UNIT – I

PARTIAL DIFFERENTIAL EQUATION

INTRODUCDTION

A differential equation defined as an equation containing derivatives of various orders and the variables.

NOTE:

If x and y are independent variables and z is a dependent variable so that $z = g(x,y)$, the following standard notation are

$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q\ ,\frac{\partial^2 z}{\partial x^2} = r\ ,\ \frac{\partial^2 z}{\partial y^2} = t\ ,\frac{\partial^2 z}{\partial x \partial y} = s$

FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

The partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.

Formation of partial Differential Equations by

Elimination of Arbitrary Constants

Let us take $g(x,y,z,a,b) = 0$ where a, b are arbitrary constants; x, y are independent variables and z is a dependent variables.

To form the partial differential equation to eliminate the arbitrary constants a and b.

Note: If the number of independent variables is equal to the number of constants to be eliminated, then after elimination of arbitrary constants the partial differential equation got will be of the first order. If the number of constants to be eliminated is more than the number of independent variables, then the resulting partial differential equation will be of the second or higher orders.

Problems:

1. Form the partial differential equations by eliminating the arbitrary constants from $z = ax + by + a^2 + b^2$

   Solution

   Given $z = ax + by + a^2 + b^2$ (1)

Differentiating equation (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = a, \; i.e. \quad p = a$$

Differentiating equation (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = b, i.e. \quad q = b$$

Substituting a = p and b = q in equation (1), we get

$z = px + qy + p^2 + q^2$ which is the required partial differential equation.

2. Find the differential equation of all spheres whose centers lie on the z-axis.

Solution

The equation of a sphere whose centre lie on z-axis is

$$x^2 + y^2 + (z-c)^2 = k^2 \text{ where k and c are constants} \qquad (1)$$

.Differentiating equation (1) partially with respect to x, we get

$$2x+2(z-c)\frac{\partial z}{\partial x} = 0$$

i.e. $\quad x+(z-c)p = 0 \qquad (2)$

Differentiating equation (1) partially with respect to y, we get

$$2y+2(z-c)\frac{\partial z}{\partial y} = 0$$

i.e. $\quad y+(z-c)q = 0 \qquad (3)$

From equation (3),

$$(z-c) = -\frac{y}{q}$$

Substituting this in equation (2), we get

$$x - \frac{y}{q}p = 0$$

$$qx - yp = 0$$

$$xq = yp.$$

which is the required partial differential equation.

ELIMINATION OF ARBITRARY FUNCTIONS

Let us consider the relation $f(u,v) = 0$, where u and v are functions of x, y, z and f is an arbitrary function to be eliminated.

Differentiating, $f(u,v) = 0$ partially with respect to x and y we get the PDE of the form $P_p + Q_q = R$.

Problems

1. Form the partial differential equation by eliminating the arbitrary function ‘f ’ from

$$xy + yz + zx = f\left(\frac{z}{x+y}\right)$$

Solution

$$xy + yz + zx = f\left(\frac{z}{x+y}\right) \quad (1)$$

Differentiating (1) partially with respect to x and y, we have

$$y + yp + xp + z = f'(u)\left\{\frac{(x+y)p - z}{(x+y)^2}\right\} \quad (2)$$

and $x + yq + xq + z = f'(u)\left\{\frac{(x+y)q - z}{(x+y)^2}\right\}$ (3)

Dividing (2) by (3), we have

$$\frac{(y+z)+(x+y)p}{(z+x)+(x+y)q} = \frac{(x+y)p-z}{(x+y)q-z}$$

$$(x+y)(z+x)p - z(z+x) - z(x+y)q = (x+y)(y+z)q - z(y+z) - z(x+y)p$$

i.e. $(x+y)(x+2z)p - (x+y)(y+2z)q = z(x-y)$

which is the required partial differential equation.

2. Form the partial differential equation by eliminating the arbitrary function f and g from

z = f(2x + y) + g(3x-y)

Solution

Given $z = f(2x + y) + g(3x-y)$ (1)

Differentiating (1) partially with respect to x,

$P = f'(u)2 + g'(v)3$ where $u = 2x + y$ and $v = 3x - y$ (2)

Differentiating (1) partially with respect to y,

$q = f'(u)1 + g'(v)(-1)$ (3)

Differentiating (2) partially with respect to x and y,

$r = f''(u)4 + g''(v)9$ (4)

and $s = f''(u)2 + g''(v)(-3)$ (5)

Eliminating $f''(u)$ and $g''(v)$ from (4), (5) and (6) using determinants, we have

$$\begin{vmatrix} 4 & 9 & r \\ 2 & -3 & s \\ 1 & 1 & t \end{vmatrix} = 0$$

i.e. $5r + 5s - 30t = 0$

or $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$$

3. Find the Partial differential equation of all planes cutting equal intercepts from x and y axis

solution: **Nov.,2009**

Intercepts form of the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Given: a = b [Equal intercepts on the x and y –axis]

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Here a and c are the two arbitrary constants.

diff., (1) p.w.r to x , we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial x}=0$$

$$\frac{1}{a}+\frac{1}{c}p=0$$

$$\frac{1}{a}=\frac{-1}{c}p$$

diff. (1) p.w.r to y , we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0$$

$$\frac{1}{a}+\frac{1}{c}q=0$$

$$\frac{1}{a}=\frac{-1}{c}q$$

(2) and (3) ⇒ $\frac{-1}{c}p=\frac{-1}{c}q$

p = q

SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Equations solvable by direct integration

Problem

1. Solve $\frac{\partial^2 z}{\partial x^2}$=xy

Solution

Given $\frac{\partial^2 z}{\partial x^2}$=xy

Integrating w.r.t. ‘x’

$$\frac{\partial z}{\partial x}=\frac{x^2y}{2}+f(y)$$

Again integrating w.r.t. 'x'

$$z = \frac{x^3}{6} y + x. f(y) + g(y)$$

2. Solve $\frac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$

Solution

Given $\frac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$

Integrating w.r.t. 'x'

$$\frac{\partial z}{\partial y} = \frac{x^2}{3} + y^2 x + f(y)$$

Again integrating w.r.t. 'y'

$$z = \frac{x^3}{3} y + \frac{y^3}{3} x + f(y) + g(x)$$

STANDARD TYPES

Type 1: Equation of the form $\boldsymbol{f(p,q) = 0}$ (1)

To solve this type of equation assume the solution to be

z = ax + by + c (2)

Differentiating (2) partially w.r.t x

$$\frac{\partial z}{\partial x} = a$$

i.e. p = a

Differentiating (2) partially w.r.t y

$$\frac{\partial z}{\partial y} = b$$

q = b

Substituting for p and q in (1) we get the complete solution.

Problem

1. Solve p + q = pq

Solution

p + q = pq (1)

Let z = ax +by +c …(2) be the solution

Differentiating (2) partially w.r.t x

$$\frac{\partial z}{\partial x} = a$$

i.e. p = a

Differentiating (2) partially w.r.t y

$$\frac{\partial z}{\partial y} = b$$

q = b

Substituting for a and b in (1)

a + b =ab

$\Rightarrow$b = $\frac{a}{a-1}$

Substituting for b in (2) we get

z =ax + $\frac{a}{a-1}$ y +c , this is the complete integral.

