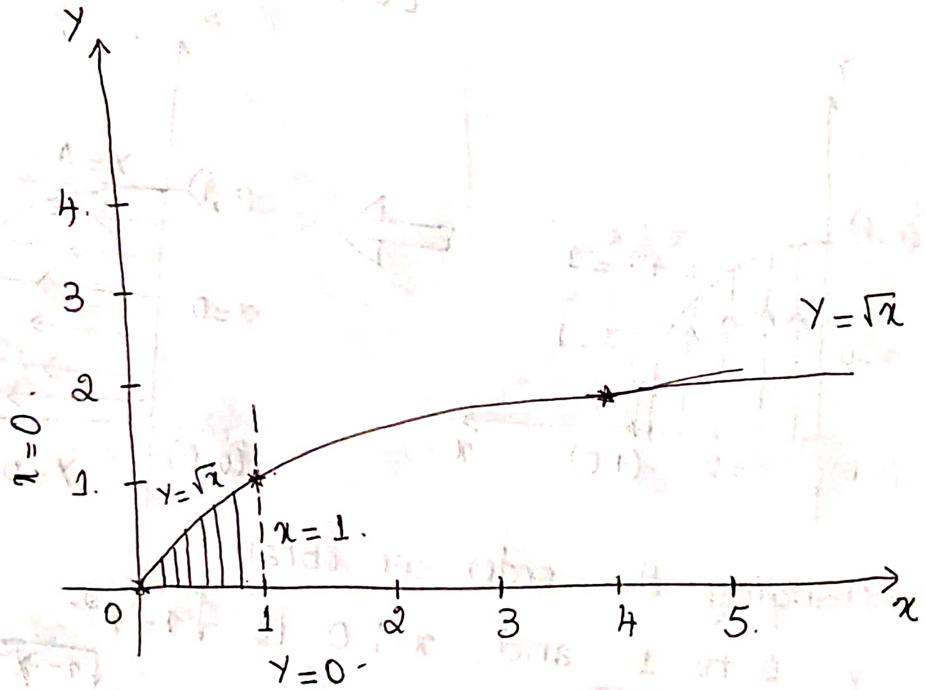
CHANGING THE ORDER OF INTEGRATION

1) Evaluate $\int_0^1 \int_0^{\sqrt{x}} (x^2+y^2)\,dy\,dx$ by changing the order of integration.

The region of integration is the area bounded by $x=0$, $x=1$, $y=0$ and $y=\sqrt{x}$.

x	0	1	4	9
y	0	1	2	3



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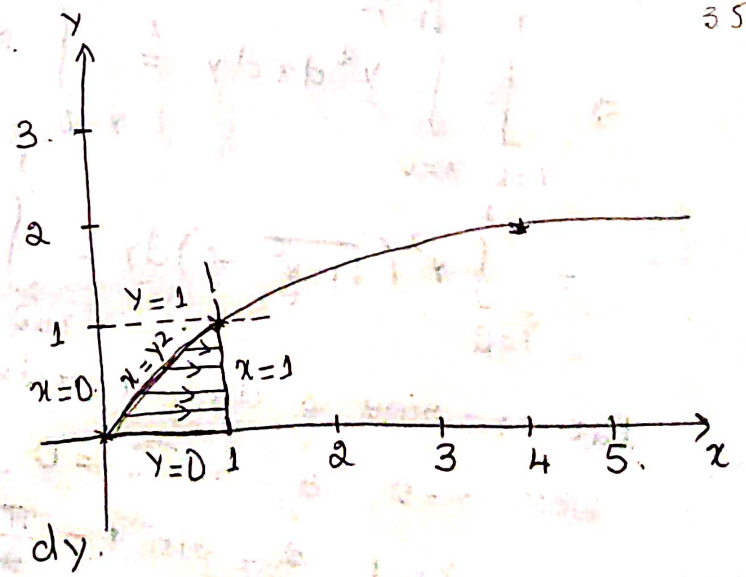
After changing order.



