Area enclosed by plane curves - Cardioid.

What is a cardioid?

Cardioid is the path traced by a point on the perimeter of a circle that rolls around a fixed circle of the same radius.

<u>Equation of a cardioid.</u>

The polar form is usually used for simplicity.

- Horizontal Cardioids : $r = a(1 \pm \cos\theta)$
- Vertical cardioids : $r = a(1 \pm \sin\theta)$.

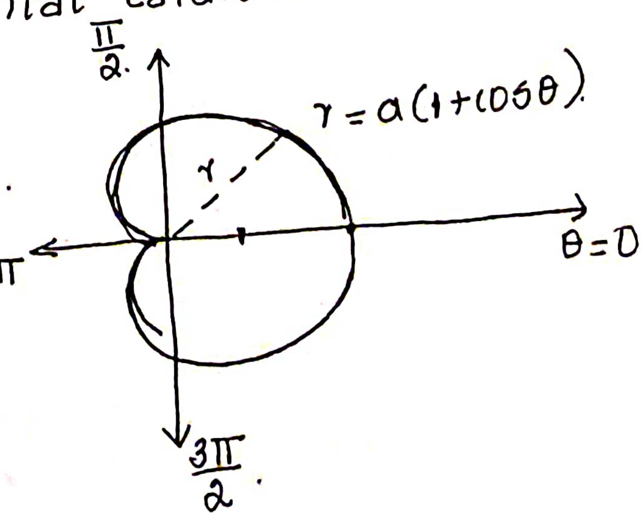
where 'a' is the radius of the circle which creates the cardioid.

<u>Exercises</u>.

1. Using double integral, find the area of the cardioid $r = a(1+\cos\theta)$.

<u>Solution</u>:

Here $r = a(1+\cos\theta)$ is a horizontal cardioid.



We observe that the curve is symmetrical about the initial line.

$\Rightarrow$ θ varies from 0 to π

and r ranges from 0 to $a(1+\cos\theta)$

$\therefore$ Area $= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta = 2\int_0^{\pi} \left.\frac{r^2}{2}\right|_0^{a(1+\cos\theta)} d\theta$.

$A = \frac{2}{2}\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta = a^2\int_0^{\pi}(1+\cos\theta)^2\,d\theta$.

$\boxed{1+\cos\theta = 2\cos^2\frac{\theta}{2}}$

$A = a^2\int_0^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta = 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta$.

Put $\frac{\theta}{2} = t \Rightarrow d\theta = 2dt$

$A = 8a^2\int_0^{\pi/2}\cos^4 t\,dt$.

$A = 8a^2\,\frac{(4-1)(4-3)}{4(4-2)}\times\frac{\pi}{2}\times\frac{1}{\cancel{1/2}}$ $\longrightarrow$ (~~coefficient~~ of θ)

$A = 8a^2\times\frac{3\times1}{4\times2}\times\frac{\pi}{2} = \frac{3\pi a^2}{2}$ sq. units.

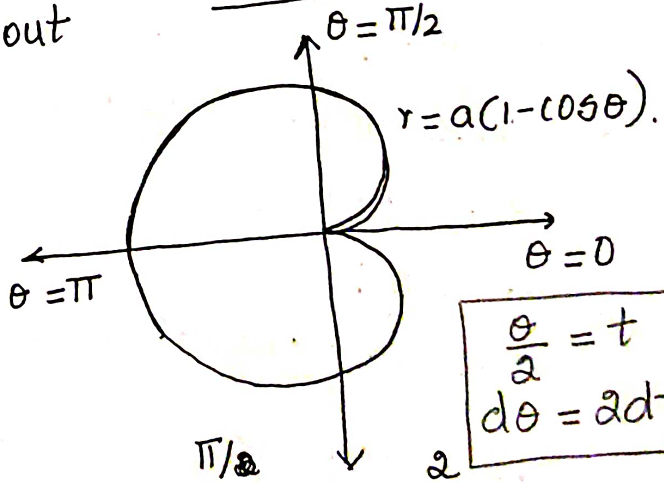
2). Find the area of the cardioid $r = a(1-\cos\theta)$.

