Engineering Mathematics

[For Semesters I and II]

Third Edition

About the Author

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Chapter 3

Analytic Functions

3.1 INTRODUCTION

Before we introduce the concept of complex variable and functions of a complex variable, the definition and properties of complex numbers, which the reader has studied in the lower classes are briefly recalled.

If x and y are real numbers and i denotes $\sqrt{-1}$, then $z = x + iy$ is called a Complex number. x is called *the real part of* z and is denoted by Re(z) or simply $R(z)$; y is called *the imaginary part of* z and is denoted by Im(z) or simply $I(z)$.

The Complex number $x - iy$ is called *the conjugate of the Complex number* $z = x + iy$ and is denoted by $\bar{z}$. Clearly $z\bar{z} = x^2 + y^2 = r^2$, where r is $|z|$, viz., the modulus of the Complex number z. Also $|\bar{z}| = |z|$; $R(z) = \dfrac{z + \bar{z}}{2}$, $I(z) = \dfrac{z - \bar{z}}{2i}$.

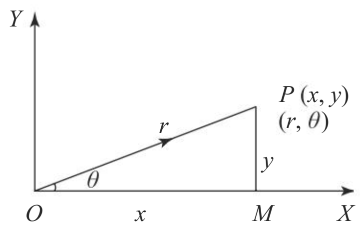


Fig. 3.1

The complex number $z = x + iy$ is geometrically represented by the point P (x, y) with reference to a pair of rectangular coordinate axes OX and OY, which are called the real axis and the imaginary axis respectively (Fig. 3.1). Corresponding to each complex number, there is a unique point in the XOY–plane and conversely, corresponding to each point in the XOY-plane, there is a unique complex number. The XOY-plane, the points in which represent complex numbers, is called the *Complex plane* or *Argand plane* or *Argand diagram.*

If the polar coordinates of the point P are (r, θ), then $r = OP = \sqrt{x^2 + y^2} =$ modulus of z or $|z|$ and $\theta = \angle MOP = \tan^{-1}\left(\dfrac{y}{x}\right) =$ amplitude of z or amp (z).

Since $x = r\cos\theta$ and $y = r\sin\theta$,

$$z = x + iy = r(\cos\theta + i\sin\theta) \quad \text{or} \quad re^{i\theta}.$$

$r(\cos\theta + i\sin\theta)$ is called the *modulus-amplitude form of* z and $re^{i\theta}$ is called *the polar form of* z.

3.2 THE COMPLEX VARIABLE

The quantity $z = x + iy$ is called a complex variable, when x and y are two independent real variables.

The Argand plane in which the variables z are represented by points is called the z-plane. The point that represents the complex variable z is referred to as the point z.

3.2.1 Function of a Complex Variable

If $z = x + iy$ and $w = u + iv$ are two complex variables such that there exists one or more values of w, corresponding to each value of z in a certain region R of the z-plane, then w is called *a function of* z and is written as $w = f(z)$ or $w = f(x + iy)$. When $w = u + iv = f(z) = f(x + iy)$, clearly u and v are functions of the variables x and y. For example, if $w = z^2$, then

$$u + iv = (x + iy)^2$$
$$= (x^2 - y^2) + i(2xy)$$

Thus $$u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$

$\therefore$ $$w = f(z) = u(x, y) + iv(x, y)$$

If z is expressed in the polar form, u and v are functions of r and θ. If, for every value of z, there corresponds a unique value of w, then w is called *a single-valued function of* z.

For example, $w = z^2$ and $w = \dfrac{1}{z}$ are single-valued functions of z.

If, for every value of z, there correspond more than one value of w, then w is called *a multiple valued function of* z. For example, $w = z^{1/4}$ and $w = \text{amp}(z)$ are multiple valued functions of z. $w = z^{1/4}$ is four valued and $w = \text{amp}(z)$ is infinitely many valued for $z \neq 0$.

3.2.2 Limit of a Function of a Complex Variable

The single valued function $f(z)$ is said to have the limit $l\ (= \alpha + i\beta)$ as z tends to z_0, if $f(z)$ is defined in a neighbourhood of z_0 (except perhaps at z_0) such that the values of $f(z)$ are as close to l as desired for all values of z that are sufficiently close to z_0, but different from z_0. We express this by writing $\lim\limits_{z \to z_0} \{f(z)\} = l$.

Note ☑ Neighbourhood of z_0 is the region of the z-plane consisting of the set of points z for which $|z - z_0| < \rho$, where ρ is a positive real number, viz., the set of points z lying inside the circle with the point z_0 as centre and ρ as radius.

Mathematically we say $\lim\limits_{z \to z_0} \{f(z)\} = l$, if, for every positive number $\in$ (however small it may be), we can find a positive number δ, such that $|f(z) - l| < \in$, whenever $0 < |z - z_0| < \delta$.

Note ☑ In real variables, $x \to x_0$ implies that x approaches x_0 along the x-axis or a line parallel to the x-axis (in which x is varying) either from left or from right. In complex variables, $z \to z_0$ implies that z approaches z_0 along any path (straight or curved) joining the points z and z_0 that lie in the z-plane, as shown in the Fig. 3.2.

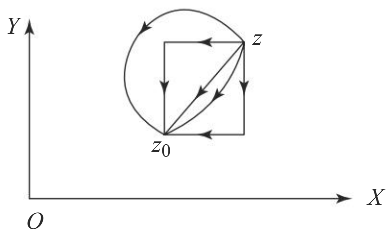


Fig. 3.2

Thus, in order that $\lim\limits_{z \to z_0} \{f(z)\}$ may exist, $f(z)$ should approach the same value l, when z approaches z_0 along all paths joining z and z_0.

3.2.3 Continuity of $f(z)$

The single valued function $f(z)$ is said to be continuous at a point z_0, if

$$\lim_{z \to z_0} \{f(z)\} = f(z_0).$$

This means that if a function $f(z)$ is to be continuous at the point z_0, the value of $f(z)$ at z_0 and the limit of $f(z)$ as $z \to z_0$ must exist (as per the definition given above) and these two values must be equal.

A function $f(z)$ is said to be *continuous in a region R* of the z-plane, if it is continuous at every point of the region.

If $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$, then $u(x, y)$ and $v(x, y)$ will be continuous at (x_0, y_0) and conversely. The function $\phi(x, y)$ is said to be continuous at the point (x_0, y_0), if $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} [\phi(x, y)] = \phi(x_0, y_0)$, in whatever manner $x \to x_0$ and $y \to y_0$.

3.2.4 Derivative of $f(z)$

The single valued function $f(z)$ is said to be *differentiable* at a point z_0, if

$$\lim_{\Delta z \to 0}\left[\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right] \text{ exists} \tag{1}$$

This limit is called *the derivative of* $f(z)$ *at* z_0 *and is denoted as* $f'(z_0)$. On putting $z_0 + \Delta z = z$ or $\Delta z = z - z_0$, we may write

$$f'(z_0) = \lim_{z \to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\right].$$

On putting $z_0 = z$ in (1), we get

$$f'(z) = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}\right]$$

3.2.5 Analytic Function

A single valued function $f(z)$ is said to be analytic at the point z_0, if it possesses a derivative at z_0 and at every point in some neighbourhood of z_0.

A function $f(z)$ is said to be analytic in a region R of the z-plane, if it is analytic at every point of R.

An analytic function is also referred to as *a regular function* or *a holomorphic function*.

A point, at which a function $f(z)$ is not analytic, viz., does not possess a derivative, is called *a singular point* or *singularity* of $f(z)$.

3.2.6 Cauchy-Riemann Equations

We shall now derive two conditions (usually referred to as necessary conditions) that are necessarily satisfied when a function $f(z)$ is analytic in a region R of the z-plane.

Theorem

If the function $f(z) = u(x, y) + iv(x, y)$ is analytic in a region R of the z-plane, then (i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist and (ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at every point in that region.

Proof

$f(z) = u(x, y) + iv(x, y)$ is analytic in R.

$\therefore$ $f'(z)$ exists at every point z in R(by definition)

i.e.,

$$L = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}\right] \text{ exists} \tag{1}$$

i.e., L takes the same value, when $\Delta z \to 0$ along all paths.

In particular, L takes the same value when $\Delta z \to 0$ along two specific paths QRP and QSP shown in Fig. 3.3. (2)

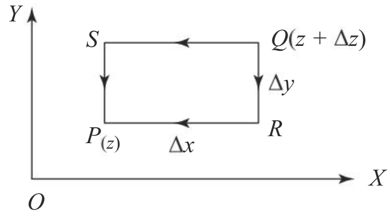


Fig. 3.3

It is evident that when z takes the increment Δz, x and y take the increments Δx and Δy respectively; hence we may write

$$L = \lim_{(\Delta x + i\Delta y)\to 0}\left[\frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i\Delta y}\right]$$

Let us now find the value of L, say L_1, corresponding to the path QRP, viz., by letting $\Delta y \to 0$ first and then by letting $\Delta x \to 0$.

Thus $$L_1 = \lim_{\Delta x\to 0}\left[\frac{\{u(x+\Delta x, y) + iv(x+\Delta x, y)\} - \{u(x\ y) + iv(x\ y)\}}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[\left\{\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x}\right\} + i\left\{\frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}\right\}\right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{3}$$

(by the definition of partial derivatives.)

Now we shall find the value of L, say L_2, corresponding to the path QSP, viz., by letting $\Delta x \to 0$ first and then by letting $\Delta y \to 0$.

Thus $$L_2 = \lim_{\Delta y\to 0}\left[\frac{\{u(x, y+\Delta y) + iv(x, y+\Delta y)\} - \{u(x, y) + iv(x, y)\}}{i\Delta y}\right]$$

$$= \lim_{\Delta y\to 0}\left[\frac{1}{i}\left\{\frac{u(x, y+\Delta y) - u(x, y)}{\Delta y}\right\} + \left\{\frac{v(x, y+\Delta y) - v(x, y)}{\Delta y}\right\}\right]$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{4}$$

(by the definition of partial derivatives.)

L exists [by (1)]

$\therefore$ L_1 and L_2 exist at every point in R.

i.e. $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ exist at every point in R (i)

L exists uniquely [by (2)]

$\therefore \quad L_1 = L_2$

i.e., $\quad \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y}$, by (3) and (4).

$\therefore \quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

and $\quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ at every point in R (ii)

Note ☑

1. The two equations given in (ii) above are called *Cauchy-Riemann equations* which will be hereafter referred to as C.R. equations.
2. When $f(z)$ is analytic, $f'(z)$ exists and is given by L, viz., by L_1 or L_2. Thus $f'(z) = u_x + iv_x$ or $v_y - iu_y$, where u_x, u_y, v_x, v_y denote the partial derivatives.
3. If $w = f(z)$, then $f'(z)$ is also denoted as $\dfrac{dw}{dz}$.

   Thus
   $$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u+iv) = \frac{\partial w}{\partial x}$$

   Also
   $$\frac{dw}{dz} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial}{\partial y}(u+iv) = -i\frac{\partial w}{\partial y}$$
4. In the above theorem, we have assumed that $w = f(z)$ is analytic and then derived the two conditions that necessarily followed. However the two conditions do not ensure the analyticity of the function $w = f(z)$. In other words, the two conditions are *not sufficient* for the analyticity of the function $w = f(z)$.
   The sufficient conditions for the function $w = f(z)$ to be analytic in a region R are given in the following theorem.

Theorem

The single valued continuous function $w = f(z) = u(x, y) + iv(x, y)$ is analytic in a region R of the z-plane, if the four partial derivatives u_x, v_x, u_y and v_y have the following features: (i) They exist, (ii) They are continuous and (iii) They satisfy the C.R. equations $u_x = v_y$ and $u_y = -v_x$ at every point of R.

Proof

Consider $\Delta u = u\,(x + \Delta x, y + \Delta y) - u\,(x, y)$

$$= [u\,(x + \Delta x, y + \Delta y) - u\,(x, y + \Delta y) + u\,(x, y + \Delta y) - u\,(x, y)]$$

$$= \Delta x \cdot u_x(x + \theta_1\Delta x, y + \Delta y) + \Delta y \cdot u_y(x, y + \theta_2\Delta y),$$

where $0 < \theta_1, \theta_2 < 1$, (by using the mean-value theorem.)

u_x is a continuous function of x and y, by (ii).

$\therefore$ $$u_x(x + \theta_1\Delta x, y + \Delta y) = u_x(x, y) + \in_1, \text{ where}$$

$$\in_1 \to 0 \text{ as } \Delta x \quad \text{and} \quad \Delta y \to 0 \text{ or as } \Delta z \to 0.$$

u_y is a continuous function of x and y, by (ii).

$\therefore$ $$u_y\,(x, y + \theta_2\Delta y) = u_y\,(x, y) + \in_2, \text{ where}$$

$$\in_2 \to 0 \text{ as } \Delta z \to 0$$

$\therefore$ $$\Delta u = \Delta x\,[u_x\,(x, y) + \in_1] + \Delta y[u_y\,(x, y) + \in_2],$$

where $\in_1$ and $\in_2 \to 0$ as $\Delta z \to 0$. (1)

Similarly, using the continuity of the partial derivatives v_x and v_y, we have

$$\Delta v = \Delta x\,[v_x(x, y) + \in_3] + \Delta y\,[v_y(x, y) + \in_4],$$

where $$\in_3 \text{ and } \in_4 \to 0 \text{ as } \Delta z \to 0 \tag{2}$$

Now $\Delta w = \Delta u + i\Delta v$

$$= (u_x + iv_x)\,\Delta x + (u_y + iv_y)\,\Delta y + (\in_1 + i\in_3)\,\Delta x + (\in_2 + i\in_4)\,\Delta y,$$

using (1) and (2).

$$= (u_x + iv_x)\,\Delta x + (u_y + iv_y)\,\Delta y + \eta_1\Delta x + \eta_2\Delta y, \text{ where}$$

$$\eta_1 = (\in_1 + i\in_3) \quad \text{and} \quad \eta_2 = (\in_2 + i\in_4) \to 0 \text{ as } \Delta z \to 0 \tag{3}$$

Using (iii) in (3), i.e., putting $v_y = u_x$ and $u_y = -\,v_x$ in (3), we have

$$\Delta w = (u_x + iv_x)\,\Delta x + (-v_x + iu_x)\,\Delta y + \eta_1\Delta x + \eta_2\Delta y$$

$$= u_x(\Delta x + i\Delta y) + iv_x(\Delta x + i\Delta y) + \eta_1\Delta x + \eta_2\Delta y$$

$$= (u_x + iv_x)\,\Delta z + \eta_1\Delta x + \eta_2\Delta y$$

$\therefore$ $$\frac{\Delta w}{\Delta z} = u_x + iv_x + \eta_1\frac{\Delta x}{\Delta z} + \eta_2\frac{\Delta y}{\Delta z} \tag{4}$$

Now $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$

$\therefore$ $$\left|\frac{\Delta x}{\Delta z}\right| \le 1 \quad \text{and} \quad \left|\frac{\Delta y}{\Delta z}\right| \le 1$$

$\therefore$ $$\left|\eta_1\frac{\Delta x}{\Delta z} + \eta_2\frac{\Delta y}{\Delta z}\right| \le |\eta_1| + |\eta_2|$$

$$\therefore \qquad \left(\eta_1 \frac{\Delta x}{\Delta z} + \eta_2 \frac{\Delta y}{\Delta z}\right) \to 0 \text{ as } \Delta z \to 0 \; [\because \; \eta_1 \text{ and } \eta_2 \to 0 \text{ as } \Delta z \to 0] \qquad (5)$$

Now taking limits on both sides of (4) and using (5), we have

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u_x + iv_x \,.$$

i.e., the derivative of $w = f(z)$ exists at every point in R.
i.e., $w = f(z)$ is analytic in the region R.

3.2.7 C.R. Equations in Polar Coordinates

When z is expressed in the polar form $re^{i\theta}$, we have already observed that u and v, where $w = u + iv$, are functions of r and θ. In this case, we shall derive the C.R. equations satisfied by $u(r, \theta)$ and $v(r, \theta)$, assuming that $w = u(r, \theta) + iv(r, \theta)$ is analytic.

Theorem

If the function $w = f(z) = u(r, \theta) + iv(r, \theta)$ is analytic in a region R of the z-plane, then (i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exit and (ii) they satisfy the C.R. equations, viz.,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \text{ at every point in that region.}$$

Proof

$$f(z) = u(r, \theta) + iv(r, \theta) \text{ is analytic in } R.$$

$\therefore \; f'(z)$ exists at every point z in R (by definition)

$$\text{i.e., } L = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right] \text{ exists} \qquad (1)$$

$$\text{i.e., } L = \lim_{\Delta(re^{i\theta}) \to 0}\left[\frac{\{u(r+\Delta r, \theta+\Delta\theta) + iv(r+\Delta r, \theta+\Delta\theta)\} - \{u(r,\theta) + iv(r,\theta)\}}{\Delta\left(re^{i\theta}\right)}\right]$$

exists.

i.e., L take the same value, in whatever manner $\Delta z = \Delta(re^{i\theta}) \to 0$.

In particular, L takes the same value, corresponding to the two ways given below in which $\Delta z \to 0$. (2)

$$\begin{aligned}\Delta z &= \Delta(re^{i\theta}) \\ &= e^{i\theta} \cdot \Delta r, \text{ if } \theta \text{ is kept fixed}\end{aligned}$$

$\therefore$ When $\Delta z \to 0$, $\Delta r \to 0$, if θ is kept fixed. Let us find the value of L, say L_1, corresponding to this way of Δz tending to zero.

Thus $L_1 = \lim\limits_{\Delta r \to 0}\left[\left\{\dfrac{u(r+\Delta r,\theta)-u(r,\theta)}{e^{i\theta}\Delta r}\right\}+i\left\{\dfrac{v(r+\Delta r,\theta)-v(r,\theta)}{e^{i\theta}\Delta r}\right\}\right]$

$$= e^{-i\theta}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right) \tag{3}$$

(by the definition of partial derivatives.)

Now $\qquad \Delta z = \Delta\,(re^{i\theta})$

$\qquad\qquad = re^{i\theta}\, i\Delta\theta$, if r is kept fixed.

$\therefore$ When $\Delta z \to 0$, $\Delta\theta \to 0$, if r is kept fixed.

We shall now find the value of $L = L_2$, corresponding to this way of Δz tending to zero.

Thus $\quad L_2 = \lim\limits_{\Delta\theta \to 0}\left[\left\{\dfrac{u(r,\theta+\Delta\theta)-u(r,\theta)}{r\,e^{i\theta}\cdot i\Delta\theta}\right\}+i\left\{\dfrac{v(r,\theta+\Delta\theta)-v(r,\theta)}{r\,e^{i\theta}\cdot i\Delta\theta}\right\}\right]$

$$= \frac{1}{r}e^{-i\theta}\left(-i\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial\theta}\right) \tag{4}$$

(by the definition of partial derivatives.)

Since L exists [by (1)], L_1 and L_2 exist at every point in R.

i.e. $\quad \dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial\theta}, \dfrac{\partial v}{\partial r}, \dfrac{\partial v}{\partial\theta}$ exists at every point in R. $\hfill$ (i)

Since L exists uniquely [by (2)],

$$L_1 = L_2$$

i.e. $\quad e^{-i\theta}\left(\dfrac{\partial u}{\partial r}+i\dfrac{\partial v}{\partial r}\right)=\dfrac{1}{r}e^{-i\theta}\left(-i\dfrac{\partial u}{\partial\theta}+\dfrac{\partial v}{\partial\theta}\right)$, from (3) and (4).

$\therefore \qquad \dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial\theta}$

and $\qquad \dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}$ at every point in R $\hfill$ (ii)

Note ☑

1. When $f(z)$ is analytic, $f'(z)$ exists and is given by L, viz., by L_1 or L_2. Thus, when $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic,

   $f'(z) = e^{-i\theta}(u_r + iv_r)$ or $\dfrac{1}{r}e^{-i\theta}(v_\theta - iu_\theta)$,

   where u_r, u_θ, v_r, v_θ denote the partial derivatives.
2. If $w = f(z)$, then

$$\frac{dw}{dz} = e^{-i\theta} \cdot \frac{\partial}{\partial r}(u+iv) = e^{-i\theta}\frac{\partial w}{\partial r}.$$

Also
$$\frac{dw}{dz} = -\frac{i}{r}e^{-i\theta}(u_\theta + iv_\theta)$$
$$= -\frac{i}{r}e^{-i\theta}\frac{\partial}{\partial\theta}(u+iv)$$
$$= -\frac{i}{r}e^{-i\theta}\frac{\partial w}{\partial\theta}.$$

3. The two conditions derived in the theorem do not ensure the analyticity of $w = f(z)$. The sufficient conditions for the analyticity of $w = f(z) = u\,(r, \theta) + iv\,(r, \theta)$ in R are given below without proof.
   (i) u_r, u_θ, v_r, v_θ must exist
   (ii) they must be continuous;
   (ii) they must satisfy the C.R. equations in polar co-ordinates at every point in the region R.

WORKED EXAMPLE 3(a)

Example 3.1 Find $\lim\limits_{z\to 0} f(z)$, when $f(z) = \dfrac{x^2 y}{x^2 + y^2}$.

Let us find the limit of $f(z)$ corresponding to any one manner, say, by letting $y \to 0$ first and then by letting $x \to 0$.

Thus
$$\lim_{\substack{y\to 0\\ x\to 0}}[f(z)] = \lim_{x\to 0}\left(\frac{0}{x^2}\right) = \lim_{x\to 0}(0) = 0.$$

This does not mean that $\lim\limits_{z\to 0}[f(z)]$ exists and is equal to 0.

However, we proceed to verify whether $\lim\limits_{z\to 0}[f(z)]$ can be 0, as per the mathematical definition, which states that $\lim\limits_{z\to 0}[f(z)] = 0$, if we can find a δ such that $|f(z) - 0| < \in$, whenever $0 < |z - 0| < \delta$.

Now, using polar coordinates,

$$|f(z) - 0| = \left|\frac{r^3\cos^2\theta\sin\theta}{r^2(\cos^2\theta + \sin^2\theta)}\right|$$
$$= r\,|\cos^2\theta|\,|\sin\theta|$$
$$< r, \text{ since } |\cos\theta| < 1 \text{ and } |\sin\theta| < 1$$

$\therefore$
$$|f(z) - 0| < \in, \text{when } r < \frac{\in}{2}$$

i.e., when $\quad |z-0|<\dfrac{\epsilon}{2}$

Thus, taking $\delta \le \dfrac{\epsilon}{2}$, we have proved that

$$|f(z)-0|<\epsilon, \text{ when } |z-0|<\delta.$$

$\therefore \lim\limits_{z\to 0}\left(\dfrac{x^2 y}{x^2+y^2}\right)$ exists and is equal to 0.

Example 3.2 If $f(z)=\dfrac{x^3 y(y-ix)}{x^6+y^2}$ $(z\neq 0)$ and $f(0)=0$, prove that $\dfrac{\{f(z)-f(0)\}}{z}$ $\to 0$ as $z\to 0$ along any radius vector, but not as $z\to 0$ in any manner.

Now $$\frac{f(z)-f(0)}{z}=\frac{x^3 y(y-ix)}{(x^6+y^2)(x+iy)}=\frac{-ix^3 y}{x^6+y^2} \qquad (1)$$

A radius vector is a line through the pole (origin) and hence its equation is $y=mx$. To take the limit of (1) as $z\to 0$ along any radius vector, we put $y=mx$ in (1) and then let $x\to 0$.

Thus $$\lim_{\substack{y=mx\\x\to 0}}\left[\frac{-ix^3 y}{x^6+y^2}\right]=\lim_{x\to 0}\left[\frac{-imx^4}{x^2(x^4+m^2)}\right]$$

$$=\lim_{x\to 0}\left[\frac{-imx^2}{x^4+m^2}\right]=0, \text{ for various values of } m.$$

$$\therefore \quad \left\{\frac{f(z)-f(0)}{z}\right\}\to 0, \text{ as } z\to 0 \text{ along any radius vector.}$$

Now let us find the limit of (1) by moving along the curve $y=x^3$ and approaching the origin.

Thus $$\lim_{z\to 0}\left\{\frac{f(z)-f(0)}{z}\right\}=\lim_{\substack{y=x^3\\x\to 0}}\left[\frac{-ix^3 y}{x^6+y^2}\right]$$

$$=\lim_{x\to 0}\left[\frac{-ix^6}{x^6+x^6}\right]=-\frac{i}{2}\neq 0.$$

Note ☑ Since the limiting values are not unique, the limit does not exist.

Example 3.3 Prove that the function $f(z)$, where $f(z)=\dfrac{x^2(1+i)-y^2(1-i)}{x+y}$, when $z\neq 0$ and

$$f(z) = 0, \text{ when } z = 0$$

is continuous at $z = 0$.

$$f(z) = \left(\frac{x^2 - y^2}{x+y}\right) + i\left(\frac{x^2 + y^2}{x+y}\right)$$

Consider $\quad L_1 = \lim\limits_{z\to 0}[f(z)] = \lim\limits_{\substack{y\to 0\\ x\to 0}}[f(z)] = \lim\limits_{x\to 0}[(1+i)\,x] = 0$

$$L_2 = \lim_{z\to 0}[f(z)] = \lim_{\substack{x\to 0\\ y\to 0}}[f(z)] = \lim_{y\to 0}[(-1+i)\,y] = 0$$

$$L_3 = \lim_{z\to 0}[f(z)] = \lim_{\substack{y\to mx\\ x\to 0}}[f(z)] = \lim_{x\to 0}\left[\left(\frac{1-m^2}{1+m} + i\frac{1+m^2}{1+m}\right)x\right] = 0$$

$$L_4 = \lim_{z\to 0}[f(z)] = \lim_{\substack{y\to\phi(x)\\ x\to 0}}[f(z)] \tag{1}$$

where $y = \phi(x)$ is any curve through the origin.

$\therefore \qquad \phi(0) = 0$

By Taylor's series expansion, we get

$$y = \phi(x) = \frac{\phi'(0)}{1!}x + \frac{\phi''(0)}{2!}x^2 + \frac{\phi'''(0)}{3!}x^3 + \ldots\infty$$

since $\qquad \phi(0) = 0.$

$$= x\left[\frac{\phi'(0)}{1!} + \frac{\phi''(0)}{2!}x + \frac{\phi'''(0)}{3!}x^2 + \ldots\infty\right]$$

$$= x \cdot \psi(x), \text{ where } \psi(x) \to \phi'(0) \text{ as } x \to 0 \tag{2}$$

Using (2) in (1), we have

$$L_4 = \lim_{x\to 0}\left[\left\{\frac{1-\psi^2(x)}{1+\psi(x)} + i\frac{1+\psi^2(x)}{1+\psi(x)}\right\}x\right]$$

$$= 0$$

Thus $\lim\limits_{z\to 0}[f(z)]$ takes the same value 0, in whatever manner $z \to 0$.

$\therefore \qquad \lim\limits_{z\to 0}[f(z) = 0 = f(0).$

$\therefore \quad f(z)$ is continuous at $z = 0$.

Example 3.4 Show that the function $f(z)$ is discontinuous at the origin ($z = 0$), given that

$$f(z) = \frac{xy(x-2y)}{x^3 + y^3}, \text{ when } z \neq 0$$

$$= 0, \text{ when } z = 0.$$

Consider

$$\lim_{z\to 0}[f(z)] = \lim_{\substack{y\to mx\\ x\to 0}}[f(z)] = \lim_{x\to 0}\left[\frac{m(1-2m)x^3}{(1+m^3)x^3}\right] = \frac{m(1-2m)}{1+m^3}$$

Thus $\lim\limits_{z\to 0}$ $[f(z)]$ depends on the value of m and hence does not take a unique value.

$\therefore$ $\lim\limits_{z\to 0}$ $[f(z)]$ does not exist.

$\therefore f(z)$ is discontinuous at the origin.

