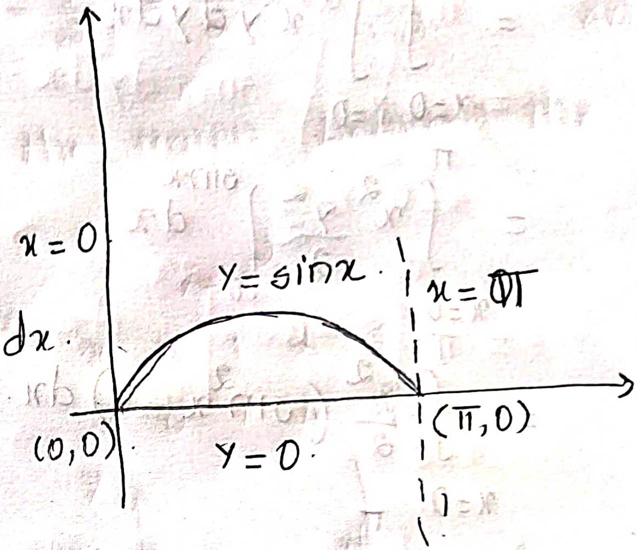
Type 4: Identifying the ~~area~~ region of integration.

1. Evaluate $\iint_D xy \, dx \, dy$ where D is the region bounded by the curve $y = \sin x$, x-axis and the line segment $0 \le x \le \pi$.

Solution:

$$\iint_D xy \, dx \, dy = \int_{x=0}^{\pi} \int_{y=0}^{\sin x} xy \, dy \, dx.$$



$$= \int_{x=0}^{\pi} x \cdot \frac{y^2}{2} \Big|_0^{\sin x} dx = \frac{1}{2} \int_{x=0}^{\pi} x(\sin^2 x - 0) \, dx.$$

$$= \frac{1}{2} \int_0^{\pi} x \sin^2 x \, dx$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= \frac{1}{2} \int_0^{\pi} x \left(\frac{1 - \cos 2x}{2}\right) dx$$

$$= \frac{1}{4} \int_0^{\pi} (x - x\cos 2x) \, dx = \frac{1}{4} \left[\frac{x^2}{2} \Big|_0^{\pi} - \int_0^{\pi} x \cos 2x \, dx \right]$$

$$= \frac{1}{4} \left[\frac{(\pi^2 - 0)}{2} - \left\{ x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} dx \right|_0^{\pi} \right.$$

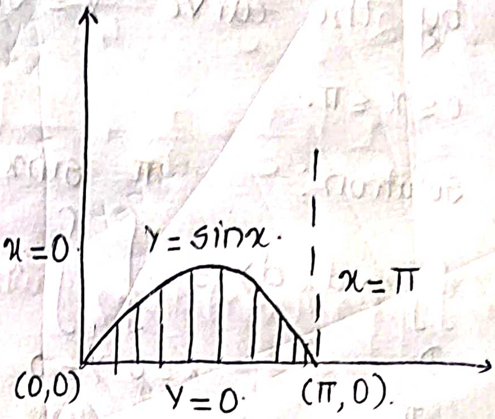
$$= \frac{1}{4} \left[\frac{\pi^2}{2} - \left\{ \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \right|_0^{\pi} \right.$$

$$= \frac{1}{4} \left[\frac{\pi^2}{2} - \left\{ \frac{\pi \sin 2\pi}{2} + \frac{\cos 2\pi}{4} - \left\{ 0 + \frac{\cos 0}{4} \right\} \right] \right.$$

$$= \frac{1}{4} \left[\frac{\pi^2}{2} - \left\{ 0 + \frac{1}{4} - \frac{1}{4} \right\} \right] = \frac{\pi^2}{8}.$$

2. $\iint_D x^2 y \, dx\, dy$ where D is the region bounded by $y = \sin x$, x-axis, $0 \le x \le \pi$.

