Engineering Mathematics
[For Semesters I and II]

Third Edition

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Chapter 5

Laplace Transforms

5.1 INTRODUCTION

The Laplace transform is a powerful mathematical technique useful to the engineers and scientists, as it enables them to solve linear differential equations with given initial conditions by using algebraic methods. The Laplace transform technique can also be used to solve systems of differential equations, partial differential equations and integral equations. Starting with the definition of Laplace transform, we shall discuss below the properties of Laplace transforms and derive the transforms of some functions which usually occur in the solution of linear differential equations.

5.1.1 Definition

If $f(t)$ is a function of t defined for all $t \geq 0$, $\int_0^\infty e^{-st} f(t)\, \mathrm{d}t$ is defined as the *Laplace transform of* $f(t)$, provided the integral exists.

Clearly the integral is a function of the parameters s. This function of s is denoted as $\bar{f}(s)$ or $F(s)$ or $\phi(s)$. Unless we have to deal with the Laplace transforms of more than one function, we shall denote the Laplace transform of $f(t)$ as $\phi(s)$. Sometimes the letter 'p' is used in the place of s.

The Laplace transform of $f(t)$ is also denoted as $L\{f(t)\}$, where L is called the Laplace transform operator.

Thus $$L\{f(t)\} = \phi(s) = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t.$$

The operation of multiplying $f(t)$ by e^{-st} and integrating the product with respect to t between 0 and ∞ is called *Laplace transformation*.

The function $f(t)$ is called the *Laplace inverse transform* of $\phi(s)$ and is denoted by $L^{-1}\{\phi(s)\}$.

Thus $f(t) = L^{-1}\{\phi(s)\}$, when $L\{f(t)\} = \phi(s)$

Note ☑

1. The parameter s used in the definition of Laplace transform is a real or complex number, but we shall assume it to be a real positive number sufficiently large to ensure the existence of the integral that defines the Laplace transform.
2. Laplace transforms of all functions do not exist. For example, L (tan t) and $L(e^{t^2})$ do not exist. We give below the sufficient conditions (without proof) for the existence of Laplace transform of a function $f(t)$:

Conditions for the existence of Laplace transform If the function $f(t)$ defined for $t \geq 0$ is

(i) piecewise continuous in every finite interval in the range $t \geq 0$, and
(ii) of the exponential order, then $L\{f(t)\}$ exists.

Note ☑

1. A function $f(t)$ is said to be piecewise continuous in the finite interval $a \leq t \leq b$, if the interval can be divided into a finite number of sub-intervals such that (i) $f(t)$ is continuous at every point inside each of the sub-intervals and (ii) $f(t)$ has finite limits as t approaches the end points of each sub-interval from the interior of the sub-interval.
2. A function $f(t)$ is said to be of the exponential order, if $|f(t)| \leq Me^{\alpha t}$, for all $t \geq 0$ and some constants M and α or equivalently, if $\lim_{t\to\infty}\{e^{-\alpha t} f(t)\} =$ a finite quantity.

Most of the functions that represent physical quantities and that we encounter in differential equations satisfy the conditions stated above and hence may be assumed to have Laplace transforms.

5.2 LINEARITY PROPERTY OF LAPLACE AND INVERSE LAPLACE TRANSFORMS

$$L\{k_1 f_1(t) \pm k_2 f_2(t)\} = k_1 L\{f_1(t)\} \pm k_2 L\{f_2(t)\},$$

where k_1 and k_2 are constants.

Proof: $L\{k_1 f_1(t) \pm k_2 f_2(t)\} = \int_0^\infty \{k_1 f_1(t) \pm k_2 f_2(t)\} e^{-st}\,\mathrm{d}t$

$$= k_1 \int_0^\infty f_1(t)\cdot e^{-st}\,\mathrm{d}t \pm k_2 \int_0^\infty f_2(t)\cdot e^{-st}\,\mathrm{d}t$$

$$= k_1 \cdot L\{f_1(t)\} \pm k_2 \cdot L\{f_2(t)\}.$$

Thus L is a linear operator.

As a particular case of the property, we get

$L\{k f(t)\} = kL\{f(t)\}$, where k is a constant.

If we take $L\{f_1(t)\} = \phi_1(s)$ and $L\{f_2(t)\} = \phi_2(s)$,
the above property can be written in the following form.

$$L\{k_1 f_1(t) \pm k_2 f_2(t)\} = k_1\phi_1(s) \pm k_2\phi_2(s).$$

$$\therefore \quad L^{-1}\{k_1\phi_1(s) \pm k_2\,\phi_2(s)\} = k_1\cdot f_1(t) \pm k_2\cdot f_2(t)$$

$$= k_1\cdot L^{-1}\{\phi_1(s) \pm k_2\cdot L^{-1}\{\phi_2(s)\}$$

Thus L^{-1} is also a linear operator

As a particular case of this property, we get

$$L^{-1}\{k\phi(s)\} = k\,L^{-1}\{\phi(s)\}, \text{ where } k \text{ is a constant.}$$

Note ☑

1. $L\{f_1(t)\cdot f_2(t)\} \neq L\{f_1(t)\}\cdot L\{f_2(t)\}$ and
   $L^{-1}\{\phi_1(s)\cdot\phi_2(s)\} \neq L^{-1}(\phi_1(s)\} \times L^{-1}\{\phi_2(s)\}$
2. Generalising the linearity properties,

we get (i) $$L\left\{\sum_{r=1}^{n} k_r\, f_r\,(t)\right\} = \sum_{r=1}^{n} k_r\cdot L\{f_r(t)\}$$

(ii) $$L^{-1}\left\{\sum_{r=1}^{n} k_r\,\phi_r\,(s)\right\} = \sum_{r=1}^{n} k_r\cdot L^{-1}\{\phi_r(s)\}$$

Using (i), we can find Laplace transform of a function which can be expressed as a linear combination of elementary functions whose transforms are known.

Using (ii), we can find inverse Laplace transform of a function which can be expressed as a linear combination of elementary functions whose inverse transforms are known.

5.3 LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

1. $L\{k\} = \dfrac{k}{s}$, s > 0, where k is a constant,

$$L\{k\} = \int_0^{\infty} k\,e^{-st}\,\mathrm{d}t, \text{by definition.}$$

$$= k\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{k}{-s}(0-1)[\because e^{-st} \to 0 \text{ as } t \to \infty, \text{if } s > 0]$$

$$= \frac{k}{s}.$$

In particular, $L(0) = 0$ and $L(1) = \dfrac{1}{s}$

$\therefore \qquad L^{-1}\left\{\dfrac{1}{s}\right\} = 1.$

2. $L\{e^{-at}\} = \dfrac{1}{s+a}$, where a is a constant,

$$L\{e^{-at}\} = \int_0^\infty e^{-st} \cdot e^{-at}\, \mathrm{d}t$$

$$= \int_0^\infty e^{-(s+a)t}\, \mathrm{d}t = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{-(s+a)}(0-1), \text{ if } (s+a) > 0$$

$$= \frac{1}{s+a}, \text{ if } s > -a.$$

Inverting, we get $L^{-1}\left(\dfrac{1}{s+a}\right) = e^{-at}$.

3. $L\{e^{at}\} = \dfrac{1}{s-a}$, where a is a constant, if $s - a > 0$ or $s > a$.

Changing a to $-a$ in (2), this result follows. The corresponding inverse result is

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

4. $L(t^n) = \dfrac{\overline{|(n+1)}}{s^{n+1}}$, if $s > 0$ and $n > -1$.

$$L(t^n) = \int_0^\infty e^{-st} \cdot t^n\, \mathrm{d}t$$

$$= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \cdot \frac{\mathrm{d}x}{s}, \text{ on putting } st = x$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x}\, x^n\, \mathrm{d}x$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } s > 0 \text{ and } n + 1 > 0$$

[by definition of Gamma function]

In particular, if n is a positive integer,

$$\Gamma(n+1) = n!$$

$\therefore$ $$L(t^n) = \frac{n!}{s^{n+1}}, \text{ if } s > 0 \text{ and } n \text{ is a positive integer.}$$

Inverting, we get $L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$ or

$$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{1}{n!}t^n$$

Changing n to $n - 1$, we get $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$, if n is a positive integer.

If $n > 0$, then $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{\Gamma(n)}t^{n-1}$

In particular, $L(t) = \frac{1}{s^2}$ or $L^{-1}\left(\frac{1}{s^2}\right) = t.$

5. $L(\sin at) = \frac{a}{s^2 + a^2}$

$$L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2}(-s \sin at - a \cos at)\right]_0^\infty$$

$$= -\frac{s}{s^2 + a^2}\left(e^{-st} \sin at\right)_0^\infty - \frac{a}{s^2 + a^2}\left(e^{-st} \cos at\right)_0^\infty$$

$$= \frac{a}{s^2 + a^2}$$

[$\because$ $e^{-st} \sin at$ and $e^{-st} \cos at$ tend to zero at $t \to \infty$, if $s > 0$]

Inverting this result we get $L^{-1}\left(\frac{a}{s^2 + a^2}\right) = \sin at.$

6. $L(\cos at) = \dfrac{s}{s^2 + a^2}$

$$L(\cos at) = \int_0^{\infty} e^{-st} \cos at \, \mathrm{d}t$$

$$= \left[\frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^{\infty}$$

$$= -\frac{s}{s^2 + a^2} \left(e^{-st} \cos at \right)_0^{\infty} + \frac{a}{s^2 + a^2} \left(e^{-st} \sin at \right)_0^{\infty}$$

$$= \frac{s}{s^2 + a^2}, \text{ as per the results stated above.}$$

Inverting the above result we get $L^{-1}\left(\dfrac{s}{s^2 + a^2} \right) = \cos at$.

Aliter

$$L(\cos at + i \sin at) = L(e^{iat})$$

$$= \frac{1}{s - ia}, \text{ by result (3).}$$

$$= \frac{s + ia}{s^2 + a^2}$$

i.e., $L(\cos at) + iL(\sin at) = \dfrac{s}{s^2 + a^2} + i \dfrac{a}{s^2 + a^2}$, by linearity property.

Equating the real parts, we get $L(\cos at) = \dfrac{s}{s^2 + a^2}$.

Equating the imaginary parts, we get $L(\sin at) = \dfrac{a}{s^2 + a^2}$

7. $L(\sinh at) = \dfrac{a}{s^2 - a^2}$

$$L(\sinh at) = L\left[\frac{1}{2} (e^{at} - e^{-at}) \right]$$

$$= \frac{1}{2} \left[L(e^{at}) - L(e^{-at}) \right], \text{ by linearity property.}$$

$$= \frac{1}{2} \left(\frac{1}{s - a} - \frac{1}{s + a} \right), \text{ if } s > a \text{ and } s > -a.$$

$$= \frac{a}{s^2 - a^2}, \text{ if } s > |a|$$

Aliter

$$L(\sinh at) = -\,iL(\sin i\,at)\ [\because \sin i\theta = i\sinh\theta]$$

$$= -i\cdot\frac{ia}{s^2+i^2a^2}, \text{ by result (5)}$$

$$= \frac{a}{s^2-a^2}$$

Inverting the above result we get $L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at.$

8. $L(\cosh at) = \frac{s}{s^2-a^2}$

$$L(\cosh at) = L\left[\frac{1}{2}\left(e^{at}+e^{-at}\right)\right]$$

$$= \frac{1}{2}\left[L\left(e^{at}\right)+L\left(e^{-at}\right)\right], \text{ by linearity property,}$$

$$= \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right), \text{ if } s > |a|.$$

$$= \frac{s}{s^2-a^2}, \text{ if } s > |a|.$$

Aliter

$$L(\cosh at) = L(\cos iat)\ [\because \cos i\theta = \cosh\theta]$$

$$= \frac{s}{s^2+i^2a^2}, \text{ by result (6)}$$

$$= \frac{s}{s^2-a^2}$$

Inverting the above result we get $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at.$

5.4 LAPLACE TRANSFORMS OF SOME SPECIAL FUNCTIONS

5.4.1 Definition

The function $f(t) = \begin{cases} 0, \text{ when } t < a \\ 1, \text{ when } t > a, \text{ where } a \geq 0 \end{cases}$, is called *Heavyside's unit step function* and is denoted by $u_a(t)$ or $u(t-a)$

In particular, $u_0(t) = \begin{cases} 0, & \text{when } t < 0 \\ 1, & \text{when } t > 0 \end{cases}$

Now $L\{u_a(t)\} = \int_0^\infty e^{-st}\, u_a(t)\, \mathrm{d}t$

$$= \int_0^a e^{-st} u_a(t)\, \mathrm{d}t + \int_a^\infty e^{-st} u_a(t)\, \mathrm{d}t$$

$$= \int_a^\infty e^{-st}\, \mathrm{d}t, \text{ by the definition of } u_a(t)$$

$$= \left(\frac{e^{-st}}{-s}\right)_a^\infty = \frac{e^{-as}}{s}, \text{ assuming that } s > 0.$$

In particular, $L\{u_0(t)\} = \dfrac{1}{s}$, which is the same as $L(1)$.

Inverting the above result, we get $L^{-1}\left\{\dfrac{e^{-as}}{s}\right\} = u_a(t)$.

5.4.2 Definition

$\lim\limits_{h\to 0}\{f(t)\}$, where $f(t)$ is defined by

$$f(t) = \begin{cases} \dfrac{1}{h}, & \text{when } a - \dfrac{h}{2} \le t \le a + \dfrac{h}{2} \\ 0, & \text{otherwise} \end{cases}$$ is called *Unit Impulse*

Function or Dirace Delta Function and is denoted by $\delta_a(t)$ or $\delta\,(t - a)$.

Now $L\{\delta_a(t)\} = L\left[\lim\limits_{h\to 0}\{f\,(t)\}\right]$,where $f(t)$ is taken as given in the definition

$$= \lim_{h\to 0} L\{f\,(t)\}$$

$$= \lim_{h\to 0} \int_0^\infty e^{-st} f(t)\, \mathrm{d}t$$

$$= \lim_{h\to 0} \int_{a-\frac{h}{2}}^{a+\frac{h}{2}} e^{-st} \cdot \frac{1}{h}\, \mathrm{d}t$$

$$= \lim_{h\to 0}\left[\frac{1}{h} \cdot \left(\frac{e^{-st}}{-s}\right)_{a-\frac{h}{2}}^{a+\frac{h}{2}}\right]$$

$$= \lim_{h \to 0} \left[\frac{e^{-s\left(a - \frac{h}{2}\right)} - e^{-s\left(a + \frac{h}{2}\right)}}{sh} \right]$$

$$= e^{-as} \lim_{h \to 0} \left[\frac{e^{sh/2} - e^{-sh/2}}{sh} \right]$$

$$= e^{-as} \lim_{h \to 0} \left[\frac{2 \sinh\left(\frac{sh}{2}\right)}{sh} \right]$$

$$= e^{-as} \lim_{h \to 0} \left[\frac{s \cosh\left(\frac{sh}{2}\right)}{s} \right], \text{ by L' Hospital's rule.}$$

$$= e^{-as} \lim_{h \to 0} \left[\cosh \frac{sh}{2} \right] = e^{-as}$$

Aliter

$$f(t) = \frac{1}{h}\left[u_{a-\frac{h}{2}}(t) - u_{a+\frac{h}{2}}(t) \right], \text{since}$$

$u_{a-\frac{h}{2}}(t) = 1$, when $t > a - \dfrac{h}{2}$ and $u_{a+\frac{h}{2}}(t) = 0$,

when $\quad t < a + \dfrac{h}{2}$ and hence $u_{a-\frac{h}{2}}(t) - u_{a+\frac{h}{2}}(t) = 1$

when $\quad a - \dfrac{h}{2} < t < a + \dfrac{h}{2}$

$$\therefore \quad L\{f(t)\} = \frac{1}{h}\left[L\left\{u_{a-\frac{h}{2}}(t)\right\} - L\left\{u_{a+\frac{h}{2}}(t)\right\} \right]$$

$$= \frac{1}{h}\left[\frac{e^{-\left(a-\frac{h}{2}\right)s}}{s} - \frac{e^{-\left(a+\frac{h}{2}\right)s}}{s} \right]$$

$$= e^{-as} \, \frac{2 \sinh\left(\frac{sh}{2}\right)}{sh}$$

$$\therefore \qquad L\{\delta_a(t)\} = \lim_{h\to 0} L\{f(t)\} = e^{-as} \cdot \lim_{h\to 0}\left[\frac{2\sinh\left(\frac{sh}{2}\right)}{sh}\right]$$

$$= e^{-as}$$

Inverting the above result, we get

$L^{-1}\{e^{-as}\} = \delta_a(t)$. When $a \to 0$, $L^{-1}\{1\} = \delta(t)$.

5.5 PROPERTIES OF LAPLACE TRANSFORMS

1. Change of Scale Property

If $L\{f(t)\} = \phi(s)$, then $L\{f(at)\} = \frac{1}{a}\phi\left(\frac{s}{a}\right)$ and $L\left\{f\left(\frac{t}{a}\right)\right\} = a\phi(as)$.

Proof

By definition, $\phi(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (1)

and $L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$ [$\because f(at)$ is a function of t]

$$= \int_0^\infty e^{-s\frac{x}{a}} f(x)\frac{dx}{a}$$, putting $x = at$ and making necessary changes.

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)x} \cdot f(x)\,dx \qquad (2)$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)t} \cdot f(t)\,dt$$, changing the dummy variable x as t.

Now, comparing (1) and (2), we note that the integral in (2) is the same as the integral in (1) except that 's' in integral in (1) is replaced by $\left(\frac{s}{a}\right)$ in the integral in (2).

$\therefore$ When the integral in (1) is equal to $\phi(s)$, that in (2) is equal to $\phi(s/a)$.

Thus $\qquad L\{f(at)\} = \frac{1}{a}\phi(s/a) \qquad (3)$

Changing a to $\frac{1}{a}$ in (3) or proceeding as in the proof given above, we have

$$L\{f(t/a)\} = a\,\phi(as)\cdot$$

2. First Shifting Property

If $L\{f(t)\} = \phi(s)$, then $L\{e^{-at}f(t)\} = \phi(s+a)$

and $L\{e^{at}f(t)\} = \phi(s-a)$.

Proof

By definition, $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \phi(s) \tag{1}$$

and $$L\{e^{-at}f(t)\} = \int_0^\infty e^{-st}[e^{-at}f(t)]\,\mathrm{d}t$$

[$\because e^{-at}f(t)$ is a function of t]

$$= \int_0^\infty e^{-(s+a)t} f(t)\,\mathrm{d}t \tag{2}$$

$= \phi(s+a)$, comparing the integrals in (1) and (2)

Changing a to $-a$ in the above result,
we get $L\{e^{at} f(t)\} = \phi(s-a)$.

Note ☑

1. The above property can be rewritten as a working rule (formula) in the following way:

$$L\{e^{-at} f(t)\} = \phi(s+a)$$
$$= [\phi(s)]_{s \to s+a}$$
$$= L\{f(t)\}_{s \to s+a}$$

‘$s \to s+a$’ means that s is replaced by $(s+a)$.
Thus, to find the Laplace transform of the product of two factors, one of which is e^{-at}, we ignore e^{-at}, find the Laplace transform of the other factor as a function of s and change s into $(s+a)$ in it.
Similarly,
$$L\{e^{at}f(t)\} = L\{f(t)_{s \to s-a}$$

2. The above property can be stated in terms of the inverse Laplace operator as follows:

If $L^{-1}\{\phi(s)\} = f(t)$, then $L^{-1}\{\phi(s+a)\} = e^{-at} \cdot f(t)$
From this form of the property, we get the following working rule:

$$L^{-1}\{\phi(s+a)\} = e^{-at} \cdot L^{-1}\{\phi(s)\}$$

This means that if we wish to find the Laplace inverse transform of a function that can be identified as a function of $(s+a)$, we have to find the Laplace inverse transform of the corresponding function of s and multiply it with e^{-at}.
Similarly,

$$L^{-1}\{\phi(s-a)\} = e^{at} L^{-1}\{\phi(s)\}.$$

3. The above property is called so, as it concerns shifting on the s-axis by a (or $-a$), i.e., replacing s by $s + a$ (or $s - a$).
   The second shifting property, that follows, concerns shifting on the t-axis by $-a$ i.e., replacing t by $t - a$.

3. Second Shifting Property

If $L\{f(t)\} = \phi(s)$, then $L\{f(t-a)u_a(t)\} = e^{-as}\phi(s)$,
where a is a positive constant and $u_a(t)$ is the unit step function.

Proof

By definition, $L\{f(t-a)u_a(t)\} = \int_0^\infty e^{-st} f(t-a)u_a(t)\,dt$

$$= \int_0^a e^{-st} f(t-a)u_a(t)\,dt + \int_a^\infty e^{-st} f(t-a)u_a(t)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt \text{ , by the definition of } u_a(t).$$

$$= \int_0^\infty e^{-s(x+a)} f(x)\,dx \text{, putting } t - a = x \text{ and effecting consequent changes}$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt \text{ , changing the dummy variable } x \text{ as } t.$$

$$= e^{-as}\phi(s)$$

Note ☑

1. Rewriting the above property, we get the following working rule:
$$L\{f(t-a)\,u_a(t)\} = e^{-as}\,L\{f(t)\}$$
2. The above property can be stated in terms of the inverse Laplace operator as follows:

If $L^{-1}\{\phi(s)\} = f(t)$, then $L^{-1}\{e^{-as}\,\phi(s)\} = f(t-a)\,u_a(t)$.

From this form of the property, we get the following working rule.

$$L^{-1}\{e^{-as}\,\phi(s)\} = L^{-1}\{\phi(s)\}_{t \to t-a} \cdot u_a(t).$$

Thus if we wish to find the Laplace inverse transform of the product of two factors, one of which is e^{-as}, we ignore e^{-as}, find the Laplace inverse transform of the other factor as a function of t, replace t by $(t - a)$ in it and multiply by $u_a(t)$.

WORKED EXAMPLE 5(a)

Example 5.1 Find the Laplace transforms of the following functions:

(i) $$f(t)=\begin{cases}(t-1)^2, & \text{for } t>1\\ 0, & \text{for } 0<t<1\end{cases}$$

(ii) $$f(t)=\begin{cases}e^{kt}, & \text{for } 0<t<a\\ 0, & \text{for } t>a\end{cases}$$

(iii) $$f(t)=\begin{cases}\sin\omega t, & \text{for } 0<t<\dfrac{\pi}{\omega}\\ 0, & \text{for } t>\dfrac{\pi}{\omega}\end{cases}$$

(iv) $$f(t)=\begin{cases}t, & \text{for } 0<t<4\\ 5, & \text{for } t>4\end{cases}$$

$$L\{f(t)\}=\int_0^\infty e^{-st}\, f(t)\,\mathrm{d}t$$

(i) $$L\{f(t)\}=\int_0^1 e^{-st}\cdot o\,\mathrm{d}t+\int_1^\infty (t-1)^2\, e^{-st}\,\mathrm{d}t$$

$$=\left[(t-1)^2\cdot\left(\frac{e^{-st}}{-s}\right)-2(t-1)\left(\frac{e^{-st}}{s^2}\right)+2\left(\frac{e^{-st}}{-s^3}\right)\right]_1^\infty$$

$$=0+\frac{2}{s^3}e^{-s}=\frac{2}{s^3}e^{-s}$$

Aliter

$$(t-1)^2\, u_1(t)=\begin{cases}0, & \text{for } 0<t<1\\ (t-1)^2 & \text{for } t>1\end{cases}$$

Thus $f(t) = (t-1)^2 u_1(t)$

$\therefore$ $L\{f(t)\} = e^{-s}\cdot L(t^2)$, by the second shifting property.

$$=\frac{2}{s^3}e^{-s}$$

(ii) $$L\{f(t)\}=\int_0^a e^{-st}\cdot e^{kt}\,\mathrm{d}t+\int_a^\infty e^{-st}\cdot 0\,\mathrm{d}t$$

$$=\left[\frac{e^{-(s-k)t}}{-(s-k)}\right]_0^a=\frac{1}{s-k}\{1-e^{-a(s-k)}\}$$

Aliter

Consider $e^{kt}\{1-u_a(t)\}=\begin{cases} e^{kt}, & \text{for } 0<t<a \\ 0, & \text{for } t>a \end{cases}$

Thus $\quad f(t) = e^{kt} - e^{kt}u_a(t)$

$$= e^{kt} - e^{k(t-a)+ka}\cdot u_a(t)$$

$\therefore \quad L\{f(t)\} = L(e^{kt}) - e^{ak}\cdot L\{e^{k(t-a)}\cdot u_a(t)\}$

$$= \frac{1}{s-k} - e^{ak}\cdot e^{-as}\cdot L\{e^{kt}\}$$, by the second shifting property

$$= \frac{1}{s-k} - e^{-a(s-k)}\cdot\frac{1}{s-k}$$

$$= \frac{1}{s-k}\{1-e^{-a(s-k)}\}.$$

(iii) $\quad L\{f(t)\} = \int_0^{\pi/\omega} e^{-st}\sin\omega t\,\mathrm{d}t$

$$= \left[\frac{e^{-st}}{s^2+\omega^2}\left(-s\sin\omega t - \omega\cos\omega t\right)\right]_0^{\pi/\omega}$$

$$= \frac{\omega}{s^2+\omega^2}\left(1+e^{-\pi s/\omega}\right)$$

(iv) $\quad L\{f(t)\} = \int_0^4 te^{-st}\,\mathrm{d}t + \int_4^\infty 5e^{-st}\,\mathrm{d}t.$

$$= \left[t\cdot\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right]_0^4 + 5\left(\frac{e^{-st}}{-s}\right)_4^\infty$$

$$= -\frac{4}{s}e^{-4s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2} + \frac{5}{s}e^{-4s}$$

$$= \frac{1}{s}e^{-4s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2}$$

Aliter

$$t\{1-u_4(t)\}+5u_4(t) = \begin{cases} t, & \text{for } 0<t<4 \\ 5, & \text{for } t>4 \end{cases}$$

Thus $\quad f(t) = t - (t-5)\,u_4(t)$

$$= t - (t-4)\,u_4(t) + u_4(t)$$

$\therefore$
$$L(f(t)\} = L(t) - e^{-4s}\cdot L(t) + L\{u_4(t)\}$$
$$= \frac{1}{s^2} - \frac{1}{s^2}e^{-4s} + \frac{1}{s}e^{-4s}.$$

Example 5.2 Find the Laplace transforms of the following functions:

(i) $\frac{1+2t}{\sqrt{t}}$, (ii) $\sin\sqrt{t}$, (iii) $\frac{\cos\sqrt{t}}{\sqrt{t}}$.

(i)
$$L\left\{\frac{1+2t}{\sqrt{t}}\right\} = L(t^{-1/2}) + 2L(t^{1/2})$$
$$= \frac{\overline{|(1/2)}}{s^{1/2}} + 2\frac{\overline{|(3/2)}}{s^{3/2}}$$
$$= \frac{\sqrt{\pi}}{\sqrt{s}} + \frac{2\cdot\frac{1}{2}\sqrt{\pi}}{s\sqrt{s}} \quad \left[\because \overline{|(1/2)} = \sqrt{\pi} \quad \text{and} \quad \overline{|(n+1)} = n\overline{|(n)}\right]$$
$$= \sqrt{\frac{\pi}{s}}\left(1+\frac{1}{s}\right).$$

(ii)
$$\sin\sqrt{t} = \sqrt{t} - \frac{\left(\sqrt{t}\right)^3}{3!} + \frac{\left(\sqrt{t}\right)^5}{5!} - \ldots.\infty$$
$$= t^{1/2} - \frac{1}{3!}t^{3/2} + \frac{1}{5!}t^{5/2} - \ldots.\infty$$

$\therefore$
$$L\left(\sin\sqrt{t}\right) = L(t^{1/2}) - \frac{1}{3!}L(t^{3/2}) + \frac{1}{5!}L(t^{5/2}) - \ldots.\infty$$
$$= \frac{\overline{|(3/2)}}{s^{3/2}} - \frac{1}{3!}\frac{\overline{|(5/2)}}{s^{5/2}} + \frac{1}{5!}\frac{\overline{|(7/2)}}{s^{7/2}} - \ldots.\infty$$
$$= \frac{1}{s^{3/2}}\left[\frac{1}{2}\overline{|(1/2)} - \frac{1}{3!}\cdot\frac{3}{2}\cdot\frac{1}{2}\overline{|(1/2)}\cdot\frac{1}{s} + \frac{1}{5!}\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\overline{|(1/2)}\cdot\frac{1}{s^2} - \ldots.\infty\right]$$
$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{1!}\cdot\left(\frac{1}{4s}\right) + \frac{1}{2!}\cdot\left(\frac{1}{4s}\right)^2 - \ldots..\infty\right]$$
$$= \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s}$$

(iii)
$$\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}}\left[1 - \frac{\left(\sqrt{t}\right)^2}{2!} + \frac{\left(\sqrt{t}\right)^4}{4!} - \ldots..\infty\right]$$
$$= t^{-1/2} - \frac{1}{2!}t^{1/2} + \frac{1}{4!}t^{3/2} - \ldots..\infty$$

$$\therefore \qquad L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right)=L(t^{-1/2})-\frac{1}{2!}L\left(t^{1/2}\right)+\frac{1}{4!}L\left(t^{3/2}\right)-.....\infty$$

$$=\frac{\overline{|(1/2)}}{s^{1/2}}-\frac{1}{2!}\frac{\overline{|(3/2)}}{s^{3/2}}+\frac{1}{4!}\frac{\overline{|(5/2)}}{s^{5/2}}-.....\infty$$

$$=\sqrt{\frac{\pi}{s}}\left[1-\frac{1}{2!\,2s}+\frac{1}{4!}\frac{3\cdot 1}{2\cdot 2s^2}-.....\infty\right]$$

$$=\sqrt{\frac{\pi}{s}}\left[1-\frac{1}{1!}\cdot\left(\frac{1}{4s}\right)+\frac{1}{2!}\cdot\left(\frac{1}{4s}\right)^2-.....\infty\right]$$

$$=\sqrt{\frac{\pi}{s}}\,e^{-1/4s}$$

Example 5.3 Find the Laplace transforms of the following functions:

(i) $(t^3+3e^{2t}-5\sin 3t)e^{-t}$; (ii) $(1+te^{-t})^3$;
(iii) $e^{-2t}\cosh^3 2t$; (iv) $\cosh at \cos at$;
(v) $\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t$

(i) $L\{(t^3+3e^{2t}-5\sin 3t)e^{-t}\}$

$= L(t^3+3e^{2t}-5\sin 3t)_{s\to s+1}$, by first shifting property.

$$=\left(\frac{3!}{s4}+\frac{3}{s-2}-5\cdot\frac{3}{s^2+9}\right)_{s\to s+1}$$

$$=\frac{6}{(s+1)^4}+\frac{3}{(s-1)}-\frac{15}{s^2+2s+10}$$

(ii) $L(1+te^{-t})^3 = L(1+3t\,e^{-t}+3t^2\,e^{-2t}+t^3\,e^{-3t})$

$= L(1)+3L(t)_{s\to s+1}+3L(t^2)_{s\to s+2}+L(t^3)_{s\to s+3}$

$$=\frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}$$

(iii) $$L(e^{-2t}\cosh^3 2t)=L\left\{e^{-2t}\cdot\left(\frac{e^{2t}+e^{-2t}}{2}\right)^3\right\}$$

$$=\frac{1}{8}L\{e^{-2t}(e^{6t}+3e^{2t}+3e^{-2t}+e^{-6t})\}$$

$$=\frac{1}{8}L\{e^{4t}+3+3e^{-4t}+e^{-8t}\}$$

$$=\frac{1}{8}\left(\frac{1}{s-4}+\frac{3}{s}+\frac{3}{s+4}+\frac{1}{s+8}\right)$$

(iv) $$L(\cosh at\cos at) = L\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\cos at\right\}$$

$$= \frac{1}{2}\left[L(\cos at)_{s\to s-a} + L(\cos at)_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{s^2+2a^2-2as} + \frac{s+a}{s^2+2a^2+2as}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2+2a^2+2as)+(s+a)(s^2+2a^2-2as)}{(s^2+2a^2)^2-4a^2s^2}\right]$$

$$= \frac{s^3}{s^4+4a^4}.$$

(v) $$L(\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t)$$

$$= L\left\{\left(\frac{e^{t/2}-e^{-t/2}}{2}\right)\sin\frac{\sqrt{3}}{2}t\right\}$$

$$= \frac{1}{2}\left[L\left(\sin\frac{\sqrt{3}}{2}t\right)_{s\longrightarrow s-\frac{1}{2}} - L\left(\sin\frac{\sqrt{3}}{2}t\right)_{s\longrightarrow s+\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{3}/2}{(s-1/2)^2+3/4} - \frac{\sqrt{3}/2}{(s+1/2)^2+3/4}\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right]$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{s}{s^4+s^2+1}$$

Example 5.4 Find the Laplace transforms of the following functions:
(i) $e^{at}\cos(bt+c)$; (ii) $e^{-2t}\cos^2 3t$; (iii) $e^t\sin^3 2t$;
(iv) $e^{-t}\sin 2t\cos 3t$; (v) $e^{3t}\sin 2t\sin t$.