Example 3.5 Show that the function $f(z)$ is discontinuous at $z = 0$, given that

$$f(z) = \frac{2xy^2}{x^2+3y^4}, \text{ when } z \neq 0 \quad \text{and} \quad f(0) = 0.$$

Consider $\lim\limits_{z\to 0}[f(z)] = \lim\limits_{\substack{y=mx\\ x\to 0}}[f(z)]$

$$= \lim_{x\to 0}\left(\frac{2m^2x}{1+3m^4x^2}\right) = 0$$

Note ☑

Just because the value of $\lim\limits_{z\to 0}$ $[f(z)]$, when $z \to 0$ along the line $y = mx$ is 0, we should not conclude that the limit exists and hence $f(z)$ is continuous at $z = 0$.)

Now let us take the limit by approaching 0 along the curve $x = y^2$.

Then $\lim\limits_{z\to 0}[f(z)] = \lim\limits_{\substack{x=y^2\\ y\to 0}}[f(z)]$

$$= \lim_{y\to 0}\left[\frac{2y^4}{y^4+3y^4}\right] = \frac{1}{2} \neq 0.$$

$\therefore$ $\lim\limits_{z\to 0}$ $[f(z)]$ does not exist and hence $f(z)$ is not continuous.

Example 3.6 Show that the function $|z|^2$ is continuous and differentiable at the origin, but it is not analytic at any point.

Let $f(z) = |z|^2$. Then $f(0) = 0$.

Now $|f(z) - 0| = |z|^2 = r^2$, where $z = re^{i\theta}$

$< \in$, whenever $r < \in$, if

$\in$ is sufficiently small and positive

i.e. $|f(z) - 0| < \in$, whenever $0 < |z| < \in$.

$\therefore \lim\limits_{z\to 0} [f(z)]$ exists and is equal to $0 = f(0)$

$\therefore f(z)$ is continuous at the origin.

Also $$f'(0) = \lim_{\Delta z\to 0}\left\{\frac{f(0+\Delta z)-f(0)}{\Delta z}\right\}$$

$$= \lim_{\Delta z\to 0}\left\{\frac{|\Delta z|^2}{\Delta z}\right\}$$

$$= \lim_{\Delta z\to 0}\left\{\Delta \bar{z}\right\} \qquad [\because |\Delta z|^2 = \Delta z\cdot\Delta\bar{z}]$$

$$= 0 \qquad \{\because \text{ When } \Delta x + i\Delta y \to 0,\ \Delta x - i\Delta y \to 0\}$$

$\therefore f(z)$ is differentiable at the origin.

Now $$\lim_{\Delta z\to 0}\left\{\frac{f(z+\Delta z)-f(z)}{\Delta z}\right\} = \lim_{\Delta z\to 0}\left\{\frac{|z+\Delta z|^2-|z|^2}{\Delta z}\right\}$$

$$= \lim_{\Delta z\to 0}\left[\frac{(z+\Delta z)(\bar{z}+\Delta\bar{z})-z\bar{z}}{\Delta z}\right]$$

$$= \lim_{\Delta z\to 0}\left[z\frac{\Delta\bar{z}}{\Delta z}+\bar{z}+\Delta\bar{z}\right]$$

$$= \lim_{\Delta z\to 0}\left[(x+iy)\frac{(\Delta x-i\Delta y)}{\Delta x+i\Delta y}+(x-iy)+(\Delta x-i\Delta y)\right]$$

Let us find the value of this limit by making $\Delta z \to 0$ in two different manners.

$$L_1 = \lim_{\substack{\Delta y\to 0\\ \Delta x\to 0}}\left[(x+iy)\frac{(\Delta x-i\Delta y)}{\Delta x+i\Delta y}+(x-iy)+(\Delta x-i\Delta y)\right]$$

$$= \lim_{\Delta x\to 0}\left[(x+iy)+(x-iy)+\Delta x\right] = 2x.$$

$$L_2 = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[(x+iy)\frac{(\Delta x-i\Delta y)}{\Delta x+i\Delta y}+(x-iy)+(\Delta x-i\Delta y)\right]$$

$$= \lim_{\Delta y\to 0}\left[-(x+iy)+(x-iy)-i\Delta y\right] = -2iy$$

$L_1 \neq L_2$ for all values of x and y.

$\therefore f(z)$ is not differentiable at any point $z \neq 0$.
$\therefore f(z)$ is not analytic at any point $z \neq 0$.

Though $f(z)$ is differentiable at $z = 0$, it is not differentiable at any point in the neighbourhood of $z = 0$.

$\therefore f(z)$ is not analytic even at the origin.

Hence $\qquad f(z) = |z|^2$ is not analytic at any point.

Example 3.7 Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied at the origin.

$$f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}\ . \quad \therefore \quad u(x, y) = \sqrt{|xy|}; v(x, y) = 0\ .$$

$$u_x(0,0) = \left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{\Delta x \to 0}\left[\frac{u(\Delta x, 0) - u(0,0)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{0-0}{\Delta x}\right] = 0$$

$$u_y(0,0) = \lim_{\Delta y \to 0}\left[\frac{u(0, \Delta y) - u(0,0)}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0}\left[\frac{0-0}{\Delta y}\right] = 0$$

$$v_x(0,0) = \lim_{\Delta x \to 0}\left[\frac{v(\Delta x, 0) - v(0,0)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{0-0}{\Delta x}\right] = 0$$

$$v_y(0,0) = \lim_{\Delta y \to 0}\left[\frac{v(0, \Delta y) - v(0,0)}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0}\left[\frac{0-0}{\Delta y}\right] = 0\cdot$$

Clearly $\qquad u_x = v_y$ and $u_y = -v_x$ at the origin.

i.e., C.R. equations are satisfied at the origin.

Now $$\lim_{\Delta z \to 0}\left[\frac{f(0+\Delta z) - f(0)}{\Delta z}\right] = \lim_{\Delta z \to 0}\left[\frac{\sqrt{|\Delta x \cdot \Delta y| - 0}}{\Delta x + i\Delta y}\right]$$

$$= \lim_{\substack{\Delta y = m\Delta x \\ \Delta x \to 0}}\left[\frac{\sqrt{|m|\Delta x^2|}}{\Delta x(1+im)}\right]$$

$$= \frac{\sqrt{|m|}}{1+im}$$

The limit is not unique, since it depends on m. $\therefore f'(0)$ does not exist.
Hence $f(z)$ is not regular at the origin.

Note ☑ This problem means that C.R. equations are not sufficient for the analyticity of the function.

Example 3.8 Prove that the following functions are analytic and also find their derivatives using the definition.

(i) z^3; (ii) e^{-z}; (iii) $\sin z$; (iv) $\cosh z$;
(v) z^n (n is a positive integer); (vi) $\log z$.

(i) Let $f(z) = u + iv = z^3 = (x + iy)^3$

$$= (x^3 - 3xy^2) + i\,(3x^2y - y^3)$$

$$\therefore \quad u = x^3 - 3xy^2\ ;\ v = 3x^2y - y^3$$

$$u_x = 3x^2 - 3y^2\ ;\ v_x = 6xy$$

$$u_y = -\,6xy\ ;\ v_y = 3x^2 - 3y^2$$

Obviously u_x, u_y, v_x, v_y exist for finite values of x and y (i.e., everywhere in the finite plane).

They are continuous everywhere, since they are polynomials in x and y.

Also $u_x = v_y = 3x^2 - 3y^2$ and $v_x = -\,u_y = 6xy$ i.e., C.R. equations are satisfied for all finite values of x and y.

All the three sufficient conditions that ensure the analyticity of $f(z)$ are satisfied everywhere.

$\therefore$ $f(z)$ is analytic everywhere.

Now $f'(z) = u_x + iv_x$

$$= 3x^2 - 3y^2 + i6xy$$

$$= 3(x + iy)^2$$

$$= 3\,z^2.$$

(ii) Let $f(z) = u + iv = e^{-z} = e^{-(x\,+\,iy)}$

$$= e^{-x}\,(\cos y - i \sin y)$$

$$\therefore \quad u = e^{-x} \cos y\ ;\ v = -\,e^{-x} \sin y$$

$$u_x = -\,e^{x} \cos y;\ v_x = e^{-x} \sin y$$

$$u_y = -\,e^{-x} \sin y;\ v_y = -\,e^{-x} \cos y$$

u_x, u_y, v_x, v_y exist everywhere, are continuous everywhere ($\because$ e^{-x}, $\sin y$ and $\cos y$ are continuous functions) and satisfy C.R. equations everywhere.

$\therefore f(z)$ is analytic everywhere.

Now $f'(z) = u_x + iv_x$

$$= -e^{-x}\cos y + ie^{-x}\sin y$$
$$= -e^{-x}(\cos y - i\sin y)$$
$$= -e^{-x}\,.\,e^{-iy} = -e^{-(x+iy)} = -e^{-z}.$$

(iii) Let $\quad f(z) = \sin z$

i.e. $\quad u + iv = \sin(x + iy)$

$$= \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i\cos x \sinh y.$$

$\therefore \quad u = \sin x \cosh y\,;\; v = \cos x \sinh y$

$$u_x = \cos x \cosh y\,;\; v_x = -\sin x \sinh y$$
$$u_y = \sin x \sinh y\,;\; v_y = \cos x \cosh y$$

The four partial derivatives are products of circular and hyperbolic functions.

$\therefore$ They exist, are continuous and satisfy C.R. equations everywhere.

$\therefore$ $f(z)$ is analytic everywhere.

Now $\quad f'(z) = u_x + iv_x$

$$= \cos x \cosh y - i\sin x \sinh y$$
$$= \cos x \cos iy - \sin x \sin iy$$
$$= \cos(x + iy)$$
$$= \cos z.$$

(iv) Let $\quad f(z) = \cosh z = \cos iz$

i.e. $\quad u + iv = \cos(ix - y)$

$$= \cosh x \cos y + i\sinh x \sin y$$

$\therefore \quad u = \cosh x \cos y\,;\; v = \sinh x \sin y$

$$u_x = \sinh x \cos y\,;\; v_x = \cosh x \sin y$$
$$u_y = -\cosh x \sin y\,;\; v_y = \sinh x \cos y$$

$\therefore$ u_x, u_y, v_x, v_y exist, are continuous and satisfy C.R. equations everywhere.

$\therefore$ $f(z)$ is analytic everywhere.

Now $\quad f'(z) = u_x + iv_x$

$$= \sinh x \cos y + i\cosh x \sin y$$
$$= \sinh(x + iy) = \sinh z.$$

(v) Let $\quad f(z) = u(r, \theta) + iv(r, \theta) = z^n = (re^{i\theta})^n$

$$= r^n(\cos n\theta + i\sin n\theta)$$

$\therefore \quad u = r^n \cos n\theta\,; \quad v = r^n \sin n\theta$

$$u_r = nr^{n-1}\cos n\theta\,; \quad v_r = nr^{n-1}\sin n\theta$$
$$u_\theta = -nr^n \sin n\theta\,; \quad v_\theta = nr^n \cos n\theta$$

$u_r, u_\theta, v_r, v_\theta$ exist and are continuous for finite values of r and hence everywhere.

Also $$u_r = \frac{1}{r} v_\theta = nr^{n-1} \cos n\theta$$

and $$v_r = -\frac{1}{r} u_\theta = nr^{n-1} \sin n\theta$$

i.e., C.R. equations are also satisfied everywhere.

$\therefore$ $f(z)$ is analytic everywhere.

Now
$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) \\ &= e^{-i\theta} \cdot nr^{n-1} (\cos n\theta + i \sin n\theta) \\ &= nr^{n-1} e^{-i\theta} \cdot e^{in\theta} \\ &= nr^{n-1} e^{i(n-1)\theta} \\ &= n[re^{i\theta}]^{n-1} = nz^{n-1}. \end{aligned}$$

(vi) Let
$$\begin{aligned} f(z) &= u + iv = \log z = \log (r\, e^{i\theta}) \\ &= \log r + i\theta \end{aligned}$$

$\therefore$
$$u = \log r; \quad v = \theta$$
$$u_r = \frac{1}{r}; \quad v_r = 0$$
$$u_\theta = 0; \quad v_\theta = 1$$

$\therefore$ $u_r, u_\theta, v_r, v_\theta$ exist, are continuous and satisfy C.R. equations everywhere except at $r = 0$ i.e. $z = 0$

$\therefore$ $f(z)$ is analytic everywhere except at $z = 0$.

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) \\ &= e^{-i\theta}\left(\frac{1}{r} + i \cdot 0\right) \\ &= \frac{1}{re^{i\theta}} = \frac{1}{z},\ z \neq 0. \end{aligned}$$

3.2.8 Important Note

In the above problems, we observe that the derivatives of some of the elementary functions of a complex variable are similar to those of corresponding functions of a real variable. This is true with respect to other elementary functions also.

This is due to the fact that the definitions of $f'(z)$ in Complex Calculus and $f'(x)$ in Real Calculus are identical, except that there is a slight difference in the interpretation of the concerned limits.

Also, due to the same reason, the rules of differentiation, such as the sum rule, product rule, quotient rule and function of function rule are the same as those in Real Calculus.

Hence, when a function $f(z)$ is known to be analytic, it can be differentiated in the ordinary manner as if z is a real variable.

Further since integration is the inverse operation of differentiation, $\int f(z)dz$ can be evaluated as in Real Calculus, using the usual formulas and rules of integration. However the arbitrary constant of integration need not be a real constant; it may be a complex constant.

Example 3.9 Find where each of the following functions ceases to be analytic.

(i) $\dfrac{z}{(z^2-1)}$; (ii) $\dfrac{z^2-4}{z^2+1}$; (iii) $\dfrac{z+i}{(z-i)^2}$; (iv) $z^3 - 4z - 1$;

(v) $\tan^2 z$.

(i) Let $$f(z)=\frac{z}{z^2-1}$$

$\therefore$ $$f'(z)=\frac{(z^2-1)\cdot 1-z\cdot 2z}{(z^2-1)^2}=\frac{-(z^2+1)}{(z^2-1)^2}$$

$f(z)$ is not analytic, where $f'(z)$ does not exist, i.e., where $f'(z) \to \infty$.

$f'(z) \to \infty$, if $(z^2 - 1)^2 = 0$, i.e., if $z = \pm 1$.

$\therefore$ $f(z)$ is not analytic at the points $z = \pm 1$.

(ii) Let $$f(z)=\frac{z^2-4}{z^2+1}$$

$\therefore$ $$f'(z)=\frac{(z^2+1)\cdot 2z-(z^2-4)\cdot 2z}{(z^2+1)^2}=\frac{10z}{(z^2+1)^2}$$

$\therefore$ $f(z)$ is not analytic where $z^2 + 1 = 0$

i.e. at the points $z = \pm i$.

(iii) Let $$f(z)=\frac{z+i}{(z-i)^2}$$

$\therefore$ $$f'(z)=\frac{(z-i)^2\cdot 1-(z+i)\cdot 2(z-i)}{(z-i)^4}$$

$$=\frac{-(z+3i)}{(z-i)^3}\to\infty,\text{ at } z=i$$

$\therefore$ $f(z)$ is not analytic at $z = i$.

(iv) Let $f(z) = z^3 - 4z - 1$

$\therefore$ $f'(z) = 3z^2 - 4$, that exists everywhere.

$\therefore$ $f(z)$ is analytic everywhere.

(v) Let $f(z) = \tan^2 z$

$$\therefore \quad f'(z) = 2\tan z \sec^2 z = \frac{2\sin z}{\cos^3 z}$$

$\therefore f'(z) \to \infty$, when $\cos^3 z = 0$, i.e., when $z = \dfrac{(2n-1)\pi}{2}$

$\therefore f(z)$ is not analytic at $z = \dfrac{(2n-1)\pi}{2}$; $n = 1, 2, 3, \ldots$

Example 3.10 Prove that every analytic function $w = u(x, y) + iv(x, y)$ can be expressed as a function of z alone.

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy$$

$$\therefore \quad x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

Hence u and v and also w may be considered as a function of z and $\bar{z}$

Consider
$$\begin{aligned}
\frac{\partial w}{\partial \bar{z}} &= \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} \\
&= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) \\
&= \left(\frac{1}{2}u_x - \frac{1}{2i}u_y\right) + i\left(\frac{1}{2}v_x - \frac{1}{2i}v_y\right) \\
&= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x) \\
&= 0, \text{ by C.R. equations, as } w \text{ is analytic.}
\end{aligned}$$

This means that w is independent of $\bar{z}$, i.e. w is a function of z alone.

This means that if $w = u(x, y) + iv(x, y)$ is analytic, it can be rewritten as a function of $(x + iy)$. Equivalently a function of $\bar{z}$ cannot be an analytic function of z.

Note ☑ Let the analytic function w be given by

$$w = u(x, y) + iv(x, y) \tag{1}$$

$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \tag{2}$$

In order to get w as a function of z alone, we may replace $\bar{z}$ by z in the R.H.S. of (2).

Then
$$w = u(z, 0) + iv(z, 0) \tag{3}$$

Equation (3) can be obtained from (1) by replacing x by z and y by 0.

Thus we get the following rule known as *Milne-Thomson rule.* If a function of x and y can be expressed as a function of $z = x + iy$, it can be done by simply replacing x by z and y by 0 in the given function.

Example 3.11 Show that $u + iv = \dfrac{x - iy}{x - iy + a}$ $(a \neq 0)$ is not an analytic function of z, whereas $u - iv$ is such a function at all points where $z \neq -a$.

Now $u + iv = \dfrac{\bar{z}}{\bar{z} + a}$ = a function of $\bar{z}$. Since a function of $\bar{z}$ cannot be analytic, $(u + iv)$ is not an analytic function of z.

$u - iv$ = conjugate of $(u + iv) = \dfrac{z}{z + a} = f(z)$, say.

$f(z)$ is a function of z alone and $f'(z) = \dfrac{a}{(z+a)^2}$, that exists everywhere except at $z = -a$.

$\therefore$ $f(z)$ is analytic, except at $z = -a$.

Example 3.12 Show that an analytic function with

(i) constant real part is a contant; and
(ii) constant modulus is a constant.

Let $f(z) = u + iv$ be the analytic function.

(i) Given $u = \text{constant} = c$, say

$\therefore$ $u_x = 0$ and $u_y = 0$.

By C.R. equations, $v_y = u_x = 0$ and
$v_x = -u_y = 0$

Since the partial derivatives of v with respect to both x and y are zero,

v is a constant $= c'$, say

$$\therefore \quad f(z) = c + ic'$$
$$= \text{constant.}$$

(ii) Given $|f(z)| = \sqrt{u^2 + v^2} = c$

$$\therefore \quad u^2 + v^2 = c^2 \qquad (1)$$

Differentiating (1) partially with respect to x and y, we get

$$2u\,u_x + 2v\,v_x = 0$$

and $$2u\,u_y + 2v\,v_y = 0$$

i.e., $$u\,u_x + v\,v_x = 0 \qquad (2)$$

and $$-u\,v_x + v\,u_x = 0 \qquad \text{(by C.R. equations)}$$

or $$v\,u_x - u\,v_x = 0 \tag{3}$$

(2) and (3) form a system of two homogeneous algebraic equations in the unknowns u_x and v_x.

The system possesses only a trivial solution, since $\begin{vmatrix} u & v \\ v & -u \end{vmatrix} = -(u^2 + v^2) \neq 0$

$\therefore$ Solution of equations (2) and (3) is $u_x = 0$ and $v_x = 0$

Now $$f'(z) = u_x + iv_x = 0$$

$\therefore$ $$f(z) = \text{a constant.}$$

EXERCISE 3(a)

Part A

(Short Answer Questions)

1. Explain briefly the concept of the limit of a function of a complex variable.
2. What is the basic difference between the limit of a function of a real variable and that of a complex variable.
3. Define the continuity of a function of a complex variable.
4. When is a function of a complex variable said to be differentiable at a point?
5. Define analytic function of a complex variable.
6. State the Cauchy-Riemann equations in Cartesian Coordinates satisfied by an analytic function.
7. State the sufficient conditions that will ensure the analyticity of a function
$$w = f(z) = u(x, y) + iv(x, y)$$
8. State the Cauchy-Riemann equations in polar coordinates satisfied by an analytic function.
9. If $w = u\,(x, y) + iv\,(x, y)$ is an analytic function of z, prove that
$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}.$$
10. If $w = u\,(r, \theta) + iv\,(r, \theta)$ is an analytic function of z, prove that $e^{i\theta}\dfrac{\mathrm{d}w}{\mathrm{d}z}$
$$= \frac{\partial w}{\partial r} = -\frac{i}{r}\frac{\partial w}{\partial \theta}.$$

Prove that the following functions are not analytic:

11 $f(z) = \bar{z}$;

12. $f(z) = \dfrac{x+iy}{x^2+y^2}$;

13. $f(z) = e^x(\cos y - i\sin y)$
14. $f(z) = \cos x \cosh y + i \sin x \sinh y$

15. $f(z) = \log\left(x^2 + y^2\right) + i\,2\cot^{-1}\left(\dfrac{y}{x}\right).$

Determine where the Cauchy-Riemann equations are satisfied for the following functions:

16. $x^2 + iy^2$;

17. $(x^3 - 3y^2x) + i(3x^2y - y^3)$;

18. $\dfrac{x}{x^2 + y^2} + i\dfrac{y}{x^2 + y^2}$;

19. $xy^2 + iyx^2$;

20. $\dfrac{z-1}{z+1}$.

21. Find the value of a, b, c, d so that the function $f(z) = (x^2 + axy + by^2) + i\,(cx^2 + dxy + y^2)$ may be analytic.
22. Determine p such that the function $f(z) = \dfrac{1}{2}\log\left(x^2 + y^2\right) + i\tan^{-1}\left(\dfrac{px}{y}\right)$ is analytic.
23. If $u + iv$ is analytic, show that $v - iu$ and $-v + iu$ are also analytic.
24. If $u + iv$ is analytic, show that $v + iu$ is not analytic.

Part B

25. Given that $f(z) = \dfrac{xy(x-y)}{x^2+y^2}$ and $f(0) = 0$, show that $\lim\limits_{z\to 0}\left[f(z)\right]$ exists.
26. Show that $\lim\limits_{z\to 0}\left[f(z)\right]$ does not exist, if $f(z) = \dfrac{xy}{x^2 + 4y^2}$ and $f(0) = 0$.
27. Show that $\lim\limits_{z\to 0}\left[\dfrac{xy^2}{x^2 + y^4}\right]$ does not exist, even though the function approaches the same limit along every straight line through the origin.
28. Prove that the function $f(z) = \dfrac{xy(y-ix)}{x^2+y^2}$, when $z \neq 0$ and $f(z) = 0$, when $z = 0$ is continuous at the origin.
29. Given that $f(z) = \dfrac{x^3 - y^3}{x^3 + y^3}$ and $f(0) = 0$, show that $f(z)$ is not continuous at $z = 0$.
30. If $f(z) = \dfrac{x^2 y}{x^4 + y^2}$, $z \neq 0$ and $f(0) = 0$, show that $f(z)$ is not continuous at $z = 0$.
31. Show that the function $f(z)$ is not continuous at $z = 0$, if $f(z) = \dfrac{xy}{2x^2 + y^2}$, $z \neq 0$ and $f(0) = 0$.
32. Show that the function $f(x, y)$ is discontinuous at $(0, 0)$, given that

$$f(x,y) = \frac{x^4\,y(y-x)}{\left(x^8 + y^2\right)(y+x)} \text{ and } f(0,\,0) = 0.$$

33. Show that the function $f(z)$ defined by

$$f(z)=\frac{xy(y-ix)}{x^2+y^2},\ z\neq 0 \qquad \text{and} \qquad f(0)=0$$

is not analytic at the origin, though it satisfies Cauchy-Riemann equations at the origin.

34. Show that function $f(z)$ defined by $f(z)=\dfrac{x^2y^3(x-iy)}{x^6+y^{10}},\ z\neq 0$ and $f(0)=0$

is not analytic at the origin, though it satisfies Cauchy-Riemann equations at the origin.

35. Prove that the function $f(z)$ defined by $f(z)=\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2},\ z\neq 0$ and

$f(0)=0$ is continuous at the origin. Prove also that the Cauchy-Riemann equations are satisfied by $f(z)$ at $z=0$ and yet $f'(z)$ does not exist at $z=0$.

Prove that the following functions are analytic and also find their derivatives using definition.

36. $f(z)=z^2$; 37. $f(z)=e^z$; 38. $f(z)=\cos z$; 39. $f(z)=\sinh z$.

40. If $f(z)$ and $\overline{f(z)}$ are both analytic, show that $f(z)$ is a constant.

3.3 PROPERTIES OF ANALYTIC FUNCTIONS

3.3.1 Definition

$\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$ is known as *Laplace equation* in two dimensions.

$\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$ is called *the Laplacian operator* and is denoted as ∇^2.

Using this operator, the Laplace equation is usually written as $\nabla^2\phi=0$.
It is recalled, from Vector Calculus, that

$$\nabla\equiv\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z} \text{ and hence}$$

$$\nabla^2=\nabla\cdot\nabla=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \text{ in three dimensions.}$$

The Laplace equation in polar co-ordinates is defined as $\dfrac{\partial^2\phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}=0.$

Property 1

The real and imaginary parts of an analytic function $w=u+iv$ satisfy the Laplace equation in two dimensions, viz. $\nabla^2u=0$ and $\nabla^2v=0$.

Since $w = u + iv$ is analytic in some region of the z-plane, u and v satisfy Cauchy-Riemann equations.

i.e. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

Differentiating both sides of (1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{3}$$

Differentiating both sides (2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{4}$$

The second order mixed partial derivatives $\frac{\partial^2 v}{\partial x \partial y}$ and $\frac{\partial^2 v}{\partial y \partial x}$ are equal, when they are continuous.

Assuming the continuity of $\frac{\partial^2 v}{\partial y \partial x}$ and $\frac{\partial^2 v}{\partial y \partial x}$ and adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

i.e. u satisfies Laplace equation or u is a solution of the Laplace equation $\nabla^2 \phi = 0$. Similarly, differentiating (1) partially with respect to y and (2) partially with respect to x and adding $\left(\text{with the assumption of continuity of } \frac{\partial^2 u}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y \partial x}\right)$, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

i.e. v also satisfies Laplace equation or v is also a solution of the Laplace equation $\nabla^2 \phi = 0$.

Note ☑ A real function of two real variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called a *harmonic* function.

If we assume the continuity of the second order partial derivatives of u and v, the above property means that when $w = u + iv$ is analytic, u and v are harmonic. Conversely, when u and v are any two harmonic functions, chosen at random, $u + iv$ need not be analytic.

If u and v are harmonic functions such that $u + iv$ is analytic, then each is called *the conjugate harmonic function* of the other.

Property 2

The real and imaginary parts of an analytic function $w = u(r, \theta) + iv(r, \theta)$ satisfy the Laplace equation in polar coordinates.

Since $w = u(r, \theta) + iv(r, \theta)$ is analytic in some region of the z-plane, u and v satisfy Cauchy-Riemann equations in polar coordinates.

i.e. $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{1}$$

and $$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \tag{2}$$

or $$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \tag{3}$$

Differentiating (1) partially with respect to r,

we get $$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial^2 v}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial v}{\partial \theta} \tag{4}$$

Differentiating (3) partially with respect to θ,

we get $$\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = -\frac{\partial^2 v}{\partial \theta\,\partial r} \tag{5}$$

Using (1), (4) and (5), we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\left(\frac{\partial^2 v}{\partial r\,\partial \theta} - \frac{\partial^2 v}{\partial \theta\,\partial r}\right) = 0,$$

assuming the continuity of the mixed derivatives.

Thus u satisfies Laplace equation in polar coordinates.

Similarly we can prove that v also satisfies Laplace equation in polar co-ordinates.

Property 3

If $w = u(x, y) + iv(x, y)$ is an analytic function, the curves of the family $u(x, y) = a$ and the curves of the family $v(x, y) = b$, cut orthogonally, where a and b are varying constants.

Consider a representative member of the family $u(x, y) = a$, corresponding to $a = a_1$.

i.e. $$u(x, y) = a_1$$

Taking differentials on both sides, we get

$$\mathrm{d}u = 0$$

i.e. $$\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y = 0$$

$\therefore$ $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = m_1,$$ say, where m_1 is the slope of the curve $u(x, y) = a_1$ at (x, y).

Similarly, considering a typical member of the second family whose equation is $v(x, y) = b_1$, we can get

$$m_2 = \frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$, where m_2 is the slope of the curve $v(x, y) = b_1$ at (x, y).

