UNIT – IV FOURIER TRANSFORMS

INTRODUCTION

The development of mathematical representation of periodic phenomena using complex numbers leads to complex form of the Fourier series representation of periodic function. The representation of periodic signals as a linear combination of harmonically related complex exponentials can be extended to develop a representation of aperiodic signals as linear combination of complex exponentials. This leads to Fourier Transforms.

Fourier Transform is widely used in the theory of communication engineering, wave propagation and other branches of applied mathematics.

FOURIER INTEGRAL THEOREM (AU 2005,2008,2009,2010,2011)

If f(x) is piecewise continuous, has piecewise continuous derivatives in every finite interval in (-∞,∞) and absolutely integrable in (-∞,∞) , then

f(x) = $\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\, dt\, ds$ --------**(i)**

R.H.S is called the Fourier Complex integral or Fourier Complex integral representation of f(x). OR f(x) = $\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\, cos\{s(x-t)\}\, dt\, ds$---**(ii)**

R.H.S is called the Fourier integral or Fourier integral representation of f(x)

RESULTS:1. $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ if f(-x)=f(x) that is f(x) is an even function

= 0 if f(-x)=-f(x) that is f(x) is an odd function

2. $e^{i\theta} = cos\theta + isin\theta$, $e^{-\infty} = 0$, $e^{0} = 1$, $e^{\infty} = \infty$ & Cos (a-b)=cos a cos b+sin a sin b and so on

3. $\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$.Bernoulli's theorem where u &v are function of x.

4. sin 0 = 0 , cos 0 =1 , cos nπ=$(-1)^n$where n=0,1,2,3 ,… sin nπ =0 , cos$\frac{(2n-1)\pi}{2}$ =0 for all n & sin $\frac{(2n-1)\pi}{2} = (-1)^{(n+1)}$for n=1,2,3,…

5. $\int e^{ax}cosbxdx = \frac{e^{ax}}{a^2+b^2}\{acosbx + bsinbx\}$ also $\int_0^{\infty} e^{-ax}cosbxdx = \frac{a}{a^2+b^2}$

$\int e^{-ax}sinbxdx = \frac{e^{-ax}}{a^2+b^2}\{-asinbx - bcosbx\}$ also

$\int_0^{\infty} e^{-ax}sinbxdx = \frac{b}{a^2+b^2}$

NOTE :

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\ cos\{s(x-t)\}\ dt\ ds$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\{cos\ sx\ cos\ st + sin\ sx\ sin\ st\}\ dt\ ds$$

$$= \frac{1}{\pi}\int_0^{\infty} cos\ sx(\int_{-\infty}^{\infty} f(t)cos\ s\ dt\ )ds += \frac{1}{\pi}\int_0^{\infty} sin\ sx(\int_{-\infty}^{\infty} f(t)sin\ st\ dt)\ ds$$

NOTE :

1. If f(x) (or f(t)) is even, then f(t) cos st is an even function of t and f(t) sin st is an odd function of t.

f(x) $= \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)cos\ sx\ cos\ st\ dt\ ds$ R.H.S is called the Fourier Cosine integral of f(x)

2. If f(x) (or f(t)) is odd, then f(t) cos st is an odd function of t and f(t) sin st is an even function of t.

f(x) $= \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)sin\ sx\ sin\ st\ dt\ ds$ R.H.S is called the Fourier sine integral of f(x)

Problems:

1. Show that f(x)=1 , 0<x<∞ can not be represented by a Fourier integral.

Sol: $\int_0^{\infty}|f(x)|dx = \int_0^{\infty} 1dx$ =(X) $_0^{\infty}$= ∞ and this value tends to ∞ as x→ ∞ . that is $\int_0^{\infty} f(x)dx$ is not convergent. Hence f(x)=1 can not be represented by a Fourier integral.

2. Using Fourier integral formula, prove that $e^{-x}cosx = \frac{2}{\pi}\int_0^{\infty}\frac{(\lambda^2+2)cos\lambda x}{\lambda^4+4}d\lambda$

Sol: Since the right hand side integrand contains Cosine term, We shall use cosine integral formula.

$$f(x) = \frac{2}{\pi}\int_0^\infty cos\lambda x\left(\int_0^\infty f(t)cos\lambda t dt\right) d\lambda$$ Here f(t)= $e^{-t}cost$

$$=\frac{2}{\pi}\int_0^\infty cos\lambda x\left(\int_0^\infty e^{-t}costcos\lambda tdt\right)d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty cos\lambda x\left(\int_0^\infty e^{-t}\left\{\frac{1}{2}[cos(1+\lambda)t + cos(1-\lambda)t]\right\}dt\right)d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \Big(cos\lambda x\left\{\frac{e^{-t}}{1+(1+\lambda)^2}[-cos(1+\lambda)t + (1+\lambda)sin(1+\lambda)t]\right\}_0^\infty + \left\{\frac{e^{-t}}{1+(1-\lambda)^2}[-cos(1-\lambda)t + (1-\lambda)sin(1-\lambda)t]\right\}_0^\infty\Big) d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \left(cos\lambda x\left\{0 - \frac{1}{1+(1+\lambda)^2}(-1)\right\}+\left\{0 - \frac{1}{1+(1-\lambda)^2}(-1)\right\}\right) d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty cos\lambda x\left\{\frac{1+(1-\lambda)^2+1+(1+\lambda)^2}{(1+(1-\lambda)^2)(1+(1+\lambda)^2)}\right\}d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty cos\lambda x\left\{\frac{\lambda^2-\boldsymbol{2\lambda}+2+\lambda^2+\boldsymbol{2\lambda}+2}{(\lambda^2-2\lambda+2)(\lambda^2+2\lambda+2)}\right\}d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty cos\lambda x\left\{\frac{2\lambda^2+4}{(\lambda^2+2)^2-4\lambda^2}\right\}d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty cos\lambda x\left\{\frac{2(\lambda^2+2)}{\lambda^4+4\lambda^2-4\lambda^2+4}\right\}d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty cos\lambda x\left\{\frac{(\lambda^2+2)}{\lambda^4+4}\right\}d\lambda$$

3. Using the Fourier integral representation, show that

(i) $\int_0^\infty \frac{wsinxw}{1+w^2}dw = \frac{\pi}{2}e^{-x}$,x>0 (ii) $\int_0^\infty \frac{cosxw}{1+w^2}dw = \frac{\pi}{2}e^{-x}$,x>0 (ii) $\int_0^\infty \frac{sinw\, cosxw}{w}dw = \frac{\pi}{2}$ 0≤x<1

Sol: (i). Fourier sine integral for f(x) is f(x)= $\frac{2}{\pi}\int_0^\infty sinwx\left\{\int_0^\infty f(t)sinwtdt\right\} dw$ Here f(t)= e^{-t}

$$f(x) = \frac{2}{\pi}\int_0^\infty sinwx\left\{\int_0^\infty e^{-t}sinwtdt\right\} dw$$

$$= \frac{2}{\pi}\int_0^{\infty} sinwx \left\{\frac{e^{-t}}{1+w^2}[-sinwt - wcoswt]\right\}_0^{\infty} dw$$

$$= \frac{2}{\pi}\int_0^{\infty} sinwx \frac{w}{1+w^2} dw$$

$$\int_0^{\infty} \frac{wsinwx}{1+w^2} dw = \frac{\pi}{2} f(x) = f(x) = \left\{\frac{\pi}{2} e^{-x}, x > 0\right.$$ where f(x)= $e^{-x}, x>0$

(ii).Fourier Cosine integral for f(x) is f(x)=$\frac{2}{\pi}\int_0^{\infty} coswx\{\int_0^{\infty} f(t)coswtdt\}\, dw$ Here f(t)=e^{-t}

$$f(x) = \frac{2}{\pi}\int_0^{\infty} coswx\{\int_0^{\infty} f(t)coswtdt\}\, dw$$

$$= \frac{2}{\pi}\int_0^{\infty} coswx\{\int_0^{\infty} e^{-t}coswtdt\}\, dw$$

$$= \frac{2}{\pi}\int_0^{\infty} coswx \left\{\frac{e^{-t}}{1+w^2}(-coswt + wsinwt)\right\}_0^{\infty} dw$$

$$= \frac{2}{\pi}\int_0^{\infty} coswx \frac{1}{1+w^2} dw$$

$$\int_0^{\infty} coswx \frac{1}{1+w^2} dw = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-x} \quad , x \geq 0$$

(iii). Fourier integral formula for f(x) is

f(x) $= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)cosw(t-x)\, dtdw$ where f(x)=1 for 0<x<1

$$= \frac{1}{\pi}\int_0^{\infty}\{\int_{-\infty}^{0} 0 + \int_0^{1} 1cosw(t-x)\, dt + \int_1^{\infty} 0\}dw$$

$$= \frac{1}{\pi}\int_0^{\infty} \left\{\frac{sinw(t-x)}{w}\right\}_0^1 dw$$

$$= \frac{1}{\pi}\int_0^{\infty} \left[\frac{sinw(1-x)}{w} - \frac{sinw(-x)}{w}\right] dw$$

$$= \frac{1}{\pi}\int_0^{\infty} \left[\frac{sinw(1-x)+sinwx}{w}\right] dw$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{2\sin\frac{w}{2}\cos\frac{(w-2wx)}{2}}{w}\,dw$$

$$\int_0^{\infty}\frac{\sin\frac{w}{2}\cos\frac{(w-2wx)}{2}}{w}\,dw=\frac{\pi}{2}f(x)$$

FOURIER TRANSFORMS

DEFINITION

$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$ is called the **Fourier transform** of f(x) and F is the Fourier transform operator, where s is used as the transform variable. Also it is denoted as F(s)

NOTE: Sometimes the letter p or w is used as the transform variable and it is obtained from Fourier complex integral representation of f(x)

DEFINITION (AU 2001, 2007, 2010, 2011)

The $f(x)=F^{-1}[F(s)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$ is called **the inverse Fourier transform** of F(s).

NOTE: $F[f(x)]$ & $f(x)=F^{-1}[F(s)]$ together is called as Fourier transform pair.

OR IF $F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$ **THEN** $f(x)=F^{-1}[F(s)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$

DEFINITION

$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos sx\,dx$ is called as **the Fourier cosine transform** of f(x) and it is also denoted as $F_c(s)$

$f(x)=F_c^{-1}[F_c(f(x)]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}F_c(f(x)\cos sx\,ds$ is called as the **Inverse Fourier cosine transform** of f(x)

NOTE: $F_c[f(x)]$ & $f(x)=F^{-1}[Fc[f(x)]]$ together is called as **Fourier Cosine transform pair**. Obtained from Fourier Cosine integral representation of f(x).