$$= \int_{x=0}^{\pi} \int_{y=0}^{\sin x} x^2 y \, dy \, dx$$



$$= \int_{x=0}^{\pi} x^2 \frac{y^2}{2} \Big|_0^{\sin x} dx$$

$$= \int_{x=0}^{\pi} \frac{x^2}{2} (\sin^2 x - 0) \, dx$$

$$= \frac{1}{2} \int_{x=0}^{\pi} x^2 \frac{(1 - \cos 2x)}{2} dx = \frac{1}{4}\left[\int_0^{\pi} x^2 dx - \int_0^{\pi} x^2 \cos 2x \, dx \right]$$

$u = x^2$ $\quad v = \cos 2x$

$u' = 2x$ $\quad v_1 = \frac{\sin 2x}{2}$

$u'' = 2$ $\quad v_2 = \frac{-\cos 2x}{4}$

$\quad v_3 = \frac{-\sin 2x}{8}$

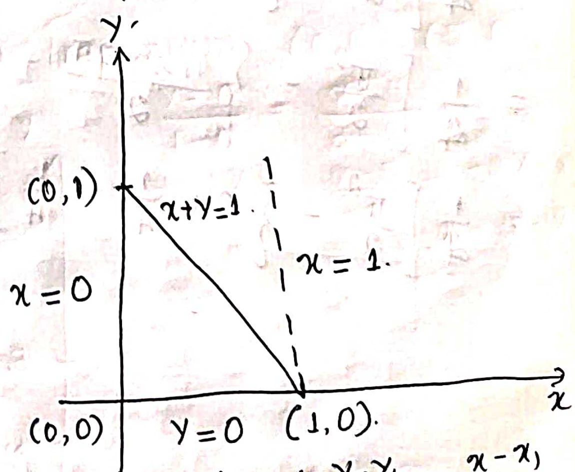
$$= \frac{1}{4}\left[\frac{x^3}{3}\Big|_0^{\pi} - \left\{ \frac{x^2 \sin 2x}{2} + \frac{2x \cdot \cos 2x}{4} + 2\left(-\frac{\sin 2x}{8}\right) \right\}_0^{\pi} \right]$$

$$= \frac{1}{4}\left[\frac{(\pi^3 - 0)}{3} - \left\{ \frac{\pi^2 \sin 2\pi}{2} + \frac{\pi \cos 2\pi}{2} - \frac{\sin 2\pi}{8} - (0) \right\} \right]$$

$$= \frac{1}{4}\left[\frac{\pi^3}{3} - \frac{\pi}{2} \right] = \frac{\pi}{24}(2\pi^2 - 3) //.$$

3). $\iint_D x^2 y \, dx\, dy$ where D is the triangle with the vertices $(0,0)$, $(1,0)$, $(0,1)$.

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} x^2 y \, dy \, dx$$



$$= \int_{x=0}^{1} x^2 \frac{y^2}{2} \Big|_0^{(1-x)} dx$$

$$= \frac{1}{2} \int_0^1 x^2 \left[(1-x)^2 - 0^2 \right] dx$$

$$= \frac{1}{2} \int_0^1 x^2 (1-x)^2 dx$$

Eqn. of a line : $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$.

$\frac{y - 0}{1 - 0} = \frac{x - 1}{1 - 0} \Rightarrow y = x - 1$.

$\Rightarrow \boxed{x + y = 1}$

$$= \frac{1}{2}\int_0^1 x^2(1-2x+x^2)\,dx = \frac{1}{2}\int_0^1 (x^2-2x^3+x^4)\,dx.$$

