $y = \pm a$ and $z = \pm a$, in which the plane $z = -a$ is cut.

$$\text{Stoke's theorem is } \int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl } \bar{F}\cdot d\bar{S}$$

Here

$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$

$$= (x^2 - 2zx)\bar{i} + (y^2 - 2xy)\bar{j} + (z^2 - 2yz)\bar{k}$$

$\therefore$ Stoke's theorem takes the form

$$\int_C (y^2 z\,dx + z^2 x\,dy + x^2 y\,dz) = \iint_S [(x^2 - 2zx)\bar{i} + (y^2 - 2xy)\bar{j} + (z^2 - 2yz)\bar{k}]\cdot d\bar{S} \qquad (1)$$

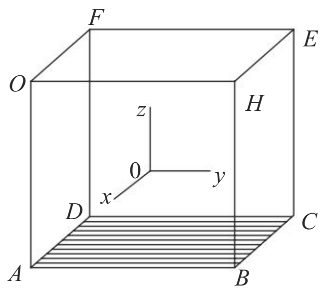

Fig. 2.18

The open cube S is bounded by the square ABCD that lies in the plane $z = -a$ (Fig. 2.18)

$$\text{L.S. of (1)} = \int_{(z=-a)} (y^2 z\,dx + z^2 x\,dy + x^2 y\,dz)$$

$$= \int_{ABCD} (-ay^2 dx + a^2 x\,dy) \qquad [\because dz = 0, \text{ as } z = -a]$$

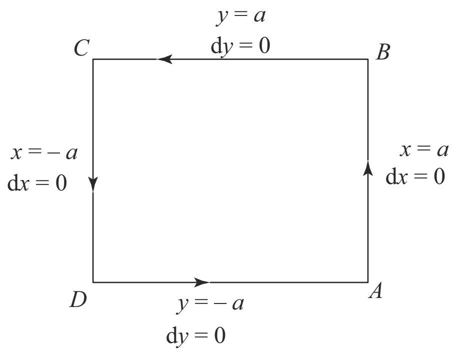

Fig. 2.19

$$\text{L.S. of (1)} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} (-ay^2\, dx + a^2\, x\, dy) \,\big(\text{Fig. 2.19}\big)$$

$$= \int_{-a}^{a} a^3 dy - \int_{a}^{-a} a^3 dx - \int_{a}^{-a} a^3 dy - \int_{-a}^{a} a^3 dx$$

$$= 4\, a^4.$$

$$\text{R.S. of (1)} = \iint\limits_{\binom{x=-a}{\hat{n}=-\bar{i}}} + \iint\limits_{\binom{x=a}{\hat{n}=\bar{i}}} + \iint\limits_{\binom{y=-a}{\hat{n}=-\bar{j}}} + \iint\limits_{\binom{y=a}{\hat{n}=\bar{j}}}$$

$$+ \iint\limits_{\binom{z=a}{\hat{n}=\bar{k}}} [(x^2 - 2zx)\, \bar{i} + (y^2 - 2xy)\, \bar{j} + (z^2 - 2yz)\, \bar{k}] \cdot \hat{n} dS$$

$$= \iint\limits_{x=-a} (2zx - x^2)\, dS + \iint\limits_{x=a} (x^2 - 2zx)\, dS + \iint\limits_{y=-a} (2xy - y^2)\, dS$$

$$+ \iint\limits_{y=a} (y^2 - 2xy)\, dS + \iint\limits_{z=a} (z^2 - 2yz)\, dS$$

$$= \iint (-2az - a^2)\, dy\, dz + \iint (a^2 - 2az)\, dy\, dz$$

$$+ \iint (-2ax - a^2)\, dz\, dx + \iint (a^2 - 2ax)\, dz\, dx$$

$$+ \iint (a^2 - 2ay)\, dx\, dy$$ [using the equations of the planes to simplify the integrands]

$$= \int_{-a}^{a}\int_{-a}^{a} -4az\, dy\, dz + \int_{-a}^{a}\int_{-a}^{a} -4ax\, dz\, dx$$

$$+ \int_{-a}^{a}\int_{-a}^{a} (a^2 - 2ay)\, dx\, dy$$

$$= 0 + 0 + 4a^4 = 4a^4.$$

Thus Stokes' theorem is verified.

