Change of variables from Cartesian to Polar form.

Put $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$.

Note :

1. If $x>0, y>0$ is mentioned then $0 \le \theta \le \frac{\pi}{2}$
2. If $x>0, y>0$ is absent then $0 \le \theta \le 2\pi$.

Exercises

1. $\iint_D (x^2+y^2)\,dx\,dy$ where D is the disc $x^2+y^2 \le a^2$.

by converting cartessian to polar forms.

Solution: Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

Given; D is the disc $x^2+y^2 \le a^2$. $\Rightarrow$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta \le a^2$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) \le a^2$.

$\Rightarrow r^2 \le a^2 \Rightarrow 0 \le r \le a$ and $0 \le \theta \le 2\pi$.

$$\iint_D (x^2+y^2)\,dx\,dy = \int_{\theta=0}^{2\pi}\int_0^a r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^a d\theta$$

$$= \int_0^{2\pi} \frac{a^4}{4}\,d\theta = \frac{a^4}{4}\int_0^{2\pi} d\theta$$

$$= \frac{a^4}{4}\,\theta\Big|_0^{2\pi} = \frac{a^4}{4}(2\pi - 0).$$

$$= \frac{\pi a^4}{2}.$$

2) $\iint_D \sqrt{x^2+y^2}\, dx\, dy$ where D is the circle $x^2+y^2=a^2$.

by changing cartesian to polar.

Solution :

Put $x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta\ ;\ 0 \le \theta \le 2\pi,\ 0 \le r \le a$.

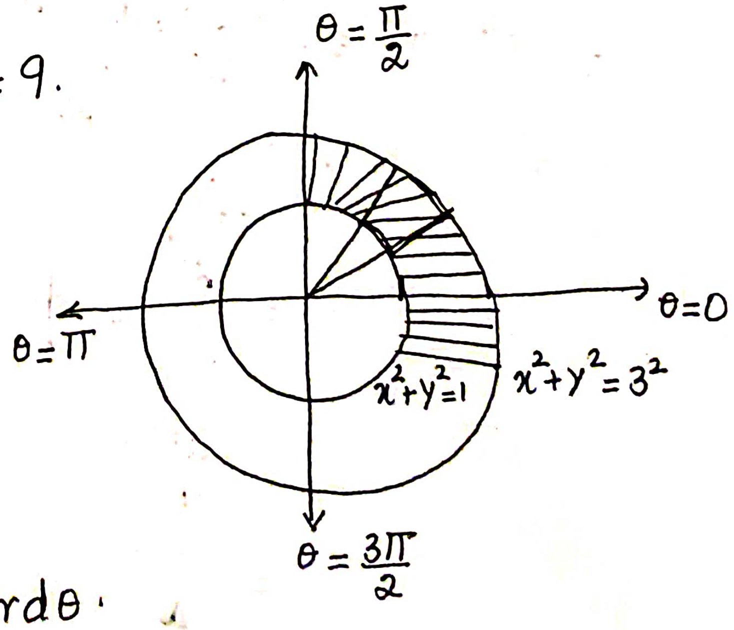
$$\iint_D \sqrt{x^2+y^2}\, dx\, dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \sqrt{r^2}\; r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} r^2\,dr\,d\theta = \int_0^{2\pi} \frac{r^3}{3}\Big|_0^a\, d\theta.$$

$$= \int_0^{2\pi} \frac{a^3}{3}\, d\theta = \frac{a^3}{3}\,\theta\Big|_0^{2\pi} = \frac{a^3}{3}(2\pi - 0).$$

$$= \frac{2\pi a^3}{3}.$$

** 3). $\iint_D e^{x^2+y^2}\, dx\, dy$ where D is the region between the circles $x^2+y^2=1,\ x^2+y^2=9$.



Solution :

Put $x = r\cos\theta,\ y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

Here θ : 0 to 2π.

r : 1 to 3.

$$\iint_D e^{x^2+y^2} dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=1}^{3} e^{r^2}\, r\, dr\,d\theta.$$

We observe $\int f'(t)\, f(t)\, dt$.

$$= \int_{\theta=0}^{2\pi} \int_{t=1}^{9} e^t \frac{dt}{2} \cdot d\theta$$

Put $r^2 = t$

$\Rightarrow 2r\,dr = dt \Rightarrow r\,dr = \frac{dt}{2}$.