OR **IF** $F_c[f(x)] = \int_0^\infty f(x) \cos sx\, dx$ THEN $f(x) = F_c^{-1}[F_c(f(x)] = \frac{2}{\pi}\int_0^\infty F_c(f(x) \cos sx\, ds$

DEFINITION

$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \sin sx\, dx$ is called as the **Fourier sine transform** of f(x) and it is also denoted as $F_s(s)$

$f(x) = F_s^{-1}[F_s(f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(f(x)) \sin sx\, ds$ is called as the **Inverse Fourier sine transform** of f(x)

NOTE: $F_s[f(x)]$ & $f(x) = F^{-1}[F_s[f(x)]]$ Together is called as **Fourier sine transform pair**. Obtained from Fourier sine integral representation of f(x). **OR** **IF** $F_s[f(x)] = \int_0^\infty f(x) \sin sx\, dx$ THEN $f(x) = F_s^{-1}[F_s(f(x)] = \frac{2}{\pi}\int_0^\infty F_c(f(x) \sin sx\, ds$

PROPERTIES OF FOURIER TRANSFORMS

1. LINEARITY PROPERTY: F is a linear operator. $F[(c_1 f_1(x) + c_2 f_2(x))] = c_1 F[(f_1(x))] + c_2 F[(f_2(x))]$, where c_1 and c_2 are constants.

Proof: $F[(c_1 f_1(x) + c_2 f_2(x))] = \int_{-\infty}^{\infty} (c_1 f_1(x) + c_2 f_2(x)) e^{isx} dx$

$$= c_1 \int_{-\infty}^{\infty} (f_1(x)) e^{isx} dx + c_2 \int_{-\infty}^{\infty} (f_2(x)) e^{isx} dx$$

$$= c_1 F[(f_1(x))] + c_2 F[(f_2(x))]$$

2. CHANGE OF SCALE PROPERTY: If $F[f(x)] = F[s]$, then $F[f(ax)] = \frac{1}{|a|} F\left[\frac{s}{a}\right]$

(AU 2006,2009, 2010,2011)

$F[f(ax)] = \int_{-\infty}^{\infty} f(ax)\, e^{isx} dx$ put ax=tand assuming that a>0

$$F[f(ax)] = \int_{-\infty}^{\infty} f(t)\, e^{is\frac{t}{a}} \frac{dt}{a} = \frac{1}{a} F[\frac{s}{a}]$$

But $$F[f(ax)] = \int_{\infty}^{-\infty} f(t)\, e^{is\frac{t}{a}} \frac{dt}{a} = -\frac{1}{a} F[\frac{s}{a}], \text{ if } a<0$$

Therefore $$F[f(ax)] = \frac{1}{|a|} F[\frac{s}{a}]$$

3. Shifting property (AU 1999,2000,2001,2006,2007,2008,20102011,2012)

i. $F[f(x-a)] = e^{ias} F[s]$ ii. $F[e^{iax} f(x)] = F[s+a]$

Proof: we know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{ias} dx$$

Put t = x-a , $x \rightarrow -\infty$ implise $t \rightarrow -\infty$

dt =dx, $x \rightarrow \infty$ implise $t \rightarrow \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{is(t+a)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{ist} e^{isa} dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{ist} dt$$

$$= e^{isa} F[f(t)]$$

$F[f(x-a)] = e^{isa} F[s]$ Hence proved (i)

ii. We know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left[e^{iax}f(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax}f(x)\,e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,e^{i(s+a)x}dx$$

$$F\left[e^{iax}f(x)\right] = F(s+a)$$

Hence proved (ii).

4. MODULATION THEOREM: If $F[f(x)] = F[s]$, then $F[f(x)cosax] = \frac{1}{2}\{F[s+a] + F[s-a]\}$ (AU 2001, 2003, 2010)

Proof: $F[f(x)cosax] = \int_{-\infty}^{\infty} f(x)\,cosax\,e^{isx}dx$

$$= \int_{-\infty}^{\infty} f(x)\,\frac{e^{iax}+e^{-iax}}{2}\,e^{isx}dx$$

$$=\frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)e^{iax}\,e^{isx}dx + \int_{-\infty}^{\infty} f(x)e^{-iax}\,e^{isx}dx\right]$$

$$=\frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)e^{(a+s)ix}\,dx + \int_{-\infty}^{\infty} f(x)e^{(s-a)ix}\,dx\right]$$

$$F[f(x)cosax] = \frac{1}{2}\{F[s+a] + F[s-a]\}$$

5. TRANSFORM OF DERIVATIVES: If f(x) is continuous, f'(x) is piecewise continuously differentiable(x) and f'(x) are absolutely integrble in $(-\infty,\infty)$ and limit x tends to $-\infty$ and ∞ of f(x) =0 , then $F[f'(x)] = isF[s]$

Proof: By the first three conditions given, $F[f(x)]$ and $F[f'(x)]$ exist.

$$F[f'(x)] = \int_{-\infty}^{\infty} f'(x)e^{isx}dx$$

$$= \left\{e^{-isx}f(x)\right\}_{-\infty}^{\infty} + is\int_{-\infty}^{\infty} e^{-isx}f(x)dx$$, on integration by parts.

$= 0 + is\, F[f(x)]$, by the given condition

$$F[f'(x)] = isF[s]$$

NOTE: $F\{f^{(n)}(x)\} = [is]^n F[s]$ where $F[s] = F[f(x)]$

6. If $F[s] = F[f(x)]$ then

(i). $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F[s]$

(ii). F$\left[\frac{d^n}{ds^n}\{f(x)\}\right] = (-is)^n F[s]$ if f, f', f'', ….f^{n-1}→0 as x→+∞ or - ∞

(iii) $F\left[x^k f^m(x)\right] = (-1)^{k+m} \frac{d^k}{ds^k}[s^m F\{f(x)\}]$

Proof:

(i) $\frac{d^n}{ds^n}[F(s)] = \frac{d^n}{ds^n}\left[\int_{-\infty}^{\infty} f(x) e^{isx} dx\right]$

$= \int_{-\infty}^{\infty} f(x)\left[\frac{\partial^n}{\partial s^n} e^{isx}\right] dx$ by Leibnitz's rule for constants limits

$= \int_{-\infty}^{\infty} f(x)\left[(ix)^n e^{isx}\right] dx$

$\frac{d^n}{ds^n}[F(s)] = (i)^n \int_{-\infty}^{\infty} f(x)\left[(x)^n e^{isx}\right] dx$

$\frac{d^n}{ds^n}[F(s)] = (i)^n \; F[x^n f(x)]$

$F[x^n f(x)] = \frac{1}{i^n} \frac{d^n}{ds^n}[F(s)]$ since $\frac{1}{i} = -i$ therefore $\left(\frac{1}{i}\right)^n = (-i)^n$

$F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F[s]$

(ii)$F[f'(x)] = \int_{-\infty}^{\infty} f'(x) e^{isx} dx$

$=\int_{-\infty}^{\infty} e^{isx} d(f(x))$

$=\{e^{-isx} f(x)\}_{-\infty}^{\infty}$ $-$ is $\int_{-\infty}^{\infty} e^{isx} f(x)dx$, on integration by parts.

= 0 – is $\int_{-\infty}^{\infty} e^{isx} f(x)dx$

$F[f'(x)]$ = -is F(s) similarly F $\left[\frac{d^n}{ds^n}\{f(x)\}\right]$ = $(-is)^n F[s]$

(iii) $F[x^k f^m(x)]$ = $(-i)^k \frac{d^k}{ds^k}\{F[f^m(x)]\}$ since $F[x^n f(x)]$ = $(-i)^n \frac{d^n}{ds^n} F[s]$

$=(-i)^k \frac{d^k}{ds^k}\{(-is)^m F[s]\}$ since F$\left[\frac{d^n}{ds^n}\{f(x)\}\right]$ = $(-is)^n F[s]$

$F[x^k f^m(x)]$ $=(-i)^{k+m} \frac{d^k}{ds^k}\{(s)^m F[s]\}$

CONVOLUTION: (AU 2000, 2003, 2008) The convolution of two functions f(x) and g(x) over the interval (-∞, ∞) is defined as f* $g = \int_{-\infty}^{\infty} f(u)g(x-u)du$

Convolution Theorem for Fourier Transforms: (or Faltung theorem): The Fourier Transform of the convolution of f(x) and g(x) is the product of their Fourier Transforms

.$F[f(x) * g(x)]$ =F$[f(x)]$.G $[g(x)]$

Proof:

$F[f(x) * g(x)]$ = F$\left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right]$

$=\int_{-\infty}^{\infty} e^{isx}$ $\left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right]$ dx

$= \int_{-\infty}^{\infty} f(u)$ $\left[\int_{-\infty}^{\infty} e^{isx} g(x-u)dx\right]$ du (By changing the order of integration where both variables x and u limits are constants)

Put x-u=t, dx=dt,t tends to -∞ to ∞

$=\int_{-\infty}^{\infty} f(u)\ \left[\int_{-\infty}^{\infty} e^{is(u+t)} g(t)dt\right]$ du

$=\int_{-\infty}^{\infty} f(u)\, e^{isu}\ \left[\int_{-\infty}^{\infty} e^{ist} g(t)dt\right]$ du

$=\int_{-\infty}^{\infty} f(u)\, e^{isu}$ F$[g(t)]$ du Treating u and t as dummy variable

$F[f(x) * g(x)] =$ F$[f(x)]$ F$[g(x)]$ that is $F[f(x) * g(x)] =$ F(s).G(s)

NOTE: (i) $[f(x) * g(x)] = F^{-1}\,[F(s).G(s)]$

CONJUGATE SYMMETRY PROPERTY: If F$[f(x)]$ = F$[s]$, then $F\left[\overline{f(-x)}\right] = \overline{F(s)}$

Proof: F$[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{isx} dx = F(s)$

$\overline{F(s)} = \int_{-\infty}^{\infty} f(x)\, e^{-isx} dx$ Putting –x=t , dx=-dt and t varies from ∞ to - ∞

$=\int_{\infty}^{-\infty} f(-t)\, e^{ist} (-dt)$

$=\int_{-\infty}^{\infty} f(-t)\, e^{ist} dt$ by definite integral property

$\overline{F(s)} = F\left[\overline{f(-x)}\right]$ treating t as the dummy variable

PARSEVAL'S IDENTITY THEOREM: If F$[f(x)]$ = F$[s]$, then $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$ (**AU 2002,2007,2008,2010, 2011 Statement)**

Proof: By convolution theorem we know that $F[f(x) * g(x)] =$ F(s).G(s)

That is $[f(x) * g(x)] = F^{-1}\,[F(s).G(s)]$

$\int_{-\infty}^{\infty} f(u) g(x-u) du = \int_{-\infty}^{\infty} [F(s).G(s)]\, e^{-isx} ds$

Put x=0, we get $\int_{-\infty}^{\infty} f(u) g(-u) du = \int_{-\infty}^{\infty} [F(s).G(s)]\, ds$

Now taking $g(u) = \overline{f(-u)}$ then $g(-u) = \overline{f(u)}$ also $f(u)\overline{f(u)} = |f(u)|^2$

F $[g(u)] = F\left[\overline{f(-u)}\right] = \overline{F(s)}$ implies that G(s)= $\overline{F(s)}$

$\int_{-\infty}^{\infty} f(u)\overline{f(u)}du = \int_{-\infty}^{\infty}\left[F(s).\overline{F(s)}\,\right]ds$

Hence

$$\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F(s)|^2ds$$

PROPERTIES OF FOURIER SINE AND COSINE TRANSFORMS

1. CHANGE OF SCALE PROPERTY :

$F_C\{f(ax)\} = \frac{1}{a}F_C\left\{\frac{s}{a}\right\}$ and $F_s\{f(ax)\} = \frac{1}{a}F_s\left\{\frac{s}{a}\right\}$

