Analytic Functions

Complex Variable

$x+iy$ is a complex variable and is denoted by z.

$z = x+iy$ where $i = \sqrt{-1}$.

Function of a complex variable

If $z = x+iy$ and $w = u+iv$ are two complex variables, and if for each value of z in a given region R of the complex plane there corresponds one or more values of w, then w is said to be a function of z and is denoted by $w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where $u(x,y)$ and $v(x,y)$ are real functions of the real variables x and y

Single-valued function.

If for each value of z in R there is only value of w, then w is called a single valued function of z.

Examples: $w = z^2$, $w = \frac{1}{z}$

Multiple-valued function

If there is more than one value of w corresponding to a given value of z, then w is called a multiple-valued function.

Example: $w = z^{1/2}$

Neighbourhood of a point z_0 :

is a sufficiently small circular region, excluding the points on the boundary, with centre at z_0.

Analytic Functions : (Holomorphic or regular).

A function $f(z)$ which is single-valued and possesses a unique derivative w.r.t. z at all points of a region R, is called an analytic function of z.

Entire Function : (Integral function).

A function which is analytic everywhere in the finite plane is called an entire function.

Example: $e^z, \sin z, \cos z$.

The necessary condition for $f(z)$ to be analytic :

Consider a complex function $f(z) = u(x,y) + iv(x,y)$.

$f(z)$ is said to be analytic in a region R if.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

i.e. $u_x = v_y$ and $v_x = -u_y$

The above equations are called Cauchy-Riemann equations.

Sufficient conditions for $f(z)$ to be analytic :

If the partial derivatives u_x, u_y, v_x, v_y are all continuous and $u_x = v_y$, $u_y = -v_x$ then $f(z)$ is analytic

2) Check the analyticity of $\log z$. or show that $f(z) = \log z$ is analytic everywhere except at the origin. and find its derivatives.

Solution :

Let $z = re^{i\theta}$.

$f(z) = \log z = \log(re^{i\theta}) = \log r + \log e^{i\theta} = \log r + i\theta$.

When $r = 0$, $f(z) = \log 0$ is $-\infty$.

$\therefore$ $f(z)$ is not defined at the origin and hence $f(z)$ is not differentiable.

At points other than the origin.

$u(r,\theta) = \log r$, $v(r,\theta) = \theta$.

$u_r = \frac{1}{r}$ $\quad$ $v_r = 0$

$u_\theta = 0$. $\quad$ $v_\theta = 1$.

$\therefore$ $u_r = \frac{1}{r} v_\theta$, $v_r = -\frac{1}{r} u_\theta$.

$\Rightarrow$ $\log z$ satisfies the C-R equations.

Further $u_r, u_\theta, v_r, v_\theta$ are continuous everywhere except at $z=0$

$\therefore$ $f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{\frac{1}{r} + i(0)}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}$.

$\left\{ f(z) = re^{i\theta} \Rightarrow \frac{\partial f}{\partial r} = e^{i\theta} \right\}$

Polar form of C-R equations

$z = x + iy$

$x = r\cos\theta, \ y = r\sin\theta$

$\Rightarrow z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$

C-R equations : $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}.$

Exercises

1. If $w = e^z$ then show that the function $w = e^z$ is analytic everywhere in the complex plane and hence find $\dfrac{dw}{dz}$.

Solution: Let $z = x + iy$

$\Rightarrow w = f(z) = e^{x+iy} = e^x . e^{iy} = e^x(\cos y + i\sin y).$

$\Rightarrow u + iv = e^x\cos y + i e^x \sin y$

$\Rightarrow u = e^x \cos y \qquad v = e^x \sin y$

$\dfrac{\partial u}{\partial x} = e^x \cos y \qquad \dfrac{\partial v}{\partial x} = e^x \sin y$

$\dfrac{\partial u}{\partial y} = -e^x \sin y. \qquad \dfrac{\partial v}{\partial y} = e^x \cos y$

$u_x = v_y = e^x \cos y, \quad u_y = -v_x = -e^x \sin y$

$\Rightarrow$ C-R equations are satisfied

Further, $e^x, \cos y, \sin y$ are continuous and therefore u_x, v_x, u_y, v_y are continuous everywhere.

$\therefore f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y = e^{x+iy} = e^z$

Test the analyticity of $f(z) = z$.

Formula: De Movire's theorem. $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow e^{in\theta} = \cos n\theta + i\sin n\theta$

$u + iv = w = (re^{i\theta})^n = r^n e^{in\theta}$.