When $r=1,\ t=1$

$r=3,\ t=9$.

$$= \frac{1}{2} \int_0^{2\pi} e^t \Big|_1^9 d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (e^9 - e^1) d\theta = \frac{e^9 - e}{2} \int_0^{2\pi} d\theta.$$

$$= \frac{e^9 - e}{2} \theta \Big|_0^{2\pi} = \frac{2\pi(e^9 - e)}{2} = \pi(e^9 - e).$$

4). $\iint_D \frac{xy}{(x^2+y^2)^2} dx\,dy$ where $D: 1 < x^2+y^2 < 9$.

Solution: Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

Given $D: 1 < x^2+y^2 < 9 \Rightarrow 1 < r^2 < 9 \Rightarrow 1 < r < 3$.

And $0 \le \theta \le 2\pi$.

$$\iint_D \frac{xy}{(x^2+y^2)^2} dx\,dy = \int_{\theta=0}^{2\pi} \int_{r=1}^{3} \frac{r\cos\theta \times r\sin\theta}{(r^2)^2} \times r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{2\pi} \int_{r=1}^{3} \frac{r^3 \cos\theta \sin\theta}{r^4} dr\,d\theta = \int_0^{2\pi} \int_{r=1}^{3} \frac{\cos\theta \sin\theta}{r} dr\,d\theta.$$

$$= \int_0^{2\pi} \sin\theta\cos\theta\,d\theta \cdot \int_{r=1}^{3} \frac{1}{r} dr = \int_0^{2\pi} \frac{\sin 2\theta}{2} d\theta \cdot \log r \Big|_1^3.$$

$$= \frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right)\Big|_0^{2\pi} \times (\log 3 - \log 1) = -\frac{1}{2}\left[\cos 4\pi - \cos 0\right] \times (\log 3 - 0)$$

$$= -\frac{\log 3}{2} \{1 - 1\} = 0.$$

5) $\iint\limits_D \frac{x^2y^2}{(x^2+y^2)^2}\,dx\,dy$. where $D: 1 < x^2+y^2 < 2$.

Solution: Put $x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$.

Here $D: 1 < x^2+y^2 < 2 \Rightarrow 1 < r^2 < 2 \Rightarrow \boxed{1 < r < \sqrt{2}}$.

And, $0 \le \theta \le 2\pi$.

$$\iint\limits_D \frac{x^2y^2}{(x^2+y^2)^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{2}} \frac{r^2\cos^2\theta \cdot r^2\sin^2\theta}{(r^2)^2}\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{2}} r\cos^2\theta\cdot\sin^2\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \cos^2\theta\cdot\sin^2\theta\,d\theta \cdot \int_{r=1}^{\sqrt{2}} r\,dr$$

$$= 4\int_0^{\pi/2} \cos^2\theta\sin^2\theta\,d\theta \times \left.\frac{r^2}{2}\right|_1^{\sqrt{2}}$$

$$= 4 \cdot \frac{(2-1)(2-1)}{4(4-2)} \times \frac{\pi}{2} \times \frac{1}{2}(2-1)$$

$$= \frac{4}{8} \times \frac{\pi}{4} = \frac{\pi}{8}.$$

Formula:

$$\int_0^{2a} f(\theta)\,d\theta = 2\int_0^{a} f(\theta)\,d\theta$$
$$= 4\int_0^{a/2} f(\theta)\,d\theta.$$

Reduction Formula

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta$$
$$= \frac{(m-1)(m-3)\ldots(n-1)(n-3)}{(m+n)(m+n-2)\ldots} \times \frac{\pi}{2}$$

when m and n are even.

6). $\iint_D \sqrt{x^2+y^2}\, dxdy$ where D is the unit disc in I quadrant.

Solution : Put $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = r\,dr\,d\theta$.

