UNIT II

FOURIER SERIES

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions.

The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 – 1830). Fourier series has its application in problems pertaining to Heat conduction, acoustics, etc. The subject matter may be divided into the following sub topics.

FOURIER SERIES

- Series with arbitrary period
- Half-range series
- Complex series
- Harmonic Analysis

FORMULA FOR FOURIER SERIES

Consider a real-valued function f(x) which obeys the following conditions called Dirichlet's conditions:

1. f(x) is defined in an interval (a,a+2*l*), and f(x+2*l*) = f(x) so that f(x) is a periodic function of period 2*l*.
2. f(x) is continuous or has only a finite number of discontinuities in the interval (a,a+2*l*).
3. f(x) has no or only a finite number of maxima or minima in the interval (a,a+2*l*).

Also, let $a_0 = \frac{1}{l}\int_a^{a+2l} f(x)dx$

$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx,\ n = 1,2,3,.....$ (2) $b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$ n=1,2,3...

Then, the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x \qquad (4)$$

is called the Fourier series of f(x) in the interval (a, a+2*l*). Also, the real numbers a_0, a_1, a_2, ...a_n, and b_1, b_2, ...b_n are called the Fourier coefficients of f(x). The formulae (1), (2) and (3) are called Euler's formulae.

It can be proved that the sum of the series (4) is f(x) if f(x) is continuous at x. Thus we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x .... \qquad (5)$$

Suppose f(x) is discontinuous at x, then the sum of the series (4) would be $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$ Where $f(x^+)$ and $f(x^-)$ are the values of f(x) immediately to the right and to the left of f(x) respectively.

<u>Particular Cases:</u>Case (i)

Suppose a=0. Then f(x) is defined over the interval (0,2*l*). Formulae (1), (2), (3) reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx,\ n=1,2,\ldots\infty \qquad (6)$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (0,2*l*).If we set *l*=π, then f(x) is defined over the interval (0, 2π). Formulae (6) reduce to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx, \quad n=1, 2\ldots\infty$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx \qquad n=1, 2\ldots\infty$$

Also, in this case, (5) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx \qquad (8)$$

Case (ii) Suppose a=-*l*. Then f(x) is defined over the interval (-*l*,*l*). Formulae (1), (2) (3) reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \qquad n=1, 2\ldots\infty \qquad (9)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx, \quad n=1, 2\ldots\infty$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (-*l*, *l*).If we set *l* =π, then f(x) is defined over the interval (-π,π). Formulae (9) reduce to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, \qquad n=1, 2\ldots\infty \qquad (10)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad \text{n=1, 2...}\infty$$

Putting $l = \pi$ in (5), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

PARTIAL SUMS

The Fourier series gives the exact value of the function. It uses an infinite number of terms which is impossible to calculate. However, we can find the sum through the partial sum S_N defined as follows:

$$S_N(x) = a_0 + \sum_{n=1}^{n=N}\left[a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x\right]$$ Where N takes positive integral values.

In particular, the partial sums for N=1, 2 are

$$S_1(x) = a_0 + a_1\cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right)$$

$$S_2(x) = a_0 + a_1\cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right) + a_2\cos\left(\frac{2\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right)$$

If we draw the graphs of partial sums and compare these with the graph of the original function f(x), it may be verified that $S_N(x)$ approximates f(x) for some large N.

<u>Some useful results:</u>1. The following rule called Bernoulli's generalized rule of integration by parts is useful in

evaluating the Fourier coefficients.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 + ...$$

Here u', u' ... are the successive derivatives of u and

$$v_1 = \int v\,dx, v_2 = \int v_1\,dx, ...$$

We illustrate the rule, through the following examples:

$$\int x^2 \sin nx\,dx = x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)$$

$$\int x^3 e^{2x}\,dx = x^3\left(\frac{e^{2x}}{2}\right) - 3x^2\left(\frac{e^{2x}}{4}\right) + 6x\left(\frac{e^{2x}}{8}\right) - 6\left(\frac{e^{2x}}{16}\right)$$

2. The following integrals are also useful:

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

3. If 'n' is integer, then

Sin nπ = 0, cosnπ = $(-1)^n$, sin2nπ = 0, cos2nπ=1

HALF-RANGE FOURIER SERIES:

The Fourier expansion of the periodic function f(x) of period 2l may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of f(x) in the interval (0,l) which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Sine series:

Suppose f(x) = φ(x) is given in the interval (0, l). Then we define f(x) = - φ(-x) in (-l, 0). Hence f(x) becomes an odd function in (-l, l). The Fourier series then is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (11)$$

where
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The series (11) is called half-range sine series over (0, l).

Putting l=π in (11), we obtain the half-range sine series of f(x) over (0,π) given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Cosine series:

Let us define

$$f(x) = \begin{cases} \phi(x) & \text{in } (0, l) \\ \phi(-x) & \text{.....given in } (-l,0) \end{cases}$$

.....in order to make the function even.

Then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad (12)$$

where,

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

The series (12) is called half-range cosine series over $(0,l)$

Putting l = π in (12), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad n = 1,2,3,\ldots..$$

HARMONIC ANALYSIS

The Fourier series of a **known** function f(x) in a given interval may be found by finding the Fourier coefficients. The method described cannot be employed when f(x) is not known explicitly, but defined through the values of the function at some equidistant points. In such a case, the integrals in Euler's formulae cannot be evaluated. Harmonic analysis is the process of finding the Fourier coefficients numerically.

To derive the relevant formulae for Fourier coefficients in Harmonic analysis, we employ the following result:

The mean value of a continuous function f(x) over the interval (a,b) denoted by **[f(x)]** is defined as

$$[f(x)] = \frac{1}{b-a}\int_a^b f(x)dx .$$

The Fourier coefficients defined through Euler's formulae, (1), (2), (3) may be redefined as

$$a_0 = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)dx\right] = 2[f(x)]$$

$$a_n = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] = 2\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right]$$

$$b_n = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right] = 2\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right]$$

Using these in (5), we obtain the Fourier series of f(x). The term $a_1\cos x+b_1\sin x$ is called the first harmonic or fundamental harmonic, the term $a_2\cos 2x+b_2\sin 2x$ is called the second harmonic and so on. The amplitude of the first harmonic is $\sqrt{a_1^2+b_1^2}$ and that of second harmonic is $\sqrt{a_2^2+b_2^2}$ and so on.

PART –A

1. **State Dirichlet's conditions.(Nov'03) (May-13)(Apr-10)**

(or)

State the Sufficient condition for the function f(x) expressed as Fourier series

(i) f(x) is single valued periodic and well defined except possibly at a finite number of points.s

(ii) f(x) has atmost a finite number of finite discontinuous and no infinite discontinuous

(iii) f(x) has atmost a finite number of maxima and minima.

2. **If f(x) = x^2 + x is expressed as a fourier series in the interval (-2,2) to which value thisseries converges at x = 2.(May'03)**

f(x) is discontinuous at x = 2 also x=2 is an end point .

Therefore f(2) = $\frac{f(-2)+f(2)}{2} = \frac{4-2+4+2}{2} = 4$

3. **Define Harmonic Analysis(may-05,Dec-08) (May-13)**
The process of finding the Fourier series for a function given by numerical values is known as Harmonic analysis

4. **State Parseval's identity for the half range cosine expansion of f(x) in (0,1)(Dec'06)**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$\int_0^1 |f(x)|^2 dx = \frac{a_0^2}{2} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

5. State whether y = tanx can be expanded as a Fourirerseries.If so how? If not why? (Dec 98,05,08)

tanx cannot be expanded as a Fourirer series. Since tanx not satisfies Dirichlet's conditions.