$u + iv = r^n(\cos n\theta + i\sin n\theta)$.

$u = r^n \cos n\theta$ $\qquad$ $v = r^n \sin n\theta$.

$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta$ $\qquad$ $\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$

$\frac{\partial u}{\partial \theta} = r^n(-n\sin n\theta)$. $\qquad$ $\frac{\partial v}{\partial \theta} = nr^n \cos n\theta$.

$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Hence C-R equations are satisfied and the partial derivatives are continuous.

$\Rightarrow z^n$ is analytic.

6) Show that $f(z) = \frac{1}{z}$ is analytic everywhere except $z = 0$.

Proof: $z = re^{i\theta} \Rightarrow \frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$.

When $z = 0$, $r = 0$ and so $f(z)$ is not defined at $z = 0$.

$u = \frac{\cos\theta}{r}$ $\qquad$ $v = -\frac{\sin\theta}{r}$.

$u_r = -\frac{\cos\theta}{r^2}$, $u_\theta = -\frac{\sin\theta}{r}$ $\qquad$ $v_r = \frac{\sin\theta}{r^2}$, $v_\theta = -\frac{\cos\theta}{r}$.

$u_r = \frac{1}{r}v_\theta$, $v_r = -\frac{1}{r}u_\theta$

C-R equations are satisfied. Hence $\frac{1}{z}$ is analytic everywhere except $z = 0$.

5). Test the analyticity of $f(z) = z^n$.

Formula: De Movire's theorem: $e^{i\theta} = \cos\theta + i\sin\theta$.

$\Rightarrow e^{in\theta} = \cos n\theta + i\sin n\theta$

$$u + iv = w = (re^{i\theta})^n = r^n e^{in\theta}.$$

$$u + iv = r^n(\cos n\theta + i\sin n\theta).$$

$u = r^n \cos n\theta$ $\qquad$ $v = r^n \sin n\theta$.

$\dfrac{\partial u}{\partial r} = nr^{n-1}\cos n\theta$ $\qquad$ $\dfrac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$

$\dfrac{\partial u}{\partial \theta} = r^n(-n\sin n\theta)$. $\qquad$ $\dfrac{\partial v}{\partial \theta} = nr^n \cos n\theta$.

$\therefore \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$, $\quad \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

Hence C-R equations are satisfied. and the partial derivatives are continuous.

$\Rightarrow z^n$ is analytic.

6) Show that $f(z) = \frac{1}{z}$ is analytic everywhere except $z = 0$.

Proof: $z = re^{i\theta} \Rightarrow \frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$.

When $z = 0$, $r = 0$ and so $f(z)$ is not defined at $z = 0$.

$u = \dfrac{\cos\theta}{r}$ $\qquad$ $v = -\dfrac{\sin\theta}{r}$.

$u_r = -\dfrac{\cos\theta}{r^2}$, $u_\theta = -\dfrac{\sin\theta}{r}$ $\qquad$ $v_r = \dfrac{\sin\theta}{r^2}$, $v_\theta = -\dfrac{\cos\theta}{r}$

$u_r = \frac{1}{r}v_\theta$, $v_r = -\frac{1}{r}u_\theta$

$\therefore$ C-R equations are satisfied. Hence $\frac{1}{z}$ is analytic everywhere except $z = 0$.

3. Check whether $w = \bar{z}$ is analytic everywhere.

Let $w = u + iv$, $z = x + iy \Rightarrow \bar{z} = x - iy$.

$\Rightarrow u + iv = x - iy$

$\Rightarrow u = x, \quad v = -y$.

$u_x = 1, \quad v_x = 0$

$u_y = 0 \quad v_y = -1$.

$\Rightarrow u_x \neq v_y$

Hence, C-R equations are not satisfied.

$\therefore f(z) = \bar{z}$ is no where analytic.

4. Test the analyticity of the function $w = \sin z$.

Prerequisites : $\cos iy = \cosh y$, $\sin iy = i \sinh y$,

$\frac{d}{dy}(\cosh y) = \sinh y$, $\frac{d}{dy}(\sinh y) = \cosh y$.

$\sin(A+B) = \sin A \cos B + \cos A \sin B$.

$u + iv = w = f(z) = \sin z = \sin(x + iy)$.

$u + iv = \sin x \cos(iy) + \cos x \sin(iy)$.

$u + iv = \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y \qquad v = \cos x \sinh y$.

$u_x = \cos x \cosh y \qquad v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y \qquad v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$.