D is a unit disc in I quadrant

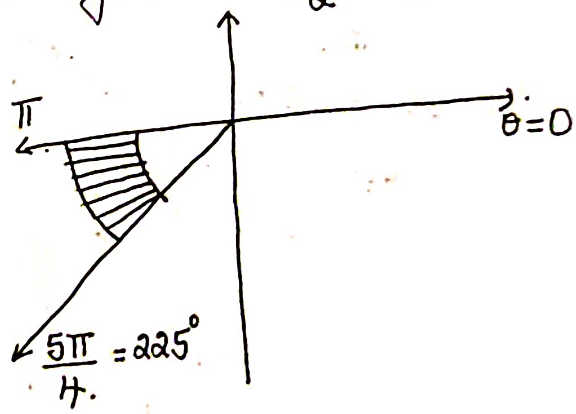
$\Rightarrow$ $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 1$. (unit disc) .

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} \sqrt{r^2}\,.\, r\,dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} r^2\,dr\,d\theta .$$

$$= \int_{\theta=0}^{\pi/2} d\theta \int_{r=0}^{1} r^2\,dr = \theta \Big|_0^{\pi/2} \times \frac{r^3}{3}\Big|_0^1 = \frac{\pi}{2} \times \frac{1}{3} = \frac{\pi}{6} .$$

(3) 7) $\iint_D \frac{xy}{x^2+y^2}\, dxdy$ where D is the region

$\pi < \theta < \frac{5\pi}{4}$, $\frac{1}{4} < x^2+y^2 < \frac{1}{2}$.



Solution :

Put $x = r\cos\theta$, $y = r\sin\theta$

$dxdy = r\,dr\,d\theta$.

Here $\frac{1}{4} < r^2 < \frac{1}{2}$ $\Rightarrow$ $\frac{1}{2} < r < \frac{1}{\sqrt{2}}$.

and $\pi < \theta < \frac{5\pi}{4}$.

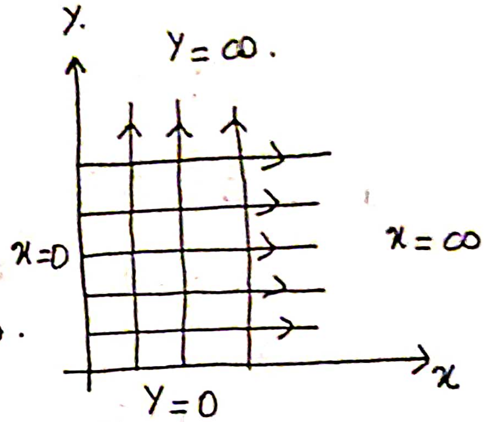
$$= \int_{\theta=\pi}^{5\pi/4} \int_{r=1/2}^{1/\sqrt{2}} \frac{r\cos\theta \times r\sin\theta}{r^2} \times r\,dr\,d\theta = \int_{\theta=\pi}^{5\pi/4} \int_{1/2}^{1/\sqrt{2}} \cos\theta\sin\theta \,.\, r\,dr\,d\theta .$$

$$= \int_{\theta=\pi}^{5\pi/4} \cos\theta\sin\theta\, d\theta \,.\, \int_{1/2}^{1/\sqrt{2}} r\,dr . = \int_{\theta=\pi}^{5\pi/4} \frac{\sin 2\theta}{2}\, d\theta \,.\, \frac{r^2}{2}\Big|_{1/2}^{1/\sqrt{2}}$$

$$= \frac{1}{2}\left(-\frac{\cos 2\theta}{2}\Big|_{\pi}^{\frac{5\pi}{4}}\right.\ \frac{1}{2}\left\{\frac{1}{2} - \frac{1}{4}\right\} = -\frac{1}{4}\left[\cos\frac{5\pi}{2} - \cos 2\pi\right] \times \frac{1}{2} \times \frac{1}{4}.$$

$$= -\frac{1}{4}(0-1) \times \frac{1}{8} = \frac{1}{32}.$$

(*) 8) $\displaystyle\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x^2+y^2)}\,dx\,dy.$