6.If f(x) = sinhx is defined in $-\pi < x < \pi$,write the value of fourirer constant a_0 and a_n? (Dec 08)

f(x) = sinhx f(-x) = sinh(-x)=-sinhx

sinhx is an odd function => $a_0 = 0, a_n = 0$

7.Define the root mean squae value of f(x) (Dec 08, May-12)

The root mean squae value of f(x) =y over the interval (a,b) is $y = \sqrt{\frac{\int_a^b \{f(x)\}^2 dx}{b-a}}$

8.Find the Fourirer series Constant b_n for x sinx in (-π , π)(m-06)

$b_n = 0$,since functions is even function

9. If f(x) is an odd function defined in (-l , l) .What are the values of a_0 and a_n (Dec 05)

If f(x) is an odd function , $a_0 = 0, a_n = 0$

10. State Parseval's identity for the half range cosine expansion of f(x) in (0,2l) . (Dec'04)

$$\frac{1}{2l}\int_0^{2l} (f(x))^2\, dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

11.What are the constant term a_0 and the coefficient of cosnx, a_n in the Fourirer series Expansion of f(x)= x-x ^{3} in (-π , π)(Dec-04)

f(x)= x-x ^{3}

f(-x)= - x-(-x) ^{3} = - f(x)

It is an odd function, => $a_0 = 0, a_n = 0$

12.If f(x) is discontinuous at x = a,what does its Fourirer series represent at that point (may-07,Dec-04) f(a) = $\frac{f(a-)+f(a+)}{2}$

13.If $f(x) = \begin{cases} \cos x, 0 \le x \le \pi \\ 0, \pi \le x \le 2\pi \end{cases}$ **find the sum of the fourirer series of f(x) at x= π**

Ans: : $f(\pi) = \frac{f(\pi-)+f(\pi+)}{2} = \frac{\cos\pi+50}{2} = \frac{-1+50}{2} == \frac{49}{2}$

14. $\boldsymbol{If\ x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\right)\cos nx}$, $\&\ Deduce\ that\ \boldsymbol{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\dots. = \frac{\pi^2}{6}}\quad \boldsymbol{(Apr-10)}$

Soln: $Put\ x = \pi\ we\ get,\ \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\right)\cos n\pi = 4\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\right)\cos n\pi = \pi^2 - \frac{\pi^2}{3}$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\dots. = \frac{\pi^2}{6} \qquad (\because \cos n\pi = (-1)^n)$$

15 $\boldsymbol{Obtain\ the\ first\ term\ of\ fourier\ series\ f(x) = x^2\ in\ \ -\pi < x < \pi\ (Dec-09)}$

Soln: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{2\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{\pi^2}{3}$

16. $\boldsymbol{Find\ RMS\ of\ f(x) = x^2\ in\ (-l,l)\ (Dec-10)}$

Soln: $\boldsymbol{\frac{1}{2l}\int_{-l}^{l} x^4 dx = \frac{4l^4}{36} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16l^4}{n^4\pi^4} \Rightarrow \frac{2}{2l}\left[\frac{x^5}{5}\right]_0^l}$ — — $\sum_{n=1}$ — — — ....

17. $\boldsymbol{Find\ the\ sum\ of\ fourier\ series\ \ f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2, & 1 \le x \le 2 \end{cases}\ at\ x = 0\quad (Dec-2011)}$

Soln: $x = 0\ is\ a\ point\ of\ discontinuity\ in\ the\ extreem\ of\ the\ given\ intervals$

$$Sum = \frac{f(0)+f(2)}{2} = \frac{0+2}{2} = 1$$

18. $\boldsymbol{Find\ the\ cosine\ series\ for\ \ f(x) = x\sin x\ \ in\ (0,\pi)\&}\qquad (Dec-2011)$

$$\boldsymbol{Deduce\ that\ 1 + 2\left(\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots \dots.\right) = \frac{\pi}{2}}$$

Soln: xsinx = 1 - $\frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{n^2-1}\right)\cos nx$

put x= $\frac{\pi}{2}$ [point of continuity]

$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 + \frac{1}{2}\cos\frac{\pi}{2} - 2\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{n^2-1}\right)\cos\frac{n\pi}{2}$

$\frac{\pi}{2} = 1 - 2\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{n^2-1}\right)\cos\frac{n\pi}{2}$

$\frac{\pi}{2} = 1 - 2\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{(n-1)(n+1)}\right)\cos\frac{n\pi}{2}$

$= 1 - 2\left(\frac{1}{1.3}(-1) + \frac{-1}{2.4}(0) + \frac{1}{3.5}(1) + \cdots\right)$

$= 1 + 2\left(\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots\right)$

19. $\boldsymbol{Give\ the\ expression\ for\ the\ fourier\ series\ coefficient\ b_n\ for\ the\ function\ f(x)}$

$$\boldsymbol{defined\ in\ (-2,2)}\qquad \boldsymbol{(Apr-2011)}$$

Soln: $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$

20. $\boldsymbol{Without\ Finding\ a_0, a_n, b_n\ find\ the\ fourier\ series\ for\ f(x) = x^2\ in\ (0,\pi)}$

$$\boldsymbol{Find\ the\ value\ of\ \left(\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right)\ (Apr - 2011)}$$

Soln: $\frac{1}{\pi}\int_0^{\pi} x^4\,dx = \frac{1}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi^5}{5}\right] = \frac{\pi^4}{5} = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ $\quad (\because By\ Parsvel`s\ Identity)$

21. $\boldsymbol{Find\ the\ constant\ term\ in\ the\ expression\ of\ cos^2x\ as\ the\ F.S\ in\ (-\pi,\pi)(May - 12)}$

Soln: Given f(x) = cos^2x = $\frac{1+\cos 2x}{2}$

W.K.T. f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

To find $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx = \frac{2}{\pi}\int_0^{\pi}\cos^2 x\,dx$

$$= \frac{2}{\pi}\int_0^{\pi}(1+\cos 2x)/2\,dx = \frac{1}{\pi}\int_0^{\pi}(1+\cos 2x)\,dx$$

$$= \frac{1}{\pi}\left[x + \sin 2x/2\right]_0^{\pi} = \frac{1}{\pi}\left[(\pi+0)-(0+0)\right]$$

$$= 1$$

$\therefore Constant\ term\ = \frac{a_0}{2} = \frac{1}{2}$

PART-B

1) Obtain the Fourier expansion of f(x) = x² over the interval (-π,π). Deduce that

$$(i)\ \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \infty\quad (ii)\ \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4}\ldots\ldots\ldots\ldots\ldots. = \frac{\pi^4}{90}$$

(m/J 07, N/D 07, dec 08, jun 09)

May-2001,Nov-2005,May-13

Solution:

The function f(x) is even. Hence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \frac{2\pi^2}{3}$$

or

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx,$$ since f(x)cosnx is even

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx dx$$

Integrating by parts, we get

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4(-1)^n}{n^2}$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$$ since f(x)sinnx is odd.

Thus

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence, $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots..$$

$$(ii)\ \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \ldots\ldots\ldots\ldots\ldots. = \frac{\pi^4}{90}$$

Using Parsvel`s Identity

$$a_0^2 = \frac{\pi^4}{9},\ a_n^2 = \left(\frac{4(-1)^n}{n^2}\right)^2 = \frac{16}{n^4},\ [f(x)]^2 = x^4$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{2}{2\pi}\int_{0}^{\pi} x^4 dx = \frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4} \Rightarrow \frac{1}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \frac{1}{\pi}\left[\frac{\pi^5}{5}\right] = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{5} - \frac{\pi^4}{9} = 8\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \frac{4\pi^4}{45} = 8\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \ldots\ldots\ldots\ldots\ldots$$

2) $\mathbf{f(x)} = |x|$ **over** $(-\pi,\pi)$; **Deduce that** $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\infty$ &

$$\textbf{Deduce } \boldsymbol{that} \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8} \quad (\boldsymbol{May} - \mathbf{12})$$

Soln: $f(-x) = |-x| = |x| = f(x) \Rightarrow f(x)$ *is even function so* $b_n = 0$

$$\text{W. K. T } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, where\ a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx,\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)cosnxdx,$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx, \Rightarrow \frac{2}{\pi}\int_0^{\pi} xdx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}[\pi^2 - 0] = \frac{\pi^2}{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)cosnxdx = \frac{2}{\pi}\int_0^{\pi} x\,cosnxdx = \frac{2}{\pi}\left[\left(x\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi n^2}[(-1)^n - 1] \Rightarrow a_n = \begin{cases} \frac{-4}{\pi n^2} & if\ n\ is\ odd \\ 0, if\ n\ is\ even \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty}\frac{-4}{\pi n^2}\cos nx \ldots\ldots\ldots\ldots (1)$$