C-R. equations are satisfied. and all four partial derivatives are continuous Hence, $\sin z$ is analytic.

Property 1(a) The real and imaginary parts of an analytic function $w = u(r,\theta) + iv(r,\theta)$. satisfy the Laplace equation in polar co-ordinates.

Property 2 :

If $w = u(x,y) + iv(x,y)$ is an analytic function the curves of the family $u(x,y) = a$ and the curves of the family $v(x,y) = b$ cut orthogonally where a and b are varying constants. (i.e. $m_1 m_2 = -1$)

Property 3 : An analytic function with constant modulus is constant

Property 4 : An analytic function whose real part is constant must be a constant itself.

Property 5 : An analytic function with constant imaginary part is constant.

Property 6 : If $f(z)$ and $\overline{f(z)}$ are analytic in a region D, then $f(z)$ is a constant in that region D.

Harmonic Conjugates

1). Laplace equation : $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is known as Laplace equation in two dimensions.

Laplacian operator : $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian operator, denoted by ∇^2.

In polar co-ordinates, $\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$.

Properties

1. The real and imaginary parts of an analytic function $w = u + iv$ satisfy the Laplace equation.

   i.e. $\nabla^2 u = 0$ and $\nabla^2 v = 0$.

   In other words, the real and imaginary parts of an analytic function are harmonic functions.

   The converse need not be true.

2. Harmonic function or Potential function.

   A real function of two variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called a harmonic function

3. Conjugate harmonic function

   If u and v are harmonic functions such that $u + iv$ is analytic, then each is called the conjugate harmonic function of the other.

Exercises based on Harmonic Conjugate.

1. If $f(z) = e^z$ then show that u and v are harmonic functions.

Solution: $z = x+iy$, $u+iv = f(z)$.

$u+iv = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y$.

$u = e^x\cos y$ $\qquad$ $v = e^x\sin y$

$u_x = e^x\cos y$ $\quad$ $u_y = -e^x\sin y$ $\quad$ $v_x = e^x\sin y$ $\quad$ $v_y = e^x\cos y$

$u_{xx} = e^x\cos y$. $\quad$ $u_{yy} = -e^x\cos y$. $\quad$ $v_{xx} = e^x\sin y$ $\quad$ $v_{yy} = -e^x\sin y$.

$u_{xx} + u_{yy} = e^x\cos y - e^x\cos y = 0$. $\quad$ $v_{xx} + v_{yy} = e^x\sin y - e^x\sin y = 0$.

2. Find the value of m if $u = 2x^2 - my^2 + 3x$ is harmonic

Solution: $u = 2x^2 - my^2 + 3x$.

$u_x = 4x+3$, $\quad u_y = -2my$

$u_{xx} = 4$. $\quad u_{yy} = -2m$.

Given $u_{xx} + u_{yy} = 0 \Rightarrow 4 - 2m = 0 \Rightarrow \boxed{m = 2}$.

Construction of Analytic function using Milne-Thomson method.

Case 1: If real part 'u' is given.

Step 1: $f(z) = u + iv$.

$\Rightarrow f'(z) = u_x + iv_x$.

Since $v_x = -u_y$.

$f'(z) = u_x - iu_y$

Step 2: Put $x = z$ and $y = 0$.

$f'(z) = u_x(z,0) - iu_y(z,0)$.

Step 3: Integrate w.r.t. z.

$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + C$

where C is a complex constant.

Case 2: If imaginary part 'v' is given.

Step 1: $f(z) = u(x,y) + iv(x,y)$

$\Rightarrow f'(z) = u_x(x,y) + iv_x(x,y)$

$u_x = v_y$

$\Rightarrow f'(z) = v_y(x,y) + iv_x(x,y)$

Step 2: Put $x = z$, $y = 0$.

$f'(z) = v_y(z,0) + iv_x(z,0)$.

Step 3: Integrate w.r.t. z.

$f(z) = \int v_y(z,0)\,dz + i\int v_x(z,0)\,dz + C$

Exercises

1. Find the function w such that $w = u + iv$ is analytic, if $u = e^x \sin y$.

Solution: $u = e^x \sin y$

$u_x = e^x \sin y$, $u_y = e^x \cos y$.

$u_x(z,0) = e^z \sin 0$, $u_y(z,0) = e^z \cos 0$.

$= 0$ $\quad u_y(z,0) = e^z$

w.k.t $f'(z) = u_x - iu_y \Rightarrow f(z) = \int 0\,dz - i\int e^z\,dz$

$f(z) = -ie^z + C$.