DOUBLE INTEGRALS

TYPE 1

1. Evaluate $\int_1^2\int_2^3 x^2y\,dx\,dy$.

$$= \int_{y=1}^{2}\int_{x=2}^{3} x^2y\,dx\,dy$$

$$= \int_{y=1}^{2} y\left\{\int_{x=2}^{3} x^2dx\right\}dy$$

$$= \int_{y=1}^{2} y\left\{\frac{x^3}{3}\Big|_2^3\right\}dy$$

$$= \int_{y=1}^{2} \frac{y}{3}(3^3-2^3)\,dy$$

$$= \int_{y=1}^{2} \frac{y}{3}(27-8)\,dy$$

$$= \frac{19}{3}\int_{y=1}^{2} y\,dy$$

$$= \frac{19}{3}\,\frac{y^2}{2}\Big|_1^2$$

$$= \frac{19}{6}\left[2^2-1^2\right]$$

$$= \frac{19}{6}(4-1) = \frac{19\times 3}{6}$$

$$= \frac{19}{2}.$$

2. Evaluate $\int_1^a\int_1^b \frac{dx\,dy}{xy}$

$$= \int_{y=1}^{a}\int_{x=1}^{b} \frac{dx\,dy}{xy}$$

$$= \int_{y=1}^{a} \frac{1}{y}\,\log x\Big|_1^b\,dy$$

$$= \int_{y=1}^{a} \frac{1}{y}\left\{\log b - \log 1\right\}dy$$

$$= \int_{y=1}^{a} \frac{\log b - 0}{y}\,dy$$

$$= \log b\int_{y=1}^{a}\frac{dy}{y} = \log b.\ \log y\Big|_1^a$$

$$= \log b(\log a - \log 1)$$

$$= \log b(\log a - 0) = \log a\,\log b.$$

3. $\iint_D e^{x+y}\,dx\,dy$ where

$$D: \begin{cases} -1<x<1 \\ -1<y<1. \end{cases}$$

$$= \int_{y=-1}^{1}\int_{x=-1}^{1} e^x.e^y\,dx\,dy.$$

$$= \int_{y=-1}^{1} e^y.\ e^x\Big|_{x=-1}^{1}\,dy.$$

$$= \int_{y=-1}^{1} e^y(e^1-e^{-1})\,dy.$$

$$= \int_{y=-1}^{1} e^y(e-(e^{-1})\,dy.$$

$$= (e-\tfrac{1}{e})\ e^y\Big|_{-1}^{1} = (e-\tfrac{1}{e})(e^1-e^{-1})$$

$$= (e-\tfrac{1}{e})^2$$

4. $\int_0^1\int_0^1 e^{x+y}\,dx\,dy$

$$= \int_{y=0}^{1}\int_{x=0}^{1} e^x.e^y\,dx\,dy.$$

$$= \int_{y=0}^{1} e^y.\,e^x\Big|_0^1\,dy = \int_0^1 e^y(e^1-e^0)\,dy.$$

$$= (e-1)\int_0^1 e^y\,dy$$

$$= (e-1)\; e^y\Big|_0^1$$

$$= (e-1)(e^1-e^0)$$

$$= (e-1)^2.$$

5) $\displaystyle\int_0^1\int_0^{\pi/2} r^3\sin^2\theta\, d\theta\, dr$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r^3\sin^2\theta\, d\theta\, dr$$

$$= \int_0^1 r^3\,dr \cdot \int_0^{\pi/2}\sin^2\theta\, d\theta$$

$$\int_0^{\pi/2}\sin^2\theta\, d\theta = \frac{\pi}{4}$$

$$= \frac{r^4}{4}\Big|_0^1 \times \frac{\pi}{4} = \frac{1}{4}\times\frac{\pi}{4}$$