Putting $x = 0$, *is a point of continuity* (1) *becomes*

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}\cos(0) \Rightarrow -\frac{\pi}{2} = -\frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

3) Find the Half Range Cosine series $\mathbf{f(x) = (\pi - x^2)}$ ***in*** $(\mathbf{0},\boldsymbol{\pi})$ $(\boldsymbol{May} - \mathbf{12})$

Soln: *Let the required half range cosine series be* $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \ldots\ldots\ldots (1)$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x^2)\,dx = \frac{2}{\pi}\left[\pi x - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\pi^2 - \frac{\pi^3}{3}\right] = \frac{2}{3}[\pi(3-\pi)],$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)cosnx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x^2)cosnx\,dx$$

$$= \frac{2}{\pi}\left[\frac{(\pi - x^2)\sin nx}{n} - 2x\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\left(0 - \frac{2\pi(-1)^n}{n^2} + 0\right) - (0 - 0 + 0)\right] = \frac{2}{n^2}[-2(-1)^n] = \frac{-4}{n^2}(-1)^n$$

$$f(x) = \frac{\frac{2}{3}[\pi(3-\pi)]}{2} + \sum_{n=1}^{\infty} \frac{-4}{n^2}(-1)^n \cos n\pi = \frac{[\pi(3-\pi)]}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x$$

4) *Find the complex form of fourier series for the function $f(x) = \cos ax$ in $(-\pi, \pi)$, Where a is not an integer* $(\boldsymbol{May - 13})$

Given $f(x) = \cos ax$ *in* $(-\pi, \pi)$

$$Formula: f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \text{ where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx} dx$$

$$To\ find\ c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax\, e^{-inx} dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{(-in)^2 + a^2}\{(-in)\cos ax + a\sin ax)\}\right]_{-\pi}^{\pi}$$

$$\frac{1}{2\pi}\left[\frac{e^{-inx}}{i^2n^2 + a^2}\{(-in)\cos ax + a\sin ax)\}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{e^{-inx}}{-n^2 + a^2}\{(-in)\cos ax + a\sin ax)\}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{a^2 - n^2}\{(-in)\cos a\pi + a\sin a\pi)\} - \frac{e^{in\pi}}{a^2 - n^2}\{(-in)\cos(-a\pi) + a\sin(-a\pi)\}\right]$$

$$c_n = \frac{1}{2\pi}\left[\frac{(-1)^n}{a^2 - n^2}\{2a\sin a\pi\}\right] = \frac{a\sin a\pi}{\pi}\left[\frac{(-1)^n}{a^2 - n^2}\right]$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{a\sin a\pi}{\pi}\left[\frac{(-1)^n}{a^2 - n^2}\right] e^{inx} = \frac{a\sin a\pi}{\pi} \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n}{a^2 - n^2}\right] e^{inx}$$

5) *Find the Fourier cosine transform of* $(\boldsymbol{x-1})^2$ ***in*** $\boldsymbol{0} < x < 1$ &

Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots..$ $(\boldsymbol{May - 13})$

Soln: Let the required half range cosine series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ .... (1)

$$a_0 = 2\int_0^1 f(x)\, dx = 2\int_0^1 (x-1)^2\, dx = 2\left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{2}{3}[0 - (-1)] = \frac{2}{3}$$

$$a_n = 2\int_0^1 f(x) \cos n\pi x\, dx = 2\int_0^1 (x-1)^2 \cos n\pi x\, dx$$

$$= 2\left[\frac{(x-1)^2 \sin n\pi x}{\pi n} - 2(x-1)\frac{\cos n\pi x}{\pi^2 n^2} + 2\frac{\sin n\pi x}{\pi^3 n^3}\right]_0^1$$

$$a_n = 2\left[(0-0+0)-\left(0-\frac{2}{\pi^2 n^2}+0\right)\right] = 2\left[\frac{2}{\pi^2 n^2}\right] = \frac{4}{\pi^2 n^2}$$

$$f(x) = \frac{\frac{2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\cos n\pi x = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

Putting $x = 0$ *is a point of continuous,* $f(0) = 1$

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(0) \Rightarrow 1 - \frac{1}{3} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \frac{2}{3}\left(\frac{\pi^2}{4}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \ldots$$

6) Obtain the half − range cosine series for $f(x) = \begin{cases} kx, 0 \le x \le \frac{l}{2} \\ k(l-x), \frac{l}{2} \le x \le l \end{cases}$ $(\boldsymbol{May - 13})$

$$Soln: f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \ldots\ldots(1), \quad Where \quad a_o = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_0 = \frac{2}{l}\left[\int_0^{\frac{l}{2}} kx\,dx + \int_{\frac{l}{2}}^{l} k(l-x)\,dx\right] = \frac{2}{l}\left[\int_0^{\frac{l}{2}} kx\,dx + \int_{\frac{l}{2}}^{l} kl\,dx - \int_{\frac{l}{2}}^{l} kx\,dx\right]$$

$$a_0 = \frac{2}{l}\left[k\left(\frac{x^2}{2}\right)_0^{\frac{l}{2}} + kl(x)_{\frac{l}{2}}^{l} - k\left(\frac{x^2}{2}\right)_{\frac{l}{2}}^{l}\right] = \frac{2}{l}\left[\frac{k}{3}\left(\frac{l^2}{4}-0\right) + kl\left(l-\frac{l}{2}\right) - \frac{k}{2}\left(l^2-\frac{l^2}{4}\right)\right]$$

$$a_0 = \frac{2}{l}\left[\frac{kl^2}{12} + \frac{kl^2}{2} - \frac{3kl^2}{8}\right] = 2kl\left[\frac{1}{12} + \frac{1}{2} - \frac{3}{8}\right] = kl\left[\frac{1}{6} + 1 - \frac{3}{4}\right]$$

$$a_0 = kl\frac{[4+24-18]}{24} = kl\left(\frac{10}{24}\right) = \frac{5kl}{12}$$

$$a_n = \frac{2}{l}\left[\int_0^{\frac{l}{2}} kx\cos\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$a_n = \frac{2}{l}\left[kx\left(\frac{l}{n\pi}\right)\left(\sin\frac{n\pi x}{l}\right) - k\left(\frac{l^2}{n^2\pi^2}\right)\left(-\cos\frac{n\pi x}{l}\right)\right]_0^{\frac{l}{2}}$$

$$+\frac{2}{l}[k(l-x)\left(\frac{l}{n\pi}\right)\left(\sin\frac{n\pi x}{l}\right)-(-k)\left(\frac{l^2}{n^2\pi^2}\right)\left(-\cos\frac{n\pi x}{l}\right)]_{\frac{l}{2}}^{l}$$

$$a_n=\frac{2}{l}[(\frac{k}{2}\left(\frac{l}{n\pi}\right)\left(\sin\frac{n\pi}{2}\right)+k\left(\frac{l^2}{n^2\pi^2}\right)\left(\cos\frac{n\pi}{2}\right))-(0+k\left(\frac{l^2}{n^2\pi^2}\right))]$$

$$+\frac{2}{l}[(0-k\left(\frac{l^2}{n^2\pi^2}\right)(-1)^n)-(\frac{k}{2}\left(\frac{l}{n\pi}\right)\left(\sin\frac{n\pi}{2}\right))-k\left(\frac{l^2}{n^2\pi^2}\right)\left(\cos\frac{n\pi}{2}\right))$$

$$a_n=\frac{2}{l}[\frac{k}{2}\left(\frac{l}{n\pi}\right)\left(\sin\frac{n\pi}{2}\right)+k\left(\frac{l^2}{n^2\pi^2}\right)\left(\cos\frac{n\pi}{2}\right)-k\left(\frac{l^2}{n^2\pi^2}\right)-k\left(\frac{l^2}{n^2\pi^2}\right)(-1)^n-\frac{k}{2}\left(\frac{l}{n\pi}\right)\left(\sin\frac{n\pi}{2}\right)$$
$$+k\left(\frac{l^2}{n^2\pi^2}\right)\left(\cos\frac{n\pi}{2}\right)]$$

$$a_n=\frac{2}{l}[2k\left(\frac{l^2}{n^2\pi^2}\right)\left(\cos\frac{n\pi}{2}\right)+-k\left(\frac{l^2}{n^2\pi^2}\right)-k\left(\frac{l^2}{n^2\pi^2}\right)(-1)^n]$$

$$a_n=\frac{2}{l}[(k\left(\frac{l^2}{n^2\pi^2}\right))\left(2\cos\frac{n\pi}{2}+[(-1)^n+1]\right)]=\frac{2kl}{n^2\pi^2}\left(2\cos\frac{n\pi}{2}+[(-1)^n+1]\right)$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}=\frac{5kl}{24}+\sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}\left(2\cos\frac{n\pi}{2}+[(-1)^n+1]\right)\cos\frac{n\pi x}{l}$$

$$f(x)=\frac{5kl}{24}+\frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\cos\frac{n\pi}{2}+[(-1)^n+1]\right)\cos\frac{n\pi x}{l}$$