(i) $$L\{e^{at}\cos(bt+c)\} = L\{\cos(bt+c)\}_{s\longrightarrow s-a}$$

$$= L\{\cos c\cos bt - \sin c\sin bt\}_{s\longrightarrow s-a}$$

$$= \frac{(s-a)\cos c}{(s-a)^2+b^2} - \frac{b\sin c}{(s-a)^2+b^2}$$

(ii) $L\{e^{-2t}\cos^2 3t\} = L(\cos^2 3t)_{s \to s+2}$

$$= L\left\{\frac{1}{2}(1+\cos 6t)\right\}_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{1}{s+2} + \frac{s+2}{(s+2)^2+36}\right]$$

(iii) $L\{e^{t}\sin^3 2t\} = L(\sin^3 2t)_{s \to s-1}$

$$= L\left\{\frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t\right\}_{s\to s-1}$$

$$= \left(\frac{3}{4}\cdot\frac{2}{s^2+4} - \frac{1}{4}\cdot\frac{6}{s^2+36}\right)_{s\to s-1}$$

$$= \frac{3}{2}\left(\frac{1}{s^2-2s+5} - \frac{1}{s^2-2s+37}\right)$$

(iv) $L\{e^{-t}\sin 2t\cos 3t\} = L(\sin 2t\cos 3t)_{s \to s+1}$

$$= L\left\{\frac{1}{2}(\sin 5t - \sin t)\right\}_{s\to s+1}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]_{s\to s+1}$$

$$= \frac{1}{2}\left\{\frac{5}{s^2+2s+26} - \frac{1}{s^2+2s+2}\right\}$$

(v) $L\{e^{3t}\sin 2t\sin t\} = L(\sin 2t\sin t)_{s \to s-3}$

$$= L\left\{\frac{1}{2}(\cos t - \cos 3t)\right\}_{s\to s-3}$$

$$= \frac{1}{2}\left\{\frac{s}{s^2+1} - \frac{s}{s^2+9}\right\}_{s\to s-3}$$

$$= \frac{1}{2}\left\{\frac{s-3}{s^2-6s+10} - \frac{s-3}{s^2-6s+18}\right\}$$

Example 5.5 Find the Laplace transforms of the following functions:
(i) $(t-1)^2 \cdot u_1(t)$; (ii) $\sin t \cdot u_\pi(t)$; (iii) $e^{-3t}\cdot u_2(t)$

(i) $L\{(t-1)^2\, u_1(t)\} = e^{-s}\cdot L\{t^2\}$, by the second shifting property

$$= \frac{2}{s^3}e^{-s}$$

(ii) $L\{\sin t \cdot u_\pi(t)\} = L\{\sin(t-\pi+\pi)\cdot u_\pi(t)\}$

$$= -L\{\sin(t-\pi)\cdot u_\pi(t)\}$$

$$= -e^{-s\pi} L(\sin t)$$

$$= -\frac{e^{-s\pi}}{s^2+1}$$

(iii) $L\{e^{-3t} u_2(t)\} = L\{e^{-3(t-2)-6} \cdot u_2(t)\}$

$$= e^{-6} \cdot e^{-2s} \cdot L\{e^{-3t}\} = \frac{e^{-(2s+6)}}{s+3}.$$

Example 5.6 Find the Laplace transforms of the following functions:

(i) $t \sin at$; (ii) $t \cos at$; (iii) $te^{-4t} \sin 3t$

(i) $L(t \sin at) = L\{t \times \text{I.P. of } e^{iat}\}$

$$= \text{I.P. of } L\{t\, e^{iat}\}$$

$$= \text{I.P. of } L(t)_{s \to s-ia}$$

$$= \text{I.P. of } \frac{1}{(s-ia)^2} = \text{I.P. of } \frac{(s+ia)^2}{(s^2+a^2)^2}$$

$$= \text{I.P. of } \left[\frac{s^2-a^2}{(s^2+a^2)^2} + i\frac{2as}{(s^2+a^2)^2}\right]$$

$$= \frac{2as}{(s^2+a^2)^2}$$

(ii) $L(t \cos at) = \text{R.P. of } L\{te^{iat}\}$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

(iii) $L(t\, e^{-4t} \sin 3t) = \text{I.P. of } L\,\{t\, e^{-4t}\, e^{i3t}\}$

$$= \text{I.P. of } L\,\{t \cdot e^{-(4-i3)t}\}$$

$$= \text{I.P. of } L(t)_{s \to (s+4-i3)}$$

$$= \text{I.P. of } \frac{1}{(s+4-i3)^2} = \text{I.P. of } \frac{(s+4+i3)^2}{\{(s+4)^2+9\}^2}$$

$$= \text{I.P. of } \frac{\{(s+4)^2-9\} + i\,6(s+4)}{\{(s+4)^2+9\}^2}$$

$$= \frac{6\,(s+4)}{\{(s+4)^2+9\}^2}$$

Example 5.7

(i) Assuming $L(\sin t)$ and $L(\cos t)$, find the Laplace transforms of $L\left(\sin\frac{t}{2}\right)$ and $L(\cos 3t)$.

(ii) Given that $L(t\sin t)=\frac{2s}{\left(s^2+1\right)^2}$ and $L(t\cos t)=\frac{s^2-1}{\left(s^2+1\right)^2}$, find $L(t\sin at)$ and $L\left(t\cos\frac{t}{a}\right)$.

(i) $L(\sin t)=\frac{1}{s^2+1}$

$$\therefore \quad L\left(\sin\frac{t}{2}\right)=2\cdot\frac{1}{(2s)^2+1}$$

$$\left[\because L\left\{f\left(\frac{t}{a}\right)\right\}=a\cdot L\{f(t)\}_{s\to as}, \text{by the charge of scale property}\right]$$

$$=\frac{2}{4s^2+1}$$

$$L(\cos t)=\frac{s}{s^2+1}$$

$$\therefore \quad L(\cos 3t)=\frac{1}{3}\cdot\frac{s/3}{(s/3)^2+1}$$

$$\left[\because L\{f(at)\}=\frac{1}{a}L\{f(t)\}_{s\to\frac{s}{a}}, \text{by the charge of scale property}\right]$$

$$=\frac{s}{s^2+9}.$$

(ii) Given that $L(t\sin t)=\frac{2s}{\left(s^2+1\right)^2}$

$$\therefore \quad L(t\sin at)=\frac{1}{a}L(at\sin at)$$

$$=\frac{1}{a}\cdot\frac{1}{a}\cdot\frac{2s/a}{\left(s^2/a^2+1\right)^2}, \text{by change of scale property}\cdot$$

$$=\frac{2as}{\left(s^2+a^2\right)^2}$$

Given that $L(t\cos t)=\dfrac{s^2-1}{(s^2+1)^2}$

$$\therefore \qquad L\left(t\cos\frac{t}{a}\right)=a\cdot L\left(\frac{t}{a}\cos\frac{t}{a}\right)$$

$$=a\cdot a\cdot\frac{(as)^2-1}{\left[(as)^2+1\right]^2}\text{, by change of scale property}$$

$$=\frac{a^4s^2-a^2}{(a^2s^2+1)^2}\ \text{ or }\ \frac{s^2-\dfrac{1}{a^2}}{\left(s^2+\dfrac{1}{a^2}\right)^2}.$$

Example 5.8 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{1}{(s+2)^{5/2}}e^{-s}$; (ii) $\dfrac{e^{-2s}}{(2s-3)^3}$; (iii) $\dfrac{s\,e^{-s}}{(s-3)^5}$; (iv) $\dfrac{(3a-4s)}{s^2+a^2}e^{-bs}$;

(v) $\dfrac{(s+4)}{s^2-4}e^{-4s}$

(i) From the second shifting property, we have

$$L^{-1}\{e^{-as}\phi(s)\}=L^{-1}\{\phi(s)_{t\to t-a}\cdot u_a(t)$$

$$\therefore L^{-1}\left\{e^{-s}\cdot\frac{1}{(s+2)^{5/2}}\right\}=L^{-1}\left\{\frac{1}{(s+2)^{5/2}}\right\}_{t\to t-1}\cdot u_1(t) \qquad (1)$$

Now $L^{-1}\left\{\dfrac{1}{(s+2)^{5/2}}\right\}=e^{-2t}\cdot L^{-1}\left\{\dfrac{1}{s^{5/2}}\right\}$

[$\because L^{-1}\{\phi(s+2)=e^{-2t}L^{-1}\{\phi(s)\}$, by the first shifting property.]

$$=e^{-2t}\cdot\frac{1}{\overline{|(5/2)}}\cdot t^{3/2} \qquad \left[\because L^{-1}\left\{\frac{1}{s^n}=\frac{1}{\overline{|(n)}}t^{n-1}\right\}\right]$$

$$=e^{-2t}\cdot\frac{1}{\frac{3}{2}\cdot\frac{1}{2}\cdot\overline{|(1/2)}}\cdot t^{3/2} \qquad \left[\because\overline{|(n)}=(n-1)\overline{|(n-1)}\right]$$

$$=\frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}}e^{-2t} \qquad \left[\because\overline{|(1/2)}=\sqrt{\pi}\right] \qquad (2)$$

Inserting (2) in (1), we have

$$L^{-1}\left\{e^{-s}\frac{1}{(s+2)^{5/2}}\right\}=\frac{4}{3\sqrt{\pi}}(t-1)^{3/2}e^{-2(t-1)}\cdot u_1(t)$$

or
$$= \begin{cases} 0, & \text{when } t < 1 \\ \dfrac{4}{3\sqrt{\pi}}(t-1)^{3/2} \cdot e^{-2(t-1)}, & \text{when } t > 1 \end{cases}$$

(ii) $$L^{-1}\left\{\frac{e^{-2s}}{(2s-3)^3}\right\} = L^{-1}\left\{\frac{1}{(2s-3)^3}\right\}_{t \longrightarrow t-2} \cdot u_2(t) \qquad (1)$$

Now $$L^{-1}\left\{\frac{1}{(2s-3)^3}\right\} = \frac{1}{8}L^{-1}\left\{\frac{1}{(s-3/2)^3}\right\}$$
$$= \frac{1}{8}e^{\frac{3}{2}t}\, L^{-1}\left\{\frac{1}{s^3}\right\}$$
$$= \frac{1}{8}e^{\frac{3}{2}t} \cdot \frac{1}{2!}t^2 = \frac{1}{16}e^{\frac{3}{2}t}\, t^2 \qquad (2)$$

Using (2) in (1), we get

$$L^{-1}\left\{\frac{e^{-2s}}{(2s-3)^3}\right\} = \frac{1}{16}\, e^{\frac{3}{2}(t-2)}\, (t-2)^2 \cdot u_2(t)$$

(iii) $$L^{-1}\left\{\frac{s\,e^{-s}}{(s-3)^5}\right\} = L^{-1}\left\{\frac{s}{(s-3)^5}\right\}_{t \longrightarrow t-1} \cdot u_1(t) \qquad (1)$$

Now $$L^{-1}\left\{\frac{s}{(s-3)^5}\right\} = L^{-1}\left\{\frac{(s-3)+3}{(s-3)^5}\right\}$$
$$= e^{3t}L^{-1}\left\{\frac{s+3}{s^5}\right\}$$
$$= e^{3t}L^{-1}\left\{\frac{1}{s^4}+\frac{3}{s^5}\right\}$$
$$= e^{3t}\left\{\frac{1}{3!}t^3 + 3\cdot\frac{1}{4!}t^4\right\}$$
$$= \frac{1}{24}e^{3t}(4t^3+3t^4) \qquad (2)$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{s}{(s-3)^5}e^{-s}\right\} = \frac{1}{24}e^{3(t-1)}\,\{4(t-1)^3+3(t-1)^4\}\cdot u_1(t)$$

(iv) $$L^{-1}\left\{\frac{(3a-4s)}{s^2+a^2}e^{-bs}\right\} = L^{-1}\left\{\frac{3a-4s}{s^2+a^2}\right\}_{t \longrightarrow t-b} \cdot u_b(t) \qquad (1)$$

Now $L^{-1}\left\{\dfrac{3a-4s}{s^2+a^2}\right\}=3L^{-1}\left\{\dfrac{a}{s^2+a^2}\right\}-4L^{-1}\left\{\dfrac{s}{s^2+a^2}\right\}$

$$= 3 \sin at - 4 \cos at \qquad (2)$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{(3a-4s)}{s^2+a^2}e^{-bs}\right\}=[3\sin a(t-b)-4\cos a(t-b)]u_b(t)$$

(v) $L^{-1}\left\{\dfrac{(s+4)}{s^2-4}e^{-4s}\right\}=L^{-1}\left\{\dfrac{s+4}{s^2-4}\right\}_{t\to t-4}\cdot u_4(t) \qquad (1)$

Now $L^{-1}\left\{\dfrac{s+4}{s^2-4}\right\}=L^{-1}\left\{\dfrac{s}{s^2-4}\right\}+2L^{-1}\left\{\dfrac{2}{s^2-4}\right\}$

$$= \cosh 2t + 2 \sinh 2t \qquad (2)$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{(s+4)}{s^2-4}e^{-4s}\right\}=\{\cosh 2(t-4)+2\sinh 2(t-4)\}\cdot u_4(t)$$

Example 5.9 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{e^{-s}}{(s-2)(s+3)}$; (ii) $\dfrac{e^{-2s}}{s^2(s^2+1)}$; (iii) $\dfrac{e^{-s}}{s(s^2+4)}$; (iv) $\dfrac{e^{-3s}}{s^2+4s+13}$;

(v) $\dfrac{(s+1)e^{-\pi s}}{s^2+2s+5}$.

(i) $L^{-1}\left\{\dfrac{e^{-s}}{(s-2)(s+3)}\right\}=L^{-1}\left\{\dfrac{1}{(s-2)(s+3)}\right\}_{t\to t-1}\cdot u_1(t) \qquad (1)$

Now to find $L^{-1}\left\{\dfrac{1}{(s-2)(s+3)}\right\}$, we resolve $\dfrac{1}{(s-2)(s+3)}$ into partial fractions and then use the linearity property of L^{-1} operator.

Let $\dfrac{1}{(s-2)(s+3)}=\dfrac{A}{s-2}+\dfrac{B}{s+3}$

Then $A(s+3)+B(s-2)=1$

By the usual procedure, we get $A=\dfrac{1}{5}$, $B=-\dfrac{1}{5}$

$$\therefore\ L^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\}=L^{-1}\left\{\frac{1/5}{s-2}-\frac{1/5}{s+3}\right\}$$

$$=\frac{1}{5}L^{-1}\left(\frac{1}{s-2}\right)-\frac{1}{5}L^{-1}\left(\frac{1}{s+3}\right)$$

$$=\frac{1}{5}(e^{2t}-e^{-3t}) \tag{2}$$

Putting (2) in (1), we have

$$L^{-1}\left\{\frac{e^{-s}}{(s-2)(s+3)}\right\}=\frac{1}{5}\left\{e^{2(t-1)}-e^{-3(t-1)}\right\}\cdot u_1(t)$$

(ii) $$L^{-1}\left\{\frac{e^{-2s}}{s^2(s^2+1)}\right\}=L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}_{t\longrightarrow t-2}\cdot u_2(t) \tag{1}$$

Now $$\frac{1}{s^2(s^2+1)}=\frac{1}{u(u+1)}=\frac{1}{u}-\frac{1}{u+1}=\frac{1}{s^2}-\frac{1}{s^2+1}$$

$\therefore$ $$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}=L^{-1}\left(\frac{1}{s^2}\right)-L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= t - \sin t \tag{2}$$

Inserting (2) in (1), we get

$$L^{-1}\left\{\frac{e^{-2s}}{s^2(s^2+1)}\right\}=\{(t-2)-\sin(t-2\}u_2(t)$$

(iii) $$L^{-1}\left\{\frac{e^{-s}}{s(s^2+4)}\right\}=L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}_{t\longrightarrow t-1}\cdot u_1(t) \tag{1}$$

Let $$\frac{1}{s(s^2+4)}=\frac{A}{s}+\frac{Bs+C}{s^2+4}$$

Then $A(s^2 + 4) + s\,(Bs + c) = 1$

$\therefore$ $$A=\frac{1}{4}, B=-\frac{1}{4} \text{ and } C = 0$$

$\therefore$ $$L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}=L^{-1}\left[\frac{\frac{1}{4}}{s}-\frac{\frac{1}{4}s}{s^2+4}\right]$$

$$=\frac{1}{4}(1-\cos 2t) \tag{2}$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{e^{-s}}{s(s^2+4)}\right\}=\frac{1}{4}\{1-\cos 2(t-1)\}\cdot u_1(t)$$

(iv) $$L^{-1}\left\{\frac{e^{-3s}}{s^2+4s+13}\right\}=L^{-1}\left\{\frac{1}{s^2+4s+13}\right\}_{t\to t-3}\cdot u_3(t) \quad (1)$$

Now $$L^{-1}\left\{\frac{1}{s^2+4s+13}\right\}=L^{-1}\left\{\frac{1}{(s+2)^2+3^2}\right\}$$

$$=e^{-2t}\cdot L^{-1}\left\{\frac{1}{s^2+3^2}\right\}, \text{ by the shifting property}$$

$$=\frac{1}{3}e^{-2t}\,L^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$=\frac{1}{3}e^{-2t}\sin 3t \quad (2)$$

Using (2) in (1), we get

$$L^{-1}\left\{\frac{e^{-3s}}{s^2+4s+13}\right\}=\frac{1}{3}e^{-2(t-3)}\cdot\sin 3(t-3)\cdot u_3(t)$$

(v) $$L^{-1}\left\{\frac{(s+1)e^{-\pi s}}{s^2+2s+5}\right\}=L^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\}_{t\to t-\pi}\cdot u_\pi(t) \quad (1)$$

Now $$L^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\}=L^{-1}\left\{\frac{(s+1)}{(s+1)^2+2^2}\right\}$$

$$=e^{-t}\cdot L^{-1}\left\{\frac{s}{s^2+2^2}\right\}, \text{ by the shifting property}$$

$$= e^{-t}\cos 2t \quad (2)$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{(s+1)e^{-\pi s}}{s^2+2s+5}\right\}=e^{-(t-\pi)}\cdot\cos 2(t-\pi)\cdot u_\pi(t)\,.$$

Example 5.10 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{s^2+s-2}{s(s+3)(s-2)}$; (ii) $\dfrac{2s^2+5s+2}{(s-2)^4}$; (iii) $\dfrac{s}{(s+1)^2(s^2+1)}$;

(iv) $\dfrac{1}{s^2(s^2+1)(s^2+9)}$; (v) $\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$.

(i) To find $L^{-1}\left\{\dfrac{s^2+s-2}{s(s+3)(s-2)}\right\}$, we resolve the given function of s into partial fractions and then use the linearity property of L^{-1} operator.

Let $\dfrac{s^2+s-2}{s(s+3)(s-2)}=\dfrac{A}{s}+\dfrac{B}{s+3}+\dfrac{C}{s-2}$

We find, by the usual procedure, that

$$A=\frac{1}{3},\ B=\frac{4}{15}\quad\text{and}\quad C=\frac{2}{5}.$$

$$\therefore \qquad L^{-1}\left\{\frac{s^2+s-2}{s(s+3)(s-2)}\right\}=L^{-1}\left[\frac{1/3}{s}+\frac{4/15}{s+3}+\frac{2/5}{s-2}\right]$$

$$=\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}.$$

(ii) To resolve $\dfrac{2s^2+5s+2}{(s-2)^4}$ into partial fractions, we put $s-2=x$, so that

$s=x+2$.

Then
$$\frac{2s^2+5s+2}{(s-2)^4}=\frac{2(x+2)^2+5(x+2)+2}{x^4}$$

$$=\frac{2x^2+13x+20}{x^4}$$

$$=\frac{2}{x^2}+\frac{13}{x^3}+\frac{20}{x^4}$$

$$=\frac{2}{(s-2)^2}+\frac{13}{(s-2)^3}+\frac{20}{(s-2)^4}$$

$$\therefore\ L^{-1}\left\{\frac{2s^2+5s+2}{(s-2)^4}\right\}=2L^{-1}\left\{\frac{1}{(s-2)^2}\right\}+13L^{-1}\left\{\frac{1}{(s-2)^3}\right\}+20L^{-1}\left\{\frac{1}{(s-2)^4}\right\}$$

$$=2e^{2t}\cdot L^{-1}\left(\frac{1}{s^2}\right)+13e^{2t}\cdot L^{-1}\left(\frac{1}{s^3}\right)+20e^{2t}\cdot L^{-1}\left(\frac{1}{s^4}\right)$$

$$=e^{2t}\left[2\cdot t+\frac{13}{2!}t^2+\frac{20}{3!}t^3\right]$$

$$=e^{2t}\left(2t+\frac{13}{2}t^2+\frac{10}{3}t^3\right)$$

$$= \frac{1}{6} e^{2t} (12t + 39t^2 + 20t^3)$$

(iii) Let $$\frac{s}{(s+1)^2 \left(s^2+1\right)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C\,s+D}{s^2+1}$$

$\therefore$ $$A(s+1)\,(s^2+1) + B(s^2+1) + (Cs+D)\,(s+1)^2 = s$$

By the usual procedure, we find that

$$A = 0;\quad B = -\frac{1}{2};\quad C = 0 \quad \text{and} \quad D = \frac{1}{2}$$

$\therefore$ $$L^{-1}\left\{\frac{s}{(s+1)^2\left(s^2+1\right)}\right\} = L^{-1}\left\{\frac{-1/2}{(s+1)^2} + \frac{1/2}{s^2+1}\right\}$$

$$= -\frac{1}{2} e^{-t} \cdot L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{t}{2} e^{-t} + \frac{1}{2} \sin t.$$

(iv) Since $\dfrac{1}{s^2\left(s^2+1\right)\left(s^2+9\right)}$ is a function of s^2, we put $s^2 = u$, we resolve

$\dfrac{1}{u\,(u+1)\,(u+9)}$ into partial fractions and then replace u by s^2.

Now let $$\frac{1}{u\,(u+1)\,(u+9)} = \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u+9}$$

By the usual procedure, we find that $A = \dfrac{1}{9},\ B = -\dfrac{1}{8}$ and $C = \dfrac{1}{72}$

$\therefore$ $$L^{-1}\left\{\frac{1}{s^2\left(s^2+1\right)\left(s^2+9\right)}\right\} = L^{-1}\left\{\frac{1/9}{s^2} - \frac{1/8}{s^2+1} + \frac{1/72}{s^2+9}\right\}$$

$$= \frac{1}{9} t - \frac{1}{8} \sin t + \frac{1}{216} \sin 3t.$$

(v) To resolve $\dfrac{s}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$ into partial fractions, we first resolve

$\dfrac{1}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$ into partial fractions as shown in (iv).

Thus $$\frac{1}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)} = \frac{1}{(u+1)\,(u+4)\,(u+9)}$$

$$= \frac{A}{u+1} + \frac{B}{u+4} + \frac{C}{u+9}, \text{ say.}$$

We find that $A=\dfrac{1}{24}$, $B=-\dfrac{1}{15}$ and $C=\dfrac{1}{40}$.

$$\therefore \quad \frac{1}{(s^2+1)(s^2+4)(s^2+9)}=\frac{1/24}{s^2+1}-\frac{1/15}{s^2+4}+\frac{1/40}{s^2+9}$$

$$\therefore \quad \frac{s}{(s^2+1)(s^2+4)(s^2+9)}=\frac{1}{24}\cdot\frac{s}{s^2+1}-\frac{1}{15}\cdot\frac{s}{s^2+4}+\frac{1}{40}\cdot\frac{s}{s^2+9}$$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right\}=\frac{1}{24}L^{-1}\left(\frac{s}{s^2+1}\right)-\frac{1}{15}L^{-1}\left(\frac{s}{s^2+4}\right)+\frac{1}{40}L^{-1}\left(\frac{s}{s^2+9}\right)$$

$$=\frac{1}{24}\cos t-\frac{1}{15}\cos 2t+\frac{1}{40}\cos 3t$$

Example 5.11 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{14s+10}{49s^2+28s+13}$; (ii) $\dfrac{2s^3+4s^2-s+1}{s^2(s^2-s+2)}$;

(iii) $\dfrac{1}{s^3-a^3}$; (iv) $\dfrac{1}{s^4+4}$;

(v) $\dfrac{s}{s^4+s^2+1}$

(i) $$L^{-1}\left\{\frac{14s+10}{49s^2+28s+13}\right\}=\frac{14}{49}L^{-1}\left\{\frac{s+\frac{5}{7}}{s^2+\frac{4}{7}s+\frac{13}{49}}\right\}$$

$$=\frac{2}{7}L^{-1}\left\{\frac{s+\frac{5}{7}}{\left(s+\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2}\right\}$$

$$=\frac{2}{7}L^{-1}\left\{\frac{\left(s+\frac{2}{7}\right)+\frac{3}{7}}{\left(s+\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2}\right\}$$

$$=\frac{2}{7}e^{-\frac{2}{7}t}\cdot L^{-1}\left\{\frac{s+\frac{3}{7}}{s^2+\left(\frac{3}{7}\right)^2}\right\}$$

$$= \frac{2}{7} e^{-\frac{2}{7}t} \left(\cos \frac{3}{7} t + \sin \frac{3}{7} t \right)$$

(ii) Let $$\frac{2s^3 + 4s^2 - s + 1}{s^2 (s^2 - s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 - s + 2}$$

$\therefore$ $$As(s^2 - s + 2) + B(s^2 - s + 2) + (Cs + D)s^2 = 2s^3 + 4s^2 - s + 1.$$

By the usual procedure, we find that

$$A = \frac{1}{4}, \quad B = \frac{1}{2}, C = \frac{9}{4} \text{ and } D = \frac{13}{4}$$

$\therefore$ $$L^{-1}\left\{ \frac{2s^3 + 4s^2 - s + 1}{s^2 \left(s^2 - s + 2\right)} \right\} = -\frac{1}{4} L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left\{ \frac{\frac{9}{4}s + \frac{13}{4}}{s^2 - s + 2} \right\}$$

$$= -\frac{1}{4} + \frac{t}{2} + L^{-1}\left\{ \frac{\frac{9}{4}\left(s - \frac{1}{2}\right) + \frac{35}{8}}{\left(s - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \right\}$$

$$= -\frac{1}{4} + \frac{t}{2} + e^{t/2} L^{-1}\left\{ \frac{9}{4} \cdot \frac{s}{s^2 + \left(\frac{\sqrt{7}}{2}\right)^2} + \frac{\frac{5\sqrt{7}}{4} \cdot \frac{\sqrt{7}}{2}}{s^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \right\}$$

$$= -\frac{1}{4} + \frac{t}{2} + e^{t/2} \left(\frac{9}{4} \cos \frac{\sqrt{7}}{2} t + \frac{5\sqrt{7}}{4} \sin \frac{\sqrt{7}}{2} t \right)$$

(iii) $$\frac{1}{s^3 - a^3} = \frac{1}{(s - a)\left(s^2 + as + a^2\right)}$$

$\therefore$ Let $$\frac{1}{s^3 - a^3} = \frac{A}{s - a} + \frac{Bs + C}{s^2 + as + a^2}$$

$\therefore$ $$A(s^2 + as + a^2) + (s - a)(Bs + C) = 1$$

By the usual procedure, we find that

$$A = \frac{1}{3a^2}, B = -\frac{1}{3a^2} \quad \text{and} \quad C = -\frac{2}{3a}.$$

$$\therefore \qquad L^{-1}\left\{\frac{1}{s^3-a^3}\right\} = \frac{1}{3a^2}L^{-1}\left(\frac{1}{s-a}\right) + L^{-1}\left\{\frac{-\frac{1}{3a^2}s-\frac{2}{3a}}{s^2+as+a^2}\right\}$$

$$= \frac{1}{3a^2}e^{at} - \frac{1}{3a^2}L^{-1}\left\{\frac{s+2a}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right\}$$

$$= \frac{1}{3a^2}e^{at} - \frac{1}{3a^2}e^{-\frac{at}{2}}\cdot L^{-1}\left\{\frac{s+\frac{3}{2}a}{s^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right\}$$

$$= \frac{1}{3a^2}\left[e^{at} - e^{-\frac{at}{2}}\left\{\cos\frac{\sqrt{3}}{2}at + \sqrt{3}\,\sin\frac{\sqrt{3}}{2}at\right\}\right]$$

(iv) $$\frac{1}{s^4+4} = \frac{1}{\left(s^4+4s^2+4\right)-4s^2} = \frac{1}{\left(s^2+2\right)^2-(2s)^2}$$

$$= \frac{1}{\left(s^2+2s+2\right)\left(s^2-2s+2\right)}$$

$\therefore$ Let $$\frac{1}{s^4+4} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2-2s+2}$$

$$\therefore \qquad (As+B)(s^2-2s+2) + (Cs+D)(s^2+2s+2) = 1$$

By the usual procedure, we get

$$A = 1/8, \quad B = 1/4, \quad C = -1/8 \quad \text{and} \quad D = 1/4$$

$$\therefore \qquad L^{-1}\left\{\frac{1}{s^4+4}\right\} = L^{-1}\left\{\frac{\frac{1}{8}s+\frac{1}{4}}{s^2+2s+2}\right\} - L^{-1}\left\{\frac{\frac{1}{8}s-\frac{1}{4}}{s^2-2s+2}\right\}$$

$$= \frac{1}{8}L^{-1}\left\{\frac{(s+1)+1}{(s+1)^2+1}\right\} - \frac{1}{8}L^{-1}\left\{\frac{(s-1)-1}{(s-1)^2+1}\right\}$$

$$= \frac{1}{8}\left[e^{-t}\cdot L^{-1}\left\{\frac{s+1}{s^2+1}\right\} - e^{t}\cdot L^{-1}\left\{\frac{s-1}{s^2+1}\right\}\right]$$

$$= \frac{1}{8}\left[e^{-t}\ (\cos t + \sin t) - e^{t}\ (\cos t - \sin t)\right]$$

$$= \frac{1}{4}\left[\left(\frac{e^{t}+e^{-t}}{2}\right)\sin t - \left(\frac{e^{t}-e^{-t}}{2}\right)\cos t\right]$$

$$= \frac{1}{4}\ (\sin t \cosh t - \cos t \sinh t)$$

(v) $$\frac{s}{s^4+s^2+1} = \frac{s}{\left(s^4+2s^2+1\right)-s^2} = \frac{s}{\left(s^2+1\right)^2 - s^2}$$

$$= \frac{s}{\left(s^2+s+1\right)\left(s^2-s+1\right)}$$

$\therefore$ Let $$\frac{s}{s^4+s^2+1} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2-s+1}$$

$\therefore$ $$(As+B)\,(s^2-s+1) + (Cs+D)\,(s^2+s+1) = s$$

By the usual procedure, we get

$$A = 0,\quad B = \frac{-1}{2},\quad C = 0,\quad \text{and}\quad D = \frac{1}{2}.$$

$\therefore$ $$L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s^2-s+1}\right\} \frac{-1}{2}L^{-1}\left\{\frac{1}{s^2+s+1}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{2}\left[e^{t/2}L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - e^{-t/2}\cdot L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\left\{\left(e^{t/2}-e^{-t/2}\right)\sin\frac{\sqrt{3}}{2}t\right\}$$

$$= \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\ \sinh\ \frac{t}{2}.$$

Example 5.12

(i) If $L^{-1}\left\{\dfrac{s}{\left(s^2+1\right)^2}\right\} = \dfrac{t}{2}\sin t$, find $L^{-1}\left\{\dfrac{s}{\left(s^2+a^2\right)^2}\right\}$

(ii) Given that $L^{-1}\left\{\frac{s^2+4}{\left(s^2-4\right)^2}\right\}=t\cosh 2t$, find $L^{-1}\left\{\frac{s^2+1}{\left(s^2-1\right)^2}\right\}$

(iii) Given that $L^{-1}\left\{\frac{1}{\left(s^2+4\right)^2}\right\}=\frac{1}{16}(\sin 2t-2t\cos 2t)$, find $L^{-1}\left\{\frac{1}{\left(s^2+9\right)^2}\right\}$.