Now
$$m_1 m_2 = \frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} \cdot \frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$

$$= \frac{\left(\frac{\partial v}{\partial y}\right)}{-\left(\frac{\partial v}{\partial x}\right)} \cdot \frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$, by C.R. equations, since $(u + iv)$ is analytic.

$$= -1$$

This is true at the point of intersection of the two curves $u(x, y) = a_1$ and $v(x, y) = b_1$ also.

Thus a typical member of the family $u(x, y) = a$ cuts orthogonally a typical member of the family $v(x, y) = b$.

$\therefore$ Every member of the family $u(x, y) = a$ cuts orthogonally every member of the family $v(x, y) = b$.

Note ☑ The two families are said to be *orthogonal trajectories* of each other.

Property 4

If $w = u(r, \theta) + iv(r, \theta)$ is an analytic function, the curves of the family $u(r, \theta) = a$ cut orthogonally the curves of the family $v(r, \theta) = b$, where a and b are arbitrary constants.

Proceeding as in property (3), we get

$$\left(\frac{d\theta}{dr}\right)_1 = -\frac{\left(\frac{\partial u}{\partial r}\right)}{\left(\frac{\partial u}{\partial \theta}\right)} \tag{1}$$

and
$$\left(\frac{d\theta}{dr}\right)_2 = -\frac{\left(\frac{\partial v}{\partial r}\right)}{\left(\frac{\partial v}{\partial \theta}\right)} \tag{2}$$

In polar coordinates the condition for orthogonality of two curves is

$$\left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)_1 \cdot \left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)_2 = -1\,.$$

From (1) and (2), we have

$$\left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)_1 \cdot \left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)_2 = \frac{r\dfrac{\partial u}{\partial r}}{\dfrac{\partial u}{\partial \theta}} \cdot \frac{r\dfrac{\partial v}{\partial r}}{\dfrac{\partial v}{\partial \theta}}$$

$$= \frac{r\cdot\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}}{\dfrac{\partial u}{\partial \theta}} \cdot \frac{r\left(-\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\right)}{\dfrac{\partial v}{\partial \theta}},$$

(by C.R. equations in polar coordinates.)

$$= -1.$$

Hence the property follows.

3.3.2 Construction of an Analytic Function, When Its Real or Imaginary Part is Known

Method 1

Let $u\,(x, y)$, the real part of the analytic function $f(z) = u\,(x, y) + iv\,(x, y)$ be known. In this method, we first find $v\,(x, y)$ and then find $u\,(x, y) + iv\,(x, y)$ as a function of z. Since $u\,(x, y)$ is given, $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ can be found out.

$$\mathrm{d}v = \frac{\partial v}{\partial x}\,\mathrm{d}x + \frac{\partial v}{\partial y}\,\mathrm{d}y$$

$$= \left(-\frac{\partial u}{\partial y}\right)\mathrm{d}x + \left(\frac{\partial u}{\partial x}\right)\mathrm{d}y \tag{1}$$

The expression $(M\mathrm{d}x + N\mathrm{d}y)$ is an exact differential if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

This condition for exactness is satisfied by the R.H.S. expression of (1), as

$$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

i.e., $$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which is true by property (1) discussed above.

$\therefore$ R.H.S. of (1) is an exact differential. Now, integrating both sides of (1), we get

$v = \int\left[\left(-\frac{\partial u}{\partial y}\right)\mathrm{d}x + \left(\frac{\partial u}{\partial x}\right)\mathrm{d}y\right] + c$, where c is an arbitrary (real) constant of integration.

Then $f(z) = u(x, y) + iv(x, y)$ is found out by using Milne-Thomson rule.

Note ☑ If $v(x, y)$ is given, we can find $u(x, y)$ first by a similar procedure and then find $f(z)$.

Method 2 (Milne-Thomson method)

Let $u(x, y)$ be the real part of the analytic function $f(z) = u(x, y) + iv(x, y)$. In this method, we first find $f'(z)$ as a function of z and then find $f(z)$ by ordinary integration. The imaginary part of $f(z)$ gives $v(x, y)$.

Since $u(x, y)$ is given, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ can be found out.

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \qquad [\because f(z) \text{ is analytic}]$$

$$= \frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}$$

$$= u_x(x, y) - iu_y(x, y)$$

$$= u_x(z, 0) - iu_y(z, 0), \text{ by Milne-Thomson rule.}$$

$\therefore\ f(z) = \int\left[u_x(z, 0) - iu_y(z, 0)\right]\mathrm{d}z + c$, where c is an arbitrary (imaginary) contant of integration.

Separating the real and imaginary parts of $f(z)$, we can find $v(x, y)$.

Note ☑

1. The real part of $f(z)$ obtained should be identical to the given $u(x, y)$.
2. If $v(x, y)$ is given, we can first find

$$f(z) = \int\left[v_y(z, 0) + iv_x(z, 0)\right]\mathrm{d}z + c,$$

by a similar procedure and then find $u(x, y)$ by separation of $f(z)$.

3.3.3 Applications

Properties (1) and (3) of analytic functions discussed above provide solutions to a number of flow and field problems.

If we consider two dimensional steady flow such as fluid flow, electric current flow and heat flow, the paths of fluid particles are called *stream lines* and their orthogonal trajectories are called *equipotential lines.* In the study of two dimensional irrotational motion of an incompressible fluid in planes parallel to the xy-plane, if $\overline{v}$ represents the velocity of a fluid particle, we can find a function $\phi(x, y)$ such that

$$\bar{v} = \frac{\partial \phi}{\partial x}\bar{i} + \frac{\partial \phi}{\partial y}\bar{j}$$

The function $\phi(x, y)$ which gives the velocity components is called the *velocity potential.* The function $\psi(x, y)$, which is such that $\phi(x, y) + i\psi(x, y)$ is analytic, is called the *stream function.*

The function $w = f(z) = \phi(x, y) + i\psi(x, y)$, which represents the flow pattern is called the *Complex potential.*

The curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are called *equipotential lines* and *stream lines* respectively.

In the study of electrostatics and gravitational fields, the curves $\phi(x, y) = a$ and $\Psi(x, y) = b$ are respectively called *equipotential lines* and *lines of force.*

In heat flow problems, the curves $\phi(x, y) = a$ and $\Psi(x, y) = b$ are respectively called *isothermals* and *heat flow lines.*

WORKED EXAMPLE 3(b)

Example 3.1 Prove that the following function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Also find the conjugate harmonic function v and the corresponding analytic function $(u + iv)$

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$u_x = 3x^2 - 3y^2 + 6x;\ u_{xx} = 6x + 6$$

$$u_y = -6xy - 6y\ ;\ u_{yy} = -6x - 6$$

$\therefore$ $$u_{xx} + u_{yy} = 0 \text{ and so } u \text{ is a harmonic function.}$$

Since v is the conjugate harmonic of u, $u + iv$ is analytic.

$\therefore$ By C.R. equations, $u_x = v_y$ and $u_y = -v_x$

Now $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= (6xy + 6y)\,dx + (3x^2 - 3y^2 + 6x)\,dy$$

$\therefore$ $$v = \int\left[(6xy + 6y)\,dx + (3x^2 - 3y^2 + 6x)\,dy\right] + c$$

$$= \int (M\,dx + N\,dy) + c, \text{ say.}$$

To evaluate the integral in the R.H.S., we integrate all the terms in M partially with respect to x, also integrate only those terms in N not containing x and add them.

Thus $$v = 3x^2y + 6xy - y^3 + c$$

Let $\quad w = u + iv$

$= (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + \text{i}\,(3x^2y + 6xy - y^3 + c)$

$= z^3 + 3z^2 + 1 + ic$, by Milne-Thomson rule.

Example 3.2 In a two dimensional fluid flow, find if $xy(x^2 - y^2)$ can represent the stream function. If so, find the corresponding velocity potential and also the complex potential.

If $\psi = xy(x^2 - y^2)$ represents the stream function, it should be the imaginary part of an analytic function and hence harmonic.

$$\psi = x^3y - xy^3$$

$$\psi_x = 3x^2y - y^3; \qquad \psi_{xx} = 6xy$$

$$\psi_y = x^3 - 3xy^2; \qquad \psi_{yy} = -6xy$$

$\therefore\ \psi_{xx} + \psi_{yy} = 0.$

i.e., ψ is a harmonic function.

$\therefore\ \psi$ can represent the stream function. Let ϕ be the corresponding velocity potential. Then $\phi + i\psi$ is analytic.

$\therefore \qquad \phi_x = \psi_y \quad \text{and} \quad \phi_y = -\psi_x \qquad$ (by C.R. equations)

Now
$$\text{d}\phi = \frac{\partial \phi}{\partial x}\text{d}x + \frac{\partial \phi}{\partial y}\text{d}y$$

$$= \frac{\partial \psi}{\partial y}\text{d}x - \frac{\partial \psi}{\partial x}\text{d}y$$

$$= (x^3 - 3xy^2)\,\text{d}x - (3x^2y - y^3)\,\text{d}y$$

$\therefore$
$$\phi = \int\left[(x^3 - 3xy^2)\,\text{d}x - (3x^2y - y^3)\,\text{d}y\right]$$

$$= \frac{x^4}{4} - \frac{3}{2}x^2y^2 + \frac{y^4}{4} + c$$

If $f(z)$ represents the complex potential,

$$f(z) = \phi + i\psi$$

$$= \left(\frac{x^4}{4} - \frac{3}{2}x^2y^2 + \frac{y^4}{4} + c\right) + i(x^3y - xy^3)$$

$$= \frac{1}{4}z^4 + c \qquad \text{(by Milne-Thomson rule)}$$

Example 3.3 Find if $\phi = (x - y)(x^2 + 4xy + y^2)$ can represent the equipotential for an electric field. Find the corresponding complex potential $w = \phi + i\psi$ and also ψ, if possible.

If ϕ represents the equipotential for an electric field, it should be the real part of an analytic function and hence harmonic.

$$\phi = (x - y)(x^2 + 4xy + y^2)$$
$$\phi_x = (x - y)(2x + 4y) + (x^2 + 4xy + y^2)$$
$$\phi_{xx} = 4x + 2y + 2x + 4y = 6x + 6y$$
$$\phi_y = (x - y)(4x + 2y) + (x^2 + 4xy + y^2)(-1)$$
$$\phi_{yy} = -4y - 2x - 4x - 2y = -6x - 6y$$
$$\therefore \quad \phi_{xx} + \phi_{yy} = 0$$

i.e., ϕ is a harmonic function.

$\therefore$ ϕ can represent the equipotential of an electric field.

The corresponding complex potential

$$w = \phi + i\,\psi \text{ is analytic}$$

$$\therefore \quad \frac{dw}{dz} = \phi_x + i\psi_x$$
$$= \phi_x - i\phi_y$$
$$= [(x - y)(2x + 4y) + (x^2 + 4xy + y^2)]$$
$$- i[(x - y)(4x + 2y) - (x^2 + 4xy + y^2)]$$
$$= 3z^2 - i\cdot 3z^2$$
$$\therefore \quad w = \int 3(1-i)z^2 dz + ic$$
$$= (1-i)z^3 + ic$$

Now
$$w = \phi + i\psi = (1 - i)(x + iy)^3 + ic$$
$$= (1 - i)(x^3 + 3ix^2y - 3xy^2 - iy^3) + ic$$
$$= (x^3 - 3xy^2 + 3x^2y - y^3) + i(3x^2y - y^3 - x^3 + 3xy^2 + c)$$
$$\therefore \quad \psi = 3(x^2y + xy^2) - (x^3 + y^3) + c.$$

Note ☑ The value of ϕ obtained from w is the same as the given value of ϕ.

Example 3.4 Prove that $v = \log[(x-1)^2 + (y-2)^2]$ is harmonic in every region which does not include the point (1, 2). Find the corresponding analytic function $w = u + iv$ and also u.

$$v = \log[(x - 1)^2 + (y - 2)^2]$$

$$v_x = \frac{2(x-1)}{(x-1)^2 + (y-2)^2}; \qquad v_{xx} = \frac{\left[(x-1)^2 + (y-2)^2\right].2 - 4(x-1)^2}{\left[(x-1)^2 + (y-2)^2\right]^2}$$

$$v_y = \frac{2(y-2)}{(x-1)^2+(y-2)^2}; \quad v_{yy} = \frac{\left[(x-1)^2+(y-2)\right]^2 \cdot 2 - 4(y-2)}{\left[(x-1)^2+(y-2)^2\right]^2}$$

v_x, v_y, v_{xx} and v_{yy} do not exist at the point (1, 2).
But in every region not containing (1, 2),

$$\begin{aligned} v_{xx}+v_{yy} &= \frac{2\left\{(y-2)^2-(x-1)^2\right\}}{\left[(x-1)^2+(y-2)^2\right]^2} + \frac{2\left\{(x-1)^2-(y-2)^2\right\}}{\left[(x-1)^2+(y-2)^2\right]^2} \\ &= 0 \end{aligned}$$

$\therefore$ v is harmonic in every region not containing the point (1, 2).
$w = u + iv$ is the corresponding analytic function.

$$\therefore \quad \begin{aligned} \frac{dw}{dz} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \text{(by C.R. equations)} \\ &= \frac{2(y-2)+i2(x-1)}{(x-1)^2+(y-2)^2} \\ &= \frac{-4+i2(z-1)}{(z-1)^2+4}, \qquad \text{(by Milne-Thomson rule.)} \\ &= \frac{2i\{(z-1)+2i\}}{(z-1+2i)\,(z-1-2i)} \\ &= \frac{2i}{z-1-2i} \end{aligned}$$

$$\therefore \quad \begin{aligned} w &= \int \frac{2i}{z-1-2i}\,dz + c \\ &= 2i\log(z-1-2i)+c. \end{aligned}$$

i.e.

$$\begin{aligned} u+iv &= 2i\log\{(x-1)+i(y-2)\}+c \\ &= 2i\left[\frac{1}{2}\log\left\{(x-1)^2+(y-2)^2\right\} + i\tan^{-1}\left(\frac{y-2}{x-1}\right)\right]+c \\ &= \left[-2\tan^{-1}\left(\frac{y-2}{x-1}\right)+c\right] + i\log\left\{(x-1)^2+(y-2)^2\right\} \end{aligned}$$

$\therefore \qquad u = -2\tan^{-1}\left(\dfrac{y-2}{x-1}\right) + c$, where c is a real constant. Also we note that v is the same as the given value.

Example 3.5 Find v such that $w = u + iv$ is an analytic function of z, given that $u = e^{x^2-y^2} \cdot \cos 2xy$. Hence find w.

$$u = e^{(x^2-y^2)} \cdot \cos 2xy$$

$$u_x = e^{(x^2-y^2)}\,(-2y\sin 2xy) + 2x\,e^{(x^2-y^2)} \cdot \cos 2xy$$

$$u_y = e^{(x^2-y^2)}\,(-2x\sin 2xy) - 2y\,e^{(x^2-y^2)} \cdot \cos 2xy$$

$$\begin{aligned} \mathrm{d}v &= (v_x\,\mathrm{d}x + v_y\,\mathrm{d}y) \\ &= -u_y\,\mathrm{d}x + u_x\,\mathrm{d}y \end{aligned}$$

$$\therefore \qquad v = \int \left[e^{(x^2-y^2)} \cdot 2x\sin 2xy + e^{(x^2-y^2)} \cdot 2y\cos 2xy\right]\mathrm{d}x + \left[-e^{(x^2-y^2)} \cdot 2y\sin 2xy + e^{(x^2-y^2)} \cdot 2x\cos 2xy\right]\mathrm{d}y + c$$

To evaluate $\int[M\mathrm{d}x + N\mathrm{d}y]$, where $M\mathrm{d}x + N\mathrm{d}y$ is an exact differential in which N does not contain any term independent of x, it is enough to evaluate just $\int M\mathrm{d}x$ treating y as a constant.

$$\therefore \qquad v = \int e^{(x^2-y^2)}\,2x\sin 2xy\,\mathrm{d}x + \int e^{(x^2-y^2)} \cdot 2y\cos 2xy\,\mathrm{d}x + c$$

$$= \int e^{(x^2-y^2)} \cdot 2x\sin 2xy\,\mathrm{d}x + \int e^{(x^2-y^2)} \cdot \mathrm{d}\{\sin 2xy\} + c,$$

treating y as a constant.

$$= \int e^{(x^2-y^2)} \cdot 2x\sin 2xy\,\mathrm{d}x + e^{(x^2-y^2)}\sin 2xy - \int \sin 2xy \cdot e^{(x^2-y^2)} \cdot 2x\mathrm{d}x + c,$$

integrating by parts.

$$= e^{(x^2-y^2)}\sin 2xy + c$$

Now $\qquad w = u + iv$

$$= e^{(x^2-y^2)}\cos 2xy + i\,\{e^{(x^2-y^2)}\sin 2xy + c\}$$

$$= e^{z^2} + c$$

(by Milne-Thomson rule.)

Example 3.6 Find the analytic function $w = u + iv$,

if $\quad v = e^{2x}(x\cos 2y - y\sin 2y)$. Hence find u.

$$v = e^{2x}(x\cos 2y - y\sin 2y)$$

$$v_x = e^{2x}\,.\cos 2y + 2e^{2x}(x\cos 2y - y\sin 2y)$$

$$v_y = e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y)$$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u_x + iv_x = v_y + iv_x, \text{ (by C.R. equations.)}$$

$$= 0 + i[2ze^{2z} + e^{2z}], \text{ (by Milne-Thomson rule.)}$$

$\therefore$ $$w = i\int\left(2ze^{2z} + e^{2z}\right)\mathrm{d}z + c, \text{ where } c \text{ is real}$$

$$= i\left[\left(2z\cdot\frac{e^{2z}}{2} - 2\cdot\frac{e^{2z}}{4}\right) + \frac{e^{2z}}{2}\right] + c, \text{ by Bernoulli's formula}$$

$$= iz\,e^{2z} + c$$

i.e., $$u + iv = i(x + iy)\,e^{2(x+iy)} + c$$

$$= (ix - y)\,e^{2x}(\cos 2y + i\sin 2y) + c$$

$$= [-(x\sin 2y + y\cos 2y)\,e^{2x} + c] + i(x\cos 2y - y\sin 2y)\,e^{2x}$$

$\therefore$ $$u = -(x\sin 2y + y\cos 2y)\,e^{2x} + c.$$

Example 3.7 Determine the analytic function $f(z) = u + iv$, given that $3u + 2v = y^2 - x^2 + 16xy$.

$$3u + 2v = y^2 - x^2 + 16xy$$

$\therefore$ $$3u_x + 2v_x = -2x + 16y \qquad (1)$$

and $$3u_y + 2v_y = 2y + 16x$$

i.e., $$2u_x - 3v_x = 2y + 16x \qquad (2)$$

(by C.R. equations.)

Solving (1) and (2), we get

$$u_x = 2x + 4y \quad \text{and} \quad vx = -4x + 2y$$

$$f'(z) = u_x + iv_x$$

$$= (2x + 4y) + i(-4x + 2y)$$

$$= 2z - i4z, \qquad \text{(by Milne-Thomson rule.)}$$

$\therefore$ $$f(z) = \int(2z - i4z)\,\mathrm{d}z + c_1 + ic_2$$

$$= z^2 - 2iz^2 + c_1 + ic_2$$

$$= (1 - 2i)z^2 + c_1 + ic_2$$

Since $3u + 2v$ does not contain any constant,

$$3c_1 + 2c_2 = 0 \quad \therefore c_2 = -\frac{3}{2}c_1$$

$\therefore f(z) = (1-2i)z^2 + \left(1 - \frac{3}{2}i\right)c_1$, where c_1 is a real constant.

Example 3.8 Determine the analytic function $f(z) = P + iQ$, given that $P - Q = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ and $f(\pi/2) = 0$.

$$f(z) = P + iQ$$

$\therefore$ $$if(z) = -Q + iP$$

$\therefore$ $$(1+i)f(z) = (P-Q) + i(P+Q)$$

i.e., $$\phi(z) = u + iv, \text{ say.}$$

Thus, if we construct the analytic function $\phi(z)$ with $u = P - Q$ as the real part,

then $$f(z) = \frac{1}{1+i}\phi(z).$$

$$u = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$$

$\therefore$ $$u_x = \frac{1}{2}\left[\frac{(\cos x - \cosh y)\cdot(\cos x - \sin x) + \sin x\cdot(\cos x + \sin x - e^{-y})}{(\cos x - \cosh y)^2}\right]$$

$$u_y = \frac{1}{2}\left[\frac{(\cos x - \cosh y)e^{-y} + \sinh y\,(\cos x + \sin x - e^{-y})}{(\cos x - \cosh y)^2}\right]$$

Now $$\phi'(z) = u_x + iv_x$$

$$= u_x - iu_y, \qquad \text{(by C.R. equations)}$$

$$= \frac{(\cos z - 1)(\cos z - \sin z) + \sin z\,(\cos z + \sin z - 1) - i(\cos z - 1)}{2(\cos z - 1)^2}$$

$$= \frac{(1+i)(1-\cos z)}{2(1-\cos z)^2} = \frac{1+i}{2(1-\cos z)} = \left(\frac{1+i}{4}\right)\operatorname{cosec}^2\left(\frac{z}{2}\right)$$

$\therefore$ $$\phi(z) = \left(\frac{1+i}{4}\right)\int \operatorname{cosec}^2\left(\frac{z}{2}\right) + dz + c$$

$$= -\left(\frac{1+i}{4}\right)2\cot\left(\frac{z}{2}\right) + c$$

i.e., $$f(z) = \frac{1}{2}\cot\frac{z}{2} + c'$$

Since $$f\left(\frac{\pi}{2}\right)=0, \quad c'-\frac{1}{2}\cot\frac{\pi}{4}=0 \qquad \therefore\ c'=\frac{1}{2}$$

$\therefore$ $$f(z)=\frac{1}{2}\left(1-\cot\frac{z}{2}\right)$$

Example 3.9 Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally, when $w = u + iv = z^3$.

$$u + iv = z^3 = (x + iy)^3$$
$$= x^3 + 3ix^2y - 3xy^2 - iy^3$$

$\therefore$ $$u = x^3 - 3xy^2 \quad \text{and} \quad v = 3x^2y - y^3$$

Consider the family of curves $u = c_1$

i.e. $$x^3 - 3xy^2 = c_1 \qquad (1)$$

Differentiating (1) with respect to x, we get

$$3x^2-3\left(y^2+2xy\frac{dy}{dx}\right)=0$$

$\therefore$ $$\frac{dy}{dx}=m_1=\frac{x^2-y^2}{2xy}$$

Consider the family of curves $v = c_2$.

i.e. $$3x^2y - y^3 = c_2 \qquad (2)$$

Differentiating (2) with respect to x, we get

$$3\left(2xy+x^2\frac{dy}{dx}\right)-3y^2\frac{dy}{dx}=0$$

$\therefore$ $$\frac{dy}{dx}=m_2=-\frac{2xy}{x^2-y^2}$$

$$m_1m_2 = -1$$

$\therefore$ The families of curves $u = c_1$ and $v = c_2$ cut orthogonally.

Example 3.10 If $u = x^2-y^2$ and $v=-\dfrac{y}{x^2+y^2}$ prove that both u and v satisfy Laplace equations, but that $(u + iv)$ is not a regular function of z.

$$u = x^2 - y^2$$

$\therefore$ $$u_x = 2x; \quad u_{xx} = 2; \quad u_y = -2y; \quad u_{yy} = -2$$

$\therefore$ $$u_{xx} + u_{yy} = 0$$

i.e., u satisfies Laplace equation.

$$v = -\frac{y}{x^2+y^2}$$

$$v_x = \frac{2xy}{\left(x^2+y^2\right)^2}; \qquad v_{xx} = 2y\left[\frac{\left(x^2+y^2\right)\cdot 1 - x\cdot 2\left(x^2+y^2\right)\cdot 2x}{\left(x^2+y^2\right)^4}\right]$$

$$= \frac{2y\left(y^2-3x^2\right)}{\left(x^2+y^2\right)^3}$$

$$v_y = -\left[\frac{\left(x^2+y^2\right)\cdot 1 - 2y^2}{\left(x^2+y^2\right)^2}\right] = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$v_{yy} = \frac{\left(x^2+y^2\right)^2\cdot 2y - \left(y^2-x^2\right)\cdot 2\left(x^2+y^2\right)\cdot 2y}{\left(x^2+y^2\right)^4}$$

$$= \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$\therefore \quad v_{xx} + v_{yy} = 0$

i.e. v satisfies Laplace equation.

Now $\quad u_x \neq v_y \quad$ and $\quad u_y \neq -v_x$

i.e. C.R. equations are not satisfied by u and v.

Hence $u + iv$ is not a regular (analytic) function of z.

Note ☑ The reason for the above situation is that u and v are not the real and imaginary parts of the same analytic function. In fact, u is the real part of z^2 and v is the imaginary part of $\frac{1}{z}$.

Example 3.11 If $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is an analytic function of $z = x + iy$.

u and v are harmonic functions.

$\therefore u_{xx}, u_{xy}, u_{yy}$ and v_{xx}, v_{xy}, v_{yy} are all continuous (and hence exist) (1)

and $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ (2)

Consider $P + iQ = (u_y - v_x) + i(u_x + v_y)$

$$P_x = u_{yx} - v_{xx} \quad \text{and} \quad P_y = u_{yy} - v_{xy}$$

$$Q_x = u_{xx} + v_{yx} \quad \text{and} \quad Q_y = u_{xy} + v_{yy}.$$

The four partial derivatives P_x, P_y, Q_x and Q_y exist and are continuous in the region R [by (1)].

$$P_x = Q_y, \text{ if } u_{yx} - v_{xx} = u_{xy} + v_{yy}$$

i.e. if $v_{xx} + v_{yy} + (u_{xy} - u_{yx}) = 0$

i.e. if $v_{xx} + v_{yy} = 0$ [by (1)],

which is true, by (2).

$$P_y = -Q_x, \text{ if } u_{yy} - v_{xy} = -(u_{xx} + v_{yx})$$

i.e. if $u_{xx} + u_{yy} - (v_{xy} - v_{yx}) = 0$

i.e. if $u_{xx} + u_{yy} = 0$ [by (1)],

which is true, by (2).

Thus the C.R. equations are satisfied by P and Q (3)

$\therefore$ By (1) and (3), $P + iQ$ is analytic.

Example 3.12 Show that the families of curves $r^n = \alpha \sec n\theta$ and $r^n = \beta \operatorname{cosec} n\theta$ intersect orthogonally, where α and β are arbitrary constants.

The given equations can be rewritten as $r^n \cos n\theta = \alpha$ and $r^n \sin n\theta = \beta$.

i.e. $u(r, \theta) = \alpha$ and $v(r, \theta) = \beta$, say.

Now
$$\begin{aligned} u(r, \theta) + iv(r, \theta) &= r^n (\cos n\theta + i \sin n\theta) \\ &= r^n e^{in}\theta \\ &= (re^{i\theta})^n \text{ or } z^n, \text{ which is an analytic function.} \end{aligned}$$

$\therefore$ By property (4), the families of curves $u(r, \theta) = \alpha$ and $v(r, \theta) = \beta$ cut orthogonally.

