SRM INSTITUTE OF SCIENCE AND TECHNOLOGY
RAMAPURAM CAMPUS
DEPARTMENT OF MATHEMATICS

Year/Sem : II/III
Branch: Common to All branches

Unit I – Partial Differential Equations

1.Form PDE of $z = ax + a^2y^2 + b$ by eliminating arbitrary constants

(a) $\boldsymbol{q = 2p^2y}$ (b) $q = 2py$ (c) $p = 2q^2y$ (d) $p = 2qy$

Solution:
Given $z = ax + a^2y^2 + b$ ------(1)

Differentiate (1) partially with respect to $x \; and \; y$,

$$p = a \quad ---\rightarrow \; (2)$$

$$q = 2a^2y \; --\rightarrow (3)$$

Substituting (2) in (3) we get $q = 2p^2y$ which is the required PDE.

2. Form PDE of $2z = (ax + y)^2 + b$, by eliminating arbitrary constants
(a) $px + qy = p^2$ (b) $py + qx = q^2$ **(c)** $\boldsymbol{px + qy = q^2}$ (d) $p^2x + q^2y = q^2$

Solution:
Given $2z = (ax + y)^2 + b$ ------(1)

Differentiate (1) partially with respect to $x \; and \; y$,

$$2p = 2a(ax + y) \quad ---\rightarrow \; (2)$$

$$2q = 2(ax + y) \quad --\rightarrow (3)$$

Dividing (2) by (3) we get $\frac{p}{q} = a \;\; substituting \; in \; (3) \; we \; get$

$px + qy = q^2$ which is the required PDE.

3. Form PDE of $z = axe^y + \frac{a^2e^{2y}}{2} + b$, by eliminating arbitrary constants

(a) $p = xq + p^2$ **(b)** $\boldsymbol{q = xp + p^2}$ (c) $p = xq + q^2$ (d) $q = x + p^2$

Solution

Given $z = axe^y + \frac{a^2e^{2y}}{2} + b$ ------(1)

Differentiate (1) partially with respect to $x \; and \; y$,

$$p = ae^y \quad ---\rightarrow \ (2)$$

$$q = axe^y + a^2e^{2y} \quad --\rightarrow (3)$$

Substituting (2) in (3) we get $q = xp + p^2$ which is the required PDE.

4. Form PDE of $z = f(x^2 - y^2)$ by eliminating arbitrary function

(a) $\boldsymbol{yp + xq = 0}$ (b) $yp - xq = 0$ (c) $yq + xp = 0$ (d) $yq - xp = 0$

Solution:

Given $z = f(x^2 - y^2) --\rightarrow (1)$

Differentiate (1) partially with respect to $x \; and \; y$,

$$p = 2xf^{`}(x^2 - y^2) --\rightarrow (2)$$

$$q = -2yf^{`}(x^2 - y^2) --\rightarrow (3)$$

From (2) & (3) $f^{`}(x^2 - y^2) = \frac{p}{2x}$ & $f^{`}(x^2 - y^2) = \frac{q}{-2y}$

So, $\frac{p}{2x} = \frac{q}{-2y} \Rightarrow yp + xq = 0 \; which \; is \; the \; required \; PDE.$

5. Form PDE of $\varphi(x^2 + y^2 + z^2, lx + my + nz) = 0$

(a) $\boldsymbol{\frac{2x+2zp}{2y+2zq} = \frac{l+np}{m+nq}}$ (b) $\frac{2y+2zq}{2x+2zp} = \frac{l+np}{m+nq}$

(c) $\frac{2x+2zp}{2y+2zq} = \frac{m+nq}{l+np}$ (d) $\frac{2x-2zp}{2y-2zq} = \frac{l+np}{m+nq}$

Solution:

$$x^2 + y^2 + z^2 = \varphi(\, lx + my + nz) --\rightarrow (1)$$

Differentiate Partially (1) with respect to $x \; and \; y$

$$2x + 2zp = (l + np)\varphi^{`}(lx + my + nz) --\rightarrow (2)$$

$$2y + 2zq = (m + nq)\varphi^{`}(lx + my + nz) --\rightarrow (3)$$

$\frac{(2)}{(3)} \Rightarrow \frac{2x+2zp}{2y+2zq} = \frac{l+np}{m+nq}$ which is the required PDE