(i) By change of scale property,

$$L\{f(at)\}=\frac{1}{a}\phi\left(\frac{s}{a}\right)$$

$$\therefore \quad L^{-1}\{\phi(s/a)\}=a\,L^{-1}\{\phi(s)\}_{t\to at} \qquad (1)$$

Given $$L^{-1}\left\{\frac{s}{\left(s^2+1\right)^2}\right\}=\frac{t}{2}\sin t$$

$$\therefore \quad L^{-1}\left\{\frac{s/a}{\left(\frac{s^2}{a^2}+1\right)^2}\right\}=a\cdot\frac{at}{2}\sin at, \text{ by } \qquad (1)$$

i.e., $$a^3L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{a^2}{2}t\sin at$$

$$\therefore \quad L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{t}{2a}\sin at.$$

(ii) By change of scale property,

$$L\{f(t/a)\}=a\,\phi(as)$$

$$\therefore \quad L^{-1}\{\phi(as)\}=\frac{1}{a}L^{-1}(\phi(s)\}_{t\to t/a} \qquad (2)$$

Given $$L^{-1}\left\{\frac{s^2+4}{\left(s^2-4\right)^2}\right\}=t\cosh 2t$$

$$\therefore \quad L^{-1}\left\{\frac{(2s)^2+4}{[(2s)^2-4]^2}\right\}=\frac{1}{2}\cdot\frac{t}{2}\cosh t, \text{ by } \qquad (2)$$

i.e., $$\frac{1}{4}L^{-1}\left\{\frac{s^2+1}{\left(s^2-1\right)^2}\right\}=\frac{t}{4}\cosh t$$

$$\therefore \quad L^{-1}\left\{\frac{s^2+1}{\left(s^2-1\right)^2}\right\}=t\cosh t.$$

(iii) Given $L^{-1}\left\{\dfrac{1}{\left(s^2+4\right)^2}\right\}=\dfrac{1}{16}(\sin 2t-2t\cos 2t)$

$$\therefore \qquad L^{-1}\left\{\frac{1}{\left[\left(\frac{2s}{3}\right)^2+4\right]^2}\right\}=\frac{3}{2}L^{-1}\left\{\frac{1}{\left(s^2+4\right)^2}\right\}_{t\to\frac{3}{2}t}, \text{ by} \qquad (2)$$

i.e. $$\frac{81}{16}L^{-1}\left\{\frac{1}{\left(s^2+9\right)^2}\right\}=\frac{3}{32}(\sin 3t-3t\cos 3t)$$

$$\therefore \qquad L^{-1}\left\{\frac{1}{\left(s^2+9\right)^2}\right\}=\frac{1}{54}(\sin 3t-3t\cos 3t)$$

EXERCISE 5(a)

Part A

(Short Answer Questions)

1. Define Laplace transform.
2. State the conditions for the existence of Laplace transform of a function.
3. Give two examples for a function for which Laplace transform does not exist.
4. State the change of scale property in Laplace transformation.
5. State the first shifting property in Laplace transformation.
6. State the second shifting property in Laplace transformation.
7. Find the Laplace transform of unit step function.
8. Find the Laplace transforms of unit impulse function.
9. Find $L\{f(t)\}$, if $f(t)=\begin{cases} e^{2t}, & \text{for } 0<t<1 \\ 0, & \text{for } t>1 \end{cases}$
10. Find $L\{f(t)\}$, if $f(t)=\begin{cases} 0, & \text{for } 0<t<1 \\ t, & \text{for } 1<t<2 \\ 0, & \text{for } t>2 \end{cases}$
11. Find $L\{f(t)\}$, if $f(t)=\begin{cases} \sin 2t, & \text{for } 0<t<\pi \\ 0, & \text{for } t>\pi \end{cases}$
12. Find $L\{f(t)\}$, if $f(t)=\begin{cases} \cos t, & \text{for } 0<t<2\pi \\ 0, & \text{for } t>2\pi \end{cases}$

13. Find $L\{f(t)\}$, if $f(t)=\begin{cases} 0, & \text{for } 0<t<\frac{2\pi}{3} \\ \cos\left(t-\frac{2\pi}{3}\right), & \text{for } t>\frac{2\pi}{3} \end{cases}$

14. Find $L\{f(t)\}$, if $f(t)=\begin{cases} \sin t, & \text{for } 0<t<\pi \\ t, & \text{for } t>\pi \end{cases}$

15. Find $L\left(\sqrt{t}\right)$ and $L\left(\frac{1}{\sqrt{\pi t}}\right)$

16. Find $L\{(t-3)\,u_3(t)\}$ and $L\{u_1(t)\sin\pi(t-1)\}$.

17. Find $L\{t^2 u_2(t)\}$

Find the Laplace transforms of the following functions:

18. $(at+b)^3$

19. $\sin(\omega t+\theta)$

20. $\sin^2 3t$

21. $\cos^3 2t$

22. $\sin 2t\cos t$

23. $\cos 3t\cos 2t$

24. $\sinh^3 t$

25. $\cosh^2 2t$

26. $(t+1)^2 e^{-t}$

27. $e^{-2t}\left(\cos 3t-\frac{2}{3}\sin 3t\right)$

28. $e^t\left(\cosh 2t+\frac{1}{2}\sinh 2t\right)$

29. $\sin t\sinh t$

30. $t^2\cosh t$

31. $\sqrt{e^{3(t+2)}}$

Find the Laplace inverse transforms of the following functions:

32. $e^{-as}/s^2\ (a>0)$.

33. $(e^{-2s}-e^{-3s})/s$

34. $\frac{e^{-2s}}{s-3}$

35. $\frac{se^{-s}}{s^2+9}$

36. $\frac{1+e^{-\pi s}}{s^2+1}$

Find $f(t)$ if $L\{f(t)\}$ is given by the following functions:

37. $\frac{1}{(s+1)^{3/2}}$

38. $\frac{s^2+2s+3}{s^3}$

39. $\frac{1}{(2s-3)^4}$

40. $\frac{s}{(s-2)^5}$

41. $\frac{2s+3}{s^2+4}$

42. $\frac{s+6}{s^2-9}$

43. $\frac{1}{s(s+a)}$

44. $\frac{1}{s^2+2s+5}$

45. $\frac{s-3}{s^2-6s+10}$

Part B

Find the Laplace transforms of the following functions:

46. $e^{3t}(2t+3)^3$

47. $e^{at}(1+2at)/\sqrt{t}$

48. $e^{-t}\sinh^3 t$

49. $\sin at\cosh at - \cos at\sinh at$

50. $(e^t\sin t)^2$

51. $\left(\dfrac{\cos 2t}{e^t}\right)^3$

52. $e^{-kt}\sin(\omega t+\theta)$

53. $e^{-t}\sin 3t\cos t$

54. $e^t \cos t\cos 2t\cos 3t$

55. $e^{-2t}\sin 2t\sin 3t\sin 4t$

56. $t\cos 2t$

57. $te^{-t}\cos t$

58. $t\,e^{2t}\sin 3t$

59. Given that $L(t\sin 2t)=\dfrac{4s}{(s^2+4)^2}$, find $L(t\sin t)$

60. Given that $L(t\cos 3t)=\dfrac{s^2-9}{(s^2+9)^2}$, find $L(t\cos 2t)$.

Find the Laplace inverse transforms of the following function:

61. $\dfrac{s^2+1}{s^3+3s^2+2s}$

62. $\dfrac{4s^2-3s+5}{(s+1)(s^2-3s+2)}$

63. $\dfrac{s^2-3s+5}{(s+1)^3}$

64. $\dfrac{7s-11}{(s+1)(s-2)^2}$

65. $\dfrac{1}{(s^2+s)^2}$

66. $\dfrac{s^4-8s^2+31}{(s^2+1)(s^2+4)(s^2+9)}$

67. $\dfrac{2s}{(s^2+1)(s^2+2)(s^2+3)}$

68. $\dfrac{2s-9}{s^2+6s+34}$

69. $\dfrac{s+1}{s^2+s+1}$

70. $\dfrac{ls+m}{as^2+bs+c}$

71. $\dfrac{3s^2-16s+26}{s(s^2+4s+13)}$

72. $\dfrac{1}{s^3+1}$

73. $\dfrac{1}{s^4+4a^4}$

74. $\dfrac{s}{s^4+4}$

75. $\dfrac{s^2}{s^4+64}$

76. $\dfrac{4s^3}{4s^4+1}$

77. $\dfrac{s^2+1}{s^4+s^2+1}$

78. If $L^{-1}\left\{\frac{s}{\left(s^2-1\right)^2}\right\}=\frac{1}{2}t\sinh t$, find $L^{-1}\left\{\frac{s}{\left(s^2-a^2\right)^2}\right\}$

79. If $L^{-1}\left\{\frac{1}{\left(s^2+9\right)^2}\right\}=\frac{1}{54}(\sin 3t-3t\cos 3t)$, find $L^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\}$.

80. Given that $L^{-1}\left\{\frac{s}{\left(s^2+9\right)^2}\right\}=\frac{t}{6}\sin 3t$, find $L^{-1}\left\{\frac{s}{\left(s^2+4\right)^2}\right\}$

5.6 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

5.6.1 Definition

A function $f(t)$ is said to be a *periodic function*, if there exists a constant P (>0) such that $f(t+P)=f(t)$, for all values of t. Now $f(t+2P)=f(t+P+P)=f(t+P)=f(t)$, for all t. In general, $f(t+nP)=f(t)$, for all t, when n is an integer (positive or negative).

P is called the *period of the function*.

Unlike other functions whose Laplace transforms are expressed in terms of an integral over the semi-infinite interval $0 \le t < \infty$, the Laplace transform of a periodic function $f(t)$ with period P can be expressed in terms of the integral of $e^{-st}f(t)$ over the finite interval $(0, P)$, as established in the following theorem.

Theorem

If $f(t)$ is a piecewise continuous periodic function with period P, then

$$L\{f(t)\}=\frac{1}{1-e^{-Ps}}\cdot\int_0^P e^{-st}f(t)\,dt.$$

Proof:

By definition, $L\{f(t)\}=\int_0^\infty e^{-st}f(t)\,dt$

$$=\int_0^P e^{-st}f(t)\,dt+\int_P^\infty e^{-st}f(t)\,dt \qquad (1)$$

In the second integral in (1), put $t=x+P$, $\therefore$ $dt=dx$ and the limits for x become 0 and ∞.

$$\therefore \qquad \int_P^\infty e^{-st}f(t)\,dt=\int_0^\infty e^{-s(x+P)}f(x+P)\,dx$$

$$=e^{-sP}\cdot\int_0^\infty e^{-sx}f(x)\,dx \qquad [\because f(x+P)=f(x)]$$

$$= e^{-sP} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \text{ , on changing the dummy variable } x \text{ to } t.$$

$$= e^{-sP} \cdot L\{f(t)\} \tag{2}$$

By putting (2) in (1), we have

$$L\{f(t)\} = \int_0^P e^{-st} f(t)\,\mathrm{d}t + e^{-sP} \cdot L\{f(t)\}$$

$$\therefore \quad (1 - e^{-Ps})\, L\{f(t)\} = \int_0^P e^{-st} f(t)\,\mathrm{d}t$$

$$\therefore \quad L\{f(t)\} = \frac{1}{1 - e^{-Ps}} \int_0^P e^{-st} f(t)\,\mathrm{d}t$$

5.7 DERIVATIVES AND INTEGRALS OF TRANSFORMS

The following two theorems, in which we differentiate and integrate the transform function $\phi(s) = L\{f(t)\}$ with respect to s, will help us to find $L\{t f(t)\}$ and $L\left\{\frac{1}{t} f(t)\right\}$ respectively. Repeated differentiation and integration of $\phi(s)$ will enable us to find $L\{t^n f(t)\}$ and $L\left\{\frac{1}{t^n} f(t)\right\}$, where n is a positive integer.

Theorem

If $\quad L\{f(t)\} = \phi(s), \quad$ then $L\{t f(t)\} = -\phi'(s)$.

***Proof*:**

Given: $\quad L\{f(t)\} = \phi(s)$

i.e.

$$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \phi(s) \qquad \cdots(1)$$

Differentiating both sides of (1) with respect to s,

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}s} \phi(s) \qquad \cdots(2)$$

Assuming that the conditions for interchanging the two operations of integration with respect to t and differentiation with respect to s in (2) are satisfied, we have

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\left\{e^{-st}\right\} f(t)\,\mathrm{d}t = \phi'(s)$$

i.e. $$\int_0^\infty -t\, e^{-st} f(t)\, dt = \phi'(s)$$

i.e. $$\int_0^\infty e^{-st} [t\, f(t)]\, dt = -\phi'(s)$$

i.e. $$L\{t\, f(t)\} = -\phi'(s)$$

Corollary

Differentiating both sides of (1) n times with respect to s, we get

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\phi(s) \quad \text{or} \quad (-1)^n \phi^{(n)}(s)\,.$$

Note ☑

1. The above theorem can be rewritten as a working rule in the following manner

$$L\{t\, f(t)\} = -\frac{d}{ds}\phi(s)$$
$$= -\frac{d}{ds} L\{f(t)\}$$

Thus, to find the Laplace transform of the product of two factors, one of which is 't', we ignore 't' and find the Laplace transform of the other factor as a function of s; then we differentiate this function of s with respect to s and multiply by (-1).

Extending the above rule,

$$L\{t^2\, f(t)\} = (-1)^2 \frac{d^2}{ds^2} L\{f(t)\} \text{ and in general}$$

$$L\{t^n\, f(t)\} = (-1)^n \frac{d^n}{ds^n} L\{f(t)\}\,.$$

2. The above theorem can be stated in terms of the inverse Laplace operator as follows:

If $\quad L^{-1}\{\phi(s)\} = f(t)$,

then $\quad L^{-1}\{\phi'(s)\} = -t\, f(t)$.

From this form of the theorem, we get the following working rule:

$$L^{-1}\{\phi(s)\} = -\frac{1}{t} L^{-1}\{\phi'(s)\}$$

This rule is applied when the inverse transform of the derivative of the given function can be found out easily. In particular, the inverse transforms of functions of s that contain logarithmic functions and inverse tangent and cotangent functions can be found by the application of this rule.

Theorem

If $L\{f(t)\} = \phi(s)$, then $L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \phi(s)\,\mathrm{d}s$, provided $\lim_{t\to 0}\left\{\frac{1}{t} f(t)\right\}$ exists.

Proof:

Given: $L\{f(t)\} = \phi(s)$

i.e. $$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \phi(s) \tag{1}$$

Integrating both sides of (1) with respect to s between the limits s and ∞, we have

$$\int_s^\infty \left[\int_0^\infty e^{-st}\, f(t)\,\mathrm{d}t\right]\mathrm{d}s = \int_s^\infty \phi(s)\,\mathrm{d}s \tag{2}$$

Assuming that the conditions for the change of order of integration in the double integral on the left side of (2) are satisfied, we have

$$\int_0^\infty \left[\int_s^\infty e^{-st}\,\mathrm{d}s\right] f(t)\,\mathrm{d}t = \int_s^\infty \phi(s)\,\mathrm{d}s$$

i.e. $$\int_0^\infty \left[\frac{e^{-st}}{-t}\right]_{s=s}^{s=\infty} f(t)\,\mathrm{d}t = \int_s^\infty \phi(s)\,\mathrm{d}s$$

i.e. $$\int_0^\infty -\frac{1}{t}(0 - e^{-st})\, f(t)\,\mathrm{d}t = \int_s^\infty \phi(s)\,\mathrm{d}s,$$

assuming that $s > 0$

i.e. $$\int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]\mathrm{d}t = \int_s^\infty \phi(s)\,\mathrm{d}s$$

i.e. $$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \phi(s)\,\mathrm{d}s$$

Corollary

$$L\left\{\frac{1}{t^2} f(t)\right\} = L\left[\frac{1}{t}\left\{\frac{1}{t} f(t)\right\}\right]$$
$$= \int_s^\infty \left[\int_s^\infty \phi(s)\,d\,s\right]\mathrm{d}s$$
$$= \int_s^\infty \int_s^\infty \phi(s)\,\mathrm{d}s\,\mathrm{d}s$$

Generalising this result, we get

$$L\left\{\frac{1}{t^n} f(t)\right\} = \int_s^\infty \int_s^\infty \dots \int_s^\infty \phi(s)\,(\mathrm{d}s)^n$$

Note ☑

1. The above theorem can be rewritten as a working rule as given below:

$$L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty L\{f(t)\}\,\mathrm{d}s$$

Thus, to find the Laplace transform of the product of two factors, one of which is $\frac{1}{t}$, we ignore $\frac{1}{t}$, find the Laplace transform of the other factor as a function of s and integrate this function of s with respect to s between the limits s and ∞.
Extending the above rule. We get;

$$L\left\{\frac{1}{t^2} f(t)\right\} = \int_s^\infty \int_s^\infty L\{f(t)\}\,\mathrm{d}s\,\mathrm{d}s \text{ and in general}$$

$$L\left\{\frac{1}{t^n} f(t)\right\} = \int_s^\infty \int_s^\infty \ldots \int_s^\infty L\{f(t)\}(\mathrm{d}s)^n .$$

2. The above theorem can be stated in terms of the inverse Laplace operator as follows:

If $\quad L^{-1}\{\phi(s)\} = f(t)$,

then $\quad L^{-1}\left[\int_s^\infty \phi(s)\,\mathrm{d}s\right] = \frac{1}{t} f(t)$.

From this form of the theorem, we get the following working rule:

$$L^{-1}\{\phi(s)\} = t \cdot L^{-1}\left[\int_s^\infty \phi(s)\,\mathrm{d}s\right]$$

This rule is applied when the inverse transform of the integral of the given function with respect to s between the limits s and ∞ can be found out easily.

In particular, the inverse transforms of proper rational functions whose numerators are first degree expressions in s and denominators are squares of second degree expressions in s can be found by applying this rule

WORKED EXAMPLE 5(b)

Example 5.1 Find the Laplace transform of the "saw-tooth wave" function $f(t)$ which is periodic with period 1 and defined as $f(t) = kt$, in $0 < t < 1$.

The graph of $f(t)$ is shown in Fig. 5.1 below. If the period of the function $f(t)$ is P, the function will be defined as $f(t) = \frac{k}{P} t$ in $0 < t < P$.

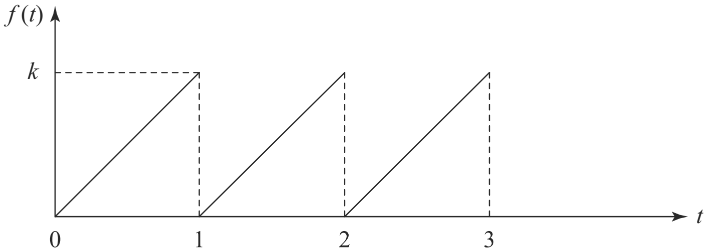

Fig. 5.1

By the formula for the Laplace transform of a periodic function $f(t)$ with period P,

$$L\{f(t)\} = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)\,\mathrm{d}t$$

$\therefore$ For the given function,

$$L\{f(t)\} = \frac{1}{1-e^{-s}} \int_0^1 kt\,e^{-st}\,\mathrm{d}t$$

$$= \frac{k}{1-e^{-s}} \left[t\left(\frac{e^{-st}}{-s}\right) - 1\cdot\left(\frac{e^{-st}}{s^2}\right) \right]_0^1$$

$$= \frac{k}{1-e^{-s}} \left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{k}{1-e^{-s}} \left[\frac{\left(1-e^{-s}\right)}{s^2} - \frac{e^{-s}}{s} \right]$$

$$= \frac{k}{s^2} - \frac{ke^{-s}}{s(1-e^{-s})}$$

Example 5.2 Find the Laplace transform of the "square wave" function $f(t)$ defined by

$$f(t) = k \text{ in } 0 \le t \le a$$

$$= -k \text{ in } a \le t \le 2a$$

and $\quad f(t+2a) = f(t)$ for all t.

$f(t+2a) = f(t)$ means that $f(t)$ is periodic with period $2a$. The graph of the function is shown in Fig. 5.2.

For a periodic function $f(t)$ with period P,

$$L\{f(t)\} = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)\,\mathrm{d}t$$

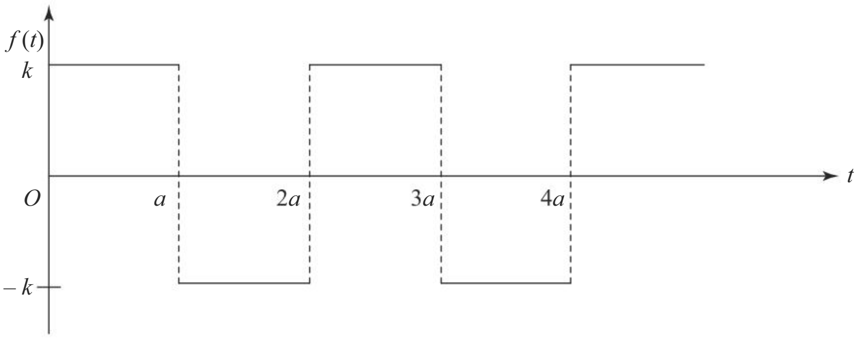

Fig. 5.2

∴ For the given function;

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\left[\int_0^a k\,e^{-st}\,dt + \int_a^{2a}(-k)\,e^{-st}\,dt\right]$$

$$= \frac{k}{1-e^{-2as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$

$$= \frac{k}{s(1-e^{-2as})}[1 - e^{-as} - e^{-as} + e^{-2as}]$$

$$= \frac{k(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})}$$

$$= \frac{k(1-e^{-as})}{s(1+e^{-as})} = \frac{k}{s}\frac{(e^{as/2}-e^{-as/2})}{(e^{as/2}+e^{-as/2})}$$

$$= \frac{k}{s}\tanh\left(\frac{as}{2}\right)$$

Example 5.3 Find the Laplace transform of "triangular wave function $f(t)$ whose graph is given below in Fig. 5.3.

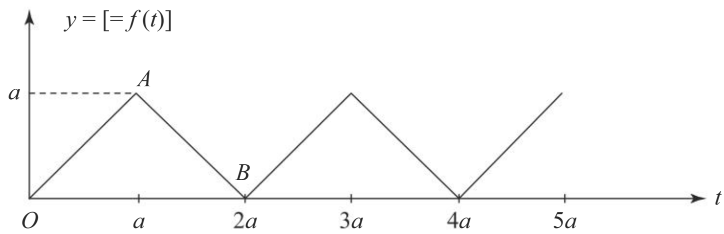

Fig. 5.3

From the graph it is obvious that $f(t)$ is periodic with period $2a$.

Let us find the value of $f(t)$ in $0 \le t \le 2a$, by finding the equations of the lines OA and AB.

OA passes through the origin and has a slope 1.

$\therefore$ Equation of OA is $y = t$, in $0 \le t \le a$

AB passes through the point $B(2a, 0)$ and has a slope -1.

$\therefore$ Equation of AB is $y - 0 = (-1)(t - 2a)$

or $y = 2a - t$ in $a \le t \le 2a$.

Thus the definition of $f(t)$ [$= y$] can be taken as

$$f(t) = t, \text{ in } 0 \le t \le a$$

$$= 2a - t, \text{ in } a \le t \le 2a$$

and $f(t + 2a) = f(a)$.

Now
$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, \mathrm{d}t$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a te^{-st}\, \mathrm{d}t + \int_a^{2a} (2a-t)e^{-st}\, \mathrm{d}t \right]$$

$$= \frac{1}{1-e^{-2as}} \left[t\left\{\left(\frac{e^{-st}}{-s}\right) - 1\cdot\left(\frac{e^{-st}}{s^2}\right)\right\}_0^a + \left\{(2a-t)\left(\frac{e^{-st}}{-s}\right) + 1\cdot\left(\frac{e^{-st}}{s^2}\right)\right\}_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{a}{s}e^{-as} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{a}{s}e^{-as} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1 - 2e^{-as} + e^{-2as}}{s^2(1-e^{-2as})} = \frac{(1-e^{-as})^2}{s^2\,(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1\,(1-e^{-as})}{s^2\,(1+e^{-as})} = \frac{1}{s^2}\left(\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}\right)$$

$$= \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

Example 5.4 Find the Laplace transform of the "half-sine wave rectifier" function $f(t)$ whose graph is given in Fig. 5.4.

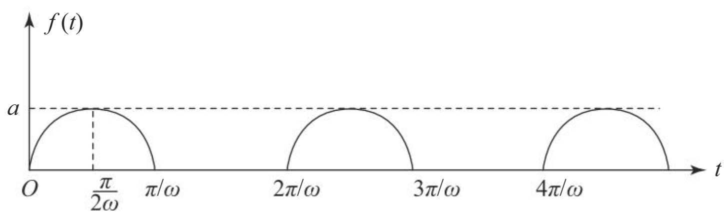

Fig. 5.4

From the graph, it is obvious that $f(t)$ is a periodic function with period $2\pi/\omega$. The graph of $f(t)$ in $0 \le t \le \pi/\omega$ is a sine curve that passes through (0, 0), $\left(\dfrac{\pi}{2\omega}, a\right)$ and $\left(\dfrac{\pi}{\omega}, 0\right)$

$\therefore$ The definition of $f(t)$ is given by

$$\begin{aligned} f(t) &= a \sin \omega t, \text{ in } 0 \le t \le \pi/\omega \\ &= 0, \text{ in } \pi/\omega \le t \le 2\pi/\omega \end{aligned}$$

and $$f\left(t+\frac{2\pi}{\omega}\right) = f(t).$$

Now
$$\begin{aligned} L\{f(t)\} &= \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)\, \mathrm{d}t \\ &= \frac{a}{1-e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, \mathrm{d}t \\ &= \frac{a}{1-e^{-2\pi s/\omega}} \cdot \left[\frac{e^{-st}}{s^2+\omega^2}(-s \sin \omega t - \omega \cos \omega t\right]_0^{\pi/\omega} \\ &= \frac{a}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left[\omega e^{-\pi s/\omega} + \omega\right] \\ &= \frac{\omega a\left(1+e^{-\pi s/\omega}\right)}{(s^2+\omega^2)\left(1-e^{-2\pi s/\omega}\right)} = \frac{\omega a}{(s^2+\omega^2)\left(1-e^{-\pi s/\omega}\right)} \end{aligned}$$

Example 5.5 Find the Laplace transform of the "full-sine wave rectifier" function $f(t)$, defined as

$$f(t) = |\sin \omega t|, \ t \ge 0$$

We note that
$$\begin{aligned} f(t+\pi/\omega) &= |\sin \omega (t+\pi/\omega)| \\ &= |\sin \omega t| \\ &= f(t) \end{aligned}$$

$\therefore f(t)$ is periodic with period π/ω.

Also $f(t)$ is always positive. The graph of $f(t)$ is the sine curve as shown in Fig. 5.5.

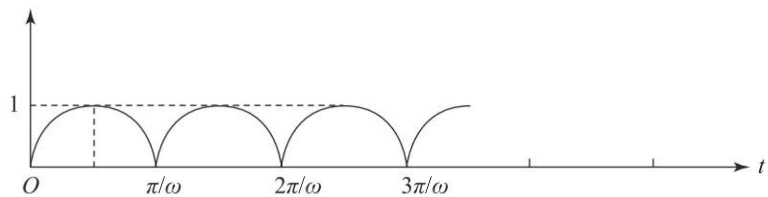

Fig. 5.5

Now $$L\{f(t)\} = \frac{1}{1-e^{-\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} |\sin \omega t| \, dt$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt \left[\because \sin \omega t > 0 \text{ in } 0 \le t \le \pi/\omega\right]$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{1}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}\left(\omega e^{-\pi s/\omega} + \omega\right) = \frac{\omega}{s^2+\omega^2}\left(\frac{1+e^{-\pi s/\omega}}{1-e^{-\pi s/\omega}}\right)$$

$$= \frac{\omega}{s^2+\omega^2}\left(\frac{e^{\pi s/2\omega}+e^{-\pi s/2\omega}}{e^{\pi s/2\omega}-e^{-\pi s/2\omega}}\right), \text{ on integration and simplification}$$

$$= \frac{\omega}{s^2+\omega^2}\coth\left(\frac{\pi s}{2\omega}\right)$$

Example 5.6 Find the Laplace transforms of the following functions:
(i) $t\cosh^3 t$; (ii) $t\cos 2t\cos t$; (iii) $t\sin^3 t$; (iv) $(t\sin at)^2$.