$$= \int_{y=0}^{1} \int_{x=y^2}^{1} (x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{1} \left. \frac{x^3}{3} + y^2 x \right|_{y^2}^{1} dy$$

$$= \int_0^1 \frac{(1-y^6)}{3} + y^2(1-y^2)\,dy$$

$$= \left. \frac{y}{3} - \frac{y^7}{21} + \frac{y^3}{3} - \frac{y^5}{5} \right|_0^1 = \frac{(1-0)}{3} - \frac{(1-0)}{21} + \frac{(1-0)}{3} - \frac{(1-0)}{5}$$

$$= \frac{1}{3} - \frac{1}{21} + \frac{1}{3} - \frac{1}{5} = \frac{2}{3} - \frac{26}{105} = \frac{70-26}{105} = \frac{44}{105}$$

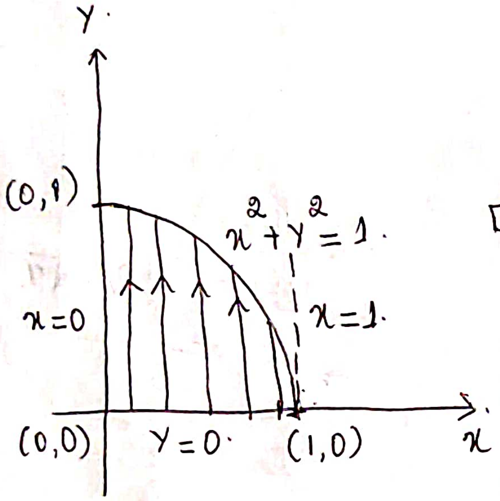
2) Evaluate the double integral by changing the order of integration. $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dy\, dx$.

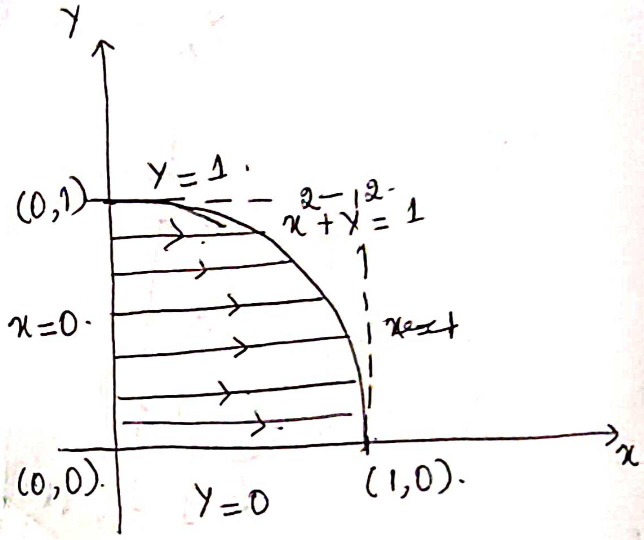
The region of integration is bounded by

$x=0, \quad x=1, \quad y=0, \quad y=\sqrt{1-x^2}$.

$\Rightarrow y^2 = 1-x^2 \Rightarrow x^2+y^2=1$.

[Circle with centre at (0,0) and radius = 1]





Changing the order we obtain

y : 0 to 1 and x : 0 to $\sqrt{1-y^2}$

$$\Rightarrow \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y^2\, dx\, dy = \int_{y=0}^{1} y^2 x \Big|_0^{\sqrt{1-x^2}} dy$$

$$= \int_{y=0}^{1} y^2\left(\sqrt{1-y^2}-0\right)dy = \int_{y=0}^{1} y^2\sqrt{1-y^2}\, dy$$

Reduction formula.

Put $y = \sin\theta \Rightarrow dy = \cos\theta\, d\theta$.

When $y=0$, $\theta = \sin^{-1}0 = 0$

$y=1$, $\theta = \sin^{-1}1 = \frac{\pi}{2}$.