6.Form a PDE of $z = f(x^2 + y^2)$ $by\ eliminating\ the\ arbitrary\ function$

(a) $px = yq$ (b) $p + y = xq$ (c) $py = x + q$ **(d)** $\boldsymbol{py = xq}$

Solution:

Given $z = f(x^2 + y^2) --\rightarrow (1)$

Differentiate partially (1) with respect to $x\ and\ y\ we\ get$

$$p = 2x\, f^{`}(x^2 + y^2)$$
$$q = 2y\, f^{`}(x^2 + y^2)$$

Therefore, $\frac{p}{q} = \frac{x}{y}$ $\Rightarrow py = xq\ which\ is\ the\ required\ PDE$

7. Solve $p^2 + q^2 = 4$

(a) $\boldsymbol{z = ax \pm \sqrt{4 - a^2} + c}$ (b) $z = ax \pm \sqrt{4 + a^2} + c$

(c) $z = ax \pm \sqrt{4 - a} + c$ (d) $z = ax \pm \sqrt{4 + a} + c$

Solution:

Given $p^2 + q^2 = 4 - -\rightarrow (1)$

Let us assume that $z = ax + by + c --\rightarrow (2)$ be a solution of (1)

Partially differentiating (2) with respect to $x\ and\ y$ we get

$$p = a, q = b$$

Substituting above in (1) we get

$$a^2 + b^2 = 4 --\rightarrow (3)$$

From (3) we get $b = \pm\sqrt{4 - a^2}$

Substituting in (2) we get $z = ax \pm \sqrt{4 - a^2} + c$ which is the complete integral of (1). There is no singular integral of this type $f(p, q) = 0$

8. Find the complete integral of $p = q$

(a) $z = a(x - y) + c$ (b) $z = 2ax + c$

(c) $z = ax + y + c$ **(d)** $\boldsymbol{z = a(x + y) + c}$

Solution:

Given $p = q --\rightarrow (1)$

Let us assume that $z = ax + by + c --\rightarrow (2)$ be a solution of (1)

Partially differentiating (2) with respect to $x\ and\ y$ we get

$$p = a, q = b$$

Substituting above in (1) we get

$$a = b --\rightarrow (3)$$

Substituting in (2) we get $z = a(x + y) + c$ which is the complete integral of (1).

***9.**Find the complete solution of* $\sqrt{p} + \sqrt{q} = 1$

(a) $z = ax + (1 + \sqrt{a})^{\frac{1}{2}}y + c$ **(b)** $\boldsymbol{z = ax + (1 - \sqrt{a})^{\frac{1}{2}}y + c}$

(c) $z = ax - (1 - \sqrt{a})^{\frac{1}{2}}y + c$ (d) $z = ax - (1 + \sqrt{a})^{\frac{1}{2}}y + c$

Solution:

Given $\sqrt{p} + \sqrt{q} - 1 = 0 \quad --\rightarrow (1)$

The Complete Solution is given by $z = ax + by + c \quad --\rightarrow (2)$

Replace $p\ by\ a$ and $q\ by\ b\ \ in\ (1), we\ get\ \sqrt{a} + \sqrt{b} - 1 = 0$

$\sqrt{b} = 1 - \sqrt{a} \Rightarrow b = (1 - \sqrt{a})^{\frac{1}{2}}$

Substituting in (2), we get $z = ax + (1 - \sqrt{a})^{\frac{1}{2}}y + c$ which is required complete solution.