(i) $$L\{t\cosh^3 t\} = L\left\{t\left(\frac{e^t+e^{-t}}{2}\right)^3\right\}$$

$$= \frac{1}{8}L\{t(e^{3t}+3e^t+3e^{-t}+e^{-3t})\} \tag{1}$$

$$= -\frac{1}{8}\frac{d}{ds}L(e^{3t}+3e^t+3e^{-t}+e^{-3t})$$

$$= -\frac{1}{8}\frac{d}{ds}\left\{\frac{1}{s-3}+\frac{3}{s-1}+\frac{3}{s+1}+\frac{1}{s+3}\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{(s-3)^2}+\frac{3}{(s-1)^2}+\frac{3}{(s+1)^2}+\frac{1}{(s+3)^2}\right\}$$

Note ☑ After getting step (1), we could have applied the first shifting property and got the same result.

(ii) $$L(t\cos 2t\cos t) = L\left\{\frac{t}{2}(\cos 3t+\cos t)\right\}$$

$$= \frac{1}{2}\left[-\frac{d}{ds}L(\cos 3t+\cos t)\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left(\frac{s}{s^2+9}+\frac{s}{s^2+1}\right)$$

$$= -\frac{1}{2}\left[\frac{s^2+9-2s^2}{(s^2+9)^2}+\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-9}{(s^2+9)^2}+\frac{s^2-1}{(s^2+1)^2}\right]$$

(iii) $$L(t\sin^3 t) = L\left\{\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right\}$$

$$= -\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}s}\{L(3\sin t - \sin 3t)\}$$

$$= -\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+1}-\frac{3}{s^2+9}\right)$$

$$= -\frac{3}{4}\left\{-\frac{2s}{(s^2+1)^2}+\frac{2s}{(s^2+9)^2}\right\}$$

$$= \frac{3}{2}s\left\{-\frac{1}{(s^2+1)^2}-\frac{1}{(s^2+9)^2}\right\}$$

(iv) $$L\{(t\sin at)^2\} = L\left[t^2\left(\frac{1-\cos 2at}{2}\right)\right]$$

$$= \frac{1}{2}(-1)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\{L(1-\cos 2at)\}$$

$$= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left\{\frac{1}{s}-\frac{s}{s^2+4a^2}\right\}$$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left\{-\frac{1}{s^2}+\frac{s^2-4a^2}{(s^2+4a^2)^2}\right\}$$

$$= \frac{1}{2}\left[\frac{2}{s^3}+\frac{(s^2+4a^2)^2\cdot 2s-(s^2-4a^2)\cdot 2(s^2+4a^2)\cdot 2s}{(s^2+4a^2)^4}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3}+\frac{2s(12a^2-s^2)}{(s^2+4a^2)^3}\right]$$

$$= \frac{1}{s^3}+\frac{s(12a^2-s^2)}{(s^2+4a^2)^3}$$

Example 5.7 Use Laplace transforms to evaluate the following;

(i) $\int_0^\infty te^{-2t}\sin 3t\,\mathrm{d}t$; (ii) $\int_0^\infty te^{-3t}\cos 2t\,\mathrm{d}t$

(i) $$\int_0^\infty e^{-st}(t\sin 3t)\,\mathrm{d}t = L(t\sin 3t)$$ (1), by definition

Now $$L(t\sin 3t) = -\frac{d}{ds}L(\sin 3t)$$
$$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$
$$= \frac{6s}{(s^2+9)^2} \qquad (2)$$

Putting (2) in (1), we have

$$\int_0^\infty te^{-st}\sin 3t\,dt = \frac{6s}{(s^2+9)^2}, s > 0 \qquad (3)$$

Putting $s = 2$ in (3), we get

$$\int_0^\infty te^{-2t}\sin 3t\,dt = \frac{12}{169}.$$

(ii) $\int_0^\infty e^{-st}(t\cos 2t)\,dt = L(t\cos 2t)$, by definition (1)

Now $$L(t\cos 2t) = -\frac{d}{ds}L(\cos 2t)$$
$$= -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$
$$= \frac{s^2-4}{(s^2+4)^2} \qquad (2)$$

Inserting (2) in (1), we have

$$\int_0^\infty te^{-st}\cos 2t\,dt = \frac{s^2-4}{(s^2+4)^2}, s > 0 \qquad (3)$$

Putting $s = 3$ in (3), we get

$$\int_0^\infty te^{-3t}\cos 2t\,dt = \frac{5}{169}.$$

Example 5.8 Find the Laplace transforms of the following functions:

(i) $te^{-4t}\sin 3t$; (ii) $t\cosh t\cos t$; (iii) $te^{-2t}\sinh 3t$ (iv) $t^2e^{-t}\cos t$.

(i) $L\{te^{-4t}\sin 3t\} = [L(t\sin 3t)]_{s\to s+4}$ (1)

(by the first shifting property.)

Now $$L(t\sin 3t) = -\frac{\mathrm{d}}{\mathrm{d}s}L(\sin 3t)$$
$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right)$$
$$= \frac{6s}{(s^2+9)^2} \quad (2)$$

Using (2) in (1), we have
$$L\{te^{-4t}\sin 3t\} = \left[\frac{6s}{(s^2+9)^2}\right]_{s\to s+4}$$
$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

Note ☑ The same problem has been solved by using an alternative method in Worked Example (6) in Section 5(a).

(ii) $$L\{t\cosh t\cos t\} = L\left\{\frac{t}{2}(e^t+e^{-t})\cos t\right\}$$
$$= \frac{1}{2}[L(t\cos t)_{s\to s-1} + L(t\cos t)_{s\to s+1}] \quad (1)$$

Now $$L(t\cos t) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+1}\right)$$
$$= \frac{s^2-1}{(s^2+1)^2} \quad (2)$$

Using (2) in (1), we have
$$L(t\cosh t\cos t) = \frac{1}{2}\left[\frac{(s-1)^2-1}{(s^2-2s+2)^2} + \frac{(s+1)^2-1}{(s^2+2s+2)^2}\right]$$
$$= \frac{1}{2}\left[\frac{s^2-2s}{(s^2-2s+2)^2} + \frac{s^2+2s}{(s^2+2s+2)^2}\right]$$

(iii) $$L\{t\,e^{-2t}\sinh 3t\} = L\{t\sinh 3\,t\}_{s\to s+2} \quad (1)$$

Now $$L(t\sinh 3t) = -\frac{\mathrm{d}}{\mathrm{d}s}L(\sinh 3t)$$
$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2-9}\right)$$
$$= \frac{6s}{(s^2-9)^2} \quad (2)$$

Using (2) in (1), we have

$$L\{te^{-2t}\sinh 3t\} = \frac{6(s+2)}{\{(s+2)^2-9\}^2} = \frac{6(s+2)}{(s-1)^2(s+5)^2}$$

Aliter

$$L\{te^{-2t}\sinh 3t\} = L\left\{te^{-2t}\cdot\frac{1}{2}(e^{3t}-e^{-3t})\right\}$$
$$= \frac{1}{2}L\{te^{t}-te^{-5t}\}$$
$$= \frac{1}{2}\left\{\frac{1}{(s-2)^2}-\frac{1}{(s+5)^2}\right\}$$
$$= \frac{1}{2}\left\{\frac{12s+24}{(s-1)^2(s+5)^2}\right\} = \frac{6(s+2)}{(s-1)^2(s+5)^2}$$

(iv) $L\{t^2e^{-t}\cos t\} = [L(t^2\cos t)]_{s\to s+1}$ (1)

Now $$L(t^2\cos t) = (-1)^2\frac{d^2}{ds^2}L(\cos t)$$
$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$
$$= \frac{d}{ds}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}$$
$$= \frac{2(s^3-3s)}{(s^2+1)^3} \quad (2)$$

Using (2) in (1), we have

$$L\{t^2e^{-t}\cos t\} = \frac{2\{(s+1)^3-3(s+1)\}}{(s^2+2s+2)^3}$$

Example 5.9 Find the inverse Laplace transforms of the following functions:

(i) $\log\left(1-\dfrac{a}{s}\right)$; (ii) $\log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)$;

(iii) $\log\dfrac{s^2+1}{s(s+1)}$; (iv) $s\log\left(\dfrac{s-1}{s+1}\right)+k$ (k is a constant)

(i) $L^{-1}\{\phi(s)\} = -\dfrac{1}{t}L^{-1}\{\phi'(s)\}$ (1)

$$\therefore\ L^{-1}\log\left(1-\frac{a}{s}\right) = L^{-1}\log\left(\frac{s-a}{s}\right) = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s-a}{s}\right)\right]$$

$$= -\frac{1}{t} L^{-1} \frac{d}{ds} \{\log (s-a) - \log s\}$$

$$= -\frac{1}{t} L^{-1} \left(\frac{1}{s-a} - \frac{1}{s} \right)$$

$$= -\frac{1}{t} (e^{at} - 1) = \frac{1}{t} (1 - e^{at})$$

(ii) By rule (1),

$$L^{-1} \log\left(\frac{s^2+a^2}{s^2+b^2} \right) = -\frac{1}{t} L^{-1} \frac{d}{ds} [\log(s^2+a^2) - \log(s^2+b^2)]$$

$$= -\frac{1}{t} L^{-1} \left\{ \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2} \right\}$$

$$= \frac{2}{t} (\cos bt - \cos at)$$

(iii) By rule (1),

$$L^{-1} \log\left[\frac{s^2+1}{s(s+1)} \right] = -\frac{1}{t} L^{-1} \frac{d}{ds} [\log (s^2+1) - \log s - \log(s+1)]$$

$$= -\frac{1}{t} L^{-1} \left\{ \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1} \right\}$$

$$= -\frac{1}{t} (2\cos t - 1 - e^{-t})$$

$$= \frac{1}{t} (1 + e^{-t} - 2\cos t)$$

(iv) By rule (1),

$$L^{-1} \left[s \log\left(\frac{s-1}{s+1} \right) + k \right] = -\frac{1}{t} L^{-1} \frac{d}{ds} [s \log(s-1) - s \log (s+1)] + L^{-1}(k)$$

$$= -\frac{1}{t} L^{-1} \left[\frac{s}{s-1} + \log(s-1) - \frac{s}{s+1} - \log(s+1) \right] + k\delta(t)$$

$$= -\frac{1}{t} L^{-1} \left[\frac{s}{s-1} - \frac{s}{s+1} \right] - \frac{1}{t} L^{-1} [\log(s-1) - \log(s+1)] + k\delta(t)$$

$$= -\frac{1}{t} L^{-1} \left[\frac{2s}{s^2-1} \right] - \frac{1}{t} \left(-\frac{1}{t} \right) L^{-1} \left[\frac{1}{s-1} - \frac{1}{s+1} \right] + k\delta(t)$$

Note ☑

$\left[[L^{-1}\left(\frac{s}{s-1}\right)\right.$ and $L^{-1}\left(\frac{s}{s+1}\right)$ do not exist, as $\frac{s}{s-1}$ and $\frac{s}{s+1}$ are improper rational functions. Hence we have simplified $\left(\frac{s}{s-1}-\frac{s}{s+1}\right)$ as $\frac{2s}{s^2-1}$, which is a proper rational function.]

$$= -\frac{2}{t}\cosh t + \frac{1}{t^2}(e^t - e^{-t}) + k\,\delta(t)$$
$$= \frac{2}{t^2}\sinh t - \frac{2}{t}\cosh t + k\,\delta(t)$$

Example 5.10 Find the inverse Laplace transforms of the following functions:

(i) $\cot^{-1}(as)$; (ii) $\tan^{-1}\left(\frac{s+a}{b}\right)$;

(iii) $\cot^{-1}\left(\frac{2}{s+1}\right)$; (iv) $\tan^{-1}\left(\frac{2}{s^2}\right)$

(i) $$L^{-1}\{\phi(s)\} = -\frac{1}{t}L^{-1}\{\phi'(s)\} \quad (1)$$

$$\therefore \quad L^{-1}\{\cot^{-1}(as)\} = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}(as)\right]$$
$$= -\frac{1}{t}L^{-1}\left(\frac{-a}{1+a^2s^2}\right)$$
$$= \frac{a}{t}L^{-1}\left(\frac{1}{1+a^2s^2}\right)$$
$$= \frac{1}{t}L^{-1}\left\{\frac{1/a}{s^2+(1/a)^2}\right\} = \frac{1}{t}\sin\frac{t}{a}.$$

(ii) $$L^{-1}\left\{\tan^{-1}\left(\frac{s+a}{b}\right)\right\} = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{s+a}{b}\right)\right]$$
$$= -\frac{1}{t}L^{-1}\left[\frac{1/b}{1+\left(\frac{s+a}{b}\right)^2}\right]$$
$$= -\frac{1}{t}L^{-1}\left[\frac{b}{(s+a)^2+b^2}\right]$$

$$= -\frac{1}{t} e^{-at} \sin bt$$

(iii) $$L^{-1}\left\{\cot^{-1}\left(\frac{2}{s+1}\right)\right\} = -\frac{1}{t} L^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}s}\cot^{-1}\left(\frac{2}{s+1}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\left\{-\frac{1}{1+\frac{4}{(s+1)^2}}\right\}\left\{-\frac{2}{(s+1)^2}\right\}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{2}{(s+1)^2+4}\right]$$

$$= -\frac{1}{t} e^{-t} \sin 2t.$$

(iv) $$L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = -\frac{1}{t} L^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}s}\tan^{-1}\left(\frac{2}{s^2}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+\frac{4}{s^4}}.\left(\frac{-4}{s^3}\right)\right]$$

$$= \frac{4}{t} L^{-1}\left[\frac{s}{s^4+4}\right] \qquad (2)$$

Consider $$\frac{s}{s^4+4} = \frac{s}{(s^2+2)^2-(2s)^2}$$

$$= \frac{s}{(s^2-2s+2)\,(s^2+2s+2)}$$

$$= \frac{1}{4}\left[\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right],$$

by resolving into partial fractions

$$= \frac{1}{4}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore\ L^{-1}\left(\frac{s}{s^4+4}\right) = \frac{1}{4}\left[e^{t}\sin t - e^{-t}\sin t\right]$$

$$= \frac{1}{2}\sin t \sinh t \qquad (3)$$

Using (3) in (2), we have

$$L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = \frac{2}{t}\sin t \sinh t$$

Example 5.11 Find the Laplace transforms of the following functions:

(i) $\dfrac{\sinh t}{t}$; (ii) $\dfrac{e^{-at}-e^{-bt}}{t}$; (iii) $\dfrac{e^{at}-\cos bt}{t}$; (iv) $\dfrac{2\sin 2t \sin t}{t}$;

(v) $\left(\dfrac{\sin t}{t}\right)^2$.

(i) $$L\left\{\frac{f(t)}{t}\right\}=\int_s^\infty L\{f(t)\}\,\mathrm{d}s \qquad (1)$$

$\therefore$ $$L\left\{\frac{\sinh t}{t}\right\}=\int_s^\infty L(\sinh t)\,\mathrm{d}s$$

$$=\int_s^\infty \frac{1}{s^2-1}\,\mathrm{d}s$$

$$=\left[\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty$$

$$=\left[\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)\right]_{s\to\infty}-\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)$$

$$=\left[\frac{1}{2}\log\left(\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right)\right]_{s\to\infty}+\frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

$$=\frac{1}{2}\log 1+\frac{1}{2}\log\left(\frac{s+1}{s-1}\right)=\frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

(ii) $$L\left\{\frac{e^{-at}-e^{bt}}{t}\right\}=\int_s^\infty L(e^{-at}-e^{-bt})\,\mathrm{d}s\text{, by rule } (1)$$

$$=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)\mathrm{d}s$$

$$=\left[\log\left(\frac{s+a}{s+b}\right)\right]_{s\to\infty}-\log\left(\frac{s+a}{s+b}\right)$$

$$=\left[\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)\right]_{s\to\infty}+\log\left(\frac{s+b}{s+a}\right)$$

$$= \log 1 + \log\left(\frac{s+b}{s+a}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

(iii) $$L\left\{\frac{e^{at} - \cos bt}{t}\right\} = \int_s^\infty L\,(e^{at} - \cos bt)\,\mathrm{d}s, \text{ by rule (1)}$$

$$= \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2+b^2}\right)\mathrm{ds}$$

$$= \left[\log(\mathrm{s}-\mathrm{a}) - \frac{1}{2}\log(\mathrm{s}^2 + b^2)\right]_s^\infty$$

$$= \left[\log\left(\frac{\mathrm{s}-\mathrm{a}}{\sqrt{\mathrm{s}^2+b^2}}\right)\right]_{s\to\infty} - \log\left(\frac{\mathrm{s}-\mathrm{a}}{\sqrt{\mathrm{s}^2+b^2}}\right)$$

$$= \left[\log\left(\frac{1-\mathrm{a}/s}{\sqrt{1+b^2/s^2}}\right)\right]_{s\to\infty} + \log\left(\frac{\sqrt{\mathrm{s}^2+b^2}}{\mathrm{s}-\mathrm{a}}\right)$$

$$= \log 1 + \log\sqrt{\frac{\mathrm{s}^2+b^2}{(\mathrm{s}-\mathrm{a})^2}}$$

$$= \frac{1}{2}\log\left\{\frac{\mathrm{s}^2+b^2}{(\mathrm{s}-\mathrm{a})^2}\right\}$$

(IV) $$L\left\{\frac{2\sin 2t \sin t}{t}\right\} = \int_s^\infty L(2\sin 2t \sin t)\,\mathrm{d}s, \text{ by rule (1)}$$

$$= \int_s^\infty L(\cos t - \cos 3t)\,\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{s}{s^2+1} - \frac{s}{s^2+9}\right)\mathrm{d}s$$

$$= \left[\frac{1}{2}\log\left(\frac{s^2+1}{s^2+9}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2+9}\right)\right]_{s\to\infty} - \frac{1}{2}\log\left(\frac{s^2+1}{s^2+9}\right)$$

$$= \frac{1}{2}\left[\log\left(\frac{1+\dfrac{1}{s^2}}{1+9/s^2}\right)\right]_{s\to\infty} + \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$$

$$= \frac{1}{2}\log 1 + \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$$

(v) $$L\left\{\frac{f(t)}{t^2}\right\} = \int_s^\infty\int_s^\infty L\{f(t)\}\,\mathrm{d}s\,\mathrm{d}s \qquad (2)$$

$\therefore$ $$L\left\{\frac{\sin^2 t}{t^2}\right\} = \int_s^\infty\int_s^\infty L(\sin^2 t)\,\mathrm{d}s\,\mathrm{d}s$$

$$= \int_s^\infty\int_s^\infty L\left\{\frac{1-\cos 2t}{2}\right\}\mathrm{d}s\,\mathrm{d}s$$

$$= \frac{1}{2}\int_s^\infty\int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+4}\right)\mathrm{d}s\,\mathrm{d}s$$

$$= \frac{1}{2}\int_s^\infty \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)\mathrm{d}s,$$

by putting $a = 0$ and $b = 2$ in (iii) above.

$$= \frac{1}{4}\left[\left\{s\log\left(\frac{s^2+4}{s^2}\right)\right\}_s^\infty - \int_s^\infty s\left\{\frac{2s}{s^2+4}-\frac{2}{s}\right\}\mathrm{d}s\right]$$

by integrating by parts.

$$= \left[\frac{s}{4}\log\left(\frac{s^2+4}{s^2}\right)\right]_{s\to\infty} - \frac{s}{4}\log\left(\frac{s^2+4}{s^2}\right) + \frac{1}{4}\int_s^\infty \frac{8}{s^2+4}\mathrm{d}s$$

$$= L + \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) - \left(\cot^{-1}\frac{s}{2}\right)_s^\infty, \text{ say}$$

$$= L + \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) + \cot^{-1}\left(\frac{s}{2}\right) \qquad (3)$$

Now $$L = \log\left(1+\frac{4}{s^2}\right)^{s/4}$$

$$= \log\left[\left(1+\frac{4}{s^2}\right)^{s^2/4}\right]^{\frac{1}{s}}$$

$$\lim_{s\to\infty}(L) = \log\left[\lim_{s\to\infty}\left\{\left(1+\frac{4}{s^2}\right)^{s^2/4}\right\}^{\frac{1}{s}}\right]$$

$$= \log(e^0) = \log 1 = 0 \tag{4}$$

Using (4) in (3), we have

$$L\left(\frac{\sin^2 t}{t^2}\right) = \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) + \cot^{-1}\left(\frac{s}{2}\right).$$

Example 5.12 Use Laplace transforms to evaluate the following:

(i) $\displaystyle\int_0^\infty \frac{e^{-t}\sin\sqrt{3}\,t}{t}\,\mathrm{d}t$; (ii) $\displaystyle\int_0^\infty \frac{\sin^2 t}{t e^t}\mathrm{d}t$;

(iii) $\displaystyle\int_0^\infty \left(\frac{\cos at - \cos bt}{t}\right)\mathrm{d}t$; (iv) $\displaystyle\int_0^\infty \left(\frac{e^{-2t} - e^{-4t}}{t}\right)\mathrm{d}t.$

(i) $$\int_0^\infty \frac{e^{-t}\sin\sqrt{3}\,t}{t}\,\mathrm{d}t = \left[\int_0^\infty e^{-st}\left(\frac{\sin\sqrt{3}\,t}{t}\right)\mathrm{d}t\right]_{s=1}$$

$$= L\left(\frac{\sin\sqrt{3}\,t}{t}\right)_{s=1} \tag{1}$$

Now $$L\left(\frac{\sin\sqrt{3}\,t}{t}\right) = \int_s^\infty L(\sin\sqrt{3}\,t)\,\mathrm{d}s$$

$$= \int_s^\infty \frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\,\mathrm{d}s$$

$$= \left[\sqrt{3}\left(-\frac{1}{\sqrt{3}}\right)\cot^{-1}\left(\frac{s}{\sqrt{3}}\right)\right]_s^\infty$$

$$= 0 + \cot^{-1}\left(\frac{s}{\sqrt{3}}\right) \tag{2}$$

Using (2) in (1), we have

$$\int_s^\infty \frac{e^{-1}\sin\sqrt{3}t}{t}\,\mathrm{d}t = \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}.$$

(ii)
$$\int_0^\infty \frac{\sin^2 t}{t\,e^t}\,\mathrm{d}t = \int_0^\infty e^{-t}\left(\frac{\sin^2 t}{t}\right)\mathrm{d}t$$
$$= \left[\int_0^\infty e^{-st}\left(\frac{\sin^2 t}{t}\right)\mathrm{d}t\right]_{s=1}$$
$$= L\left(\frac{\sin^2 t}{t}\right)_{s=1} \qquad (1)$$

Now
$$L\left(\frac{\sin^2 t}{t}\right) = L\left(\frac{1-\cos 2t}{2t}\right)$$
$$= \frac{1}{2}\int_s^\infty L(1-\cos 2t)\,\mathrm{d}s$$
$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\mathrm{d}s$$
$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$
$$= \frac{1}{2}\left(\log\sqrt{\frac{s^2}{s^2+4}}\right)_{s\to\infty} - \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$
$$= \frac{1}{2}\left(\log\sqrt{\frac{1}{1+4/s^2}}\right)_{s\to\infty} + \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$
$$= \frac{1}{2}\log 1 + \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$
$$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right) \qquad (2)$$

Using (2) in (1), we have

$$\int_0^\infty \frac{\sin^2 t}{t\,e^t}\,\mathrm{d}t = \frac{1}{2}\log\sqrt{5} = \frac{1}{4}\log 5$$

(iii)
$$\int_0^\infty \left(\frac{\cos at - \cos bt}{t}\right)\mathrm{d}t = \left[\int_0^\infty e^{-st}\left(\frac{\cos at - \cos bt}{t}\;\mathrm{d}t\right)\right]_{s=0}$$
$$= L\left\{\frac{\cos at - \cos bt}{t}\right\}_{s=0} \qquad (1)$$

Now $$L\left\{\frac{\cos at-\cos bt}{t}\right\}=\int_s^\infty L(\cos at-\cos bt)\,\mathrm{d}s$$

$$=\int_s^\infty\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)\mathrm{d}s$$

$$=\left[\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$=\left[\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_{s\to\infty}-\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$=\log\sqrt{\frac{s^2+b^2}{s^2+a^2}} \qquad (2)$$

Using (2) in (1), we have

$$\int_0^\infty\left(\frac{\cos at-\cos bt}{t}\right)\mathrm{d}t=\log\left(\frac{b}{a}\right).$$

(iv) $$\int_0^\infty\left(\frac{e^{-2t}-e^{-4t}}{t}\right)\mathrm{d}t=\left[\int_0^\infty e^{-st}\left(\frac{e^{-2t}-e^{-4t}}{t}\right)\mathrm{d}t\right]_{s=0}$$

$$=L\left(\frac{e^{-2t}-e^{-4t}}{t}\right)_{s=0} \qquad (1)$$

Now $$L\left(\frac{e^{-2t}-e^{-4t}}{t}\right)=\int_s^\infty L(e^{-2t}-e^{-4t})\,\mathrm{d}s$$

$$=\int_s^\infty\left(\frac{1}{s+2}-\frac{1}{s+4}\right)\mathrm{d}s$$

$$=\left[\log\left(\frac{s+2}{s+4}\right)\right]_s^\infty$$

$$=\left[\log\left(\frac{s+2}{s+4}\right)\right]_{s\to\infty}-\log\left(\frac{s+2}{s+4}\right)$$

$$=\log\left(\frac{s+4}{s+2}\right) \qquad (2)$$

Using (2) in (1), we have

$$\int_0^\infty\left(\frac{e^{-2t}-e^{-4t}}{t}\right)\mathrm{d}t=\log 2$$

Example 5.13 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{s}{(s^2+a^2)^2}$; (ii) $\dfrac{s}{(s^2-4)^2}$;

(iii) $\dfrac{4(s-1)}{(s^2-2s+5)^2}$; (iv) $\dfrac{s^2-3}{(s^2+4s+5)^2}$;

(v) $\dfrac{s+1}{(s^2+2s-8)^2}$

(i) $$L^{-1}\{\phi(s)\} = t\cdot L^{-1}\int_s^\infty \phi(s)\,\mathrm{d}s \quad (1)$$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = t\cdot L^{-1}\int_s^\infty \frac{s}{(s^2+a^2)^2}\,\mathrm{d}s$$

$$= tL^{-1}\int_{s^2+a^2}^\infty \frac{1}{2}\frac{\mathrm{d}x}{x^2}, \text{ on putting } s^2+a^2 = x$$

$$= \frac{t}{2}L^{-1}\left(-\frac{1}{x}\right)_{s^2+a^2}^\infty$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$= \frac{t}{2a}\sin at.$$

(ii) $$L^{-1}\left\{\frac{s}{(s^2-4)^2}\right\} = t\cdot L^{-1}\int_s^\infty \frac{s}{(s^2-4)^2}\,\mathrm{d}s$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{s^2-4}\right), \text{ as in (i) above.}$$

$$= \frac{t}{4}\sinh 2t$$

Note ☑ The inverse transform in this case can also found out by resolving the given function into partial fractions.

(iii) $$L^{-1}\left\{\frac{4(s-1)}{(s^2-2s+5)^2}\right\} = 4L^{-1}\left\{\frac{s-1}{\left[(s-1)^2+2^2\right]^2}\right\}$$

$$= 4e^t L^{-1}\left\{\frac{s}{(s^2+2^2)^2}\right\}, \text{ by the first shifting property}$$

$$= 4e^t \frac{t}{4} \sin 2t \text{, as in problem (i)}$$

$$= te^t \sin 2t.$$

(iv) $$L^{-1}\left\{\frac{s^2-3}{(s^2+4s+5)^2}\right\} = L^{-1}\left[\frac{(s^2+4s+5)-(4s+8)}{(s^2+4s+5)^2}\right]$$

$$= L^{-1}\left\{\frac{1}{s^2+4s+5}\right\} - 4L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} - 4L^{-1}\left[\frac{s+2}{\{(s+2)^2+1\}^2}\right]$$

$$= e^{-2t}\sin t - 4e^{-2t}\frac{t}{2}\sin t \text{, as in problem (i)}$$

$$= e^{-2t}(1-2t)\sin t.$$

(v) $$L^{-1}\left\{\frac{s+1}{(s^2+2s-8)^2}\right\} = L^{-1}\left\{\frac{s+1}{[(s+1)^2-3^2]^2}\right\}$$

$$= e^{-t}\cdot L^{-1}\left\{\frac{s}{(s^2-3^2)^2}\right\} ,$$

$$= e^{-t}\cdot\frac{t}{6}\sinh 3t \text{, proceeding as in problem (ii)}$$

$$= \frac{t}{6}e^{-t}\sinh 3t$$

EXERCISE 5(b)

***Part** A*

(Short Answer Questions)

1. State the formula for the Laplace transform of a periodic function.
2. Find the Laplace transform of $f(t) = t$, in $0 < t < 1$ if $f(t+1) = f(t)$
3. State the relation between the Laplace transforms of $f(t)$ and $t \cdot f(t)$.
4. State the relation between the inverse Laplace transforms of $\phi(s)$ and $\phi'(s)$.
5. State the relation between the Laplace transforms of $f(t)$ and $\frac{1}{t}f(t)$.
6. State the relation between the inverse Laplace transform of $\phi(s)$ and its integral.