$$= \int_{\theta=0}^{\pi/2} \sin^2\theta\sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta = \int_{\theta=0}^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$$

$$= \frac{(2-1)(2-1)}{4(4-2)} \times \frac{\pi}{2} = \frac{\pi}{16}.$$

3). Evaluate by changing the order of integration

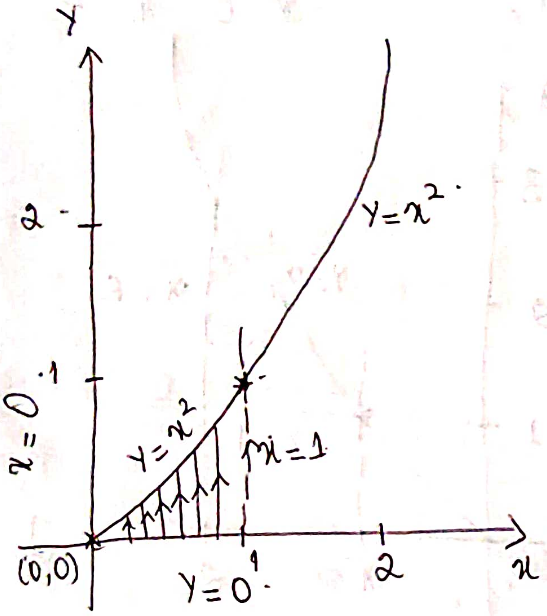
$$\int_0^1 \int_0^{x^2} (x^2+y^2)\, dy\, dx.$$

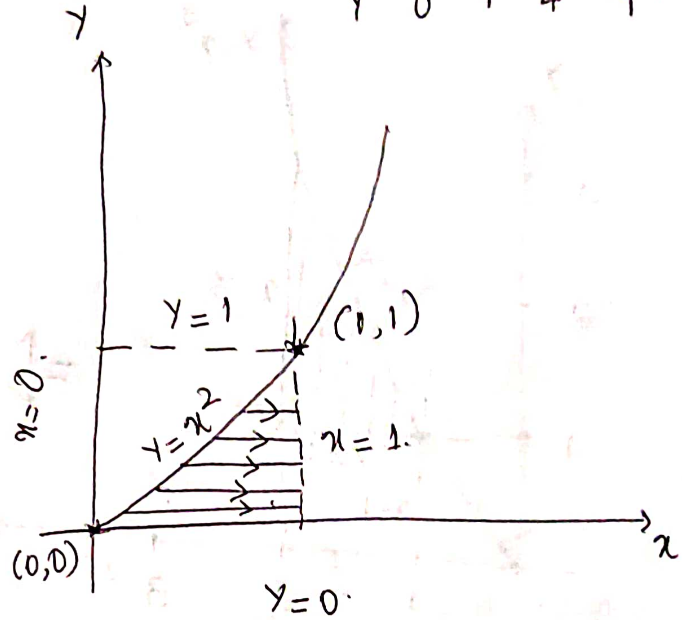
The region of integration is bounded by

$x=0$, $x=1$, $y=0$, $y=x^2$.

$y = x^2$

x	0	1	2	3
y	0	1	4	9





$$= \int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} (x^2+y^2)\, dx\, dy.$$

$$= \int_{y=0}^{1} \left. \frac{x^3}{3} + y^2 x \right|_{\sqrt{y}}^{1} dy = \int_0^1 \left. \frac{x^3}{3} + y^2 x \right|_{\sqrt{y}}^{1} dy.$$

$$= \int_0^1 \left\{ \frac{(1-y^{3/2})}{3} + y^2(1-\sqrt{y}) \right\} dy$$

$$= \int_0^1 \left(\frac{1}{3} - \frac{y^{3/2}}{3} + y^2 - y^{5/2} \right) dy$$

$$= \frac{1}{3} y - \frac{1}{3} \frac{y^{5/2}}{5/2} + \frac{y^3}{3} - \left. \frac{y^{7/2}}{7/2} \right|_0^1$$

$$= \frac{1}{3} - \frac{2}{15} + \frac{1}{3} - \frac{2}{7} = \frac{2}{3} - 2 \times \frac{22}{105} = \frac{26}{105}.$$