2. Solve $\sqrt{p} + \sqrt{q} = 1$

Solution

$\sqrt{p} + \sqrt{q} = 1$ (1)

Let z = ax +by +c …(2) be the solution

Differentiating (2) partially w.r.t x

$\frac{\partial z}{\partial x} = a$ i.e. p = a

Differentiating (2) partially w.r.t y

$\frac{\partial z}{\partial y} = b$ i.e. q = b

Substituting for a and b in (1)

$\sqrt{a} + \sqrt{b} = 1$

$\Rightarrow \sqrt{b} = 1 - \sqrt{a}$

$\therefore\ b = (1 - \sqrt{a})^2$

Substituting for b in (2) we get

$z = ax + (1 - \sqrt{a})^2 y + c$ is the complete solution.

Type 2: Equation of the form **z = ax +by +** $\boldsymbol{f(p,q)}$ (1)

This form is called as Clairaut's form

To solve this type of equation assume the solution to be

z = ax + by + c (2)

Differentiating (2) partially w.r.t x

$\frac{\partial z}{\partial x} = a$ i.e. p = a

Differentiating (2) partially w.r.t y

$\frac{\partial z}{\partial y} = b$ i.e. q = b

Substituting for p and q in (1) we get the complete solution.

For type (2) singular solution also exists. The solution which does not contain any arbitrary constant is called as singular solution.

Problem

1. Solve $z = px + qy + p^2 - q^2$

Solution

$z = px + qy + p^2 - q^2$ (1)

Let $z = ax + by + c$ …(2) be the solution

Differentiating (2) partially w.r.t x

$\frac{\partial z}{\partial x} = a$ i.e. $p = a$

Differentiating (2) partially w.r.t y

$\frac{\partial z}{\partial y} = b$ i.e. $q = b$

Substituting for p and q in (1)

$z = ax + by + a^2 - b^2$ (3)

This is the complete integral.

To find the singular integral, differentiate (3) partially w.r.t 'a'

$$\frac{\partial z}{\partial a} = x + 2a$$

$\frac{\partial z}{\partial a} = 0 \Rightarrow x + 2a = 0 \Rightarrow a = \frac{-x}{2}$

Differentiating (3) partially w.r.t b

$$\frac{\partial z}{\partial b} = y - 2b$$

$\frac{\partial z}{\partial b} = 0 \Rightarrow y - 2b = 0$ $b = \frac{y}{2}$

Substituting for a and b in C.I

$z = \frac{-x}{2}x + \frac{y}{2}y + (\frac{-x}{2})^2 - (\frac{y}{2})^2$

$\therefore$ $4z = y^2 - x^2$ is the singular integral.

Type 3: Equation of the form $\boldsymbol{f(z,p,q) = 0}$

Problem

1. Solve p + q = z

   Solution

p + q = z ……(1)

Let u = x + ay

$\therefore p = \frac{dz}{du}; \quad q = a\frac{dz}{du}$

Substituting for p and q in (1) we get

$$\frac{dz}{du} + a\,\frac{dz}{du} = z$$

$$\frac{dz}{du}(1+a) = z$$

$$\frac{dz}{du} = \frac{z}{(1+a)}$$

$$\frac{dz}{z} = \frac{1}{(1+\ a)}\,du$$

Integrating we get

$\log z = \frac{1}{(1+\ a)}\ u + k$

$\log z = \frac{x+ay}{(1+a)} + k$.

2. Solve $p^2 + pq = z^2$

   Solution

$p^2 + pq = z^2$……(1)

Let u = x + ay

$\therefore p = \frac{dz}{du}; \quad q = a\frac{dz}{du}$

Substituting for p and q in (1) we get

$$\left(\frac{dz}{du}\right)^2 + \frac{dz}{du}a\frac{dz}{du} = (z)^2$$

$$\left(\frac{dz}{du}\right)^2 (1 + a) = (z)^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{(z)^2}{(1 + a)}$$

$$\frac{dz}{z} = \frac{1}{\sqrt{(1 + a)}}du$$

Integrating we get

log z = $\frac{1}{\sqrt{(1+a)}}u + k$

log z = $\frac{x+ay}{\sqrt{(1+a)}} + k$.

Type 4: Equation of the form $\boldsymbol{f_1(x,p) = f_2(y,q)}$

1. Solve $p - x^2 = q + y^2$

Solution

$$p - x^2 = q + y^2$$

Let $p - x^2 = q + y^2 = k(say)$

$p - x^2 = k$ and $q + y^2 = k$

$p = k + x^2$ and $q = k - y^2$

We know that dz = pdx + qdy (1)

Substituting for p and q in (1)

dz $=(k+x^2)dx+(k-y^2)dy$

Integrating we get,

$z = kx + ky + \frac{x^3}{3} - \frac{y^3}{3} + c$

EQUATIONS REDUCIBLE TO STANDARD TYPES

Type V : F$\left(x^m z^k p, y^n z^k q\right) = 0$

1. Solve $2x^4p^2 - yzq - 3z^2 = 0$

Solution: $2x^4p^2 - yzq - 3z^2 = 0$

$\div$ by z^2

$$\frac{2x^4p^2}{z^2} - \frac{yzq}{z^2} = \frac{3z^2}{z^2}$$

$$2\left(\frac{x^2p}{z}\right)^2 - \frac{yq}{z} = 3 \qquad (1)$$

This can be reduced to standard type by the following substitution.

X = x^{-1},Y$=log(y)$ and Z $= logz$

P $= \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X}$

Z$= logz$

$$\frac{\partial Z}{\partial z} = \frac{1}{z}$$

X = $x^{-1} \Rightarrow x = \frac{1}{X}$

$$\frac{\partial x}{\partial X} = \frac{-1}{X^2} = \frac{-1}{(x^{-1})^2} = -x^2$$

$$P = \frac{1}{z}p(-x^2) = \frac{-x^2p}{z} \qquad (2)$$

$$Q = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial Y}$$

Y= log y $\Rightarrow y = e^Y$

$$\frac{\partial y}{\partial Y} = e^Y = e^{logy} = y$$

$\therefore$ Q= $\frac{yq}{z}$ (3)

Substituting (2) and (3) in (1 we get,

$2P^2 - Q = 3.$ (4)

This is type (1)

Let z = ax +by +c …(5) be the solution

Differentiating (5) partially w.r.t x

$\frac{\partial Z}{\partial X} = a$ i.e. P = a

Differentiating (2) partially w.r.t y

$\frac{\partial Z}{\partial Y} = b$ i.e. Q = b

Substituting for P and Q in (4) we get the complete solution.