Find the Laplace transforms of the following functions:

7. $\frac{1}{2a}t\sin at$
8. $t\cos at$
9. $\sin kt - kt\cos kt$

10. $\sin kt + kt\cos kt$ 11. $\dfrac{t}{a^2}(1-\cos at)$

12. $\cos kt - \dfrac{1}{2}kt\sin kt.$

Find the inverse Laplace transforms of the following functions:

13. $\log\left(\dfrac{s+1}{s-1}\right)$ 14. $\log\left(\dfrac{s+1}{s}\right)$ 15. $\log\left(1+\dfrac{a}{s}\right)$

16. $\log\left(\dfrac{s+a}{s+b}\right)$ 17. $\log\left(\dfrac{s}{s-1}\right)$ 18. $\log\left(\dfrac{s^2+1}{s^2+4}\right)$

19. $\cot^{-1} s$ 20. $\tan^{-1}\dfrac{a}{s}$

Find the Laplace transforms of the following functions:

21. $\dfrac{\sin at}{t}$ 22. $\dfrac{1-e^{-t}}{t}$ 23. $\dfrac{1-e^{t}}{t}$

24. $\dfrac{1-\cos at}{t}$ 25. $\dfrac{\sin^2 t}{t}$

Part B

Find the Laplace transforms of the following periodic functions:

26. $$\begin{aligned} f(t) &= E, \text{ in } 0 \le t < \frac{1}{E} \\ &= 0, \text{ in } \frac{1}{E} \le t < \frac{2\pi}{n}, \end{aligned}$$

given that $f\left(t+\dfrac{2\pi}{n}\right) = f(t)$

27. $$\begin{aligned} f(t) &= E, \text{ in } 0 \le t < \frac{T}{2} \\ &= -E, \text{ in } T/2 \le t < T. \end{aligned}$$

given that $f(t+T) = f(t)$

28. $f(t) = e^t$, in $0 < t < 2\pi$ and $f(t+2\pi) = f(t)$

29. $f(t) = \sin\left(\dfrac{t}{2}\right)$, in $0 < t < 2\pi$ and $f(t+2\pi) = f(t)$

30. $f(t) = |\cos \omega t|$, $t \ge 0$

$$\left[\text{Hint: } f(t) \text{ is periodic with period } (\pi/\omega) \text{ and } \int_0^{\pi/\omega} e^{-st}\,|\cos\omega t|\,dt = \int_0^{\pi/2\omega} e^{-st}\cos\omega t\,dt + \int_{\pi/2\omega}^{\pi/\omega} e^{-st}(-\cos\omega t)\,dt\right]$$

31. $$f(t) = t, \text{ in } 0 < t < \pi$$
$$= 0, \text{ in } \pi < t < 2\pi,$$
given that $f(t + 2\pi) = f(t)$

32. $$f(t) = \sin t, \text{ in } 0 < t < \pi$$
$$= 0, \text{ in } \pi < t < 2\pi,$$
given that $f(t + 2\pi) = f(t)$.

33. $$f(t) = 0, \text{ in } 0 < t < \frac{\pi}{\omega}$$
$$= -\sin \omega t, \text{ in } \frac{\pi}{\omega} < t < \frac{2\pi}{\omega},$$
given that $f\left(t + \frac{2\pi}{\omega}\right) = f(t)$.

34. $$f(t) = t, \text{ in } 0 < t < \pi$$
$$= 2\pi - t, \text{ in } \pi < t < 2\pi,$$
given that $f(t + 2\pi) = f(t)$.

35. $$f(t) = t, \text{ in } 0 < t < \pi$$
$$= \pi - t, \text{ in } \pi < t < 2\pi,$$
given that $f(t + 2\pi) = f(t)$.

Find the Laplace transforms of the following functions:

36. $t \sinh^3 t$
37. $t \cos^3 2t$
38. $t \sin 3t \sin 5t$
39. $t \sin 5t \cos t$
40. $(t \cos 2t)^2$
41. $t^2 \sin t \cos 2t$
42. $t^2 e^{-2t} \sin 3t$
43. $te^{3t} \cos 4t$
44. $t^2 e^{-3t} \cosh 2t$
45. $t \sinh 2t \sin 3t$

Find the inverse Laplace transforms of the following functions:

46. $\log\left(1 + \frac{a^2}{s^2}\right)$
47. $\log \frac{s^2 + a^2}{(s+b)^2}$
48. $\log \frac{(s-2)^2}{s^2+1}$
49. $s \log\left(\frac{s-a}{s+a}\right) + a$.
50. $\tan^{-1}\left(\frac{1}{2s}\right)$
51. $\tan^{-1}\left(\frac{s+2}{3}\right)$
52. $\cot^{-1}\left(\frac{a}{s+b}\right)$
53. $\tan^{-1}(s^2)$

Find the values of the following integrals, using Laplace transforms:

54. $\int_0^\infty t\, e^{-2t} \cos 2t \, dt$
55. $\int_0^\infty t^2\, e^{-t} \sin t \, dt$
56. $\int_0^\infty \left(\frac{e^{-t} - e^{-3t}}{t}\right) dt$

57. $\displaystyle\int_0^\infty \frac{(1-\cos t)\, e^{-t}}{t}\, \mathrm{d}t$ 58. $\displaystyle\int_0^\infty \left(\frac{e^{-at}-\cos bt}{t}\right) \mathrm{d}t$ 59. $\displaystyle\int_0^\infty \frac{e^{-\sqrt{2}t}\sin t \sinh t}{t}\, \mathrm{d}t$

Find the Laplace transforms of the following functions:

60. $\dfrac{1-e^{-t}}{t}$ 61. $\dfrac{1-\cos at}{t}$ 62. $\left(\dfrac{\sin 2t}{\sqrt{t}}\right)^2$

63. $\dfrac{\sin 3t \sin t}{t}$ 64. $\dfrac{1-\cos t}{t^2}$

Find the Laplace inverse transforms of the following functions:

65. $\dfrac{s}{(s^2+1)^2}$ 66. $\dfrac{s}{(s^2-a^2)^2}$ 67. $\dfrac{s-2}{(s^2-4s+5)^2}$

68. $\dfrac{(s-a)^2}{(s^2-a^2)^2}$ 69. $\dfrac{s^2+8s+16}{(s^2+6s+10)^2}$ 70. $\dfrac{s+4}{(s^2+8s+15)^2}$

5.8 LAPLACE TRANSFORMS OF DERIVATIVES AND INTEGRALS

In the following two theorems we find the Laplace transforms of the derivatives and integrals of a function $f(t)$ in terms of the Laplace transform of $f(t)$. These results will be used in solving differential and integral equations using Laplace transforms.

Theorem

If $f(t)$ is continuous in $t \geq 0$, $f'(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and $f(t)$ and $f'(t)$ are of the exponential order, then
$L\{f'(t)\} = sL\{f(t)\} - f(0)$

Proof:

The given conditions ensure the existence of the Laplace transforms *of* $f(t)$ and $f'(t)$.

By definition, $\displaystyle L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, \mathrm{d}t$

$$= \int_0^\infty e^{-st}\mathrm{d}\,[f(t)]$$

$$= \left[e^{-st}\cdot f(t)\right]_0^\infty - \int_0^\infty (-s)\, e^{-st}\, f(t)\, \mathrm{d}t, \text{on integration by parts.}$$

$$= \lim_{t\to\infty}\, [e^{-st} f(t)] - f(0) + s\cdot L\{f(t)$$

$$= 0 - f(0) + sL\,\{f(t)\}\ [\because f(t) \text{ is of the exponential order}]$$

$$= sL\,\{f(t)\} - f(0) \qquad (1)$$

Corollary 1

In result (1) if we replace $f(t)$ by $f'(t)$ we get

$$L\{f''(t)\} = sL\{f'(t)\} - f'(0)$$
$$= s[sL\{f(t)\} - f(0)] - f'(0), \quad \text{again by (1)}$$
$$= s^2 L\{f(t)\} - sf(0) - f'(0) \quad (2)$$

Note ☑

1. Result (2) holds good, if $f(t)$ and $f'(t)$ are continuous in $t \geq 0$, $f''(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and $f(t), f'(t)$ and $f''(t)$ are of the exponential order.

Corollary 2

Repeated application of (1) gives the following result:

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \quad (3)$$

Note ☑

2. Result (3) holds good, if $f(t)$ and its first $(n-1)$ derivatives are continuous in $t \geq 0$, $f^{(n)}(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and $f(t), f'(t), \cdots, f^{(n)}(t)$ are of the exponential order.

3. If we take $L\{f(t)\} = \phi(s)$, result (1) becomes

$$L\{f'(t)\} = s\phi(s) - f(0) \quad (4)$$

If we further assume that $f(0) = 0$, the result becomes

$$L\{f'(t)\} = s\phi(s) \quad (5)$$

In terms of the inverse Laplace operator, (5) becomes

$$L^{-1}\{s\phi(s)\} = f'(t) \quad (6)$$

From result (6), we get the following working rule:

$$L^{-1}\{s\phi(s) = \frac{d}{dt} L^{-1}\{\phi(s), \text{ provided that}$$
$$f(0) = L^{-1}\{\phi(s)\}_{t=0} = 0.$$

Thus, to find the inverse transform of the product of two factors, one of which is 's', we ignore 's', and find the inverse transform of the other factor; we call it $f(t)$, verify that $f(0) = 0$ and get $f'(t)$, which is the required inverse transform.

4. In a similar manner, from result (2) we get

$$L^{-1}\{s^2\phi(s)\} = \frac{d^2}{dt^2} L^{-1}\{\phi(s)\}, \text{ provided that}$$
$$f(0) = 0 \text{ and } f'(0) = 0, \text{ where}$$
$$f(t) = L^{-1}\{\phi(s)\}.$$

Theorem

If $f(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and is of the exponential order, then

$$L\left[\int_0^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}L\{f(t)\}.$$

Proof:

Let $\qquad g(t) = \int_0^t f(t)\,\mathrm{d}t$

$\therefore \qquad g'(t) = f(t)$

Under the given conditions, it can be shown that the Laplace transforms of both $f(t)$ and $g(t)$ exist.

Now by the previous theorem,

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$

i.e.,
$$s\cdot L\left[\int_0^t f(t)\,\mathrm{d}t\right] - \int_0^0 f(t)\,\mathrm{d}t = L\{f(t)\}$$

$\therefore$
$$L\left[\int_0^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}L\{f(t)\} \tag{1}$$

Corollary

$L\left[\int_0^t\int_0^t f(t)\,\mathrm{d}t\,\mathrm{d}t\right] = \frac{1}{s^2}L\{f(t)\}$, as explained below.

Let $\qquad \int_0^t f(t)\,\mathrm{d}t = g(t)$.

Then, by result (1) above,

$$L\int_0^t g(t)\,\mathrm{d}t = \frac{1}{s}L\{g(t)\}$$

i.e.,
$$L\left[\int_0^t\int_0^t f(t)\,\mathrm{d}t\,\mathrm{d}t\right] = \frac{1}{s}L\int_0^t f(t)\mathrm{d}t$$

$$= \frac{1}{s}\cdot\frac{1}{s}L\{f(t)\}\text{, again by} \tag{1}$$

$$= \frac{1}{s^2}L\{f(t)\} \tag{2}$$

Generalising (2), we get

$$L\left[\int_0^t\int_0^t\cdots\int_0^t f(t)\,(\mathrm{d}t)^n\right] = \frac{1}{s^n}L\{f(t)\} \tag{3}$$

Note ☑

1. If we put $L\{f(t)\} = \phi(s)$, result (1) becomes

$$L\left[\int_0^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}\phi(s) \tag{4}$$

Result (4) can be expressed, in terms of L^{-1} operator, as

$$L^{-1}\left\{\frac{1}{s}\phi(s)\right\} = \int_0^t f(t)\,\mathrm{d}t \tag{5}$$

From (5), we get the following rule:

$$L^{-1}\left\{\frac{1}{s}\phi(s)\right\} = \int_0^t L^{-1}\{\phi(s)\}\,\mathrm{d}t\ .$$

Thus, to find the inverse Laplace transform of the product of two factors, one of which is $\frac{1}{s}$, we ignore $\frac{1}{s}$, find the inverse transform of the other factor and integrate it with respect to t between the limits 0 and t.

2. In a similar manner, from (2) above, we get

$$L^{-1}\left\{\frac{1}{s^2}\phi(s)\right\} = \int_0^t \int_0^t L^{-1}\{\phi(s)\}\,\mathrm{d}t\,\mathrm{d}t\,.$$

3. $$L\left[\int_a^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}L\{f(t)\} + \frac{1}{s}\int_a^0 f(t)\,\mathrm{d}t$$

If we let $$g(t) = \int_a^t f(t)\,\mathrm{d}t \text{ and } g'(t) = f(t),$$

we get $$L\{g'(t)\} = sL\{g(t)\} - g(0)$$

i.e., $$L\{f(t)\} = sL\left[\int_a^t f(t)\,\mathrm{d}t\right] - \int_a^0 f(t)\,\mathrm{d}t$$

or $$L\left[\int_a^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}L\{f(t)\} + \frac{1}{s}\int_a^0 f(t)\,\mathrm{d}t.$$

5.9 INITIAL AND FINAL VALUE THEOREMS

We shall now consider two results, which are derived by applying the theorem on Laplace transform of the derivative of a function.

The first result, known as the initial value theorem, gives a relation between $\lim_{t\to 0}[f(t)]$ and $\lim_{s\to\infty}[s\phi(s)]$, where $\phi(s) = L\{f(t)\}$

The second result, known as the final value theorem, gives a relation between $\lim_{t\to\infty}[f(t)]$ and $\lim_{s\to 0}[s\phi(s)]$.

5.9.1 Initial Value Theorem

If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L\{f(t)\} = \phi(s)$, then

$$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[s\phi(s)]$$

Proof:

We know that $\quad L\{f'(t)\} = s\phi(s) - f(0)$

$\therefore$

$$s\phi(s) = L\{f'(t)\} + f(0)$$

$$= \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t + f(0)$$

$\therefore$

$$\lim_{s\to\infty}[s\phi(s) = \lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t + f(0),$$

$$= \int_0^\infty \lim_{s\to\infty}\{e^{-st} f'(t)\}\,\mathrm{d}t + f(0),$$

assuming that the conditions for the interchange of the operations of integration and taking limit hold.

i.e. $\quad \lim_{s\to\infty}[s\phi(s)] = 0 + f(0)$

$$= \lim_{t\to 0}[f(t)].$$

5.9.2 Final Value Theorem

If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L\{f(t)\} = \phi(s)$, then $\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[s\phi(s)]$, provided all the singularities of $\{s\phi(s)]$ are in the left half plane $Rl(s) < 0$.

Proof:

We know that $\quad L\{f'(t)\} = s\phi(s) - f(0)$

$\therefore$

$$s\phi(s) = L\{f'(t)\} + f(0)$$

$$= \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t + f(0)$$

$$\therefore \qquad \lim_{s\to 0}[s\phi(s)] = \lim_{s\to 0}\int_0^\infty e^{-st} f'(t)\mathrm{d}t + f(0$$

$$= \int_0^\infty \lim_{s\to 0}\{e^{-st} f'(t)\}\mathrm{d}t + f(0),$$ assuming that the conditions for the interchange of the operations of integration and taking limit hold.

i.e. $$\lim_{s\to 0}[s\phi(s)] = \int_0^\infty f'(t)\mathrm{d}t + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= \lim_{t\to\infty}[f(t)] - f(0) + f(0)$$

Thus $$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[s\phi(s)]$$

5.10 THE CONVOLUTION

Another result, which is of considerable practical importance, is the convolution theorem that enables us to find the inverse Laplace transform of the product of $\bar{f}(s)$ and $\bar{g}(s)$ in terms of the inverse transforms of $\bar{f}(s)$ and $\bar{g}(s)$.

Definition The *convolution* or *convolution integral* of two function $f(t)$ and $g(t)$, defined in $t \geq 0$, is defined as the integral

$$\int_0^t f(u)\, g(t-u)\, \mathrm{d}u$$

It is denoted as $f(t) * g(t)$ or $(f * g)(t)$

i.e. $$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\,\mathrm{d}u$$

$$= \int_0^t f(t-u)\, g[t-(t-u)]\,\mathrm{d}u,$$ on using the result

$$\int_0^t \phi(u)\,\mathrm{d}u = \int_0^t \phi(t-u)\,\mathrm{d}u$$

$$= \int_0^t g(u) f(t-u)\,\mathrm{d}u$$

$$= g(t) * f(t).$$

Thus the convolution product is commutative.

5.10.1 Convolution Theorem

If $f(t)$ and $g(t)$ are Laplace transformable,

then $$L\{f(t) * g(t)\} = L\{f(t)\} \cdot L\{g(t)\}$$

Proof:

By definition,
$$L\{f(t)*g(t)\} = \int_0^\infty e^{-st}\{f(t)*g(t)\}\, dt,$$
$$= \int_0^\infty e^{-st}\left[\int_0^t f(u)g(t-u)du\right] dt,$$

by the definition of convolution.

$$= \int_0^\infty \int_0^t e^{-st} f(u)\, g(t-u) du\, dt \tag{1}$$

The region of integration for the double integral (1) is bounded by the lines $u = 0$, $u = t$, $t = 0$ and $t = \infty$ and is shown in the Fig. 5.6.

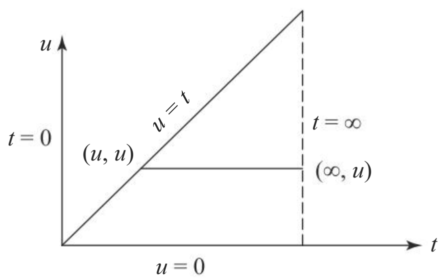

Fig. 5.6

Changing the order of integration in (1), we get,

$$L\{f(t)*g(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)\, g(t-u)\, dt\, du \tag{2}$$

In the inner integral in (2), on putting $t - u = v$ and making the consequent changes, we get,

$$L\{f(t)*g(t)\} = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u) g(v)\, dv\, du$$
$$= \int_0^\infty e^{-su} f(u)\left[\int_0^\infty e^{-sv} g(v) dv\right] du$$

$$= \int_0^\infty e^{-su} f(u)\mathrm{d}u \cdot \int_0^\infty e^{-sv}\, g(v)\mathrm{d}v$$

$$= \int_0^\infty e^{-st} f(t)\mathrm{d}t \cdot \int_0^\infty e^{-st} g(t)\mathrm{d}t,$$

on changing the dummy variables u and v.

$$= L\{f(t)\} \cdot L\{g(t)\}$$

Note ☑ If $L\{f(t)\} = \bar{f}(s)$ and $L\{g(t)\} = \bar{g}(s)$, the convolution theorem can be put as

$$L\{f(t) * g(t)\} = \bar{f}(s) \cdot \bar{g}(s) \qquad (3)$$

In terms of the inverse Laplace operator, result (3) can be written in the following way.

$$L^{-1}\{\bar{f}(s) \cdot \bar{g}(s)\} = f(t) * g(t)$$

$$= \int_0^t f(u)\, g(t-u)\, \mathrm{d}u \qquad (4)$$

Result (4) means that the inverse Laplace transform of the ordinary product of two functions of s is equal to the convolution product of the inverses of the individual functions.

WORKED EXAMPLE 5(c)

Example 5.1 Using the Laplace transforms of derivatives, find the Laplace transforms of

(i) e^{-at}, (ii) $\sin at$,

(iii) $\cos^2 t$, (iv) t^n (n is a positive integer)

(i) $L\{f'(t)\} = sL\{f(t)\} - f(0)$ (1)

Putting $f(t) = e^{-at}$ in (1), we get

$L(-ae^{-at}) = sL(e^{-at}) - 1$

i.e. $(s + a)\, L(e^{-at}) = 1$

$$\therefore \quad L(e^{-at}) = \frac{1}{s+a}$$

(ii) $L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$ (2)

Putting $f(t) = \sin at$ in (2), we get

$L(-a^2 \sin at) = s^2 L(\sin at) - s \times 0 - a$

i.e. $(s^2 + a^2)\, L\,(\sin at) = a$

$$\therefore \quad L(\sin at) = \frac{a}{s^2 + a^2}$$

(iii) Putting $f(t) = \cos^2 t$ in (1), we get

$$L\,(-\,2 \cos t \sin t) = sL\,(\cos^2 t) - 1$$

i.e. $$s \cdot L\,(\cos^2 t) = 1 - L\,(\sin 2t)$$

$$= 1 - \frac{2}{s^2 + 4}$$

$$= \frac{s^2 + 2}{s^2 + 4}$$

$$\therefore \quad L(\cos^2 t) = \frac{s^2 + 2}{s(s^2 + 4)}$$

(iv) $$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - f^{(n-1)}(0) \qquad (3)$$

Putting $f(t) = t^n$ in (3) and nothing that $f^{(n)}(\mathrm{t}) = n!$

and $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$, we get

$$L\{n!\} = s^n L\,(t^n)$$

i.e. $$n! L\,(1) = \mathrm{s}^n L\,(t^n)$$

i.e. $$n! \frac{1}{s} = s^n L(t^n)$$

$$\therefore \quad L(t^n) = \frac{n!}{s^{n+1}}$$

Example 5.2 Find the Laplace transform of $\sqrt{\dfrac{t}{\pi}}$ and hence find $L\left\{\dfrac{1}{\sqrt{\pi t}}\right\}$.

$$L\left\{\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{\sqrt{\pi}} L(t^{1/2}) = \frac{1}{\sqrt{\pi}} = \frac{\left\lceil(3/2)\right.}{s^{3/2}}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\frac{1}{2}\left\lceil(1/2)\right.}{s^{3/2}} = \frac{1}{2s^{3/2}} \qquad \left(\because \left\lceil(1/2)\right. = \sqrt{\pi}\right)$$

In the result $L\{f'(t)\} = sL\,\{f(t)\} - f(0)$, we put

$$f(t) = \sqrt{\frac{t}{\pi}}\text{, we get}$$

$$L\left\{\frac{1}{2\sqrt{\pi t}}\right\} = s \cdot \frac{1}{2s^{3/2}} - 0$$

$$= \frac{1}{2\sqrt{s}}$$

$\therefore$ $$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}.$$

Aliter

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = L\left\{\frac{1}{t}\cdot\sqrt{t/\pi}\right\}$$

$$= \int_s^\infty L(\sqrt{t/\pi})\ \mathrm{d}s$$

$$= \int_s^\infty \frac{1}{2s^{3/2}}\ \mathrm{d}s$$

$$= \left(-\frac{1}{\sqrt{s}}\right)_s^\infty = \frac{1}{\sqrt{s}}.$$

Example 5.3 Using the Laplace transforms of the derivatives find

(i) $L(t \cos at)$ and hence $L(\sin at - at \cos at)$ and $L(\cos at - at \sin at)$

(ii) $L(t \sinh at)$ and hence $L\ (\sinh at + at \cosh at)$ and $L(\cosh at + \frac{1}{2} at \sinh at)$.

(i) $$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0)\text{-}f'(0) \tag{1}$$

Put $f(t) = t \cos at$ in (1)

Then $$f'(t) = \cos at - at \sin at \text{ and } f'(0) = 1$$

$$f''(t) = -a^2 t \cos at - 2a \sin at.$$

$\therefore$ $$L\{-a^2 t \cos at - 2a \sin at\} = s^2 L\ \{t \cos at\} - 1$$

$[\because f(0) = 0]$ and $f'(0) = 1$

i.e. $$(s^2 + a^2)\ L(t \cos at) = 1 - 2a\ L(\sin at)$$

$$= 1 - \frac{2a^2}{s^2 + a^2}$$

$$= \frac{s^2 - a^2}{s^2 + a^2}$$

$\therefore$ $$L(t \cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

Now $$L(\sin at - at \cos at) = \frac{a}{s^2 + a^2} - \frac{a(s^2 - a^2)}{(s^2 + a^2)^2}$$

$$= \frac{a\left\{(s^2+a^2)-(s^2-a^2)\right\}}{(s^2+a^2)^2}$$

$$= \frac{2a^3}{(s^2+a^2)^2}$$

Taking $f(t) = t \cos at$ in the result

$L\{f'(t)\} = sL\{f(t)\} - f(0)$, we get

$L\{\cos at - at \sin at\} = sL\,(t \cos at) - 0$

$$= \frac{s(s^2-a^2)}{(s^2+a^2)^2}$$

(ii) Put $f(t) = t \sinh at$ in (1).

Then $f'(t) = \sinh at + at \cosh at$ and $f'(0) = 0$

$f''(t) = a^2 t \sinh at + 2a \cosh at$

$\therefore\ L\{a^2 t \sinh at + 2a \cosh at\}$

$= s^2 L(t \sinh at)$ $[\because f(0) = 0 = f'(0)]$

i.e. $(s^2 - a^2)\, L(t \sinh at) = 2a\, L(\cosh at)$

$$= \frac{2as}{s^2-a^2}$$

$$\therefore \qquad L(t \sinh at) = \frac{2as}{(s^2-a^2)^2}$$

In the result $L\{f'(t)\} = sL\,\{f(t)\} - f(0)$, if we put $f(t) = t \sinh at$, we have

$$L(\sinh at + at \cosh at) = \frac{2as^2}{(s^2-a^2)^2} \quad [\because f(0) = 0]$$

Now $$L\left(\cosh at + \frac{1}{2} at \sinh at\right)$$

$$= L(\cosh at) + \frac{a}{2} L(t \sinh at)$$

$$= \frac{s}{s^2-a^2} + \frac{a}{2} \cdot \frac{2as}{(s^2-a^2)^2}$$

$$= \frac{s(s^2-a^2)+a^2 s}{(s^2-a^2)^2} = \frac{s^3}{(s^2-a^2)^2}.$$

Example 5.4 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{s}{(s+2)^4}$; (ii) $\dfrac{s^2}{(s-2)^3}$;

(iii) $\dfrac{s}{s^2+4s+5}$; (iv) $\dfrac{s}{(s+2)(s+3)}$;

(v) $\dfrac{s}{(s^2+1)(s^2+4)}$

(i) $L^{-1}\{s\phi(s)\} = \dfrac{d}{dt}L^{-1}\{\phi(s)\}$, provided $L^{-1}\{\phi(s)\}$ vanishes at $t = 0$ (1)

To find $L^{-1}\left\{\dfrac{s}{(s+2)^4}\right\}$, let us first find $f(t) = L^{-1}\left\{\dfrac{1}{(s+2)^4}\right\}$ and then apply rule (1)

Now
$$f(t) = e^{-2t}L^{-1}\left(\frac{1}{s^4}\right)$$
$$= e^{-2t}\frac{1}{3!}t^3 = \frac{1}{6}t^3e^{-2t}$$

We note that $f(0) = 0$

$\therefore$ By(1),
$$L^{-1}\left\{\frac{s}{(s+2)^4}\right\} = \frac{d}{dt}\left(\frac{1}{6}t^3e^{-2t}\right)$$
$$= \frac{1}{6}(-2t^3e^{-2t} + 3t^2e^{-2t})$$
$$= \frac{1}{6}t^2e^{-2t}(3-2t).$$

(ii) $L^{-1}\{s^2\phi(s)\} = \dfrac{d^2}{dt^2}L^{-1}\{\phi(s)\}$, provided

$f(0)=0\ f'(0)=0$, where $f(t) = L^{-1}\{\phi(s)\}$. (2)

To find $L^{-1}\left\{\dfrac{s^2}{(s-2)^3}\right\}$, we shall find $f(t) = L^{-1}\left\{\dfrac{1}{(s-2)^3}\right\}$ and then apply rule (2).

Now
$$f(t) = e^{2t}L^{-1}\left(\frac{1}{s^3}\right)$$
$$= e^{2t}\cdot\frac{1}{2!}t^2 = \frac{1}{2}t^2e^{2t}$$
$$f'(t) = \frac{1}{2}(2t^2e^{2t} + 2te^{2t})$$

We note that $f(0) = 0$ and $f'(0) = 0$

$\therefore$ By (2),
$$L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\} = \frac{d^2}{dt^2}\left\{\frac{1}{2}t^2e^{2t}\right\}$$

$$= \frac{d}{dt}[(t^2 + t)e^{2t}]$$
$$= 2(t^2 + t)e^{2t} + (2t + 1)e^{2t}$$
$$= (2t^2 + 4t + 1)e^{2t}$$

(iii) To find $L^{-1}\left\{\frac{s}{s^2 + 4s + 5}\right\}$,

we shall find $f(t) = L^{-1}\left\{\frac{1}{s^2 + 4s + 5}\right\}$ and then apply the rule (1).

Now $f(t) = L^{-1}\left\{\frac{1}{(s + 2)^2 + 1}\right\}$

$= e^{-2t} \sin t$

and $f(0) = 0.$

$\therefore$ By (l), $L^{-1}\left\{\frac{s}{s^2 + 4s + 5}\right\} = \frac{d}{dt}(e^{-2t} \sin t)$

$= e^{-2t} \cos t - 2e^{-2t} \sin t$

$= e^{-2t} (\cos t - 2 \sin t)$

(iv) To find $L^{-1}\left\{\frac{s}{(s + 2)(s + 3)}\right\}$, we shall find $f(t) = L^{-1}\left\{\frac{1}{(s + 2)(s + 3)}\right\}$ and then apply the rule (1).

Now $f(t) = L^{-1}\left\{\frac{1}{(s + 2)(s + 3)}\right\}$

$= L^{-1}\left\{\frac{1}{s + 2} - \frac{1}{s + 3}\right\}$,

by resolving the function into partial fractions.

$= e^{-2t} - e^{-3t}$

and $f(0) = 0.$

$\therefore$ By (1), $L^{-1}\left\{\frac{s}{(s + 2)(s + 3)}\right\} = \frac{d}{dt}(e^{-2t} - e^{-3t})$

$= 3e^{-3t} - 2e^{-2t}$

(v) To find $L^{-1}\left\{\frac{s}{(s^2 + 1)(s^2 + 4)}\right\}$, we shall find $f(t) = L^{-1}\left\{\frac{1}{(s^2 + 1)(s^2 + 4)}\right\}$

and then apply the rule (1).

Now $$f(t)=L^{-1}\left\{\frac{1/3}{s^2+1}-\frac{1/3}{s^2+4}\right\},$$

by resolving the function into partial fractions.

$$=\frac{1}{3}\sin t-\frac{1}{6}\sin 2t$$

and $f(0) = 0$

$\therefore$ By (1), $$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{3}\sin t-\frac{1}{6}\sin 2t\right\}$$

$$=\frac{1}{3}(\cos t-\cos 2t)$$

Note ☑ We have solved the problems in the above example by using the working rule derived from the theorem on Laplace transforms of derivatives. They can be solved by elementary methods, such as partial fraction methods, discussed in Section 5(a) also.

Example 5.5 Find the inverse Laplace transforms of the following functions.

(i) $\dfrac{s^2}{(s^2+a^2)^2}$; (ii) $\dfrac{s^3}{(s^2+a^2)^2}$;

(iii) $\dfrac{(s+1)^2}{(s^2+2s+5)^2}$; (iv) $\dfrac{s^2}{(s^2-4)^2}$;

(v) $\dfrac{(s-3)^2}{(s^2-6s+5)^2}$.

(i) Let $$f(t)=L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$=\frac{t}{2a}\sin at$$ [Refer to Worked Example (13) (i) in Section 5(b)]

We note that $f(0) = 0$

Now $L^{-1}\{s\phi(s)\}=\dfrac{\mathrm{d}}{\mathrm{d}t}L^{-1}\{\phi(s)\}$, provided $f(0) = 0$,

where $f(t) = L^{-1}\{\phi(s)\}$ (1)

$$\therefore \quad L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}=L^{-1}\left\{s\cdot\frac{s}{(s^2+a^2)^2}\right\}$$

$$=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{2a}\sin at\right)$$, by rule (1)

$$= \frac{1}{2a}(\sin at + at \cos at) \qquad (2)$$

(ii) Let $$f(t) = L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{1}{2a}(\sin at + at \cos at), \text{ by (2)}$$

we note that $f(0) = 0$

Now $$L^{-1}\left\{\frac{s^3}{(s^2+a^2)^2}\right\} = L^{-1}\left\{s \cdot \frac{s^2}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{dt}\left[\frac{1}{2a}(\sin at + at \cos at)\right], \text{ by rule (1)}$$

$$= \frac{1}{2}(2 \cos at - at \sin at).$$

(iii) $$L^{-1}\left\{\frac{(s+1)^2}{(s^2+2s+5)^2}\right\} = L^{-1}\left[\frac{(s+1)^2}{\{(s+1)^2+2^2\}^2}\right]$$

$$= e^{-t} L^{-1}\left\{\frac{s^2}{(s^2+2^2)^2}\right\}, \text{ by the first shifting property}$$

$$= \frac{1}{4} e^{-t}(\sin 2t + 2t \cos 2t), \text{ by (2)}$$

(iv) Let $$f(t) = L^{-1}\left\{\frac{s}{(s^2-4)^2}\right\}$$

$$= \frac{t}{4} \sinh 2t$$

[Refer to Worked Example (13) (ii) in Section 5(b)]

We note that $f(0) = 0$.