4). Evaluate by changing the order of integration

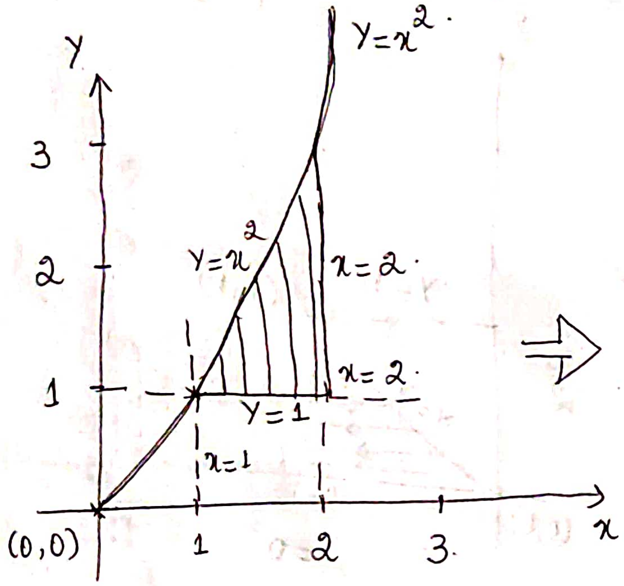
$$\int_1^2 \int_1^{x^2} (x^2+y^2)\, dy\, dx .$$

The region of integration is bounded by

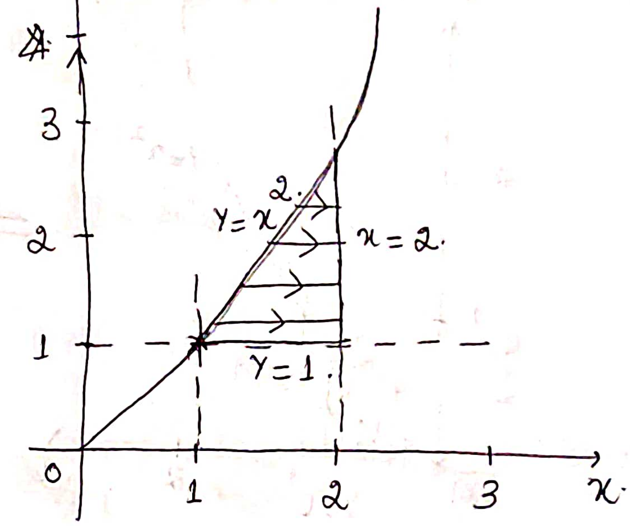
$x=1,\ x=2,\ y=1,\ y=x^2.$

$y = x^2$

x	0	1	2	3
y	0	1	4	9



$\Rightarrow$



The region of integration is

$y=1$ to $y=4$, $x=\sqrt{y}$ to $x=2$.

$$= \int_{y=1}^{4} \int_{\sqrt{y}}^{2} (x^2+y^2)\, dx\, dy = \int_{y=1}^{4} \left. \frac{x^3}{3} + y^2 x \right|_{\sqrt{y}}^{2} dy$$

$$= \int_1^4 \left\{ \frac{(2^3 - (\sqrt{y})^3)}{3} + y^2(2-\sqrt{y}) \right\} dy = \int_1^4 \frac{8 - y^{3/2}}{3} + 2y^2 - y^{5/2}\, dy$$

$$= \left. \frac{8}{3} y - \frac{y^{5/2}}{3 \times 5/2} + \frac{2y^3}{3} - \frac{y^{7/2}}{7/2} \right|_1^4$$

$$= \frac{8}{3}(4-1) - \frac{2}{15}\left((4)^{5/2} - 1\right) + \frac{2}{3}(4^3 - 1^3) - \frac{2}{7}(4^{7/2} - 1)$$