2. MODULATION THEOREM: If $F_c[f(x)] = F_c[s]$, and $F_s[f(x)] = F_s[s]$ then

(i) $F_c\{f(x)cosax\} = \frac{1}{2}\left[F_c[s+a] + F_c[s-a]\right]$

(i) $F_c\{f(x)sinax\} = \frac{1}{2}\left[F_s[s+a] + F_s[a-s]\right]$

(i) $F_s\{f(x)cosax\} = \frac{1}{2}\left[F_s[s+a] + F_s[s-a]\right]$

(i) $F_s\{f(x)sinax\} = \frac{1}{2}\left[F_c[s-a] - F_c[s+a]\right]$

3. PARSEVAL'S IDENTITY FOR FOURIER SINE AND COSINE TRANSFORMS:

If $F_c[f(x)] = F_c[s]$, and $F_s[f(x)] = F_s[s]$ then

(i) $\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F_c(s)|^2ds = \int_{-\infty}^{\infty}|F_s(s)|^2ds$

(ii) $\int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F_c[s]G_c[s]ds = \int_0^{\infty} F_c[s]G_c[s]ds$

4. DERIVATIVES FOR FOURIER SINE AND COSINE TRANSFORM:

(i) $F_c\{f'(x)\}$ = s$F_s\{f(x)\}$ - f(0)

(ii) $F_c\{f''(x)\}$ = - $s^2 F_c\{f(x)\}$ - f'(0)

(iii) $F_s\{f'(x)\}$ = - s$F_c\{f(x)\}$

(iv) $F_s\{f''(x)\}$ = - $s^2 F_s\{f(x)\}$ +sf(0)

NOTE: $F_c\{f'(x)\}$ = s $F_s\{f(x)\}$ and $F_s\{f'(x)\}$ = - s $F_c\{f(x)\}$ where f(x)→0 as x→∞

5. If $F_c[f(x)]$ =$F_c[s]$, and$F_s[f(x)]$ =$F_s[s]$ then (i) $\frac{d}{ds}\{F_c[s]\}$ = -$F_s\{xf(x)\}$

(ii) $\frac{d}{ds}\{F_s[s]\}$ = $F_c\{xf(x)\}$

Proof: (i) $F_c[f(x)] = \int_0^\infty f(x) cos\, sx\, dx$

$\frac{d}{ds}\{F_c[s]\}$ = $\int_0^\infty f(x)\left\{\frac{\partial}{\partial s} cos\, sx\right\} dx$ By Leibnit's rule for constants limits

$= \int_0^\infty f(x)\{-xsin\, sx\}\, dx$

$=- \int_0^\infty f(x)\{xsin\, sx\}\, dx$

$= - \int_0^\infty [xf(x)] sin\, sx\, dx$

$\frac{d}{ds}\{F_c[s]\}$ = -$F_s\{xf(x)\}$

(ii) $F_s[f(x)] = \int_0^\infty f(x) sin\, sx\, dx$

$\frac{d}{ds}\{F_s[s]\}$ = $\int_0^\infty f(x)\left\{\frac{\partial}{\partial s} sin\, sx\right\} dx$ By Leibnit's rule for constants limits

$=\int_0^\infty f(x)\{xcos\, sx\}\, dx$

$= \int_0^\infty [xf(x)] cos\, sx\, dx$

$\frac{d}{ds}\{F_s[s]\}$ = $F_c\{xf(x)\}$

PROBLEMS

1.Find the Fourier transform of the unit step function and unit impulse function.

SOL:The unit step function is defined as $u_a(x) = u(a-x) = \begin{cases} 0, & x < a \\ 1, & x \geq a \end{cases}$

$$F[u_a(x)] = \int_a^{\infty} e^{-isx} dx = \left\{\frac{e^{-isx}}{-is}\right\}_a^{\infty} = \frac{e^{-ias}}{is} \quad \text{since} \quad e^{-\infty} = 0$$

NOTE: $F[u_a(x)] = \frac{e^{-ias}}{is}$ & $F[u_0(x)] = \frac{1}{is} = -\frac{i}{s}$

The unit impulse function or Dirac Delta function $\delta_a(x) is$ $\lim_{h\to 0} f(x)$ where

$$f(x) = \begin{cases} \frac{1}{h}, & for\ a - \frac{h}{2} \leq x \leq a + \frac{h}{2} \\ 0 & ,elsewhere \end{cases}$$

$$F\{f(x)\} = \int_{a-\frac{h}{2}}^{a+\frac{h}{2}} \frac{1}{h} e^{-isx} dx$$

$$= \frac{1}{h}\left\{\frac{e^{-isx}}{-is}\right\}_{a-\frac{h}{2}}^{a+\frac{h}{2}}$$

$$= \frac{1}{ihs}\left\{e^{-is\left(a-\frac{h}{2}\right)} - e^{-is\left(a+\frac{h}{2}\right)}\right\}$$

$$F\,[f(x)] = e^{-ias} \frac{sin\left(\frac{hs}{2}\right)}{\left(\frac{hs}{2}\right)}$$

$$F\,[\delta_a(x)] = \lim_{h\to 0} e^{-ias} \frac{sin\left(\frac{hs}{2}\right)}{\left(\frac{hs}{2}\right)}$$

$$F\,[\delta_a(x)] = \left\{\text{since } \lim_{h\to 0} \frac{sin\left(\frac{hs}{2}\right)}{\left(\frac{hs}{2}\right)} = 1\right\} e^{-ias} \quad \& \quad F\,[\delta_0(1)] =$$

2. Find the Fourier transform of f(x) if $f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$ Where a is a positive real number, hence deduce that (i) $\int_0^{\infty}\frac{sint}{t}dt=\frac{\pi}{2}$ (ii) $\int_0^{\infty}\left[\frac{sint}{t}\right]^2$ dt=$\frac{\pi}{2}$ (AU 2003, 2004, 2005)

SOL: $F[f(x)]$ $=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$

$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(\cos sx+i\sin sx)dx$ $\{f(x)=1 in(-a,a)\ \cos sx\ is\ even\ \&\ sinsx\ is\ odd\ \}$

$=\frac{2}{\sqrt{2\pi}}\int_0^{a}(\cos sx)dx$

$\left\{since \int_{-a}^{a}f(x)dx=2\int_0^{a}f(x)dx\quad if\ f(x)\ is\ an\ even\ function \atop =0\ \ if\ f(x)\ is\ an\ 0dd\ function\right\}$

$F[f(x)]=\sqrt{\frac{2}{\pi}}\left\{\frac{\sin sx}{s}\right\}_0^{a}=\sqrt{\frac{2}{\pi}}\left\{\frac{\sin sa}{s}\right\}$ $[since \sin 0= 0 \qquad]$

By Fourier inversion Formula f(x)=$F^{-1}[F(s)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$

f(x)=$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left\{\frac{\sin sa}{s}\right\}(\cos sx-i\sin sx)ds$ $(\cos sx\ is\ even\ \&\ sinsx\ is\ odd)$

$=\frac{2}{\pi}\int_0^{\infty}\left\{\frac{\sin sa}{s}\right\}(\cos sx)ds$

$\left\{\int_{-a}^{a}f(x)dx=2\int_0^{a}f(x)dx\quad if\ f(x)\ is\ an\ even\ function \atop =0\qquad if\ f(x)\ is\ an\ 0dd\ function\right\}$

$=\frac{1}{\pi}\int_0^{\infty}\left\{\frac{2\sin sa\cos sx}{s}\right\}ds$ $\{put\ x=0 \Rightarrow f(0)=1\}$

1 = $\frac{2}{\pi}\int_0^{\infty}\left\{\frac{\sin sa}{s}\right\}ds$ $\{put\ as=t, ads=dt\ and\ s\to 0to\infty \Rightarrow t\to 0to\infty\}$

$\int_0^{\infty}\left\{\frac{\sin t}{\frac{t}{a}}\right\}\frac{dt}{a}=\frac{\pi}{2}\Rightarrow\int_0^{\infty}\left\{\frac{\sin t}{t}\right\}dt=\frac{\pi}{2}$

By Parseval's identity theorem $\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|F(s)|^2\,ds$

$$\int_{-a}^{a} 1^2 dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}}\left\{\frac{\sin sa}{s}\right\}\right]^2 ds$$

$$\{x\}_{-a}^{a} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin as}{s}\right]^2 ds \quad \left\{\begin{array}{ll}\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx & \textit{if } f(x) \textit{ is an even function} \\ = 0 & \textit{if } f(x) \textit{ is an Odd function}\end{array}\right\}$$

$$2a = \frac{4}{\pi}\int_0^{\infty}\left[\frac{\sin as}{s}\right]^2 ds \quad \{put\ as = t, ads = dt\ and\ s \to 0 to \infty \Rightarrow t \to 0 to \infty\}$$

$$\int_0^{\infty}\left[\frac{\sin t}{\frac{t}{a}}\right]^2 \frac{dt}{a} = \left[\frac{\pi}{4}\right] \quad \Rightarrow \int_0^{\infty}\left[\frac{\sin t}{t}\right]^2 dt\ [2a]\left[\frac{\pi}{4}\right]\left[\frac{2a}{a}\right] = \frac{\pi}{2}$$

3. Find the Fourier transform of f(x) given by $f(x) = \begin{cases} 1 - x^2, for |x| < 1 \\ 0, \quad for |x| > 1 \end{cases}$

Hence evaluate (i) $\int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{\pi}{4} f\left(\frac{1}{2}\right)$ show that (ii) $\int_0^{\infty} \frac{(s\cos s - \sin s)^2}{s^6} ds = \frac{\pi}{15}$ (AU 2000, 2001,2004,2005,2006,2007,2010,2011)

Sol: $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1 - x^2)[\cos sx + i\sin sx]dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^1(1 - x^2)[\cos sx]dx \quad \{since\ [(1 - x^2)\sin sx]\ is\ an\ odd\ function\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{(1 - x^2)\frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right\}_0^1 \{By\ Bernoullis\ theorem\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{0 - \frac{2\cos s}{s^2} + \frac{2\sin s}{s^3} - 0\right\}$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}}\left\{\frac{2\sin s - 2s\cos s}{s^3}\right\} = F(s)$$

By inversion formula f(x)=$F^{-1}[F(s)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$

f(x)=$F^{-1}[F(s)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left\{\frac{2sins-2scos\,s}{s^3}\right\}[cossx - isinsx]ds$

f(x)=$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\ 2\int_0^{\infty}\left\{\frac{2sins-2scos\,s}{s^3}\right\}[cossx]ds$ (since $\frac{2sins-2scos\,s}{s^3}$ is an even function therefore $\frac{2sins-2scos\,s}{s^3}\,cossx$ is an even function and $\frac{2sins-2scos\,s}{s^3}\,sinsx$ is an odd function)

