Type 2

1. Evaluate $\int_0^1 \int_1^2 x(x+y+1)\, dy\, dx$.

$$= \int_{x=0}^{1} \int_{y=1}^{2} (x^2+xy+x)\, dy\, dx.$$ → Integrate w.r.t. y keeping x constant.

$$= \int_{x=0}^{1} x^2 y + x\frac{y^2}{2} + xy \Big|_1^2 \, dx.$$

$$= \int_{x=0}^{1} \left\{x^2(2-1) + \frac{x}{2}(2^2-1^2) + x(2-1)\right\} dx$$

$$= \int_{x=0}^{1} \left(x^2 + \frac{3}{2}x + x\right) dx = \left[\frac{x^3}{3} + \frac{3}{2}\frac{x^2}{2} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{(1-0)}{3} + \frac{3}{4}(1-0) + \frac{1-0}{2} = \frac{1}{3} + \frac{3}{4} + \frac{1}{2}.$$

$$= \frac{4+9+6}{12} = \frac{19}{12}.$$

2. $\int_0^1 \int_1^2 x(x+y+1)\, dx\, dy$. → Identify the upper and lower limits.

$$= \int_{y=0}^{1} \int_{x=1}^{2} (x^2+xy+x)\, dx\, dy.$$ → Integrate w.r.t. x treating y as a constant.

$$= \int_{y=0}^{1} \frac{x^3}{3} + y\frac{x^2}{2} + \frac{x^2}{2} \Big|_1^2 \, dy.$$

$$= \int_{y=0}^{1} \frac{(2^3-1^3)}{3} + \frac{y}{2}(2^2-1^2) + \frac{(2^2-1^2)}{2}\, dy.$$

$$= \int_{y=0}^{1} \left(\frac{7}{3} + \frac{3}{2}y + \frac{3}{2}\right) dy = \frac{7}{3}y + \frac{3}{2}\frac{y^2}{2} + \frac{3}{2}y \Big|_0^1$$

$$= \frac{7}{3}(1-0) + \frac{3}{4}(1^2-0^2) + \frac{3}{2}(1-0) = \frac{7}{3} + \frac{3}{4} + \frac{3}{2} = \frac{28+9+18}{12}$$

$$= \frac{54}{12} = \frac{18}{4} = \frac{9}{2}.$$

3) $\int_0^a \int_0^b (x^2+y^2)\, dx\, dy$ → identify the upper and lower limits for x and y.

$= \int_{y=0}^{a} \int_{x=0}^{b} (x^2+y^2)\, dx\, dy$ → Integrate w.r.t. x treating y as a constant.

$= \int_{y=0}^{a} \frac{x^3}{3} + y^2 x \Big|_0^b dy$

$= \int_{y=0}^{a} \left(\frac{b^3}{3} + y^2(b-0)\right) dy$

$= \int_{y=0}^{a} \frac{b^3}{3} + by^2 \, dy = \frac{b^3}{3} y + b\frac{y^3}{3} \Big|_0^a$

$= \frac{b^3}{3}(a-0) + \frac{b}{3}(a^3-0^3) = \frac{ab^3+a^3b}{3} = \frac{ab(a+b)}{3}$.

4) $\iint_D (3x-2y^3)\, dx\, dy$ D: $\begin{cases} -5 < x < -4 \\ -3 < y < -2 \end{cases}$

$= \int_{y=-3}^{-2} \int_{x=-5}^{-4} (3x-2y^3)\, dx\, dy$ → Integrate w.r.t. x keeping y as a constant.

$= \int_{y=-3}^{-2} 3\frac{x^2}{2} - 2y^3 x \Big|_{-5}^{-4} dy$

$= \int_{-3}^{-2} \left[\frac{3}{2}(16-25) - 2y^3(-4-(-5))\right] dy$

$= \int_{-3}^{-2} \left(-\frac{27}{2} - 2y^3\right) dy = -\frac{27}{2}y - 2\frac{y^4}{4} \Big|_{-3}^{-2}$

$= -\frac{27}{2}(-2-(-3)) - \frac{1}{2}((-2)^4-(-3)^4)$

$= -\frac{27}{2}(1) - \frac{1}{2}(16-81) = \frac{-27+65}{2} = \frac{38}{2} = 19$

5) $\int_0^{\pi/2}\int_{\pi/2}^{\pi} \cos(x+y)\,dx\,dy$ → Identify the upper and lower limits for x and y.