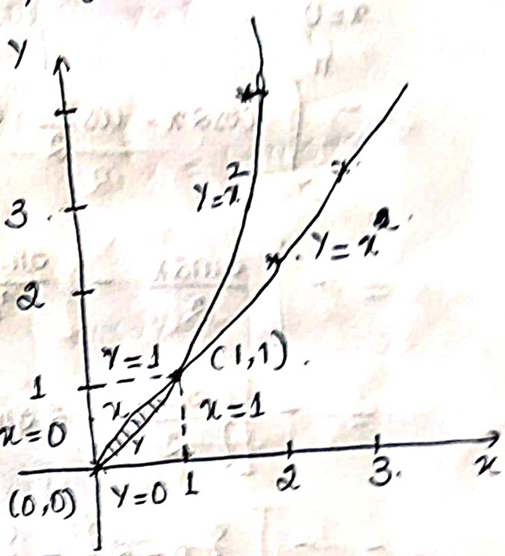
$$= \frac{1}{2}\left[\frac{x^3}{3}-\frac{2x^4}{4}+\frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left[\frac{(1-0)}{3}-\frac{(1-0)}{2}+\frac{(1-0)}{5}\right].$$

$$= \frac{1}{2}\left[\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right] = \frac{1}{2}\left[\frac{8}{15}-\frac{1}{2}\right] = \frac{1}{2}\times\frac{(16-15)}{30} = \frac{1}{60}$$

4). $\iint_D xy(x+y)\,dx\,dy$ where D is the region between the curves $y=x^2$ and $y=x$.

$y=x^2$

x	0	1	2	3
y	0	1	4	9



$$= \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dy\,dx.$$

$$= \int_{x=0}^{1} x^2\frac{y^2}{2} + xy^2\Big|_{x^2}^{x}\,dx.$$

$$= \int_{x=0}^{1} \frac{x^2}{2}(x^2-x^4) + x(x^2-x^4)\,dx.$$

$$= \int_0^1 \frac{x^4}{2}-\frac{x^6}{2}+x^3-x^5\,dx.$$

$$= \frac{x^5}{10}-\frac{x^7}{14}+\frac{x^4}{4}-\frac{x^6}{6}\Big|_0^1 = \frac{1}{10}-\frac{1}{14}+\frac{1}{4}-\frac{1}{6}.$$

$$= \frac{1}{2}\left[\frac{1}{5}+\frac{1}{2}-\frac{1}{8}-\frac{1}{7}\right] = \frac{1}{2}\left[\frac{7}{10}-\left(\frac{8+7}{56}\right)\right]$$

$$= \frac{1}{2}\left[\frac{7}{10}-\frac{15}{56}\right] = \frac{1}{2}\left[\frac{7\times 56-150}{560}\right] = \frac{1}{2}\left[\frac{392-150}{560}\right]$$

$$= \frac{1}{2}\left[\frac{242}{560}\right] = \frac{121}{560}$$

5) $\iint_D \sin y \, dx\, dy$ where D is the region bounded by the lines $2y = x$, $2x = \cancel{2}y$ and $x = \pi$.

$$= \int_{x=0}^{\pi} \int_{y=\frac{x}{2}}^{2x} \sin y \, dy\, dx$$

$$= \int_{x=0}^{\pi} -\cos y \Big|_{x/2}^{2x} dx$$

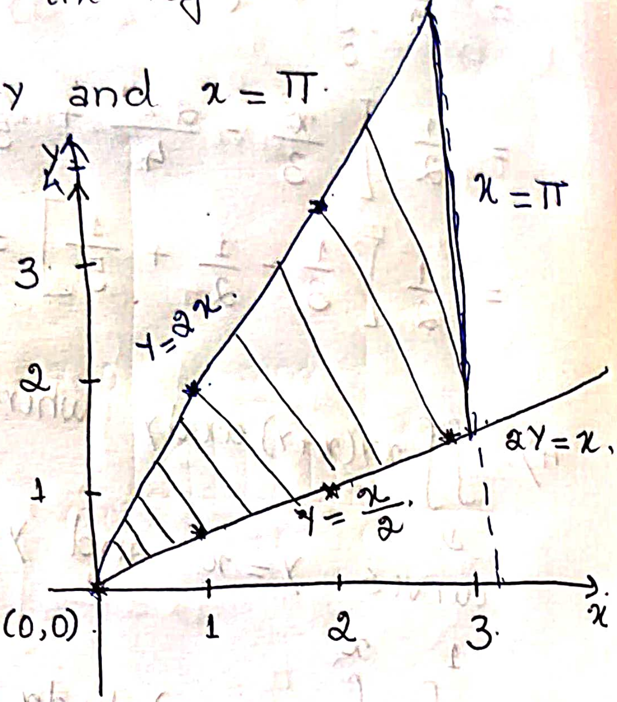
$$= -\int_{0}^{\pi} \cos 2x - \cos \frac{x}{2} \, dx$$