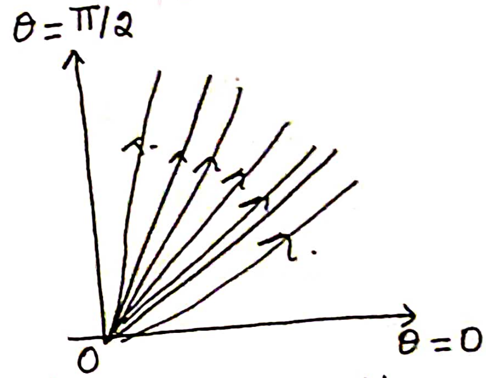


<u>Solution</u> :

Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

Here θ ranges from 0 to $\frac{\pi}{2}$.

r ranges from 0 to ∞.



$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{\pi/2} d\theta . \int_{r=0}^{\infty} r.e^{-r^2}\,dr.$$

Put $r^2 = t \Rightarrow 2r\,dr = dt$

$r\,dr = \frac{dt}{2}$.

When $r = 0$, $t = 0$

$r = \infty$, $t = \infty$.

$$= \theta\Big|_0^{\frac{\pi}{2}} \int_{t=0}^{\infty} e^{-t}\,\frac{dt}{2}.$$

$$= -\left(\frac{\pi}{2} - 0\right) \times \frac{1}{2}\left(e^{-\infty} - e^{-0}\right).$$

$$= -\frac{\pi}{2} \times \frac{1}{2}\left(\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right) = -\frac{\pi}{4}\left(\frac{1}{\infty} - \frac{1}{1}\right).$$

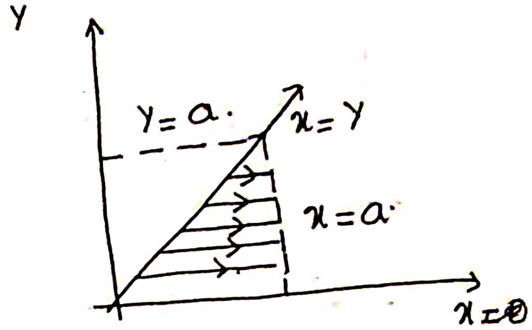
$$= \frac{\pi}{4}$$

9) **(5)

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$$

Here $x=y$ and $x=a$

$y=0$ and $y=a$.



Put $x=r\cos\theta$, $y=r\sin\theta$, $dx\,dy=r\,dr\,d\theta$.

⇓ Polar.

Here $y=0$, $y=a$.

$\Rightarrow r\sin\theta=0 \qquad \Rightarrow r\cos\theta = r\sin\theta$

$\Rightarrow \sin\theta=0 \qquad \Rightarrow \tan\theta=1$.

$\Rightarrow \theta=0. \qquad \Rightarrow \theta=\frac{\pi}{4}$.

$\therefore$ θ ranges from 0 to $\frac{\pi}{4}$.

r ranges from 0 to $x=a \Rightarrow r\cos\theta=a \Rightarrow r=\frac{a}{\cos\theta}$.

$$= \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=0}^{a/\cos\theta} \frac{r\cos\theta}{r^2}\, r\,dr\,d\theta$$

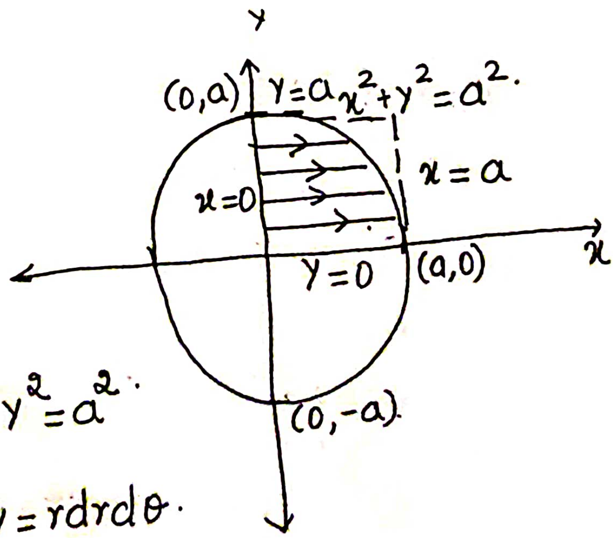
$$= \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=0}^{a/\cos\theta} \cos\theta\,dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{4}} \cos\theta\, r\Big|_0^{a/\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} \cos\theta\cdot\left(\frac{a}{\cos\theta}-0\right)d\theta = \int_{\theta=0}^{\pi/4} a\,d\theta = a\times\theta\Big|_{\theta=0}^{\frac{\pi}{4}}$$