Now $$L^{-1}\left\{\frac{s^2}{(s^2-4)^2}\right\} = L^{-1}\left\{s \cdot \frac{s}{(s^2-4)^2}\right\}$$

$$= \frac{d}{dt} L^{-1}\left\{\frac{s}{(s^2-4)^2}\right\}, \text{ by rule (1)}$$

$$= \frac{d}{dt}\left(\frac{t}{4} \sinh 2t\right)$$

$$= \frac{1}{4}(\sinh 2t + 2t \cosh 2t) \qquad (3)$$

(v) $L^{-1}\left[\dfrac{(s-3)^2}{(s^2-6s+5)^2}\right] = L^{-1}\left\{\dfrac{(s-3)^2}{\{(s-3)^2-4\}^2}\right\}$

$$= e^{3t}L^{-1}\left[\frac{s^2}{(s^2-4)^2}\right], \text{ by the first shifting property.}$$

$$= \frac{1}{4}e^{3t}(\sinh 2t + 2t \cosh 2t), \text{ by (3).}$$

Example 5.6 Find the Laplace transforms of the following functions:

(i) $\displaystyle\int_0^t te^{-4t} \sin 3t \, \mathrm{d}t$; (ii) $\displaystyle e^{-4t}\int_0^t t \sin 3t \, \mathrm{d}t$;

(iii) $\displaystyle t\int_0^t e^{-4t} \sin 3t \, \mathrm{d}t$; (iv) $\displaystyle\int_0^t \frac{e^{-t} \sin t}{t} \mathrm{d}t$;

(v) $\displaystyle e^{-t}\int_0^t \frac{\sin t}{t} \mathrm{d}t$; (vi) $\displaystyle\frac{1}{t}\int_0^t e^{-t} \sin t \, \mathrm{d}t$.

(i) $\displaystyle L\left[\int_0^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}L\{f(t)\}$ (1),

by the theorem on Laplace transform of integral

$$\therefore \quad L\left[\int_0^t te^{-4t}\sin 3t \, \mathrm{d}t\right]$$

$$= \frac{1}{s}L\{te^{-4t} \sin 3t\}$$

$$= \frac{6(s+4)}{s(s^2+8s+25)^2}$$

[Refer to Worked Example 8(i) in Section 5(b)].

(ii) $\displaystyle L\left[e^{-4t}\int_0^t t \sin 3t \, \mathrm{d}t\right] = \left[L\int_0^t t \sin 3t \, \mathrm{d}t\right]_{s\to s+4}$ (2), by the first shifting property

Now $\displaystyle L\int_0^t t \sin 3t \, \mathrm{d}t = \frac{1}{s}L(t\sin 3t)$, by rule (1)

$$= \frac{1}{s}\left[-\frac{\mathrm{d}}{\mathrm{d}s}L(\sin 3t)\right]$$

$$= -\frac{1}{s}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right)$$

$$= -\frac{1}{s} \times \frac{-3 \times 2s}{(s^2+9)^2} = \frac{6}{(s^2+9)^2} \qquad (3)$$

Using (3) in (2), we get

$$L\left[e^{-4t}\int_0^t t\sin 3t \; \mathrm{d}t\right] = \frac{6}{\{(s+4)^2+9\}^2}$$

$$= \frac{6}{(s^2+8s+25)^2}$$

(iii) $L\left[t \cdot \int_0^t e^{-4t} \; \sin 3t \; \mathrm{d}t\right]$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left[L\int_0^t e^{-4t} \sin 3t \, \mathrm{d}t\right] \qquad (4)$$

Now $\quad L\int_0^t e^{-4t}\sin 3t \; \mathrm{d}t = \frac{1}{s} L(e^{-4t} \; \sin 3t)$, by (1)

$$= \frac{1}{s}[L(\sin 3t)]_{s \to s+4}$$

$$= \frac{1}{s} \cdot \frac{3}{(s+4)^2+9}$$

$$= \frac{3}{s^3+8s^2+25s} \qquad (5)$$

Using (5) in (4), we get

$$L\left[t\int_0^t e^{-4t}\sin 3t \; \mathrm{d}t\right] = -\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{3}{s^3+8s^2+25s}\right\}$$

$$= \frac{3(3s^2+16s+25)}{(s^3+8s+25s)^2}$$

$$= \frac{3(3s^2+16s+25)}{s^2(s^2+8s+25)^2}$$

(iv) $\quad L\int_0^t \frac{e^{-t}\sin t}{t} \; \mathrm{d}t = \frac{1}{s} L\left(\frac{e^{-t}\sin t}{t}\right)$, by (1) $\qquad (6)$

Now $\quad L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty L(e^{-t}\sin t) \; \mathrm{d}s$

$$= \int_s^\infty \frac{\mathrm{d}s}{(s+1)^2+1}$$

$$= \{-\cot^{-1}(s+1)\}_s^{\infty}$$
$$= \cot^{-1}(s+1) \tag{7}$$

Using (7) in (6), we get

$$L\int_0^t \frac{e^{-t}\sin t}{t}\,\mathrm{d}t = \frac{1}{s}\cot^{-1}(s+1)$$

(v) $$L\left[e^{-t}\int_0^t \frac{\sin t}{t}\,\mathrm{d}t\right] = \left[L\int_0^t \frac{\sin t}{t}\,\mathrm{d}t\right]_{s\to s+1} \tag{8}$$

Now $$L\int_0^t \frac{\sin t}{t}\,\mathrm{d}t = \frac{1}{s}L\left(\frac{\sin t}{t}\right), \text{ by (1)}$$

$$= \frac{1}{s}\int_s^{\infty} L(\sin t)\,\mathrm{d}s$$

$$= \frac{1}{s}\int_s^{\infty}\frac{\mathrm{d}s}{s^2+1} = \frac{1}{s}\cot^{-1}s \tag{9}$$

Using (9) in (8), we get

$$L\left[e^{-t}\int_0^t \frac{\sin t}{t}\,\mathrm{d}t\right] = \frac{1}{s+1}\cot^{-1}(s+1)$$

(vi) $$L\left[\frac{1}{t}\int_0^t e^{-t}\sin t\,\mathrm{d}t\right] = \int_s^{\infty} L\left[\int_0^t e^{-t}\sin t\,\mathrm{d}t\right]\mathrm{d}s \tag{10}$$

Now $$L\int_0^t e^{-t}\sin t\,\mathrm{d}t = \frac{1}{s}L\{e^{-t}\sin t\}, \text{ by (1)}$$

$$= \frac{1}{s}\cdot\frac{1}{(s+1)^2+1} \tag{11}$$

Using (11) in (10), we get

$$L\left[\frac{1}{t}\int_0^t e^{-t}\sin t\,\mathrm{d}t\right] = \int_s^{\infty}\frac{\mathrm{d}s}{s(s^2+2s+2)}$$

$$= \int_s^{\infty}\frac{1}{2}\left[\frac{1}{s} - \frac{s+2}{s^2+2s+2}\right]\mathrm{d}s,$$

on resolving the integrand into partial fractions.

$$= \frac{1}{2}\int_s^{\infty}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right]\mathrm{d}s$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log\{(s+1)^2+1\} + \cot^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+2s+2}}\right) + \cot^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{4}\log\left(\frac{s^2+2s+2}{s^2}\right) - \frac{1}{2}\cot^{-1}(s+1).$$

Example 5.7 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{1}{s(s+2)^3}$; (ii) $\dfrac{54}{s^3(s-3)}$; (iii) $\dfrac{1}{s(s^2+4s+5)}$.

(iv) $\dfrac{1}{s^2(s^2+a^2)}$; (v) $\dfrac{1}{s^2}\left(\dfrac{s+1}{s^2+1}\right)$; (vi) $\dfrac{5s-2}{s^2(s-1)(s+2)}$;

(vii) $\dfrac{1}{(s+2)(s^2+4s+13)}$

Note ☑ All the problems in this example may be solved by resolving the given functions into partial fractions and applying elementary methods. However we shall solve them by applying the following working rule and its extensions.

$$L^{-1}\left\{\frac{1}{s}\phi(s)\right\} = \int_0^t L^{-1}(\phi(s)\,\mathrm{d}t \tag{1}$$

(i) $L^{-1}\left[\dfrac{1}{s(s+2)^3}\right] = \displaystyle\int_0^t L^{-1}\left\{\frac{1}{(s+2)^3}\right\}\mathrm{d}t$, by (1)

$$= \int_0^t e^{-2t}L^{-1}\left(\frac{1}{s^3}\right)\mathrm{d}t$$

$$= \int_0^t e^{-2t}\frac{1}{2}t^2\,\mathrm{d}t$$

$$= \frac{1}{2}\left[t^2\left(\frac{e^{-2t}}{-2}\right) - 2t\left(\frac{e^{-2t}}{4}\right) + 2\left(\frac{e^{-2t}}{-8}\right)\right]_0^t$$

by Bernoulli's formula.

$$= \frac{1}{2}\left[-e^{-2t}\left(\frac{t^2}{2}+\frac{t}{2}+\frac{1}{4}\right)+\frac{1}{4}\right]$$

$$= \frac{1}{8}[1-(2t^2+2t+1)e^{-2t}]$$

(ii) $L^{-1}\left[\dfrac{54}{s^3(s-3)}\right] = 54\int_0^t\int_0^t\int_0^t L^{-1}\left(\dfrac{1}{s-3}\right)\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$, by the extension of rule (1).

$$= 54\int_0^t\int_0^t\int_0^t e^{3t}\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$$

$$= 54\int_0^t\int_0^t\left(\frac{e^{3t}}{3}\right)_0^t\mathrm{d}t\,\mathrm{d}t$$

$$= 18\int_0^t\int_0^t(e^{3t}-1)\,\mathrm{d}t\,\mathrm{d}t$$

$$= 18\int_0^t\left(\frac{e^{3t}}{3}-t\right)_0^t\mathrm{d}t$$

$$= 6\int_0^t(e^{3t}-3t-1)\,\mathrm{d}t$$

$$= 6\left(\frac{e^{3t}}{3}-\frac{3t^2}{2}-t\right)_0^t$$

$$= 2e^{3t}-9t^2-6t-2.$$

Aliter

We can avoid the multiple integration by using the following alternative method.

$$L^{-1}\left[\frac{54}{s^3(s-3)}\right] = L^{-1}\left[\frac{54}{\left(\overline{s-3}+3\right)^3(s-3)}\right]$$

$$= 54e^{3t}L^{-1}\left\{\frac{1}{s(s+3)^3}\right\},$$

by the first shifting property.

$$= 54e^{3t}\cdot\int_0^t L^{-1}\left\{\frac{1}{(s+3)^3}\right\}\mathrm{d}t, \text{ by rule (1)}$$

$$= 54e^{3t}\int_0^t e^{-3t}\cdot L^{-1}\left\{\frac{1}{s^3}\right\}\mathrm{d}t$$

$$= 54e^{3t}\int_0^t \frac{1}{2}t^2\,e^{-3t}\,\mathrm{d}t$$

$$= 27e^{3t}\left[t^2\left(\frac{e^{-3t}}{-3}\right) - 2t\left(\frac{e^{-3t}}{9}\right) + 2\left(\frac{e^{-3t}}{-27}\right)\right]_0^t$$

$$= e^{3t}[2 - e^{-3t}(9t^2 + 6t + 2)]$$

$$= 2e^{3t} - 9t^2 - 6t - 2$$

(iii) $$L^{-1}\left\{\frac{1}{s(s^2+4s+5}\right\} = \int_0^t L^{-1}\left\{\frac{1}{s^2+4s+5}\right\}\mathrm{d}t,$$

by rule (1).

$$= \int_0^t L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}\mathrm{d}t$$

$$= \int_0^t e^{-2t}\sin t\,\mathrm{d}t$$

$$= \left[\frac{-e^{-2t}}{5}(2\sin t + \cos t)\right]_0^t$$

$$= \frac{1}{5}[1 - e^{-2t}(2\sin t + \cos t)]$$

(iv) $$L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \int_0^t\int_0^t L^{-1}\left(\frac{1}{s^2+a^2}\right)\mathrm{d}t\,\mathrm{d}t,$$

by the extension of rule (1).

$$= \int_0^t\int_0^t \frac{1}{a}\sin at\,\mathrm{d}t\,\mathrm{d}t$$

$$= \frac{1}{a}\int_0^t\left(\frac{-\cos at}{a}\right)_0^t\mathrm{d}t$$

$$= \frac{1}{a^2}\int_0^t(1-\cos at)\,\mathrm{d}t$$

$$= \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)_0^t$$

$$= \frac{1}{a^3}(at - \sin at).$$

(v) $$L^{-1}\left\{\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)\right\} = \int_0^t\int_0^t L^{-1}\left(\frac{s+1}{s^2+1}\right)\mathrm{d}t\,\mathrm{d}t,$$

by the extension of rule (1).

$$= \int_0^t \int_0^t (\cos t + \sin t)\, dt\, dt$$

$$= \int_0^t (\sin t - \cos t)_0^t\, dt$$

$$= \int_0^t (\sin t - \cos t + 1)\, dt$$

$$= (-\cos t - \sin t + t)_0^t$$

$$= 1 + t - \cos t - \sin t.$$

(vi) $$L^{-1}\left\{\frac{5s-2}{s^2(s-1)(s+2)}\right\} = \int_0^t \int_0^t L^{-1}\left\{\frac{5s-2}{(s-1)(s+2)}\right\} dt\, dt,$$

by the extension of rule (1).

$$= \int_0^t \int_0^t L^{-1}\left(\frac{1}{s-1} + \frac{4}{s+2}\right) dt\, dt,$$

by resolving the function into partial fractions.

$$= \int_0^t \int_0^t (e^t + 4e^{-2t})\, dt\, dt$$

$$= \int_0^t (e^t - 2e^{-2t})_0^t\, dt$$

$$= \int_0^t (e^t - 2e^{-2t} + 1)\, dt$$

$$= (e^t + e^{-2t} + t)_0^t = e^t + e^{-2t} + t - 2$$

(vii) $$L^{-1}\left[\frac{1}{(s+2)(s^2+4s+13)}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)\{(s+2)^2+9\}}\right]$$

$$= e^{-2t} \cdot L^{-1}\left[\frac{1}{s(s^2+9)}\right],$$ by the first shifting property.

$$= e^{-2t} \int_0^t L^{-1}\left(\frac{1}{s^2+9}\right) dt,$$ by rule (1),

$$= \frac{1}{3} e^{-2t} \int_0^t \sin 3t\, dt$$

$$= \frac{1}{3} e^{-2t} \left(\frac{-\cos 3t}{3} \right)_0^t$$

$$= \frac{1}{9} e^{-2t} (1 - \cos 3t)$$

Example 5.8 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{1}{(s^2+a^2)^2}$; (ii) $\dfrac{1}{s(s^2+a^2)^2}$; (iii) $\dfrac{1}{(s^2+2s+5)^2}$;

(iv) $\dfrac{1}{(s^2-4)^2}$; (v) $\dfrac{1}{s(s^2-4)^2}$; (vi) $\dfrac{1}{(s^2-2s-3)^2}$.

(i)
$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right]$$

$$= \int_0^t L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} \mathrm{d}t, \text{as}$$

$$L^{-1}\left\{\frac{1}{s}\phi(s)\right\} = \int_0^t L^{-1}\{\phi(s)\}\,\mathrm{d}t \quad (1)$$

$$= \int_0^t \frac{t}{2a} \sin at \,\mathrm{d}t$$

[Refer to Worked Example 13(i) in Section 5(b)]

$$= \frac{1}{2a}\left[t\left(\frac{-\cos at}{a}\right) - \left(\frac{-\sin at}{a^2}\right)\right]_0^t$$

$$= \frac{1}{2a^3}(\sin at - at\cos at) \quad (2)$$

(ii)
$$L^{-1}\left[\frac{1}{s(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s^2}\cdot\frac{s}{(s^2+a^2)^2}\right]$$

$$= \int_0^t\int_0^t L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} \mathrm{d}t\,\mathrm{d}t,$$

by the extension of rule (1)

$$= \int_0^t\int_0^t \frac{t}{2a}\sin at\,\mathrm{d}t\,\mathrm{d}t$$

$$= \int_0^t \frac{1}{2a^3}(\sin at - at\cos at)\,\mathrm{d}t, \text{ by } (2)$$

$$= \frac{1}{2a^3}\left[\frac{-\cos at}{a} - a\left\{t\left(\frac{\sin at}{a}\right) - \left(\frac{-\cos at}{a^2}\right)\right\}\right]_0^t$$

$$= \frac{1}{2a^4}\,(-2\cos at - at\sin at)_0^t$$

$$= \frac{1}{2a^4}\,(2 - 2\cos at - at\sin at).$$

(iii) $$L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\} = L^{-1}\left\{\frac{1}{\left[(s+1)^2+4\right]^2}\right\}$$

$$= e^{-t}\cdot L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= \frac{1}{16}\,e^{-t}\ (\sin 2t - 2t\cos 2t), \text{ by problem (i)}$$

(iv) $$L^{-1}\left\{\frac{1}{(s^2-4)^2}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{s}{(s^2-4)^2}\right\}$$

$$= \int_0^t L^{-1}\left\{\frac{s}{(s^2-4)^2}\right\}\mathrm{d}t, \text{ by rule (1)}$$

$$= \int_0^t \frac{t}{4}\sinh 2t\,\mathrm{d}t$$

[Refer to Worked Example 13 (ii) in Section 5(b)]

$$= \frac{1}{4}\left[t\left(\frac{\cosh 2t}{2}\right) - \left(\frac{\sinh 2t}{4}\right)\right]_0^t$$

$$= \frac{1}{16}\,(2t\cosh 2t - \sinh 2t) \qquad (3)$$

(v) $$L^{-1}\left\{\frac{1}{s(s^2-4)^2}\right\} = L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2-4)^2}\right\}$$

$$= \int_0^t\int_0^t L^{-1}\left\{\frac{s}{(s^2-4)^2}\right\}\mathrm{d}t\,\mathrm{d}t,$$

by the extension of rule (1).

$$= \int_0^t\int_0^t \frac{t}{4}\sinh 2t\,\mathrm{d}t\,\mathrm{d}t$$

$$= \int_0^t \frac{1}{16}(2t\cosh 2t - \sinh 2t)\,\mathrm{d}t, \text{ by (3)}$$

$$= \frac{1}{16}\left[2\left(t\frac{\sinh 2t}{2} - \frac{\cosh 2t}{4}\right) - \frac{\cosh 2t}{2}\right]_0^t$$

$$= \frac{1}{16}[t \sinh 2t - \cosh 2t]_0^t$$

$$= \frac{1}{16}(1 + t \sinh 2t - \cosh 2t)$$

(vi) $$L^{-1}\left\{\frac{1}{(s^2 - 2s - 3)^2}\right\} = L^{-1}\left[\frac{1}{\{(s-1)^2 - 4\}^2}\right]$$

$$= e^t \cdot L^{-1}\left[\frac{1}{(s^2 - 4)^2}\right],$$

by the first shifting property.

$$= \frac{1}{16} e^t (2t \cosh 2t - \sinh 2t),$$ by problem (iv).

Example 5.9

(i) Verify the initial and final value theorems when (*a*) $f(t) = (t + 2)^2 e^{-t}$; (*b*) $f(t) = L^{-1}\left\{\frac{1}{s(s+2)^2}\right\}$

(ii) If $L(e^{-t} \cos^2 t) = \phi(s)$, find $\lim_{s \to 0} [s\phi(s)]$ and $\lim_{s \to \infty} [s\phi(s)]$.

(iii) If $L\{f(t)\} = \frac{1}{s(s+1)(s+2)}$, find $\lim_{t \to 0} [f(t)]$ and $\lim_{t \to \infty} [f(t)]$.

(i) (a) $f(t) = (t^2 + 4t + 4)e^{-t}$

$$\therefore \quad \phi(s) = L\{f(t)\} = \frac{2}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4}{s+1}$$

$$\therefore \quad s\phi(s) = \frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}$$

Now $\lim_{t \to 0} [f(t)] = 4$ and $\lim_{s \to \infty} [s\phi(s)] = 0 + 0 + 4 = 4$

Hence the initial value theorem is verified.

Also $\lim_{t \to \infty} [f(t)] = 0$ and $\lim_{s \to 0} [s\phi(s)] = 0$

Hence the final value theorem is verified.

(i) (b) $$f(t) = L^{-1}\left\{\frac{1}{s(s+2)^2}\right\} = \int_0^t L^{-1}\left\{\frac{1}{(s+2)^2}\right\} dt$$

$$= \int_0^t te^{-21}\, dt$$

$$= \left[t\left(\frac{e^{-2t}}{-2}\right) - \left(\frac{e^{-2t}}{4}\right) \right]_0^t$$

$$= \frac{1}{4}(1 - 2t\,e^{-2t} - e^{-2t})$$

$$s\phi(s) = \frac{1}{(s+2)^2}$$

Now $\lim_{t \to 0}[f(t)] = 0 = \lim_{s \to \infty}[s\phi(s)]$

and $\lim_{t \to \infty}[f(t)] = \frac{1}{4} = \lim_{s \to 0}[s\phi(s)]$

Hence the initial and final value theorems are verified.

(ii) $L(e^{-t}\cos^2 t) = \phi(s)$

i.e., $f(t) = e^{-t}\cos^2 t$

By the final value theorem,

$$\lim_{s \to 0}[s\phi(s)] = \lim_{t \to \infty}[e^{-t}\cos^2 t] = 0.$$

By the initial value theorem,

$$\lim_{s \to \infty}[s\phi(s)] = \lim_{t \to 0}[e^{-t}\cos^2 t] = 1.$$

(iii) $L\{f(t)\} = \dfrac{1}{s(s+1)(s+2)}$

$\therefore$ $s\phi(s) = \dfrac{1}{(s+1)(s+2)}$

By the initial value theorem,

$$\lim_{t \to 0}[f(t)] = \lim_{s \to \infty}[s\phi(s)] = 0$$

By the final value theorem,

$$\lim_{t \to \infty}[f(t)] = \lim_{s \to 0}[s\phi(s)] = \frac{1}{2}$$

Example 5.10 Use convolution theorem to evaluate the following

(i) $\int_0^t u^2\, e^{-a(t-u)}\, du$; (ii) $\int_0^t \sin u \cos(t-u)\, du$.

(i) $\int_0^t u^2\, e^{-a(t-u)}\, du$ is of the form $\int_0^t f(u)\, g(t-u)\, du$, where $f(t) = t^2$ and $g(t) = e^{-at}$

i.e. $\displaystyle\int_0^t u^2\, e^{-a(t-u)}\, du = (t^2) * (e^{-at})$

$\therefore$ By convolution theorem,

$$L\left[\int_0^t u^2\, e^{-a(t-u)}\, du\right] = L(t)^2 \cdot L(e^{-at})$$

$$= \frac{2}{s^3} \cdot \frac{1}{s+a}$$

$$\int_0^t u^2\, e^{-a(t-u)}\, du = L^{-1}\left\{\frac{2}{s^3\,(s+a)}\right\}$$

$$= e^{-at} \cdot L^{-1}\left\{\frac{2}{s\,(s-a)^3}\right\}$$

$$= e^{-at} \int_0^t L^{-1}\left\{\frac{2}{(s-a)^3}\right\} d\,t$$

$$= e^{-at} \int_0^t t^2\, e^{at}\, dt$$

$$= e^{-at}\left[t^2\, \frac{e^{at}}{a} - 2t\, \frac{e^{at}}{a^2} + 2\, \frac{e^{at}}{a^3}\right]_0^t$$

$$= e^{-at}\left(\frac{t^2\, e^{at}}{a} - \frac{2t\, e^{at}}{a^2} + \frac{2e^{at}}{a^3} - \frac{2}{a^3}\right)$$

$$= \frac{1}{a^3}\,\{a^2 t^2 - 2at + 2 - 2e^{-at}\}$$

(ii) $\displaystyle\int_0^t \sin u \cos(t-u)\, du$ is of the form $\displaystyle\int_0^t f(u)\, g(t-u)\, du$,

where $f(t) = \sin t$ and $g(t) = \cos t$

i.e. $\displaystyle\int_0^t \sin u \cos(t-u)\, du = (\sin t) * (\cos t)$

$\therefore$ By convolution theorem,

$$L\left[\int_0^t \sin u \cos(t-u)\, du\right] = L(\sin t) \cdot L(\cos t)$$

$$= \frac{s}{(s^2+1)^2}$$

$$\therefore \quad \int_0^t \sin u \cos (t-u)\, du = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$= \frac{t}{2}\sin t, \text{ by Worked Example 13(i) of Section 5(b).}$$

Example 5.11 Use convolution theorem to find the inverse Laplace transforms of the following functions:

(i) $\frac{1}{(s+1)(s+2)}$; (ii) $\frac{s}{(s^2+a^2)^2}$; (iii) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$;

(iv) $\frac{4}{(s^2+2s+5)^2}$; (v) $\frac{s^2+s}{(s^2+1)(s^2+2s+2)}$.

(i) $$L^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s+2}\right\} = L^{-1}\left(\frac{1}{s+1}\right) * L^{-1}\left(\frac{1}{s+2}\right), \text{ by convolution theorem}$$

$$= e^{-t} * e^{-2t}$$

$$= \int_0^t e^{-u}\cdot e^{-2(t-u)}\, du$$

$$= e^{-2t}\int_0^t e^{u}\, du$$

$$= e^{-2t}(e^t-1) = e^{-t} - e^{-2t}$$

(ii) $$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{s}{s^2+a^2}\right\}$$

$$= L^{-1}\left(\frac{1}{s^2+a^2}\right) * L^{-1}\left(\frac{s}{s^2+a^2}\right), \text{ by convolution theorem}$$

$$= \left(\frac{1}{a}\sin at\right) * (\cos at)$$

$$= \int_0^t \frac{1}{a}\sin au \cos a(t-u)\, du$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin (2au - at)]\, du$$

$$= \frac{1}{2a}\left[(\sin at)\, u - \frac{\cos(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t \sin at - \frac{1}{2a}(\cos at - \cos at)\right]$$

$$= \frac{1}{2a} t \sin at.$$

(iii) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\}$

$$= L^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} * L^{-1}\left\{\frac{s}{s^2+b^2}\right\}, \text{ by convolution theorem}$$

$$= (\cos at) * (\cos bt)$$

$$= \int_0^t \cos au \cdot \cos b(t-u)\,\mathrm{d}u$$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u+bt] + \cos[(a+b)u-bt]\}\,\mathrm{d}u$$

$$= \frac{1}{2}\left[\frac{1}{a-b}\sin\{(a-b)u+bt + \frac{1}{a+b}\sin\{(a+b)u-bt\}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{a-b}(\sin at - \sin bt) + \frac{1}{a+b}(\sin at + \sin bt)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{a-b}+\frac{1}{a+b}\right)\sin at + \left(\frac{1}{a+b}+\frac{1}{a-b}\right)\sin bt\right]$$

$$= \frac{1}{2}\left[\frac{2a}{a^2-b^2}\sin at - \frac{2b}{a^2-b^2}\sin bt\right]$$

$$= \frac{1}{a^2-b^2}(a \sin at - b \sin bt).$$

(iv) $L^{-1}\left[\frac{4}{\left(s^2+2s+5\right)^2}\right] = L^{-1}\left[\frac{2}{\left(s^2+2s+5\right)}\cdot\frac{2}{\left(s^2+2s+5\right)}\right]$

$$= L^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} * L^{-1}\left\{\frac{2}{(s+1)^2+4}\right\},$$

by convolution theorem.

$$= (e^{-t}\sin 2t) * (e^{-t}\sin 2t)$$

$$= \int_0^t e^{-u}\sin 2u \cdot e^{-(t-u)} \cdot \sin 2(t-u)\,\mathrm{d}u$$

$$= \frac{1}{2} e^{-t} \int_0^t [\cos(4u - 2t) - \cos 2t]\, dv$$

$$= \frac{1}{2} e^{-t} \left[\frac{\sin(4u - 2t)}{4} - (\cos 2t) \cdot u \right]_0^t$$

$$= \frac{1}{2} e^{-t} \left[\frac{1}{4} (\sin 2t + \sin 2t) - t \cos 2t \right]$$

$$= \frac{1}{4} e^{-t} (\sin 2t - 2t \cos 2t).$$

(v) $$L^{-1} \left[\frac{s^2 + s}{(s^2+1)(s^2+2s+2)} \right] = L^{-1} \left[\frac{s+1}{s^2+2s+2} \cdot \frac{s}{s^2+1} \right]$$

$$= L^{-1} \left\{ \frac{s+1}{(s+1)^2 + 1} \right\} * L^{-1} \left(\frac{s}{s^2+1} \right)$$

$$= (e^{-t} \cos t) * (\cos t)$$

$$= \int_0^t e^{-u} \cos u \cos(t - u)\, du$$

$$= \frac{1}{2} \int_0^t e^{-u} [\cos t + \cos(2u - t)]\, du$$

$$= \frac{1}{2} \cos t \left(-e^{-u} \right)_0^t + \frac{1}{2} \cdot \frac{1}{5} \left[e^{-u} \{ -\cos(2u - t) + 2 \sin(2u - t) \} \right]_0^t$$

$$= \frac{1}{2} \cos t (1 - e^{-t}) + \frac{1}{10} \left[e^{-t} (2 \sin t - \cos t) + (2 \sin t + \cos t) \right]$$

$$= \frac{1}{5} e^{-t} (\sin t - 3 \cos t) + \frac{1}{5} (\sin t + 3 \cos t).$$

EXERCISE 5(c)

Part A

(Short Answer Questions)

1. State the relation between the Laplace transforms of $f(t)$ and $f'(t)$. Under what conditions does this relation hold good?
2. Express $L^{-1}\{s\phi(s)\}$ interms of $L^{-1}\{\phi(s)\}$. State the condition for the validity of your answer.
3. Express $L\left[\int_0^t \int_0^t f(t)\, dt\, dt \right]$ in terms of $L\{f(t)\}$.

4. State the relation between $L^{-1}\{\phi(s)\}$ and $L^{-1}\left\{\frac{1}{s^2}\phi(s)\right\}$.
5. State the initial value theorem in Laplace transforms.
6. State the final value theorem in Laplace transforms.
7. Define the convolution product of two functions and prove that it is commutative.
8. Verify whether $1 * g(t) = g(t)$, when $g(t) = t$.
9. State convolution theorem in Laplace transforms.

