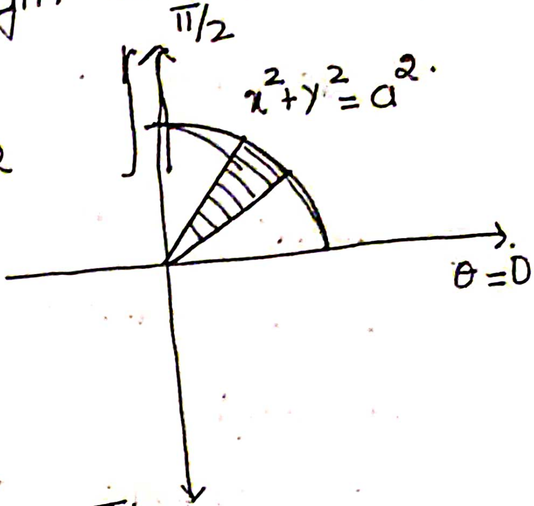
Area enclosed by plane curves - polar co-ordinates.

1. Find the area of a circle of radius 'a' by double integration in polar co-ordinates.

Solution :

Area of a circle with centre at origin and radius 'a'.



A/c Cartesian form : $x^2+y^2=a^2$.

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$\Rightarrow r^2 = a^2$

$\Rightarrow r = \pm a \,,\; r = a$:

r varies from 0 to a

θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore\; A = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\,dr\,d\theta = 4\int_{\theta=0}^{\pi/2} \frac{r^2}{2}\Big|_0^a = 2\int_0^{\pi/2} a^2\,d\theta.$$

$$A = 2a^2\,\theta\Big|_0^{\pi/2} = 2a^2\times\frac{\pi}{2} = \pi a^2 \text{ sq.units}.$$

Note :

1. $r = a$ is the equation of a circle with centre at $(0,0)$ and radius 'a'.
2. $r = 2a\cos\theta$ is the equation of the circle with centre at $(a,0)$ and radius 'a'.
3. [Cartesian equivalent : $(x-a)^2 + (y)^2 = a^2$]

$\cos^2\theta\, r^2 - 2ar\cos\theta + a^2 + r^2\sin^2\theta = a^2$

$\Rightarrow r^2 = 2ar\cos\theta \Rightarrow r = 2a\cos\theta$

3. $r = 2b\sin\theta$. ∴ equation of a circle with centre at $(0, b)$ and radius 'b''

{ cartesian equivalent : $x^2 + (y-b)^2 = b^2$ }.

4. $r = 2a\cos\theta + 2b\sin\theta$; equation of a circle with centre at (a, b). and radius $= \sqrt{a^2+b^2}$.

2). Calculate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

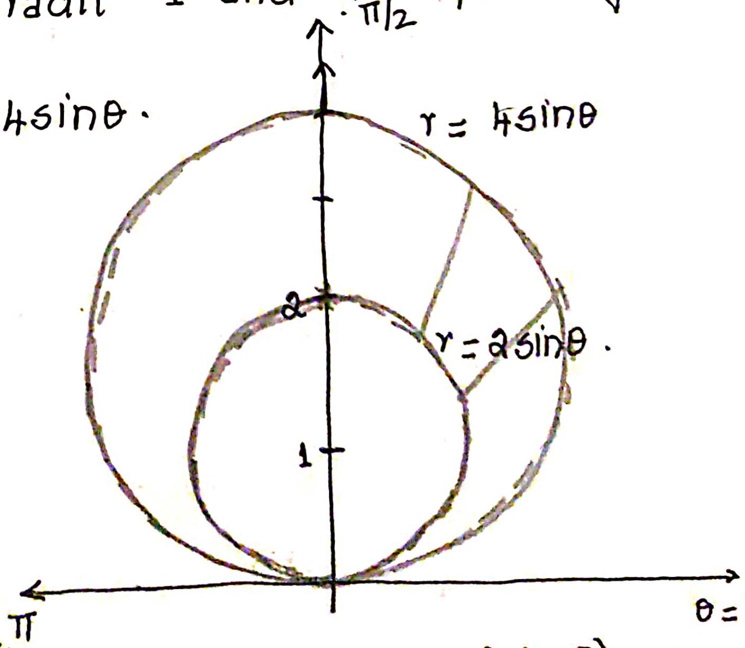
Solution :

Given $r = 2\sin\theta$, $r = 4\sin\theta$.