Example 3.13 If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

Let $f(z) = u+iv$

$\therefore$ $|f(z)|^2 = u^2 + v^2$ and $|f'(z)|^2 = u_x^2 + v_x^2$

Since $f(z)$ is a regular (analytic) function,

$$u_x = v_y \text{ and } u_y = -v_x \text{ (C.R. equations)} \tag{1}$$

and
$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0 \text{ (Laplace equations)} \tag{2}$$

Now
$$\frac{\partial}{\partial x}(u^2) = 2u\,u_x$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2(u\,u_{xx} + u_x^2)$$

Similarly,
$$\frac{\partial^2}{\partial y^2}(u^2) = 2(u\,u_{yy} + u_y^2),$$

$$\frac{\partial^2}{\partial x^2}(v^2) = 2(vv_{xx} + v_x^2) \quad \text{and} \quad \frac{\partial^2}{\partial y^2}(v^2) = 2(vv_{yy} + v_y^2)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v^2$$

$$= 2u\,(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2) + 2v(v_{xx} + v_{yy}) + 2(v_x^2 + v_y^2)$$

$$= 2(u_x^2 + u_y^2 + v_x^2 + v_y^2), \text{ by} \qquad (2)$$

$$= 2(u_x^2 + v_x^2 + v_x^2 + u_x^2), \text{ by} \qquad (1)$$

$$= 4(u_x^2 + v_x^2)$$

$$= 4|f'(z)|^2.$$

Example 3.1 If $f(z) = u + iv$ is a regular function of z, prove that $\nabla^2\{\log |f(z)|\} = 0.$

$f(z) = (u + iv)$ is analytic.

$$\therefore \quad u_x = v_y \quad \text{and} \quad u_y = -v_x \text{ (C.R. equations)} \qquad (1)$$

$$\text{and} \quad u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0 \text{ (Laplace equations)}$$

$$\log|f(z)| = \frac{1}{2}\log(u^2 + v^2) \qquad (2)$$

$$\therefore \quad \frac{\partial}{\partial x}\log|f(z)| = \frac{1}{2}\cdot\left(\frac{2uu_x + 2v\cdot v_x}{u^2 + v^2}\right) = \frac{uu_x + v\,v_x}{u^2 + v^2}$$

$$\frac{\partial^2}{\partial x^2}\log|f(z)| = \frac{\left[\left(u^2 + v^2\right)\left\{uu_{xx} + u_x^2 + vv_{xx} + v_x^2\right\} - (uu_x + vv_x)(2uu_x + 2vv_x)\right]}{\left(u^2 + v^2\right)^2}$$

$$= \frac{1}{u^2 + v^2}\{u\,u_{xx} + v\,v_{xx} + u_x^2 + v_x^2\} - \frac{2}{\left(u^2 + v^2\right)^2}(u\,u_x + v\,v_x)^2 \qquad (3)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\log|f(z)|| = \frac{1}{u^2 + v^2}\{u\,u_{yy} + vv_{yy} + u_y^2 + v_y^2\} - \frac{2}{\left(u^2 + v^2\right)^2}(u\,u_y + v\,v_y)^2 \qquad (4)$$

Adding (3) and (4), we get

$$\nabla^2\left\{\log|f(z)|\right\} = \frac{1}{u^2 + v^2}[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2]$$

$$- \frac{2}{\left(u^2 + v^2\right)^2}[(u\,u_x + v\,v_x)^2 + (u\,u_y + v\,v_y)^2]$$

$$= \frac{1}{u^2 + v^2}[2(u_x^2 + v_x^2)] - \frac{2}{\left(u^2 + v^2\right)^2}[(u\,u_x + v\,v_x)^2 + (-u\,v_x + v\,u_x)^2],$$

by (1) and (2).

$$= \frac{2\left(u_x^2 + v_x^2\right)}{u^2 + v^2} - \frac{2}{\left(u^2 + v^2\right)^2} \{u^2 \left(u_x^2 + v_x^2\right) + v^2 \left(u_x^2 + v_x^2\right)\}$$

$$= \frac{2\left(u_x^2 + v_x^2\right)}{u^2 + v^2} - \frac{2\left(u^2 + v^2\right)\left(u_x^2 + v_x^2\right)}{\left(u^2 + v^2\right)^2}$$

$$= 0$$

Example 3.15 Find the equation of the orthogonal trajectories of the family of curves given by $3x^2y + 2x^2 - y^3 - 2y^2 = a$, where a is an arbitrary constant.

If $w = u + iv$ is analytic, the families of curves $u = a$ and $v = b$ are known as the orthogonal trajectories of each other. The given family can be assumed as $u = a$. We have to find the equation $v = b$, such that $u + iv$ is analytic.

$$u = 3x^2y + 2x^2 - y^3 - 2\,y^2$$

$$u_x = 6_{xy} + 4x\,; \quad u_y = 3x^2 - 3y^2 - 4y$$

$$\mathrm{d}v = v_x\,\mathrm{d}x + v_y\,\mathrm{d}y$$

$$= - u_y\,\mathrm{d}x + u_x\,\mathrm{d}y, \qquad \text{(by C.R. equations)}$$

$$= (3y^2 + 4y - 3x^2)\,\mathrm{d}x + (6xy + 4x)\,\mathrm{d}y$$

$$\therefore \qquad v = \int\left[\left(3y^2 + 4y - 3x^2\right)\mathrm{d}x + \left(6xy + 4x\right)\mathrm{d}y\right]$$

$$= 3xy^2 + 4xy - x^3$$

$\therefore$ The required equation of the orthogonal trajectories is $3xy^2 + 4xy - x^3 = b$, where b is an arbitrary constant.

EXERCISE 3(b)

Part A

(Short Answer Questions)

1. State any two properties of an analytic function.
2. Define a harmonic function and give an example.
3. How are analytic function and harmonic function related?
4. Write down the Laplace equations in two-dimensional cartesian and polar coordinates.
5. What do you mean by conjugate harmonic function? Find the conjugate harmonic of x.

   Verify whether the following function's are harmonic.

6. xy; 7. $e^x \sin y$; 8. $x^2 + y^2$; 9. $\cos x \sinh y$;
10. $e^y \cosh x$.

    The following functions are harmonic. Find the corresponding conjugate harmonic functions:

11. $x^2 - y^2$; 12. $e^x \cos y$; 13. $\sin x \cosh y$; 14. $2x(1 - y)$;
15. $\log (x^2 + y^2)$

Find the analytic function $f(z) = u + iv$, given that

16. $v = \text{amp}(z)$; 17. $u = y^2 - x^2$; 18. $u = e^y \cos x$; 19. $v = \sinh x \sin y$;

20. $u = \dfrac{x}{x^2 + y^2}$.

Part B

21. Prove that the function $u = x(x^2 - 3y^2) + (x^2 - y^2) + 2xy$ is harmonic. Also find the conjugate harmonic function v and the corresponding analytic function $(u + iv)$.
22. Prove that the function $v = 3x^2y + x^2 - y^3 - y^2$ is harmonic. Also find the conjugate harmonic function u and the corresponding analytic function $(u + iv)$.
23. Show that $\phi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$ can represent the velocity potential in an incompressible fluid flow. Also find the corresponding stream function and complex potential.
24. Show that $\psi = x^2 - y^2 - 3x - 2y + 2xy$ can represent the stream function of an incompressible fluid flow. Also find the corresponding velocity potential and complex potential.
25. Show that the equation $x^3y - xy^3 + xy + x + y = c$ can represent the path of electric current flow in an electric field. Also find the complex electric potential and the equation of the potential lines.
26. Find the analytic function $w = u + iv$, if $u = e^x (x \sin y + y \cos y)$. Hence find v.
27. Find the analytic function $w = u + iv$, if $v = e^{-x} (x \cos y + y \sin y)$. Hence find u.
28. Find the analytic function $w = u + iv$, if $u = e^{-2xy} \cdot \sin (x^2 - y^2)$. Hence find v.
29. Find the analytic function $w = u + iv$, if $v = e^{-2y} (y \cos 2x + x \sin 2x)$. Hence find u.
30. Find the analytic function $f(z) = u + iv$, given that $u + v = \dfrac{2x}{x^2 + y^2}$ and $f(1) = i$.
31. Find the analytic function $f(z) = u + iv$, given that $2u - 3v = 3y^2 - 2xy - 3x^2 + 3y - x$ and $f(0) = 0$.
32. Find the analytic function $f(z) = P + iQ$, if $P - Q = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.
33. Find the analytic function $f(z) = P + iQ$, if $Q = \dfrac{\sin x \sinh y}{\cos 2x + \cosh 2y}$, if $f(0) = 1$.
34. Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally, when $w = u + iv = z^4$.

35. Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally, when $w = u+iv=\frac{1}{z}\cdot$

36. Show that the families of curves $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ cut orthogonally, where a and b are arbitrary constants.

37. Find the equation of the orthogonal trajectories of the family of curves given by $2x - x^3 + 3xy^2 = a$.

38. Prove that $u = e^{-y} \cos x$ and $v = e^{-x} \sin y$ satisfy Laplace equations, but that $(u + iv)$ is not an analytic function of z.

39. If $f(z)$ is an analytic function of z in any domain, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f'(z)|^2\,|f(z)|^{p-2}\,.$$

40. If $f(z)$ is an analytic function of z in a region R, prove that

$$\nabla^2\{Re\, f(z)\}^2 = \nabla^2\{\text{Im}\, f(z)\}^2 = 2\,|f'(z)|^2\,.$$

3.4 CONFORMAL MAPPING

3.4.1 Mapping

A continuous real function $y = f(x)$ can be represented graphically by a curve in the Cartesian xy-plane. Similarly a continuous real function $z = f(x, y)$ is represented graphically by a surface in three dimensional space.

In the same fashion, if we wish to represent a function of the complex variable $w = f(z)$ or $u + iv = f(x + iy)$, a four-dimensional space is required, since $w = f(z)$ involves four real variables two independent variables x and y and two dependent variables u and v. As it is not possible, we make use of two complex planes for the two variables z and w. These are called the z-plane and the w-plane respectively. In the z-plane, the point $z = x + iy$ is plotted and in the w-plane, the point $w = u + iv$ is plotted.

A function $w = f(z)$ is not, as usual, represented by a locus of points in the four-dimensional space, but by a correspondence between points of the z-plane and points of the w-plane. To each point (x, y) in the z-plane, the function $w = f(z)$ determines a point (u, v) in the w-plane if $f(z)$ is a single-valued function. If the point z moves along some curve C in the z-plane, the corresponding point w will, in general, move along a curve C' in the w-plane. Similarly if the point z moves over a region R in the Z-plane, the corresponding point w moves over a region R' in the w-plane. The correspondence thus defined is called a *mapping* or *transformation* of elements (points, curves or regions) in the z-plane onto elements in the w-plane. The function $w = f(z)$ is called the *mapping* or *transformation function.* The corresponding points, curves or regions in the two planes are called the *image* of each other.

To visualise the nature of a function $f(z)$, we study the properties of the mapping defined by $w = f(z)$. To get a clear idea of the mapping given by $w = f(z)$, we usually consider the images of lines parallel to either co-ordinate axis, of concurrent lines passing through the origin, of concentric circles $|z|$ = constant and of regions enclosed by such curves in the z-plane. Also we can investigate the maps onto the z-plane of lines parallel to the u-axis and v-axis of the w-plane. The images of $u = c_1$ and $v = c_2$ that lie in the z-plane are called *the level curves* of u and v.

3.4.2 Conformal Mapping

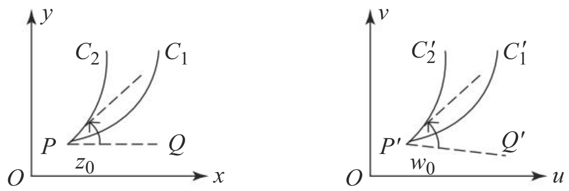


Fig. 3.4

Consider the transformation $w = f(z)$, where $f(z)$ is a single valued function of z. Under this transformation, a point z_0 and any two curves c_1 and c_2 passing through z_0 in the z-plane will be mapped onto a point w_0 and two curves c'_1 and c'_2 in the w-plane. If the angle between c_1 and c_2 at z_0 is the same as the angle between c'_1, and c'_2 at w_0, both in magnitude and sense, then the transformation $w = f(z)$ is said to be *conformal* at the point z_0. The formal definition is given as follows.

3.4.3 Definition

A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense, is said to be *conformal* at that point. A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be *isogonal* at that point.

The conditions under which the transformation $w = f(z)$ is conformal are given by the following theorem:

Theorem

If $f(z)$ is analytic and $f'(z) \neq 0$ in a region R of the z-plane, then the mapping performed by $w = f(z)$ is conformal at all points of R.

Proof

Let z_0 be a point in the region R of the z-plane where $f(z)$ is analytic and let $f'(z_0) \neq 0$.

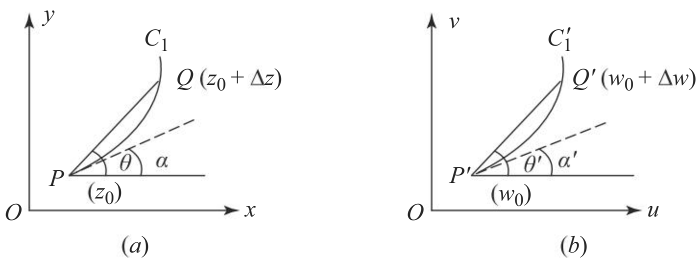


Fig. 3.5

Let c_1 be a continuous curve through z_0 and c'_1 be its image through w_0.

Let Q be a neighbouring point $z_0 + \Delta z$ on c_1 and Q' be its image $w_0 + \Delta w$ on c'_1. Let the chords PQ and $P'Q'$ make angles θ and θ' with the x-axis and u-axis respectively. Then Δz is a complex number whose modulus is the length PQ $(= r)$ and amplitude is θ.

$\therefore$ $$\Delta z = r\, e^{i\theta}$$

Similarly, $$\Delta w = r'\, e^{i\theta'}$$

Since $f(z)$ is analytic at z_0,

$$f'(z_0) = \lim_{\Delta z \to 0} \left\{ \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right\} = \lim_{\Delta z \to 0} \left(\frac{\Delta w}{\Delta z} \right) \text{exists} \tag{1}$$

Since $f'(z_0) \neq 0$, it can be expressed in the polar form as $R\, e^{i\phi}$.

$\therefore$ From (1), we get

$$R\, e^{i\phi} = \lim_{\Delta z \to 0} \left[\frac{r'\, e^{i\theta'}}{r\, e^{i\theta}} \right] = \lim_{r \to 0} \left[\left(\frac{r'}{r} \right) e^{i(\theta' - \theta)} \right]$$

$\therefore$ $$R = \lim_{r \to 0} \left(\frac{r'}{r} \right) \tag{2}$$

and $$\phi = \lim_{r \to 0} (\theta' - \theta) \tag{3}$$

Let the angle made by the tangent to c_1 at z_0 with the x-axis be α and that to c'_1 at w_0 with the u-axis be α'.

When $r \to 0$ or $\Delta z \to 0$, Q approaches P and hence the chord PQ tends to be the tangent at z_0 to the curve c_1 and so $\theta \to \alpha$. Similarly $\theta' \to \alpha'$.

Hence, from (3), we get

$$\phi = \alpha' - \alpha \tag{4}$$

If c_2 is another curve through z_0 in the z-plane and c'_2 is its image through w_0 in the w-plane and if the tangents to c_2 and c'_2 at z_0 and w_0 make angles β and β' with the x-axis and the u-axis respectively, then

$$\phi = \beta' - \beta \tag{5}$$

From (4) and (5), we get

$$\alpha' - \alpha = \beta' - \beta \quad \text{or} \quad \beta - \alpha = \beta' - \alpha'.$$

i.e. the angle between c_1 and c_2 is equal to the angle between c'_1 and c'_2, both in magnitude and sense. This means that the mapping $w = f(z)$ preserves angles between any two curves through the point z_0.

$\therefore$ The mapping $w = f(z)$ is conformal at z_0.

Note ☑

1. If $f(z_0) = 0$, it cannot be expressed in the polar form, since the amp $\{f(z_0)\}$ is undefined. Hence the proof of the theorem is not valid. Thus, though $f(z)$ may be analytic at z_0, the mapping $w = f(z)$ will not be conformal at z_0, if $f(z_0) = 0$.

[See Worked Example 3.3 in Section 3(c)]

2. The point, at which the mapping $w = f(z)$ is not conformal, i.e. $f'(z) = 0$, is called *a critical point* of the mapping.

   It is known that if the transformation $w = f(z)$ is conformal at a point, the inverse transformation $z = f^{-1}(w)$ is also conformal at the corresponding point.

   The critical points of $z = f^{-1}(w)$ are given by $\dfrac{\mathrm{d}z}{\mathrm{d}w} = 0$. Hence the critical

   points of the transformation $w = f(z)$ are given by $\dfrac{\mathrm{d}w}{\mathrm{d}z} = 0$ and $\dfrac{\mathrm{d}z}{\mathrm{d}w} = 0$

   [See Worked Example (3.3) below].

3. From equation (2) in the proof of the theorem, we get $R = \dfrac{r'}{r}$ approximately, i.e. $r' = Rr$

   or $$|\Delta w| = |f'(z_0)| \cdot |\Delta z|.$$

   This means that, under the transformation, an infinitesimal length Δz in the neighbourhood of z_0 is magnified by the factor $|f'(z_0)|$. Consequently, infinitesimal areas near z_0 in the z-plane are magnified by the factor $|f'(z_0)|^2$.

4. From equation (4) in the proof of the theorem, we get

   $$\alpha' = \alpha + \phi \quad \text{or} \quad \alpha + \text{amp}\,[f'(z_0)]$$

   This means that, under the transformation, the tangent to a curve through z_0 is rotated through an angle $\phi = \text{amp}\,[f'(z_0)]$, i.e. the direction of a curve through z_0 is rotated through an angle $\phi = \text{amp}\,[f'(z_0)]$.

5. From Notes (3) and (4), we observe that the image of a small figure near z_0 under the mapping $w = f(z)$ can be obtained by rotating it through an angle $= \text{amp}[f'(z_0)]$ and by magnifying it by a factor $= |f'(z_0)|$. Hence the shape of the image of a small figure near z_0 is approximately the same as that of the original figure under a conformal transformation.

3.5 SOME SIMPLE TRANSFORMATIONS

1. Translation

The transformation $w = z + c$, where c is a complex constant, represents a translation.

Let $z = x + iy$, $w = u + iv$ and $c = a + ib$

Then $$u + iv = x + iy + a + ib$$

$$\therefore \quad u = x + a \quad \text{and} \quad v = y + b$$

These two equations may be called the transformation equations.

Hence the image of any point (x, y) in the z-plane is the point $(x + a, y + b)$ in the w-plane.

If we assume that the w-plane is super-imposed on the z-plane, we observe that the point (x, y) and hence any figure is shifted by a distance $|c| = \sqrt{a^2 + b^2}$ in the direction of c i.e., translated by the vector representing c. Hence this transformation transforms a circle into an equal circle. Also the corresponding regions in the z- and w-planes will have the same shape, size and orientation.

2. Magnification

The transformation $w = cz$, where c is a real constant, represents magnification. The transformation equations are given by

$$u + iv = c\,(x + iy)$$

i.e., $\quad u = cx \quad \text{and} \quad v = cy$

$\therefore$ The image of the point (x, y) is the point (cx, cy).

Hence the size of any figure in the z-plane is magnified c times, but there will be no change in the shape and orientation. This transformation also transforms circles into circles.

3. Magnification and Rotation

The transformation $w = cz$, where c is a complex constant, represents both magnification and rotation.

Let $\quad z = r\,e^{i\theta}, \quad w = R\,e^{i\phi} \quad \text{and} \quad c = \rho e^{i\alpha},$

Then
$$Re^{i\phi} = (\rho\, e^{i\alpha})(r\, e^{i\theta})$$
$$= (\rho\, r) \cdot e^{i\,(\theta + \alpha)}$$

$\therefore$ The transformation equations are

$$R = \rho r \quad \text{and} \quad \phi = \theta + \alpha.$$

Thus the point (r, θ) in the z-plane is mapped onto the point $(\rho r, \theta + \alpha)$. This means that the magnitude of the vector representing z is magnified by $\rho = |c|$ and its direction is rotated through angle $\alpha = \text{amp}\,(c)$. Hence the transformation consists of a magnification and a rotation. Clearly circles in the z-plane are mapped onto circles by this transformation. Also every region in the z-plane is mapped onto a similar region by this transformation.

4. Magnification, Rotation and Translation

The general linear transformation $w = az + b,$ where a and b are complex constants, represents magnification, rotation and translation. The transformation $w = az + b$ can be considered as the combination of the two simple transformations $w_1 = az$ and $w = w_1 + b$.

$w_1 = az$ represents magnification by $|a|$ and rotation through amp (a).

$w = w_1 + b$ represents translation by the vector representing b.

Thus any figure in the z-plane will undergo magnification, rotation and translation by the transformation $w = az + b$. In particular, circles will be mapped into circles by this transformation.

5. Inversion and Reflection

The transformation $w = \dfrac{1}{z}$ represents inversion with respect to the unit circle $|z| = 1$, followed by reflection in the real axis.

[The *inverse* of a point P with respect to a circle with centre O and radius r is defined as the point P' on OP such that $OP \cdot OP' = r^2$]

Let $\qquad z = re^{i\theta} \quad \text{and} \quad w = Re^{i\phi}$

Then $\qquad w = \dfrac{1}{z} \quad \text{gives} \quad Re^{i\phi} = \dfrac{1}{r}e^{-i\theta}$

∴ The transformation equations are

$$R = \frac{1}{r} \quad \text{and} \quad \phi = -\theta.$$

Thus the image of the point (r, θ) in the z-plane is $\left(\dfrac{1}{r}, -\theta\right)$ under this transformation

If we assume that the w-plane is super-imposed on the z-plane and that P is (r, θ) and P' is $\left(\dfrac{1}{r}, \theta\right)$, then $OP' = \dfrac{1}{OP}$.

i.e., $\qquad OP \cdot OP' = 1$

∴ P' is the inverse of P with respect to the unit circle $|z| = 1$, as shown in Fig. 3.6.

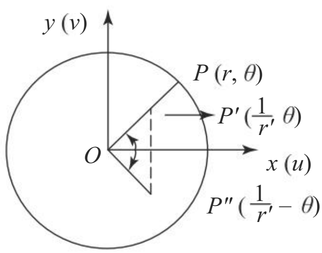


Fig. 3.6

If we consider the point $P''\left(\dfrac{1}{r}, -\theta\right)$, it is the reflection of the point $P'\left(\dfrac{1}{r}, \theta\right)$ in the real axis. Thus the transformation $w = \dfrac{1}{z}$ consists of an inversion of z with respect to the unit circle $|z| = 1$, followed by reflection in the real axis.

Also it is observed that the interior (exterior) of the unit circle $|z| = 1$ is mapped onto the exterior (interior) of the unit circle $|w| = 1$.

3.6 SOME STANDARD TRANSFORMATIONS

1. The Transformation $w = z^2$

$$w = z^2 \quad \therefore \quad u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$$

Hence the transformation equations are

$$u = x^2 - y^2 \tag{1}$$

and $$v = 2xy \tag{2}$$

Consider a line parallel to the x-axis, given by the equation

$$y = b \tag{3}$$

The equation of the image of (3), which will be an equation in u and v, is got by eliminating x and y from (1), (2) and (3).

Using (3) in (1) and (2), we have

$$u = x^2 - b^2 \tag{4}$$

and $$v = 2bx \tag{5}$$

Eliminating x from (4) and (5), we have

$$u = \left(\frac{v}{2b}\right)^2 - b^2, \text{ i.e. } v^2 = 4b^2(u + b^2) \tag{6}$$

Equation (6) represents in the w-plane a parabola whose vertex is $(-b^2, 0)$, focus is $(0, 0)$ and axis lies along the u-axis and which is open to the right. [Fig. 3.7]

If b is regarded as an arbitrary constant or parameter, (3) represents a family of lines parallel to the x-axis. In this case, (6) represents a system of parabolas, all having the origin as the common focus, i.e. equation (6) represents a family of confocal parabolas.

Consider the equation $x = a$ (7)

This represents a line parallel to the y-axis. The image of line (7) is got by eliminating x and y from (1), (2) and (7).

Thus the image of (7) is given by the following equations

$$u = a^2 - y^2$$

and $$v = 2ay$$

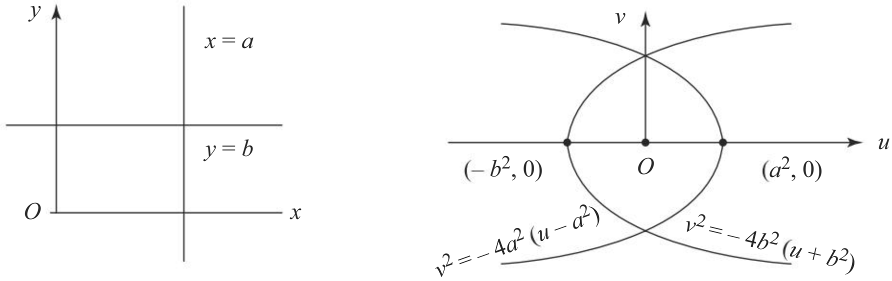
i.e. by the equation

$$u = a^2 - \left(\frac{v}{2a}\right)^2$$

or $$v^2 = -4a^2(u - a^2) \tag{8}$$

Equation (8) represents in the w-plane a parabola, whose vertex is $(a^2, 0)$, focus is $(0, 0)$ and axis lies along the u-axis and which is open to the left. [Fig. 3.7]

If a is regarded as an arbitrary constant or parameter, (7) represents a family of lines parallel to the y-axis and (8) represents a family of confocal parabolas with the common focus at the origin.



(a) (b)

Fig. 3.7

Thus a system of lines parallel to either coordinate axis is mapped onto a system of confocal parabolas under the transformation $w = z^2$, with the exceptions $x = 0$ and $y = 0$.

The map of $y = 0$ is given by $u = x^2$ and $v = 0$, i.e., by $v = 0$, where $u > 0$.

i.e., the map of the entire x-axis is the positive part or the right half of the u-axis.

The map of $x = 0$ is given by $u = -y^2$ and $v = 0$, i.e., by $v = 0$, where $u < 0$.
i.e., the map of the entire y-axis is the negative part or the left half of the u-axis.

Using polar forms of z and w, i.e. putting

$$z = re^{i\theta} \quad \text{and} \quad w = Re^{i\phi} \text{ in } w = z^2, \text{ we have}$$

$$R\, e^{i\phi} = (r\, e^{i\theta})^2 = r^2 e^{i2\theta}$$

$\therefore$ The transformation equations are

$$R = r^2 \quad \text{and} \quad \phi = 2\theta.$$

Now $r = a$ represents a family of concentric circles in the z-plane. Its map is given by $R = a^2$, that represents a family of concentric circles in the w-plane. $\theta = \alpha$ represents a family of concurrent lines through the origin in the z-plane. Its map is given by $\phi = 2\alpha$, that represents a family of concurrent lines through the origin in the w-plane.

Consider now $u = c$, that represents a family of lines parallel to the v-axis. The image of $u = c$ is given by $x^2 - y^2 = c^2$, that represents a family of rectangular hyperbolas whose principal axes lie along the coordinate axes in the z-plane. Consider $v = d$, that represents a family of lines parallel to the u-axis. The image of $u = d$ is given by $xy = \dfrac{d}{2}$, that represents a family of rectangular hyperbolas whose asymptotes are the coordinate axes in the z-plane.

Finally for the mapping function $w = z^2$,

$$\frac{\mathrm{d}w}{\mathrm{d}z} = 2z$$

$$= 0, \text{ at } z = 0.$$

Hence the transformation $w = z^2$ is conformal at all points in the z-plane except at the origin, i.e. $z = 0$ is the critical point of the transformation $w = z^2$.

2. The Transformation $w = e^z$

$$w = e^z \therefore \frac{dw}{dz} = e^z \neq 0 \text{ for any } z.$$

$\therefore$ The transformation $w = e^z$ is conformal at all points in the z-plane.

Putting $z = x + iy$ and $w = R\, e^{i\phi}$ in $w = e^z$, we get

$$R\, e^{i\phi} = e^{x+iy}$$

$$= e^x \cdot e^{iy}$$

$\therefore$ The transformation equations are

$$R = e^x \tag{1}$$

and
$$\phi = y \tag{2}$$

The image of the system of parallel lines $x = a$ in the z-plane is given by $R = e^a$, that represents a family of concentric circles in the w-plane.

The image of the system of parallel lines $y = b$ in the z-plane is given by $\phi = b$, that represents a family of concurrent lines through the origin in the w-plane. In particular, the image of the y-axis i.e. $x = 0$ is the unit circle $R = 1$ or $|w| = 1$. The image of $x > 0$ is given by $R > 1$ or $|w| > 1$ and the image of $x < 0$ is given by $R < 1$ or $|w| < 1$. i.e. the region lying on the right side of the y-axis in the z-plane is mapped onto the exterior of the unit circle $|w| = 1$ in the w-plane.