Using the Laplace transforms of the derivatives find the Laplace transforms of the following functions:

10. e^{at} 11. $\cos at$ 12. $\sin^2 t$

Find the inverse Laplace transforms of the following functions:

13. $\dfrac{s}{(s+2)^3}$ 14. $\dfrac{s^2}{(s-1)^3}$

15. $\dfrac{s}{(s-a)^2+b^2}$ 16. $\dfrac{s}{(s+1)(s+2)}$

Find the Laplace transforms of the following functions:

17. $\displaystyle\int_0^t \frac{\sin t}{t}\,dt$ 18. $\displaystyle\int_0^t \frac{1-e^t}{t}\,dt$ 19. $\displaystyle\int_0^t \frac{1-2\cos t}{t}\,dt$

20. $\displaystyle\int_0^t t e^{-t}\,dt$ 21. $\displaystyle\int_0^t e^{-t}\sin t\,dt$ 22. $\displaystyle\int_0^t t\,\sin t\,dt$

Find the inverse Laplace transforms of the following functions:

23. $\dfrac{1}{s(s+a)}$ 24. $\dfrac{1}{s^2(s+1)}$

25. $\dfrac{1}{s(s^2-a^2)}$ 26. $\dfrac{1}{s(s^2+1)}$

27. If $L\{f(t)\} = \dfrac{s+3}{(s+1)(s+2)}$, find $\lim\limits_{t\to 0}(f(t)\}$ and $\lim\limits_{t\to\infty}\{f(t)\}$.

28. If $L^{-1}\{\phi(s)\} = \frac{1}{2}(1-2e^{-t}+e^{-2t})$, find $\lim\limits_{s\to 0}(s\phi(s)\}$ and $\lim\limits_{s\to\infty}\{s\phi(s)\}$.

29. Show that $1*1*1*\ldots\ldots*1$ (n times) $= \dfrac{t^{n-1}}{(n-1)!}$, where * denotes Convolution.

30. If $L\{f(t)\} = \dfrac{1}{\sqrt{s^2+1}}$, evaluate $\displaystyle\int_0^t f(u)f(t-u)\,du$.

Use convolution theorem to find the inverse Laplace transforms of the following functions:

31. $\dfrac{1}{(s-1)(s+3)}$ 32. $\dfrac{1}{(s-1)^2}$

33. $\dfrac{1}{s(s^2+a^2)}$ 34. $\dfrac{1}{s^2(s+1)}$

35. $2/(s+1)(s^2+1)$.

Part B

36. Using the Laplace transforms of the derivatives find $L(t \sin a t)$ and hence find $L(2 \cos at - a t \sin a t)$ and $L(\sin a t + a t \cos a t)$.
37. Using the Laplace transforms of the derivatives, find $L(t \cosh a t)$ and hence find $L(\sinh a t + a t \cosh a t)$ and $L(a t \cosh a t - \sinh a t)$.
38. Find $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$ and hence find $L^{-1}\left\{\dfrac{s}{(s+a)(s+b)}\right\}$
39. Find $L^{-1}\left\{\dfrac{1}{(s-1)(s-2)(s-3)}\right\}$ and hence find $L^{-1}\left\{\dfrac{s^2}{(s-1)(s-2)(s-3)}\right\}$.
40. Find $L^{-1}\left\{\dfrac{1}{(s^2+a^2)(s^2+b^2)}\right\}$ and hence find $L^{-1}\left\{\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right\}$ and $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$
41. Given that $L^{-1}\left\{\dfrac{s}{(s^2+4)^2}\right\} = \dfrac{t}{4}\sin 2t$, find $L^{-1}\left\{\dfrac{s^2}{(s^2+4)^2}\right\}$, $L^{-1}\left\{\dfrac{s^3}{(s^2+4)^2}\right\}$ and $L^{-1}\left\{\dfrac{(s+3)^2}{(s^2+6s+13)^2}\right\}$
42. Given that $L^{-1}\left\{\dfrac{s}{(s^2-a^2)^2}\right\} = \dfrac{t}{2a}\sinh at$, find $L^{-1}\left\{\dfrac{s^2}{(s^2-a^2)^2}\right\}$, $L^{-1}\left\{\dfrac{s^3}{(s^2-a^2)^2}\right\}$ and $L^{-1}\left\{\dfrac{s+a}{s(s+2a)}\right\}^2$.
43. Given that $L^{-1}\left\{\dfrac{1}{s^4+4}\right\} = \dfrac{1}{4}(\sin t \cosh t - \cos t \sinh t)$, find $L^{-1}\left\{\dfrac{s}{s^4+4}\right\}$,

$L^{-1}\left\{\dfrac{s^2}{s^4+4}\right\}$ and $L^{-1}\left\{\dfrac{s^3}{s^4+4}\right\}$.

Find the Laplace transforms of the following functions:

44. $\displaystyle\int_0^t t\, e^t \sin t \, dt$

45 . $\displaystyle e^t \int_0^t t \sin t \, dt$

46. $\displaystyle t\int_0^t e^t \sin t \, dt$

47. $\displaystyle\int_0^t \frac{e^{-2t} \sin 3t}{t} dt$

48. $\displaystyle e^{-2t}\int_0^t \frac{\sin 3t}{t} dt$

49. $\displaystyle \frac{1}{t}\int_0^t e^{-2t} \sin 3t \, dt$

Find the inverse Laplace transforms of the following functions:

50. $\displaystyle\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)$ $\left[\textbf{Hint}\text{: Consider the function as } \dfrac{s-1}{s(s^2+s)}\right]$

51. $\displaystyle\frac{4s+7}{s^2\,(2s+3)\,(3s+5)}$

52. $\displaystyle\frac{1}{s\,(s^2+6s+25)}$

53. $\displaystyle\frac{1}{(s+1)\,(s^2+2s+2)}$

54. $\displaystyle\frac{1}{s^2}\left(\frac{s-2}{s^2+4}\right)$

55. $\displaystyle\frac{1}{(s^2+9)^2}$

56. $\displaystyle\frac{1}{s\,(s^2+9)^2}$

57. $\displaystyle\frac{1}{(s^2+6s+10)^2}$

58. $\displaystyle\frac{1}{(s^2-a^2)^2}$

59. $\displaystyle\frac{1}{s\,(s^2-a^2)^2}$

60. $\displaystyle\frac{1}{(s^2+4s)^2}$

Verify the initial and final value theorems when

61. $f(t) = (2t+3)^2 e^{-4t}$

62. $f(t) = L^{-1}\left\{\dfrac{1}{s(s+4)^3}\right\}$

63. Use convolution theorem to evaluate

$$\int_0^t e^{-u} \sin (t-u)\, du$$

64. Evaluate $\displaystyle\int_0^t \cos a\,u \cosh a\,(t-u)\, d\,u$, using convolution theorem.

Use convolution theorem to find the inverse of the following functions:

65. $\displaystyle\frac{s}{(s^2+4)\,(s^2+9)}$

66. $\displaystyle\frac{s^2}{(s^2+a^2)^2}$

67. $\displaystyle\frac{1}{(s^2+4)^2}$

68. $\displaystyle\frac{10}{(s+1)\,(s^2+4)}$

69. $\dfrac{1}{s^2(s+1)^3}$ 70. $\dfrac{1}{s^4+4}$

5.11 SOLUTIONS OF DIFFERENTIAL AND INTEGRAL EQUATIONS

As mentioned in the beginning, Laplace transform technique can be used to solve differential (both ordinary and partial) and integral equations. We shall apply this method to solve only ordinary linear differential equations with constant coefficients and a few integral and intergo-differential equations. The advantage of this method is that it gives the particular solution directly. This means that there is no need to first find the general solution and then evaluate the arbitrary constants as in the classical approach.

5.11.1 Procedure

1. We take the Laplace transforms of both sides of the given differential equation in $y\,(t)$, simultaneously using the given initial conditions. This gives an algebraic equation in $\bar{y}\,(s) = L\{y\,(t)\}$.

Note☑ $L\{y^{(n)}(t)\} = s^n\ \bar{y}\,(s) - s^{n-1}y(0) - s^{n-2}y'(0) \ldots\ldots\ y^{(n-1)}\ (0).$

2. We solve the algebraic equation to get $\bar{y}\,(s)$ as a function of s.
3. Finally we take $L^{-1}\ \{\bar{y}\,(s)\}$ to get $y(t)$. The various methods we have discussed in the previous sections will enable us to find $L^{-1}\ \{\bar{y}\,(s)\}$.
   The procedure is illustrated in the worked examples given below:

WORKED EXAMPLE 5(d)

Example 5.1 Using Laplace transform, solve the following equation

$L\,\dfrac{\mathrm{d}i}{\mathrm{d}t} + Ri = E\,e^{-at}$; $i\,(0) = 0$, where L, R, E and a are constants.

Taking Laplace transforms of both sides of the given equation, we get,

$$L \cdot L\{i'(t)\} + RL\{i\,(t)\} = EL\ \{e^{-at}\}$$

i.e., $$L\{s\,\bar{i}\ (s) - i\,(0)\} + R\bar{i}\,(s) = \frac{E}{s+a},\ \text{where } \bar{i}\,(s) = L\{i\,(t)\}$$

i.e., $$(Ls + R)\,\bar{i}\ (s) = \frac{E}{s+a}$$

$\therefore$ $$\bar{i}\,(s) = \frac{E}{(s+a)\,(Ls+R)}$$

$$= E\left\{\frac{\left(\dfrac{1}{R-aL}\right)}{s+a}+\frac{\left(\dfrac{1}{aL-R}\right)}{s+R/L}\right\}$$

Taking inverse Laplace transforms

$$i\,(t) = \frac{E}{R-aL}\left[L^{-1}\left\{\frac{1}{s+a}\right\}-L^{-1}\left\{\frac{1}{s+R/L}\right\}\right]$$

$$= \frac{E}{R-aL}\,(e^{-at}-e^{-Rt/L})$$

Example 5.2 Solve $y''-4y'+8y=e^{2t}$, $y(0)=2$ and $y'(0)=-2$.
Taking Laplace transforms of both sides of the given equation, we get

$$[s^2\,\bar{y}(s)-s\,y(0)-y'(0)]-4[s\,\bar{y}(s)-y(0)]+8\bar{y}(s)=\frac{1}{s-2}$$

i.e., $$(s^2-4s+8)\,\bar{y}(s)=\frac{1}{s-2}+(2s-10)$$

$\therefore$ $$\bar{y}(s)=\frac{1}{(s-2)\left(s^2-4s+8\right)}+\frac{2s-10}{s^2-4s+8}$$

$$=\frac{A}{s-2}+\frac{Bs+C}{s^2-4s+8}+\frac{2s-10}{s^2-4s+8}$$

$$=\frac{\frac{1}{4}}{s-2}+\frac{-\frac{1}{4}s+\frac{1}{2}}{s^2-4s+8}+\frac{2s-10}{s^2-4s+8}$$

$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{7}{4}s-\frac{19}{2}}{s^2-4s+8}$$

$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$$

$$y=\frac{1}{4}L^{-1}\left(\frac{1}{s-2}\right)+e^{2t}L^{-1}\left\{\frac{\frac{7}{4}s-6}{s^2+4}\right\}$$

$$=\frac{1}{4}e^{2t}+e^{2t}\left(\frac{7}{4}\cos 2t-3\sin 2t\right)$$

$$=\frac{1}{4}e^{2t}\,(1+7\cos 2t-12\sin 2t)$$

Example 5.3 Solve $y'' - 2y' + y = (t+1)^2$, $y(0) = 4$ and $y'(0) = -2$.
Taking Laplace transforms of both sides of the given equation, we get,

$$[s^2\bar{y}(s) - sy(0) - y'(0)] - 2[s\bar{y}(s) - y(0)] + \bar{y}(s) = L(t^2 + 2t + 1)$$

i.e., $$(s^2 - 2s + 1)\,\bar{y}(s) - 4s + 10 = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

i.e., $$\bar{y}(s) = \frac{4s-10}{(s-1)^2} + \frac{1}{s(s-1)^2} + \frac{2}{s^2(s-1)^2} + \frac{2}{s^3(s-1)^2}$$

$$= \frac{4}{s-1} - \frac{6}{(s-1)^2} + \frac{1}{s(s-1)^2} + \frac{2}{s^2(s-1)^2} + \frac{2}{s^3(s-1)^2}$$

$$\therefore\ y = 4e^t - 6te^t + \int_0^t te^t\,dt + 2\int_0^t\int_0^t te^t\,dt\,dt + 2\int_0^t\int_0^t\int_0^t te^t\,dt\,dt\,dt$$

$$= 4e^t - 6te^t + (te^t - e^t + 1) + 2\int_0^t (te^t - e^t + 1)\,dt + 2\int_0^t\int_0^t (te^t - e^t + 1)\,dt\,dt$$

$$= 3e^t - 5te^t + 1 + 2(te^t - 2e^t + t + 2) + 2\int_0^t (te^t - 2e^t + t + 2)\,dt$$

$$= -e^t - 3te^t + 2t + 5 + 2\left(te^t - 3e^t + \frac{t^2}{2} + 2t + 3\right)$$

$$= -7e^t - te^t + t^2 + 6t + 11$$

Example 5.4 Solve $y'' + 4y = \sin \omega t$, $y(0) = 0$ and $y'(0) = 0$.

Taking Laplace transforms of both sides of the equation, we get

$$[s^2\bar{y}(s) - sy(0) - y'(0)] + 4\bar{y}(s) = \frac{\omega}{s^2 + \omega^2}$$

$$\therefore \qquad \bar{y}(s) = \frac{\omega}{(s^2+4)(s^2+\omega^2)} \tag{1}$$

$$= \frac{\omega}{\omega^2 - 4}\left(\frac{1}{s^2+4} - \frac{1}{s^2+\omega^2}\right)$$

Inverting, we have,

$$y = \frac{1}{\omega^2 - 4}\left(\frac{\omega}{2}\sin 2t - \sin \omega t\right), \text{ if } \omega \neq 2.$$

If $\omega = 2$, from (1), we have

$$\bar{y}(s) = \frac{2}{(s^2+4)^2}$$

$$\therefore \qquad y = 2L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= \frac{1}{8}(\sin 2t - 2t\cos 2t)$$

[Refer to Worked Example 8(i) in Section 5(c).

Example 5.5 Solve the equation $y'' + y' - 2y = 3 \cos 3t - 11 \sin 3t$, $y(0) = 0$ and $y'(0) = 6$.

Taking Laplace transforms and using the giving initial conditions, we get

$$(s^2 + s - 2)\,\bar{y}(s) = \frac{3s}{s^2+9} - \frac{33}{s^2+9} + 6$$

$$= \frac{6s^2 + 3s + 21}{s^2+9}$$

$$\therefore \quad \bar{y}(s) = \frac{6s^2+3s+21}{(s^2+9)(s+2)(s-1)},$$

$$= \frac{As+B}{s^2+9} + \frac{C}{s+2} + \frac{D}{s-1}$$

$$= \frac{3}{s^2+9} - \frac{1}{s+2} + \frac{1}{s-1} \quad \text{by the usual procedure}$$

$$\therefore \quad y = \sin 3t - e^{-2t} + e^t.$$

Example 5.6 Solve the equation $(D^2 + 4D + 13)\,y = e^{-t}\sin t$, $y = 0$ and $D\,y = 0$ at $t = 0$, where $D \equiv \frac{d}{dt}$.

Taking Laplace transforms and using the given initial conditions, we get

$$(s^2 + 4s + 13)\,\bar{y}(s) = \frac{1}{s^2+2s+2}$$

$$\therefore \quad \bar{y}(s) = \frac{1}{(s^2+2s+2)(s^2+4s+13)}$$

$$= \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+4s+13}$$

$$= \frac{1}{85}\left[\frac{-2s+7}{s^2+2s+2} + \frac{2s-3}{s^2+4s+13}\right],$$

on finding the constants A, B, C, D by the usual procedure.

$$= \frac{1}{85}\left[\frac{-2(s+1)+9}{(s+1)^2+1} + \frac{2(s+2)-7}{(s+2)^2+9}\right]$$

$$\therefore \quad y = \frac{1}{85}\left[e^{-t}\{-2\cos t + 9\sin t\} + e^{-2t}\left\{2\cos 3t - \frac{7}{3}\sin 3t\right\}\right]$$

Example 5.7 Solve the equation $(D^2 + 6D + 9)\,x = 6\,t^2 e^{-3t}$, $x = 0$ and $Dx = 0$ at $t = 0$.

Taking Laplace transforms and using the initial conditions, use get,

$$(s^2 + 6s + 9)\,\bar{x}(s) = \frac{12}{(s+3)^3}$$

$$\therefore \qquad \bar{x}(s) = \frac{12}{(s+3)^5}$$

$$\therefore \qquad x = e^{-3t} \cdot L^{-1}\frac{12}{s^5}.$$

$$= \frac{1}{2}t^4\,e^{-3t}$$

Example 5.8 Solve the equation $y'' + 9y = \cos 2t$, $y(0) = 1$ and $y(\pi/2) = -1$.

Note ☑ In all the problems discussed above the values of y and y' at $t = 0$ were given. Hence they are called initial conditions. In fact, the differential equation with such initial conditions is called an *initial value problem.*

But in this problem, the value of y at $t = 0$ and $t = \pi/2$ are given. Such conditions are called boundary conditions and the differential equation itself is called a *boundary value problem.*

As $y'(0)$ is not given, it will be assumed as a constant, which will be evaluated towards the end using the condition $y(\pi/2) = -1$.

Taking Laplace transforms, we get

$$s^2\bar{y}(s) - s\,y(0) - y'(0) + 9\,\bar{y}(s) = \frac{s}{s^2+4}$$

i.e. $$(s^2 + 9)\,\bar{y}(s) = \frac{s}{s^2+4} + s + A, \text{ where } A = y'(0).$$

$$\therefore \qquad \bar{y}(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{1}{5}\left\{\frac{s}{s^2+4} - \frac{s}{s^2+9}\right\} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$\therefore \qquad y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$$

Given $$y\left(\frac{\pi}{2}\right) = -1$$

i.e. $$-1 = -\frac{1}{5} - \frac{A}{3} \quad \therefore A = \frac{12}{5}$$

$$\therefore \qquad y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t\,.$$

Example 5.9 Find the general solution of the following equation

$$y'' - 2ky' + k^2y = f(t).$$

Taking Laplace transforms, we get

$$[s^2\bar{y}(s) - sy(0) - y'(0)] - 2k[s\bar{y}(s) - y(0)] + k^2\bar{y}(s) = \bar{f}(s)$$

i.e. $$(s^2 - 2ks + k^2)\,\bar{y}(s) = sy(0) + [y'(0) - 2ky(0)] + \bar{f}(s)$$

i.e. $$(s-k)^2\,\bar{y}(s) = As + B + \bar{f}(s)$$

[As y (0) and y'(0) are not given, they are assumed as arbitrary contants]

$\therefore$ $$\bar{y}(s) = \frac{As}{(s-k)^2} + \frac{B}{(s-k)^2} + \frac{\bar{f}(s)}{(s-k)^2}$$

$$= \frac{A(s-k) + (Ak+B)}{(s-k)^2} + \frac{\bar{f}(s)}{(s-k)^2}$$

$$= \frac{C_1}{s-k} + \frac{C_2}{(s-k)^2} + \frac{\bar{f}(s)}{(s-k)^2}$$, where $C_1 = A$ and $C_2 = Ak + B$

$\therefore$ $$y = C_1 e^{kt} + C_2\, t\, e^{kt} + L^{-1}\left\{\bar{f}(s)\cdot\frac{1}{(s-k)^2}\right\}$$

i.e. $$y = (C_1 + C_2 t)\, e^{kt} + f(t) * t\, e^{kt}$$

i.e. $$y = (C_1 + C_2 t)\, e^{kt} + \int_0^t f(t-u)\, u e^{ku}\, du.$$

Example 5.10 Solve the equation $(D^3 + D)\,x = 2$, $x = 3$, $D\,x = 1$ and $D^2x = -2$ at $t = 0$.

Taking Laplace transforms and using the initial conditions, we get

$$(s^3 + s)\,\bar{y}(s) = 3s^2 + s + 1 + \frac{2}{s}$$

$\therefore$ $$\bar{y}(s) = \frac{3s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s\left(s^2+1\right)} + \frac{2}{s^2\left(s^2+1\right)}$$

$\therefore$ $$y = 3\cos t + \sin t + \int_0^t \sin t\, dt + 2\int_0^t \int_0^t \sin t\, dt\, dt$$

$$= 3\cos t + \sin t + (1 - \cos t) + 2\int_0^t (1 - \cos t)\, dt$$

$$= 2\cos t + \sin t + 1 + 2(t - \sin t)$$

$$= 2\cos t - \sin t + 1 + 2t.$$

Example 5.11 Solve the equation $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} = 0$, $y = \frac{dy}{dx} = 2$ and $\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 1$ at $x = 0$.

Note ☑ Change in the independent variable from t to x makes no difference in the procedure.)

Taking Laplace transforms and using the initial conditions, we get

$$(s^4 - s^3)\bar{y}(s) = 2s^3 - s$$

$$\therefore \qquad \bar{y}(s) = \frac{2s^2 - 1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}.$$

i.e. $$\bar{y}(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s-1}$$

$$\therefore \qquad y = 1 + t + e^t$$

Example 5.12 Solve the simultaneous differential equation $\frac{dy}{dt} + 2x = \sin 2t$ and $\frac{dy}{dt} + 2x = \cos 2t$, $x(0) = 1$, $y(0) = 0$.

Taking Laplace transforms of both sides of the given equation and using the given initial conditions, we get

$$s\bar{y}(s) + 2\bar{x}(s) = \frac{2}{s^2 + 4} \qquad (1)$$

and $$s\bar{x}(s) - 2\bar{y}(s) = \frac{s}{s^2 + 4} + 1 \qquad (2)$$

Solving (1) and (2), we have

$$\bar{x}(s) = \frac{1}{s^2 + 4} + \frac{s}{s^2 + 4} \text{ and } \bar{y}(s) = -\frac{2}{s^2 + 4}$$

$$\therefore \qquad x = \frac{1}{2}\sin 2t + \cos 2t \text{ and } y = -\sin 2t.$$

Example 5.13 Solve the simultaneous equations

$2x' - y' + 3x = 2t$ and $x' + 2y' - 2x - y = t^2 - t$, $x(0) = 1$ and $y(0) = 1$.

Taking Laplace transforms of both the equations, we get

$$2[s\bar{x}(s) - 1] - [s\bar{y}(s) - 1] + 3\bar{x}(s) = \frac{2}{s^2} \text{ and}$$

$$[s\bar{x}(s)-1]+2[s\,\bar{y}(s)-1]-2\bar{x}(s)-\bar{y}(s)=\frac{2}{s^3}-\frac{1}{s^2}$$

i.e. $$(2s+3)\bar{x}(s)-s\bar{y}(s)=\frac{2}{s^2}+1 \qquad (1)$$

and $$(s-2)\,\bar{x}(s)+(2s-1)\,\bar{y}(s)=\frac{2}{s^3}-\frac{1}{s^2}+3 \qquad (2)$$

Solving (1) and (2), we have

$$\bar{x}(s)=\frac{3}{s(s+1)(5s-3)}+\frac{5s-1}{(s+1)(5s-3)}$$

$$=\left(\frac{-1}{s}+\frac{3/8}{s+1}+\frac{25/8}{5s-3}\right)+\left(\frac{3/4}{s+1}+\frac{5/4}{5s-3}\right)$$

$$=-\frac{1}{s}+\frac{9/8}{s+1}+\frac{7/8}{s-3/5}$$

∴ $$x=-1+\frac{9}{8}e^{-t}+\frac{7}{8}e^{3/5t} \qquad (3)$$

Eliminating y' from the given equations,
we get $5x'+4x-y=t^2+3t$

∴ $$y=5x'+4x-t^2-3t$$

$$=5\left(-\frac{9}{8}e^{-t}+\frac{21}{40}e^{\frac{3}{5}t}\right)+4\left(-1+\frac{9}{8}e^{-t}+\frac{7}{8}e^{\frac{3}{5}t}\right)$$

$-t^2-3t,$ on using (3)

i.e. $$y=-\frac{9}{8}e^{-t}+\frac{49}{8}e^{\frac{3}{5}t}-t^2-3t-4\,.$$

Example 5.14 Solve the simultaneous equations

$Dx+Dy=t$ and $D^2x-y=e^{-t}$; n$x=3$,
$Dx=-2$ and $y=0$ at $t=0$.

Taking Laplace transformed of both the equations, we get

$$s\bar{x}(s)-3+s\bar{y}(s)=\frac{1}{s^2} \text{ and}$$

$$s^2\bar{x}(s)-3s+2-\bar{y}(s)=\frac{1}{s+1}$$

i.e. $$\bar{x}(s)+\bar{y}(s)=\frac{1}{s^3}+\frac{3}{s} \qquad (1)$$

and $$s^2\bar{x}(s)-\bar{y}(s)=\frac{1}{s+1}+3s-2 \qquad (2)$$

Solving (1) and (2), we have

$$\bar{x}(s)=\frac{1}{(s+1)\left(s^2+1\right)}+\frac{3}{s\left(s^2+1\right)}+\frac{1}{s^3\left(s^2+1\right)}+\frac{3s-2}{s^2+1}$$

$$=\frac{\frac{1}{2}}{s+1}+\frac{\left(\frac{1}{2}-\frac{1}{2}s\right)}{s^2+1}+\frac{3}{s\left(s^2+1\right)}+\frac{1}{s^3\left(s^2+1\right)}+\frac{3s}{s^2+1}-\frac{2}{s^2+1}$$

$$=\frac{\frac{1}{2}}{s+1}-\frac{\frac{3}{2}}{s^2+1}+\frac{\frac{5}{2}s}{s^2+1}+\frac{3}{s\left(s^2+1\right)}+\frac{1}{s^3\left(s^2+1\right)}$$

$\therefore$
$$x=\frac{1}{2}e^{-t}-\frac{3}{2}\sin t+\frac{5}{2}\cos t+3\int_0^t \sin t\,\mathrm{d}t+\int_0^t\int_0^t\int_0^t \sin t\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$$

$$=\frac{1}{2}e^{-t}-\frac{3}{2}\sin t+\frac{5}{2}\cos t+3\left(1-\cos t\right)+\frac{t^2}{2}+\cos t-1$$

i.e.
$$x=\frac{1}{2.}e^{-t}-\frac{3}{2}\sin t+\frac{1}{2}\cos t+\frac{t^2}{2}+2 \tag{3}$$

$\therefore$
$$x''=\frac{1}{2}e^{-t}+\frac{3}{2}\sin t-\frac{1}{2}\cos t+1 \tag{4}$$

From the given second equation we have

$$y=x''-e^{-t}$$

$$=1-\frac{1}{2}e^{-t}+\frac{3}{2}\sin t-\frac{1}{2}\cos t.$$

Example 5.15 Solve the simultaneous equations

$D^2x - Dy = \cos t$ and $Dx + D^2y = -\sin t$; $x = 1$,
$Dx = 0$, $y = 0$, $Dy = 1$ at $t = 0$.

Taking the Laplace transforms of both equations, we get

$$s^2\bar{x}(s)-s\bar{y}(s)=\frac{s}{s^2+1}+s \tag{1}$$

and
$$s\bar{x}(s)+s^2\bar{y}(s)=2-\frac{1}{s^2+1} \tag{2}$$

Solving (1) and (2), we have

$$\bar{x}(s)=\frac{s}{s^2+1}+\frac{2}{s\left(s^2+1\right)}+\frac{s}{\left(s^2+1\right)^2}-\frac{1}{s\left(s^2+1\right)^2}$$

$$=\frac{s}{s^2+1}+\frac{2}{s\left(s^2+1\right)}+\frac{s^2-1}{s\left(s^2+1\right)^2}$$

and $$\bar{y}(s)=\frac{1}{s^2+1}-\frac{2}{\left(s^2+1\right)^2}$$

$\therefore$ $$x=\cos t+2\int_0^t \sin t\,\mathrm{d}t+\int_0^t\left\{L^{-1}\left(\frac{1}{s^2+1}\right)-2L^{-1}\frac{1}{\left(s^2+1\right)^2}\right\}\mathrm{d}t$$

$$=\cos t+2(1-\cos t)+\int_0^t(\sin t-\sin t+t\cos t)\,\mathrm{d}t$$

$$=1+t\sin t$$

and $$y=\sin t-2\times\frac{1}{2}(\sin t-t\cos t)$$

$$=t\cos t.$$

Example 5.16 Show that the solution of the equation

$L\frac{\mathrm{d}i}{\mathrm{d}t}+Ri+\frac{1}{C}\int_0^t i\,\mathrm{d}t=E, i(0)=0$ [where L, R, E are constants] is given by

$$i=\begin{cases}\frac{E}{\omega L}e^{-at}\sin\omega t, \text{if } \omega^2>0\\ \frac{E}{L}te^{-at}, \text{if } \omega=0\\ \frac{E}{kL}e^{-at}\sinh kt, \text{if } \omega^2<0\end{cases}$$

where $$a=\frac{R}{2L},\ \omega^2=\frac{1}{LC}-\frac{R^2}{4L^2}\quad\text{and}\quad k^2=-\omega^2.$$

Note ☑ The given equation is an integro-differential equation, as the unknown (dependent variable) i occurs within the integral and differential operations.]