$$= 8 - \frac{2}{15}(32-1) + \frac{2}{3}(64-1) - \frac{2}{7}(2^7 - 1).$$

$$= \frac{1006}{105}$$

5) $\int_0^1 \int_y^{\sqrt{y}} xy\,dx\,dy$ by changing the order of integration

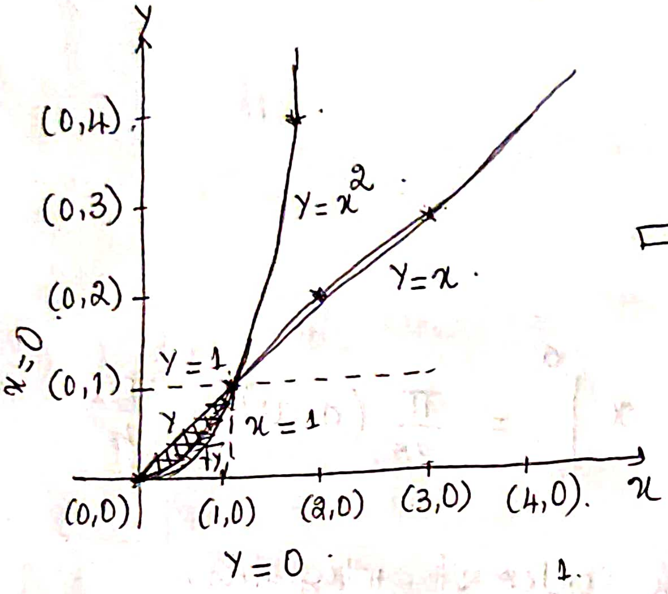
The region of integration is

$y = 0,\ y = 1,\ x = y,\ x = \sqrt{y}$

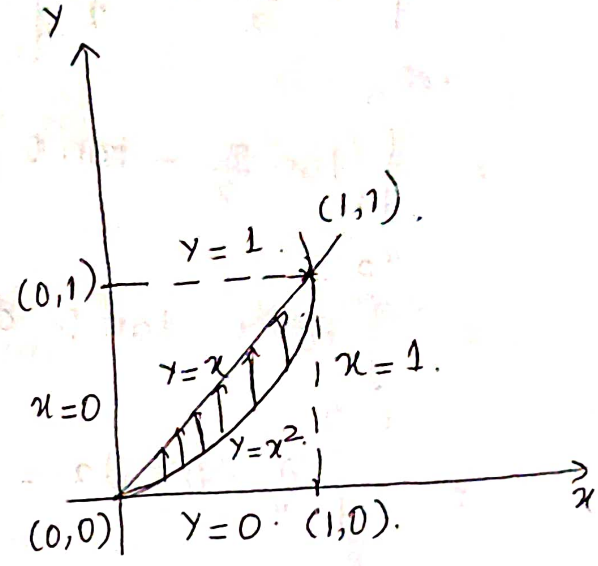
$\Rightarrow y = 0,\ y = 1,\ x = y,\ y = x^2$

$y = x^2$

x	0	1	2	3
y	0	1	4	9



$\Rightarrow$



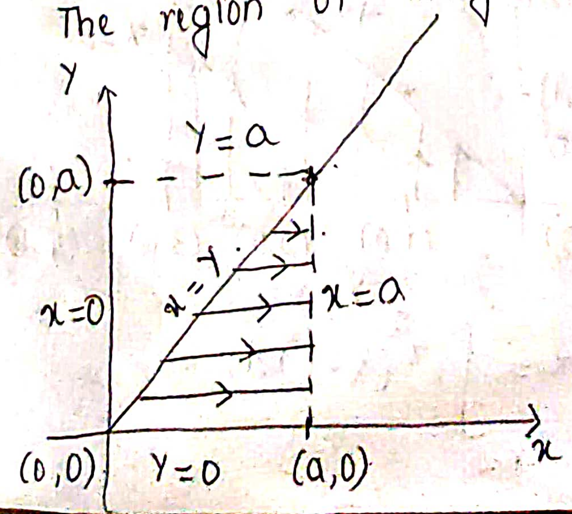
$$= \int_{x=0}^{1} \int_{y=x^2}^{x} xy\,dy\,dx = \int_{x=0}^{1} x \left.\frac{y^2}{2}\right|_{x^2}^{x} dx = \frac{1}{2}\int_{x=0}^{1} x\cdot(x^2 - x^4)\,dx$$

$$= \frac{1}{2}\int_0^1 (x^3 - x^5)\,dx = \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^6}{6}\right|_0^1$$