$log = \frac{a}{x} + (2a^2 - 3)logy + c.$

LAGRANGES LINEAR TYPE:

An equation of the form $P_p + Q_q = R$ where P,Q,,R are functions of x, y, z is called as Lagranges linear equation. To solve this type of equation, consider the auxiliary equation.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Consider two of these at a time such that it is integrable. The solution is $\varphi(C_1, C_2) = 0$

1. Solve x^2 p + y^2q= z^2

Solution

The equation of the form $P_p + Q_q = R$

$$\text{where} \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The Auxiliary equation is $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$

Consider $\frac{dx}{x^2} = \frac{dy}{y^2}$

Integrating both sides we get,

$$\frac{1}{y} - \frac{1}{x} = C_1$$

Consider 2nd and 3rd

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

Integrating both sides we get,

$$\frac{1}{z} - \frac{1}{y} = C_2$$

$\therefore$ The solution is $\varphi(C_1, C_2) = 0$

$$\varphi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0.$$

2. Solve $(y + z)p - (x + z)q = x - y$

Solution

The equation of the form $P_p + Q_q = R$

$$\text{where} \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The Auxiliary equation is $\quad \frac{dx}{y+z} = \frac{dy}{-x-z} = \frac{dz}{x-y}$

Taking the multipliers 1, 1, 1 each term in subsidiary equations $= \frac{dx+dy+dz}{y+z-x-z+x-y}$

$$= \frac{dx+dy+dz}{0}$$

$\Rightarrow dx + dy + dz = 0$

Integrating we get,

$$x + y + z = C_1$$

Taking the multipliers as x, y, -z

$$\frac{xdx}{x(y+z)} = \frac{ydy}{y(-x-z)} = \frac{-zdz}{-z(x-y)}$$

$$\frac{xdx+ydy-zdz}{xy+xz-yx-yz-zx+zy}$$

$$\frac{xdx+ydy-zdz}{0}$$

$\Rightarrow xdx + ydy - zdz = 0$

Integrating

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = C_2$$

$\therefore$ The solution is $\varphi(C_1, C_2) = 0$

$$\varphi\left(x + y + z, \frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2}\right) = 0.$$

HIGHER ORDER PARTIAL DIFFERENTIAL EQUATION

An equation of the form $a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = F(x,y)$ is called as homogeneous second order partial differential equation.

This equation can be written as

$$(aD^2 + bDD' + cD'^2)z = F(x,y)$$

The solution of this consists of two parts

1. Complementary function(C.F)
2. Particular integral (P.I)

To find the CF consider the auxiliary equation by replacing D by m and D' by 1.

$Am^2 + bm + c = 0$

Solving this we get two roots.

Case 1: Let the roots be m_1, m_2, then the CF is

$$z = f_1(y + m_1x) + f_2(y + m_2x)$$

Case 2: Let the roots be m, m then the CF is

$$z = f_1(y + mx) + xf_2(y + mx)$$

Problem

1. Solve $(D^2 + 2DD' + D'^2)z = e^{2x+3y}$

Solution: The A.E is $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$ $\therefore$ m = -1, -1

$\therefore$ The CF is $z = f_1(y - x) + xf_2(y - x)$

P.I:

$$z = \frac{1}{D^2+2DD'+D'^2} e^{2x+3y}$$

$$= \frac{1}{2^2+2(2)(3)+3^2} e^{2x+3y}$$

$$= \frac{1}{25} e^{2x+3y}$$

$\therefore$ The solution is z = C.F + P.I

$$z = f_1(y - x) + xf_2(y - x) + \frac{1}{25} e^{2x+3y}$$

2. Solve $(D^2 + 3DD' - 4D'^2)z = \sin(x + 5y)$

Solution: The A.E is $m^2 + 3m - 4 = 0$

$(m + 4)(m - 1) = 0$ $\therefore$ m = -4, 1

$\therefore$ The CF is $z = f_1(y - 4x) + f_2(y + x)$

P.I:

$$z = \frac{1}{D^2+3DD'-D'^2}\sin(x+5y) \qquad D^2 = -a^2 = -1,$$

$$= \frac{1}{-1+3(-5)-4(-25)}\sin(x+5y) \quad D'^2 = -b^2 = -25$$

$$= \frac{1}{84}\sin(x+5y) \qquad DD' = -ab = -5$$

∴ The solution is z = C.F + P.I

$$z = f_1(y-4x) + f_2(y+x) + \frac{1}{84}\sin(x+5y)$$

3. Solve $(D^2 - 7DD' + 6D'^2)z = xy$

Solution: The A.E is $m^2 - 7m + 6 = 0$

$(m-6)(m-1) = 0$ ∴m = 6, 1

∴ The CF is $z = f_1(y+6x) + f_2(y+x)$

P.I:

$$z = \frac{1}{D^2-7DD'+6D'^2}xy$$

$$= \frac{1}{D^2\left[1-\frac{7DD'}{D^2}+\frac{6D'^2}{D^2}\right]}xy$$

$$= \frac{1}{D^2}\left[1+\frac{7D'}{D}-\frac{6D'^2}{D^2}\right]xy$$

$$= \frac{1}{D^2}\left[xy+\frac{7}{D}D'(xy)\right]$$

$$= \frac{x^3}{6}y + \frac{7x^4}{24}$$

∴ The solution is z = C.F + P.I

$$z = f_1(y+6x) + f_2(y+x) + \frac{x^3}{6}y + \frac{7x^4}{24}$$

4. Solve $(D^2 + DD' - 2D'^2)z = y\sin x$

Solution: The A.E is $m^2 + m - 2 = 0$

$(m+2)(m-1)=0$ $\therefore$ m = -2, 1

$\therefore$ The CF is $z = f_1(y-2x) + f_2(y+x)$

P.I:

$$z = \frac{1}{D^2+DD'-2D'^2}\ y\sin x$$

$$= \frac{1}{(D-D')(D+2D')}\ y\sin x$$

$$= \frac{1}{(D-D')} \int (c+2x)\sin x\,.\,dx$$ Replacing y by c + 2x

$$= \frac{1}{(D-D')} [(c+2x)(-\cos x) - (2)(-\sin x)]$$

$$= \frac{1}{(D-D')} [-(c+2x)\cos x + 2\sin x]$$

$$= \frac{1}{(D-D')} [-y\cos x + 2\sin x]$$

$$= \int [-(c-x)\cos x + 2\sin x] dx$$

$$= -y\sin x - \cos x$$

$\therefore$ The solution is z = C.F + P.I

$$z = f_1(y-2x) + f_2(y+x) + -y\sin x - \cos x$$

Non Homogeneous equations:

A partial differential equation in which order of the derivatives is not same is called as non homogeneous equations.