Taking Laplace transforms of the given equation, we get

$$Ls\bar{i}(s)+R\bar{i}(s)+\frac{1}{Cs}\bar{i}(s)=\frac{E}{s}$$

i.e. $$\left(LCs^2+RCs+1\right)\bar{i}(s)=EC$$

$\therefore$ $$\bar{i}(s)=\frac{EC}{LCs^2+RCs+1}$$

$$=\frac{E}{L}\cdot\frac{1}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

$$= \frac{E}{L} \cdot \frac{1}{\left(s + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

$$= \frac{E}{L} \cdot \frac{1}{(s+a)^2 + \omega^2}, \text{if } \omega^2 > 0$$

$\therefore$ $$i(t) = \frac{E}{L\omega} \cdot e^{-at} \sin \omega t$$

If $\omega = 0$,

$$\bar{i}(s) = \frac{E}{L} \cdot \frac{1}{(s+a)^2}$$

$\therefore$ $$i(t) = \frac{E}{L} t e^{-at}$$

If $\omega^2 < 0$ and $\omega^2 = -k^2$,

$$\bar{i}(s) = \frac{E}{L} \cdot \frac{1}{(s+a)^2 - k^2}$$

$\therefore$ $$i(t) = \frac{E}{Lk} e^{-at} \sinh kt.$$

Example 5.17 Solve the simultaneous equations

$3x' + 2y' + 6x = 0$ and

$$y' + y + 3\int_0^t x \, dt = \cos t + 3 \sin t, \quad x(0) = 2 \quad \text{and} \quad y(0) = -3.$$

Taking Laplace transforms of both the equations, we get

$$3[s\bar{x}(s) - 2] + 2[s\bar{y}(s) + 3] + 6\bar{x}(s) = 0 \text{ and}$$

$$s\bar{y}(s) + 3 + \bar{y}(s) + \frac{3}{s}\bar{x}(s) = \frac{s}{s^2+1} + \frac{3}{s^2+1}$$

i.e. $$(3s+6)\bar{x}(s) + 2s\bar{y}(s) = 0 \quad (1)$$

and $$\frac{3}{s}\bar{x}(s) + (s+1)\bar{y}(s) = \frac{s+3}{s^2+1} - 3 \quad (2)$$

Solving (1) and (2) for $\bar{x}(s)$, we have

$$[3(s+2)(s+1) - 6]\bar{x}(s) = -\frac{2s(s+3)}{s^2+1} + 6s$$

i.e. $$\bar{x}(s) = -\frac{2}{3} \cdot \frac{1}{s^2+1} + \frac{2}{s+3}$$

$\therefore \qquad x = -\frac{2}{3}\sin t + 2e^{-3t}.$

Solving (1) and (2) for $\bar{y}(s)$, we have

$$\bar{y}(s) = \frac{3s^2+5s-2}{(s+3)\left(s^2+1\right)} = -\left[\frac{1}{s+3} + \frac{2s-1}{s^2+1}\right]$$

$\therefore \qquad y = -\,e^{-3t} - 2\cos t + \sin t.$

Example 5.18 Solve the integral equation

$$y(t) = \frac{t^2}{2} - \int_0^t u\, y(t-u)\,\mathrm{d}u$$

Noting that the integral in the given equation is a convolution type integral and taking Laplace transforms, we get

$$\bar{y}(s) = \frac{1}{s^3} - L(t)\cdot L\{y(t)\}$$

$$= \frac{1}{s^3} - \frac{1}{s^2}\,\bar{y}(s)$$

$\therefore \qquad \left(\frac{1+s^2}{s^2}\right)\bar{y}(s) = \frac{1}{s^3} \quad \text{or} \quad \bar{y}(s) = \frac{1}{s\left(1+s^2\right)}$

$\therefore \qquad y(t) = \int_0^t \sin t\,\mathrm{d}t$

$$= 1 - \cos t.$$

Example 5.19 Solve the integral equation

$$y(t) = a\sin t - 2\int_0^t y(u)\cos(t-u)\,\mathrm{d}u.$$

Taking Laplace transforms,

$$\bar{y}(s) = \frac{a}{s^2+1} - 2L\{y(t)\}\cdot L(\cos t)$$

i.e. $\qquad \bar{y}(s) = \frac{a}{s^2+1} - \frac{2s}{s^2+1}\cdot\bar{y}(s)$

i.e. $\qquad \frac{(s+1)^2}{s^2+1}\,\bar{y}(s) = \frac{a}{s^2+1}$

i.e. $$\bar{y}(s) = \frac{a}{(s+1)^2}$$

$\therefore$ $$y(t) = a\,t\,e^{-t}.$$

Example 5.20 Solve the integro-differential equation

$$y'(t) = t + \int_0^t y(t-u)\cos u\,\mathrm{d}u,\ y(0) = 4$$

Taking Laplace transforms,

$$s\,\bar{y}(s) - 4 = \frac{1}{s^2} + \frac{s}{s^2+1}\bar{y}(s)$$

i.e. $$s\left(1 - \frac{1}{s^2+1}\right)\bar{y}(s) = \frac{1}{s^2} + 4$$

i.e. $$\bar{y}(s) = \frac{(s^2+1)(1+4s^2)}{s^5}$$
$$= \frac{4}{s} + \frac{5}{s^3} + \frac{1}{s^5}$$

$\therefore$ $$y(t) = 4 + \frac{5}{2}t^2 + \frac{1}{24}t^4.$$

EXERCISE 5(d)

Part A

(Short Answer Questions)

Using Laplace transforms, solve the following equations:

1. $x' + x = 2\sin t,\ x(0) = 0$
2. $x' - x = e^t,\ x(0) = 0$
3. $y' - y = t,\ y(0) = 0$
4. $y' + y = 1,\ y(0) = 0$
5. $y + \int_0^t y(t)\,\mathrm{d}t = e^{-t}$
6. $x + \int_0^t x(u)\,\mathrm{d}u = t^2 + 2t$

7. $x+\int_0^t x(t)\,\mathrm{d}t=\cos t+\sin t$

8. $x-2\int_0^t x(t)\,\mathrm{d}t=1$

9. $y=1+2\int_0^t e^{-2u}\,y(t-u)\,\mathrm{d}u$

10. $y=1+\int_0^t y(u)\,\sin(t-u)\,\mathrm{d}u$

11. $f(t)=\cos t+\int_0^t e^{-u}\,f(t-u)\,\mathrm{d}u$

12. $y(t)=t+\int_0^t \sin u\,y(t-u)\,\mathrm{d}u$

Part B

Solve the following differential equations, using Laplace transforms:

13. $x'' + 3x' + 2x = 2(t^2 + t + 1)$, $x(0) = 2$, $x'(0) = 0$.
14. $y'' - 3y' - 4y = 2e^{-t}$, $y(0) = y'(0) = 1$.
15. $x'' + 4x' + 3x = 10 \sin t$, $x(0) = x'(0) = 0$.
16. $(D^2 + D - 2)\,y = 20 \cos 2t$, $y = -1$, $Dy = 2$ *at* $t = 0$
17. $x'' + 4x' + 5x = e^{-2t}(\cos t - \sin t)$, $x(0) = 1$, $x'(0) = -3$.
18. $y'' + 2y' + 2y = 8\,e^t \sin t$, $y(0) = y'(0) = 0$.
19. $x'' - 2x' + x = t^2\,e^t$, $x(0) = 2$, $x'(0) = 3$.
20. $y'' + y = t \cos 2t$, $y(0) = y'(0) = 0$.
21. $x'' + 9x = 18t$, $x(0) = 0$, $x(\pi/2) = 0$.
22. $y'' + 4y' = \cos 2t$, $y(\pi) = 0$, $y'(\pi) = 0$.
23. $(D^2 + a^2)\,x = f(t)$.
24. $(D^3 - D)\,y = 2 \cos t$, $x = 3$, $Dx = 2$, $D^2x = 1$ *at* $t = 0$.
25. $x''' - 3x'' + 3x' - x = 16\,e^{3t}$, $x(0) = 0$, $x'(0) = 4$, $x''(0) = 6$.
26. $(D^4 - a^4)\,y = 0$, $y(0) = 1$, $y'(0) = y''(0) = y'''(0) = 0$.

Solve the following simultaneous equations, using Laplace transforms:

27. $x' - y = e^t$; $y' + x = \sin t$, given that $x(0) = 1$ and $y(0) = 0$.
28. $x' - y = \sin t$; $y' - x = -\cos t$, given that $x = 2$ and $y = 0$ for $t = 0$.
29. $x' + 2x - y = -6t$; $y' - 2x + y = -30t$, given that $x = 2$ and $y = 3$ at $t = 0$.
30. $Dx + Dy + x - y = 2$; $D^2x + Dx - Dy = \cos t$, given that $x = 0$, $Dx = 2$ and $y = 1$ at $t = 0$.
31. $D^2x + y = -5 \cos 2t$; $D^2y + x = 5 \cos 2t$, given that $x = Dx = Dy = 1$ and $y = -1$ at $t = 0$.

32. $x' + y' - x = 2e^t + e^{-t}; 2x' + y + 3\int_0^t y \, dt = 2e^t(t+3)$, given that $x(0) = -1$ and $y(0) = 2$.

Solve the following integral equations, using Laplace transforms:

33. $x' + 3x + 2\int_0^t x \, dt = t, x(0) = 0.$

34. $y' + 4y + 5\int_0^t y \, dt = e^{-t}, y(0) = 0.$

35. $x' + 2x + \int_0^t x \, dt = \cos t, x(0) = 1.$

36. $y' + 4y + 13\int_0^t y \, dt = 3e^{-2t} \sin 3t, y(0) = 3.$

37. $x(t) = 4t - 3\int_0^t x(u) \sin(t-u) \, du$.

38. $y(t) = e^{-t} - 2\int_0^t y(u) \cos(t-u) \, du.$

39. $\int_0^t \frac{x(u)}{\sqrt{t-u}} \, du = 1 + t + t^2 \cdot$

40. $\int_0^t y(u)\, y(t-u) \, du = 2y(t) + t - 2$

ANSWERS

Exercise 5(a)

3. $\tan t; e^{t^2}$

9. $\frac{1}{s-2}\{1 - e^{-(s-2)}\}.$

10. $\left(\frac{1}{s} + \frac{1}{s^2}\right)e^{-s} - \left(\frac{2}{s} + \frac{1}{s^2}\right)e^{-2s}$

11. $(1 - e^{-\pi s}) \cdot \frac{2}{s^2 + 4}$

12. $(1 - e^{-2\pi s}) \cdot \frac{2}{s^2 + 1}.$

13. $\dfrac{s}{s^2+1}\cdot e^{-2\pi s/3}$.

14. $\dfrac{1}{s^2+1}(1+e^{-\pi s})+\left(\dfrac{\pi}{s}+\dfrac{1}{s^2}\right)e^{-\pi s}$.

15. $\dfrac{1}{2s}\sqrt{\dfrac{\pi}{s}};\ \dfrac{1}{\sqrt{s}}$.

16. $\dfrac{1}{s^2}e^{-3s};\ \dfrac{\pi}{s^2+\pi^2}e^{-s}$.

17. $\left(\dfrac{2}{s^3}+\dfrac{4}{s^2}+\dfrac{4}{s}\right)e^{-2s}$.

18. $\dfrac{6a^3}{s^4}+\dfrac{6a^2b}{s^3}+\dfrac{3ab^2}{s^2}+\dfrac{b^3}{s}$.

19. $\dfrac{1}{s^2+\omega^2}(\omega\cos\theta+s\sin\theta)$.

20. $\dfrac{1}{2}\left(\dfrac{1}{s}-\dfrac{s}{s^2+36}\right)$

21. $\dfrac{1}{4}\left(\dfrac{3s}{s^2+4}-\dfrac{s}{s^2+36}\right)$.

22. $\dfrac{1}{2}\left(\dfrac{3}{s^2+9}-\dfrac{1}{s^2+1}\right)$.

23. $\dfrac{1}{2}\left(\dfrac{s}{s^2+25}+\dfrac{s}{s^2+1}\right)$.

24. $\dfrac{1}{8}\left(\dfrac{1}{s-3}-\dfrac{1}{s-1}+\dfrac{3}{s+1}-\dfrac{1}{s+3}\right)$

25. $\dfrac{1}{4}\left(\dfrac{1}{s-2}+\dfrac{3}{s+2}+\dfrac{2}{s}\right)$.

26. $\dfrac{2}{(s+1)^3}+\dfrac{2}{(s+1)^2}+\dfrac{1}{s+1}$.

27. $\dfrac{s}{(s+2)^2+9}$.

28. $\dfrac{1}{4}\left\{\dfrac{3}{s-3}+\dfrac{1}{s+1}\right\}$.

29. $\dfrac{2s}{s^4+4}$

30. $\dfrac{1}{(s-1)^3}+\dfrac{1}{(s+1)^3}$

31. $\dfrac{2e^3}{2s-3}$

32. $(t-a)\,u_a(t)$.

33. $u_2(t)-u_3(t)$.

34. $e^{3(t-2)}.\ u_2(t)$

35. $\cos 3\,(t-1)\,u_1\,(t)$

36. $\sin t+\sin(t-\pi)\,u_\pi(t)$.

37. $2\sqrt{\dfrac{t}{\pi}}\,e^{-t}$.

38. $\dfrac{1}{2}(2+4t+3t^2)$.

39. $\dfrac{1}{96}t^3\,e^{3t/2}$

40. $\dfrac{1}{12}t^3\,(2+t)\,e^{2t}$.

41. $2\cos 2t+\dfrac{3}{2}\sin 2t$.

42. $\cosh 3t+2\sinh 3t$.

43. $\dfrac{1}{a}(1-e^{-at})$.

44. $\dfrac{1}{2}e^{-t}\sin 2t$.

45. $e^{3t}\cos t$.

46. $\dfrac{8}{(s-3)^3}+\dfrac{12}{(s-3)^2}+\dfrac{9}{s-3}$

47. $\sqrt{\dfrac{\pi}{s-a}}\cdot\dfrac{s}{s-a}$

48. $\dfrac{1}{8}\left\{\dfrac{1}{s-2}-\dfrac{3}{s}+\dfrac{3}{s+2}-\dfrac{1}{s+4}\right\}$.

49. $\dfrac{4a^3}{s^4+4a^4}$ 50. $\dfrac{2}{(s-2)\left(s^2-4s+8\right)}$.

51. $\dfrac{(s+3)}{4}\left\{\dfrac{3}{(s-3)^2+4}+\dfrac{1}{(s+3)^2+36}\right\}$

52. $\dfrac{\omega\cos\theta+(s+k)\sin\theta}{(s+k)^2\ \omega^2}$. 53. $\dfrac{3\left(s^2+2s+9\right)}{\left(s^2+2s+5\right)\left(s^2+2s+17\right)}$.

54. $\dfrac{(s-1)}{4}\left\{\dfrac{1}{(s-1)^2}+\dfrac{1}{(s-1)^2+4}+\dfrac{1}{(s-1)^2+16}+\dfrac{1}{(s+1)^2+36}\right.$.

55. $\dfrac{1}{4}\left\{\dfrac{1}{(s+2)^2}+\dfrac{3}{(s+2)^2+9}+\dfrac{5}{(s+2)^2+25}-\dfrac{9}{(s+2)^2+81}\right\}$.

56. $\dfrac{s^2-4}{\left(s^2+4\right)^2}$. 57. $\dfrac{s\,(s+2)}{\left(s^2+2s+2\right)^2}$

58. $\dfrac{6\,(s-2)}{\left(s^2-4s+13\right)^2}$ 59. $\dfrac{2s}{\left(s^2+1\right)^2}$.

60. $\dfrac{s^2-4}{\left(s^2+4\right)^2}$. 61. $\dfrac{1}{2}-2e^t+\dfrac{5}{2}e^{-2t}$.

62. $2e^{-t}-3e^t+5e^{2t}$. 63. $\left(1-5t+\dfrac{9}{2}t^2\right)e^{-t}$.

64. $-2e^{-t}+2e^{2t}+te^{2t}$. 65. $-2+t+2e^{-t}+t\,e^{-t}$.

66. $\sin t-\dfrac{1}{2}\sin 2t+\dfrac{1}{3}\sin 3t$. 67. $\cos t-2\cos\sqrt{2}\,t+\cos\sqrt{3}\,t$.

68. $e^{-3t}(2\cos 5t-3\sin 5t)$. 69. $\dfrac{1}{\sqrt{3}}e^{-t/2}\left(\sqrt{3}\cos\dfrac{\sqrt{3}}{2}t+\sin\dfrac{\sqrt{3}}{2}t\right.$.

70. $\dfrac{1}{a}e^{-bt/2a}\cdot\left\{\cos\left(\dfrac{\lambda t}{2a}\right)+\left(\dfrac{m}{l}-\dfrac{b}{2a}\right)\cdot\dfrac{2a}{\lambda}\sin\left(\dfrac{\lambda t}{2a}\right)\right.$.

if $b^2-4ac<0$ and $=-\lambda^2$.

$\dfrac{1}{a}e^{-bt/2a}\cdot\left\{\cosh\left(\dfrac{kt}{2a}\right)+\left(\dfrac{m}{l}-\dfrac{b}{2a}\right)\cdot\dfrac{2a}{k}\sinh\left(\dfrac{kt}{2a}\right)\right\}$,

if $b^2-4ac>0$ and $=k^2$.

$\dfrac{1}{a}e^{-bt/2a}\left\{1+\left(\dfrac{m}{l}-\dfrac{b}{2a}\right)t\right\}$, if $b^2-4ac=0$.

71. $2 + e^{-2t}(\cos 3t + 2\sin 3t)$.

72. $\frac{1}{3}\left[e^{-t} - e^{t/2}\left(\cos\frac{\sqrt{3}}{2}t - \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)\right]$.

73. $\frac{1}{4a^3}(\sin at \cosh at - \cos at \sinh at)$.

74. $\frac{1}{2}\sin t \sinh t$

75. $\frac{1}{4}(\sinh 2t \cos 2t + \cosh 2t \sin 2t)$.

76. $\cos\frac{t}{2}\cosh\frac{t}{2}$.

77. $\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t \cosh\frac{t}{2}$.

78. $\frac{1}{2a}t\sinh at$

79. $\frac{1}{2}(\sin t - t\cos t)$.

80. $\frac{1}{4}t\sin 2t$.

Exercise 5(b)

2. $\frac{1}{s^2} - \frac{1}{s(e^s - 1)}$.

7. $\frac{s}{(s^2 - a^2)^2}$.

8. $\frac{s^2 - a^2}{(s^2 + a^2)^2}$.

9. $\frac{2k^3}{(s^2 + k^2)^2}$.

10. $\frac{2ks^2}{(s^2 + k^2)^2}$.

11. $\frac{3s^2 + a^2}{s^2(s^2 + a^2)^2}$.

12. $\frac{s^3}{(s^2 + k^2)^2}$.

13. $\frac{2\sinh t}{t}$.

14. $\frac{1 - e^{-t}}{t}$

15. $\frac{1 - e^{-at}}{t}$.

16. $\frac{e^{-bt} - e^{-at}}{t}$

17. $\frac{e^t - 1}{t}$.

18. $\frac{2}{t}(\cos 2t - \cos t)$.

19. $\frac{\sin t}{t}$.

20. $\frac{\sin at}{t}$.

21. $\cot^{-1}(s/a)$.

22. $\log\left(\dfrac{s+1}{s}\right)$.

23. $\log\left(\dfrac{s-1}{s}\right)$.

24. $\dfrac{1}{2}\log\left(\dfrac{s^2+a^2}{s^2}\right)$.

25. $\dfrac{1}{4}\log\left(\dfrac{s^2+4}{s^2}\right)$.

26. $\dfrac{E\left(1-e^{-s/E}\right)}{s\left(1-e^{-2\pi s/n}\right)}$.

27. $\dfrac{E}{s}\tanh\left(\dfrac{sT}{4}\right)$.

28. $[1-e^{-2\pi(s-1)}]/(s-1)(1-e^{-2\pi s})$.

29. $\dfrac{2}{4s^2+1}\coth(\pi s)$.

30. $\dfrac{1}{s^2+\omega^2}\left\{s+\omega\ \text{cosech}\left(\dfrac{\pi s}{2\omega}\right)\right\}$.

31. $\dfrac{1}{1-e^{-2\pi s}}\left\{\dfrac{1}{s^2}\left(1-e^{-\pi s}\right)-\dfrac{\pi}{s}e^{-\pi s}\right\}$.

32. $1/(s^2+1)(1-e^{-\pi s})$.

33. $\omega/(s^2+\omega^2)(e^{\pi s/\omega}-1)$.

34. $\dfrac{1}{s^2}\tanh\left(\dfrac{\pi s}{2}\right)$.

35. $\left[\dfrac{1}{s^2}(e^{-\pi s}-1)^2+\dfrac{\pi}{s}e^{-\pi s}(e^{-\pi s}-1)\right]/(1-e^{-2\pi s})$

36. $\dfrac{1}{8}\left[\dfrac{1}{(s-3)^2}-\dfrac{3}{(s-1)^2}+\dfrac{3}{(s+1)^2}-\dfrac{1}{(s+3)^2}\right]$.

37. $\dfrac{1}{4}\left[\dfrac{3(s^2-4)}{(s^2+4)^2}+\dfrac{(s^2-36)}{(s^2+36)^2}\right]$

38. $\dfrac{1}{2}\left[\dfrac{(s^2-4)}{(s^2+4)^2}-\dfrac{(s^2-64)}{(s^2+64)^2}\right]$

39. $s\left[\dfrac{s}{(s^2+36)^2}+\dfrac{s}{(s^2+4)^2}\right]$.

40. $\dfrac{1}{s^3}-\dfrac{s(48-s^2)}{(s^2+16)^3}$.

41. $\dfrac{9(s^2-3)}{(s^2+9)^3}+\dfrac{1-3s^2}{(s^2+1)^3}$.

42. $\dfrac{18(s^2+4s+1)}{(s^2+4s+13)^3}$.

43. $\dfrac{s^2-6s-7}{(s^2-6s+25)^2}$.

44. $\dfrac{1}{(s+1)^3}+\dfrac{1}{(s+5)^3}$.

45. $3\left[\dfrac{s-2}{(s^2-4s+13)^2}-\dfrac{s+2}{(s^2+4s+13)^2}\right]$.

46. $\dfrac{2}{t}(1-\cos at)$

47. $\dfrac{2}{t}(e^{-bt}-\cos at)$.

48. $\dfrac{2}{t}(\cos t-e^{2t})$.

49. $\dfrac{2}{t^2}\sinh at-\dfrac{2a}{t}\cosh at+a\,\delta(t)$.

50. $\dfrac{1}{t}\sin\left(\dfrac{t}{2}\right)$.

51. $-\dfrac{1}{t}\cdot e^{-2t}\sin 3t$.

52. $-\dfrac{1}{t}e^{-bt}\sin at$.

53. $-\dfrac{2}{t}\sin\dfrac{t}{\sqrt{2}}\sinh\dfrac{t}{\sqrt{2}}$.

54. 0.

55. $\dfrac{1}{2}$

56. log 3.

57. $\dfrac{1}{2}\log 2$.

58. $\log\left(\dfrac{b}{a}\right)$.

59. $\dfrac{\pi}{8}$.

60. $\log\left(\dfrac{s+1}{s}\right)$.

61. $\dfrac{1}{2}\log\left(\dfrac{s^2+a^2}{s^2}\right)$.

62. $\dfrac{1}{4}\log\left(\dfrac{s^2+16}{s^2}\right)$.

63. $\dfrac{1}{4}\log\left(\dfrac{s^2+16}{s^2+4}\right)$.

64. $s\log\dfrac{s}{\sqrt{s^2+1}}+\cot^{-1}s$.

65. $\dfrac{t}{2}\sin t$.

66. $\dfrac{t}{2a}\sinh at$.

67. $\dfrac{t}{2}e^{2t}\sin t$.

68. $\dfrac{1}{a}\sin at-t\sin at$.

69. $e^{-3t}(1+t)\sin t$.

70. $\dfrac{t}{2}e^{-4t}\sinh t$.

Exercise 5(c)

8. No; $1*t=\dfrac{t^2}{2}\neq t$.

10. $\dfrac{1}{s-a}$.

11. $\dfrac{s}{s^2+a^2}$

12. $\dfrac{2}{s(s^2+4)}$.

13. $t(1-t)\,e^{-2t}$.

14. $e^t\left(\dfrac{t^2}{2}+2t+1\right)$.

15. $\dfrac{1}{b}e^{at}(a\sin bt+b\cos bt)$.

16. $2e^{-2t}-e^{-t}$.

17. $\frac{1}{s}\cot^{-1} s.$ 18. $\frac{1}{s}\log\left(\frac{s-1}{s}\right)$ 19. $\frac{1}{s}\log\left(\frac{s^2+1}{s}\right).$

20. $\frac{1}{s(s+1)^2}.$ 21. $\frac{1}{s(s^2+2s+2)}.$ 22. $\frac{2}{\left(s^2+1\right)^2}.$

23. $\frac{1}{a}(1-e^{-at}).$ 24. $e^{-at}+t-1.$ 25. $\frac{1}{a^2}(\cosh at-1).$

26. $1-\cos t.$ 27. 1;0. 28. $\frac{1}{2}$; 0 30. $\sin t$

31. $\frac{1}{4}(e^t-e^{-3t}).$ 32. $t\,e^t.$ 33. $\frac{1}{a^2}(1-\cos at).$

34. $e^{-t}+t-1.$ 35. $\sin t-\cos t+e^{-t}.$

36. $\frac{2as}{\left(s^2+a^2\right)^2};\frac{2s^3}{\left(s^2+a^2\right)^2};\frac{2as^2}{\left(s^2+a^2\right)^2}$

37. $\frac{s^2+a^2}{\left(s^2-a^2\right)^2};\frac{2as^2}{\left(s^2-a^2\right)^2};\frac{2a^3}{\left(s^2-a^2\right)^2}.$

38. $\frac{1}{a-b}(e^{-bt}-e^{-at});\ \frac{1}{a-b}(ae^{-at}-be^{-bt}).$

39. $\frac{1}{2}e^t-e^{2t}+\frac{1}{2}e^{3t};\frac{1}{2}e^t-4e^{2t}+\frac{9}{2}e^{3t}.$

40. $\frac{1}{a^2-b^2}\left(\frac{1}{b}\sin bt-\frac{1}{a}\sin at\right);\ \frac{1}{a^2-b^2}(\cos bt-\cos at);$

$\frac{1}{a^2-b^2}(a\sin at-b\sin bt).$

41. $\frac{1}{4}(\sin 2t+2t\sin 2t);\ (\cos 2t-t\sin 2t);\frac{t}{4}e^{-3t}\sin 2t.$

42. $\frac{1}{2a}(\sinh at+at\cosh at);\frac{1}{2}(at\sinh at+2\cosh at);\ \frac{t}{2a}e^{-at}\sinh at.$

43. $\frac{1}{2}\sin t\sinh t;\ \frac{1}{2}(\sin t\cosh t+\cos t\sinh t);\ \cos t\cosh t.$

44. $\frac{2(s-1)}{s\left(s^2-2s+2\right)^2}.$ 45. $\frac{2}{\left(s^2-2s+2\right)^2}.$ 46. $\frac{3s^2-4s+2}{s^2\left(s^2-2s+2\right)^2}.$

47. $\frac{1}{s}\cot^{-1}\left(\frac{s+2}{3}\right).$ 48. $\frac{1}{s+2}\cot^{-1}\left(\frac{s+2}{3}\right).$

49. $\frac{3}{26}\log\left(\frac{s^2+4s+13}{s^2}\right)-\frac{6}{13}\cot^{-1}\left(\frac{s+2}{3}\right).$

50. $2 - t - 2e^{-t}$.

51. $\frac{4}{9}e^{-3t/2} - \frac{3}{25}e^{-5t/3} + \frac{7}{15}t - \frac{73}{225}$.

52. $\frac{1}{100}[4 - e^{-3t}(3\sin 4t + 4\cos 4t)]$.

53. $e^{-t}(1 - \cos t)$.

54. $\frac{1}{4}(1 - 2t - \cos 2t + \sin 2t)$.

55. $\frac{1}{54}(\sin 3t - 3t\cos 3t)$.

56. $\frac{1}{162}(2 - 2\cos 3t - 3t\sin 3t)$.

57. $\frac{1}{2}e^{-3t}(\sin t - t\cos t)$.

58. $\frac{1}{2a^3}(at\cosh at - \sinh at)$.

59. $\frac{1}{2a^4}(2 - 2\cosh at + at\sinh at)$.

60. $\frac{1}{16}e^{-2t}(2t\cosh 2t - \sinh 2t)$.

63. $\frac{1}{2}(\cos t - \sin t - e^{-t})$.

64. $\frac{1}{2a}(\sin at + \sinh at)$.

65. $\frac{1}{5}(\cos 2t - \cos 5t)$

66. $\frac{1}{2a}(\sin at + at\cos at)$.

67. $\frac{1}{16}(\sin 2t - 2t\cos 2t)$.

68. $2e^{-t} + \sin 2t - 2\cos 2t$.

69. $\frac{e^{-t}}{2}(t^2 + 4t + 6) + t - 3$.

70. $\frac{1}{4}(\sin t\cosh t - \cos t\sinh t)$.

Exercise 5(d)

1. $x = e^{-t} - \cos t + \sin t$.
2. $x = te^{t}$.
3. $y = e^{t} - t - 1$.
4. $y = 1 - e^{-t}$.
5. $y = (1 - t)e^{-t}$.
6. $x = 2t$.
7. $x = \cos t$.
8. $x = e^{2t}$.
9. $y = 1 + 2t$.
10. $y = 1 + \frac{t^2}{2}$.
11. $f(t) = \cos t + \sin t$.
12. $y = t + \frac{t^3}{6}$.
13. $x = t^2 - 2t + 3 - e^{-2t}$.
14. $y = \frac{1}{25}(13e^{-t} - 10t\,e^{-t} + 12e^{4t})$
15. $x = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} - 2\cos t + \sin t$.
16. $y = \frac{2}{3}e^{-2t} + \frac{4}{3}e^{t} - 3\cos 2t + \sin 2t$.
17. $x = e^{-2t}\left[\cos t - \frac{3}{2}\sin t + \frac{t}{2}(\sin t + \cos t)\right]$

18. $y = 2\,(\sin t \cosh t - \cos t \sinh t)$.

19. $x = \left(\dfrac{t^4}{12} + t + 2\right) e^t$.

20. $y = -\dfrac{5}{9}\sin t + \dfrac{4}{9}\sin 2t - \dfrac{t}{3}\cos 2t$.

21. $x = 2t + \pi \sin 3t$.

22. $y = \dfrac{1}{4}(t - \pi)\sin 2t$.

23. $x = A\cos at + B\sin at + \dfrac{1}{a}\displaystyle\int_0^t \sin au \cdot f(t-u)\,\mathrm{d}u$.

24. $y = 3\sinh t + \cosh t - \sin t + 2$.

25. $x = 2e^{3t} - 5t^2 e^t - 2e^t$.

26. $y = \dfrac{1}{2}(\cos at + \cosh at)$.

27. $x = \dfrac{1}{2}(e^t + 2\sin t + \cos t - t\cos t)$;

$y = \dfrac{1}{2}(-e^{-t} - \sin t + \cos t + t\sin t)$.

28. $x = 2\cosh t$; $y = 2\sinh t - \sin t$.

29. $x = 1 + 2t - 6t^2 + e^{-3t}$; $y = 4 - 2t - 12t^2 - e^{-3t}$.

30. $x = t + \sin t$; $y = t + \cos t$.

31. $x = \sin t + \cos 2t$; $y = \sin t - \cos 2t$.

32. $x = t\,e^t - e^{-t}$; $y = e^t + e^{-t}$.

33. $x = \dfrac{1}{2}(1 + e^{-2t}) - e^{-t}$.

34. $y = -\dfrac{1}{2}e^{-t} + \dfrac{1}{2}e^{-2t}(\cos t + 3\sin t)$.

35. $x = \dfrac{1}{2}\{(1-t)e^{-t} + \cos t\}$.

36. $y = e^{-2t}\left\{3\cos 3t - \dfrac{7}{3}\sin 3t + \dfrac{3}{2}t\sin 3t + t\cos 3t\right\}$.

37. $x = t + \dfrac{3}{2}\sin 2t$.

38. $y(t) = e^{-t}(1-t)^2$.

39. $x(t) = \dfrac{1}{\pi}\left(t^{-1/2} + 2t^{1/2} + \dfrac{8}{3}t^{3/2}\right)$

40. $y(t) = 1$.