SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 2 – Vector Calculus

Part – B (Each question carries 3 Marks)

1. Find $\nabla\phi$ if $\phi = \log\left(x^2+y^2+z^2\right)$.

Solution

$$\nabla\phi = \vec{i}\ \frac{\partial\phi}{\partial x} + \vec{j}\ \frac{\partial\phi}{\partial y} + \vec{k}\ \frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}\left(\log(x^2+y^2+z^2)\right) + \vec{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)$$

$$= \vec{i}\frac{2x}{(x^2+y^2+z^2)} + \vec{j}\frac{2y}{(x^2+y^2+z^2)} + \vec{k}\frac{2z}{(x^2+y^2+z^2)}$$

$$= \frac{2}{x^2+y^2+z^2}\left(x\vec{i}+y\vec{j}+z\vec{k}\right) = \frac{2\vec{r}}{r^2} \qquad \because (\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}\ \&\ r^2 = x^2+y^2+z^2)$$

2. Find the unit normal vector to the surface $x^2+y^2=z$ at the point $(1,-2,5)$.

Solution

Given

$$\phi = x^2+y^2-z$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi \text{ at } (1,-2,5) = 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{4+16+1} = \sqrt{21}$$

Unit Normal vector is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}-4\vec{j}-\vec{k}}{\sqrt{21}}$$

3. Prove that $curl(grad\phi) = 0$.

Solution

$$grad\phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$Curl\,(grad\,\varphi) = \nabla\times\nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial y}\right) - \vec{j}\left(\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial x}\right) + \vec{k}\left(\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial y\,\partial x}\right)$$

$= 0\vec{i} + 0\vec{j} + 0\vec{k}$ **(Since mixed partial derivatives are equal.)**

4. Find $curl\vec{F}$ if $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$.

Solution

Given $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$

$$curl\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{i}\,(0-y) - \vec{j}(z-0) + \vec{k}\,(0-x)$$

$$= -y\,\vec{i} - z\,\vec{j} - x\,\vec{k}$$

5. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.

Solution

Given $\phi = x^2y^2z^4$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2xy^2z^4\vec{i} + 2x^2yz^4\vec{j} + 4x^2y^2z^3\vec{k}$$

$$\nabla\phi \text{ at } (3, 1, -2) = 96\vec{i} + 288\vec{j} - 288\,\vec{k}$$

$$|\nabla\phi| = \sqrt{96^2 + 288^2 + 288^2} = \sqrt{175104}$$

The directional derivative is maximum in the direction $\nabla\phi$ and the magnitude of this maximum is $|\nabla\phi| = \sqrt{175104}$.

6. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at (1, – 2 , – 1) in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution

Given $\phi = x^2yz + 4xz^2$,

$$\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}, \quad |\vec{a}| = \sqrt{4+1+4} = 3$$

$$\nabla\phi = (2xyz + 4z^2)\vec{i} + x^2z\vec{j} + (x^2y + 8xz)\vec{k}$$

$$(\nabla\phi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

$$\text{D.D.} = \nabla\phi.\frac{\vec{a}}{|\vec{a}|} = (8\vec{i} - \vec{j} - 10\vec{k}).\frac{2\vec{i} - \vec{j} - 2\vec{k}}{3} = \frac{37}{3}$$

7. Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at P (1, 2 , 3) in the direction of line PQ where Q is (5, 0, 4).