$\Rightarrow \quad 2b_1 = 2 \quad , \quad 2b_2 = 4$

$\quad\quad b_1 = 1. \quad\quad b_2 = 2.$

$\Rightarrow$ Both are circles with centres $(0,1)$ and $(0,2)$, with radii 1 and 2 respectively.



∴ θ ranges from 0 to π

r ranges from $2\sin\theta$ to $4\sin\theta$.

$$\text{Area} = \int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta.$$

$$= \int_0^{\pi} \left.\frac{r^4}{4}\right|_{2\sin\theta}^{4\sin\theta} d\theta.$$

$$= \frac{1}{4}\int_0^{\pi}\{4^4\sin^4\theta - 2^4\sin^4\theta\}\,d\theta$$

$$= \int_0^{\pi}(64-4)\sin^4\theta\,d\theta = 60\times 2\int_0^{\pi/2}\sin^4\theta\,d\theta = 120 \times \frac{(4-1)(4-3)}{4(4-2)} \times \frac{\pi}{2}.$$

$$= \frac{45\pi}{2} \text{ sq. units.}$$

3. Find the area of the region outside the inner circle $r = 2\cos\theta$ and inside the outer circle $r = 4\cos\theta$.

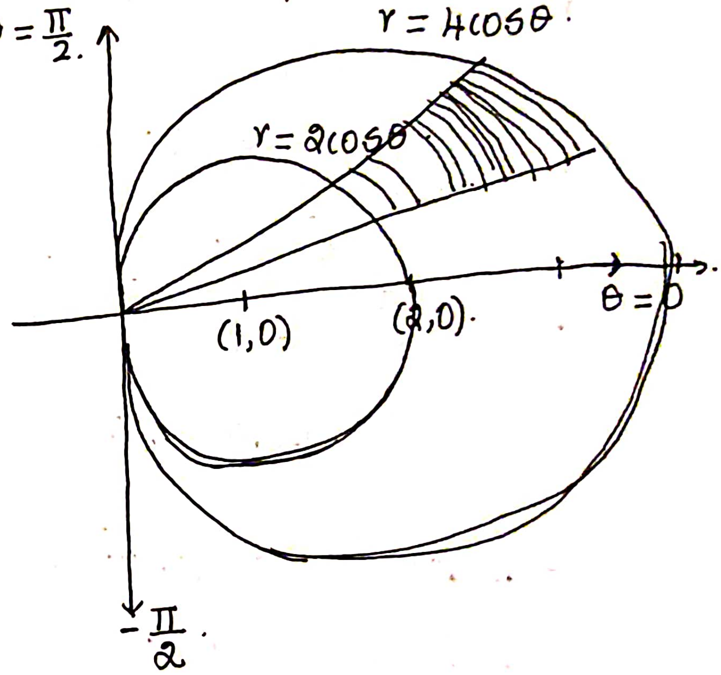
Solution:

$r = 2\cos\theta$ and $r = 4\cos\theta$ are circles with radius

$2a_1 = 2$ and $2a_2 = 4$

$\Rightarrow a_1 = 1$ $\quad a_2 = 2$. respectively.

with centres at $(1,0)$ and $(2,0)$ respectively.



Required area

$$A = \iint r\,dr\,d\theta$$

$$A = \int_{-\pi/2}^{\pi/2} \int_{2\cos\theta}^{4\cos\theta} r\,dr\,d\theta$$

$$A = \int_{-\pi/2}^{\pi/2} \left.\frac{r^2}{2}\right|_{2\cos\theta}^{4\cos\theta} d\theta$$

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (16\cos^2\theta - 4\cos^2\theta)\,d\theta$$

$$A = \frac{1}{2} \times 2 \int_0^{\pi/2} 12\cos^2\theta\,d\theta = 12\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$A = 12 \times \frac{(2-1)}{2} \times \frac{\pi}{2} = 3\pi \text{ square units.}$$