10. Find the Singular integral of $z = px + qy + pq$

(a) $z = xy$ (b) $z = -\frac{x}{y}$

(c) $\boldsymbol{z = -xy}$ (d) $z = \frac{x}{y}$

Solution:

The Complete Integral is $z = ax + by + ab --\rightarrow (1)$

Partially differentiating (1) with respect to $`a`\ and\ `b`\ \ and\ equating\ to\ 0$

$$\frac{\partial z}{\partial a} = x + b = 0 --\rightarrow (2)$$

$$\frac{\partial z}{\partial b} = y + a = 0 --\rightarrow (3)$$

From (2) and (3) $a = -y, b = -x\ substituting\ in\ (1)$

$$z = -xy - xy + xy$$

So, $z = -xy\ \ is\ required\ singular\ integral.$

11.Solve $p+q=1$

(a) $f(x+y,y-z)=0$ (b) $f(x-y,y+z)=0$

(c) $\boldsymbol{f(x-y,y-z)=0}$ (d) $f(x+y,y+z)=0$

Solution:

$$p+q=1\ in\ the\ form\ of\ pP+qQ=R$$

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{1}$$

Comparing 1st and 2nd & 2nd and 3rd term and integrating

$$\int dx=\int dy \quad \& \quad \int dy=\int dz$$

$$x-y=c_1,\quad y-z=c_2$$

Solution is $f(x-y,y-z)=0$

12.Find the complete integral of $p^2=qz$

(a) $z=ke^{a(x-ay)}$ (b) $z=ke^{x+ay}$

(c) $z=ke^{x-ay}$ **(d)** $\boldsymbol{z=ke^{a(x+ay)}}$

Solution:

Given $p^2=qz--\rightarrow(1)$ which of the form $f(p,q,z)=0$

Let $u=x+ay$ $,where\ `a`\ is\ arbitary\ constant.$

Replace $p\ by\frac{dz}{du}\ and\ q\ by\ a\left(\frac{dz}{du}\right)\ in\ (1)$, we get

$$\frac{dz}{du}=az$$

$$\frac{dz}{z}=adu$$

Integrating both sides , $logz=au+c\quad(or)\ z=ke^{au}$

The complete integral is given by $z=ke^{a(x+ay)}$

13. Find the complete integral of $z=pq$

(a) $\boldsymbol{4az=(x+ay+b)^2}$ (b) $4az=(x+ay+b)^3$

(c) $4az=(x-ay+b)^2$ (d) $4az=(x-ay+b)^3$

Solution:

Given $z=pq--\rightarrow(1)$ which of the form $f(p,q,z)=0$

Let $u=x+ay$ $,where\ `a`\ is\ arbitary\ constant.$

Replace $p\ by\frac{dz}{du}\ and\ q\ by\ a\left(\frac{dz}{du}\right)\ in\ (1)$, we get

$$z = a\left(\frac{dz}{du}\right)^2$$
$$\frac{dz}{du} = \pm\sqrt{\frac{z}{a}}$$
$$\frac{dz}{\sqrt{z}} = \pm\frac{du}{\sqrt{a}}$$

Integrating on both sides, $\pm 2\sqrt{az} = u + k$

Squaring on both sides and substituting $u = x + ay\ we\ get$

$4az = (x + ay + b)^2$ which is the required complete integral.

14. Solve $p^2 + q^2 = x + y$

(a) $z = \frac{2}{3}(x+k)^{\frac{3}{2}} + \frac{3}{2}(y-k)^{\frac{3}{2}}$ (b) $z = \frac{3}{2}(x+k)^{\frac{3}{2}} + \frac{2}{3}(y-k)^{\frac{3}{2}}$

(c) $\boldsymbol{z = \frac{2}{3}(x+k)^{\frac{3}{2}} + \frac{2}{3}(y-k)^{\frac{3}{2}}}$ (d) $z = \frac{3}{2}(x+k)^{\frac{3}{2}} + \frac{3}{2}(y-k)^{\frac{3}{2}}$

Solution:

The given problem can be written as

$$p^2 - x = y - q^2 \dashrightarrow (1)$$

This is of the form $f_1(x,p) = f_2(y,q)$ and there is no singular integral for this type. We will find the complete integral.