Similarly, the region lying on the left side of the y-axis in the z-plane is mapped onto the interior of the unit circle $|w| = 1$ in the w-plane.

The image of the entire x-axis, i.e. $y = 0$ is given by $\phi = 0$, i.e., the positive part of the u-axis.

Similarly the image of the line $y = \pi$ is given by $\phi = \pi$, i.e., the negative part of the u-axis.

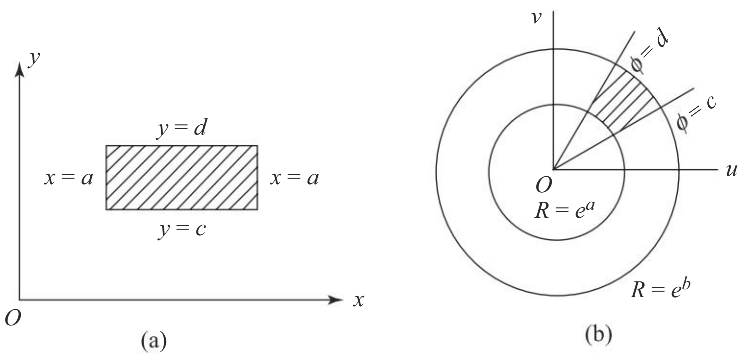


Fig. 3.8

Finally we note that the image of the rectangular region in the z-plane defined by $a \leq x \leq b$ and $c \leq y \leq d$ is the annular region in the w-plane defined by $e^a \leq R \leq e^b$ and $c \leq \phi \leq d$. The corresponding regions are shaded in the Fig. 3.8.

3. The Transformation $w = \sin z$

$$w = \sin z \quad \therefore \quad \frac{dw}{dz} = \cos z$$

$$= 0,$$

$$\text{when } z = \frac{(2n-1)\pi}{2}, \text{ where } n \text{ is an integer.}$$

$\therefore$ The transformation $w = \sin z$ is conformal at all points except at $z = \dfrac{(2n-1)\pi}{2}$.

i.e. the critical points of the transformation are $z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$

Putting $z = x + iy$ and $w = u + iv$ in $w = \sin z$, we get

$$u + iv = \sin(x + iy)$$

$$= \sin x \cosh y + i \cos x \sinh y.$$

$\therefore$ The transformation equations are

$$u = \sin x \cosh y \tag{1}$$

and
$$v = \cos x \sinh y \tag{2}$$

Consider the family of lines parallel to the x-axis, given by

$$y = b \tag{3}$$

where b is an arbitrary constant.

Image of (3) is given by

$$u = \sin x \cosh b \tag{3a}$$

and
$$v = \cos x \sinh b \tag{4}$$

Eliminating x from (3a) and (4), we get the equation of the image of (3) as

$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1 \tag{5}$$

Equation (5) represents a family of ellipses whose principal axes lie along the u- and v-axes, centres are at the origin and semi axes are of lengths $\cosh b$ and $|\sinh b|$.

The foci of these ellipses are at the points $(\pm 1, 0)$ [$\because$ The foci of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are $(\pm ae, 0)$, i.e. $(\pm\sqrt{a^2 - b^2}, 0)$]

The coordinates of the foci of the family of ellipses (5) are independent of the parameter b.

This means that all the members of the family (5) have the same two points as foci.

Hence (5) represents a family of confocal ellipses. [Fig. 3.9]

Thus a family of lines parallel to the x-axis in the z-plane is mapped onto a family of confocal ellipses in the w-plane, with the exception of $y = 0$, i.e, the x-axis itself.

Now the map of $y = 0$ is given by

$$u = \sin x \quad \text{and} \quad v = 0$$

i.e. by $\quad -1 \le u \le 1 \quad$ and $\quad v = 0$

i.e. the map of the entire x-axis is the segment of the u-axis, lying between -1 and $+1$.

Consider again the segment of the line $y = b$ within the range $-\pi/2 \le x \le \pi/2$. For these values of x, $\cos x$ is positive.

Hence, from (4) i.e. $v = \cos x \sinh b$, we see that $v > 0$ when $b > 0$ [as $\sinh b > 0$ when $b > 0$] and $v < 0$ when $b < 0$ [as $\sinh b < 0$ when $b < 0$].

Thus the image of the segment of the line $y = b$ within $-\pi/2 \le x \le \pi/2$ is the upper or lower half of the ellipse $\dfrac{u^2}{\cosh^2 b} + \dfrac{v^2}{\sinh^2 b} = 1$, according as $b > 0$ or < 0.

Now consider the family of lines parallel to the y-axis, given by $x = a$ (6)
where a is an arbitrary constant.

Image of (6) is given by

$$u = \sin a \cosh y \tag{7}$$

and

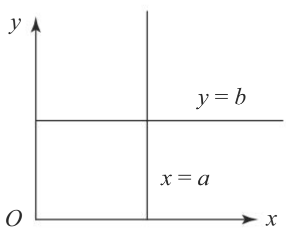
$$v = \cos a \sinh y \tag{8}$$

Eliminating y from (7) and (8), we get the image of (6) as

$$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1 \tag{9}$$

(9) Represents a family of confocal hyperbolas with common foci at $(\pm 1, 0)$ [$\because$ the foci of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are $(\pm ae, 0)$, i.e. $(\pm\sqrt{a^2 + b^2},\ 0)$] [Fig. 3.9]

Thus a family of lines parallel to the y-axis in the z-plane is mapped onto a family of confocal hyperbolas in the w-plane, with the exceptions of the y-axis and the lines $x = \pm\dfrac{\pi}{2}$.



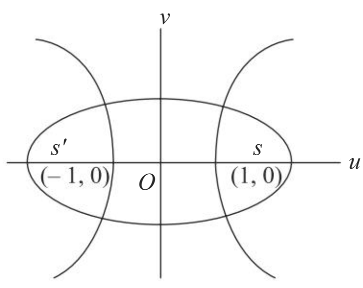


Fig. 3.9

The map of $x = 0$ is given by

$$u = 0 \quad \text{and} \quad v = \sinh y$$

i.e.

$$u = 0 \quad \text{and} \quad v > 0 \text{ (when } y > 0) \text{ and}$$
$$u = 0 \quad \text{and} \quad v < 0 \text{ (when } y < 0)$$

Thus the upper and lower halves of the y-axis are mapped onto the upper and lower halves of the v-axis under the transformation $w = \sin z$.

Consider the map of the line $x = \pi/2$, which is given by $u = \cosh y$ and $v = 0$ i.e. by $u \geq 1$ and $v = 0$

Similarly, the map of the line $x = -\pi/2$ is given by $u = -\cosh y$ and $v = 0$ i.e. by $u \leq -1$ and $v = 0$

Thus the part of the u-axis for which $u \geq 1$ and the part of the u-axis for which $u \leq -1$ are the images of the lines $x = \dfrac{\pi}{2}$ and $x = -\dfrac{\pi}{2}$ respectively.

In conclusion, if we note the family of ellipses (5) and the family of hyperbolas (9) have the same two points $(\pm 1, 0)$ as common foci, we can say that in general the transformation $w = \sin z$ maps the system of lines parallel to either coordinate axis in the z-plane into a system of confocal conics in the w-plane.

4. The Transformation $w = \cosh z$

$$w = \cosh z \quad \therefore \quad \frac{dw}{dz} = \sinh z$$

$$= -i \sin iz$$

$$= 0, \text{ when } z = i.\, n\pi, \text{ where } n \text{ is an integer.}$$

$\therefore$ The transformation $w = \cosh z$ is conformal at all points except the critical points $z = \pm i\pi, \pm i2\pi, \ldots$

Putting $z = x + iy$ and $w = u + iv$ in

$w = \cosh z$, we get

$$u + iv = \cos i\,(x + iy)$$

$$= \cos\,(ix - y)$$

$$= \cosh x \cos y + i \sinh x \sin y$$

$\therefore$ The transformation equations are

$$u = \cosh x \cos y \tag{1}$$

and
$$v = \sinh x \sin y \tag{2}$$

The image of the family of parallel lines $y = b$ (3)

is the family of hyperbolas

$$\frac{u^2}{\cos^2 b} - \frac{v^2}{\sin^2 b} = 1 \tag{4}$$

The foci of (4) are $(\pm 1, 0)$

Thus the family of lines parallel to the x-axis in the z-plane is mapped onto a family of confocal hyperbolas in the w-plane with the exception $y = 0$, $y = \pi$ and $y = \pi/2$.

The map of $y = 0$ is given by

$$u = \cosh x \quad \text{and} \quad v = 0.$$

When $x > 0$, $\cosh x$ takes values from 1 to ∞.

$\therefore$ The image of the positive part of the x-axis is the portion of the u-axis for which $u \geq 1$.

The map of $y = \pi$ is given by

$$u = -\cosh x \quad \text{and} \quad v = 0$$

$\therefore$ The image of the positive part of the line $y = \pi$ is the portion of the u-axis for which $u \leq -1$.

The map of $y = \pi/2$ is given by

$$u = 0 \quad \text{and} \quad v = \sinh x$$

$\therefore$ The images of the positive and negative parts of the line $y = \dfrac{\pi}{2}$ are the positive and negative parts of the v-axis respectively.

The image of the family of parallel lines $x = a$ (5), is the family of ellipses

$$\frac{u^2}{\cosh^2 a} + \frac{v^2}{\sinh^2 a} = 1 \tag{6}$$

The foci of (6) are ($\pm$ 1, 0).

Thus the family of lines parallel to the y-axis in the z-plane is mapped onto a family of confocal ellipses in the w-plane with the exception of the y-axis itself.

The image of $x = 0$ is given by

$$u = \cos y \quad \text{and} \quad v = 0$$

i.e. by $$-1 \leq u \leq 1 \quad \text{and} \quad v = 0$$

i.e. the image of the entire y-axis is the segment of the u-axis, for which $-1 \leq u \leq 1$. Thus, in general, the transformation $w = \cosh z$ maps the system of lines parallel to either coordinate axis in the z-plane into a system of confocal conics in the w-plane.

5. The Transformation $w = z + \dfrac{k^2}{z}$, where k is real and positive

$$w = z + \frac{k^2}{z} \qquad \therefore \quad \frac{dw}{dz} = 1 - \frac{k^2}{z^2}$$

$$= 0, \qquad \text{when } z = \pm k$$

$\therefore$ The transformation $w = z + \dfrac{k^2}{z}$ is conformal at all points of the z-plane except at $z = \pm k$.

Putting $z = r\,e^{i\theta}$ and $w = u + iv$ in $w = z + \dfrac{k^2}{z}$, we get

$$u + iv = re^{i\theta} + \frac{k^2}{r}e^{-i\theta}$$

$$= \left(r + \frac{k^2}{r}\right)\cos\theta + \left(r - \frac{k^2}{r}\right)\sin\theta$$

$\therefore$ The transformation equations are

$$u = \left(r + \frac{k^2}{r}\right)\cos\theta \tag{1}$$

and
$$v = \left(r - \frac{k^2}{r}\right)\sin\theta \tag{2}$$

Consider a family of concentric circles with centre at the origin in the z-plane, given by the polar equation $r = a$, (3)

where a is a parameter. The image of (3) is given by

$$u = \left(a + \frac{k^2}{a}\right)\cos\theta \text{ and } v = \left(a - \frac{k^2}{a}\right)\sin\theta.$$

Eliminating θ from these equations, the equation of the image of the family (3) is

$$\frac{u^2}{\left(a + \frac{k^2}{a}\right)^2} + \frac{v^2}{\left(a - \frac{k^2}{a}\right)^2} = 1 \tag{4}$$

Equation (4) represents a family of ellipses whose centres are at the origin, principal axes lie along the u- and v-axes and foci are at the points

$$\left(\pm\sqrt{\left(a + \frac{k^2}{a}\right)^2 - \left(a - \frac{k^2}{a}\right)^2},\ 0,\quad \text{i.e. } (\pm 2k,\ 0).\right.$$

The co-ordinates of the foci do not depend on a.

Hence (4) represents a family of confocal ellipses

Thus a family of concentric circles with centre at the origin in the z-plane is mapped onto a family of confocal ellipses, with the exception of $r = k$.

The image of the circle $r = k$ is given by $u = 2k\cos\theta$ and $v = 0$ (from (1) and (2) i.e. $-2k \le u \le 2k$ and $v = 0$.

Thus the image of the circle $r = k$ is the segment of the u–axis, given by $-2k \le u \le 2k$.

Consider a family of concurrent lines through the origin in the z- plane, given by the polar equation $\theta = \alpha$ (5)

where α is a parameter. The image of (5) is given by

$$u = \left(r + \frac{k^2}{r}\right)\cos\alpha \quad \text{and} \quad v = \left(r - \frac{k^2}{r}\right)\sin\alpha.$$

Eliminating r from these equations, the equation of the image of the family (5) is

$$\frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha} = 4k^2$$

or
$$\frac{u^2}{4k^2\cos^2\alpha} - \frac{v^2}{4k^2\sin^2\alpha} = 1 \tag{6}$$

Equation (6) represents a family of hyperbolas whose centres are at the origin, principal axes lie along the u- and v-axes and foci are at the points

$$\left(\pm\sqrt{4k^2\cos^2\alpha+4k^2\sin^2\alpha}\,,0\right),\ \text{i.e. } (\pm\, 2k,\, 0)$$

The coordinates of the foci do not depend on α. Hence (6) represents a family of confocal hyperbolas.

Thus a family of concurrent lines through the origin in the z-plane is mapped onto a family of confocal hyperbolas, with the exceptions of $\theta = 0$, $\theta = \pi$ and $\theta = \pi/2$. The image of $\theta = 0$ is given by

$$u = r + \frac{k^2}{r} \quad \text{and} \quad v = 0$$

i.e. $$u = \left(\sqrt{r} - \frac{k}{\sqrt{r}}\right)^2 + 2k \quad \text{and} \quad v = 0$$

i.e. $$u > 2k \quad \text{and} \quad v = 0$$

Thus the image of the positive part of the x-axis is the part of the u-axis for which $u > 2k$.

Similarly the image of $\theta = \pi$, i.e. the negative part of the x-axis is that part of the u-axis for which $u < -2k$.

The images of the lines $\theta = \pi/2$ and $\theta = \dfrac{3\pi}{2}$ are given by $u = 0$.

Hence the image of the y-axis is the v-axis.

WORKED EXAMPLE 3(c)

Example 3.1 Find the image of the circle $|z| = 2$ under the transformation (i) $w = z + 3 + 2i$,
(ii) $w = 3z$, (iii) $w = \sqrt{2}\, e^{i\pi/4} z$ and (iv) $w = (1 + 2i)z + (3 + 4i)$

(i) The equation of the given circle $|z| = 2$ in the Cartesian from is $\sqrt{x^2 + y^2} = 2$ or $x^2 + y^2 = 4$ (1)

The mapping function is $w = z + 3 + 2i$

i.e. $$u + iv = x + iy + 3 + 2i$$

$\therefore$ The transformation equations are

$$u = x + 3 \tag{2}$$

and $$v = y + 2 \tag{3}$$

Eliminating x and y from (1), (2) and (3), we get the equation of the image.
From (2), $x = u - 3$; from (3), $y = v - 2$.

Using these values of x and y in (1), the required equation of the image is

$$(u-3)^2+(v-2)^2=4.$$

(ii) The transformation is $w = 3z$

i.e. $u + iv = 3\,(x + iy)$

$\therefore \quad u = 3x \qquad (4)$

and $\quad v = 3y \qquad (5)$

Eliminating x and y from (1), (4) and (5), the equation of the image of (1) is obtained as

$$\left(\frac{u}{3}\right)^2+\left(\frac{v}{3}\right)^2=4 \quad \text{or} \quad u^2+v^2=36.$$

(iii) The transformation is $w=\sqrt{2}\cdot e^{i\pi/4}\cdot z$

i.e., $u+iv=\sqrt{2}(\cos\pi/4+i\sin\pi/4)z$

$\qquad =(1+i)(x+iy)$

$\therefore \quad u=x-y \qquad (6)$

and $\quad v=x+y \qquad (7)$

From (6) and (7), w get $x=\dfrac{u+v}{2}$ and $y=\dfrac{v-u}{2}$.

Using these values of x and y in (1), the image of (1) is obtained as

$$\left(\frac{v+u}{2}\right)^2+\left(\frac{v-u}{2}\right)^2=4$$

i.e. $\qquad u^2+v^2=8.$

(iv) The transformation is $w = (1 + 2i)\,z + (3 + 4i)$

i.e. $u + iv = (1 + 2i)\,(x + iy) + 3 + 4i$

$\qquad = (x - 2y + 3) + \mathrm{i}\,(2x + y + 4)$

$\therefore \quad u = x - 2y + 3 \qquad (8)$

and $\quad v = 2x + y + 4 \qquad (9)$

Solving (8) and (9), we get

$$x=\frac{u+2v-11}{5} \quad \text{and} \quad y=\frac{v-2u+2}{5}$$

Using these values of x and y in (1), the image of (1) is obtained as

$$(u-3)^2+(v-4)^2=20 \qquad (10)$$

Aliter

The transformation is $w - (3 + 4i) = (1 + 2i)z$

$\therefore \qquad |w-(3+4i)| = |1+2i|\,|z|$

$\therefore$ The map of $|z| = 2$ is given by

$|w-(3+4i)| = 2\sqrt{5}$, which is a circle whose centre is the point $(3+4i)$ and radius equal to $2\sqrt{5}$ and which is the same as the circle given by (10).

Example 3.2

(a) Find the image of the triangular region in the z-plane bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ under the transformation (i) $w = 2z$ and (ii) $w = e^{i\pi/4}\,.\,z$

(b) Find the image of the rectangular region in the z-plane bounded by the lines $x = 0$, $y = 0$, $x = 2$ and $y = 1$ under the transformation

(i) $\quad w = z + 2 - i \quad$ and $\quad$ (ii) $\; w = (1+2i)z + (1+i)$.

(a) (i) $\quad w = 2z$ i.e., $u + iv = 2(x + iy)$

$\therefore \qquad u = 2x \quad$ and $\quad v = 2y$

$\therefore$ The images of $x = 0$, $y = 0$ and $x + y = 1$ are respectively $u = 0$, $v = 0$ and $u + v = 2$. The corresponding regions in the z-and w-planes are shown in the Figs 3.10 (a) and (b) respectively.

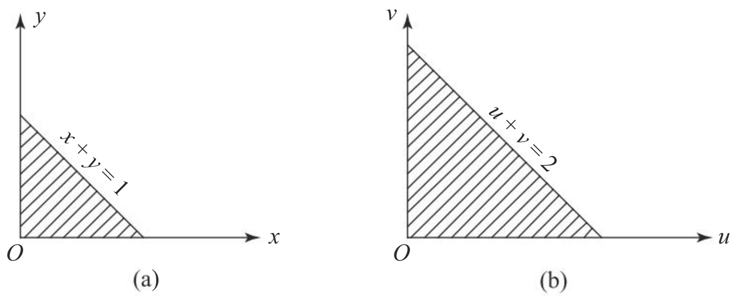


Fig. 3.10

(a) (ii) $\quad w = e^{i\pi/4}\cdot z \quad$ i.e., $u + iv = \dfrac{1}{\sqrt{2}}(1+i)\,(x+iy)$

$$\therefore \qquad u = \frac{1}{\sqrt{2}}(x-y) \quad \text{and} \quad v = \frac{1}{\sqrt{2}}(x+y)$$

On solving, we get,

$$x = \frac{1}{\sqrt{2}}(u+v) \quad \text{and} \quad y = \frac{1}{\sqrt{2}}(v-u)$$

$\therefore$ The maps of $x = 0$ and $y = 0$ are respectively

$$u + v = 0 \quad \text{and} \quad u = v.$$

The map of $x + y = 1$ is $v = \dfrac{1}{\sqrt{2}}$.

The corresponding regions in the z-plane and w-plane are shown in the Figs 3.11 (a) and (b) respectively.

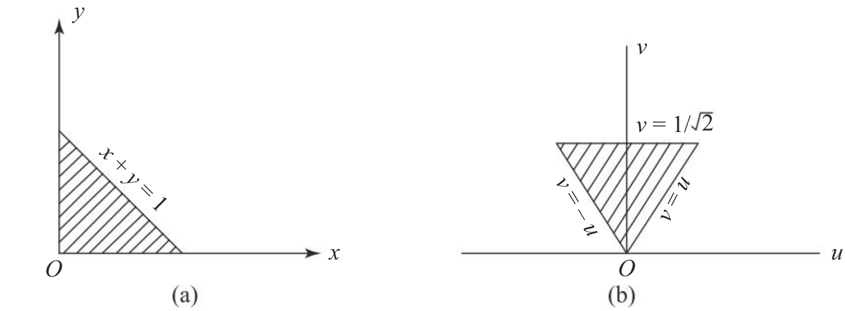


Fig. 3.11

(b) (i) $$w = z + 2 - i, \text{ i.e. } u + iv = x + iy + 2 - i$$

i.e. $$u = x + 2 \quad \text{and} \quad v = y - 1$$

The vertices of the given rectangle in the z-plane are (0, 0), (2, 0), (2, 1) and (0, 1).
The images of these points in the w-plane are (2, – 1), (4, – 1), (4, 0) and (2, 0) respectively. The corresponding regions in the two planes are shown in the Figs 3.12 (a) and (b)

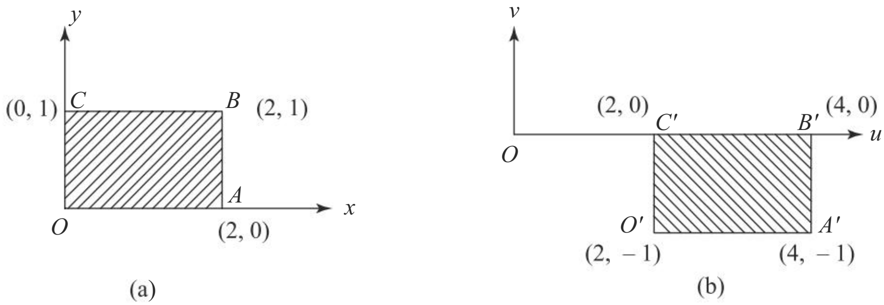


Fig. 3.12

(b) (ii) $$w = (1 + 2i)z + (1 + i)$$

i.e. $$u + iv = (1 + 2i)(x + iy) + (1 + i)$$

The image of (0, 0) is given by

$$u + iv = (1 + 2i)(0 + i0) + 1 + i;$$
$$= 1 + i, \text{ i.e. the point } (1, 1).$$

The image of (2, 0) is given by

$$u + iv = (1 + 2i)(2 + i0) + 1 + i$$
$$= 3 + 5i, \text{ i.e. the point } (3, 5).$$

The image of (2, 1) is given by

$$u + iv = (1 + 2i)(2 + i) + 1 + i$$
$$= 1 + 6i, \text{ i.e. the point } (1, 6)$$

The image of (0, 1) is given by

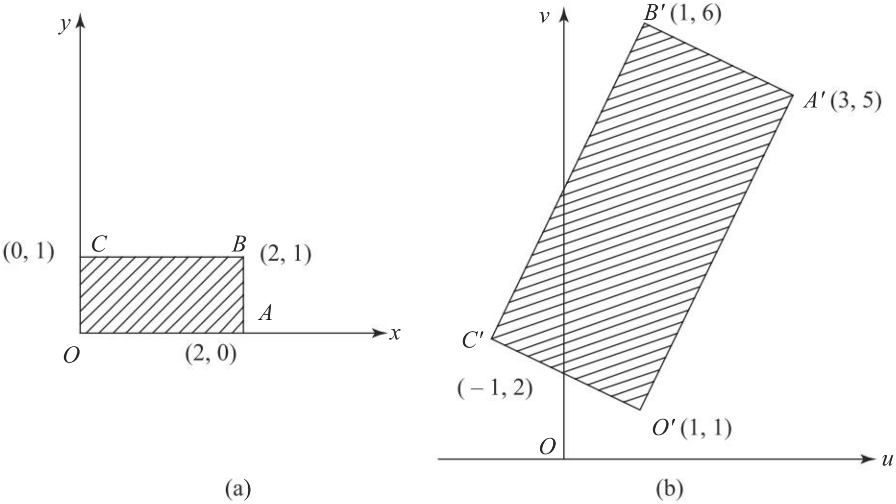


Fig. 3.13

$$u + iv = (1 + 2i)(0 + i) + 1 + i.$$
$$= -1 + 2i, \text{ i.e. the point } (-1, 2).$$

The corresponding regions in the two planes are shown in the Figs 3.13 (a) and (b).

Example 3.3 Find the critical points of the transformations

(i) $w = z^2$ and (ii) $w = \dfrac{1}{z}$. Give one example each to show that the mapping given by these functions is not conformal at the critical points.

(i) $$w = z^2 \quad \therefore \quad \frac{dw}{dz} = 2z$$

$$\therefore \quad \frac{dw}{dz} = 0 \quad \text{at} \quad z = 0.$$

$\therefore$ $z = 0$ is the critical point of the transformation $w = z^2$.

Using polar coordinates, $w = z^2$ becomes

$$Re^{i\phi} = r^2 e^{i2\theta}$$

$$\therefore \quad R = r^2 \tag{1}$$

$$\text{and} \quad \phi = 2\theta \tag{2}$$

Consider two lines passing through the origin in the z-plane, given by the polar equations $\theta = \alpha$ and $\theta = \beta\ (\beta > \alpha)$.

Angle between these two lines = $\beta - \alpha$.

The images of these two lines are

$$\phi = 2\alpha \quad \text{and} \quad \phi = 2\beta \quad \text{[from (2)]}$$

Angle between these two image lines = $2\beta - 2\alpha$. Thus angle between any two curves, i.e. lines through $z = 0$ is not preserved in magnitude.

$\therefore$ The mapping $w = z^2$ is not conformal at $z = 0$.

(ii) $$w = \frac{1}{z} \quad \therefore \quad \frac{dw}{dz} = -\frac{1}{z^2} \to \infty \text{ as } z \to 0.$$

$\therefore$ $w = 1/z$ is not analytic at $z = 0$ and hence the mapping $w = 1/z$ is not conformal at $z = 0$.

i.e. $z = 0$ is the critical point of the transformation.

Using polar co-ordinates, $w = \dfrac{1}{z}$ becomes

$$Re^{i\phi} = \frac{1}{r} e^{i\theta}$$

$\therefore$ $$R = \frac{1}{r} \tag{3}$$

and $$\phi = -\theta \tag{4}$$

From (4), we see that the images of the two lines $\theta = \alpha$ and $\theta = \beta$ $(\beta > \alpha)$ in the z-plane are $\phi = -\alpha$ and $\phi = -\beta$.

Angle between the lines $\theta = \alpha$ and $\theta = \beta$ is $(\beta - \alpha)$ and the angle between the image lines is $(\alpha - \beta)$.

Thus the angle between any two curves, i.e. the lines through $z = 0$ is not preserved in sense.

$\therefore$ The mapping $w = \dfrac{1}{z}$ is not conformal at $z = 0$.

Example 3.4 Show that the transformation $w = \dfrac{1}{z}$ transforms, in general, circles and straight lines into circles and straight lines. Point out the circles and straight lines that are transformed into straight lines and circles respectively.

$$w = \frac{1}{z} \quad \therefore \quad z = \frac{1}{w} \text{ i.e. } x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$\therefore$ The transformation equations are

$$x = \frac{u}{u^2 + v^2} \tag{1}$$

and $$y = \frac{-v}{u^2 + v^2} \tag{2}$$

Consider the equation

$$a(x^2 + y^2) + bx + cy + \mathrm{d} = 0 \tag{3}$$

Equation (3) represents a circle not passing through the origin if $a \neq 0$ and $d \neq 0$, a circle passing through the origin if $a \neq 0$ and $\mathrm{d} = 0$, a straight line not passing through the origin if $a = 0$ and $\mathrm{d} \neq 0$ and a straight line passing through the origin, if $a = 0$ and $\mathrm{d} = 0$.