$$= a\left(\frac{\pi}{4}-0\right) = \frac{\pi a}{4}.$$

10)

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\,dx\,dy.$$



Cartesian : $y=0$, $y=a$.

$x=0$, $x=\sqrt{a^2-y^2} \Rightarrow x^2+y^2=a^2$.

Polar: $x=r\cos\theta$, $y=r\sin\theta$, $dx\,dy=r\,dr\,d\theta$.

where $\theta : 0$ to $\frac{\pi}{2}$ and $r : 0$ to a.

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\sin\theta \cdot \sqrt{r^2}\; r\,dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a} r^3 \sin\theta\, dr\, d\theta.$$

$$= \int_{r=0}^{a} r^3\, dr \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta = \left.\frac{r^4}{4}\right|_0^a \times \left.(-\cos\theta)\right|_0^{\pi/2}.$$

$$= -\frac{a^4}{4}\left(\cos\frac{\pi}{2} - \cos 0\right) = -\frac{a^4}{4}(0-1) = \frac{a^4}{4}.$$

(6) 11). $\displaystyle\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx.$

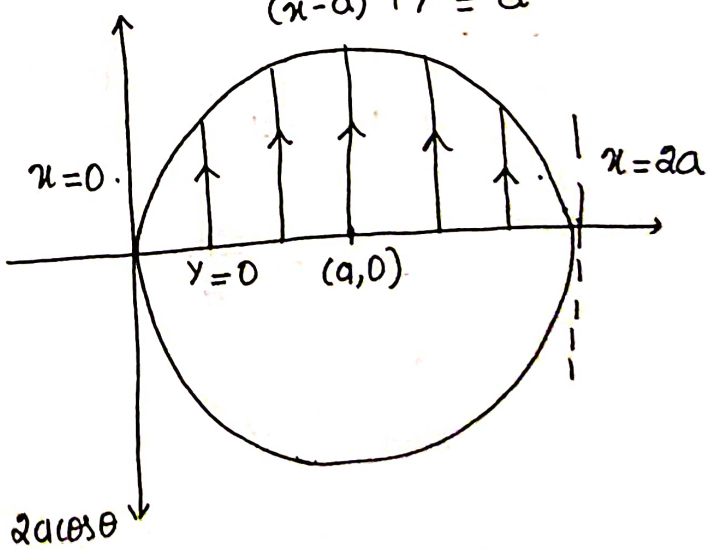
Cartesian: $x = 0$, $x = 2a$.

$y = 0$, $y = \sqrt{2ax - x^2}$

$\Rightarrow x^2 + y^2 - 2ax = 0 \Rightarrow (x-a)^2 + y^2 = a^2.$

Equation of circle with centre at $(a,0)$.

and radius $r = a$.



Polar: $\theta : 0$ to $\frac{\pi}{2}$.

$x^2 + y^2 = 2ax.$

$r^2 = 2a\cos\theta\, r.$

$\Rightarrow r^2 - 2ar\cos\theta = 0$

$\Rightarrow r(r - 2a\cos\theta) = 0.$

$\Rightarrow r = 0,\ r = 2a\cos\theta.$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} \frac{r\cos\theta}{\sqrt{r^2}}\, r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r\cos\theta \cdot dr\, d\theta.$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta \cdot \frac{r^2}{2}\Big|_0^{2a\cos\theta} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos\theta\,(4a^2\cos^2\theta - 0)\, d\theta.$$

$$= \frac{4}{2}\int_0^{\pi/2} a^2\cos^3\theta\, d\theta = 2a^2\int_0^{\pi/2}\cos^3\theta\, d\theta.$$

$$= 2a^2\, \frac{(3-1)}{3\times(3-2)} \times 1.$$

$$= \frac{4a^2}{3}.$$

Reduction Formula.

$$\int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \frac{(n-1)(n-3)\dots}{n(n-2)(n-4)\dots}\times 1$$

when n is odd.