Horizontal cardioid.

The curve is symmetrical about the initial line.

Here θ varies from 0 to π

r ranges from 0 to $a(1-\cos\theta)$.



$\boxed{\begin{aligned}\frac{\theta}{2} &= t\\ d\theta &= 2dt\end{aligned}}$

Area $A = 2\int_0^{\pi}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta$

$= 2\int_0^{\pi}\left.\frac{r^2}{2}\right|_0^{a(1-\cos\theta)} d\theta = a^2\int_0^{\pi}(1-\cos\theta)^2\,d\theta = a^2\int_0^{\pi/2}\left(2\sin^2\frac{\theta}{2}\right)d\theta$.

$= 4a^2\times2\int_0^{\pi/2}\sin^4 t\,dt = 8a^2\,\frac{(4-1)(4-3)}{4(4-2)}\times\frac{\pi}{2}\times\frac{1}{\cancel{1/2}}$.

$\frac{\theta}{2} = t$

$d\theta = 2dt$

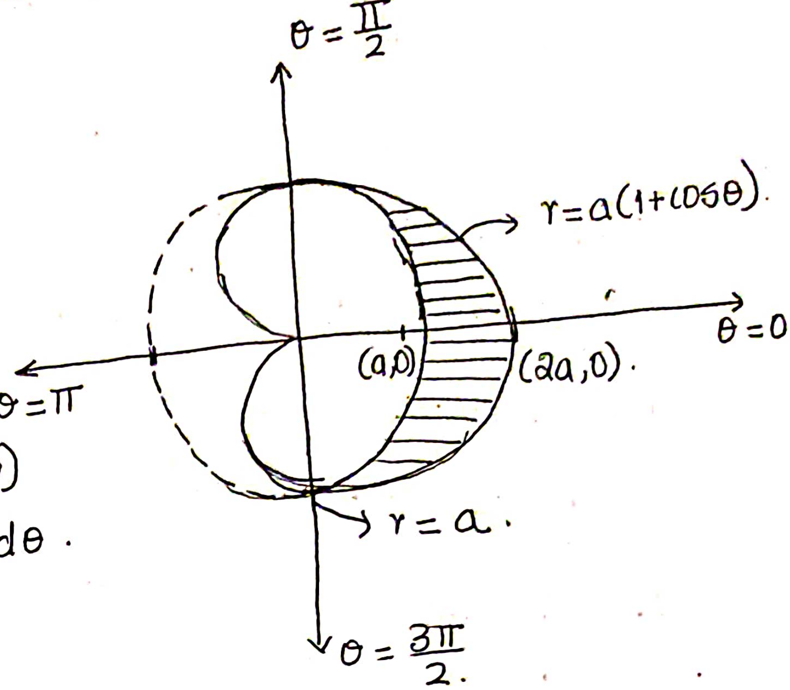
$\theta = \frac{\pi}{2}, t = \frac{\pi}{4}$

$\theta = 0, t = 0$

$= 16a^2\pi\times\frac{3\times1}{4\times2\times2} = \frac{3\pi a^2}{2}$ sq. units.