$\int_0^{\infty}\frac{2sins-2scos\,s}{s^3}\cos sxds=\frac{\pi}{2}f(x)$ ---(i) put x=1/2 in (i) so $f\left(\frac{1}{2}\right)=1-\frac{1}{4}=\frac{3}{4}$and

$$\int_0^{\infty}\frac{sins-scos\,s}{s^3}\cos\left(\frac{s}{2}\right)ds=\frac{\pi}{4}f\left(\frac{1}{2}\right) = \left(\frac{\pi}{4}\right)\left(\frac{3}{4}\right) = \frac{3\pi}{16}$$

Sol:(ii)By Parseval's identity theorem $\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|F(s)|^2\,ds$

$$\int_{-1}^{1}(1-x^2)^2dx=\int_{-\infty}^{\infty}\left[\sqrt{\frac{2}{\pi}}\frac{2(sins-scos\,s)}{s^3}\right]^2ds$$

$$2\int_0^1(1-2x^2+x^4)dx = \frac{8}{\pi}\int_{-\infty}^{\infty}\left[\frac{sins-scos\,s}{s^3}\right]^2ds$$

$$\left[x-2\frac{x^3}{3}+\frac{x^5}{5}\right]_0^1 = \frac{4}{\pi}2\int_0^{\infty}\left[\frac{sins-scos\,s}{s^3}\right]^2ds$$

$$\left[1-\frac{2}{3}+\frac{1}{5}\right] = \quad = \frac{8}{\pi}\int_0^{\infty}\left[\frac{sins-scos\,s}{s^3}\right]^2ds$$

Therefore $\int_0^{\infty}\left[\frac{sins-scos\,s}{s^3}\right]^2ds=\left(\frac{\pi}{8}\right)\left(\frac{15-10+3}{15}\right) = \left(\frac{\pi}{8}\right)\left(\frac{8}{15}\right) = \frac{\pi}{15}$

4. Find the Fourier transform of f(x) given by $f(x)=\begin{cases}1-|x|, for\,|x| < 1\\ 0, \quad for\ |x| > 1\end{cases}$

Hence evaluate (i) $\int_0^\infty \left[\frac{\sin t}{t}\right]^2 dt = \pi/2$ (ii) $\int_0^\infty \left[\frac{\sin t}{t}\right]^4 dt = \pi/3$

(AU 2001,2005,2006,2007,2008,2009,2010,2011,2012)

Sol: $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)[cossx + isinsx]dx \ \left\{|x| = \begin{cases}-x, & x<0\\ x, & x\ge 0\end{cases} \text{ is an even function}\right\}$

$=\frac{2}{\sqrt{2\pi}}\int_0^1(1-x)[cossx]dx \qquad \{since\ [(1-|x|)sinsx]\ is\ an\ odd\ function\}$

$=\sqrt{\frac{2}{\pi}}\left\{(1-x)\frac{sinsx}{s}-(-1)\left(\frac{-cossx}{s^2}\right)\right\}_0^1 \ \{By\ Bernoullis\ theorem\}$

$= \sqrt{\frac{2}{\pi}}\left[\left\{0-\frac{cos\,s}{s^2}\right\}-\left\{0-\frac{1}{s^2}\right\}\right] \quad \{since\ \ sin0 = 0, cos0 = 1\}$

$F[f(x)] = \sqrt{\frac{2}{\pi}}\left[\left\{\frac{1-cos\,s}{s^2}\right\}\right] = F[s]$

By using inverse Fourier transform $f(x)=F^{-1}[F(s)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$

$f(x)=F^{-1}[F(s)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left\{\frac{1-cos\,s}{s^2}\right\}[cossx - isinsx]ds$

$f(x)=\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\ 2\int_0^\infty\left\{\frac{1-cos\,s}{s^2}\right\}[cossx]ds$ (since $\left\{\frac{1-cos\,s}{s^2}\right\}$ is an even function therefore $\left\{\frac{1-cos\,s}{s^2}\right\} cossx$ is an even function and $\left\{\frac{1-cos\,s}{s^2}\right\} sinsx$ is an odd function)

$$\int_0^\infty \left\{\frac{1-\cos s}{s^2}\right\} \cos sx\, ds = \frac{\pi}{2} f(x) \; \{Put\ x = 0 then f(0) = 1\}$$

$$\int_0^\infty \left\{\frac{2\sin^2\left(\frac{s}{2}\right)}{s^2}\right\} ds = \frac{\pi}{2}$$

$$\frac{1}{2}\int_0^\infty \left\{\frac{\sin^2\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2}\right\} ds = \frac{\pi}{2} \quad \left\{Put\ t = \frac{s}{2} then\ dt = \frac{ds}{2} and\ s \to 0 to \infty \Rightarrow t \to 0 to \infty\right\}$$

$$\int_0^\infty \left[\frac{\sin t}{t}\right]^2 dt = \frac{\pi}{2}$$

By Parseval's identity theorem $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$

$$\int_{-1}^{1} [1-|x|]^2 dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}}\left\{\frac{1-\cos s}{s^2}\right\}\right]^2 ds$$

$$2\int_0^1 (1-2x+x^2)dx = \frac{2}{\pi}\int_{-\infty}^{\infty} \left[\frac{1-\cos s}{s^2}\right]^2 ds$$

$$\left[x - 2\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 = \frac{1}{\pi} 2 \int_0^\infty \left[\frac{1-\cos s}{s^2}\right]^2 ds$$

$$\left[1 - 1 + \frac{1}{3}\right] = \quad = \frac{2}{\pi}\int_0^\infty \left[\frac{2\sin^2\left(\frac{s}{2}\right)}{s^2}\right]^2 ds$$

$$\int_0^\infty \left[\left(\frac{1}{2}\right)\frac{\sin^2\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2}\right]^2 ds = \left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right) \left\{Put\ t = \frac{s}{2} then\ dt = \frac{ds}{2} and\ s \to 0 to \infty \Rightarrow t \to 0 to \infty\right\}$$

$$\int_0^\infty \left(\frac{1}{4}\right)\left[\frac{\sin^2(t)}{(t)^2}\right]^2 2dt = \left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right)$$

Therefore $\int_0^\infty \left[\frac{\sin t}{t}\right]^4 dt = \frac{\pi}{3}$

5. Find the Fourier transform of f(x) given by $f(x)=\begin{cases} a^2-x^2, for|x| < a \\ 0, \quad for\ |x| > a \end{cases}$

Show that (i) $F[f(x)] = 2\sqrt{\frac{2}{\pi}}\left[\frac{sinas-ascosas}{s^3}\right]$ (ii) $\int_0^\infty \left[\frac{sint-tcost}{t^3}\right]$ dt= $\frac{\pi}{4}$ (iii) $\int_0^\infty \frac{(sint-tcost)^2}{t^6}dt = \frac{\pi}{15}$ (AU 1996,2001,2004,2008,2009,2010,2011, 2012)

Sol: F$[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)[cossx+isinsx]dx$

$=\frac{2}{\sqrt{2\pi}}\int_{0}^{a}(a^2-x^2)[cossx]dx \quad \{since\ [(a^2-x^2)sinsx]\ is\ an\ odd\ function\}$

=

$\sqrt{\frac{2}{\pi}}\left\{(a^2-x^2)\frac{sinsx}{s}-(-2x)\left(\frac{-cossx}{s^2}\right)+(-2)\left(\frac{-sinsx}{s^3}\right)\right\}_0^a \{By\ Bernoullis\ theorem\}$

$=\sqrt{\frac{2}{\pi}}\left\{0-\frac{2acos\,sa}{s^2}+\frac{2sin\,sa}{s^3}-0\right\}$

F$[f(x)]$=2$\sqrt{\frac{2}{\pi}}\left\{\frac{sinsa-ascos\,sa}{s^3}\right\}$=F(s)

By inversion formula f(x)=$F^{-1}[F(s)]$=$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$

f(x)=$F^{-1}[F(s)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left\{\frac{2sinas-2sacos\,sa}{s^3}\right\}[cossx-isinsx]ds$

f(x)= $\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\ 2\int_0^\infty\left\{\frac{2sinas-2sacos\,sa}{s^3}\right\}[cossx]ds$ (since $\frac{2sinas-2sacos\,as}{s^3}$ is an even function therefore $\frac{2sinas-2sacos\,as}{s^3}\,cossx$ is an even function and $\frac{2sinas-2sacos\,as}{s^3}\,sinsx$ is an odd function)

$$\int_0^\infty \frac{2sinas-2sacos\,as}{s^3}\cos sxds=\frac{\pi}{2}f(x) \qquad Put\ x=0 then f(0)=a^2$$

$$\int_0^\infty \frac{sinas-ascos\,sa}{s^3}ds=\frac{\pi}{4}f(0)=\frac{\pi a^2}{4}$$

$\{Put\ t=as \Rightarrow\ dt=adsand\,s\to 0to\infty \Rightarrow t\to 0to\infty\}$

$$\int_0^\infty \frac{sint-tcos\,t}{\left(\frac{t}{a}\right)^3}\frac{dt}{a}=\frac{\pi a^2}{4} \quad\Rightarrow\quad \int_0^\infty \frac{sint-tcos\,t}{t^3}dt=\frac{\pi}{4}$$

By Parseval's identity theorem $\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|F(s)|^2\,ds$

$$\int_{-a}^{a}\left(a^2-x^2\right)^2dx=\int_{-\infty}^{\infty}\left[\sqrt{\frac{2}{\pi}}\frac{2(sinas-sacos\,as)}{s^3}\right]^2ds$$

$$2\int_0^a(a^4-2a^2x^2+x^4)dx=\frac{8}{\pi}\int_{-\infty}^{\infty}\left[\frac{sinas-sacos\,as}{s^3}\right]^2ds$$

$$\left[a^4x-2a^2\frac{x^3}{3}+\frac{x^5}{5}\right]_0^a=\frac{4}{\pi}2\int_0^\infty\left[\frac{sinas-sacos\,as}{s^3}\right]^2ds$$

$$\left[a^5-\frac{2a^5}{3}+\frac{a^5}{5}\right]==\frac{8}{\pi}\int_0^\infty\left[\frac{sinas-sacos\,as}{s^3}\right]^2ds$$

$$\int_0^\infty\left[\frac{sinas-sacos\,as}{s^3}\right]^2ds=a^5\left(\frac{\pi}{8}\right)\left(\frac{15-10+3}{15}\right)=\left(\frac{\pi}{8}\right)\left(\frac{8a^5}{15}\right)=\frac{\pi a^5}{15}$$

$$\int_0^\infty\left[\frac{sint-tcos\,t}{\left(\frac{t}{a}\right)^3}\right]^2\frac{dt}{a}=\frac{\pi a^5}{15}\ \{substituting\ t=as\Rightarrow\ dt=adsand\ s\to 0to\infty\Rightarrow\ t\to 0to\infty\}$$

$$\int_0^\infty\left[\frac{sint-tcos\,t}{t^3}\right]^2dt=\frac{\pi}{15}$$

6. Find the Fourier transform of f(x) given by $f(x)=\begin{cases} a-|x|, for|x| & < a \\ 0\,, \quad for\ |x| & > a \end{cases}$

Hence evaluate (i) $\int_0^{\infty}\left[\frac{\sin t}{t}\right]^2 dt$ (ii) $\int_0^{\infty}\left[\frac{\sin t}{t}\right]^4 dt$ (AU 1996,2008,2011)