7) Find the Fourier series $\mathbf{f(x)=(\pi-x)^2}$ *in* $\mathbf{(0,2\pi)}$ $\quad$ $\boldsymbol{(May-12)}$

Soln: $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$, $\quad$ *where* $a_o=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx,\ b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx$$

$$a_o=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\,dx=\frac{1}{\pi}\int_0^{2\pi}(\pi^2+x^2-2\pi x)\,dx$$

$$a_o=\frac{1}{\pi}\left[\pi^2x+\frac{x^3}{3}-2\pi\left(\frac{x^2}{2}\right)\right]_0^{2\pi}=\frac{1}{\pi}[2\pi^3+\frac{\pi^3}{3}-\pi(4\pi^2)]=\frac{1}{\pi}\left[\frac{6\pi^3+\pi^3-12\pi^3}{3}\right]$$

$$a_o=\frac{1}{\pi}\left[-\frac{5\pi^3}{3}\right]=-\frac{5\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$$

$$a_n=\frac{1}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n}+2(x-\pi)\frac{\cos nx}{n^2}-2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$a_n=\frac{1}{\pi}\left[\left(0+\frac{2\pi}{n^2}+0\right)-\left(0-\frac{2\pi}{n^2}+0\right)\right]\Rightarrow a_n=\frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]=\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$$b_n=\frac{1}{\pi}\left[(\pi-x)^2\frac{-\cos nx}{n}+2(x-\pi)\frac{\sin nx}{n^2}+2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$b_n=\frac{1}{\pi}\left[\left(-\frac{\pi^2}{n}+0+\frac{2}{n^3}\right)-\left(-\frac{\pi^2}{n}+0+\frac{2}{n^3}\right)\right]=0$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=-\frac{5\pi^2}{6}+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx+(0)\sin nx\right)$$

$$f(x)=\frac{-5\pi^2}{6}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

8) Find the Fourier series $f(x)=\begin{cases}\mathbf{0}, & -\pi\le x\le 0\\ \sin x, & 0\le x\le\pi\end{cases}$ &

$$\textbf{Evluate that }\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots.=?\quad(\textbf{Dec}-\textbf{2011})$$

$$Soln:\ f(x)=\begin{cases}0, & -\pi\le x\le 0\\ \sin x, & 0\le x\le\pi\end{cases},\qquad let\ f(x)=\begin{cases}\varphi_1(x), & \pi\le x\le 0\\ \varphi_2(x), & 0\le x\le\pi\end{cases}$$

$$Where\ \varphi_1(x)=0\ \varphi_2(x)=\sin x,\ Here\ \varphi_1(-x)=0\neq\begin{cases}\varphi_2(x)\\ -\varphi_2(x)\end{cases}$$

$$\therefore f(x) is\ neither\ even\ nor\ odd\ in\ [-\pi,\pi]$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),\qquad where\ a_o=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

$$a_o=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{\pi}\sin x\,dx=\frac{1}{\pi}[-\cos x]_0^{\pi}=-\frac{1}{\pi}[\cos\pi-\cos 0]=-\frac{1}{\pi}(-2)=\frac{2}{\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}\sin x\cos nx\,dx=\frac{1}{2\pi}\int_0^{\pi}(\sin(n+1)x-\sin(n-1)x)\,dx$$

$$a_n = \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} = \frac{1}{2\pi}\left[\left(\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$a_n = \frac{1}{2\pi}\left[(-1)^n\left\{\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right\} + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$a_n = \frac{1}{2\pi}\left[-\frac{2}{n^2-1}\left[1+(-1)^n\right]\right], n \neq 1$$

$$a_n = \begin{cases} 0, \mathit{if\ n\ is\ odd} \\ \dfrac{-2}{\pi(n^2-1)}, \mathit{if\ n\ is\ even} \end{cases}$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{\pi} \sin 2x \, dx = \frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{4\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx = \frac{1}{2\pi}\int_0^{\pi} (\cos(n-1)x - \cos(n+1)x) \, dx$$

$$b_n = \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n-1}\right]_0^{\pi} = \frac{1}{2\pi}[(0-0)+(0-0)]$$

$$b_n = \frac{1}{2\pi}[0] = 0, n \neq 1$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx = \frac{1}{\pi}\int_0^{\pi} \frac{(1-\cos 2x)}{2} \, dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$b_1 = \frac{1}{2\pi}[(\pi-0)-(0-0)] = \frac{1}{2}$$

$$f(x) = \frac{a_0}{2} + ((a_1 \cos x + b_1 \sin x) + \sum_{n=even}^{\infty} (a_n \cos nx) + \sum_{n=2}^{\infty} b_n \sin nx)$$

$$f(x) = \frac{1}{\pi} + (((0)\cos x + \frac{1}{2}\sin x) + \sum_{n=even}^{\infty} \frac{-2}{\pi(n^2-1)} \cos nx + (0) \sin nx)$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=even}^{\infty} \frac{\cos nx}{(n^2-1)} \Rightarrow f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{((2n)^2-1)}$$

$$\mathit{Put}\ n = 2n$$

Put x=0 is a point of continuous,

$$f(0) = \sin 0 = 0, \qquad \Rightarrow \ 0 = \frac{1}{\pi} + \frac{1}{2}\sin(0) - \frac{2}{\pi}\sum_{n=even}^{\infty} \frac{\cos 2n(0)}{(n^2-1)}$$

$$\frac{-1}{\pi} = \frac{-2}{\pi}\sum_{n=even}^{\infty} \frac{1}{(n+1)(n-1)} \Rightarrow \frac{1}{2} = \sum_{n=even}^{\infty} \frac{1}{(n+1)(n-1)}$$

9) **Find the Fourier series of** $f(x) = \begin{cases} -1+x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi. \end{cases}$

Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \cdots \ldots = \frac{\pi}{4}$ **(May'07,05,2011)**

Sol: Given f(-x) = $\begin{cases} -1 - x, in - \pi < -x < 0 \\ 1 - x, in 0 < -x < \pi \end{cases}$

f(-x) = $-\begin{cases} 1 + x, in \pi > x > 0 \\ -1 + x, in 0 > x > -\pi \end{cases}$ ie., f(-x) = $-\begin{cases} -1 + x, in - \pi < x < 0 \\ 1 + x, in 0 < x < \pi \end{cases}$

=>f(-x) = - f(x) => f(x) is an odd function . Therefore

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$ & $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx = 0$

Let the required series be $f(x) = \sum_{n=1}^{\infty} b_n sinnx$ where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)sinnxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} (x+1)sinnxdx$$

$$= \frac{2}{\pi}\left\{(x+1)\left[\frac{-cosnx}{n}\right] - (1)\left[\frac{-sinnx}{(n)^2}\right]\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(\pi+1)\frac{(-1)^n}{n} + \frac{1}{n}\right]\{\cos n\pi = (-1)^n 1, \cos 0 = 1, \sin n\pi = 0, , \sin 0 = 0\}$$

$$b_n = \frac{2}{n\pi}[1 - (\pi+1)(-1)^n]$$

Therefore $f(x) = \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{[1-(\pi+1)(-1)^n]}{n} sinnx$ -------- (1)