Case 1: If $(D-m_1D'-\alpha_1)(D-m_2D'-\alpha_2) = F(x,y)$ then C.F $z = e^{\alpha_1 x} f_1(y+m_1x) + e^{\alpha_2 x} f_2(y+m_2x)$

Case 2: If $(D-mD'-\alpha)^2 = F(x,y)$ then C.F

$$z = e^{\alpha x} f_1(y+mx) + xe^{\alpha x} f_2(y+m_2x)$$

Problem

1. Solve $(D-D' - 1)(D-D' - 2) = e^{2x-y}$

Solution: C.F: $(D-D' - 1)(D-D' - 2) = 0$

$m_1 = 1, c_1 = 1, m_2 = 1, c_1 = 2,$

C.F is $z = e^x f_1(y + x) + e^{2x} f_2(y + x)$

P.I:

$$z = \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y}$$

$$= \frac{1}{(2-(-1)-1)(2-(-1)-2)} e^{2x-y}$$

$$= \frac{1}{2} e^{2x-y}$$

$\therefore$ The solution is z = C.F + P.I

$$z = e^x f_1(y + x) + e^{2x} f_2(y + x) + \frac{1}{2} e^{2x-y}$$

1. **Solve $(D^3 - 2D^2D')z = 0$ [NOV 2009]**

   Solution: Given $(D^3 - 2D^2D')z = 0$

   The auxiliary equation is $m^3 - 2m^2 = 0$ $[Replace\ D\ by\ m\ and\ D'\ by\ 1$

   $$m^2(m - 2) = 0$$

   m = 0, 0 ; m = 2

   C.F = $\phi_1(y + 0x) + x\phi_2(y + 0x) + \phi_3(y + 2x) = 0$

Hence , the general solution is z = C.F $\phi_1(y) + x\phi_2(y) + \phi_3(y + 2x)$

2. **Classify the partial differential equation $4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ [NOV 2009]**

   Solution: Given $4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

   $4u_{xx} - u_t = 0$

   Second order p.d.e in the function 'u' of the form

   $$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Here, A = 4, B = 0, C = 0

$$B^2 - 4AC = 0$$

Therefore the given equation is a parabolic equation.

3. Form the Partial differential equation by eliminating constants a and b from

$\boldsymbol{z = (x^2 + a^2)(y^2 + b^2)}$ **[APR 2009]**

Solution: Given $z = (x^2 + a^2)(y^2 + b^2)$..(1)

$$p = \frac{\partial z}{\partial x} = (2x)\,(y^2 + b^2)$$

$$\frac{p}{2x} = y^2 + b^2$$

$$q = \frac{\partial z}{\partial y} = (x^2 + a^2)\,(2y)$$

$$\frac{q}{2y} = (x^2 + a^2) \text{ ………………..(3)}$$

Sub (2) & (3) in (1) , we get the required p.d.e

$$z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right) = \frac{pq}{4xy}$$

4xyz = pq

i.e., pq = 4xyz

4. Solve the partial differential equation pq = x [APR 2009,APR2011]

Solution :

Given pq = x ....(1)

This equation is of the form $f(x, p, q) = 0$

Assume that $q = a$,

Then ap = x

p = $\frac{x}{a}$

$dz = pdx + qdy$

$= \frac{x}{a}dx + ady$

Integrating , $z = \frac{1}{a}\frac{x^2}{2} + ay + b$ *is the complete solution.*

5.Find the particular integral of $(D^2 - 2DD' + D'^2)z = e^{x-y}$ [NOV 2010]

Solution:

$P.I = \frac{1}{D^2-2DD'+D'^2} e^{x-y}$ (Replace D by 1 and D' by -1)

$= \frac{1}{1+2+1} e^{x-y} \quad = \frac{1}{4} e^{x-y}$

6. Solve the equation $(D - D')^3 z = 0$ **[APRIL 2011 & NOV 2011]**

Solution: The Auxillary equation is $(m-1)^3 = 0$

$$m = 1,1,1$$

General solution is $z = \phi_1(y + x) + x\phi_2(y + x) + x^2\phi_3(y + x)$

7. Find the p.d.e of the family of spheres having their centers on the z-axis**.** **[NOV 2011]**

Solution: Let the centre of the sphere be (0,0,c) a point on the z axis and k its radius.

Its equation is $(x-0)^2 + (y-0)^2 + (z-c)^2 = k^2$

i.e., $x^2 + y^2 + (z-c)^2 = k^2$ .... (1)

Here, c and k are arbitrary constants

diff (1) p.w.r to x, we get

$2x+2(z-c)\frac{\partial z}{\partial x} = 0$

$x+p(z-c) = 0$

$2y+2q(z-c) = 0$

$y+q(z-c) = 0$ .... (3)

Eliminate c from (2) & (3) we get

$$(2) \Rightarrow x+p(z-c) = 0$$

$$z-c = \frac{-x}{p}$$

$$(3) \Rightarrow z-c = \frac{-y}{q}$$

$$\frac{-x}{p} = \frac{-y}{q}$$

$$qx = py$$

8. Form the partial differential equation by eliminating the arbitrary function from $z^2 - xy = f\left(\frac{x}{z}\right)$ **[MAY 2012]**

Solution: Given: $z^2 - xy = f\left(\frac{x}{z}\right)$

Diff. (1) p w.r.t x we get ,

$$2z\frac{\partial z}{\partial x} - y = f'\left(\frac{x}{z}\right)\left[\frac{z(1) - x\frac{\partial z}{\partial x}}{z^2}\right]$$

$$2zp - y = f'\left(\frac{x}{z}\right)\left[\frac{z-xp}{z^2}\right] \text{..} (2)$$

Diff. (1) p w.r.t. y we get,

$$2z\frac{\partial z}{\partial y} - x = f'\left(\frac{x}{z}\right)\left[\frac{z(0) - x\frac{\partial z}{\partial y}}{z^2}\right]$$

$$2zq - x = f'\left(\frac{x}{z}\right)\left[\frac{-xq}{z^2}\right] \text{..} (3)$$

Divide (2) by (3) $\Rightarrow \frac{2zp-y}{2zq-x} = \frac{z-xp}{-xq}$

$$-2xzpq + xyq = 2z^2q - 2xzpq - xz + x^2p$$

$xyq = 2z^2q - xz + x^2p$

$x^2\ p + 2z^2q - xyq = xz$

$x^2\ p - (xy - 2z^2)q = xz$

9.Solve $(D^2 - 7\ DD' + 6\ D'^2)\ z = 0$ **[MAY 2012]**

Solution: Given $(D^2 - 7\ DD' + 6\ D'^2)\ z = 0$

The Auxillaryi equation is $m^2 - 7m + 6 = 0$ [Replace D by m and D' by 1]

$(m - 6)(m - 1) = 0$

$m = 6$, $m = 1$

$z = \phi_1(y + 6x) + \phi_2(y + x)$

10. Eliminate the arbitrary function 'f' from $z = f\left(\frac{y}{x}\right)$ *and* form the p.d.e

[NOV 2012]

Solution:

Given $z = f\left(\frac{y}{x}\right)$ ……………………………………………………(1)

diff (1) p.w.r. to x , we get

$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\ \frac{-y}{x^2}$ ……………………………………………………(2)

diff (1) p.w.r to y, we get

$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$……………………………………………………(3)

Divide (2) by (3) $\Rightarrow \frac{p}{q} = \frac{-y}{x}$

Therefore $px = -\ qy$

i.e., $px + qy = 0$ is required p.d.e

11. Solve $(D - 1)(D - D' + 1)\ z = 0$ **[NOV 2012]**

Solution : Given (D – 1) (D – D' + 1) z = 0

(D – 0D' - 1) (D – D' – (-1)) z = 0

by working rule,

If (D – xD' – c) z = 0, then z = $e^{cx} f(y + mx)$ *where* f is arbitrary

Here $m_1 = 0$, $c_1 = 1$

$m_2 = 1$, c_2= -1

Therefore z = $e^{x} f_1(y + 0x) + e^{-x} f_2(y + x)$

z = $e^{x} f_1(y) + e^{-x} f_2(y + x)$

12. Form the PDE from $(x – a)^2 + (y- b)^2 + z^2 = r^2$ **[MAY 2013]**

Solution: $(x – a)^2 + (y – b)^2 + z^2 = r^2$ ....(1)

