COMPLEX INTEGRATION - SUMMARY.

1. Connected region.

A region in which any two points in it can be connected by a curve which lies entirely within the region is called connected region.

2. Simply connected region.

A curve which does not cross itself is called a simple closed curve.

A region in which every closed curve in it encloses points of the region is called a simply connected region.

In other words, a region which has no holes is called simply connected region.

3. Contour integral.

An integral along a simple closed curve is called a contour integral.

Note: In case of closed paths, the positive direction is anti clock-wise.

4. Cauchy's integral theorem or Cauchy's fundamental theorem.

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C, then $\int_C f(z)dz = 0$.

5. Cauchy's integral formula

If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if 'a' is any point within C, then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz. \Rightarrow \int_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a).$$

6. Cauchy's integral formula for derivative.

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2}\, dz.$$

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a).$$

7. Taylor's series.

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \dots$$

↳ $a = 0$, Maclaurin's series.

$$f(z) = f(0) + f'(0).z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \dots$$

8. Laurent's series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}.$$: Analytic + Principal

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}}\, dz, \quad b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}}\, dz.$$

<u>Prerequisites</u>

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n.$$

, valid only if $|x| < 1$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n.$$

9. Zeros of an analytic function

If a function $f(z)$ is analytic at a region R, is zero at a point $z = z_0$ in R then z_0 is called a zero $f(z_0) = 0$.

Example: $f(z) = z-1$; zero is $z-1=0 \Rightarrow z=1$.

10. Simple zero.

If $f(z_0) = 0$ but $f'(z_0) \neq 0$ then $z=z_0$ is called a simple zero. or zero of first order.

11. Singularity of $f(z)$ or singular points.

If at a point $z=z_0$, $f(z)$ fails to be analytic then z_0 is called singular point of $f(z)$.

Example: $f(z) = \frac{1}{(z-2)^2}$, $z=2$ is a singular point.

12. Types of singularities.

i. Isolated singularity. ii. Non-isolated singularities.

- a. Removable singularity : $\lim_{z \to z_0} f(z) =$ finite number.
- b. Pole : $\lim_{z \to a} f(z) = \infty$.
- c. Essential singularity : Neither removable nor a pole.

Example : $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \cdots$

13. Residues :

If $z = z_0$ is an singular isolated point of $f(z)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}.$$

The co-efficient of $\frac{1}{z-z_0}$ is called residue.

a. Residue at simple pole

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z-z_0) f(z).$$

b. Residue at a pole of order m.

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

14. Cauchy's residue theorem.

If $f(z)$ is analytic at all points inside and on a simple closed curve C, except for a finite number of isolated singularities $z_1, z_2, \dots, z_n$ inside C, then $\int_C f(z)\,dz = 2\pi i \times$ [Sum of the residues of $f(z)$ at $z_1, z_2, \dots, z_n$]

15. Contour integration.

a). Type I : $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta \Rightarrow \int_C f(z) \frac{dz}{iz}$, $C : |z| = 1$.

$z = e^{i\theta}$, $\frac{1}{z} = e^{-i\theta}$, $d\theta = \frac{dz}{iz}$.

$\cos\theta = \frac{z^2+1}{2z}$, $\sin\theta = \frac{z^2-1}{2iz}$.

i. $$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad a > |b|.$$

ii. $$\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}, \quad a > |b|.$$

iii. $$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad a > |b|.$$

iv. $$\int_0^{2\pi} \frac{\cos m\theta}{a+b\cos\theta}\, d\theta = \frac{4\pi}{b} \cdot \frac{\alpha^m}{(\alpha \mp \beta)}$$

$$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}, \quad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}.$$

Type II : $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx \Rightarrow \int_C f(z)\, dz$; C is the upper half of the semi-circle Γ with bounding diameter $[-R, R]$.

i. $$\int_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}, \quad a, b > 0.$$

ii. $$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}, \quad a, b > 0.$$

iii. $$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, \quad a > 0.$$

iv. $$\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}\, a^3}.$$

Type III : $\int_{-\infty}^{\infty} f(x) \cos nx\, dx$, $\int_{-\infty}^{\infty} f(x) \sin nx\, dx$.

$\sin nx = \text{I.P.} \{e^{inx}\}$.

$\cos nx = \text{R.P.} \{e^{inx}\}$