Put $x = \frac{\pi}{2}$ $\{x = \frac{\pi}{2}$ *is a point of continuity*$\}$

$$\therefore [f(x) at (x = \frac{\pi}{2})] = 1 + \frac{\pi}{2}$$

$\therefore$ (1) => $1 + \frac{\pi}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[1-(\pi+1)(-1)^n]}{n} sinn\frac{\pi}{2}$

$(\frac{2+\pi}{2}) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[1-(\pi+1)(-1)^n]}{n} sinn\frac{\pi}{2}$

$(\frac{2+\pi}{2}) = \frac{2}{\pi}[\frac{2+\pi}{1} - \frac{2+\pi}{3} + \frac{2+\pi}{5} - \frac{2+\pi}{7} + \cdots]$

$\left(\frac{2+\pi}{2}\right) = \frac{2(2+\pi)}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$

$\left(\frac{\pi}{4}\right) = \left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$

10) *Find F.S of $f(x) = x(\pi - x)$ in $(0, 2\pi)$* & *Deduce* $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \ldots = \frac{\pi^2}{8}$ ***(Apr − 11)***

Soln: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, *where* $a_o = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_o = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x(\pi - x)\,dx = \frac{1}{\pi}\int_0^{2\pi}(x\pi - x^2)\,dx$$

$$a_o = \frac{1}{\pi}\left[\pi\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{4\pi^3}{2} - \frac{8\pi^3}{3}\right] = \frac{1}{\pi}\left[\frac{6\pi^3 - 8\pi^3}{3}\right] = -\frac{2\pi^3}{3\pi} = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(x\pi - x^2)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[(x\pi - x^2)\frac{\sin nx}{n} + (\pi - 2x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left[\left(0 + \frac{-3\pi}{n^2} + 0\right) - \left(0 + \frac{\pi}{n^2} + 0\right)\right] \Rightarrow a_n = \frac{1}{\pi}\left[\frac{-4\pi}{n^2}\right] = \frac{-4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(x\pi - x^2)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[(x\pi - x^2)\frac{-\cos nx}{n} + (\pi - 2x)\frac{\sin nx}{n^2} - 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left[\left\{\left(\frac{2\pi^2}{n}\right) + 0 - \frac{2}{n^3}\right\} - \left(0 + 0 - \frac{2}{n^3}\right)\right] = \frac{1}{\pi}\left[\left(\frac{2\pi^2}{n}\right) - \frac{2}{n^3} + \frac{2}{n^3}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{2\pi^2}{n}\right] = \frac{2\pi}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2}\cos nx + \frac{2\pi}{n}\sin nx\right)$$

Put $x = 0$ *is a point of discontinuity so* $f(x) = \frac{f(0) + f(2\pi)}{2}$

$$f(x) = x(\pi - x), \qquad f(0) = 0, f(2\pi) = 2\pi(\pi - 2\pi) = 2\pi(-\pi) = -2\pi^2$$

$$f(x) = \frac{f(0) + f(2\pi)}{2} = \frac{0 + (-2\pi^2)}{2} = -\pi^2$$

$$f(x) = -\pi^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2}\cos n(0) + \frac{2\pi}{n}\sin n(0)\right) \Rightarrow -\pi^2 + \frac{\pi^2}{3} = -4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{-2\pi^2}{3} = -4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots ..$$

11) Obtain the Fourier series of f(x) = 1-x^2 over the interval (-1, 1).(NOV 05, Dec-10)

The given function is even, as f(-x) = f(x). Also period of f(x) is 1-(-1) =2

Solution:

Here $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = 2\int_0^1 f(x)dx = 2\int_0^1 (1-x^2)dx = 2\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4}{3}$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx$$

$= 2\int_0^1 f(x)\cos(n\pi x)dx$ as f(x) cos(nπx) is even

$= 2\int_0^1 (1-x^2)\cos(n\pi x)dx$

Integrating parts, $a_n = 2\left[(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1 = \frac{4(-1)^{n+1}}{n^2\pi^2}$

$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx$ =0, since f(x) sin (nπx) is odd.

The Fourier series of f(x) is

$$f(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$$

12) Obtain the half − range sine series for $f(x) = \begin{cases} x, 0 \le x \le \frac{l}{2} \\ (l-x), \frac{l}{2} \le x \le l \end{cases}$ $(Apr`11, may`05)$

Soln: $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \dots\dots (1), Where\, b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

$$b_n = \frac{2}{l}\left[\int_0^{\frac{l}{2}} x \sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$b_n = \frac{2}{l}\left[x\left(\frac{l}{n\pi}\right)\left(-\cos\frac{n\pi x}{l}\right) - \left(\frac{l^2}{n^2\pi^2}\right)\left(-\sin\frac{n\pi x}{l}\right)\right]_0^{\frac{l}{2}}$$

$$+\frac{2}{l}\left[(l-x)\left(\frac{l}{n\pi}\right)\left(-\cos\frac{n\pi x}{l}\right) + \left(\frac{l^2}{n^2\pi^2}\right)\left(-\sin\frac{n\pi x}{l}\right)\right]_{\frac{l}{2}}^{l}$$

$$b_n = \frac{2}{l}\left[\left(-0 + \left(\frac{l^2}{n^2\pi^2}\right)\left(\sin\frac{n\pi}{2}\right)\right) - (-0+0)\right] + \frac{2}{l}\left[(-0+0) - \left(-0 - \left(\frac{l^2}{n^2\pi^2}\right)\left(\sin\frac{n\pi}{2}\right)\right)\right]$$

$$b_n = \frac{2}{l}\left[2\left(\frac{l^2}{n^2\pi^2}\right)\left(\sin\frac{n\pi}{2}\right)\right] = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l} = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{l}$$

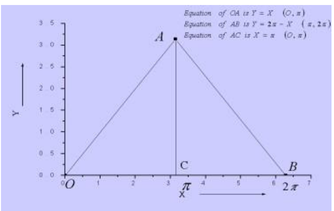
13) Obtain the Fourier expansion of

$$f(x) = \begin{cases} x, 0 \le x \le \pi \\ 2\pi - x, \pi \le x \le 2\pi \end{cases}$$ **May-2004, Dec-10**

Deduce that $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Solution:

The graph of f(x) is shown below.



Here OA represents the line f(x)=x, AB represents the line f(x)=(2π-x) and AC represents the line x=π. Note that the graph is symmetrical about the line AC, which in turn is parallel to y-axis. Hence the function f(x) is an even function.

Here,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x dx = \pi$$

Since f(x) cosnx is even.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

Also, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$, since f(x)sinnx is odd

Thus the Fourier series of f(x) is $f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx$

For x=π, we get $f(\pi) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos n\pi$

or $$\pi = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2\cos(2n-1)\pi}{(2n-1)^2}$$

Thus, $\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$

14) Obtain the half − range cosine series for f(x) = x in $(0, \pi)$

Soln: *Let the required half range cosine series be* $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ ....(1)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2}\right] = \pi,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) cosnx\,dx = \frac{2}{\pi}\int_0^{\pi} x cosnx\,dx = \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \Rightarrow a_n = \begin{cases} 0, if\ n\ is\ even \\ -\frac{4}{\pi n^2}, if\ n\ is\ odd \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} -\frac{4}{\pi n^2}\cos nx = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}\cos nx$$

15) Expand f(x) = x (π-x) as half-range sine series over the interval (0,π)

(JUN 07,09,Ap-10)

We have,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi} = \frac{4}{n^3\pi}\left[1-(-1)^n\right]$$

The sine series of f(x) is $f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1-(-1)^n\right]\sin nx$

16)

Obtain the Fourier expansion of f(x) =e^{-ax} in the interval (-π,π).