Example 2.12 Verify Stokes' theorem for $\bar{F} = -y\bar{i} + 2yz\bar{j} + y^2\bar{k}$, where S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ and C is the circular boundary on the xoy-plane.

$$\text{Curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & 2yz & y^2 \end{vmatrix}$$

$$= \bar{k}.$$

Stoke's theorem is $\displaystyle\int_C \bar{F} \cdot d\bar{r} = \iint_S \bar{F} \cdot d\bar{S}$

Here $$\oint_C (-y\,dx + 2yz\,dy + y^2\,dz) = \iint_S \bar{k} \cdot d\bar{S} \qquad (1)$$

C is the circle in the xoy-plane whose equation is $x^2 + y^2 = a^2$ and whose parametric equations are $x = a\cos\theta$ and $y = a\sin\theta$.

$\therefore$ $$\text{L.S. of (1)} = \int_{x^2+y^2=a^2} -y\,dx \qquad (\because C \text{ lies on } z = 0)$$

$$= \int_0^{2\pi} a^2 \sin^2\theta\, d\theta$$

$$= \frac{a^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$$= \pi a^2.$$

$$\text{R.S. of (1)} = \iint_S \bar{k} \cdot \hat{n}\, dS, \text{ where}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}, \text{ where } \phi = x^2 + y^2 + z^2$$

$$= \frac{2(x\bar{i} + y\bar{j} + z\bar{k})}{\sqrt{4(x^2 + y^2 + z^2)}}$$

$$= \frac{x\bar{i} + y\bar{j} + z\bar{k}}{a} \qquad [\because \text{the point } (x, y, z) \text{ lies on } \phi = a^2]$$

$\therefore$ $$\text{R.S. of (1)} = \iint_S \frac{z}{a}\, dS$$

$$= \iint_R \frac{z}{a} \frac{dx\,dy}{\hat{n}\cdot\bar{k}}, \text{ where } R \text{ is the projection of } S \text{ on the } xoy\text{-plane.}$$

$$= \iint_R dx\,dy, \text{ where } R \text{ is the region enclosed by } x^2 + y^2 = a^2.$$

$$= \pi a^2$$

Thus Stoke's theorem is verified.

Example 2.13 If S is the surface of the sphere $x^2 + y^2 + z^2 = 1$, evaluate $\iint_S (x\bar{i} + 2y\bar{j} + 3z\bar{k}) \cdot d\bar{S}$.

By Divergence theorem,

$$\iint_S \bar{F} \cdot d\bar{S} = \iiint_V (\text{div } \bar{F})\, dv$$

$$\therefore \quad \iint_S \bar{F} \cdot d\bar{S} = \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)\right] dV$$

$$= 6\iiint_V dV$$

$= 6V$, where V is the volume enclosed by S.

$$= 6 \times \frac{4}{3}\pi$$

$$= 8\pi.$$

Example 2.14 Verify Gauss divergence theorem for $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$, where S is the surface of the cuboid formed by the planes $x = 0$, $x=a$, $y=0$, $y=b$, $z = 0$ and $z = c$.

Divergence theorem is

$$\iint_S \bar{F} \cdot d\bar{S} = \iiint_V (\text{div } \bar{F})\, dV \tag{1}$$

S is made up of six plane surfaces.

$$\therefore \quad \text{L.S. of (1)} = \iint_{\substack{x=0\\ \hat{n}=-\bar{i}}} + \iint_{\substack{x=a\\ \hat{n}=\bar{i}}} + \iint_{\substack{y=0\\ \hat{n}=-\bar{j}}} + \iint_{\substack{y=b\\ \hat{n}=\bar{j}}} + \iint_{\substack{z=0\\ \hat{n}=-\bar{k}}}$$

$$+ \iint_{\substack{z=c\\ \hat{n}=\bar{k}}} (x^2\bar{i} + y^2\,\bar{j} + z^2\bar{k}) \cdot \hat{n}\, dS$$