Solution

$$\nabla\varphi = grad\ \varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = grad\ \varphi = \vec{i}\,2x + \vec{j}\,(-2y) + \vec{k}\,4z$$

$$\nabla\varphi \text{ at } (1,2,3) = 2\vec{i} - 4\vec{j} + 12\vec{k}$$

$$\vec{a} = OQ - OP = (5\vec{i} + 0\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 4\vec{i} - 2\vec{j} + \vec{k}$$

Directional derivative = $\nabla\varphi \bullet \dfrac{\vec{a}}{|\vec{a}|}$

$$= (2\vec{i} - 4\vec{j} + 12\vec{k}) \bullet \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

8. Find the angle between the normals to the surfaces $x^2 = yz$ at the points (1, 1, 1) and (2, 4, 1).

Solution

Given $\varphi = x^2 - yz$

$$\nabla\varphi = 2x\vec{i} - z\vec{j} - y\vec{k}$$

$\nabla\varphi_1 /(1,1,1) = 2\vec{i} - \vec{j} - \vec{k}$ $\qquad$ $\nabla\varphi_2 /(2,4,1) = 4\vec{i} - \vec{j} - 4\vec{k}$

$|\nabla\varphi_1| = \sqrt{4+1+1} = \sqrt{6}$ $\qquad$ $|\nabla\varphi_2| == \sqrt{16+1+16} = \sqrt{33}$

$$\cos\theta = \frac{\nabla\varphi_1 \circ \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|} = \frac{(2\vec{i} - \vec{j} - \vec{k}) \circ (4\vec{i} - \vec{j} - 4\vec{k})}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{6}\sqrt{33}}.$$

9. Find a **such that** $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\,\vec{j} + (x - y + 2z)\vec{k}$ **is solenoidal.**

Solution

Given $\nabla \circ \vec{F} = 0 \Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$

$$3 + a + 2 = 0 \Rightarrow a + 5 = 0 \;\Rightarrow a = -5$$

10. Find the constant a, b, c so that $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ **is irrotational.**

Solution

Given $\vec{F}$ is irrotational **i.e.,** $\nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\vec{i}\left(\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az)\right)$$
$$+\vec{k}\left(\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az)\right) = \vec{0}$$

$= i.e., \quad \vec{i}\,(c+1) - \vec{j}(4 - a) + \vec{k}(b - 2) = 0$

$= \therefore c + 1 = 0, 4 - a = 0, and\, b - 2 = 0$

$\Rightarrow a = 4, b = 2, c = -1$

11. If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, then find div curl $\vec{F}$.

Solution $div\ curl\,\vec{F} = \nabla.(\nabla\times\vec{F})$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}$$

$$\nabla\times\vec{F} = \vec{0}$$

$$\therefore \nabla.(\nabla\times\vec{F}) = 0$$

12. Prove that $div\,\vec{r} = 3$**.**

Solution

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$div\,\vec{r} = \nabla\bullet\vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\bullet\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

13. Show that the vector $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$ is irrotational.

Solution

Given $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$

$$curl\vec{F} = \nabla\times\vec{F} = \vec{0}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

14. If $F = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$**. Evaluate** $\int_C \vec{F} \bullet d\vec{r}$ **from (0,0,0) to (1,1,1) along the curve x = t, y = t^2, z = t^3.**

Solution

The end points are (0,0,0) and (1,1,1).

These points correspond to t = 0 and t = 1.

$\therefore$ dx = dt, dy = 2t dt, dz = $3t^2$dt

$$\int_C \vec{F} \bullet d\vec{r} = \int_C \left(3x^2 + 6y\right)dx - 14yzdy + 20xz^2 dz$$

$$= \int_0^1 (3t^2 + 6t^2)dt - 14t^5(2tdt) + 20t^7(3t^2)dt = \int_0^1 (9t^2 - 28t^6 + 60t^9)dt = 5$$

15. If $F = ax\vec{i} + by\vec{j} + cz\vec{k}$, **a, b, c are constants, show that** $\iint_S \vec{F} \bullet \hat{n}\, ds = \frac{4\pi}{3}(a + b + c)$ **where S is the surface of a unit sphere.**

Solution

W.K.T. Gauss's divergence theorem

$$\iint_S \vec{F} \bullet \hat{n} ds = \iiint_V \nabla \bullet \vec{F} dV = \iiint_V \left(\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) \right) dV$$