3. Find the area that lies inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$ by double integration.

Solution:



The region of integration is symmetrical about the initial line

$$\therefore \text{Area } A = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta.$$

$$A = 2\int_0^{\pi/2} \frac{r^2}{2}\Big|_a^{a(1+\cos\theta)} d\theta.$$

$$A = \int_0^{\frac{\pi}{2}} a^2(1+\cos\theta)^2\,d\theta = a^2\int_0^{\pi/2}(1+\cos\theta)^2\,d\theta - a^2\,d\theta$$

$$A = a^2\int_0^{\pi/2}\left(1+\cos^2\theta + 2\cos\theta\right)d\theta - a^2\,\theta\,\Big|_0^{\pi/2}$$

$$A = a^2\int_0^{\pi/2} 1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta\,d\theta - a^2\times\frac{\pi}{2}.$$

$$A = a^2\left[\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta\Big|_0^{\pi/2}\right] - \frac{\pi a^2}{2}.$$

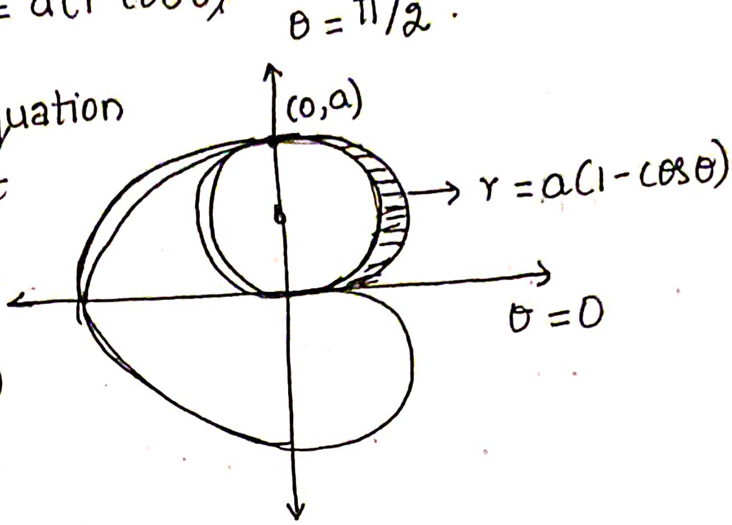
$$A = a^2\left[\frac{\pi}{2} + \frac{\pi}{4} + \frac{\sin\pi - \sin 0}{4} + 2\sin\frac{\pi}{2}\right] - \frac{\pi a^2}{2}.$$

$$A = a^2\left[\frac{\pi}{2} + \frac{\pi}{4} + 2\right] = a^2\left[\frac{3\pi}{4} + 2\right] - \frac{\pi a^2}{2} = a^4$$

$$A = \frac{\pi a^2}{2} + \frac{\pi a^2}{4} + 2a^2 - \frac{\pi a^2}{2} = \frac{a^2}{4}(\pi + 8) \text{ square units.}$$

4) Find the area inside the circle $r = a\sin\theta$ and outside the circle $r = a(1-\cos\theta)$

Here $r = a\sin\theta$ is the equation of a circle with centre at $(0, \frac{a}{2})$ and radius $\frac{a}{2}$.



The cardioid. $r = a(1-\cos\theta)$ is a horizontal cardioid.

$\therefore$ r : $a\sin\theta$ to $a(1-\cos\theta)$

θ : 0 to $\frac{\pi}{2}$.

$$\text{Area } A = \int_{\theta=0}^{\frac{\pi}{2}} \int_{a\sin\theta}^{a(1-\cos\theta)} r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left.\frac{r^2}{2}\right|_{a\sin\theta}^{a(1-\cos\theta)} d\theta.$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(a^2(1-\cos\theta)^2 - a^2\sin^2\theta\right) d\theta.$$

$$A = \frac{a^2}{2}\int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta - \sin^2\theta)\, d\theta.$$

$$A = \frac{a^2}{2}\int_0^{\pi/2} (1 - 2\cos\theta + \cos 2\theta)\, d\theta.$$

$$A = \frac{a^2}{2}\left[\left.\theta - 2\sin\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}}\right.$$

$$A = \frac{a^2}{2}\left[\frac{\pi}{2} - 2\sin\frac{\pi}{2} + \frac{\sin\pi}{2} - \{0 - 0 + 0\}\right].$$

$$A = \frac{a^2}{2}\left[\frac{\pi}{2} - 2\right] \text{ sq. units}$$