$$= \frac{\pi}{16}$$

6) $\displaystyle\int_0^{\pi/2}\int_0^a r^2\sin\theta\, dr\, d\theta$

$$= \int_{\theta=0}^{\pi/2}\sin\theta\, d\theta \int_{r=0}^{a} r^2\,dr$$

$$= -\cos\theta\Big|_0^{\pi/2}\;\frac{r^3}{3}\Big|_0^a$$

$$= -\left(\cos\frac{\pi}{2}-\cos 0\right)\left(\frac{a^3-0^3}{3}\right)$$

$$= -(0-1)(a^3/3)$$

$$= \frac{a^3}{3}$$

7) $\displaystyle\int_{x=0}^{1}\int_{y=2}^{3} xye^x\,dx\,dy$

$$= \int_{x=0}^{1} xe^x\,dx \int_{y=2}^{3} y\,dy$$

$$= \frac{y^2}{2}\Big|_2^3 \left[xe^x - e^x\right]_0^1$$

$$= \frac{(9-4)}{2}\left[(e-e)-(0-e^0)\right].$$

$$= \frac{5}{2}$$

8) $\displaystyle\int_0^1\int_0^1 xy^2e^{x^2}\,dx\,dy$

$$= \int_{y=0}^{1} y^2\,dy\int_{x=0}^{1} xe^{x^2}\,dx \longrightarrow (1)$$

$$\int_0^1 y^2\,dy = \frac{y^3}{3}\Big|_0^1 = \frac{1-0}{3} = \frac{1}{3} \hookrightarrow (2).$$

$$\int_0^1 xe^{x^2}\,dx.$$

We observe that if

$x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

$$\int xe^{x^2}dx = \int e^t\frac{dt}{2} = \frac{1}{2}e^t = \frac{e^{x^2}}{2}$$

$$\therefore \int_0^1 xe^{x^2}dx = \frac{e^{x^2}}{2}\Big|_0^1$$

$$= \frac{e^1-e^0}{2} = \frac{e-1}{2} \hookrightarrow (3).$$

Substituting (2) and (3) in (1),

$$\int_0^1\int_0^1 xy^2e^{x^2}dx\,dy = \frac{e-1}{2}\times\frac{1}{3}$$

$$= \frac{e-1}{6}$$

9). $\iint_D w\sin^{-1}z \, dz\, dw$

where $D : \begin{cases} 0 < z < 1. \\ 2 < w < 3. \end{cases}$

$$= \int_{z=0}^{1} \int_{w=2}^{3} w\sin^{-1}z \, dw\, dz.$$

$$= \int_{z=0}^{1} \sin^{-1}z \, dz \int_{w=2}^{3} w\, dw \longrightarrow (1).$$

consider $\int_{w=2}^{3} w\, dw = \left.\frac{w^2}{2}\right|_2^3 = \frac{3^2 - 2^2}{2} = \frac{9-4}{2} = \frac{5}{2} \longrightarrow (2).$

consider $\int_{z=0}^{1} \sin^{-1}z \, dz$. (use integration by parts) . ILATE.

$u = \sin^{-1}z$, $v = 1$.

$$\int_{z=0}^{1} \sin^{-1}z\, dz = \left[\sin^{-1}z \int 1\, dz - \int\int 1\, dz . \frac{1}{\sqrt{1-z^2}}\right|_0^1$$

$$= \left[z\sin^{-1}z - \int \frac{z\, dz}{\sqrt{1-z^2}} \right|_0^1$$

Put $1-z^2 = t$.

$\Rightarrow -2z\,dz = dt$

$\Rightarrow z\,dz = -\frac{dt}{2}$.

$\int \frac{dt}{2\sqrt{t}} = \sqrt{t}$.

$$= \left[z\sin^{-1}z - \int \frac{-dt}{2\sqrt{t}} \right|_0^1$$

$$= \left[z\sin^{-1}z + \sqrt{t} \right|_0^1$$

$$= \left[z\sin^{-1}z + \sqrt{1-z^2} \right|_0^1$$

$$= 1\times\sin^{-1}1 + \sqrt{1-1} - \{0 + \sqrt{1-0}\}.$$

$$= \frac{\pi}{2} + 0 - 0 - 1 = \frac{\pi}{2} - 1 \longrightarrow (3).$$

substituting (2) and (3) in (1),

$$\iint_D w\sin^{-1}z\, dz\, dw = \frac{5}{2}\left(\frac{\pi}{2} - 1\right)$$