Using (1) and (2) in (3), the image of (3) in the w-plane is given by

$$a\left\{\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}\right\}+\frac{bu}{u^2+v^2}-\frac{cv}{u^2+v^2}+\mathrm{d}=0$$

i.e. $$\mathrm{d}\,(u^2+v^2)+bu-cv+a=0 \tag{4}$$

Now Eq. (4) represents a circle not passing through the origin if $a \neq 0$ and $\mathrm{d} \neq 0$, a straight line not passing through the origin if $a \neq 0$ and $\mathrm{d} = 0$, a circle passing through the origin if $a = 0$ and $\mathrm{d} \neq 0$ and a straight line passing through the origin if $a = 0$ and $\mathrm{d} = 0$.

Thus circles not passing through the origin and straight lines passing through the origin are mapped onto similar circles and straight lines respectively.

But circles passing through the origin are mapped onto straight lines not passing the origin.

Straight lines not passing through the origin are mapped onto circles passing through the origin.

Example 3.5 Find the image of the following regions under the transformation $w = \dfrac{1}{z}$:

(i) the half-plane $x > c$, when $c > 0$

(ii) the half-plane $y > c$, when $c < 0$

(iii) the infinite strip $\dfrac{1}{4} \leq y \leq \dfrac{1}{2}$.

Also show the corresponding regions graphically.

$$w=\frac{1}{z} \quad \therefore \quad z=\frac{1}{w} \quad \text{i.e. } x+iy=\frac{u-iv}{u^2+v^2}$$

$\therefore$ The transformation equations are

$$x=\frac{u}{u^2+v^2} \tag{1}$$

and $$y=-\frac{v}{u^2+v^2} \tag{2}$$

(i) The image of the region $x > c$ is given by $\dfrac{u}{u^2+v^2} > c$ from (1).

i.e. $$c(u^2+v^2) < u \quad \text{or} \quad u^2+v^2 < \frac{u}{c} \qquad [\because c > 0]$$

i.e. $$\left(u-\frac{1}{2c}\right)^2+v^2 < \left(\frac{1}{2c}\right)^2 \tag{3}$$

Equation (3) represents the interior of the circle

$\left(u-\dfrac{1}{2c}\right)^2+v^2 = \left(\dfrac{1}{2c}\right)^2$, whose centre is $\left(\dfrac{1}{2c}, 0\right)$ and radius is $\dfrac{1}{2c}$.

The corresponding regions are shown in Figs 3.14 (a) and (b)

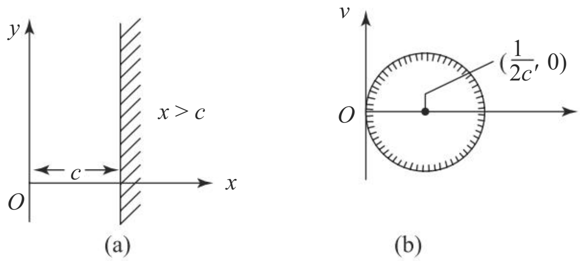


Fig. 3.14

The image of the region $y > c$ is given by

$$-\frac{v}{u^2 + v^2} > c \text{ from (2)}$$

i.e. $\quad c\,(u^2 + v^2) < -\,v \quad$ or $\quad u^2 + v^2 > \dfrac{-v}{c} \;\; [\because c < 0]$

i.e. $$u^2 + \left(v + \frac{1}{2c}\right)^2 > \left(\frac{1}{2c}\right)^2 \tag{4}$$

Equation (4) represents the exterior of the circle

$u^2 + \left(v + \dfrac{1}{2c}\right)^2 = \left(\dfrac{1}{2c}\right)^2$, whose centre is $\left(0, -\dfrac{1}{2c}\right)$ and radius is $\dfrac{1}{2|c|}$.

The corresponding regions are shown in the Figs 3.15 (a) and (b)

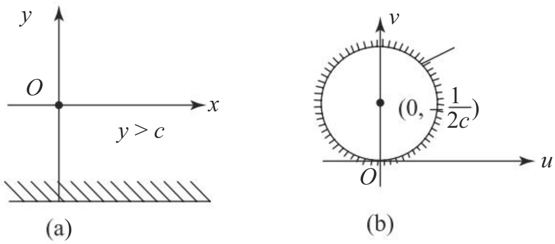


Fig. 3.15

(iii) The image of $y \geq \dfrac{1}{4}$ is given by

$$-\frac{v}{u^2 + v^2} \geq \frac{1}{4}$$

i.e. $\quad u^2 + v^2 \leq -\,4v \quad$ or $\quad u^2 + (v + 2)^2 \leq 2^2$,

i.e. the interior of the circle $u^2 + (v + 2)^2 = 2^2$.

The image of $y \leq 1/2$ is given by

$$-\frac{v}{u^2 + v^2} \leq \frac{1}{2} \text{ i.e.,}$$

$$u^2 + v^2 \geq -\,2v \quad \text{or} \quad u^2 + (v + 1)^2 \geq 1,$$

i.e. the exterior of the circle $u^2 + (v + 1)^2 = 1$.

The corresponding regions are shown in the Figs 3.16 (a) and (b)

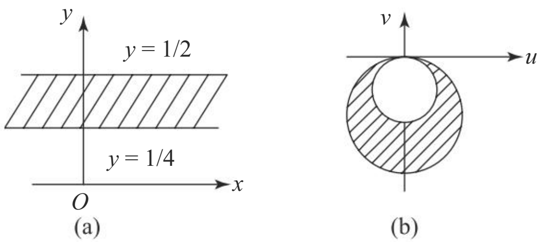


Fig. 3.16

Example 3.6

(i) Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ is the lemniscate $R^2 = \cos 2\phi$.

(ii) Show that the image of the circle $|z - 1| = 1$ under the transformation $w = z^2$ is the cardioid $R = 2(1 + \cos \phi)$.

(i) $$w = \frac{1}{z} \quad \text{or} \quad z = \frac{1}{w}$$

i.e. $$x + iy = \frac{1}{Re^{i\phi}} = \frac{1}{R}(\cos \phi - i \sin \phi)$$

∴ The transformation equations are

$$x = \frac{1}{R} \cos \phi \tag{1}$$

and $$y = -\frac{1}{R} \sin \phi \tag{2}$$

The given hyperbola is $\qquad x^2 - y^2 = 1 \qquad (3)$

Using (1) and (2) in (3), we get the image of (3) in the w-plane in polar co-ordinates

as $$\frac{1}{R^2} \cos^2 \phi - \frac{1}{R^2} \sin^2 \phi = 1$$

i.e. $\qquad R^2 = \cos 2\phi$, which is a lemniscate.

(ii) The given circle is $|z - 1| = 1$

i.e. $\qquad |x - 1 + iy| = 1$

i.e. $\qquad (x - 1)^2 + y^2 = 1 \quad$ or $\quad x^2 + y^2 - 2x = 0$

i.e. $\qquad r^2 - 2r \cos \theta = 0 \quad$ or $\quad r = 2 \cos \theta \qquad (4)$

The transformation is $w = z^2$

i.e. $\qquad R\, e^{i\phi} = r^2\, e^{i2\theta}$

∴ $\qquad R = r^2 \qquad (5)$

and $\qquad \phi = 2\theta \qquad (6)$

Eliminating r and θ from (4), (5) and (6), we get the polar equation of the image of (4).

From (4), $\qquad r^2 = 4 \cos^2 \theta$

$\qquad = 2(1 + \cos 2\theta)$

i.e. $\qquad R = 2(1 + \cos \phi)$, which is a cardioid.

Example 3.7 Find the image in the w-plane of the region of the z-plane bounded by the straight lines $x = 1$, $y = 1$ and $x + y = 1$ under the transformation $w = z^2$.

$$w = z^2 \quad \text{i.e. } u + iv = (x + iy)^2 = x^2 - y^2 + i2\,xy$$

$\therefore$ The transformation equations are

$$u = x^2 - y^2 \tag{1}$$

and
$$v = 2xy \tag{2}$$

From the discussion of the transformation $w = z^2$, we get the image of the line $x = 1$ as the parabola $v^2 = -4\,(u - 1)$ and the image of the line $y = 1$ as the parabola $v^2 = 4(u + 1)$.

The image of the line $x + y = 1$ (3)

is got by eliminating x and y from (1), (2) and (3).

Using (3) in (1) and (2), we have

$$u = x^2 - (1 - x)^2$$

and
$$v = 2x\,(1 - x)$$

i.e.
$$u = 2x - 1 \tag{4}$$

and
$$v = 2x\,(1 - x) \tag{5}$$

Eliminating x from (4) and (5), we get

$$v = (u + 1)\left\{1 - \frac{u + 1}{2}\right\}$$

i.e. $\quad v = \dfrac{1 - u^2}{2} \quad$ or $\quad u^2 = -\,2(v - 1/2)$,

which represents a parabola in the w-plane.

Thus the image of the region bounded by $x = 1$, $y = 1$ and $x + y = 1$ is the region bounded by the three parabolas $v^2 = -4(u - 1)$, $v^2 = 4(u + 1)$ and $u^2 = -2\,(v - 1/2)$ The corresponding regions in the two planes are shown in the Figs 3.17 (a) and (b).

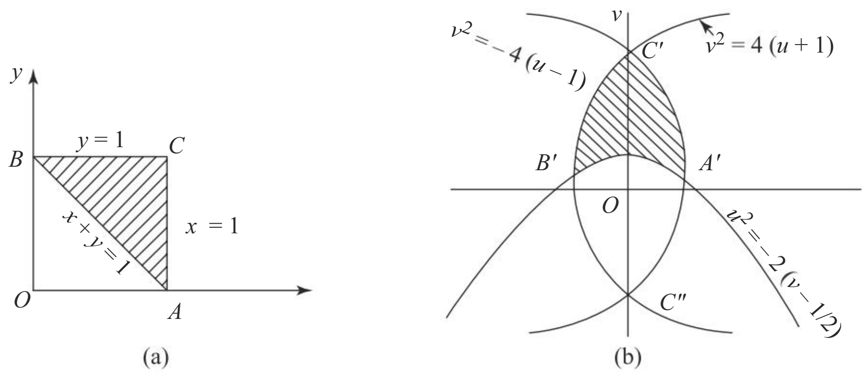


Fig. 3.17

Note ☑ The region $A'\,B'\,C''$ is also bounded by the three parabolas, but the corresponding region is that which contains the images of the points $A\,(1, 0)$, $B\,(0, 1)$ and $C\,(1, 1)$, namely the points $A'(1, 0)$, $B'(-1, 0)$ and $C'(0, 2)$.

Example 3.8 Find the image of the rectangular region bounded by the lines (i) $x = 1$, $x = 3$, $y = 1$ and $y = 2$ and (ii) $u = 1$, $u = 3$, $v = 1$ and $v = 2$ under the transformation $w = z^2$.

(i) Proceeding as in the discussion of the transformation $w = z^2$, we find that the images of the lines $x = 1$, $x = 3$, $y = 1$ and $y = 2$ are respectively the parabolas $v^2 = -4(u - 1)$, $v^2 = -36(u - 9)$, $v^2 = 4(u + 1)$ and $v^2 = 16(u + 4)$.

The given rectangular region in the z–plane lies in the first quadrant, bounded by $\theta = 0$ and $\theta = \pi/2$ in polar form.

Hence the image region lies in the upper half of the w-plane, since the images of $\theta = 0$ and $\theta = \pi/2$ are respectively $\phi = 0$ and $\phi = \pi$, as one of the transformation equations of $w = z^2$ in the polar form is $\phi = 2\theta$. The corresponding regions are shown in the Figs 3.18 (a) and (b)

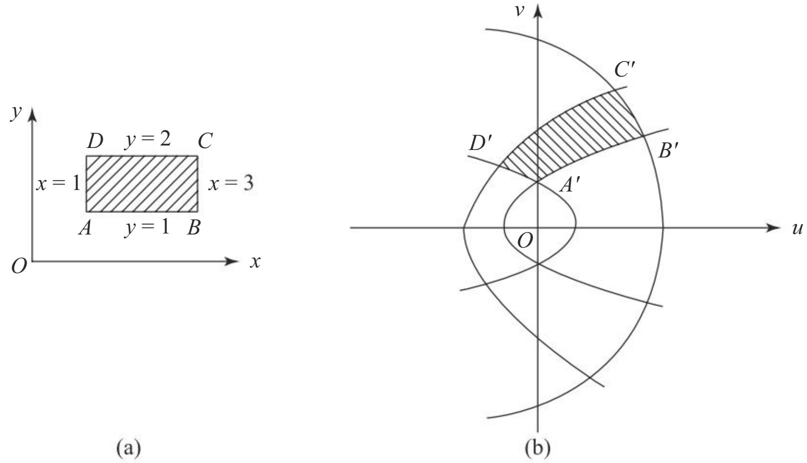


Fig. 3.18

(ii) The transformation equations corresponding to $w = z^2$ are $u = x^2 - y^2$ and $v = 2xy$.

$\therefore$ The images of $u = 1$, $u = 3$, $v = 1$ and $v = 2$ are respectively the rectangular hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 3$, $xy = 1/2$ and $xy = 1$ in the z-plane.

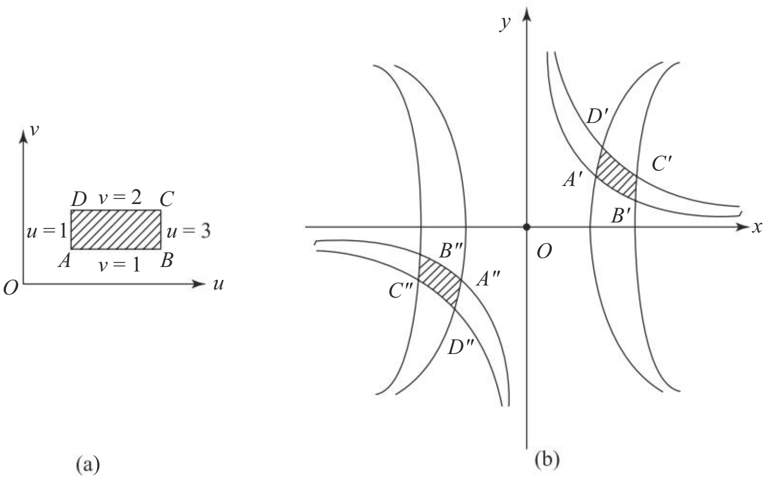


Fig. 3.19

The required image region is that bounded by these four hyperbolas. There are two regions bounded by these hyperbolas as shown in the Figs 3.19 (a) and (b).

Both the region in the z-plane are valid images of the given rectangular region in the w-plane.

This is because we are transforming from the w-plane to the z-plane by means of the transformation $z = w^{1/2}$, which is a two valued function.

Example 3.9 Find the images of the following under the transformation $w = e^z$:

(i) the line $y = x$, (ii) the segment of the y-axis, given by $0 \le y \le \pi$, (iii) the left half of the strip $0 \le y \le \pi$ and (iv) the right half of the strip $0 \le y \le \pi$.

The transformation equations of the mapping $w = e^z$ are given by

$$Re^{i\phi} = e^{x+iy} = e^x \cdot e^{iy}$$

i.e. $$R = e^x \qquad (1)$$

and $$\phi = y \qquad (2)$$

(i) The image of the line $y = x$ is

$$\log R = \phi \quad \text{or} \quad R = e^{\phi},$$

which is the polar equation of an equiangular spiral.

(ii) The image of the y-axis i.e., $x = 0$ is $R = 1$ [from (1)] i.e., the unit circle $|w| = 1$. The image of the $y = 0$ is $\phi = 0$ and that of the line $y = \pi$ is $\phi = \pi$.

$\therefore$ The image of the region $0 \le y \le \pi$ is the region defined by $0 \le \phi \le \pi$, i.e. the upper half of the w-plane.

Hence the image of the segment of $x = 0$, between $y = 0$ and $y = \pi$ is the semicircle $|w| = 1$, $v \ge 0$, as shown in the Figs 3.20 (a) and (b).

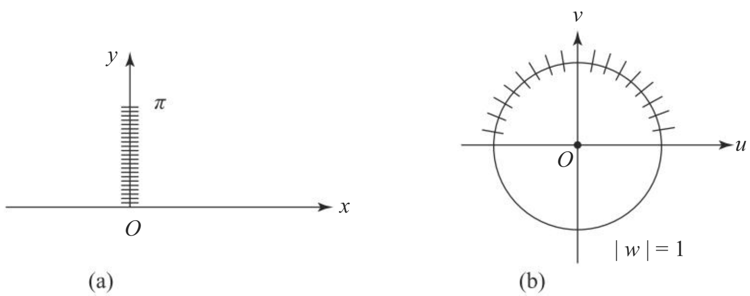


Fig. 3.20

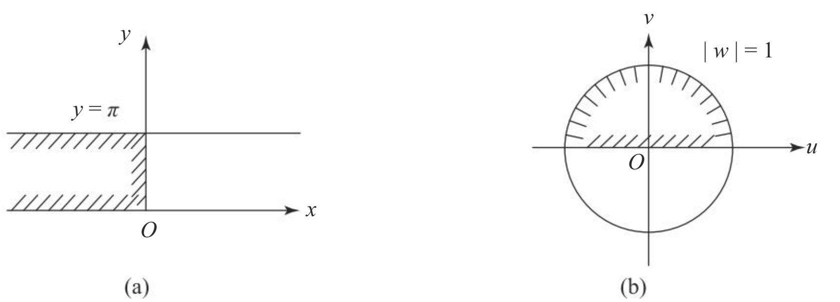


Fig. 3.21

(iii) The left half of the strip $0 \le y \le \pi$ is given by $0 \le y \le \pi$ and $x \le 0$.

$\therefore$ The image is given by $0 \le \phi \le \pi$ and $R \le 1$ or $|w| \le 1$, i.e. the interior of the unit circle $|w| = 1$ lying in the upper half of the w–plane.

The corresponding images are shown in the Figs 3.21 (a) and (b)

(iv) Similarly the right half of the strip $0 \le y \le \pi$ is mapped onto the exterior of the unit circle $|w| = 1$ lying in the upper half of the w-plane Figs 3.22 (a) and (b).

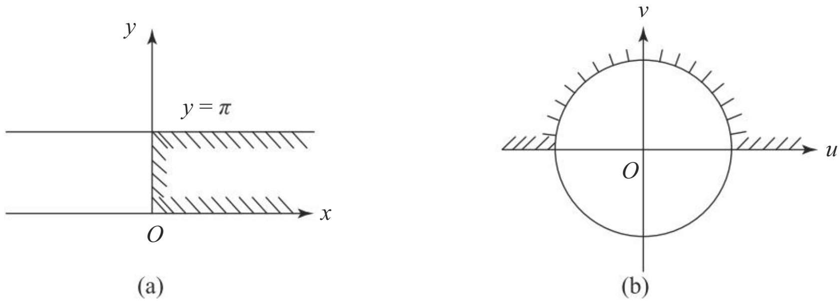


Fig. 3.22

Example 3.10 Find the maps of the boundary and interior of the rectangle formed by $x = \pm\dfrac{\pi}{2}$ and $y = \pm\, c$ in the z-plane under the transformation $w = \sin z$.

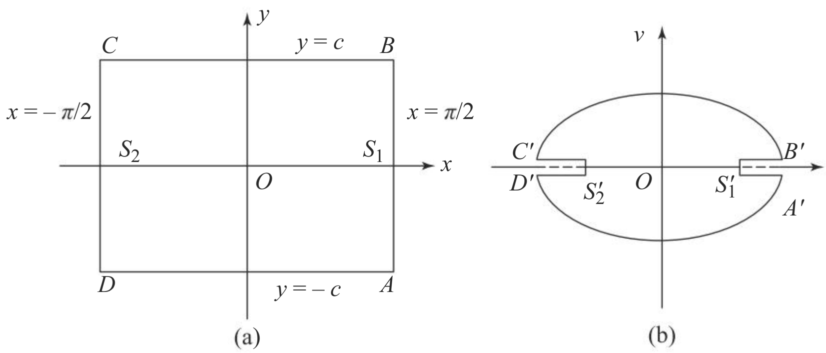


Fig. 3.23

The transformation equations of the mapping function $w = \sin z$ are

$$u = \sin x \cosh y \tag{1}$$

and
$$v = \cos x \sinh y \tag{2}$$

The part AS_1 of the side AB of the given rectangle is given by $x = \pi/2$ and $-c \le y \le 0$. Its image is given by

$$u = \cosh y \quad \text{and} \quad v = 0 \text{ [Fig. 3.23]}$$

Now u is a decreasing function of y in $-c \le y \le 0$, since $\dfrac{du}{dy} = \sinh y < 0$, when $y < 0$. Hence the image of AS_1 is given by $v = 0$ and $\cosh c \ge u \ge 1$, i.e. by the segment $A'S'_1$ of the u-axis in the w-plane. Similarly the image of $S_1 B$ is given by

$v = 0$ and $1 \le u \le \cosh c$, i.e. by the segment S'_1B' (which is the same as $S'_1 A'$) of the u-axis.

The image of the line segment BC, i.e. $y = c$, $\frac{\pi}{2} \ge x \ge -\pi/2$ is given by

$$u = \sin x \cosh c \text{ and}$$
$$v = \cos x \sinh c, \; -\pi/2 \le x \le \pi/2$$

i.e. the image of BC is the elliptic arc

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1, \; v > 0$$

[When $-\pi/2 \le x \le \pi/2$ and $c > 0$, $\cos x > 0$ and $\sinh c > 0$].

i.e. the image of BC is the upper half $B'C'$ of the ellipse

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1 \tag{3}$$

Similarly the images of the line segments CS_2, S_2D and DA of the z-plane are the line segments $C'S'_2$, S'_2D' (which is the same as S'_2C') and the lower half $D'A'$ of the ellipse (3).

Thus the image of the boundary of the rectangle consists of the ellipse and the two segments of the u-axis.

Hence the image of the rectangular region is the interior of the ellipse in the w-plane.

Example 3.11 Show that the transformation $w = \cos z$ maps the segment of the x-axis given by $0 \le x \le \pi/2$ into the segment $0 \le u \le 1$ of the u-axis. Show also that it maps the strip $y \ge 0$, $0 \le x \le \pi/2$ into the fourth quadrant of the w-plane.

The transformation equation of $w = \cos z$ are given by $u + iv = \cos(x + iy)$

$$= \cos x \cosh y - i \sin x \sinh y$$

i.e.
$$u = \cos x \cosh y \tag{1}$$
$$v = -\sin x \sinh y \tag{2}$$

The image of the x-axis, i.e. $y = 0$ is given by

$u = \cos x$ and $v = 0$

Since we consider the segment of the x–axis given by $0 \le x \le \pi/2$, $0 \le u \le 1$

Thus the image of the segment of $y = 0$, $0 \le x \le \pi/2$, is the segment of $v = 0$, $0 \le u \le 1$. The boundaries of the given strip are $x = 0$, $x = \pi/2$ and $y = 0$.

The image of $x = 0$ is given by $u = \cosh y$ and $v = 0$, i.e. $u \ge 1$ and $v = 0$.

The image of $x = \pi/2$ is given by $u = 0$ and $v = \sinh y$.

Since $y \ge 0$ for the given strip, $v \le 0$

Thus the image of $x = \pi/2$, $y \ge 0$ is $u = 0$, $v \le 0$

The image of $y = 0$, $0 \le x \le \pi/2$ is $v = 0$, $0 \le u \le 1$.

Thus the image of the given region is the region bounded by

$$v = 0, \; 0 \le u \le 1; \quad v = 0, \; u \ge 1 \text{ and } u = 0, \; v \le 0$$

i.e.
$$v = 0, \; u \ge 0 \quad \text{and} \quad u = 0, \; v \le 0$$

i.e. the fourth quadrant in the w-plane. The corresponding regions are shown in the Figs. 3.24 (a) and (b)

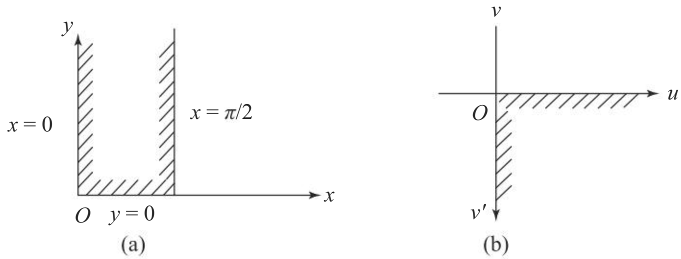


Fig. 3.24

Example 3.12 Find the image of the region defined by $0 \le x \le 3$, $-\pi/4 < y < \pi/4$ under the transformation $w = \sinh z$.

The transformation equations of $w = \sinh z$ are given by $u + iv = \sinh(x + iy)$

$$= -i \sin i(x + iy)$$
$$= -i \sin i(ix - y)$$
$$= \sinh x \cos y + i \cosh x \sin y$$

i.e. $$u = \sinh x \cos y \tag{1}$$

and $$v = \cosh x \sin y \tag{2}$$

The image of $x = 0$ is given by $u = 0$ and

$v = \sin y$ i.e. $u = 0$ and

$$-1/\sqrt{2} \le v \le 1/\sqrt{2} \qquad (\because -\pi/4 \le y \le \pi/4)$$

The image of $x = 3$ is given by

$$u = \sinh 3 \cos y \text{ and } v = \cosh 3 \cdot \sin y$$

i.e. the ellipse $\dfrac{u^2}{\sinh^2 3} + \dfrac{v^2}{\cosh^2 3} = 1$, $u > 0$, since $\cos y \ge 0$.

The image of $y = -\pi/4$ is given by

$$u = \frac{1}{\sqrt{2}} \sinh x \quad \text{and} \quad v = -\frac{1}{\sqrt{2}} \cosh x$$

i.e. the lower part of rectangular hyperbola $v^2 - u^2 = \dfrac{1}{2}$ $(\because v < 0)$

Similarly the image of $y = \pi/4$ is the upper part of the rectangular hyperbola $v^2 - u^2 = 1/2$. Thus the image of the given region is the region in the right part of the v-axis, bounded by the segment of the v-axis, the ellipse and the rectangular hyperbola as shown in the Figs 3.25 (a) and (b).

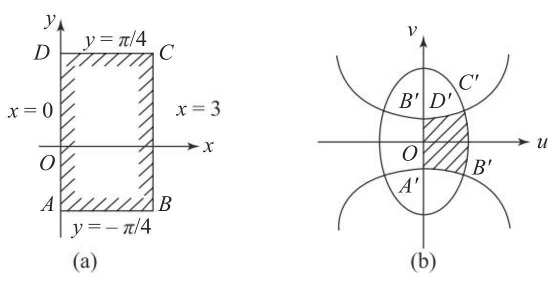


Fig. 3.25

Example 3.13 Show that the transformation $w = z + \dfrac{a^2 - b^2}{4z}$ transforms the circle $|z| = \dfrac{1}{2}(a+b)$ in the z-plane into an ellipse of semi-axes a, b in the w-plane.

In the discussion of the transformation $w = z + \dfrac{k^2}{z}$, we have proved that the image of the circle $|z| = c$, i.e., $r = c$ is the ellipse whose equation is

$$\frac{u^2}{\left(c + \dfrac{k^2}{c}\right)^2} + \frac{v^2}{\left(c - \dfrac{k^2}{c}\right)^2} = 1 \tag{1}$$

Putting $\quad k^2 = \dfrac{a^2 - b^2}{4}$ and $c = \dfrac{1}{2}(a+b)$ in (1), we get

$$c + \frac{k^2}{c} = \frac{1}{2}(a+b) + \frac{1}{2}(a-b) = a \text{ and}$$

$$c - \frac{k^2}{c} = \frac{1}{2}(a+b) - \frac{1}{2}(a-b) = b$$

$\therefore$ The image of $|z| = \dfrac{1}{2}(a+b)$, i.e., $r = \dfrac{1}{2}(a+b)$ under the transformation $w = z + \left(\dfrac{a^2 - b^2}{4}\right)/z$ is the ellipse $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$, whose semi-axes are a and b.

Example 3.14 Prove that the region outside the circle $|z| = 1$ maps onto the whole of the w-plane under the transformation $w = z + \dfrac{1}{z}$.