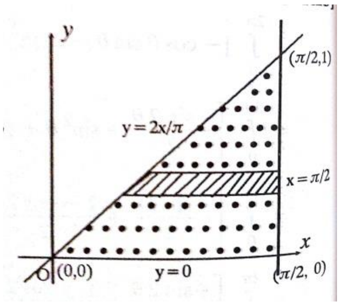
$$= \iiint_V (a + b + c)dV = (a + b + c)V = (a + b + c)\frac{4}{3}\pi(1)^3$$

$$\iint_S \vec{F} \bullet \hat{n} ds = \frac{4}{3}\pi(a + b + c)$$

16. Using Green's theorem, evaluate $\int_c (y-\sin x)\,dx+\cos x\,dy$ where c is the triangle formed by $y=0, x=\dfrac{\pi}{2}, y=\dfrac{2x}{\pi}.$

Solution

Using Green's theorem, we convert the line integral to double integral over the given region. $ie., \int_C u\,dx+v\,dy=\iint_R\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right)dxdy$



$u=y-\sin x$ $\qquad$ $v=\cos x$

$\dfrac{\partial u}{\partial y}=1$ $\qquad$ $\dfrac{\partial v}{\partial x}=-\sin x$

Hence, $\displaystyle\int_C \{(y-\sin x)dx+\cos x\,dy\}=\iint_R(-\sin x-1)dxdy$

$$=\int_0^1\int_{\pi y/2}^{\pi/2}(-\sin x-1)dxdy=\int_0^1[\cos x-x]_{\pi y/2}^{\pi/2}$$

$$=\int_0^1\left(0-\frac{\pi}{2}-\cos\frac{\pi y}{2}+\frac{\pi y}{2}\right)dy$$

$$=\left[\frac{-\pi y}{2}-\frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}}+\frac{\pi}{2}.\frac{y^2}{2}\right]_0^1=\frac{-\pi}{2}-\frac{2}{\pi}+\frac{\pi}{4}$$

$$=\frac{-\pi^2-8}{4\pi}=-\left[\frac{\pi}{4}+\frac{2}{\pi}\right].$$

17. Using Green's theorem, evaluate $\int_c (3x^2-8y^2)\,dx+(4y-6xy)\,dy$ where c is the boundary of the triangle formed by the lines $x=0, y=0, x+y=1$ in the xy plane.

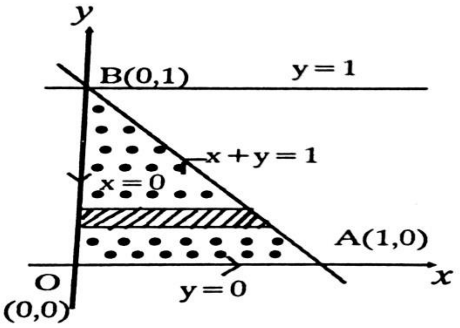
Solution

Using Green's theorem, we convert the line integral to double integral over the given region. $ie., \int_C u\,dx+v\,dy=\iint_R\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right)dxdy$

$u = 3x - 8y^2$ $\qquad$ $v = 4y - 6xy$

$\frac{\partial u}{\partial y} = -16y$ $\qquad$ $\frac{\partial v}{\partial x} = -6y$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -6y + 16y = 10y$$



Hence, $$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy = \iint_R (10y) dxdy$$

$$= 10\int_0^1 \int_0^{1-y} (y) dxdy = \int_0^1 y[x]_0^{1-y}\, dy$$

$$= 10\int_0^1 y(1-y)dy = 10\int_0^1 (y - y^2)dy$$

$$= 10\left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1$$

$$= 10\left(\frac{1}{2} - \frac{1}{3}\right).$$

$$= 10\frac{3-2}{6} = \frac{10}{6} = \frac{5}{3}$$

18. Using Gauss divergence theorem evaluate $\iiint_V \nabla \circ \vec{F} dv$ **where** $\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$ **taken over the cube bounded by the planes** $x = 0,\ x = 1, y = 0,\ y = 1, z = 0, z = 1.$