Here , a and b are two arbitrary constants.

diff. (1) p.w.r. to x, we get

$2(x – a) + 2z \frac{\partial z}{\partial x} = 0$

(x – a) + zp = 0.. (2)

diff.,(1) p.w.r to y, we get

$2 (y – b) +2z \frac{\partial z}{\partial y} = 0$

(y - b) + 2q = 0 .... (3)

Eliminate a and b from (1), (2) and (3)

(2) ⇒ x – a = - z p

(3) ⇒ y – b = - z q

(1) ⇒ $(-zp)^2 + (-zq)^2 + z^2 = r^2$

$z^2p^2 + z^2q^2 + z^2 = r^2$

$z^2 [p^2 + q^2 +1] = r^2$

which is the required p.d.e.

13. Find the complete integral of p + q = pq **[MAY 2013]**

Solution:

Given : p + q = pq ....(1)

This equation is of the form F (p,q) = 0

Hence, the trial solution is z = ax + by + c.......................... (3)

To get the complete integral of (3) we have to eliminate any one of the arbitrary constants. Since in a complete integral

number of a.c. = number of I.V.

Therefore (3) ⇒ z = ax + by +c

$p = \frac{\partial z}{\partial x} = a$

$q = \frac{\partial z}{\partial y} = b$

Therefore (1) ⇒ a + b = ab

b – ab = -a

b (1 – a) = -a

$b = \frac{-a}{1-a}$

i.e. $b = \frac{a}{a-1}$

Hence the complete solution is $z = ax + \frac{a}{a-1} + c$ ....(4)

Since number of a.c. = number of I.V

PART B

1. Solve (x^2 - yz) p + (y^2 - zx) q = z^2 - xy **[MAY 2010]**

Solution: Given (x^2 - yz) p + (y^2 - zx) q = z^2 - xy

This equation is of the form Pp + Qq = R

when P = (x^2 – yz)p + (y^2 – zx) q = (z^2 –xy)………………………………(1)

Lagrange 's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$ ……………………………………………………(2)

So, Method of grouping is not possible

Use two sets of multipliers x , y , z ; 1 , 1 , 1 each of the ratio in (2)

$= \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$

$= \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$

$= \frac{xdx+ydy+zdz}{(x+y+z)} = \frac{dx+dy+dz}{1}$

⇒ xdx + ydy + zdz = (x+y+z) d(x+y+z)

Integrating on both sides, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + \frac{a}{2}$$

$x^2 + y^2 + z^2 = (x + y + z)^2 + a$

$a = x^2 + y^2 + z^2 - (x + y + z)^2$

$a = x^2 + y^2 + z^2 - x^2 - y^2 - z^2 - 2xy - 2yz - 2zx$

$= -2 (xy + yz + zx)$

$(xy + yz + zx) = - \frac{a}{2} = u$

Using two set of multipliers 1, -1, 0 ; 0, 1 , -1 each of the ratio in (2)

$$\frac{dx - dy}{(x^2 - yz) - (y^2 - zx)} = \frac{dy - dz}{(y^2 - zx) - (z^2 - xy)}$$

$$\frac{d(x - y)}{x^2 - yz - y^2 + zx)} = \frac{dy - dz}{(y - zx - z^2 - xy)}$$

$$\frac{d(x - y)}{(x^2 - y^2) + z (x - y)} = \frac{d(y - z)}{(y^2 - z^2) + x (y - z)}$$

$$\frac{d(x - y)}{(x - y)(x + y) + z (x - y)} = \frac{d(y - z)}{(y - z)(y + z) + x(y - z)}$$

$$\frac{d(x - y)}{(x - y) (x + y + z)} = \frac{d(y - z)}{(y - z) (x + y + z)}$$

$$\frac{d(x - y)}{(x - y)} = \frac{d(y - z)}{(y - z)}$$

Integrating on both sides, we get

log (x - y) = log (y – z) + log b

log (x - y) = log [b (y – z)]

x – y = b (y – z)

$$\frac{x - y}{y - z} = b = v$$

Hence, the general solution is f $(xy + yz + zx, \frac{x-y}{y-z}) = 0$

when, f is arbitrary.

(ii) p (1+q) = qz

Solution : p (1+q) = qz..(1)

This equation is of the form f (z,p,q) = 0

Let u = x + ay

$\frac{\partial u}{1\partial x} = 1, \frac{\partial u}{\partial y} = a$

p = $\frac{dz}{du}, q = a\frac{dz}{du}$

Therefore (1) $\Rightarrow \frac{dz}{du}(1 + a\frac{dz}{du}) = a\frac{dz}{du}z$

$1 + a\frac{dz}{du} = az$

$a\frac{dz}{du}\ az - 1 \Rightarrow \frac{a}{az-1}dz = du$

i.e., du = $\frac{a}{az-1}$ dz

Integratinig on both sides,

$$\int du = \int \frac{a}{az-1}dz$$

u = log (az - 1) + log c

x + ay = log [c (az-1)]

(iii) $p^2 + q^2 = x^2 + y^2$

This is a separable equation

Therefore $p^2 - x^2 = y^2 - q^2 = a^2$ (say)

i.e., p = $\sqrt{a^2 + x^2}$ *and* $q = \sqrt{y^2 - a^2}$

we know that , dz =p dx + q dy

dz = $\sqrt{a^2 + x^2}dx + \sqrt{y^2 - a^2}dy$

Integrating $\int dz = \int \sqrt{a^2 + x^2}\,dx + \int \sqrt{y^2 - a^2}dy$

z = $\frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}sinh^{-1}(\frac{x}{a}) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}cosh^{-1}\frac{y}{a} + b$

Which is the complete solution

There is no singular integral

2. Find the partial differential equation of all planes which are at a constant distance 'K' from the origin. **[May 2010]**