→ Use $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a}$

or expand $\cos(x+y)$ using $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

Method (1):

$$= \int_{y=0}^{\pi/2}\int_{x=\pi/2}^{\pi} \cos(x+y)\,dx\,dy$$

$$= \int_{y=0}^{\pi/2} \frac{\sin(x+y)}{1}\Big|_{\pi/2}^{\pi} dy$$

$$= \int_{y=0}^{\pi/2} \left\{\sin(\pi+y) - \sin\left(y+\frac{\pi}{2}\right)\right\} dy$$

Formula: $\int \sin(a+y)\,dy = \frac{-\cos(a+y)}{1} + c$.

$$= \left[-\frac{\cos(y+\pi)}{1} + \frac{\cos(y+\pi/2)}{1}\right]_0^{\pi/2}$$

$$= -\left\{\cos\left(\frac{\pi}{2}+\pi\right) - \cos(0+\pi)\right\} + \left\{\cos(\pi/2+\pi/2) - \cos\left(0+\frac{\pi}{2}\right)\right\}$$

$$= -\left\{\cos\frac{3\pi}{2} - \cos\pi\right\} + \cos\pi - \cos\frac{\pi}{2}$$

$\cos\frac{n\pi}{2} = 0$; n is not a multiple of 2

$\cos n\pi = -1$, n is odd

$$= -\{0-(-1)\} + (-1) - 0$$

$$= -1-1$$

$$= \underline{\underline{-2}}$$

Method 2

$$= \int_0^{\pi/2}\int_{\pi/2}^{\pi} (\cos x\cos y - \sin x\sin y)\,dx\,dy$$ → Integrate w.r.t. x keeping y constant.

$$= \int_0^{\pi/2} \cos y\sin x - \sin y(-\cos x)\Big|_{\pi/2}^{\pi} dy$$

$$= \int_0^{\pi/2} \cos y\left(\sin\pi - \sin\frac{\pi}{2}\right) + \sin y\left(\cos\pi - \cos\frac{\pi}{2}\right) dy$$

$$= \int_0^{\pi/2} (-1)\cos y + \sin y(-1-0)\,dy = -\sin y - (-\cos y)\Big|_0^{\pi/2}$$

$$= -\left(\sin\frac{\pi}{2} - \sin 0\right) + \left(\cos\frac{\pi}{2} - \cos 0\right) = -1-1 = \underline{\underline{-2}}$$

6). $\iint_D \{e^{x+y} + \log(xy)\}\, dx\, dy$ $\quad D: \begin{cases} 10 < x < 11 \\ 4 < y < 5 \end{cases}$

$= \int_{x=10}^{11}\int_{y=4}^{5} \{e^x \cdot e^y + \log x + \log y\}\, dx\, dy.$ $\quad a^{m+n} = a^m \cdot a^n$

$\log(ab) = \log a + \log b.$

Integrate w.r.t. y keeping x constant.

$= \int_{x=10}^{11} e^x e^y \Big|_4^5 + \log y \cdot y - \int y \cdot \frac{1}{y}\, dy \Big|_4^5 + \log x \cdot y \Big|_4^5 \, dx.$

$= \int_{x=10}^{11} e^x (e^5 - e^4) + \{(5\log 5 - 5) - (4\log 4 - 4)\} + \log x\,(5-4).$

$= \int_{x=10}^{11} (e^5 - e^4)\, e^x\, dx + (5\log 5 - 4\log 4 - 5 + 4) + \log x \, dx.$

$= (e^5 - e^4)\, e^x \Big|_{10}^{11} + (5\log 5 - 4\log 4 - 1)\, x \Big|_{10}^{11} + \int_{10}^{11} \log x\, dx.$

$= (e^5 - e^4)(e^{11} - e^{10}) + (5\log 5 - 4\log 4 - 1)(11 - 10)$

$+ \log x \cdot x - \int x \cdot \frac{1}{x}\, dx \Big|_{10}^{11}$

(Use integration by parts).

$= (e^5 - e^4)(e^{11} - e^{10}) + (5\log 5 - 4\log 4 - 1) + 11\log 11 - 11 - \{10\log 10 - 10\}$

$= (e^5 - e^4)(e^{11} - e^{10}) + (5\log 5 - 4\log 4 - 1) + 11\log 11 - 10\log 10 - 1.$

$= (e^5 - e^4)(e^{11} - e^{10}) + 5\log 5 - 4\log 4 + 11\log 11 - 10\log 10 - 2.$