Proceeding as in the discussion of the transformation $w = z + \dfrac{k^2}{z}$, we can prove that the image of the circle $|z| = 1$, i.e. $r = 1$ is the segment of the real axis, given by $-2 \le u \le 2$ and that the image of the circle $|z| = a$, i.e. $r = a$ is the ellipse

$$\frac{u^2}{\left(a + \dfrac{1}{a}\right)^2} + \frac{v^2}{\left(a - \dfrac{1}{a}\right)^2} = 1 \tag{1}$$

The semi axes of this ellipse are $f(a) = a + 1/a$ and $g(a) = a - \dfrac{1}{a}$.

Now $f'(a) = 1 - 1/a^2 = \dfrac{a^2 - 1}{a^2} > 0$, when $a > 1$ and $g'(a) = 1 + 1/a^2 > 0$, for all a and hence when $a > 1$.

$\therefore$ f(a) and $g(a)$ are increasing functions of a, when $a > 1$.

Hence, when a increases, the semi axes of the ellipse (1) increase.
The region outside the circle $r = 1$ may be regarded as the area swept by the circle $r = a$, where $1 < a < \infty$.

Similarly the area swept by the ellipse $\dfrac{u^2}{\left(a+\dfrac{1}{a}\right)^2}+\dfrac{v^2}{\left(a-\dfrac{1}{a}\right)^2}=1, 1<a<\infty$, is the entire w-plane.

Hence the region outside the circle $r = 1$ maps onto the entire w-plane.

Example 3.15 Show that the transformation $w=\dfrac{1}{2}\left(ze^{-\alpha}+\dfrac{e^{\alpha}}{z}\right)$, where α is real, maps the upper half of the interior of the circle $|z| = e^{\alpha}$ onto the lower half of the w-plane.

The transformation equations of the mapping function $w=\dfrac{1}{2}\left(ze^{-\alpha}+\dfrac{1}{ze^{-\alpha}}\right)$ are given by

$$u+iv=\frac{1}{2}\left(re^{i\theta}\,e^{-\alpha}+\frac{1}{re^{-\alpha}}\,e^{-i\theta}\right)$$

i.e. $$u=\frac{1}{2}\left(r\,e^{-\alpha}+\frac{1}{re^{-\alpha}}\right)\cos\theta \tag{1}$$

and $$v=\frac{1}{2}\left(r\,e^{-\alpha}+\frac{1}{re^{-\alpha}}\right)\sin\theta \tag{2}$$

The boundary of interior of the circle $|z| = e^{\alpha}$ or $r = e^{\alpha}$ lying in the upper half of the z-plane consists of $\theta = 0$, $\theta = \pi$ and $r = e^{\alpha}$.

The image of $\theta = 0$ is given by

$$u=\frac{1}{2}\left(r\,e^{-\alpha}+\frac{1}{re^{-\alpha}}\right) \text{ and } v=0, \text{ from (1) and (2).}$$

i.e. $$u=\frac{1}{2}\left[\left(\sqrt{r}\;e^{-\alpha/2}-\frac{1}{\sqrt{r}\;e^{-\alpha/2}}\right)^2+2\right] \text{ and } v=0$$

i.e. $$u\geq\frac{1}{2}(0+2) \text{ and } v=0 \text{ or } u\geq 1 \text{ and } v=0.$$

Similarly the image of $\theta = \pi$ is given by $u \le -1$ and $v = 0$.

The image of $r = e^{\alpha}$ is given by

$u = \cos\theta$ and $v = 0$, from (1) and (2).

i.e., $-1 \le u \le 1$ and $v = 0$.

Thus the boundary of the semi-circle $r = e^{\alpha}$ lying in the upper half of the z-plane is the entire u-axis.

The interior of the semi-circle is given by $r < e^{\alpha}$ and $0 < \theta < \pi$.

When $r < e^{\alpha}$, $re^{-\alpha} < 1$

$$\therefore \qquad -\frac{1}{re^{-\alpha}}<-1$$

$$\therefore \qquad \left(re^{-\alpha} - \frac{1}{re^{-\alpha}}\right) < 0$$

$$\therefore \qquad \frac{1}{2}\left(re^{-\alpha} - \frac{1}{re^{-\alpha}}\right)\sin\theta < 0\text{, since } \sin\theta \text{ is positive when } 0 < \theta < \pi.$$

i.e. $\quad v < 0.$

Hence the interior of the semi-circle $|z| = e^{\alpha}$ lying in the upper half of the z-plane maps onto the lower half of the w-plane. The corresponding images are shown in the

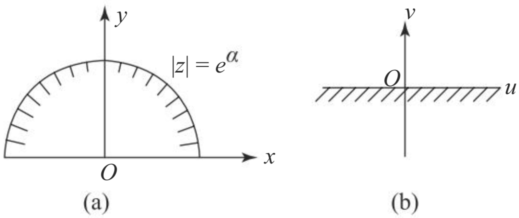


Fig. 3.26

Figs 3.26 (a) and (b).

EXERCISE 3(c)

Part A

(Short Answer Questions)

1. What do you mean by conformal mapping?
2. When is a transformation said to be isogonal? Prove that the mapping $w = \bar{z}$ is isogonal.
3. State the conditions for the transformation $w = f(z)$ to be conformal at a point.
4. Define critical point of a transformation.
5. Find the critical points of the transformation
$$w = \frac{1}{2}\left(ze^{-\alpha} + \frac{e^{\alpha}}{z}\right).$$
6. Find the critical points of the transformation $w^2 = (z - \alpha)(z - \beta)$.
7. Find the magnification factor of small lengths near $z = \pi/4$ under the transformation $w = \sin z$.

   Find the image of the circle $|z| = a$ under the following transformations

8. $w = z + 2 + 3i$ 9. $w = 2z$ 10. $w = (3 + 4i)z$
11. $w = (1 + i)z + 2 - i$ 12. $w = \dfrac{i}{2z}$
13. Find the image of the infinite strip $0 \le x \le 2$ under the transformation $w = iz$.
14. Find the image of the region that lies on the right of the y-axis under the transformation $w = iz + i$.

15. Find the image of the region $y > 1$ under the transformation $w = (1 - i)z$.
16. Find the image of the hyperbola $xy = 1$ under the transformation $w = 3z$.
17. Find the image of the hyperbola $x^2 - y^2 = a^2$ under the transformation $w = (1 + i)z$.
18. Find the image of the half-plane $y > c$, when $c > 0$ under the transformation $w = \dfrac{1}{z}$.
19. Find the image of the half-plane $x > c$, when $c < 0$ under the transformation $w = 1/z$.
20. Find the image of the line $y = mx$ under the transformation $w = \dfrac{i}{z}$.
21. Find the image of the region $0 < \theta < \pi/n$ in the z-plane under the transformation $w = z^n$, where n is a positive integer.
22. Find the images of the lines $x = 1$ and $y = 1$ under the transformation $w = iz^2$.
23. Find the image of the region $2 < |z| < 3$ under the transformation $w = z^2$.
24. Find the image of the region $\dfrac{\pi}{4} < \arg z < \dfrac{\pi}{2}$ under the transformation $w = z^2$.
25. Find the level curves of u and v under the transformation $w = -iz^2$.
26. Find the image of the infinite strip $0 \le x \le 1$ under the transformation $w = e^z$.
27. Find the image of the x-axis under the transformation $w = \cos z$.
28. Find the image of the y-axis under the transformation $w = \cos z$.
29. Find the image of the x-axis under the transformation $w = \sinh z$.
30. Find the image of the y-axis under the transformation $w = \sinh z$.
31. Find the equations of transformation for the mapping given by $w = \dfrac{a}{2}\left(z + \dfrac{1}{z}\right)$.
32. Find the images of the points whose polar co-ordinates are $(1, 0)$ and $(1, \pi)$ under the transformation $w = z + \dfrac{1}{z}$.

Part B

33. Find the image of the triangle with vertices at $z = i$, $z = 1 - i$, $z = 1 + i$ under the transformation (i) $w = 3z + 4 - 2i$ and (ii) $w = iz + 2 - i$.
34. Find the map of the square whose vertices are $z = 1 + i$, $-1 + i$, $-1 - i$ and $1 - i$ by the transformation $w = az + b$, where $a = \sqrt{2}\,(1 + i)$ and $b = 3 + 3i$.
35. Prove that the interior of the circle $|z| = a$ maps onto the interior of an ellipse in the w-plane under the transformation $w = x + \dfrac{iby}{a}$, $0 < b < a$. Is the transformation conformal?
36. Show that the transformation $w = \dfrac{1}{z}$ maps the circle $|z - 3| = 5$ onto the circle $\left|w + \dfrac{3}{16}\right| = \dfrac{5}{16}$. What is the image of the interior of the given circle in the z- plane?

37. Find the image of (i) the infinite strip $0 < y < \dfrac{1}{2c}$ and (ii) the quadrant $x > 0$, $y > 1$, under the transformation $w = \dfrac{1}{z}$.
38. Find the image of the triangle formed by the lines $y = x$, $y = -x$ and $y = 1$ under the transformation $w = z^2$.
39. Find the image of the region of the square whose vertices are $z = 0, 1, 1 + i$ and i under the transformation $w = z^2$.
40. When $a > 0$, show that $w = \exp(\pi z/a)$ transforms the infinite strip $0 \le y \le a$ onto the upper half of the w-plane.
41. Discuss the transformation $w = \log z$. Also prove that the whole of the z-plane maps onto the horizontal strip $-\pi \le v \le \pi$. (The principal value of $\log z$ is to be considered.)

    Note ☑ The properties of the transformation $w = \log z$ are identical with those of $w = e^z$, if we interchange z and w.
42. Discuss the transformation $w = \cos z$.
43. Discuss the transformation $w = \sinh z$.
44. Show that the transformation $w = \sin \dfrac{\pi z}{a}$ maps the region of the z-plane given by $y \ge 0$ and $-a/2 \le x \le a/2$ onto the upper half of the w-plane.
45. Find the image of the semi-infinite strip $0 \le x \le \pi, y \ge 0$ under the transformation $w = \cos z$.
46. Find the image of the region defined by $-\pi/4 \le x \le \pi/4$, $0 \le y \le 3$ under the transformation $w = \sin z$.
47. Show that the transformation $w = \cosh z$ maps (i) the segment of the y-axis from 0 to $\dfrac{i\pi}{2}$ onto the segment $0 \le u \le 1$ of the u-axis, (ii) the semi-infinite strip $x > 0$, $0 \le y \le \pi/2$ onto the first quadrant of the w-plane.
48. Find the image of the region in the z-plane given by $1 \le x \le 2$ and $-\pi/2 \le y \le \pi/2$ under the transformation $w = \sinh z$.
49. Show that the transformation $w = \dfrac{a}{2}\left(z + \dfrac{1}{z}\right)$ where a is a positive constant maps (i) the semi-circle $|z| = 1$ in the upper z-plane onto the segment $-a \le u \le a$ of the u-axis, (ii) the exterior of the unit circle in the upper z-plane onto the upper w-plane.
50. Show that the transformation

    $w = \dfrac{1}{2}\left(ze^{-\alpha} + \dfrac{e^{\alpha}}{z}\right)$, where α is real, maps the interior of the circle $|z| = 1$ onto the exterior of an ellipse whose major and minor axes are of lengths 2 cosh α and 2 sinh α respectively.

[**Hint**: The image of the circle $|z| = 1$ is the ellipse $\dfrac{u^2}{\cosh^2\alpha}+\dfrac{v^2}{\sinh^2\alpha}=1$. The image of any point inside $|z| = 1$, for example, the point (a, θ), where $a<1$ is a point outside the ellipse, since $|u\,(a, \theta)| > |u\,(1, \theta)|$ and $|v(a, \theta)| > |v(1, \theta)|$.]

3.7 BILINEAR AND SCHWARZ–CHRISTOFFEL TRANSFORMATIONS

3.7.1 Bilinear Transformation

The transformation $w=\dfrac{az+b}{cz+d}$, where a, b, c, d are complex constants such that $ad - bc \neq 0$ is called *a bilinear transformation.* It is also called *Mobius* or *linear fractional transformation.*

Now $\dfrac{\mathrm{d}w}{\mathrm{d}z}=\dfrac{ad-bc}{(cz+d)^2}$. If $ad - bc = 0$, every point of the z-plane becomes a critical point of the bilinear transformation.

The transformation can be rewritten as

$$w=\frac{a}{c}-\frac{(ad-bc)}{c(cz+d)}\text{. Hence, if } ad - bc = 0,$$

the transformation takes the form $w=\dfrac{a}{c}$, which has no meaning as a mapping function. Due to these reasons, we assume that $ad - bc \neq 0$. The expression $(ad - bc)$ is called the *determinant* of the bilinear transformation.

The inverse of the transformation $w=\dfrac{az+b}{cz+d}$ is $z=\dfrac{-dw+b}{cw-a}$, which is also a bilinear transformation.

The images of all points in the z-plane are uniquely found, except for the point $z=-\dfrac{d}{c}$. Similarly the image of every point in the w-plane is a unique point, except for the point $w=\dfrac{a}{c}$. If we assume that the images of the points $z=-\dfrac{d}{c}$ and $w=\dfrac{a}{c}$ are the points at infinity in the w- and z-planes respectively, the bilinear transformation becomes one-to-one between all the points in the two planes.

To discuss the transformation $w=\dfrac{az+b}{cz+d}$, (1)

we express it as the combination of simple transformations discussed in the previous section.

When $c \neq 0$, (1) can be expressed as

$$w=\frac{a}{c}+\left(\frac{bc-ad}{c}\right)\cdot\frac{1}{cz+d}$$

If we make the substitutions

$$w_1 = cz + d \tag{2}$$

$$w_2 = \frac{1}{w_1} \tag{3}$$

then $$w = Aw_2 + B \tag{4}$$

where $$A=\frac{bc-ad}{c} \quad \text{and} \quad B=\frac{a}{c}\cdot$$

The substitutions (2), (3) and (4) can be regarded as transformations from the z-plane onto the w_1-plane, from the w_1-plane onto the w_2-plane and from the w_2-plane onto the w-plane respectively.

We know that each of the transformations (2), (3) and (4) maps circles and straight lines into circles and straight lines (since straight lines may be regarded as circles of infinite radii).

Hence the bilinear transformation (1) maps circles and straight lines onto circles and straight lines, in general. When $c = 0$, (1) becomes $w=\left(\frac{a}{d}\right)z+\left(\frac{b}{d}\right)(d\neq 0)$ i.e. (1) reduces to the form $w = Az + B$. This transformation also maps circles into circles.

Thus the bilinear transformation always maps circles into circles with lines as limiting cases.

3.7.2 Definition

If the image of a point z under a transformation $w = f(z)$ is itself, then the point is called *a fixed point* or *an invariant point* of the transformation.

Thus a fixed point of the transformation $w = f(z)$ is given by $z = f(z)$.

The fixed points of the bilinear transformation $w=\frac{az+b}{cz+d}$ are given by $\frac{az+b}{cz+d}=z$.

As this is a quadratic equation in z, we will get two fixed points for the bilinear transformation.

Note ☑

1. A bilinear transformation can be uniquely found, if the images w_1, w_2, w_3 of any three points z_1, z_2, z_3 of the z-plane are given.
   Let the bilinear transformation required be

$$w=\frac{az+b}{cz+d} \tag{1}$$

(1) can be re-written as $w=\dfrac{\left(\frac{a}{d}\right)z+\left(\frac{b}{d}\right)}{\left(\frac{c}{d}\right)z+1}$ or $\dfrac{Az+B}{Cz+1}$

Since the images of z_1, z_2 and z_3 are w_1, w_2 and w_3 respectively, we have

$$w_1 = \frac{Az_1 + B}{Cz_1 + 1} \tag{2}$$

$$w_2 = \frac{Az_2 + B}{Cz_2 + 1} \tag{3}$$

and $$w_3 = \frac{Az_3 + B}{Cz_3 + 1} \tag{4}$$

Equations (2), (3) and (4) are three equations in three unknowns A, B, C. Solving them we get the values of A, B, C uniquely and hence the bilinear transformation (1) uniquely.

2. If a set of three points and their images by a bilinear transformation are given, it can be found out by using the cross-ratio property of the bilinear transformation, which is given below.

3.7.3 Definition

If z_1, z_2, z_3, z_4 are four points in the z-plane, then $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ is called the *cross- ratio of these points.*

Cross-ratio property of a bilinear transformation The cross-ratio of four points is invariant under a bilinear transformation.

i.e. if w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 respectively under a bilinear transformation, then

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

Proof

Let the bilinear transformation be $w = \dfrac{az + b}{cz + d}$

Then $$w_i - w_j = \frac{az_i + b}{cz_i + d} - \frac{az_j + b}{cz_j + d} = \frac{(ad - bc)(z_i - z_j)}{(cz_i + d)(cz_j + d)}$$

$\therefore$ $$(w_1 - w_2)(w_3 - w_4) = \frac{(ad - bc)^2 (z_1 - z_2)(z_3 - z_4)}{(cz_1 + d)(cz_2 + d)(cz_3 + d)(cz_4 + d)}$$

and $$(w_1 - w_4)(w_3 - w_2) = \frac{(ad - bc)^2 (z_1 - z_4)(z_3 - z_2)}{(cz_1 + d)(cz_2 + d)(cz_3 + d)(cz_4 + d)}$$

$\therefore$ $$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

Note ☑ To get the bilinear transformation that maps z_2, z_3, z_4 of the z-plane onto w_2, w_3, w_4, we assume that the image of z under this transformation is w and make use of the invariance of the cross-ratio of the four points z, z_2, z_3, z_4. Thus

$$\frac{(w-w_2)(w_3-w_4)}{(w-w_4)(w_3-w_2)}=\frac{(z-z_2)(z_3-z_4)}{(z-z_4)(z_3-z_2)} \tag{1}$$

(1) ensures that the images of $z = z_2, z_3, z_4$ are respectively $w = w_2, w_3, w_4$. Now, simplifying (1) and solving for w, we get the required bilinear transformation in the form $w=\dfrac{az+b}{cz+d}$.

3.8 SCHWARZ-CHRISTOFFEL TRANSFORMATION

3.8.1 Definition

The transformation that maps the boundary of a given polygon in the w-plane onto the x-axis (and hence maps the vertices of the polygon onto points on the x-axis) and the interior of the polygon onto the upper half of the z-plane is called *Schwarz-Christoffel transformation.*

Specifically, if $x_1, x_2, \ldots, x_n$ that are points on the x-axis such that $x_1 < x_2 < x_3 < \ldots < x_n$, are the images of the vertices $w_1, w_2, \ldots, w_n$ of a polygon in the w-plane and $\alpha_1, \alpha_2, \ldots \alpha_n$ are the corresponding interior angles of the polygon, then the required Schwarz-Christoffel transformation is given by

$$\frac{\mathrm{d}w}{\mathrm{d}z}=A(z-x_1)^{\frac{\alpha_1}{\pi}-1}\cdot(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots(z-x_n)^{\frac{\alpha_n}{\pi}-1},$$

where A is an arbitrary complex constant.

Proof

[The proof is in the nature of verification of the mapping of the transformation.]

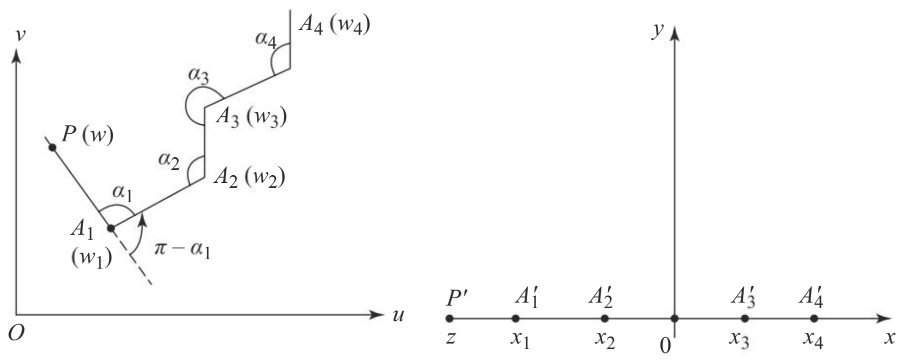


Fig. 3.27

The given transformation is

$$\mathrm{d}w=A(z-x_1)^{\frac{\alpha_1}{\pi}-1}\cdot(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots(z-x_n)^{\frac{\alpha_n}{\pi}-1}\cdot\mathrm{d}z$$

$$\therefore \text{ amp } (\mathrm{d}w) = \text{amp } (A) + \left(\frac{\alpha_1}{\pi} - 1\right) \text{amp } (z - x_1) + \left(\frac{\alpha_2}{\pi} - 1\right) \text{amp } (z - x_2) + \cdots +$$

$$\left(\frac{\alpha_n}{\pi} - 1\right) \text{amp } (z - x_n) \tag{1}$$

[$\because$ amp $(z_1 z_2 \ldots)$ = amp (z_1) + amp (z_2) + ... and amp $(z^k) = k$ amp (z)]

Let the image of $P(w)$ be $P'(z)$ as shown in Fig. 3.27.

When P moves towards A_1, P' moves towards A'_1. As long as $P'(z)$ is on the left of A'_1 (x_1), $z - x_1, z - x_2, \ldots, z - x_n$ are all negative real numbers.

$\therefore$ amp $(z - x_1)$ = amp $(z - x_2)$ = ... = amp $(z - x_n) = \pi$

But once $P'(z)$ has crossed $A'_1(x_1)$, i.e., when $x_1 < z < x_2$, $z - x_1$ is a positive real number and hence amp $(z - x_1) = 0$, while amp $(z - x_2)$ = amp $(z - x_3)$ = ... = amp $(z - x_n) = \pi$

Also amp(A) and amp(dz) do not change.

[$\because$ A is a constant and dz is positive hence amp (dz) = 0]

Thus when z croses A'_1, amp $(z - x_1)$ suddenly changes from π to 0 or undergoes an increment of $-\pi$.

$\therefore$ From (1), we get

$$\text{Increase in amp } (\mathrm{d}w) = \left(\frac{\alpha_1}{\pi} - 1\right)(-\pi) = \pi - \alpha_1 .$$

This increase is in the anticlockwise direction. This means that when $P(w)$, moving along PA_1, reaches A_1, it changes its direction through an angle $\pi - \alpha_1$ in the anticlockwise sense and then starts moving along A_1A_2. Similarly when $P(w)$ reaches A_2, it turns through an angle $\pi - \alpha_2$ and then starts moving along $A_2 A_3$.

Proceeding further, we find that as $P(w)$ moves along the boundary of the polygon, $P'(z)$ moves along the x-axis and conversely. Now when a person walks along the boundary of the polygon in the w-plane in the anticlockwise sense, the interior of the polygon lies to the left of the person. Hence the corresponding area in the z-plane should lie to the left of the person, when he or she walks along the corresponding path in the z-plane, i.e. along the x-axis from left to right. Clearly the corresponding area is the upper half of the z-plane.

Thus the interior of the polygon in the w-plane is mapped onto the upper half of the z-plane.

Note ☑

1. Integrating (1), the Schwarz-Christoffel transformation can also be expressed as $w = A\int (z - x_1)^{\frac{\alpha_1}{\pi} - 1} \cdot (z - x_1)^{\frac{\alpha_2}{\pi} - 1} \cdots (z - x_n)^{\frac{\alpha_n}{\pi} - 1} \cdot \mathrm{d}z + B$

   where B is a complex constant of integration.
2. The transformation which maps a polygon in the z-plane onto the real axis of the w-plane is got by interchanging z and w in the above transformation.
3. It is known that not more than three of the points $x_1, x_2, \ldots, x_n$ can be chosen arbitrarily.

4. It is advantageous to choose one point, say x_n, at infinity, as explained below.

If we take $A=\dfrac{\lambda}{(-x_n)^{\frac{\alpha_n}{\pi}-1}}$, where λ is a constant, the transformation can be written as

$$\frac{\mathrm{d}w}{\mathrm{d}z}=\lambda(z-x_1)^{\frac{\alpha_1}{\pi}-1}\cdot(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots(z-x_{n-1})^{\frac{\alpha_{n-1}}{\pi}-1}\cdot\left(\frac{x_n-z}{x_n}\right)^{\frac{\alpha_n}{\pi}-1}$$

As $x_n\to\infty$, the transformation reduces to

$$\frac{\mathrm{d}w}{\mathrm{d}z}=\lambda(z-x_1)^{\frac{\alpha_1}{\pi}-1}\cdot(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots\cdot(z-x_{n-1})^{\frac{\alpha_{n-1}}{\pi}-1}$$

This means that, if x_n is at infinity, the factor $(z-x_n)^{\frac{\alpha_n}{\pi}-1}$ is absent in the transformation. Thus the R.H.S. of the transformation contains one factor less than the original form.

5. Infinite open polygons can be considered as limiting cases of closed polygons.

WORKED EXAMPLE 3(d)

Example 3.1 Find the invariant points of the transformation $w=-\dfrac{2z+4i}{iz+1}$. Prove also that these two points together with any point z and its image w, form a set of four points having a constant cross ratio.

The invariant points of the transformation are given by

$$z=-\frac{2z+4i}{iz+1}$$

i.e. $\quad iz^2+3z+4i=0 \quad$ or $\quad z^2-3iz+4=0$

i.e. $\quad (z-4i)(z+i)=0$

$\therefore$ The invariant points are $4i$ and $-i$. Taking $z_1=z, z_2=w=-\dfrac{2z+4i}{iz+1}$, $z_3=4i$ and $z_4=-i$, the cross-ratio of the four points z_1, z_2, z_3 and z_4 is given by

$$(z_1,z_2,z_3,z_4)=\frac{\left(z+\dfrac{2z+4i}{iz+1}\right)(4i+i)}{(z+i)\left(4i+\dfrac{2z+4i}{iz+1}\right)}$$

$$=\frac{5i\left(iz^2+3z+4i\right)}{(z+i)(-2z+8i)}$$

$$= \frac{5i\left(iz^2 + 3z + 4i\right)}{2i\left(iz^2 + 3z + 4i\right)}$$

$= 5/2$

$=$ a constant, independent of z.

Example 3.2 Find the bilinear transformation that maps the points $1 + i, - i, 2 - i$ of the z-plane into the points $0, 1, i$ of the w-plane.

Taking $z_1 = 1 + i$, $z_2 = - i$, $z_3 = 2 - i$ and $w_1 = 0$, $w_2 = 1$, $w_3 = i$ and using the invariance of the cross-ratio (z, z_1, z_2, z_3), we have

$$\frac{(w-0)(1-i)}{(w-i)(1-0)} = \frac{(z-1-i)(-2)}{(z-2+i)(-1-2i)}$$

i.e.
$$\frac{w-i}{w} = \frac{(z-2+i)(-1-2i)(1-i)}{(z-1-i)(-2)}$$

i.e.
$$1 - \frac{i}{w} = \frac{(3+i)(z-2+i)}{2(z-1-i)}$$

$\therefore$
$$\frac{i}{w} = 1 - \frac{(3+i)z - 7 + i}{2z - 2 - 2i}$$
$$= \frac{-(1+i)z + 5 - 3i}{2z - 2 - 2i}$$

$\therefore$
$$w = \frac{(2z - 2 - 2i)}{-i\{-(1+i)z + 5 - 3i\}}$$

i.e. the required bilinear transformation is

$$w = \frac{2z - 2 - 2i}{(i-1)z - 3 - 5i}$$

Example 3.3 Find the bilinear transformation which maps the points (i) $i, - 1, 1$ of the z-plane into the points $0, 1, \infty$ of the w-plane respectively (ii) $z = 0, z = 1$ and $z = \infty$ into the points $w = i$, $w = 1$ and $w = - i$.