Solution: The equation of a plane which at a distance K from the origin is

$lx+my+nz = K$ where $l^2 + m^2 + n^2 = 1$

Let $l = \cos\alpha = a$, $m = \cos\beta$ b, $n=\cos\gamma =c$,

we get $ax+ by + cz = K$

i.e., $ax + by + \sqrt{1-a^2-b^2}z = K$..(1)

diff., (1) p.w,r to x, we get

$a + \sqrt{1-a^2-b^2}\frac{\partial z}{\partial x} = 0$

$\sqrt{1-a^2-b^2}\,p = -a$

$p = \frac{-a}{\sqrt{1-a^2-b^2}}$

diff.,(1) p,w.r to y, we get

$b + \sqrt{1-a^2-b^2}\frac{\partial z}{\partial y} = 0$

$q = \frac{-b}{\sqrt{1-a^2-b^2}}$

$$\frac{a}{p} = \frac{b}{q} = -\sqrt{1-a^2-b^2} = \lambda \; say$$

$a = p\lambda$, $b = q\lambda$, $\sqrt{1-a^2-b^2} = -\lambda$

$$\sqrt{1-p^2\lambda^2-q^2\lambda^2} = -\lambda$$

$1 - p^2\lambda^2 - q^2\lambda^2 = \lambda^2$

$1 - (p^2 + q^2)\lambda^2 = \lambda^2$

$1 = \lambda^2 + (p^2 + q^2)\lambda^2$

$\lambda = \pm \frac{1}{\sqrt{1+p^2+q^2}}$

$-\frac{1}{\lambda} = \sqrt{1 + p^2 + q^2}$

$(1) \Rightarrow p\lambda x + q\lambda y - \lambda z = K$

$(px + qy - z)\lambda = K$

$px + qy - z = \frac{K}{\lambda}$

$z = px + qy - \frac{K}{\lambda}$

$z = px + qy + K\sqrt{1 + p^2 + q^2}$

(b). (ii) Solve $(D^2+2DD'+D'^2-2D-2D')Z = \sin(x+2y)$

where $D = \frac{\partial}{\partial x}$ $and\ D' = \frac{\partial}{\partial y}$

Solution : Given $(D^2+2DD'+D'^2-2D-2D')Z = e^{2x+y} + 3$

i.e., $(D+D')(D + D' - 2)Z = e^{2x-y} + 3e^{0x+0y}$

i.e. To find the complementary function.

$(D + D')(D+D'-2)Z = 0$

i.e., $[D - (-1)D' - 0]\ [D - (-1)D' - 2]\ Z = 0$

We know that, working rule is

If $(D - m_1D' - c_1)(D - m_2D' - c_2) \dots (D - m_nD' - c_n)z = 0$

then $z = e^{c_1} f_1(y + m_1x) + e^{c_2} f_2(y + m_2x) + \cdots + e^{c_n} f_n(y + m_nx)$

Here, $m_1 = -1, c_1 = 0$

$$m_2 = -1 , c_2 = 2$$

C.F $= e^{0x} f_1[y + (-1)x] + e^{2x} f_2[y + (-1)x]$

$= f_1(y - x) + e^{2x} f_2(y - x)$

P.I $= \frac{1}{D^2+2DD'+D'^2-2D-2D'} \sin(x + 2y)$

$= \frac{1}{-1-4-4-2D-2D'} \sin(x + 2y)$

$= -\frac{1}{9+2D+2D'} \sin(x + 2y)$

$= -\frac{D}{9D+2D^2+2DD'} \sin(x + 2y)$

$= -\frac{D}{9D-2-4} \sin(x + 2y)$ $[\, since\ Replace\ D^2 by - 1^2, DD'^2 by - (1)(2)\,]$

$= -\frac{D}{9D-6}$ sin (x+2y)

$= -\frac{1}{3(3D-2)} \cos(x + 2y)$

$= -\frac{3D+2}{9D^2-4} \cos(x + 2y)$

$= -\frac{3D+2}{3(-13)} \cos(x + 2y)$

$= \frac{1}{39}[-3 \sin(x + 2y) + 2\cos(x + 2y)]$

$= -\frac{3}{39}$ sin (x+2y) + $\frac{2}{39}\cos(x + 2y)$

3.(i) Form the partial differential equation by eliminating the arbitrary function ϕ from $\phi(x^2 + y^2 + z^2 , ax+by+cz) = 0$ **[NOV 2010, MAY 2012]**

Solution:

Given u = $x^2 + y^2 + z^2$

$$\frac{\partial u}{\partial x} = 2x + 2 \quad \frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2x + 2pz$$

$$\frac{\partial u}{\partial y} = 2y + 2 \quad \frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial y} = 2y + 2zq$$

$$\frac{\partial v}{\partial x} = a + cp$$

$$\frac{\partial v}{\partial y} = b + c\,\frac{\partial z}{\partial y}$$

$$\frac{\partial v}{\partial y} = b + cq$$

Required p.d.e is given by,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2x + 2pz & a + cp \\ 2y + 2qz & b + cq \end{vmatrix} = 0$$

$\Rightarrow$ (2x + 2pz)(b + cq) – (a + cp)(2y + 2qz) = 0

$\Rightarrow$ 2bx + 2cqx +2pbz + 2pczq – 2ay – 2aqz – 2cpy – 2 cpqz = 0

(2bz – 2cy)p + (2cx – 2az)q = 2ay – 2bx

(bz - cy) p + (cz - az) q = ay –bx

(or) (cy – bz)p + (az – cx) q = bx – ay

(ii) Solve the partial differential equation

x^2 (y –z)p + y^2 (z - x)q = z^2 (x - y)

Solution:

The given equation is of the form Pp + Qq = R

where P = x^2 (y-z) , Q = y^2 (z-x), R = z^2 (x - y)

Lagrange's subsidiary esquatons are

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$ ……………………………………………….(1)

Taking the Lagrangian multipliers are $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

we get each ratio in (1),

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x\,(y-z) + y\,(z-x) + z(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

Hence,

$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating we get, logx + logy + logz = loga

log (xyz) = log a

xyz = a

Taking the Lagrangian multiples are $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$

we get each ratio in (1),

=

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{(y-z) + (z-x) + (x-y)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

Hence $\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$

Integrating we get $\int x^{-2}dx + \int y^{-2}dy + \int z^{-2}\,dz = b_1$

$$\frac{x^{-1}}{-1} + \frac{y^{-1}}{-1} + \frac{z^{-1}}{-1} = b_1$$

$$x^{-1} + y^{-1} + z^{-1} = -b_1$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = b$$

Hence, the general solution is f (a,b) = 0

i.e., f (xyz , $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$) = 0

where f is arbitrary

(ii) Solve the partial differential equation

x^2 (y –z)p + y^2 (z - x)q = z^2 (x - y)

Solution:

The Auxiliary equation is $m^3 + m^2 - 4m - 4 = 0$ [Replace D by m and D' by 1]

m^2 (m+1) - 4 (m+1) = 0

(m +1) (m +2) (m -2) = 0

⇒ m = -1, m = -2, m = 2

The roots are distinct

Therefore C.F = $\phi_1(y - x) + \phi_2\,(y - 2x) + \phi_3(y + 2x)$

P.I = $\frac{1}{D^3+D^2D'-4DD'^2-4D'^3}$ cos (2x+y) [Replace D^2 by -2^2, DD' by -2, D'^2 by -1]

= $\frac{1}{0}$ cos (2x+y) [Ordinary Rule fails]

= x $\frac{1}{3D^2+2DD'-4D^3}$ cos (2x+y)

$= x \frac{1}{-12-4+4} \cos(2x + y)$ [Replace D^2 by -2^2, DD' by -2, D'^2 by -1]

Therefore the general solution is z = C.F + P.I

$= \phi_1(y - x) + \phi_2(y - 2x) + \phi_3 \ (y + 2x) - \frac{x}{12} \cos(2x + y)$