Sol: F$[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

=

$\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a-|x|)[cossx+isinsx]dx$ $\left\{|x|=\begin{cases}-x, & x<0 \\ x, & x\geq 0\end{cases} is\ an\ even\ function\right\}$

$=\frac{2}{\sqrt{2\pi}}\int_0^a(a-x)[cossx]dx$ $\{since\ [(1-|x|)sinsx]\ is\ an\ odd\ function\}$

$=\sqrt{\frac{2}{\pi}}\left\{(a-x)\frac{sinsx}{s}-(-1)\left(\frac{-cossx}{s^2}\right)\right\}_0^a$ $\{By\ Bernoullis\ theorem\}$

$=\sqrt{\frac{2}{\pi}}\left[\left\{0-\frac{\cos sa}{s^2}\right\}-\left\{0-\frac{1}{s^2}\right\}\right]$ $\{since\ \ sin0=0, cos0=1\}$

F$[f(x)]$ = $\sqrt{\frac{2}{\pi}}\left[\left\{\frac{1-\cos sa}{s^2}\right\}\right]$ $=F[s]$

By using inverse Fourier transform f(x)=$F^{-1}[F(s)]$=$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$

f(x)=$F^{-1}[F(s)]$=$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left\{\frac{1-\cos as}{s^2}\right\}[cossx-isinsx]ds$

f(x)= $\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\ 2\int_0^{\infty}\left\{\frac{1-\cos as}{s^2}\right\}[cossx]ds$ (since $\left\{\frac{1-\cos as}{s^2}\right\}$ is an even function therefore $\left\{\frac{1-\cos as}{s^2}\right\}$ $cossx$ is an even function and $\left\{\frac{1-\cos as}{s^2}\right\}$ $sinsx$ is an odd function)

$\int_0^{\infty}\left\{\frac{1-\cos as}{s^2}\right\}\cos sx\ ds=\frac{\pi}{2}f(x)$ $\{Put\ x=0 then f(0)=a\}$

$$\int_0^\infty \left\{\frac{2\sin^2\left(\frac{as}{2}\right)}{s^2}\right\} ds = \frac{\pi}{2} a$$

$$\frac{a^2}{2}\int_0^\infty \left\{\frac{\sin^2\left(\frac{as}{2}\right)}{\left(\frac{as}{2}\right)^2}\right\} ds = \frac{\pi a}{2}$$

$$\left\{Put\ t = \frac{as}{2}\ then\ dt = \frac{ads}{2} and\ s \to 0 to \infty \Rightarrow t \to 0 to \infty\right\}$$

$$\int_0^\infty \left[\frac{\sin t}{t}\right]^2 \frac{2dt}{a} = \frac{\pi}{a} \Rightarrow \int_0^\infty \left[\frac{\sin t}{t}\right]^2 dt = \frac{\pi}{2}$$

By Parseval's identity theorem $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$

$$\int_{-a}^{a} [a - |x|]^2 dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}}\left\{\frac{1-\cos as}{s^2}\right\}\right]^2 ds$$

$$2\int_0^a (a^2 - 2ax + x^2)dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{1-\cos as}{s^2}\right]^2 ds$$

$$\left[a^2x - 2\frac{ax^2}{2} + \frac{x^3}{3}\right]_0^a = \frac{1}{\pi} 2\int_0^\infty \left[\frac{1-\cos as}{s^2}\right]^2 ds$$

$$\left[a^3 - a^3 + \frac{a^3}{3}\right] = \quad = \frac{2}{\pi}\int_0^\infty \left[\frac{2\sin^2\left(\frac{as}{2}\right)}{s^2}\right]^2 ds$$

$$\int_0^\infty \left[\left(\frac{a^2}{2}\right)\frac{\sin^2\left(\frac{as}{2}\right)}{\left(\frac{sa}{2}\right)^2}\right]^2 ds = \left(\frac{\pi}{2}\right)\left(\frac{a^3}{3}\right)\left\{Put\ t = \frac{as}{2} then\ dt = \frac{ads}{2} and\ s \to 0 to \infty \Rightarrow t \to 0 to \infty\right\}$$

$$\int_0^\infty \left(\frac{a^4}{4}\right)\left[\frac{\sin^2(t)}{(t)^2}\right]^2 \frac{2dt}{a} = \left(\frac{\pi}{2}\right)\left(\frac{a^3}{3}\right)$$

Therefore $\int_0^\infty \left[\frac{\sin t}{t}\right]^4 dt = \frac{\pi}{3}$

7. Find the Fourier transform of $e^{-a^2x^2}$. Hence (i) Prove that $e^{-\frac{x^2}{2}}$ is self reciprocal with respect to Fourier transforms and (ii) Find the Fourier cosine transform of e^{-x^2} (AU 2007 2009

Sol: $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

F

$$\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2+isx}dx$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[a^2x^2-isx+\left(\frac{is}{2a}\right)^2-\left(\frac{is}{2a}\right)^2\right]}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[a^2x^2-isx+\left(\frac{is}{2a}\right)^2\right]-\frac{s^2}{4a^2}}dx$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-\left[ax-\frac{is}{2a}\right]^2}dx$$

$\left\{put\ ax-\frac{is}{2a} = t \Rightarrow adx = dt\ and\ x \to -\infty\ to \infty \Rightarrow t \to -\infty to \infty\right\}$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-[t]^2}\frac{dt}{a}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{4a^2}}\frac{2}{a}\int_{0}^{\infty} e^{-t^2}dt$$

$\left\{put\ u = t^2 \Rightarrow du = 2tdt,\ and\ t \to 0\ to \infty \Rightarrow u \to 0\ to \infty \therefore \int_0^{\infty} e^{-t^2}dt = \int_0^{\infty} e^{-u}\frac{du}{2\sqrt{u}} = \frac{1}{2}\int_0^{\infty} e^{-u}u^{\frac{1}{2}-1}du = \frac{\sqrt{\pi}}{2}\right\}$

$$= \frac{2}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \frac{\sqrt{\pi}}{2}$$

$$\therefore F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \quad (put\ a = \tfrac{1}{\sqrt{2}})$$

$\therefore F\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{2}}$ ($\therefore e^{-\frac{x^2}{2}}$ is self reciprocal with respect to Fourier transforms)

(ii) From (i) we have $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$

$\int_{-\infty}^{\infty} e^{-a^2x^2}[\cos sx + i\sin sx]dx = \frac{\sqrt{\pi}}{a} e^{-\frac{s^2}{4a^2}}$ Equating the **real parts on bothsides** , we get $\int_{-\infty}^{\infty} e^{-a^2x^2}[\cos sx]dx = \frac{\sqrt{\pi}}{a} e^{-\frac{s^2}{4a^2}}$

$2 \int_0^{\infty} e^{-a^2x^2}[\cos sx]dx = \frac{\sqrt{\pi}}{a} e^{-\frac{s^2}{4a^2}} \Rightarrow F_c\left[e^{-a^2x^2}\right] = \frac{\sqrt{\pi}}{2a} e^{-\frac{s^2}{4a^2}}$

8. Find the Fourier cosine transform of $e^{-2x} + 3e^{-x}$

Sol: $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx$ where $f(x) = \{e^{-2x} + 3e^{-x}\}$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} \{e^{-2x} + 3e^{-x}\}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} \{e^{-2x}\}\cos sx\, dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} \{3e^{-x}\}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{\frac{e^{-2x}}{4+s^2}(-2\cos sx + s\sin sx)\right\}_0^{\infty} + 3\left\{\frac{e^{-x}}{1+s^2}(-\cos sx + s\sin sx)\right\}_0^{\infty}\right]$$

$= \sqrt{\frac{2}{\pi}} \left[\left\{0+\frac{2}{4+s^2}\right\}+\left\{0+\frac{1}{1+s^2}\right\}\right]$

$\{sin\ 0 = 0, cos\ 0 = 1, e^{-\infty} = 0\ \&e^{0} = 1\}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \left[\left\{\frac{2}{4+s^2}+\frac{1}{1+s^2}\right\}\right]$$

9. Find the Fourier sine transform of f(x) = e^{-ax} , a>0 hence deduce that

$\int_0^{\infty} \frac{s}{a^2+s^2}\ sin\ sx\ ds = \frac{\pi}{2}$ f(x) $= \frac{\pi}{2} e^{-ax}$ (AU 2007,2009, 20102012)

Sol: $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) sin\ sx\, dx$ where f(x) $= \{e^{-ax}\}$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} sin\ sx\ dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{\frac{e^{-ax}}{a^2+s^2}(-asinsx - scos\ sx)\right\}_0^{\infty}\right]$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\left\{0+\frac{s}{a^2+s^2}\right\}\right]$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\left\{\frac{s}{a^2+s^2}\right\}\right] \quad a > 0$$

By inversion formula for sine transform f(x) = $F_s^{-1}[F_s(f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(f(x) sin\ sx\ ds$

f(x) = $\sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\left\{\frac{s}{a^2+s^2}\right\}\right] sin\ sx\ ds$

f(x) $= \frac{2}{\pi} \int_0^{\infty} \frac{s}{a^2+s^2}\ sin\ sx\ ds$

$\int_0^{\infty} \frac{s}{a^2+s^2}\ sin\ sx\ ds = \frac{\pi}{2}$ f(x) $= \frac{\pi}{2} e^{-ax}$, $a > 0$

$\int_0^{\infty} \frac{x}{1^2+x^2} \sin mx\, dx == \frac{\pi}{2} e^{-m}$ By replacing s=x, x=m & a=1

10. Find the Fourier sine transform of f(x) = $\frac{1}{x}$ (AU 2008,2009)

Sol: $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$ where f(x) = = $\frac{1}{x}$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{x} \sin sx\, dx \quad \{Let\ sx = \theta, sdx = d\theta, \theta \to 0\ to \infty\}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin\theta}{\frac{\theta}{s}} \frac{d\theta}{s}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin\theta}{\theta} d\theta$$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2}\right] \qquad \left\{since \int_0^{\infty} \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2}\right\}$$

11.Find $F_c[xe^{-ax}]$ and $F_s[xe^{-ax}]$ (AU 2011)

Sol: we know that (i) $\frac{d}{ds}\{F_c[s]\}$ = - $F_s\{xf(x)\}$ & (ii) $\frac{d}{ds}\{F_s[s]\}$ = $F_c\{xf(x)\}$

Also $F_s[e^{-ax}]$ = $\sqrt{\frac{2}{\pi}} \left[\left\{\frac{s}{a^2+s^2}\right\}\right]$ and $F_c[e^{-ax}]$ = $\sqrt{\frac{2}{\pi}} \left[\left\{\frac{a}{a^2+s^2}\right\}\right]$

$$\therefore F_c[xe^{-ax}] = \frac{d}{ds}\{F_s[e^{-ax}]\}$$

$$= \frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}} \left[\left\{\frac{s}{a^2+s^2}\right\}\right]\right\}$$

$$\therefore F_c[xe^{-ax}] = \sqrt{\frac{2}{\pi}} \left\{\frac{s2s-(s^2+a^2)}{[s^2+a^2]^2}\right\} = \sqrt{\frac{2}{\pi}} \left\{\frac{(s^2-a^2)}{[s^2+a^2]^2}\right\}$$

$$F_s\{xf(x)\} \quad = -\frac{d}{ds}\{F_c[f(x)]\}$$

$$= -\frac{d}{ds}\{F_c[e^{-ax}]\}$$

$$= -\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}}\left[\left\{\frac{a}{a^2+s^2}\right\}\right]\right\}$$

$$F_s\{xe^{-ax}\} \quad = \sqrt{\frac{2}{\pi}}\left\{\frac{2as}{[s^2+a^2]^2}\right\}$$