$$= -\left[\frac{\sin 2x}{2} - \frac{\sin x/2}{1/2} \right]_0^{\pi}$$

$$= -\left[\left\{ \frac{\sin 2\pi}{2} - 2 \sin \frac{\pi}{2} \right\} - \left\{ \frac{\sin 0}{2} - 2 \sin 0 \right\} \right]$$

$$= -\left[\{0 - 2\} - \{0 - 0\} \right]$$

$$= \underline{\underline{2}}$$



$2y = x \Rightarrow y = \frac{x}{2}$

x	0	1	2	3	4
y	0	0.5	1.	1.5	2

$y = 2x$

x	0	1	2	3	4
y	0	2	4	6	8

$x = \pi = 3.1415$

** 6) $\iint_D x^2 \, dx\, dy$ where D is the trapezium $(0,0)$, $(5,0)$, $(3,1)$, $(0,1)$.

$\frac{y-0}{1-0} = \frac{x-5}{3-5}$

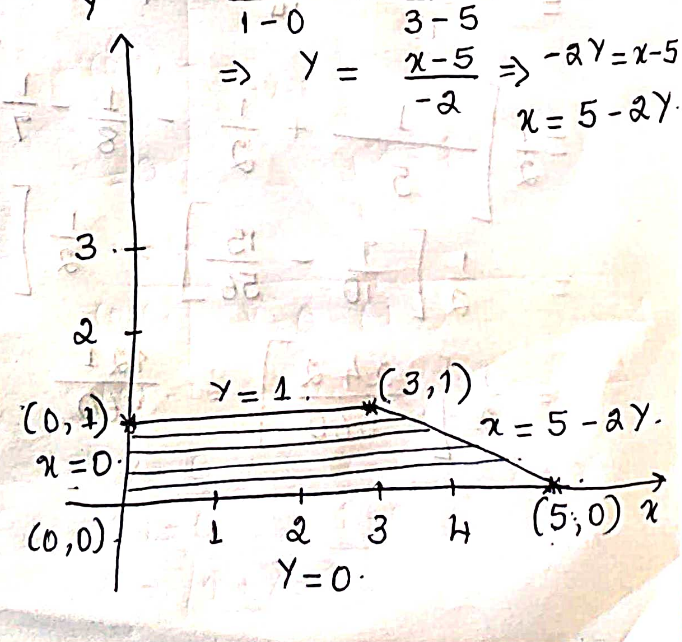
$\Rightarrow y = \frac{x-5}{-2} \Rightarrow -2y = x - 5$

$x = 5 - 2y$

$$= \int_{y=0}^{1} \int_{x=0}^{5-2y} x^2 \, dx\, dy$$

$$= \int_{y=0}^{1} \frac{x^3}{3} \Big|_0^{5-2y} dy$$

$$= \frac{1}{3} \int_0^1 (5-2y)^3 \, dy$$



Formula: $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

$$= \frac{1}{3} \left[\frac{(5-2y)^4}{4} \times \left(-\frac{1}{2}\right) \right]_0^1 \qquad \int (ay+b)^n dy = \frac{(ay+b)^{n+1}}{(n+1)a}$$

$$= -\frac{1}{24} \left[(5-2)^4 - (5-0)^4 \right].$$

$$= -\frac{1}{24} \left[81 - 625 \right] = \frac{544}{24} = \frac{68}{3}.$$

7). ~~sketch the region of is~~

**. Evaluate $\iint_D \sqrt{x}\, y \, dx \, dy$ where D is the region bounded by $x > 0,\ y > x^2,\ y < 2 - x^2$.