Solution

$\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \circ \vec{F} = 4z - 2y + y = 4z - y$$

$$\iiint_V \nabla \circ \vec{F} dv = \int_0^1\int_0^1\int_0^1 (4z - y) dx dy dz = \int_0^1\int_0^1 [4zx - yx]_0^1 dy dz = \int_0^1\int_0^1 [4z - y] dy dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz = \int_0^1 \left[4z - \frac{1}{2}\right] dz = \left[4\frac{z^2}{2} - \frac{z}{2}\right]_0^1 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

19. Using Gauss divergence theorem theorem evaluate $\iiint_V \nabla \circ \vec{F} dv$ where $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ taken over the cube bounded by the planes $x = 0,\ x = 1, y = 0,\ y = 1, z = 0, z = 1.$

Solution

$$\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \circ \vec{F} = 2x + 2y + 2z = 2(x + y + z)$$

$$\iiint_V \nabla \circ \vec{F} dv = 2\int_0^1\int_0^1\int_0^1 (x + y + z) dx dy dz = 2\int_0^1\int_0^1 \left[\frac{x^2}{2} + xy + xz\right]_0^1 dy dz = 2\int_0^1\int_0^1 \left[\frac{1}{2} + y + z\right] dy dz$$

$$= 2\int_0^1 \left[\frac{y}{2} + \frac{y^2}{2} + yz\right]_0^1 dz = 2\int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z\right] dz = 2\int_0^1 [1 + z]_0^1 dz = 2\left[z + \frac{z^2}{2}\right]_0^1$$

$$= 2\left(1 + \frac{1}{2}\right) = 2\left(\frac{3}{2}\right) = 3$$

20. Using Stokes theorem find $\iint_S curl\,\vec{F}\,ds$ where $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\,\vec{j}$ in the rectangular region of $x = 0,\ y = 0,\ x = a$ and $y = a.$

Solution Stokes theorem $\int_c \vec{F}.d\vec{r} = \iint_s curl\,\vec{F}.d\vec{s}$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\,\vec{k}$$

Here $\hat{n} = \vec{k}$

$$\iint_S curl\,\vec{F}.\hat{n}ds = \iint_S 4ydxdy = \int_0^b \int_0^a 4ydxdy = 2ab^2$$

21. Prove that the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\int_C (xdy - ydx).$$

Solution

W.K.T. Green's theorem

$$\int_C (udx + vdy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy \qquad \ldots 1$$

Here $v = \frac{x}{2}$ $\qquad u = -\frac{y}{2}$

$\frac{\partial v}{\partial x} = \frac{1}{2}$ $\qquad \frac{\partial u}{\partial y} = -\frac{1}{2}$

$$(1) \Rightarrow \int_C \left(\frac{x}{2}dy - \frac{y}{2}dx \right) = \iint_R \left(\frac{1}{2} + \frac{1}{2} \right) dxdy$$

$$\frac{1}{2}\int_C (xdy - ydx) = \iint_R dxdy$$

22. Find the area of the ellipse x = a cos θ, y = b sin θ using Green's theorem.

Solution

Given $\qquad$ x = a cos θ, $\qquad$ y = b sin θ

dx = -a sin θ dθ, $\qquad$ dy = b cos θ dθ

θ varies from 0 to 2π.

Area of the ellipse $= \frac{1}{2}\oint_C xdy - ydx$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(-b\cos\theta d\theta) - (b\sin\theta)(-a\sin\theta d\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} [ab\cos\theta\cos\theta + ab\sin\theta\sin\theta]d\theta$$

$$= \frac{ab}{2}\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)d\theta = \frac{ab}{2}\int_0^{2\pi} d\theta = \frac{ab}{2}[\theta]_{\theta=0}^{\theta=2\pi}$$

Area of the ellipse $= \frac{ab}{2}[2\pi] = \pi ab$

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