12). (7) Transform the double integral $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$ in polar form and then evaluate it.

$$= \int_{x=0}^{a} \int_{y=\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-x^2-y^2}}.$$

$x^2+y^2-ax=0$

$x^2 - 2\cdot\frac{a}{2}x + \frac{a^2}{4} - \frac{a^2}{4} + y^2 = 0$

$\Rightarrow \left(x-\frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}.$

Circle ; centre $\left(\frac{a}{2}, 0\right)$

radius: $\frac{a}{2}$.

$x=0,\ x=a \Rightarrow r\cos\theta = 0 \Rightarrow \theta = 0.$

~~$r\cos\theta = a$~~

$y = \sqrt{ax-x^2}$

$y^2 = ax - x^2$

$x^2+y^2-ax = 0.$

$r^2 - ar\cos\theta = 0.$

$\Rightarrow r = a\cos\theta$

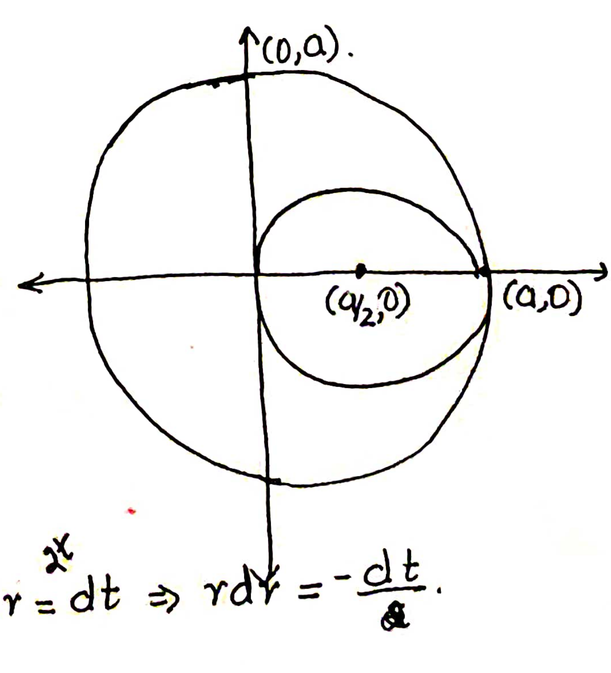
$y = \sqrt{a^2-x^2}$

$\Rightarrow y^2 = a^2 - x^2$

$\Rightarrow x^2+y^2 = a^2.$

$\Rightarrow r^2 = a^2.$

$\Rightarrow r = a.$



$$\therefore\ I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=a\cos\theta}^{a} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}.$$

$I =$. Put $a^2 - r^2 = t^2 \Rightarrow -2r\,dr = dt \Rightarrow r\,dr = -\frac{dt}{2}.$

$$= \int_0^{\pi/2} \int_{a\sin\theta}^{0} \frac{1}{t} (-t\,dt)\,d\theta$$

$$= \int_0^{\pi/2} \int_{a\sin\theta}^{0} -dt\,d\theta$$

$$= -\int_0^{\pi/2} t \Big|_{a\sin\theta}^{0} d\theta$$

$$= -\int_0^{\pi/2} 0 - a\sin\theta\,d\theta = a\int_0^{\pi/2} \sin\theta\,d\theta$$

$$= -a\cos\theta \Big|_0^{\pi/2} = -a\left\{\cos\frac{\pi}{2} - \cos 0\right\} = a\{1-0\} = a.$$

$r = a\cos\theta,$

$t^2 = a^2 - a^2\cos^2\theta$

$t^2 = a^2\sin^2\theta$

$t = a\sin\theta$

When $r = a.$

$t^2 = 0 \Rightarrow t = 0.$