4.(ii) Solve $[2D^2 - DD' - D'^2 + 6D + 3D'^2]\, z = xe^y$ **[NOV 2010, JUNE 2012]**

Solution: Given $[2D^2 - DD' - D'^2 + 6D + 3D'^2]\, z = xe^y$

$[(2D + D')(D - D') + 3(2D + D')]\, z = xe^y$

$(2D + D')(D - D' + 3)]\, z = xe^y$

To find C.F $(2D + D')(D - D' + 3)]\, z = 0$

$\Rightarrow [D - (\frac{-1}{2})D'][D - D' + 3]\, z = 0$

Here $m_1 = 1, c_1 = -3, m_2 = \frac{-1}{2}, c_2 = \frac{0}{2}$

$C.F = e^{-3x} f_1(y + x) + e^{0x} f_2(y - \frac{1}{2}x)$

$= e^{-3x} f_1(y + x) + f_2(y - \frac{1}{2}x)$

$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} \, x \, e^y$

$= e^y \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D+1)'} \, x$

$= e^y \frac{1}{2D^2 - DD' - D - D'^2 - 2D' - 1 + 6D + 3D' + 3} \, x$

$= e^y \frac{1}{2 + 5D + D' + 2D^2 - DD' - D'^2} \, x$

$= \frac{e^y}{2} \frac{1}{1 + \frac{1}{2}(5D + D' + 2D^2 - DD' - D'^2)} \, x$

$= \frac{e^y}{2} [1 + \frac{1}{2}(5D + D' + 2D^2 - DD' - D'^2)]^{-1} \, x$

$= \frac{e^y}{2} [1 - \frac{1}{2}(5D + D' + 2D^2 - DD' - D'^2)] \, x$

$= \frac{e^y}{2}[\, x - \frac{1}{2}[5(1) + 0 + 0]\,]$

$= \frac{e^y}{2}[\, x - \frac{5}{2}]$

$= \frac{e^y}{2}\ [\, 2x - 5]$

The general solution is z = C.F + P.I

$= e^{3x}\ f_1(y + x) + f_2\,(y - \frac{1}{2}x) + \quad \frac{e^y}{4}\ (2x - 5)$

5.(a) (i) Form the partial differential equation by eliminating arbitrary functions f and ϕ from z = f(x + ct) + ϕ(x - ct). **[MAY 2011]**

Solution: Given z = f(x + iy) + ϕ(x - iy) .... (1)

p $= \frac{\partial z}{\partial x} = f'(x + iy) + \phi'(x-iy)$ .... (2)

q $= \frac{\partial z}{\partial y} = if'(x + iy) - i\,\phi'(x-iy)$.. (3)

r $= \frac{\partial^2 z}{\partial x^2} = f''(x + iy) + \phi''(x-iy)$.. (4)

t $= \frac{\partial^2 z}{\partial y^2} = - f''(x + iy) - \phi''(x-iy)$.. (5)

r + t = o is the required p.d.e

(ii) Solve the partial differential equation (mz - ny)p + (nz-lz)q = ly – mx

Solution:

The given equation is of the form Pp+Qq=R

Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$.................(1)

Using two set of multipliers x, y, z; l, m, n each of the ratio in (1)

$$\frac{xdx + ydy + zdz}{x(mz - ny) + y\,(nx - lz) + z(ly - mz)} = \frac{xdx + ydy + zdz}{0}$$

$$\frac{ldx + mdy + ndz}{x(mz - ny) + y(nx - lz) + z\,(ly - mz)} = \frac{ldx + mdy + ndz}{0}$$

Hence xdx + ydy + zdz = 0 ; and l dx + m dy + n dz = 0

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2}, lx + my + nz = b$$

$x^2 + y^2 + z^2 = a$, $lx + my + nz = b$

Hence, the general integral is $f(x^2 + y^2 + z^2 , lx + my + nz) = 0$

6.(i) Solve $(D^2 - D'^2)z = e^{x-y}\sin(2x + 3y)$ **[MAY 2011]**

Solution:

The auxiliary equation is $m^2 - 1 = 0$

$\Rightarrow m^2 = 1; m = \pm 1$

C.F $= \phi_1(y + x) + \phi_2(y - x)$

P.I $= \frac{1}{D^2 - D'^2} e^{x-y}\sin(2x + 3y)$

$= e^{x-y}\frac{1}{(D+1)^2 - (D'-1)^2}\sin(2x + 3y)$

$= e^{x-y}\frac{1}{(D^2+2D+1) - (D'^2 - 2D'+1)}\sin(2x + 3y)$

$= e^{x-y}\frac{1}{(D^2 - D'^2 + 2D + 2D')}\sin(2x + 3y)$

$= e^{x-y}\frac{1}{5 + 2D + 2D'}\sin(2x + 3y)$

$= e^{x-y}\frac{1}{2(D + D') + 5}\sin(2x + 3y)$

$= e^{x-y}\frac{2(D + D') - 5}{[2(D + D')]^2 - 5^2}\sin(2x + 3y)$

$=e^{x-y}\frac{2(D+D')-5}{[4(D^2+2DD'+D'^2)]-5^2}\sin(2x+3y)$

$=e^{x-y}\frac{2(D+D')-5}{4[D^2+2DD'+D'^2]-5^2}\sin(2x+3y)$

$=e^{x-y}\frac{2D+2D'-5}{4[-4-12-9]-25}\sin(2x+3y)$

$= e^{x-y}\frac{2D+2D'-5}{-100-25}\sin(2x+3y)$

$=-\frac{e^{x-y}}{125}$[2D[sin(2x+3y)+2D'[sin(2x+3y)]-5sin(2x+3y)]

$= -\frac{e^{x-y}}{125}$[4cos(2x+3y) +6cos(2x+3y) -5sin(2x+3y)]

$= -\frac{e^{x-y}}{125}$[10cos(2x+3y) -5sin(2x+3y)]

$=\frac{e^{x-y}}{25}$[2cos(2x+3y) -sin(2x+3y)]

Therefore the complete solution is z = C.F + P.I

7. Find the singular integral if z = px+qy+ $\sqrt{1+p^2+q^2}$ **[MAY 2011, MAY 2013]**

Solution: Given : z = px+qy+ $\sqrt{1+p^2+q^2}$

This is of the form z= px + qy + f(p,q) [Clairaut's form]

Hence, the complete integral is z = ax+by+ $\sqrt{1+a^2+b^2}$

Where a nad b are arbitrary constants.