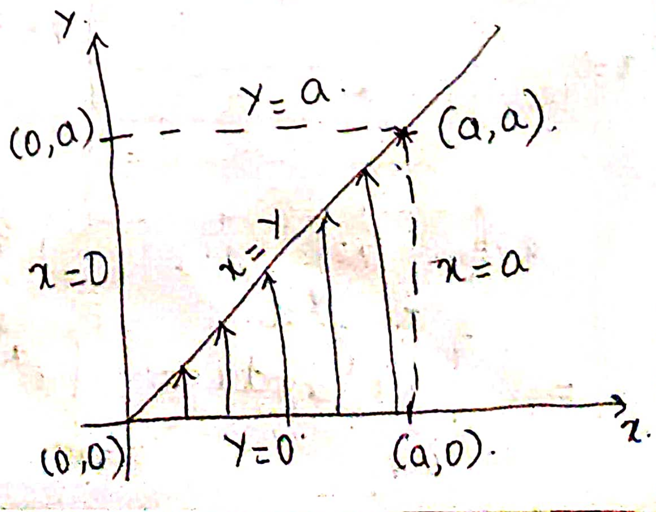
$$= \frac{1}{2}\left[\frac{(1-0)}{4} - \frac{(1-0)}{6}\right] = \frac{1}{2}\left[\frac{1}{4} - \frac{1}{6}\right] = \frac{2}{2\times 24} = \frac{1}{24}$$

6) $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$

The region of integration is $y = 0,\ y = a,\ x = y$ and $x = a$



$\Rightarrow$



After changing the order

$$= \int_{x=0}^{a}\int_{y=0}^{x} \frac{x}{x^2+y^2}\,dy\,dx.$$

$$\int \frac{dy}{a^2+y^2} = \frac{1}{a}\tan^{-1}\frac{y}{a}$$

$$= \int_{x=0}^{a} x\cdot\frac{1}{x}\tan^{-1}\frac{y}{x}\Big|_0^x\,dx.$$

$$= \int_0^a\left(\tan^{-1}\frac{x}{x} - \tan^{-1}0\right)dx.$$

$$= \int_0^a \tan^{-1}1 - \tan^{-1}0\,dx.$$

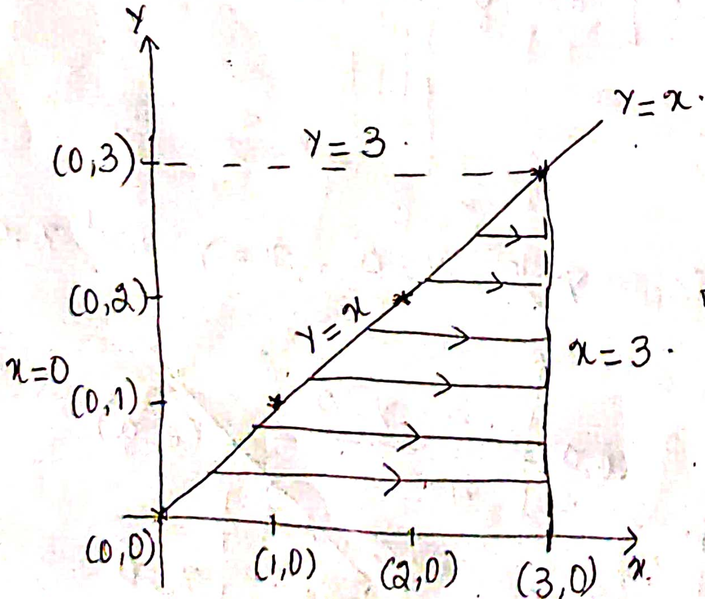
$$= \int_0^a \frac{\pi}{4} - 0\,dx = \frac{\pi}{4}x\Big|_0^a = \frac{\pi}{4}(a-0) = \frac{\pi a}{4}.$$

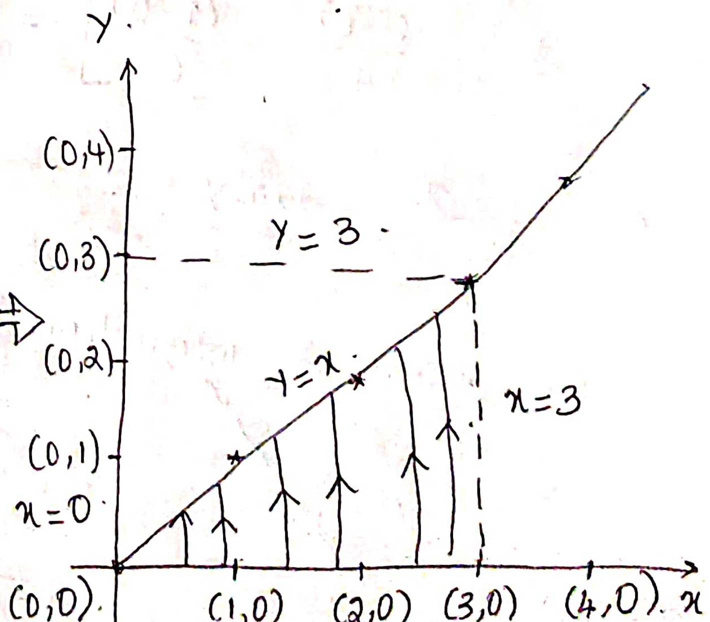
7) Evaluate by changing the order of integration.