12. Find the Fourier Sine transform of $xe^{-\frac{x^2}{2}}$

Sol: $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$F_c\left[e^{-\frac{x^2}{2}}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{x^2}{2}})\cos sx\,dx = I \quad \text{(say)}$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{x^2}{2}}\left\{\frac{\partial}{\partial s}\cos sx\right\} dx$$ (By Leibnit's rule for constants limits)

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{x^2}{2}}\{-\sin sx\}x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \{\sin sx\}\,d\left[e^{-\frac{x^2}{2}}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{\sin sx\left(e^{-\frac{x^2}{2}}\right)\right\}_0^\infty - \int_0^\infty \left(e^{-\frac{x^2}{2}}\right) s\cos sx dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \int_0^\infty \left(e^{-\frac{x^2}{2}}\right) s\cos sx dx\right]$$

$$\frac{dI}{ds} = s\sqrt{\frac{2}{\pi}}\left[\int_0^\infty \left(e^{-\frac{x^2}{2}}\right)\cos sx dx\right]$$

$\frac{dI}{ds}=-sI\Rightarrow\frac{dI}{I}=sds\Rightarrow\int\frac{dI}{I}=\int sds$

$logI=-\frac{s^2}{2}+logc\Rightarrow I=e^{-\frac{s^2}{2}}e^{logc}$ when s=0, $\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{x^2}{2}})\,dx=I$

(put $t=\frac{x^2}{2}$ $then\ dt=xdx\ \ andx\rightarrow 0to\infty\Rightarrow t\rightarrow 0to\infty$x)

$$I=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-t}\frac{dt}{\sqrt{2t}}=\sqrt{\frac{1}{\pi}}\int_0^\infty e^{-t}\,t^{\left(\frac{1}{2}-1\right)}dt=\frac{\sqrt{\pi}}{\sqrt{\pi}}=1$$

Fc$\left[e^{-\frac{x^2}{2}}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty(e^{-\frac{x^2}{2}})\cos sx\,dx=I\quad I=e^{-\frac{s^2}{2}}e^{logc}\Rightarrow$c=1

$\therefore I=e^{-\frac{s^2}{2}}$ Fc$\left[e^{-\frac{x^2}{2}}\right]=I=e^{-\frac{s^2}{2}}$

$F_s\{xf(x)\}=-\frac{d}{ds}\{F_c[f(x)]\}$

$F_s\left\{xe^{-\frac{x^2}{2}}\right\}=-\frac{d}{ds}\left\{F_c\left[e^{-\frac{x^2}{2}}\right]\right\}$

$\therefore\ F_s\left\{xe^{-\frac{x^2}{2}}\right\}=-\frac{d}{ds}\left\{\left[e^{-\frac{s^2}{2}}\right]\right\}=se^{-\left(\frac{s^2}{2}\right)}$ that is $F_s\left\{xe^{-\frac{x^2}{2}}\right\}=se^{-\left(\frac{s^2}{2}\right)}$

13. Find the Fourier cosine transform of $e^{-(x^2)}$ (AU 2006,2008,2010,2011)

Sol: F$_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

F$_c\left[e^{-x^2}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos sx\,dx$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty e^{-x^2}Real\,part\,e^{isx}\,dx$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}Real\ part\int_{-\infty}^{\infty}e^{-x^2}e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}Real\ part\int_{-\infty}^{\infty}e^{-x^2+isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}Real\ part\int_{-\infty}^{\infty}e^{-\left[x^2-isx+\left(\frac{is}{2}\right)^2-\left(\frac{is}{2}\right)^2\right]}dx$$

$$=\frac{1}{\sqrt{2\pi}}Real\ part\int_{-\infty}^{\infty}e^{-\left[x^2-isx+\left(\frac{is}{2}\right)^2\right]-\frac{s^2}{4}}dx$$

$$=\frac{1}{\sqrt{2\pi}}Real\ part\,e^{-\frac{s^2}{4}}\int_{-\infty}^{\infty}e^{-\left[x-\frac{is}{2}\right]^2}dx$$

$$\left\{put\ x-\frac{is}{2}=t\Rightarrow dx=dt\ and\ x\rightarrow-\infty\ to\infty\Rightarrow t\rightarrow-\infty to\infty\right\}$$

$$=\frac{1}{\sqrt{2\pi}}Real\ part\ e^{-\frac{s^2}{4}}\int_{-\infty}^{\infty}e^{-[t]^2}dt$$

$$=\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{4}}2\int_{0}^{\infty}e^{-t^2}dt$$

$\Big\{put\ u=t^2\Rightarrow du=2tdt,\ \ and\ t\rightarrow 0\ to\infty\Rightarrow u\rightarrow 0\ to\infty \therefore \int_0^{\infty}e^{-t^2}dt=\int_0^{\infty}e^{-u}\frac{du}{2\sqrt{u}}=\frac{1}{2}\int_0^{\infty}e^{-u}u^{\frac{1}{2}-1}du=\frac{\sqrt{\pi}}{2}\Big\}$

$$Fc\left[e^{-x^2}\right]=Real\ part\ \frac{2}{\sqrt{2\pi}}e^{-\frac{s^2}{4}}\frac{\sqrt{\pi}}{2}$$

$$\therefore\ Fc\left[e^{-x^2}\right]=\frac{1}{\sqrt{2}}e^{-\frac{s^2}{4}}$$

14. Find the Fourier sine transform of $e^{-|x|}$. Hence show that $\int_0^{\infty}\frac{x\sin ax}{1+x^2}dx=\frac{\pi}{2}e^{-a}$,a>0

Sol: Fs$[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \sin sx\, dx$

$F_s\left[e^{-|x|}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-|x|} \sin sx\, dx$

$F_s\left[e^{-|x|}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x} \sin sx\, dx$

$= \sqrt{\frac{2}{\pi}}\left[\left\{\frac{e^{-x}}{1+s^2}(-sinsx - s\cos sx)\right\}_0^\infty\right]$

$= \sqrt{\frac{2}{\pi}}\left[\left\{0 + \frac{s}{1+s^2}\right\}\right] \quad \{sino = 0 and cos0 = 1\}$

$F_s\left[e^{-|x|}\right] = \sqrt{\frac{2}{\pi}}\left[\left\{\frac{s}{1+s^2}\right\}\right]$

By inversion formula f(x) = $F_s^{-1}[F_s(f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(f(x) \sin sx\, ds$

f(x) = $\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\left\{\frac{s}{1+s^2}\right\}\right] \sin sx\, ds$

$e^{-|x|} = \frac{2}{\pi}\int_0^\infty \left[\left\{\frac{s}{1+s^2}\right\}\right] \sin sx\, ds$

$\int_0^\infty \left[\left\{\frac{s}{1+s^2}\right\}\right] \sin sx\, ds = \frac{\pi}{2} e^{-x}$ (Replacing x=a & s=x then we get)

$\int_0^\infty \left[\left\{\frac{x \sin ax}{1+x^2}\right\}\right] dx = \frac{\pi}{2} e^{-a}$

15. Find the Fourier sine transform of $\frac{e^{-ax}}{x}, a > 0$ and deduce that $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} sinsxdx = \tan^{-1}\left(\frac{b}{s}\right) - \tan^{-1}\left(\frac{a}{s}\right)$ (AU 2005,2007,2009)

Sol: Fs$[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \sin sx\, dx$

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{e^{-ax}}{x} \sin sx\, dx = I \quad \text{(say)}$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{e^{-ax}}{x}\left\{\frac{\partial}{\partial s} \sin sx\right\} dx \text{ (By Leibnit's rule for constants limits)}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{e^{-ax}}{x} x\{\cos sx\}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax} \cos sx\, dx$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\left[\left\{\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s \sin sx)\right\}_0^{\infty}\right]$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\left[\left\{0 + \frac{a}{a^2+s^2}\right\}\right]$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\left[\left\{\frac{a}{a^2+s^2}\right\}\right] \quad a > 0$$ Integrating bothsides with respect to s we get

$$I = \sqrt{\frac{2}{\pi}}\, a\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right) + c$$

To show that $\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x} \sin sx\, dx = \left[\tan^{-1}\left(\frac{b}{s}\right) - \tan^{-1}\left(\frac{a}{s}\right)\right]$

We know that $F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{e^{-ax}}{x} \sin sx\, dx = I = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\left\{\frac{s}{a^2+s^2}\right\}\right] \Rightarrow \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax} \sin sx\, dx = \sqrt{\frac{2}{\pi}}\left[\left\{\frac{s}{a^2+s^2}\right\}\right]$$

Integrating with respect to 'a' on bothsides we get

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{e^{-ax}}{-x} \sin sx\, dx = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{a}{s}\right)$$

$\Rightarrow$ $F_s\left[\frac{e^{-ax}}{x}\right] = -\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{a}{s}\right)$ similarly $F_s\left[\frac{e^{-bx}}{x}\right] = -\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{b}{s}\right)$

$\therefore$ $F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{b}{s}\right) - \tan^{-1}\left(\frac{a}{s}\right)\right]$

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ax}-e^{-bx}}{x}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{b}{s}\right) - \tan^{-1}\left(\frac{a}{s}\right)\right]$$

$$\int_0^{\infty}\frac{e^{-ax}-e^{-bx}}{x}\sin sx\,dx = \left[\tan^{-1}\left(\frac{b}{s}\right) - \tan^{-1}\left(\frac{a}{s}\right)\right]$$

16. Find the Fourier sine and cosine transform of x^{n-1} and prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.

Sol: $\Gamma n = \int_0^{\infty} e^{-x}\, x^{n-1}dx$ Replacing x by ax and changing the limits,

$$\Gamma n = \int_0^{\infty} e^{-ax}a^{n-1}\, x^{n-1}adx$$

$$= a^n\int_0^{\infty} e^{-ax}\, x^{n-1}dx$$

$\Rightarrow$ $\int_0^{\infty} e^{-ax}\, x^{n-1}dx = \frac{\Gamma n}{a^n}$, n>0 Putting a=is then

$\int_0^{\infty} e^{-isx}\, x^{n-1}dx = \frac{\Gamma n}{(is)^n}$, n>0

$\int_0^{\infty} e^{-isx}\, x^{n-1}dx = \frac{(-i)^n\Gamma n}{(s)^n}$, n>0

$$\left\{(-i)^n = \left[\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\right]^n = e^{-\frac{in\pi}{2}} = \cos n\left(\frac{\pi}{2}\right) - i\sin n\left(\frac{\pi}{2}\right)\right\}$$