Dec-2008,Apr-2010

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left\{-a\cos nx + n\sin nx\right\}\right]_{-\pi}^{\pi}$$

$$= \frac{2a}{\pi}\left[\frac{(-1)^n\sinh a\pi}{a^2+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left\{-a\sin nx - n\cos nx\right\}\right]_{-\pi}^{\pi}$$

$$= \frac{2n}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

$$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx + \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

17) **Find the fourier series of periodicity 3 for f(x) = 2x-x^2 in 0<x<3. Apr-10,May'08**

Sol : Here 2l = 3 =>$l = \frac{3}{2}$ Let the Fourierseries be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3}$$

Where $a_0 = \frac{2}{3}\int_0^3 f(x)dx$, $a_n = \frac{2}{3}\int_0^3 f(x)cos\frac{2n\pi x}{3}dx$ and $b_n = \frac{2}{3}\int_0^3 f(x)sin\frac{2n\pi x}{3}dx$

$$a_0 = \frac{2}{3}\int_0^3 (2x-x^2)dx$$

$$= \frac{2}{3}\left[\left\{x^2 - \frac{x^3}{3}\right\}_0^3\right]$$

$$= \frac{1}{\pi}[9-9] \Rightarrow a_0 = 0$$

$$a_n = \frac{2}{3}\int_0^3 f(x)cos\frac{2n\pi x}{3}dx$$

$a_n = \frac{2}{3}\int_0^3 (2x-x^2)cos\frac{2n\pi x}{3}dx$ By Bernoullis Theorem $\int udv = uv - u'v_1 + u''v_2 - \cdots$

$$= \frac{2}{3}\left\{(2x-x^2)\left[\frac{sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right] - (2-2x)\left[\frac{-cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right] + (-2)\frac{-sin\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$= \frac{2}{3}\left[\left(\frac{-9}{n^2\pi^2}\right) - \left(\frac{-9}{2n^2\pi^2}\right)\right] \{\cos 2n\pi = 1,\ \sin 2n\pi = 0,\ \cos 0 = 1 .\ \sin 0 = 0\}$$

$$= \left(\frac{-9}{n^2\pi^2}\right)\left(\frac{2}{3}\right)\left(1+\frac{1}{2}\right)$$

$$a_n = \left(\frac{-9}{n^2\pi^2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)$$

$$a_n = \left(\frac{-9}{n^2\pi^2}\right)$$

$$b_n = \frac{2}{3}\int_0^3 f(x)sin\frac{2n\pi x}{3}dx$$

$$b_n = \frac{2}{3}\int_0^3 (2x - x^2) \sin\frac{2n\pi x}{3}\, dx$$ By Bernoullis Theorem $\int u dv = uv - u'v_1 + u''v_2 - \cdots$

$$= \frac{2}{3}\left\{(2x - x^2)\left[\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right] - (2 - 2x)\left[\frac{-\sin\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right] + (-2)\frac{\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$= \frac{2}{3}\left[\left(\frac{9}{2n\pi}\right) - \left(\frac{27}{4n^3\pi^3}\right)(1 - 1)\right] \{\cos 2n\pi = 1, \sin 2n\pi = 0, \cos 0 = 1 . \sin 0 = 0\}$$

$b_n = \frac{3}{n\pi}$ Therefore $f(x) = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{3} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{3}$

18) Obtain the Complex form of Fourier series of the function f(x) = e^{-x} in the interval 0<x<2π(A/M 10) (Dec-09)

Ans : The complex form of Fourier series of the function is given by

f(x) = $\sum_{n=-\infty}^{n=\infty} C_n e^{inx}$

where $C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx} dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-x} e^{-inx} dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-(1+in)x} dx = \frac{1}{2\pi}\left\{\left[-\frac{e^{-(1+in)x}}{1+in}\right]\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{-\frac{e^{-2\pi}(e^{-i2n\pi}) + 1}{1+in}\right\}_0^{2\pi} = \frac{1}{2\pi}\left(\frac{1 - e^{-2\pi}(1)}{1+in}\right)$$

Where $e^{-i2n\pi} = \cos 2n\pi - i \sin 2n\pi = 1$

$$C_n = \left(\frac{(1 - e^{-2\pi})(1 - in)}{2\pi(1 + n^2)}\right)$$

The required complex form of the series is $f(x) = \left(\frac{1 - e^{-2\pi}}{2\pi}\right)\sum_{n=-\infty}^{n=\infty}\left(\frac{1 - in}{1 + n^2}\right) e^{inx}$

19) Obtain the Fourier expansion of

$$\textbf{f(x)} = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$ **(Nov-09)**

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_0^{\pi} x dx\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx dx + \int_{0}^{\pi} x\cos nx dx\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx dx + \int_{0}^{\pi} x\sin nx dx\right]$$

$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Fourier series is $\quad f(x) = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\frac{\left[1-2(-1)^n\right]}{n}\sin nx$

Note that the point x=0 is a point of discontinuity of f(x). Here $f(x^+) = 0$, $f(x^-) = -\pi$ at x=0. Hence

$$\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2}(0-\pi) = \frac{-\pi}{2}$$

The Fourier expansion of f(x) at x=0 becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

$$or\ \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

Simplifying we get,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

20) Find the complex form of the Fourirer Series of f(x)= e^{-x} in -1< x <1.(Nov2009)

Solution:

The complex form of the Fourirer Series of f(x)= e^{-x} in -1< x <1, is given by

$f(x) = \sum_{n=-\infty}^{\infty} Cn\, e^{inx\pi}$

$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x}\, e^{-ix\pi n}\, dx \qquad = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\, dx$

$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]$at (-1 to 1) $\qquad = \frac{1}{-2(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)x}\right]$

$= \frac{-1}{2(1+in\pi)}\left[e^{-1}(cosn\pi - i\sin n\pi) - e^{1}(cosn\pi + i\sin n\pi)\right]$

$= \frac{\cos n\pi}{-2\,(i+n\pi)}\left[e^{-1} - e^{1}\right] = \frac{\cos n\pi}{2\,(i+n\pi)}$ 2 sin h 1

$= \frac{(-1)^n \sin h1}{1+in\pi}$

Sub in f(x)

f(x)= $\sum_{-\infty}^{\infty} \frac{(-1)^n \sin h1}{1+in\pi} e^{inx\pi}$ = sinh 1 $\sum_{-\infty}^{\infty} \frac{(-1)^n (1-in\pi)}{1+n^2\pi^2} e^{inx\pi}$

21) Expand f(x)= x-x^2 in –L <X<L and using this series find the root mean square values of f(x) in the interval. (Nov-09)

The fourirer series of f(x) in (-L , L) is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x / L + \sum_{n=1}^{\infty} b_n \sin n\pi x / L$

………(1)

$$where\ a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$= \frac{1}{L}\int_{-L}^{L} (x - x^2)dx$$

$$= \frac{1}{L}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{-L}^{L}$$

$$= \frac{1}{L}\left(\frac{L^2}{2} - \frac{L^3}{3} - \frac{L^2}{2} - \frac{L^3}{3}\right)$$

$$= -\frac{2L^3}{3L}$$

$$a_0 = -\frac{2L^2}{3} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . (2)$$

$$Where\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L}(x - x^2)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_n = \frac{1}{L}\left[(x - x^2)\left(\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right) - (1 - 2x)\left(\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}\right) + (-2)\left(\frac{-\sin\left(\frac{n\pi x}{L}\right)}{\frac{n^3\pi^3}{L^3}}\right)\right]_{-L}^{L}$$

$$= \frac{1}{L}\left[\frac{(1 - 2L)cosn\pi}{\frac{n^2\pi^2}{L^2}} - \frac{(1 + 2L)cosn\pi}{\frac{n^2\pi^2}{L^2}}\right]$$

$$= \frac{1}{L}\frac{L^2}{n^2\pi^2}[-4Lcosn\pi]$$

$$a_n = \frac{4L^2}{n^2\pi^2}(-1)^{n+1} \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (3)$$

$$Where\; b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L}(x - x^2)\sin(\frac{n\pi x}{L})dx$$

$$b_n = \frac{1}{L}\left[(x - x^2)\left(\frac{-\cos(\frac{n\pi x}{L})}{\frac{n\pi}{L}}\right) - (1 - 2x)\left(\frac{-\sin(\frac{n\pi x}{L})}{\frac{n^2\pi^2}{L^2}}\right) + (-2)\left(\frac{\cos(\frac{n\pi x}{L})}{\frac{n^3\pi^3}{L^3}}\right)\right]_{-L}^{L}$$

$$b_n = \frac{1}{L}\left[(L - L^2)\left(-\frac{cosn\pi}{\frac{n\pi}{L}}\right) - \frac{2cosn\pi}{\frac{n^3\pi^3}{L^3}} - (-L - L^2)\left(-\frac{cosn\pi}{\frac{n\pi}{L}}\right) + \frac{2cosn\pi}{\frac{n^3\pi^3}{L^3}}\right]$$

$$= \frac{\frac{1}{L}(-2Lcosn\pi)}{\frac{n\pi}{L}}$$

$$b_n = \frac{2L}{n\pi}(-1)^{n+1} \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . . (4)$$

Sub (2)(3) and (4) in (1) we get final answer

$$f(x) = -\frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2}(-1)^{n+1}\cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin(\frac{n\pi x}{L})$$

22) Obtain the Fourier expansion of

f(x) = $\frac{1}{2}(\pi - x)$ **in -π < x < π**

We have,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\cos nx dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{n}$$

Using the values of a_0, a_n and b_n in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

We get,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

This is the required Fourier expansion of the given function.