$$\int_0^{3}\int_y^3 e^{x^2}\,dx\,dy.$$

The region of integration is.

$Y=0$, $Y=1$, $x=\cancel{y}$ and $x=3$.





$Y=0$ $\quad$ $Y=0$

$$= \int_{x=0}^{3}\int_{y=0}^{x} e^{x^2}\,dy\,dx = \int_{x=0}^{3} e^{x^2}\,y\Big|_0^x\,dx = \int_{x=0}^{3} e^{x^2}(x-0)\,dx.$$

$$= \int_0^3 e^{x^2} x \, dx$$

→ We observe that ~~Bernoulli~~ $f(x)$, $f'(x)$ are present in the integral.

Hence, put $x^2 = t \Rightarrow 2x\,dx = dt$

$\Rightarrow x\,dx = \frac{dt}{2}$.

$$= \int_0^9 e^t \frac{dt}{2}$$

When $x = 0$, $t = 0$

$x = 3$, $t = 9$.

$$= \frac{1}{2} e^t \Big|_0^9$$

$$= \frac{1}{2}\left(e^9 - e^0\right) = \frac{1}{2}\left(e^9 - 1\right)$$

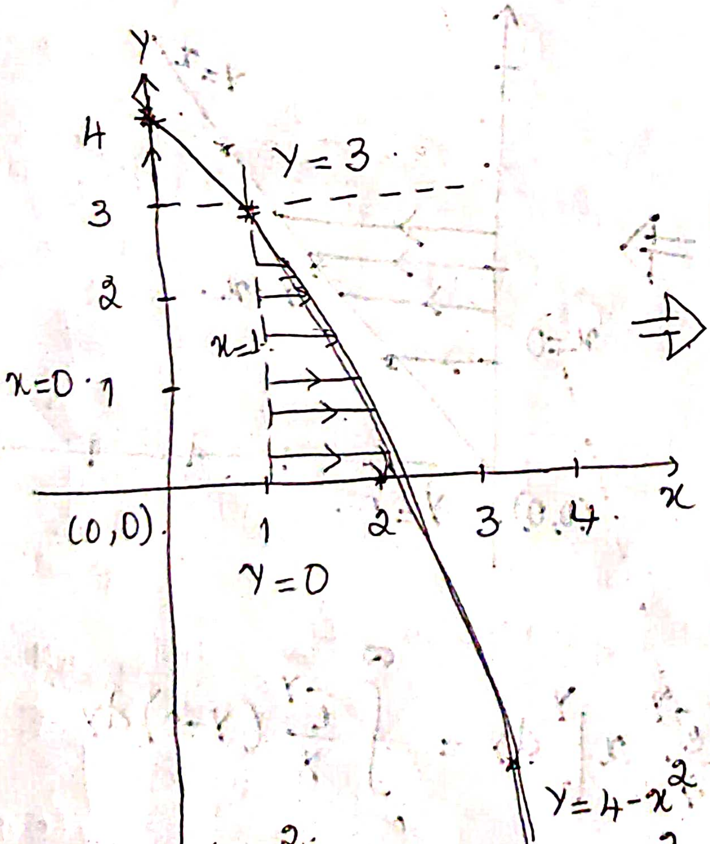
8). Evaluate by changing the order of integration.