Singular solution is found as follows:

z = ax+by+ $\sqrt{1+a^2+b^2}$ .... (1)

diff. (1) w.r.to a & b, we get

$0=x+\frac{a}{\sqrt{1+a^2+b^2}}$

diff.,(1) p.w.r. to b, we get

$0 = y + \frac{b}{\sqrt{1+a^2+b^2}}$

i.e., $x = - \frac{a}{\sqrt{1+a^2+b^2}}$

i.e., $y = \frac{b}{\sqrt{1+a^2+b^2}}$

$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$

$1 - (x^2 + y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$

$1 - x^2 - y^2 = \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2}$

$1 - x^2 - y^2 = \frac{1}{1+a^2+b^2}$

$\sqrt{1 - x^2 - y^2} = \frac{1}{\sqrt{1+a^2+b^2}}$..(I)

$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$ ....(ii)

(2) ⇒ $x = -a\sqrt{1 - x^2 - y^2}$ by (i)

(3) ⇒ $y = - b \sqrt{1 - x^2 - y^2}$ by (i)

$a = \frac{-x}{\sqrt{1-x^2-y^2}}$, $b = \frac{-y}{\sqrt{1-x^2-y^2}}$

Sub in (i), we get

$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{-y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$ by (ii)

$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$

$z = \sqrt{1 - x^2 - y^2}$

$z^2 = 1 - x^2 - y^2$

$x^2 + y^2 + z^2 = 1$ is the singular solution

To get the general integral

put $b = \phi(a)$ in (1) we get

$z = ax + \phi(a)y + \sqrt{1 + a^2 + [\phi(a)]^2}$ .... (4)

$0 = x + \phi'(a)\, y + \frac{[2a + 2\phi(a)\phi'(a)]}{2\sqrt{1 + a^2 + [\phi(a)]}}$ .... (5)

eliminate a between (4) and (5) we get the solution

8.Solve $(D^2 - 2D^2D')z = 2e^{2x} + 3x^2y$ **[DEC 2011]**

Solution: The auxiliary equation is

$m^3 - 2m^2 = 0$

$m^2(m-2) = 0$

m = 0,0,2

C.F = $\varphi_1(y) + x\varphi_2(y) + \varphi_3(y + 2x)$

$P.I_1 = \frac{1}{D^3 - 2D^2D'} 2e^{2x+0y}$ $[Replace\ D\ by\ m\ and\ D'by\ ]$

$= 2\frac{1}{8}e^{2x} = \frac{1}{4}e^{2x}$

$P.I_2 = \frac{1}{D^3 - 2D^2D'} 3x^2y$

$$= 3\frac{1}{D^3 - 2D^2D'}x^2y$$

$$= 3\frac{1}{D^2(D - 2D')}x^2$$

$$= 3\frac{1}{D^3(1 - \frac{2D'}{D})}x^2y$$

$$= 3\frac{1}{D^3}(1 - \frac{2D'}{D})^{-1}x^2y$$

$$= 3\frac{1}{D^3}[1 + \frac{2D'}{D} + (\frac{2D'}{D})^2 + \cdots]x^2y$$

$$= 3\frac{1}{D^3}[x^2y + \frac{2}{D}(x^2)] = 3\frac{1}{D^3}[x^2y + \frac{2}{3}x^3]$$

$$= 3\frac{1}{D^2}[\frac{x^3y}{3} + \frac{2}{3}\frac{x^4}{4}] = 3\frac{1}{D}[\frac{x^4y}{12} + \frac{1}{6}\frac{x^5}{5}]$$

$$= 3[\frac{x^5y}{60} + \frac{1}{30}\frac{x^6}{6}] = [\frac{x^5y}{20} + \frac{1}{60}x^6]$$

$\therefore z = C.F + P.I_1 + P.I_2$

$z = \phi_1(y) + x\,\phi_2(y) + \phi_3(y+2x) + \frac{1}{4}e^{2x} + \frac{x^5y}{20} + \frac{x^6}{60}$

9. Solve $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = e^{2x-y}$ **[DEC 2011]**

Solution :

Given $(D - D' - 1)(D - D' - 2)\,z = e^{2x-y}$

To find C.F. :

Take $(D - D' - 1)(D - D' - 2)\,z = 0$

By working rule,

If $(D - m_1D' - c_1)(D - m_2D' - c_2)\,z = 0$ then

$z = e^{c_1 x} f_1(y+m_1x) + e^{c_1 x} f_2(y+m_2x)$

Here, $m_1 = 1$, $c_1 = 1$

$m_2 = 1$, $c_2 = 2$

C.F. $= e^x f_1(y + x) + e^{2x} f_2(y + x)$

P.I. $= \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y}$

$= \frac{1}{(2+1-1)(2+1-2)} e^{2x-y}$ [Replacing D by 2, D' by -1

$= \frac{1}{2} e^{2x-y}$

z = C.F + P.I.

$= e^x f_1(y + x) + e^{2x} f_2(y + x) + \frac{1}{2} e^{2x-y}$

10. Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ **[JUNE 2013]**

Solution:

This equation is of the form P p + Q q = R

Where P = x (y^2 - z^2), Q = y (z^2 - x^2) , R = z ($x^2 - y^2$)

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e., $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$..(2)

Use Lagrange's multipliers x,y,z we get each ratio in (2)

$= \frac{xdx+ydy+zdz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)} = \frac{xdx+ydy+zdz}{0}$

$= xdx + ydy + zdz = 0$

Integrating we get

$\int xdx + \int ydy + \int zdz = 0$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2}$

i.e., $x^2 + y^2 + z^2 = a$

Use Lagrange multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, *we get* each raio in (2)

$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

i.e $\frac{1}{x}\, dx + \frac{1}{y} dy + \frac{1}{z}\, dz = 0$

Integrating we get $\int \frac{1}{x}\, dx + \int \frac{1}{y}\, dy + \int \frac{1}{z}\, dz = 0$

Log x + log y + log z = log b

Log (xyz) = log b

i.e., xyz = b

Hence the general solution is f (a,b) =0

i.e,, f ($x^2+y^2+z^2$, xyz) = 0

where f is arbitrary.

11. Solve ($D^2 - 7DD'^2 - 6D'^2$)Z = Sin (2x + y) **[JUNE 2013]**

Solution :

The auxiliary equation is m – 7m – 6 = 0

(m + 1) (m+2) (m-3) = 0

i.e., m = -1 , m = -2, m = 3

C.F = $\varphi_1(y-x)+\varphi_2(y-2x)+\varphi_3(y+3x)$

P.I = $\frac{1}{D^3-7\,DD'^2-6\,D'^3}$sin (2x+y)

= $\frac{1}{-4D-7(-2)D'-6(-1)\,D'}$sin (2x+y)

= $\frac{1}{-4D+14D'+6\,D'}$sin (2x+y)

= $\frac{1}{-4D+20D'}$ sin (2x+y)

=$\frac{-1}{4}[\frac{1}{D-5D'}]$ sin (2x+y)

= $\frac{-1}{4}[\frac{D+5D'}{D^2-25D'^2}]$ sin (2x+y)

= $\frac{-1}{4}[\frac{D+5D'}{-4-25(-1)}]$ sin (2x+y)

= $\frac{-1}{4}[\frac{D+5D'}{21}]$ sin (2x+y)

= $\frac{-1}{84}$ (D+5D') sin (2x+y)

= $\frac{-1}{84}$ [2cos (2x+y) + 5 cos(2x+y)]

= $\frac{-1}{12}$ cos (2x+y)

Therefore z = C.F + P.I

= $\varphi_1(y-x)+\varphi_2(y-2x)+\varphi_3(y+3x)$ - $\frac{-1}{12}$ cos (2x+y)