(i) $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$

i.e.,
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \tag{1}$$

To avoid the substitution of $w_3 = \infty$ in (1) directly, we put $w_3 = \dfrac{1}{w_3'}$ simplify and then put $w'_3 = 0$. Thus (1) becomes

$$\frac{(w-w_1)(w_2\,w_3'-1)}{(w\,w_3'-1)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \tag{2}$$

Using the given values $z_1 = i$, $z_2 = -1$, $z_3 = 1$, $w_1 = 0$, $w_2 = 1$ and $w'_3 = 0$, we get

$$\frac{w(-1)}{(-1)1}=\frac{(z-i)(-2)}{(z-1)(-1-i)}, \text{ i.e., } w=\frac{(1-i)(z-i)}{z-1}$$

(ii)
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$=\frac{(z-z_1)(z_2\,z_3'-1)}{(z\,z_3'-1)(z_2-z_1)}, \text{ where } z_3'=\frac{1}{z_3}$$

Using the values $z_1 = 0$, $z_2 = 1$, $z'_3 = 0$ and $w_1 = i$, $w_2 = 1$ and $w_3 = -i$, we get

$$\frac{(w-i)(1+i)}{(w+i)(1-i)}=\frac{z(-1)}{(-1)\cdot 1}$$

i.e.
$$\frac{w-i}{w+i}=\frac{(1-i)z}{1+i}$$

i.e.
$$\frac{2w}{2i}=\frac{(1-i)z+(1+i)}{(1+i)-(1-i)z}\left(=\frac{Nr+Dr}{Dr-Nr}\right)$$

$\therefore$
$$w=\frac{(1+i)z+(i-1)}{(1+i)-(1-i)z} \quad \text{or} \quad w=\frac{z+i}{iz+1}.$$

Example 3.4 If a, b are the two fixed points of a bilinear transformation, show that it can be written in the form (i) $\dfrac{w-a}{w-b}=k\left(\dfrac{z-a}{z-b}\right)$, where k is a constant, if $a \neq b$;

(ii) $\dfrac{1}{w-a}=\dfrac{1}{z-a}+c$, where c is a constant, if $a = b$.

(i) Since the images of a and b are a and b respectively, $(w, a, w_3, b) = (z, a, z_3, b)$

i.e.
$$\frac{(w-a)(w_3-b)}{(w-b)(w_3-a)}=\frac{(z-a)(z_3-b)}{(z-b)(z_3-a)}$$

i.e.
$$\frac{w-a}{w-b}=\left[\frac{(z_3-b)(w_3-a)}{(z_3-a)(w_3-b)}\right]\left(\frac{z-a}{z-b}\right)$$

i.e. $$\frac{w-a}{w-b} = k\left(\frac{z-a}{z-b}\right), \text{ where } k \text{ is a constant.}$$

(ii) Since the image of $z = a$ is $w = a$, the bilinear transformation can assumed as

$$w - a = \frac{z-a}{cz+d} \tag{1}$$

The fixed points of (1) are given by

$(z - a)\,[(cz + \mathrm{d}) - 1] = 0$

i.e. $$c\,(z-a)\left[z + \frac{d-1}{c}\right] = 0 \tag{2}$$

Since both the roots of (2) are equal to a,

$$\frac{d-1}{c} = -a \quad \therefore \quad d = -ca + 1 \tag{3}$$

Using (3) in (1), the required bilinear transformation is $w - a = \dfrac{z-a}{cz - ca + 1}$

i.e. $$\frac{1}{w-a} = \frac{c\,(z-a)+1}{z-a}$$

i.e. $$\frac{1}{w-a} = \frac{1}{z-a} + c, \text{ where } c \text{ is a constant.}$$

Example 3.5 Show that the transformation $w = \dfrac{z-1}{z+1}$ maps the unit circle in the w-plane onto the imaginary axis in the z-plane. Find also the images of the interior and exterior of the unit circle.

The image of the unit circle $|w| = 1$ is given by

$$\left|\frac{z-1}{z+1}\right| = 1, \text{ i.e. } |z - 1| = |z + 1|$$

i.e. $|(x - 1) + iy| = |(x + 1) + iy|$

i.e. $(x - 1)^2 + y^2 = (x + 1)^2 + y^2$

i.e. $-2x = 2x$ or $x = 0$, which is the imaginary axis.

The image of the interior of the unit circle i.e. $|w| < 1$ is given by $|z - 1| < |z + 1|$ i.e. $-2x < 2x$ or $x > 0$, which is the right half of the z-plane.

Similarly the image of the exterior of the circle $|w| = 1$ is the left half of the z-plane.

Example 3.6 Show that the transformation $w = \dfrac{z-i}{1-iz}$ maps (i) the interior of the circle $|z| = 1$ onto the lower half of the w-plane and (ii) the upper half of the z-plane onto the interior of the circle $|w| = 1$.

$$w = \frac{z - i}{1 - iz} \tag{1}$$

$\therefore$ $$w - iwz = z - i$$

i.e. $$z\,(1 + iw) = w + i$$

$\therefore$ $$z = \frac{w + i}{1 + iw} \tag{2}$$

(2) is the inverse transformation of (1). The interior of the circle $|\,z\,| = 1$ is given by $|\,z\,| < 1$.

From (2), the image of $|\,z\,| < 1$ is given by

$$\left|\frac{w + i}{1 + i\,w}\right| < 1\text{, i.e. } |\,w + i\,| < |\,1 + iw\,|$$

i.e. $$|\,u + i\,(v + 1)| < |\,(1 - v) + iu\,|$$

i.e. $$u^2 + (v + 1)^2 < (1 - v)^2 + u^2$$

i.e. $$2v < -\,2v \quad \text{or} \quad 4v < 0 \quad \text{or} \quad v < 0,$$

i.e., the lower half of the w-plane.

(2) can be written as

$$\begin{aligned} x + iy &= \frac{u + i\,(v + 1)}{(1 - v) + iu} \\ &= \frac{[u + i\,(v + 1)]\,[(1 - v) - iu]}{(1 - v)^2 + u^2} \\ &= \frac{u\,[1 - v + v + 1] + i\,(1 - v^2 - u^2)}{u^2 + (1 - v)^2} \end{aligned}$$

$\therefore$ $$x = \frac{2u}{u^2 + (1 - v)^2} \tag{3}$$

and $$y = \frac{1 - u^2 - v^2}{u^2 + (1 - v)^2} \tag{4}$$

The upper half of the z-plane is given by $y > 0$. Its image is given by

$$\frac{1 - u^2 - v^2}{u^2 + (1 - v)^2} > 0 \ \text{[from (4)]}$$

i.e. $$u^2 + v^2 < 1 \ \text{ or } \ |\,w\,|^2 < 1$$

i.e. $$|\,w\,| < 1$$

i.e. the interior of the circle $|\,w\,| = 1$.

Example 3.7 Find the most general bilinear transformation that maps the upper half of the z-plane onto the interior of the unit circle in the w-plane.

Let the required bilinear transformation be

$$w = \frac{az + b}{cz + d} \tag{1}$$

Since the image of $y > 0$ has to be $|w| < 1$, the boundaries of the two regions must correspond i.e. the image of $y = 0$ must be $|w| = 1$. Since three points determine a circle uniquely, we shall make three convenient points lying on $y = 0$ map into three points on $|w| = 1$. Let us assume that the three points $z = 0$, $z = \infty$ and $z = 1$ map onto points on the circle $|w| = 1$. Thus, when $z = 0$, $|w| = 1$.

$\therefore$ From (1), we get $$1 = \left|\frac{b}{d}\right|, \text{ i.e., } |b| = |d| \tag{2}$$

Rewriting (1), we have $$w = \frac{a + b/z}{c + d/z} \tag{1)'$$

When $z = \infty$, $|w| = 1$.

$\therefore$ From (1)′, we get $$\left|\frac{a}{c}\right| = 1, \quad \text{i.e. } |a| = |c| \tag{3}$$

If $a = 0$, then, from (3), we see that $c = 0$

In this case, the transformation (1) becomes $w = \frac{b}{d}$, which will map the whole of z-plane onto a single point $w = \frac{b}{d}$, which is not true. Hence $a \neq 0$ and so $c \neq 0$, from (3).

$\therefore$ $\frac{a}{c} \neq 0$, such that $\left|\frac{a}{c}\right| = 1$, from (3)

$\therefore$ $\frac{a}{c}$ may be taken as $e^{i\theta}$, where θ is real.

Again, re-writing (1), $w = \frac{a}{c}\,\frac{(z + b/a)}{(z + d/c)}$

i.e. $$w = e^{i\theta}\left(\frac{z + b/a}{z + d/c}\right)$$

Putting $\frac{b}{a} = -\alpha$ and $\frac{d}{c} = -\beta$, where α and β are complex, the required transformation becomes

$$w = e^{i\theta}\left(\frac{z - \alpha}{z - \beta}\right) \tag{4}$$

From (2) and (3), we have $\left|\dfrac{b}{a}\right| = \left|\dfrac{d}{c}\right|$

i.e. $\quad |\alpha| = |\beta| \qquad (5)$

We have assumed that the point $z = 1$ also maps onto a point on the circle $|w| = 1$.

$\therefore$ From (4), $1 = \left|e^{i\theta}\right| \left|\dfrac{1-\alpha}{1-\beta}\right|$

i.e. $\quad |1 - \alpha| = |1 - \beta| \qquad (6)$

From (5) and (6), we find that either $\alpha = \beta$ or $\bar{\alpha} = \beta$.

If we assume that $\alpha = \beta$, the transformation reduces to $w = e^{i\theta}$, which will map the whole of the z-plane into a single point, which is not true. Hence $\beta = \bar{\alpha}$.

$\therefore$ The required transformation is $w = e^{i\theta}\left(\dfrac{z-\alpha}{z-\bar{\alpha}}\right)$. Since α is arbitrary, it can be taken as any point in the upper half of the z-plane.

The image of $z = \alpha$ is $w = 0$, which lies inside $|w| = 1$.

Thus the upper half of the z-plane maps onto $|w| < 1$ by the transformation

$w = e^{i\theta}\left(\dfrac{z-\alpha}{z-\bar{\alpha}}\right)$, where α is any point in $y > 0$.

Example 3.8 Find the transformation that will map the strip $x \geq 1$ and $0 \leq y \leq 1$ of the z-plane into the half-plane (*i*) $v \geq 0$ and (ii) $u \geq 0$.

(i) Consider the isosceles triangle BAC, which is a three sided polygon. [Fig. 3.28]

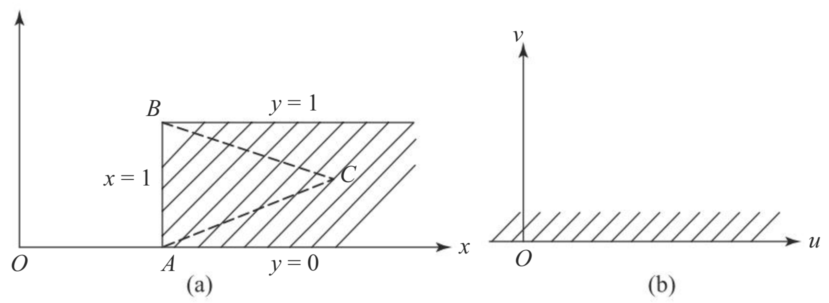


Fig. 3.28

When $C \to \infty$, the interior of this triangle becomes the region of the given strip.

In the limit, the interior angles of the polygon are $\dfrac{\pi}{2}, \dfrac{\pi}{2}$ and 0.

Let us assume that the images of B, A, C are $w = -1$, 1 and ∞ respectively. Then the required Schwarz-Christoffel transformation is

$$\frac{dz}{dw} = A\,(w+1)^{\frac{\pi}{2\pi}-1} \cdot (w-1)^{\frac{\pi}{2\pi}-1},$$

omitting the factor corresponding to $w = \infty$, as per Note (4) under the discussion of the transformation.

Note ☑ z and w are interchanged in the Schwarz-Christoffel transformation formula, as we are mapping a polygon in the z-plane onto the upper half of the w-plane.

i.e. $$\frac{dz}{dw} = A(w+1)^{-1/2} . (w-1)^{-1/2}$$

$\therefore$ $$= \frac{A}{\sqrt{w^2-1}}$$

Integrating with respect to w, we get

$$z = A \cosh^{-1} w + B$$

When $z = 1, \quad w = 1, \quad \therefore B = 1$

When $z = 1 + i, \quad w = -1 \quad \therefore 1 + i = A \cosh^{-1}(-1) + 1$

i.e. $i = A . i\pi$ $\quad [\because \cosh i\pi = \cos \pi = -1]$

$$A = \frac{1}{\pi}$$

$\therefore$ Required transformation is $z = \frac{1}{\pi} \cosh^{-1} w + 1$

i.e. $\cosh^{-1} w = \pi(z-1)$ or $w = \cosh \pi(z-1)$

(ii) Put $w' = u' + iv' = i(u + iv) = iw$

This means that $u - v$ system is rotated about the origin through $\frac{\pi}{2}$ in the positive direction giving $u' - v'$ system. Hence $v' \geq 0$ corresponds to $u \geq 0$.

$\therefore$ The required transformation that maps the given region in the z-plane onto $u \geq 0$ is

$$i\, w = \cosh \pi(z-1)$$

i.e. $$w = -i \cosh \pi(z-1)$$

Example 3.9 Find the transformation that maps the semi-infinite strip $y \geq 0, -a \leq x \leq a$ onto the upper half of the w-plane. Make the points $z = -a$ and $z = a$ correspond to the points $w = -1$ and $w = 1$.

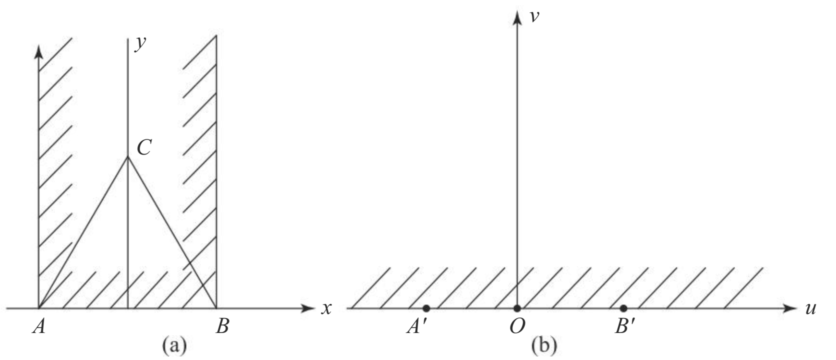


Fig. 3.29

Proceeding as in the previous example, the required transformation is

$$\frac{dz}{dw}=\frac{A}{\sqrt{w^2-1}}=\frac{iA}{i\sqrt{w^2-1}}=\frac{k}{\sqrt{1-w^2}} \quad \text{[Fig. 3.29]}$$

Note ☑ This change is made to simplify the evaluation of constants after integration.

$\therefore$ $$z=k\int\frac{dw}{\sqrt{1-w^2}}+B$$

$$= k\sin^{-1} w + B \tag{1}$$

When $z=-a,\quad w=-1$ and when $z=a$, $w=1$

$\therefore$ $k(-\pi/2)+B=-a$ and $k(\pi/2)+B=a$

Solving, we get $k=\dfrac{2a}{\pi}$ and $B=0$.

Using these values in (1), the required transformation is $z=\dfrac{2a}{\pi}\sin^{-1} w$ or

$$w=\sin\left(\frac{\pi z}{2a}\right)$$

Example 3.10 Find the transformation that maps the infinite strip $0\le v\le\pi$ of the w-plane onto the upper half of the z-plane.

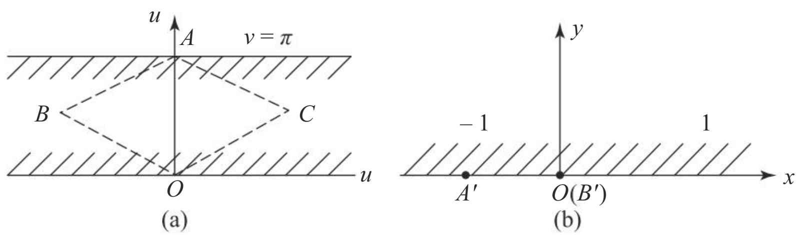


Fig. 3.30

Consider the rhombus $ABOC$, which is a four sided polygon. [Fig. 3.30]

When B and $C\to\infty$, the interior of the this rhombus becomes the region of the given strip. In the limit, the interior angles at A, B, O and C are π, O, π and O respectively.

Let us assume that the images of A, B, O, C are the points -1, 0, 1 and ∞ of the z-plane.

Then the required transformation is

$$\frac{dw}{dz}=A(z+1)^{\frac{\pi}{\pi}-1}\cdot(z-0)^{\frac{0}{\pi}-1}\cdot(z-1)^{\frac{\pi}{\pi}-1}$$

i.e. $$\frac{dw}{dz}=\frac{A}{z}\quad\therefore\ w=A\log z+B \tag{1}$$

When $w=0,\quad z=1\quad\therefore B=0$

When $w=\pi i,\quad z=-1\quad\therefore\quad A\log(-1)=\pi i$

i.e. $A\log(e^{i\pi})=\pi i$

i.e. $A(i\pi)=i\pi$ or $A=1$

Using these values in (1), the required transformation is $w=\log z$.

EXERCISE 3(d)

Part A

(Short Answer Questions)

1. Define a bilinear transformation and its determinant.
2. Find the value of the determinant of the transformation $w = \dfrac{1 - iz}{z - i}$.
3. What do you mean by the fixed point of a transformation? Does the fixed point exist only for a bilinear transformation?
4. Define the cross ratio of four points in a complex plane.
5. State the cross ratio property of a bilinear transformation.
6. What is Schwarz-Christoffel transformation? State the formula for the same.
7. What is the advantage of choosing the image of one vertex of the polygon at infinity in Schwarz-Christoffel transformation?
8. Find the invariant points of the transformations
   (i) $w = iz^2$ and (ii) $w = z^3$
   Find the invariant points of the transformations

9. $w = \dfrac{2z+6}{z+7}$ 10. $w = \dfrac{3z - 5\,i}{iz - 1}$ 11. $w = \dfrac{z - 1 - i}{z + 2}$
12. Find the condition for the invariant points of the transformation $w = \dfrac{az+b}{cz+d}$ to be equal.
13. Find all linear fractional transformations whose fixed points are – 1 and 1.
14. Find all linear fractional transformations whose fixed points are – i and i.
15. Find all linear fractional transformations without fixed points in the finite plane.
16. Find the image of the real axis of the z-plane by the transformation $w = \dfrac{1}{z+i}$.

Part B

17. Find the bilinear transformation which maps the points
    (i) $z = 0, -i, -1$ into $w = i, 1, 0$ respectively,
    (ii) $z = -i, 0, i$ into $w = -1, i, 1$ respectively.
18. Find the bilinear transformation that maps the points (i) $z = 0, -1, \infty$, into the points $w = -1, -2 - i, i$ respectively, (ii) $z = 0, -i, 2i$ into the points $w = 5i, \infty - i/3$ respectively.
19. Prove that $w = \dfrac{z}{1-z}$ maps the upper half of the z-plane onto the upper half of the w-plane. What is the image of the circle $|z| = 1$ under this transformation?
20. When the point z moves along the real axis of the z-plane from $z = -1$ to $z = +1$, find the corresponding movement of the point w in the w-plane, if $w = \dfrac{1-iz}{z-i}$.

21. Prove that, under the transformation $w = \dfrac{z-i}{iz-1}$, the region $\text{Im}(z) \geq 0$ is mapped onto the region $|w| \leq 1$. Into what region is $\text{Im}(z) \leq 0$ mapped by this transformation?

22. Find the images of (i) the segment of the real axis between $z = +1$ and $z = -1$ (ii) the interior of the circle $|z| = 1$ and (iii) the exterior of the circle $|z| = 1$ under the transformation $w = \dfrac{1+iz}{z+i}$.

23. Show that the transformation $w = \dfrac{i-z}{i+z}$ maps the circle $|z| = 1$ onto the imaginary axis of the w-plane. Find also the images of the interior and exterior of this circle.

24. Find the bilinear transformation that maps the upper half of the z-plane onto the interior of the unit circle of the w-plane in such a way that the points $z = i, \infty$ are mapped onto $w = 0, -1$.

    [***Hint:*** Use Worked Example (3.7).]

25. Find the transformation which maps the area in the z-plane within an infinite sector of angle $\dfrac{\pi}{m}$ onto the upper half of the w-plane.

    $\left[\right.$***Hint:*** The sectoral region bounded by OX and OP may be regarded as an open polygon with vertex at O and interior angle $\dfrac{\pi}{m}$. Make the origin of the z-plane correspond to the origin of the w-plane.$\left.\right]$

26. Find the transformation which maps the semi-infinite strip

    (i) $x \geq 0, 0 \leq y \leq c$ onto $v \geq 0$,

    (ii) $u \geq 0, 0 \leq v \leq \pi$ onto $y \geq 0$.

27. Find the transformation which maps the semi-infinite strip $y \geq 0, 0 \leq x \leq a$ onto the upper half of the w-plane. Make the points $z = 0$ and $z = a$ correspond to $w = +1$ and $w = -1$.

28. Find the transformation that maps the infinite strip $0 \leq y \leq k$ onto the upper half of the w-plane. Make the points $z = 0$, $z = ik$ correspond to $w = 1, -1$.

29. Find the transformation that maps the region in the z-plane above the line $y = b$, when $x < 0$ and that above the x-axis, when $x > 0$ into the upper half of the w-plane. Make the points $z = ib$ and $z = 0$ correspond to $w = -1$ and $w = 1$.

30. Find the transformation that maps the region in the w-plane above the u-axis when $u < 0$ and that above $v = b$ when $u > 0$ into the upper half of the z-plane. Make the points $w = 0$ and w $= ib$ correspond to $z = 0$ and $z = 1$.

ANSWERS

Exercise 3(a)

16. at all points on the line $y = x$ 17. at all points
18. nowhere 19. only at the origin
20. at all points except $z = -1$ 21. $a = 2, b = -1, c = -1, d = 2$.
22. $p = -1$ 36. $2z$ 37. e^z 38. $-\sin z$
39. $\cosh z$

Exercise 3(b)

5. $y + c$ 6. Yes 7. Yes 8. No
9. Yes 10. No 11. $2\,xy$ 12. $e^x \sin y$
13. $\cos x \sinh y$ 14. $x^2 - y^2 + 2y$; 15. $2\tan^{-1}\left(\frac{y}{x}\right)$ 16. $\log z$
17. $-z^2$ 18. e^{iz} 19. $\cosh z$ 20. $\frac{1}{z}$
21. $v = 3x^2y - y^3 + 2xy + y^2 - x^2 + c$; $f(z) = z^3 + z^2 - iz^2 + ic$
22. $u = x^3 - 3xy^2 - 2xy + c$; $f(z) = z^3 + iz^2 + c$.
23. $\psi = 2xy - \frac{y}{x^2 + y^2} + c$; $f(z) = z^2 + \frac{1}{z} + ic$.
24. $\phi = x^2 - y^2 - 2x + 3y - 2xy + c$; $f(z) = z^2 - 2z + i\,(z^2 - 3z)$
25. $\frac{z^4}{4} + \frac{z^2}{2} + (1+i)\,z$; $x^4 - 6x^2y^2 + y^4 + 2\,(x^2 - y^2) + 4\,(x - y) = c'$.
26. $w = -iz \cdot e^z + ic$; $v = e^x\,(y \sin y - x \cos y) + c$
27. $w = ize^{-z} + c$; $u = e^{-x}\,(x \sin y - y \cos y) + c$
28. $w = -ie^{iz^2} + ic$; $v = e^{-2xy} \cos(x^2 - y^2) + c$
29. $w = ze^{i2z} + c$; $u = e^{-2y}\,(x \cos 2x - y \sin 2x) + c$
30. $f(z) = \frac{1+i}{z} - 1$ 31. $f(z) = iz^2 - z$
32. $f(z) = \frac{\cot z}{1+i} + c$ 33. $f(z) = \frac{1}{2}(1 + \sec z)$
37. $2y + y^3 - 3x^2y = b$.

Exercise 3(c)

5. $z = \pm\, e^{\alpha}$ 6. $z = \alpha, \beta$ and $\frac{1}{2}(\alpha + \beta)$ 7. $\frac{1}{\sqrt{2}}$
8. $(u - 2)^2 + (v - 3)^2 = a^2$ 9. $|\,w\,| = 2a$ 10. $|\,w\,| = 5a$

11. $(u-2)^2+(v+1)^2=2a^2$ 12. $|w|=\dfrac{1}{2a}$ 13. $0<v<2$

14. $v>1$ 15. $u+v>2$ 16. $u\,v=9$ 17. $u\,v=a^2$

18. Interior of the circle $u^2+\left(v+\dfrac{1}{2c}\right)^2=\left(\dfrac{1}{2c}\right)^2$

19. Exterior of the circle $\left(u-\dfrac{1}{2c}\right)^2+v^2=\left(\dfrac{1}{2c}\right)^2$

20. $u=mv$ 21. Upper half of the w-plane

22. $u^2=-4(v-1)$; $u^2=4(v+1)$ 23. $4<|w|<9$

24. $\dfrac{\pi}{2}<\arg(w)<\pi$ 25. $x\,y=c_1$ and $y^2-x^2=c_2$

26. The annular region $1\le R\le e$
27. Segment of $v=0$, given by $-1\le u\le 1$
28. Part of $v=0$, given by $u\ge 1$
29. u-axis 30. Segment of $u=0$, given by $-1\le v\le 1$.

31. $u=\dfrac{a}{2}\left(r+\dfrac{1}{r}\right)\cos\theta$; $v=\dfrac{a}{2}\left(r-\dfrac{1}{r}\right)\sin\theta$.

32. $(\pm 2, 0)$
33. (i) Triangle with vertices $w=4+i, 7+i, 7-5i$;
    (ii) Triangle with vertices $w=1-i, 1, 3-2i$
34. Square with vertices $w=3+\left(3\pm 2\sqrt{2}\right)i, \left(3\mp 2\sqrt{2}\right)+3i$
35. No
36. The exterior of the image circle
37. (i) $u^2+(v+c)^2>c^2, v<0$;
    (ii) $u^2+v^2+v<0, u>0$.
38. The region enclosed by $v^2=4(u+1)$ and $u=0$
39. The region above the u-axis, bounded by the u-axis, $v^2=4(1-u)$ and $v^2=4(1+u)$.
45. The lower half of the w-plane
46. The region in the upper half of the w-plane bounded by the part of the u-axis given by $-\dfrac{1}{\sqrt{2}}\le u\le\dfrac{1}{\sqrt{2}}$, the ellipse $\dfrac{u^2}{\cosh^2 3}+\dfrac{v^2}{\sinh^2 3}=1$ and the hyperbola $u^2-v^2=\dfrac{1}{2}$.
48. The elliptic annular region bounded by $\dfrac{u^2}{\sinh^2 1}+\dfrac{v^2}{\cosh^2 1}=1$ and $\dfrac{u^2}{\sinh^2 2}+\dfrac{v^2}{\cosh^2 2}=1$, lying in the right half of the w-plane.

Exercise 3(d)

2. -2 3. No

8. (i) $z = 0$ and $z = -i$;
   (ii) $z = 0, \pm 1$.

9. $z = 1, -6$ 10. $z = i, -5i$ 11. $z = -i, \quad z = -1 + i$

12. $(a-d)^2 + 4bc = 0$ 13. $w = \dfrac{az+b}{bz+a}$ 14. $w = \dfrac{az+b}{a-bz}$

15. $w = z + a$ 16. $u^2 + v^2 + v = 0$

17. (i) $w = -i\left(\dfrac{z+1}{z-1}\right)$;
    (ii) $w = \dfrac{i - iz}{z+1}$

18. (i) $w = \dfrac{iz-2}{z+2}$;
    (ii) $w = \dfrac{3z-5i}{iz-1}$

19. The line $u = -1/2$

20. w moves along the upper half of the circle $|w| = 1$ from $w = -1$ to $w = +1$ in the clockwise sense.

21. $|w| \geq 1$

22. The lower half of the circle $|w| = 1$; $v < 0$; $v > 0$

23. $u > 0$; $u < 0$ 24. $w = \dfrac{i-z}{i+z}$. 25. $w = kz^m$

26. (i) $w = \cosh \dfrac{z}{c}$;
    (ii) $z = \cosh w$

27. $w = \cos \dfrac{\pi z}{a}$ 28. $w = e^{\pi z/k}$

29. $z = \dfrac{b}{\pi}\left(\sqrt{w^2 - 1} + \cosh^{-1} w\right)$ 30. $w = \dfrac{2ib}{\pi}\left(\sin^{-1}\sqrt{z} + \sqrt{z(1-z)}\right)$