Vector Calculus:

1. Gradient $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$.

- ∇ is a vector differential operator.
- If ϕ is a constant then $\nabla\phi = \vec{0}$
- $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$.
- $\nabla\left(\dfrac{\phi_1}{\phi_2}\right) = \dfrac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}$, $\phi_2 \neq 0$.
- $v = f(u) \Rightarrow \nabla v = f'(u)\nabla u$.

2. $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of a point $P(x,y,z)$ w.r.t. the origin.

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$.

$\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$.

↳ $\nabla r = \dfrac{\vec{r}}{r}$, $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$, $\nabla r^n = n r^{n-2}\vec{r}$,

$\nabla \log r = \dfrac{\vec{r}}{r^2}$.

3. Directional derivative: $\nabla\phi \cdot \hat{a} = \nabla\phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$.

Maximum value of directional derivative is $|\nabla\phi|$.

4. Unit tangent vector: $= \dfrac{d\vec{r}/dt}{|d\vec{r}/dt|}$.

5. Unit normal vector $= \hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$.

6. Angle between the surfaces: $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$.

7. Equation of tangent plane is $(\vec{r}-\vec{a})\cdot\nabla\phi = 0$

Equation of normal line is $(\vec{r}-\vec{a})\times\nabla\phi = \vec{0}$.

8. Divergence $\text{Div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

↳ Scalar quantity.

↳ Solenoidal if $\nabla\cdot\vec{F} = 0$.

9. Curl $\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$.

↳ Vector quantity.

↳ Irrotational if $\nabla\times\vec{F} = \vec{0}$.

* $\nabla\cdot(\nabla\times\vec{F}) = 0$.
* $\nabla\cdot\vec{r} = 3$. $\quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.
* $\nabla\times\vec{r} = \vec{0}$
* $\nabla\cdot\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$.

10. Vector identities.

* $\nabla\times(\nabla\phi) = \vec{0} \Leftrightarrow \text{curl}(\text{grad}\,\phi) = \vec{0}$.
* $\nabla\cdot(\nabla\times\vec{F}) = 0 \Leftrightarrow \text{div}(\text{curl}\,\vec{F}) = 0$.

11. If $\vec{A}$ and $\vec{B}$ are irrotational then $\vec{A}\times\vec{B}$ is solenoid[al]

* If $\nabla\phi$ and $\nabla\psi$ are irrotational then $\nabla\phi\times\nabla\psi$ is solen[oidal]

12. Physical interpretation of line integral.

$\int_A^B \vec{F}\cdot d\vec{r}$ denotes the total work done by the force $\vec{F}$ in moving/displacing a particle from A to B along C.

13. If the integral depends only on the end points but not on the path C, then $\vec{F}$ is called conservative vector field i.e. if $\vec{F}$ is expressible as the gradient of a scalar point function ϕ then F is conservative.

★ If $\vec{F}$ is conservative then curl $\vec{F}$ = curl grad ϕ = $\vec{0}$.

⇒ $\vec{F}$ is irrotational.

★. Green's theorem : line integral with double integral.

↳. The area bounded by a simple closed curve C is

$$\frac{1}{2}\int_C x\,dy - y\,dx.$$

↳. Area of ellipse $= \frac{1}{2}\int_C x\,dy - y\,dx = \pi ab$.

★★ $\int_C x\,dy - y\,dx = 2\iint_R dx\,dy = 2\{\text{Area}\}$.

★ Gauss divergence theorem : surface integral with volume.

↳. $\iint_S \vec{r}\cdot\hat{n}\,dS = 3V$. $V \to$ volume of the closed surface.

↳ $\iint_S \nabla r^2\cdot\hat{n}\,dS = 6V$.

↳ $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = 0$.

↳. $\iint_S \phi\,\hat{n}\,dS = \iiint_V \nabla\phi\,dV$.

★. Stoke's theorem : surface integral of the normal component of the curl of a vector function $\vec{F}$ over an open surface S is equal to the line integral of the tangential component of $\vec{F}$ around the closed curve C bounding S. $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$.