Let $p^2 - x = y - q^2 = k\ (say)$

Then $p = \sqrt{x+k}$, $q = \sqrt{y-k}$

We know that $dz = pdx + qdy$

Integrating both sides

$$z = \int \sqrt{x+k}\ dx + \int \sqrt{y-k}\ dy$$
$$z = \frac{2}{3}(x+k)^{\frac{3}{2}} + \frac{2}{3}(y-k)^{\frac{3}{2}}$$

which is the required solution

15.Solve $yp = 2yx + \log q$

(a) $\boldsymbol{z = (x^2 + kx) + \frac{e^{ky}}{k} + C}$ (b) $z = (x^2 - kx) - \frac{e^{ky}}{k} + C$

(c) $z = (x^2 + kx) - \frac{e^{ky}}{k} + C$ (d) $z = (x^2 - kx) + \frac{e^{ky}}{k} + C$

Solution:

The given problem can be written as

$$p - 2x = \frac{logq}{y} \dashrightarrow (1)$$

This is of the form $f_1(x,p) = f_2(y,q)$ and there is no singular integral for this type. We will find the complete integral.

$$p - 2x = \frac{logq}{y} = k\ (say)$$

i.e., $p - 2x = k$, $\frac{logq}{y} = k$

$$p = 2x + k, \quad logq - ky = 0$$

$$p = 2x + k, \quad q = e^{ky}$$

$$z = \int pdx + \int qdy$$

$$z = \int (2x + k)dx + \int e^{ky}\, dy$$

$$z = (x^2 + kx) + \frac{e^{ky}}{k} + C$$

$$which\ is\ the\ required\ solution$$

16. Solve $xp + yq = x$

(a) $\varphi\left(\frac{y}{x}, x - z\right) = 0$

(b) $\varphi(xy, x - z) = 0$

(c) $\boldsymbol{\varphi\left(\frac{x}{y}, x - z\right) = 0}$

(d) $\varphi\left(\frac{x}{y}, \frac{z}{x}\right) = 0$

Solution:

This is of Lagrange`s type of PDE where $P = x, Q = y, R = x$

The subsidiary equations are $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x}$

Taking first two $\frac{dx}{x} = \frac{dy}{y}$ $and\ integrating\ we\ get\ logx = logy + logc_1$

$$ie., \frac{x}{y} = c_1 \quad So, u = \frac{x}{y}$$

Taking first and last, $\frac{dx}{x} = \frac{dz}{x}$ $\Rightarrow dx = dz\ and\ integrating\ we\ get\ x = z + c_2$

So $v = x - z$

The Solution is given by $\varphi(u, v) = 0\ ie., \varphi\left(\frac{x}{y}, x - z\right) = 0$

17. *Solve* $(D^2 - 4DD^` - 5D^{`2})z = 0$

(a) $z = f_1(y-x) + f_2(y-5x)$ **(b)** $\boldsymbol{z = f_1(y-x) + f_2(y+5x)}$

(c) $z = f_1(y+x) + f_2(y+5x)$ (d) $z = f_1(y-x) + f_2(y-5x)$

Solution:

Replace $D\ by\ m\ and\ D^`by\ 1$

The auxiliary equation is given by $m^2 - 4m - 5 = 0$

$$\Rightarrow m = 5, -1$$

$The\ general\ solution\ is$ given by

$$z = f_1(y-x) + f_2(y+5x)$$

18. Solve $25r - 40s + 16t = 0$

(a) $\boldsymbol{z = f_1(5y+4x) + f_2(5y+4x)}$ (b) $z = f_1(5y+4x) + f_2(4y+5x)$

(c) $z = f_1(4y+5x) + f_2(5y+4x)$ (d) $z = f_1(y+4x) + f_2(5y+x)$

Solution

Since $r = \frac{\partial^2 z}{\partial x^2} = D^2 z$, $t = \frac{\partial^2 z}{\partial y^2} = D^{`2} z$ $s = \frac{\partial^2 z}{\partial x \partial y} = DD^` z$

The auxiliary equation is given by $25m^2 - 40m + 16 = 0 \Rightarrow (5m-4)^2 = 0$

So, $m = \frac{4}{5}, \frac{4}{5}$

$The\ general\ solution\ is$ given by

$$z = f_1(5y+4x) + f_2(5y+4x)$$

19. *Solve* $\frac{\partial^3 z}{\partial z^3} = 0$

(a) $z = f_1(y) + x^2 f_2(y) + x f_3(y)$ (b) $z = f_1(y) - x f_2(y) + x^2 f_3(y)$

(c)$\boldsymbol{z = f_1(y) + x f_2(y) + x^2 f_3(y)}$ (d)$z = f_1(y) + x f_2(y) - x^2 f_3(y)$