$$= \iint_{x=0} -x^2 dS + \iint_{x=a} x^2 dS + \iint_{y=0} -y^2 dS + \iint_{y=b} y^2 dS$$

$$-\iint_{z=0} z^2 dS + \iint_{z=c} z^2 dS$$ (on using the relevant values of $\hat{n}$)

$$= a^2 \int_0^c \int_0^b dy\, dz + b^2 \int_0^a \int_0^c dz\, dx + c^2 \int_0^b \int_0^a dx\, dy$$

$$= abc\,(a + b + c)$$

$$\text{R.S. of (1)} = \iiint_V (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= \int_0^c \int_0^b \int_0^a (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= \int_0^c \int_0^b (x^2 + 2y \cdot x + 2z \cdot x)_0^a \mathrm{d}y \, \mathrm{d}\, z$$

$$= \int_0^c (a^2 y + ay^2 + 2\, a\, z \cdot y)_0^b \, \mathrm{d}\, z$$

$$= \int_0^c (a^2 b + ab^2 + 2ab\, z)\, \mathrm{d}z$$

$$= a^2 bc + + ab^2 c + abc^2$$

$$= abc\,(a + b + c)$$

Thus divergence theorem is verified.

Example 2.15 Verify divergence theorem for $\overline{F} = x^2 \overline{i} + z\, \overline{j} + yz\, \overline{k}$ over the cube formed by

$$x = \pm 1, \quad y = \pm 1, \quad z = \pm 1.$$

Divergence theorem is $\iint_S \overline{F} \cdot \mathrm{d}\overline{S} = \iiint_V (\mathrm{div}\, \overline{F})\, \mathrm{d}V$ (1)

$$\text{L.S. of (1)} = \iint_{\substack{x=-1 \\ \hat{n}=-\overline{i}}} + \iint_{\substack{x=1 \\ \hat{n}=\overline{i}}} + \iint_{\substack{y=-1 \\ \hat{n}=-\overline{j}}} + \iint_{\substack{y=1 \\ \hat{n}=\overline{j}}} + \iint_{\substack{z=-1 \\ \hat{n}=-\overline{k}}}$$

$$+ \iint_{\substack{z=1 \\ \hat{n}=\overline{k}}} (x^2\, \overline{i} + z\, \overline{j} + yz\, \overline{k}) \cdot \hat{n}\, \mathrm{d}S$$

$$= \iint_{x=-1} -x^2 \mathrm{d}S + \iint_{x=1} x^2 \mathrm{d}S + \iint_{y=-1} -z\, \mathrm{d}S + \iint_{y=1} z\, \mathrm{d}S$$

$$\iint_{z=-1} -yz\, \mathrm{d}S + \iint_{z=1} yz\, \mathrm{d}S \qquad \text{(using the relevant values of } \hat{n})$$

$$= 0$$

$$\text{R.S. of (1)} = \iiint_v (2x + y)\, \mathrm{d}\, x\, \mathrm{d}y\, \mathrm{d}z$$

$$= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x + y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

$$= \int_{-1}^1 \int_{-1}^1 (2y\, \mathrm{d}y\, \mathrm{d}z)$$

$$= 0$$

Thus divergence theorem is verified.

EXERCISE 2(d)

Part A

(Short Answer Questions)