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy.$$

The region of integration is:

$y = 0$, $y = 3$, $x = 1$, $x = \sqrt{4-y} \Rightarrow x^2 = 4 - y \Rightarrow y = 4 - x^2$

x	0	1	2	3
y	4	3	0	-5



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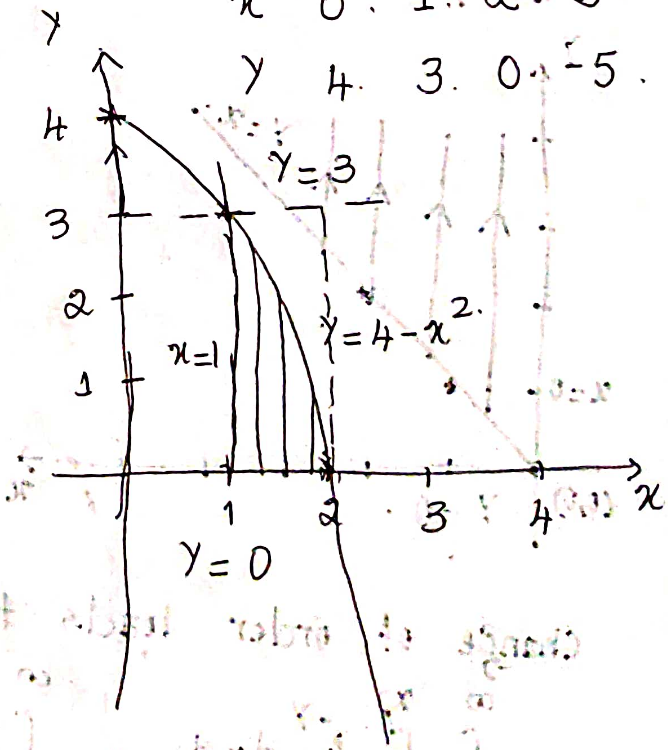
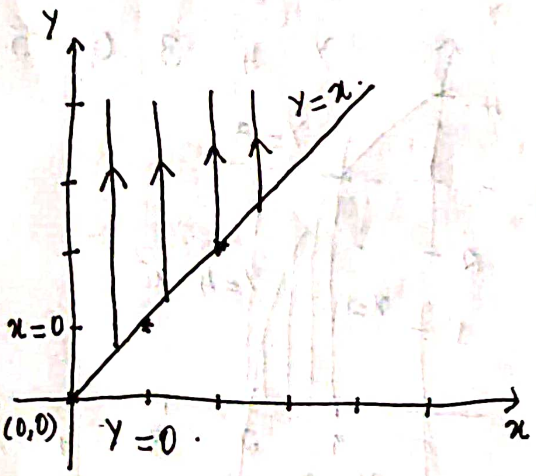


$$= \int_0^2 \int_0^{4-x^2} (x+y)\, dy\, dx = \int_0^2 \left. xy + \frac{y^2}{2} \right|_0^{4-x^2} dx$$

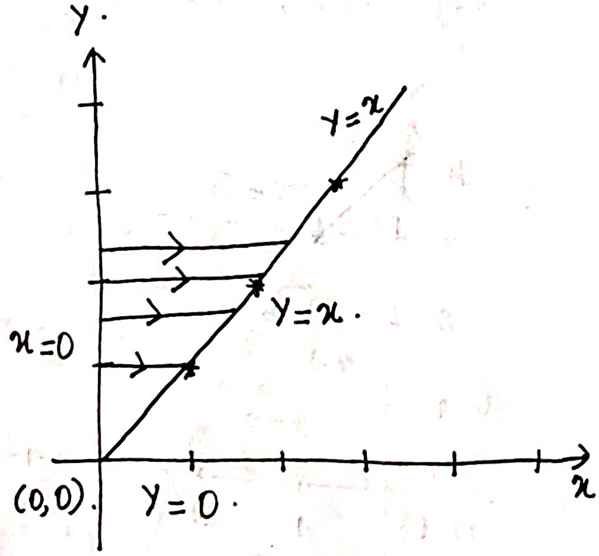
9) $\displaystyle\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx.$

The region of integration is

$x = 0,\ x = \infty,\ y = x,\ y = \infty$



$\Rightarrow$



Change of order leads to

$$= \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y}\, x\Big|_0^{y} dy = \int_0^{\infty} \frac{e^{-y}}{y}(y-0)\, dy$$

$$= \int_0^{\infty} e^{-y}\, dy = \frac{e^{-y}}{-1}\Big|_0^{\infty} = -1\left(e^{-\infty} - e^{-0}\right) = -1(0-1) = 1.$$