$\int_0^{\infty} e^{-isx}\, x^{n-1}dx = \frac{e^{-\frac{in\pi}{2}}\Gamma n}{(s)^n}$, n>0

$$\int_0^\infty e^{-isx}\, x^{n-1} dx = \frac{\Gamma n}{(s)^n} \left[cosn\left(\frac{\pi}{2}\right) - isinn\left(\frac{\pi}{2}\right)\right]$$

$$\int_0^\infty [\cos sx - isinsx]\, x^{n-1} dx = \frac{\Gamma n}{(s)^n} \left(cosn\left(\frac{\pi}{2}\right) - isinn\left(\frac{\pi}{2}\right)\right)$$

Equating the real and imaginary parts on both sides,

$$\int_0^\infty [\cos sx]\, x^{n-1} dx = \frac{\Gamma n}{(s)^n} \left(cosn\left(\frac{\pi}{2}\right)\right)$$

$$\int_0^\infty [sinsx]\, x^{n-1} dx = \frac{\Gamma n}{(s)^n} \left(sinn\left(\frac{\pi}{2}\right)\right)$$

$$\therefore F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos\left(\frac{n\pi}{2}\right) \text{ and } F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \sin\left(\frac{n\pi}{2}\right)$$

Taking $n = \frac{1}{2}$ we get $F_c\left[x^{-\left(\frac{1}{2}\right)}\right] = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \cos\left(\frac{\pi}{4}\right)$

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}} \frac{1}{\sqrt{2}} \Rightarrow F_c\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}} \quad \text{since} \left\{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right\}$$

Similarly $n = \frac{1}{2}$ substitution gives $F_s\left[x^{-\left(\frac{1}{2}\right)}\right] = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \sin\left(\frac{\pi}{4}\right)$

$$\Rightarrow F_s\left|\frac{1}{\sqrt{x}}\right| = \frac{1}{\sqrt{s}}$$

$$F_s\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}} \frac{1}{\sqrt{2}}$$

$F_c\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$ and $F_s\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$ $\Rightarrow$ that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.

17. Evaluate (i) $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$ (ii) $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$ (iii) $\int_0^\infty \frac{x^2 dx}{(a^2+x^2)^2}$

Using convolution theorem. (AU 2001,2005,2006,2009,2010,2011),

Sol: Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$ then $F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$ and

$F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2}$

By convolution theorem

By convolution theorem If $F_c[f(x)] = F_c[s]$, and $F_s[f(x)] = F_s[s]$ then

(i) $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F_c(s)|^2 ds = \int_{-\infty}^{\infty} |F_s(s)|^2 ds$

(ii) $\int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F_c[s]G_c[s]ds = \int_0^{\infty} F_s[s]G_s[s]ds$

Sol (i) $\int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F_c[s]G_c[s]ds$

$$\Rightarrow \int_0^{\infty} e^{-ax}e^{-bx}dx = \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2} ds$$

$$\int_0^{\infty} e^{-(a+b)x}dx = \frac{2}{\pi}\int_0^{\infty} \frac{ab}{(a^2+s^2)(b+s^2)} ds$$

$$\int_0^{\infty} \frac{ab}{(a^2+s^2)(b+s^2)} ds = \frac{\pi}{2}\left\{\frac{e^{-(a+b)x}}{-(a+b)}\right\}_0^{\infty}$$

$$\int_0^{\infty} \frac{ds}{(a^2+s^2)(b+s^2)} = \frac{\pi}{2ab(a+b)}$$ where a>0 and b>0.

$$\int_0^{\infty} \frac{dx}{(a^2+x^2)(b+x^2)} = \frac{\pi}{2ab(a+b)}$$

Sol(ii) $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F_c(s)|^2 ds$

$$\int_{-\infty}^{\infty} e^{-2ax}dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}\right]^2 ds$$

$$2\int_0^{\infty} e^{-2ax}dx = 2\int_0^{\infty} \frac{2}{\pi}\left\{\frac{a^2}{[a^2+s^2]^2}\right\} ds$$

$$\left\{\frac{e^{-2ax}}{-2a}\right\}_0^\infty = \frac{2}{\pi}\int_0^\infty \left\{\frac{a^2}{[a^2+s^2]^2}\right\} ds$$

$$\int_0^\infty \left\{\frac{1}{[a^2+s^2]^2}\right\} ds = \left[\frac{\pi}{2a^2}\right]\left[\frac{1}{2a}\right]$$

$$\int_0^\infty \left\{\frac{1}{[a^2+s^2]^2}\right\} ds = \left[\frac{\pi}{4a^3}\right]$$

$$\int_0^\infty \left\{\frac{1}{[a^2+x^2]^2}\right\} dx = \left[\frac{\pi}{4a^3}\right]$$

Sol(iii) $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F_s(s)|^2 ds$

$$\int_{-\infty}^{\infty} e^{-2ax} dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}}\ \frac{s}{a^2+s^2}\right]^2 ds$$

$2\int_0^\infty e^{-2ax}$ dx $=2\int_0^\infty \frac{2}{\pi}\left\{\frac{s^2}{[a^2+s^2]^2}\right\} ds$

$$\left\{\frac{e^{-2ax}}{-2a}\right\}_0^\infty = \frac{2}{\pi}\int_0^\infty \left\{\frac{s^2}{[a^2+s^2]^2}\right\} ds$$

$$\int_0^\infty \left\{\frac{1}{[a^2+s^2]^2}\right\} ds = \left[\frac{\pi}{2}\right]\left[\frac{1}{2a}\right]$$

$$\int_0^\infty \left\{\frac{s^2}{[a^2+s^2]^2}\right\} ds = \left[\frac{\pi}{4a}\right]$$

$$\int_0^\infty \left\{\frac{x^2}{[a^2+x^2]^2}\right\} dx = \left[\frac{\pi}{4a}\right]$$

18. Find the function whose sine transform is $\frac{e^{-as}}{s}$, a>0 (AU 2010,2011)

Sol: $F_s[f(x)] = \frac{e^{-as}}{s}$ The inverse Fourier sine transform of $F_s[s]$ is

f(x) = $F_s^{-1}[F_s(f(x)]$= $\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s) \sin sx\, ds$

$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{e^{-as}}{s} \sin sx\, ds$ …(i) differentiating bothsides with respect to x we get

$$f'(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{e^{-as}}{s} s\cos sx\, ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-as} \cos sx\, ds$$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{\frac{e^{-as}}{a^2+x^2}(-a\cos sx + x\sin sx)\right\}_0^{\infty}\right]$$

$$f'(x) = \sqrt{\frac{2}{\pi}}\left[\left\{0 + \frac{a}{a^2+x^2}\right\}\right]$$

$f'(x) = \sqrt{\frac{2}{\pi}}\left[\left\{\frac{a}{a^2+x^2}\right\}\right]$ $a > 0$ Integrating bothsides with respect to x we get

$f(x) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)$ +c …(ii) putting x=0 in (i) gives f(0)=0 and (ii)⇒c=0

$$\therefore f(x) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)$$

19. Find the Fourier cosine transform of e^{-4x} . Deduce

$\int_0^{\infty} \frac{\cos 2x}{x^2+16} dx = \left[\frac{\pi}{8}\right] e^{-8}$ and $\int_0^{\infty} \frac{x\sin 2x}{x^2+16} dx = \left[\frac{\pi}{2}\right] e^{-8}$

Sol: $F_c[e^{-4x}] == \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-4x} \cos sx\, dx$

$= \sqrt{\frac{2}{\pi}}\left[\left\{\frac{e^{-4x}}{16+s^2}(-4\cos sx + s\sin sx)\right\}_0^{\infty}\right]$

$F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}}\left[\left\{0+\frac{4}{16+s^2}\right\}\right]$

$F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}}\left[\left\{\frac{4}{16+s^2}\right\}\right]$ By the inverse Fourier cosine transform

f(x) = $F_c^{-1}[F_c(f(x))] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\cos sx\, ds$

f(x) = $= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\left[\left\{\frac{4}{16+s^2}\right\}\right]\cos sx\, ds$

$e^{-4x} = \frac{2}{\pi}\int_0^{\infty}\left[\left\{\frac{4}{16+s^2}\right\}\right]\cos sx\, ds$

$\int_0^{\infty}\left[\left\{\frac{1}{16+s^2}\right\}\right]\cos sx\, ds = \left[\frac{\pi}{2}\right]\left[\frac{e^{-4x}}{4}\right]$ Interchanging x as s implies that

$\int_0^{\infty}\left[\left\{\frac{\cos sx}{16+s^2}\right\}\right]ds = \left[\frac{\pi}{8}\right]e^{-4s}$ …..(i) putting s=2 in (i) we get

$\int_0^{\infty}\left[\left\{\frac{\cos 2x}{16+x^2}\right\}\right]dx = \left[\frac{\pi}{8}\right]e^{-8}$

Differentiating (i) with respect to s we get $\int_0^{\infty}\left[\left\{\frac{-x\sin sx}{16+x^2}\right\}\right]dx = \left[\frac{\pi}{8}\right]e^{-4s}(-4)$

$\Rightarrow \int_0^{\infty}\left[\left\{\frac{x\sin sx}{16+x^2}\right\}\right]dx = \left[\frac{\pi}{2}\right]e^{-4s}$ putting s=2 we get $\int_0^{\infty}\left[\left\{\frac{x\sin 2x}{16+x^2}\right\}\right]dx$
$= \left[\frac{\pi}{2}\right]e^{-8}$

20. Find the Fourier sine transform of $\frac{x}{a^2+x^2}$ and Fourier cosine transform of $\frac{1}{a^2+x^2}$

Sol: Fc$[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx$

$$Fc\left[\frac{1}{a^2+x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{1}{a^2+x^2}\cos sx\,dx = I$$ (say) …..(i)

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{1}{a^2+x^2}\left\{\frac{\partial}{\partial s}\cos sx\right\}dx$$ (By Leibnit's rule for constants limits)

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\frac{-x\sin sx}{a^2+x^2}\right]dx$$ …….(ii)

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\frac{x^2}{x}\right]\left[\frac{\sin sx}{a^2+x^2}\right]dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\frac{a^2+x^2-a^2}{x[a^2+x^2]}\right][\sin sx]\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\frac{a^2+x^2}{x[a^2+x^2]}\right][\sin sx]\,dx + a^2\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\frac{\sin sx}{x[a^2+x^2]}\right]dx$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\frac{\sin sx}{x}\right]dx + a^2\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\frac{\sin sx}{x[a^2+x^2]}\right]dx$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}}\left[\left\{\frac{\pi}{2}\right\} - a^2\left\{\int_0^{\infty}\left[\frac{\sin sx}{x[a^2+x^2]}\right]dx\right\}\right]$$ Differentiating again with respect to s

$$\frac{d^2I}{ds^2} = a^2\left\{\int_0^{\infty}\left[\frac{x\cos sx}{x[a^2+x^2]}\right]dx\right\}$$

$$\frac{d^2I}{ds^2} = a^2 I \Rightarrow \frac{d^2I}{ds^2} - a^2 I = 0$$ that is $[D^2 - a^2]I = 0$

$$\Rightarrow I = Ae^{-as} + Be^{as}$$ and $\frac{dI}{ds} = -aAe^{-as} + aBe^{as}$