1. State Green's theorem in a plane (or) the connection between a line integral and a double integral.
2. **State Stoke's theorem (or) the connection between a line integral and a surface integral.**
3. State Gauss divergence theorem (or) the connection between a surface integral and a volume integral.
4. Deduce Green's theorem in a plane from Stoke's theorem.
5. State the scalar form of Stoke's theorem.
6. Give the scalar form of divergence theorem.
7. Derive Green's identities from divergence theorem.
8. Use Stoke's theorem to prove that $\nabla \times \nabla\, \phi = 0$.
9. Use the integral theorems to prove $\nabla \cdot (\nabla \times \overline{F}) = 0$.
10. Evaluate $\oint_C (yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z)$ where C is the circle given by $x^2 + y^2 + z^2 = 1$ and $z = 0$.
11. Evaluate $\iint_S (x\,\mathrm{d}y\,\mathrm{d}z + y\,\mathrm{d}z\,\mathrm{d}x + z\,\mathrm{d}x\,\mathrm{d}y)$ over the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
12. If C is a simple closed curve and $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$, prove that $\int_C \overline{r} \cdot \mathrm{d}\overline{r} = 0$.
13. If C is a simple closed curve and ϕ is a scalar point function, prove that $\int_C \phi \nabla \phi \cdot \mathrm{d}\overline{r} = 0 \cdot$
14. If S is a closed surface and $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$, find the value $\iint_S \nabla\left(\frac{1}{r}\right) \cdot \mathrm{d}\overline{S}$.
15. If S is a closed surface enclosing a volume V, evaluate $\iint_S \nabla(r^2) \cdot \mathrm{d}\overline{S}$.
16. Evaluate $\int_C [(x - 2y)\mathrm{d}x + (3x - y)\mathrm{d}y]$, where C is the boundary of a unit square.
17. Evaluate $\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x)$, where C is the circle $x^2 + y^2 = a^2$.
18. If $\overline{A} = \text{curl}\, \overline{F}$, prove that $\iint_S \overline{A} \cdot \mathrm{d}\overline{S} = 0$, where S is any closed surface.
19. If S is any closed surface enclosing a volume V and if $\overline{A} = a\,x\overline{i} + by\,\overline{j} + c\,z\overline{k}$, prove that $\iint_S \overline{A} \cdot \mathrm{d}\overline{S} = (a + b + c)V$.
20. If $\overline{r} = x\,\overline{i} + y\overline{j} + z\overline{k}$ and S the surface of a sphere of unit radius, find $\iint_S \overline{r} \cdot \mathrm{d}\overline{S}$.

Part B

21. Verify Green's theorem in a plane with respect to $\int_C (x^2\, dx - xy\, dy)$, where C is the boundary of the square formed by $x = 0, y = 0, x = a, y = a$.
22. Verify Stoke's theorem for $\overline{F} = x^2\overline{i} + xy\overline{j}$ in the square region in the xy-plane bounded by the lines $x = 0, y = 0, x = 2, y = 2$.
23. Use Green's theorem in a plane to evaluate $\int_C \left[x^2(1+y)\, dx + (x^3 + y^3)\, dy\right]$ where C is the square formed by $x = \pm 1$ and $y = \pm 1$.
24. Use Stoke's theorem to find the value of $\int_C \overline{F} \cdot d\overline{r}$, when $\overline{F} = (xy - x^2)\,\overline{i} + x^2 y\,\overline{j}$ and C is the boundary of the triangle in the xoy-plane formed by $x = 1$, $y = 0$, and $y = x$.
25. Verify Green's theorem in a plane with respect to $\oint_C \left[(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy\right]$, where C is the boundary of the region in the xoy-plane enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = 1$.
26. Verify Stoke's theorem for $\overline{F} = (xy + y^2)\overline{i} + x^2\,\overline{j}$ in the region in the xoy-plane bound by $y = x$ and $y = x^2$.
27. Use Green's theorem in a plane to find the finite area enclosed by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.
28. Use Green's theorem in a plane to find the area of the region in the xoy-plane bounded by $y^3 = x^2$ and $y = x$.
29. Verify Stoke's theorem for $\overline{F} = (y - z + 2)\overline{i} + (yz + 4)\overline{j} - xz\overline{k}$, where S is the open surface of the cube formed by $x = 0, x = 2, y = 0, y = 2$ and $z = 2$.
30. Verify Stoke's theorem for $\overline{F} = (x^2 - y^2)\overline{i} + 2xy\overline{j} + xyz\overline{k}$ over the surface of the box bounded by the planes $x = 0, x = a, y = 0, y = b$ and $z = c$.
31. Verify Stoke's theorem for $\overline{F} = (2x - y)\,\overline{i} - yz^2\overline{j} - y^2 z\overline{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is the circular boundary in the xoy-plane.
32. Verify Gauss divergence theorem for $\overline{F} = (x^2 - yz)\overline{i} + (y^2 - zx)\overline{j} + (z^2 - xy)\overline{k}$ and the closed surface of the rectangular parallelopiped formed by $x = 0, x = 1, y = 0, y = 2, z = 0$ and $z = 3$.
33. Verify divergence theorem for (i) $\overline{F} = 4xz\overline{i} - y^2\overline{j} + yz\overline{k}$ and (ii) $\overline{F} = (2x - z)\overline{i} + x^2 y\overline{j} - xz^2\overline{k}$, when S is the closed surface of the cube formed by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
34. Use divergence theorem to evaluate $\iint_S (yz^2\overline{i} + zx^2\overline{j} + 2z^2\overline{k}) \cdot d\overline{S}$, where S is the closed surface bounded by the xoy-plane and the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ above this plane.