Solution:

The auxiliary equation is given by $m^3 = 0$

$$m = 0,0,0$$

Hence the general solution is given by

$$z = f_1(y) + x f_2(y) + x^2 f_3(y)$$

20. Find the particular integral of $\frac{\partial z}{\partial x} + \frac{\partial z}{dy} = sinx$

(a) $\boldsymbol{-cosx}$ (b) $cosx$ (c) $-sinx$ (d) $sinx$

Solution:

$P.I = \frac{1}{D+D^{`}} sinx = \int sinx\, dx = -cosx$

$P.I = -cosx$

21. Find the Particular Integral of $\left(D^2 - 2DD^{`} + D^{`2}\right)z = \cos(x - 3y)$

(a) $\boldsymbol{\frac{-1}{16} \cos(x - 3y)}$ (b) $\frac{1}{16} \cos(x - 3y)$

(c) $\frac{-1}{32} \cos(x - 3y)$ (d) $\frac{x}{16} \cos(x - 3y)$

Solution:

$$P.I = \frac{1}{D^2 - 2DD^{`} + D^{`2}} \cos(x - 3y)$$

Replace $D^2\ by - 1, D^{`2}\ by - 9, DD^{`}\ by\ 3$

$$P.I = \frac{1}{-1-6-9} \cos(x - 3y) \quad = \frac{-1}{16} \cos(x - 3y)$$

22. Find the Particular Integral of $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$

(a) $\boldsymbol{\frac{1}{2}\, e^{x+y}}$ (b) $\frac{1}{2}\, e^{x-y}$ (c) $\frac{1}{2}\, e^{2x+y}$ (d) $\frac{1}{2}\, e^{2x-y}$

Solution:

The given equation can be written as

$$\left(D^2 - 5DD^` + 6D^{`2}\right)z = e^{x+y}$$

$$P.I = \frac{1}{D^2 - 5DD^` + 6D^{`2}}\, e^{x+y}$$

$$= \frac{1}{1-5+6} e^{x+y} = \frac{1}{2}\, e^{x+y}$$

23. Solve $\left(D^3 - 6D^2D^` + 11DD^{`2} - 6D^{`3}\right)z = 0$

(a) $z = f_1(y-x) + f_2(y+2x) + f_3(y+3x)$

(b) $\boldsymbol{z = f_1(y+x) + f_2(y+2x) + f_3(y+3x)}$

(c) $z = f_1(y+x) + f_2(y-2x) + f_3(y+3x)$

(d) $z = f_1(y-x) + f_2(y+2x) + f_3(y-3x)$

Solution:

The auxiliary Equation is given by $m^3 - 6m^2 + 11m - 6 = 0$

Solving $m = 1,2,3$

C.F is $f_1(y+x) + f_2(y+2x) + f_3(y+3x)$

$$where\ f_1, f_2, f_3\ are\ arbitrary\ functions$$

Solution is $z = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$

24. Find the Particular Integral of $\left(D^2 + 3DD^` - 4D^{`2}\right)z = siny$

(a) $-\frac{1}{4}siny$ (b) $\frac{1}{4}cosy$ (c)-$\frac{1}{4}cosy$ **(d)** $\boldsymbol{\frac{1}{4}siny}$

Solution:

$P.I = \frac{1}{D^2+3DD^`-4D^{`2}}siny$

Replace D^2 by $0, D^{`2}$ $by - 1, DD^`$ by 0

$$P.I = \frac{1}{0+0-4(-1)}siny = \frac{1}{4}siny$$

25. Solve $\left(D^3 - 3D^2D^` + 4D^{`3}\right)z = 0$

(a) $z = f_1(y+x) + f_2(y+2x) + xf_3(y-2x)$

(b) $z = f_1(y+x) + f_2(y+2x) + xf_3(y+2x)$

(c) $z = f_1(y+x) + f_2(y-2x) + xf_3(y-2x)$

(d) $\boldsymbol{z = f_1(y-x) + f_2(y+2x) + xf_3(y+2x)}$

Solution:

The auxiliary equation is $m^3 - 3m^2 + 4 = 0$

The roots are $m = -1,2,2$

C.F = $f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$

Solution is $z = f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$