23) Obtain the Fourier expansion of

$$\mathbf{f(x)} = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Here,

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx dx + \int_{0}^{\pi} x\cos nx dx\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx dx + \int_{0}^{\pi} x\sin nx dx\right]$$

$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Fourier series is $\quad f(x) = = \frac{\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\frac{\left[1-2(-1)^n\right]}{n}\sin nx$

Note that the point x=0 is a point of discontinuity of f(x). Here $f(x^+) = 0$, $f(x^-) = -\pi$ at x=0. Hence

$$\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2}(0-\pi) = \frac{-\pi}{2}$$

The Fourier expansion of f(x) at x=0 becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

$$or\ \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

Simplifying we get,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

24) Obtain the Fourier expansion of

$$\mathbf{f(x)} = \begin{cases} 1 + \frac{4x}{3} & in\ -\frac{3}{2} < x \le 0 \\ 1 - \frac{4x}{3} & in\ 0 \le x < \frac{3}{2} \end{cases} \quad \text{Deduce that } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

The period of f(x) is $\frac{3}{2} - \left(\frac{-3}{2}\right) = 3$

Also $\quad$ f(-x) = f(x). $\quad$ Hence f(x) is even

$$a_0 = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)dx = \frac{2}{3/2}\int_0^{3/2} f(x)dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1-\frac{4x}{3}\right)dx = 0$$

$$a_n = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3/2}\int_0^{3/2} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left[\left(1-\frac{4x}{3}\right)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right)-\left(\frac{-4}{3}\right)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\right]_0^{3/2}$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

Also,

$$b_n = \frac{1}{3}\int_{-3/2}^{3/2} f(x)\sin\left(\frac{n\pi x}{3/2}\right)dx = 0$$

Thus

$$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{2n\pi x}{3}\right)$$

Putting x=0, we get

$$f(0) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]$$

or $$1 = \frac{8}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+......\right]$$

Thus, $$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+......$$

25) Obtain the cosine series of

$$f(x)=\begin{cases} x, 0<x<\frac{\pi}{2} \\ \pi-x, \frac{\pi}{2}<x<\pi \end{cases} \qquad over(0,\pi)$$

Here

$$a_0=\frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx+\int_{\pi/2}^{\pi}(\pi-x)\,dx\right]=\frac{\pi}{2}$$

$$a_n=\frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx+\int_{\pi/2}^{\pi}(\pi-x)\cos nx\,dx\right]$$

Performing integration by parts and simplifying, we get

$$a_n=-\frac{2}{n^2\pi}\left[1+(-1)^n-2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$=-\frac{8}{n^2\pi},\ n=2,6,10,.....$$

Thus, the Fourier cosine series is

$$f(x)=\frac{\pi}{4}-\frac{2}{\pi}\left[\frac{\cos 2x}{1^2}+\frac{\cos 6x}{3^2}+\frac{\cos 10x}{5^2}+\cdots\infty\right]$$

26) Obtain the half-range cosine series of f(x) = c-x in 0<x<c

Here

$$a_0=\frac{2}{c}\int_0^c (c-x)\,dx=c$$

$$a_n=\frac{2}{c}\int_0^c (c-x)\cos\left(\frac{n\pi x}{c}\right)dx$$

Integrating by parts and simplifying we get,

$$a_n=\frac{2c}{n^2\pi^2}\left[1-(-1)^n\right]$$

The cosine series is given by

$$f(x)=\frac{c}{2}+\frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{c}\right)$$

27) Find the first two harmonics of the Fourier series of f(x) given the following table:

x	**0**	$\pi/3$	$2\pi/3$	**π**	$4\pi/3$	$5\pi/3$	**2π**
f(x)	**1.0**	**1.4**	**1.9**	**1.7**	**1.5**	**1.2**	**1.0**

Note that the values of y = f(x) are spread over the interval 0≤ x ≤ 2π and f (0) = f (2π) = Hence the function is periodic and so we omit the last value f (2π) = 0. We prepare the following table to compute the first two harmonics. (dec 04, 06, 08,Apr-11)

x^0	y = f(x)	cosx	cos2x	sinx	sin2x	ycosx	ycos2x	ysinx	ysin2x
0	1.0	1	1	0	0	1	1	0	0
60	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	1.2124
120	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
180	1.7	-1	1	0	0	-1.7	1.7	0	0
240	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	1.299	1.299
300	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
Total						-1.1	-0.3	3.1176	-0.1732

We have

$$a_n = 2\left[f(x)\cos\left(\right) \right]$$

$$b_n = 2\left[f(x)\sin\left(\right) \right]$$

as the length of interval= 2*l* = 2π or *l*=π

Putting, n=1,2, we get

$$a_1 = 2[y\cos x] = \frac{2\sum y\cos x}{6} = \frac{2(1.1)}{6} == -0.367$$

$$a_2 = 2[y\cos 2x] = \frac{2\sum y\cos 2x}{6} = \frac{2(-0.3)}{6} = -0.1$$

$$b_1 = [y\sin x] = \frac{2\sum y\sin x}{6} = 1.0392$$

$$b_2 = [y\sin 2x] = \frac{2\sum y\sin 2x}{6} = -0.0577$$

The first two harmonics are $a_1\cos x+b_1\sin x$ and $a_2\cos 2x+b_2\sin 2x$. That is (-0.367cosx + 1.0392 sinx) and (-0.1cos2x – 0.0577sin2x)

28) Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table (Jun-12,Dec-10)

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Soln:

$$Here\ \ the\ length\ of\ the\ interval\ is\ 2l = 6, \qquad \Rightarrow l = 3$$

$$\therefore The\ fourier\ series\ can\ be\ repersented\ by$$

$$y = \frac{a_0}{2} + \left(a_1 \cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right)\right) + \left(a_2 \cos\left(\frac{2\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right)\right) + \cdots.$$

$$Here\ y = \frac{a_0}{2} + \left(a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right)\right) + \cdots \ldots \ldots (1)$$

x	y	$\cos\left(\frac{\pi x}{3}\right)$	$\sin\left(\frac{\pi x}{3}\right)$	$y\cos\left(\frac{\pi x}{3}\right)$	$y\sin\left(\frac{\pi x}{3}\right)$
0	9	1	0	9	0
1	18	0.5	0.866	9	15.588
2	24	-0.5	0.866	-12	20.785
3	28	-1	0	-28	0
4	26	-0.5	-0.866	-13	-22.517
5	20	0.5	-0.866	10	-17.321
$\sum$	125			-25	-3.465

$$a_0 = \frac{2}{n}\left(\sum y\right) = \frac{1}{3}(125) = 41.67, \qquad a_1 = \frac{2}{n}\left(\sum y \cos\left(\frac{\pi x}{3}\right)\right) = \frac{1}{3}(-25) = -8.33$$

$$b_1 = \frac{2}{n}\left(\sum y \sin\left(\frac{\pi x}{3}\right)\right) = \frac{1}{3}(-3.465) = -1.16$$

$$y = \frac{41.67}{2} + \left((-8.33)\cos\left(\frac{\pi x}{3}\right) + (-1.16)\sin\left(\frac{\pi x}{3}\right)\right)$$

$$y = 20.84 - 8.33\cos\left(\frac{\pi x}{3}\right) - 1.16\sin\left(\frac{\pi x}{3}\right)$$