35. Use divergence theorem to evaluate $\iint_S (4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}) \cdot d\bar{S}$, where S is the closed surface bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$.

ANSWERS

Exercise 2(a)

(5) $-12\bar{i} - 9\bar{j} - 16\bar{k}$ (6) $\sqrt{14}$ (7) 5

(8) $\frac{1}{\sqrt{19}}(\bar{i} + 3\bar{j} - 3\bar{k})$ (9) $\frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})$ (10) $\frac{1}{3}(2\bar{i} + 2\bar{j} - \bar{k})$

(11) 25; $-56\bar{i} + 30\bar{j} - 47\bar{k}$ (12) $\frac{14}{3}$ (13) $-\frac{11}{3}$

(14) $\pm\frac{1}{3}(-\bar{i} + 2\bar{j} + 2\bar{k})$ (15) $-\frac{27}{\sqrt{11}}$ (16) $\cos^{-1}\left\{\frac{-45}{\sqrt{2299}}\right\}$

(17) $\cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$ (18) $\cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$ (19) $\cos^{-1}\left(-\frac{3}{7\sqrt{6}}\right)$

(20) $\cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)$ (21) $2x - y - 3z + 1 = 0$ (22) $\lambda = \frac{5}{2}$; $\mu = 1$

(23) $a = -\frac{7}{3}$; $b = \frac{64}{9}$ (24) $x^2y - xz^2 + y^2z + c$ (25) $\phi = x^2yz^3 + 20$

Exercise 2(b)

(7) 3; 0. (8) 4 (9) $2(\bar{i} + \bar{j} + \bar{k})$

(14) λ can take any value (15) $\lambda = -1$ (16) $a = 4$

(17) $a = -1$, $b = 1$, $c = -1$ (21) 0; $3\bar{k}$ (22) $\frac{124}{\sqrt{21}}$

(23) 80; $80\bar{i} + 37\bar{j} + 36\bar{k}$; $27\bar{i} - 54\bar{j} + 20\bar{k}$; 0; $74\bar{i} + 27\bar{j}$

(25) $x^3y^2z^4$ (26) $x^2 + y^2 + 3xy + yz + z^2x$

(27) $a = 4$, $b = 2$, $c = -1$; $\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4zx$

(28) x^2yz^3 (30) -3.

Exercise 2(c)

(6) 3 (7) 6 (9) $\frac{a^2b^2}{2}$

(10) 0	(11) $\frac{8}{11}\bar{i} + \frac{4}{5}\bar{j} + \bar{k}$	(12) $-\pi\bar{i} + \frac{1}{2}\bar{j} + \frac{3\pi}{2}\bar{k}$
(13) $\left(2 - \frac{\pi}{4}\right)\bar{i} + \left(\pi - \frac{1}{2}\right)\bar{j}$		(14) $\frac{51}{70}$
(15) $-\frac{7}{6}$	(16) $2a^2$	(17) 8
(18) 8π	(19) 2	(20) $-\frac{19}{2}$
(21) $4\pi + 5$	(22) 0	(23) $\frac{27}{4}$
(24) πa^2	(25) 90	

Exercise 2(d)

(10) 0	(11) $4\pi a^3$	(15) $6V$
(16) 5	(17) $2\pi a^2$	(20) 4π
(23) $\frac{8}{3}$	(24) $-\frac{1}{12}$	(27) $\frac{16}{3}a^2$
(28) $\frac{1}{10}$	(34) πa^4	(35) 84π