Put s=0, $\Rightarrow A + B = \frac{1}{a}\sqrt{\frac{\pi}{2}}$ …(i) and $-aA + aB = -\sqrt{\frac{\pi}{2}}$ …(ii) since s=0 gives

$\Rightarrow$

$$I = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{a^2+x^2}dx = \left\{\sqrt{\frac{2}{\pi}}\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right\}_0^\infty = \frac{1}{a}\sqrt{\frac{2}{\pi}}\tan^{-1}[\infty] = \left[\frac{\pi}{2}\right]\frac{1}{a}\sqrt{\frac{2}{\pi}} = \frac{1}{a}\sqrt{\frac{\pi}{2}}$$

$$\Rightarrow\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}}\left[\left\{\frac{\pi}{2}\right\}\right] = -\sqrt{\frac{\pi}{2}}$$

Solving (i) and (ii) $A + B = \frac{1}{a}\sqrt{\frac{\pi}{2}}$ & $A - B = -\frac{1}{a}\sqrt{\frac{\pi}{2}}$ we get

$$A = 0 and B = \frac{2}{a}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{2\pi}}{a}$$

$$\therefore I = \frac{\sqrt{2\pi}}{a} \quad e^{as} = \text{Fc}\left[\frac{1}{a^2+x^2}\right]$$

F

$$s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

F $s\left[\frac{x}{a^2+x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin sx}{a^2+x^2}dx$) But $\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\int_0^\infty\left[\frac{-x\sin sx}{a^2+x^2}\right]$ dx

$\therefore$ F $s\left[\frac{x}{a^2+x^2}\right]$ =- $\frac{dI}{ds}$ $\Rightarrow$ F $s\left[\frac{x}{a^2+x^2}\right]$ = $\sqrt{2\pi} \quad e^{as}$

21. Find the Fourier sine transform of f(x) $= \begin{cases} \sin x & in\ 0 < x < \pi \\ 0 & in\ \pi \le x < \infty \end{cases}$

Sol: Fs$[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

Fs $[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\pi \sin x \sin sx\,dx$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\pi}[\cos(s-1)x-\cos(s+1)x]\,dx$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\left\{\frac{\sin(s-1)x}{s-1}-\frac{\sin(s+1)x}{s+1}\right\}_0^{\pi}\right]$$

$$=\sqrt{\frac{1}{2\pi}}\left[\left\{\frac{\sin(s-1)\pi}{s-1}-\frac{\sin(s+1)\pi}{s+1}\right\}\right]$$

$$=\sqrt{\frac{1}{2\pi}}\left[\left\{\frac{\sin(s\pi-\pi)}{s-1}-\frac{\sin(s\pi+\pi)}{s+1}\right\}\right]$$

$$=\sqrt{\frac{1}{2\pi}}\left[\left\{\frac{-\sin(\pi-s\pi)}{s-1}-\frac{\sin(\pi+s\pi)}{s+1}\right\}\right]$$

$$=\sqrt{\frac{1}{2\pi}}\left[\left\{\frac{-\sin(s\pi)}{s-1}-\frac{\sin(s\pi)}{s+1}\right\}\right]$$

$$=\frac{\sin(s\pi)}{\sqrt{2\pi}}\left[\left\{\frac{-s-1+s-1}{s^2-1}\right\}\right]$$

Fs$[f(x)]$ $=\frac{\sin(s\pi)}{\sqrt{2\pi}}\left[\left\{\frac{-2}{s^2-1}\right\}\right]\Rightarrow$ Fs$[f(x)]$ $=\sqrt{\frac{2}{\pi}}\frac{\sin(s\pi)}{1-s^2}$

22. Find the Fourier cosine transform of f(x) $=\begin{cases}\cos x & in\ 0<x<1\\ 0 & in\ 1\le x<\infty\end{cases}$

Sol: Fc$[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos sx\,dx$

$$=\sqrt{\frac{2}{\pi}}\int_0^{1}\cos x\cos sx\,dx$$

Fc$[f(x)]$ $=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{1}[\cos(s+1)x+\cos(1-s)x]\,dx$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\left\{\frac{\sin(s+1)x}{s+1}-\frac{\sin(1-s)x}{1-s}\right\}_0^1\right]$$

$$\therefore \text{Fc}[f(x)] = \sqrt{\frac{1}{2\pi}}\left[\left\{\frac{\sin(s+1)}{s+1}-\frac{\sin(1-s)}{1-s}\right\}\right]$$

23. Solve the integral equation $\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases}1-\lambda & in\ 0\le\lambda\le 1\\ 0 & in\ \lambda>1\end{cases}$

Hence evaluate $\int_0^\infty \left[\frac{\sin t}{t}\right]^2 dt$

Sol: $\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx = \sqrt{\frac{2}{\pi}}\begin{cases}1-\lambda & in\ 0\le\lambda\le 1\\ 0 & in\ \lambda>1\end{cases}$

$\therefore$ Fc$[f(x)] = \sqrt{\frac{2}{\pi}}\begin{cases}1-\lambda & in\ 0\le\lambda\le 1\\ 0 & in\ \lambda>1\end{cases}$ .... (i)

By the inverse Fourier cosine transform

$$f(x) = F_c^{-1}[F_c(f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos\lambda x\,d\lambda$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^1\sqrt{\frac{2}{\pi}}\,[1-\lambda]\cos\lambda x\,d\lambda$$

$$=\frac{2}{\pi}\int_0^1[\{1-\lambda\}]\cos\lambda x\,d\lambda$$

$$=\frac{2}{\pi}\left[\left\{[1-\lambda]\frac{\sin\lambda x}{x}-(-1)\left[\frac{-\cos\lambda x}{x^2}\right]\right\}_0^1\right]$$

$$=\frac{2}{\pi}\left[\left\{0-\frac{\cos x}{x^2}+\frac{1}{x^2}\right\}\right]$$

$$\therefore f(x) = \frac{2}{\pi}\left[\left\{\frac{1-\cos x}{x^2}\right\}\right] = \frac{2}{\pi x^2}2\sin^2\left(\frac{x}{2}\right)$$

Fc$[f(x)]$ = $\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\lambda x\,dx \qquad = \pi r^2$

Fc$[f(x)]$ = $\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{4}{\pi x^2}\sin^2\left(\frac{x}{2}\right)\cos\lambda x\,dx$ .....(ii)

From (i) and (ii) $\sqrt{\frac{2}{\pi}}\begin{cases}1-\lambda & in\ 0\le\lambda\le 1\\ 0 & in\ \lambda>1\end{cases} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{4}{\pi x^2}\sin^2\left(\frac{x}{2}\right)\cos\lambda x\,dx$

Now putting $\lambda = 0 \Rightarrow \frac{4}{\pi}\int_0^{\infty}\frac{\sin^2\left(\frac{x}{2}\right)}{x^2}dx = 1$

$$\int_0^{\infty}\frac{\sin^2\left(\frac{x}{2}\right)}{\frac{x^2}{4}}dx = \pi \Rightarrow \int_0^{\infty}\left[\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right]^2 dx = \pi$$

(put $t = \frac{x}{2}$ $then\ dt = \frac{dx}{2}$ $and x \to 0 to \infty \Rightarrow t \to 0 to \infty$x)

$$\int_0^{\infty}\left[\frac{\sin(t)}{(t)}\right]^2 2dt = \pi \Rightarrow \int_0^{\infty}\left[\frac{\sin(t)}{(t)}\right]^2 dt = \frac{\pi}{2}$$

24. Solve the integral equation $\int_0^{\infty} f(x)\cos\lambda x\,dx = e^{-\lambda}$ Also show that $\int_0^{\infty}\frac{\cos\lambda x}{\lambda^2+1}d\lambda = \left[\frac{\pi}{2}\right]e^{-x}$

Sol: Given $\int_0^{\infty} f(x)\cos\lambda x\,dx = e^{-\lambda}$

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\lambda x\,dx = \sqrt{\frac{2}{\pi}}\ e^{-\lambda}$$

$\therefore$ Fc$[f(x)]$ = $\sqrt{\frac{2}{\pi}}e^{-\lambda}$..................(i)

By the inverse Fourier cosine transform

f(x) = $F_c^{-1}[F_c(f(x)]= \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos \lambda x \, d\lambda$

f(x) $= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\, e^{-\lambda} \cos \lambda x \, d\lambda$

$$= \frac{2}{\pi} \int_0^\infty e^{-\lambda} \cos \lambda x \, d\lambda$$

$$= \frac{2}{\pi} \left[\left\{ \frac{e^{-\lambda}}{1+x^2} (-\cos \lambda x + x \sin \lambda x) \right\}_0^\infty \right]$$

$$= \frac{2}{\pi} \left[\left\{ 0 + \frac{1}{1+x^2} \right\} \right]$$

∴ f(x) $= \frac{2}{\pi} \left[\left\{ \frac{1}{1+x^2} \right\} \right]$

25. Solve the integral equation $\int_0^\infty f(x) \sin sx \, dx = \begin{cases} 1 for 0 \le s < 1 \\ 2 for 1 \le s < 2 \\ 0 for \quad s \ge 2 \end{cases}$

Sol:Given $\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx = \sqrt{\frac{2}{\pi}} \begin{cases} 1 for 0 \le s < 1 \\ 2 for 1 \le s < 2 \\ 0 \quad for \quad s \ge 2 \end{cases}$

∴ Fs$[f(x)] = \sqrt{\frac{2}{\pi}} \begin{cases} 1 for 0 \le s < 1 \\ 2 for 1 \le s < 2 \\ 0 \quad for \quad s \ge 2 \end{cases}$ ……(i)

By the inverse Fourier sine transform f(x) = $F_s^{-1}[F_s(f(x)]= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$

$$f(x) = \left[\left\{ \int_0^1 \sin sx ds \right\} + \left\{ \int_1^2 2 \sin sx ds \right\} \right]$$

$$=\frac{2}{\pi}\left[\left\{\frac{-cossx}{x}\right\}_0^1 + 2\left\{\frac{-cossx}{x}\right\}_1^2\right]$$

$$=\frac{2}{\pi}\left[\left\{\frac{1-\cos x}{x} - 2\,\frac{\cos 2x-\cos x}{x}\right\}\right]$$

$$\therefore f(x) = \frac{2}{\pi x}\left[\{1 + cosx - 2cos2x\}\right]$$

Shifting property

iii. F$[f(x-a)] = e^{ias}$ F$[s]$ ii. F$\left[e^{iax} f(x)\right]$ = F$[s+a]$

Proof: we know that

$$\text{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx}\,dx$$

$$\text{F}[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\, e^{ias}\,dx$$

Put t = x-a , x$\rightarrow -\infty$ implise t $\rightarrow -\infty$

dt =dx, x$\rightarrow \infty$ implise t $\rightarrow \infty$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\, e^{is(t+a)}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\, e^{ist}\, e^{isa}\,dt$$

$$= e^{isa}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\, e^{ist}\,dt$$

$$= e^{isa}\ \text{F}\,[f(t)]$$

F$[f(x-a)]$ = e^{isa} F$[s]$ Hence proved (i)

iv. We know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F\left[e^{iax} f(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx$$

$$F\left[e^{iax} f(x)\right] = F(s+a)$$

Hence proved (ii).