29) The following table gives the variations of a periodic current A over a period T :

t(secs)	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amp)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a constant part of 0.75amp. in the current A and obtain the amplitude of the first harmonic. (A/M 03, Nov09)

Note that the values of A at t=0 and t=T are the same. Hence A(t) is a periodic function of period T. Let us denote $\theta = \left(\frac{2\pi}{T}\right)t$ **. We have**

$$a_0 = 2[A]$$

$$a_1 = 2\left[A\cos\left(\frac{2\pi}{T}t\right)\right] = 2[A\cos\theta] \qquad (1)$$

$$b_1 = 2\left[A\sin\left(\frac{2\pi}{T}t\right)\right] = 2[A\sin\theta]$$

We prepare the following table:

t	$\theta = \frac{2\pi t}{T}$	A	cosθ	sinθ	Acosθ	Asinθ
0	0	1.98	1	0	1.98	0
T/6	60^0	1.30	0.5	0.866	0.65	1.1258
T/3	120^0	1.05	-0.5	0.866	-0.525	0.9093
T/2	180^0	1.30	-1	0	-1.30	0
2T/3	240^0	-0.88	-0.5	-0.866	0.44	0.7621
5T/6	300^0	-0.25	0.5	-0.866	-0.125	0.2165
Total		4.5			1.12	3.0137

Using the values of the table in (1), we get

$$a_0 = \frac{2\sum A}{6} = \frac{4.5}{3} = 1.5$$

$$a_1 = \frac{2\sum A\cos\theta}{6} = \frac{1.12}{3} = 0.3733$$

$$b_1 = \frac{2\sum A\sin\theta}{6} = \frac{3.0137}{3} = 1.0046$$

The Fourier expansion upto the first harmonic is

$$A = \frac{a_0}{2} + a_1\cos\left(\frac{2\pi t}{T}\right) + b_1\sin\left(\frac{2\pi t}{T}\right)$$

$$= 0.75 + 0.3733\cos\left(\frac{2\pi t}{T}\right) + 1.0046\sin\left(\frac{2\pi t}{T}\right)$$

The expression shows that A has a constant part 0.75 in it. Also the amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2}$ = 1.0717.

30) The displacement y of a part of a mechanism is tabulated with corresponding angular movement x^0 of the crank. Express y as a Fourier series upto the third harmonic. (Dec-11)

x^0	0	30	60	90	120	150	180	210	240	270	300	330
f(x)	1.80	1.10	0.30	0.16	0.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

Soln:

x^0	0	30	60	90	120	150	180	210	240	270	300	330	Σ
y=f(x)	1.8	1.10	0.30	0.16	0.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00	15.15
Cosx	1	0.86	0.5	0	-0.5	-0.86	-1	-0.86	-0.5	0	0.5	0.86	
Cos2x	1	0.5	-0.5	-1	-0.5	0.5	1	0.5	-0.5	-1	-0.5	0.5	
Cos3x	1	0	-1	0	1	0	-1	0	1	0	-1	0	
Sinx	0	0.5	0.86	1	0.86	0.5	0	-0.5	-0.86	-1	-0.86	-0.5	
Sin2x	0	0.86	0.86	0	-0.86	-0.86	0	0.86	0.86	0	-0.86	-0.86	
Sin3x	0	1	0	-1	0	1	0	-1	0	1	0	-1	Σ
yCosx	1.8	0.94	0.15	0	-0.25	-1.11	-2.16	-1.07	-0.65	0	0.88	1.72	0.243
yCos2x	1.8	0.55	-0.15	-0.16	-0.25	0.65	2.16	0.625	-0.65	-1.52	-0.88	1	3.175
yCos3x	1.8	0	-0.3	0	0.5	0	-2.16	0	1.3	0	-1.76	0	-0.062
ySinx	0	0.55	0.25	0.16	0.43	0.65	0	-0.625	-1.118	-1.52	-1.514	-1	-2.73
ySin2x	0	0.946	0.25	0	-0.43	-1.11	0	1.075	1.118	0	-1.514	-1.72	-1.39
ySin3x	0	1.1	0	-0.16	0	1.3	0	-1.25	0	1.52	0	-2	0.51

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$$a_0 = \frac{2}{n}(\Sigma y) = \frac{2}{12}(15.15) = 2.525, \qquad \frac{a_0}{2} = \frac{2.525}{2} = 1.2625$$

$$a_1 = \frac{2}{n}\left(\sum y\cos x\right) = 0.0405,\ a_2 = \frac{2}{n}\left(\sum y\cos 2x\right) = 0.5292$$

$$a_3 = \frac{2}{n}\left(\sum y\cos 3x\right) = -0.103,\ b_1 = \frac{2}{n}\left(\sum y\sin x\right) = -0.4548$$

$$b_2 = \frac{2}{n}\left(\sum y\sin 2x\right) = -0.2308,\ b_3 = \frac{2}{n}\left(\sum y\sin 3x\right) = 0.085$$

$$f(x) = 1.2625 + (0.0405)\cos x + (0.5292)\cos 2x - (0.103)\cos 3x$$
$$-(0.4548)\sin x - (0.2308)\sin 2x + (0.085)\sin 3x + \cdots$$

31) Express y as a Fourier series upto the third harmonic given the following values:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

The values of y at x=0,1,2,3,4,5 are given and hence the interval of x should be 0 ≤ x < 6. The length of the interval = 6-0 = 6, so that 2*l* = 6 or *l* = 3.

The Fourier series upto the third harmonic is

$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l}\right) + \left(a_2\cos\frac{2\pi x}{l} + b_2\sin\frac{2\pi x}{l}\right) + \left(a_3\cos\frac{3\pi x}{l} + b_3\sin\frac{3\pi x}{l}\right)$$

or

$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{3} + b_1\sin\frac{\pi x}{3}\right) + \left(a_2\cos\frac{2\pi x}{3} + b_2\sin\frac{2\pi x}{3}\right) + \left(a_3\cos\frac{3\pi x}{3} + b_3\sin\frac{3\pi x}{3}\right)$$

Put $\theta = \frac{\pi x}{3}$, then

$$y = \frac{a_0}{2} + (a_1\cos\theta + b_1\sin\theta) + (a_2\cos 2\theta + b_2\sin 2\theta) + (a_3\cos 3\theta + b_3\sin 3\theta) \qquad (1)$$

We prepare the following table using the given values :

x	$\theta = \frac{\pi x}{3}$	y	ycosθ	ycos2θ	ycos3θ	ysinθ	ysin2θ	ysin3θ

0	0	04	4	4	4	0	0	0
1	60^0	08	4	-4	-8	6.928	6.928	0
2	120^0	15	-7.5	-7.5	15	12.99	-12.99	0
3	180^0	07	-7	7	-7	0	0	0
4	240^0	06	-3	-3	6	-5.196	5.196	0
5	300^0	02	1	-1	-2	-1.732	-1.732	0
Total		**42**	**-8.5**	**-4.5**	**8**	**12.99**	**-2.598**	**0**

$$a_0 = 2[f(x)] = 2[y] = \frac{2\sum y}{6} = \frac{1}{3}(42) = 14$$

$$a_1 = 2[y\cos\theta] = \frac{2}{6}(-8.5) = -2.833$$

$$b_1 = 2[y\sin\theta] = \frac{2}{6}(12.99) = 4.33$$

$$a_2 = 2[y\cos 2\theta] = \frac{2}{6}(-4.5) = -1.5$$

$$b_2 = 2[y\sin 2\theta] = \frac{2}{6}(-2.598) = -0.866$$

$$a_3 = 2[y\cos 3\theta] = \frac{2}{6}(8) = 2.667$$

$$b_3 = 2[y\sin 3\theta] = 0$$

Using these in (1), we get

$$y = 7 - 2,833\cos\left(\frac{\pi x}{3}\right) + (4.33)\sin\left(\frac{\pi x}{3}\right) - 1.5\cos\left(\frac{2\pi x}{3}\right) - 0.866\sin\left(\frac{2\pi x}{3}\right) + 2.667\cos\pi x$$

This is the required Fourier series upto the third harmonic.