4) Evaluate $\iint r^2 \sin\theta\,dr\,d\theta$ where R is the semi circle $r = 2a\cos\theta$ about the initial line.

$$A = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2 \sin\theta \, dr\, d\theta .$$

$$A = \int_0^{\pi/2} \sin\theta \cdot \frac{r^3}{3}\Big|_0^{2a\cos\theta} d\theta .$$

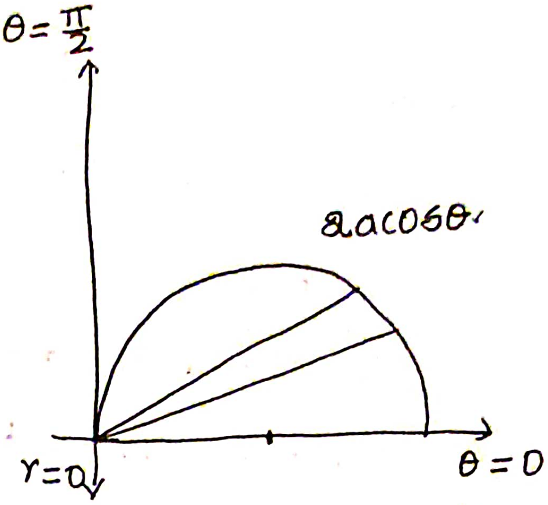
$$A = \frac{1}{3}\int_0^{\pi/2} \sin\theta \; 8a^3\cos^3\theta \, d\theta .$$

$$A = \frac{8a^3}{3}\int_0^{\pi/2} \cos^3\theta \, \sin\theta \, d\theta .$$

$$A = -\frac{8a^3}{3}\int_1^0 t^3 \, dt .$$

$$A = -\frac{8a^3}{3} \frac{t^4}{4}\Big|_1^0 = -\frac{2a^3}{3}(0-1)$$

$$A = \frac{2a^3}{3} \text{ square units.}$$



Put $\cos\theta = t \Rightarrow -\sin\theta \, d\theta = dt$

$\Rightarrow \sin\theta \, d\theta = -dt.$

$\theta = 0 ; \; t = \cos 0 = 1$

$\theta = \frac{\pi}{2} ; \; t = \cos\frac{\pi}{2} = 0 .$

5) . Evaluate $\iint r\sqrt{a^2-r^2}\, dr\, d\theta$ over the ~~ha~~ upper half of the circle $r = a\cos\theta$.

Solution :

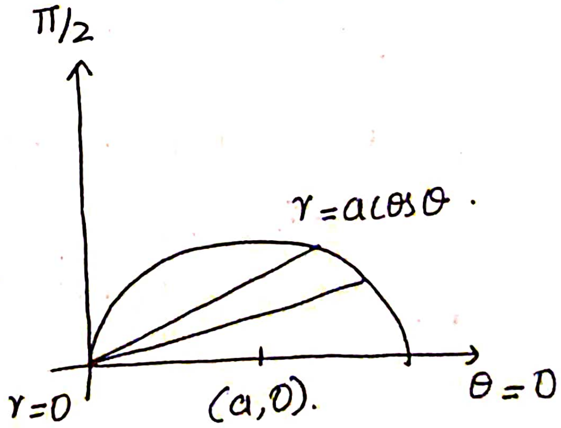
$0 \le \theta \le \frac{\pi}{2}$, $0 \le r \le a\cos\theta .$

$$I = \int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta .$$

$$I = \int_0^{\pi/2}\int_a^{a\sin\theta} \sqrt{t^2}\,(-t)\, dt\, d\theta .$$

$$I = -\int_0^{\pi/2} \frac{t^3}{3}\Big|_a^{a\sin\theta} d\theta .$$

$$I = -\frac{\cancel{a^3}\,1}{3}\int_0^{\pi/2} a^3\sin^3\theta\, d\theta - a^3 . d\theta = \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right) .$$



Put $a^2 - r^2 = t^2$

$\Rightarrow -2r\,dr = 2t\,dt$

$\Rightarrow r\,dr = -t\,dt .$

When $r = 0$, $t^2 = a^2 \Rightarrow t = a$

$r = a\cos\theta$, $t^2 = a^2\sin^2\theta .$