UNIT – III- APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

CLASSIFICATION OF SECOND ORDER QUASI –LINEAR EQUATION:

Consider the second order linear homogeneous P.D.E

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0 \qquad \text{------ (1)}$$

Where A, B, C, D, E are functions of x, y or they are real constants.

The P.D.E (1) is said to be

i. "parabolic equation" if $B^2 - 4AC = 0$
ii. "hyperbolic equation" if $B^2 - 4AC > 0$
iii. "elliptic equation" if $B^2 - 4AC < 0$

Examples:

(i) Parabolic equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$ (One dimensional hear equation)

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$ (Two dimensional hear equation)

(ii) Hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$ (One dimensional Wave equation)

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$ (Two dimensional Wave equation)

(iii) Elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ (Two dimensional Laplace equation)

Classify the following partial differential equations:

(i) $\frac{\partial^2 u}{\partial x^2} = 5\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(iii) $\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

Solution:

(i) $\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$

A=1, B=0, C=0

B^2-4AC = 0

∴ The given P.D.E is a parabola

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

A=1, B=0, C=1

B^2-4AC < 0

∴ The given P.D.E is an elliptic.

(iii) $\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

A=1, B=3, C=1

$B^2-4AC > 0$

∴ The given P.D.E is a hyperbolic

Transverse Vibrations of a Stretched String (1-dimensional wave equation)

We make the following assumptions:

1. The motion takes place entirely in one plane.
2. n this plane, each particle moves at right angles to the equilibrium position of the string.
3. The tension T caused by stretching the string before fixing it at the end points is constant at all points of the deflecting string.
4. The tension T is considered to be very large compared with the weight of the sting and hence the force of gravity is negligible.
5. The effect of friction is negligible.
6. The string is perfectly flexible.
7. The slope of the deflection curve at all points and at all instants is so small.

Definition:

The P.D.E of 1-dimensional wave equation is given by $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$,

where $a^2 = \frac{T}{m}$. This is also called the P.D.E of the vibrating string.

Solution of One Dimensional Wave Equation:

The 1-dimensional wave equation is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (1)$$

Assume that the solution of (1) is of the form

$$Y(x, t) = X(x).T(t) \qquad (2)$$

where X is a function of x only and T is a function of t only.

$\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$, where $X'' = \frac{d^2X}{dx^2}$ and $T'' = \frac{d^2T}{dt^2}$

Hence (1) becomes, $XT'' = a^2X''T$

ie. $X''/X = T''/a^2T = k$ (say), where k is any constant.

Hence $X''-k X=0$ and $T''-a^2k T=0$ (3)

Solution of these equations depends upon the nature of the value Of k.

Case1: Let $k = p^2$, a positive number.

Now the equation (3) are $X''- p^2X=0$ and $T''- a^2p^2T=0$.

Solving the ordinary differential equations we get,

$X = A e^{px} + B e^{-\lambda x}$ and $T = C e^{pat} + D e^{-pat}$.

Case 2: Let $k=-p^2$, a negative number.

Then the equations (3) are $X''+ p^2X=0$ and $T''+ a^2p^2T=0$.

Solving we get,

$X = A \cos px+ B \sin px$ and $T = C \cos pat + D \sin pat$

Case 3: Let k=0.

Now the equations (3) are $X''=0$ and $T''=0$.

Integrating, we get,

$X = Ax+B$ and $T = Ct+D$.

Thus the various possible solutions of the wave equation are

$$\mathbf{Y(x, t) = (A e^{px}+B e^{-px})(Ce^{pat}+D e^{-pat})} \quad \text{(I)}$$

$$\mathbf{Y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat)} \quad \text{(II)}$$

$$\mathbf{Y(x, t) = (Ax+B)(Ct+D)} \quad \text{(III)}$$

Out of these solutions, we have to select that particular solution which suite the physical nature of the problem and the given boundary conditions. In the case of vibration of string, it is evident that y must be a periodic function of x and t. Hence we select the solution II as the probable solution of the wave equation. The constants are determined by using the boundary conditions in the problem. In doing problems, we shall select the solution II directly.

One Dimensional Heat Flow Equation

Assumptions:

(i) Heat flow from a higher to lower temperature.

(ii) The amount of heat required for a change in temperature in a body is proportional to the mass of the body and the change in temperature.

(iii) The rate of flow of heat in an area is proportional to the area and the temperature gradient normal.

Definition

The 1-dimensional heat flow equation is defined by

$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, $where\, \alpha^2 = \frac{k}{c\ell}$ is called the diffusivity of the material of the bar.

Solution of 1-Dimensional Heat flow equation

We have to solve the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ where\, \alpha^2 = \frac{k}{c\ell} \qquad (1)$$

Assume that the solution of (1) is of the form

$$u(x, t) = X(x).T(t) \qquad (2)$$

where X is a function of x only and T is a function of t only.

Then (1) becomes, XT' $=\alpha^2$X''T where $X'' = \frac{d^2X}{dx^2}$ $and\ \ T' = \frac{dT}{dt}$

ie. X''/X = T'/α^2T = k (say), where k is any constant.

Hence X''-kX=0 and T'-α^2kT=0 (3)

Solution of these equations depends upon the nature of the value of k.

Case1: Let $k=p^2$, a positive number.

Now the equation (3) are $X''- p^2X=0$ and $T'- \alpha^2p^2T=0$.

Solving the ordinary differential equations we get,

$X=Ae^{px}+Be^{-\lambda x}$ and $T= Ce^{\alpha^2p^2t}$

Case 2: Let $k = -\lambda^2$, a negative number.

Then the equations (3) are $X''+ p^2X=0$ and $T'+ \alpha^2p^2T=0$.

Solving we get,

$X=A\cos px+B\sin px$ and $T= Ce^{-\alpha^2p^2t}$

Case 3: Let $k=0$.

Now the equations (3) are $X''=0$ and $T'=0$.

Integrating, we get,

$X=Ax+B$ and $T=C$.

Thus the various possible solutions of the wave equation are

$$\mathbf{u = (Ae^{px}+Be^{-px})\, Ce^{\alpha^2p^2t}} \qquad \mathbf{(I)}$$

$$\mathbf{u = (A\cos px+B\sin px)\, Ce^{-\alpha^2p^2t}} \qquad \mathbf{(II)}$$

$$\mathbf{u = (Ax+B)\, C} \qquad \mathbf{(III)}$$

Out of these solutions, we have to select that particular solution which suite the physical nature of the problem and the given boundary conditions. As we are concerned with heat conduction, u(x,t) must decrease with increase of time. Therefore, out of the three solutions, we select the (II) solution to suit the physical nature of the problem. In the steady state conditions, when the temperature no longer varies with time, the solution of the diffusion equation (I) will be the last solution(III).

PART A

Problem 1 In the one dimensional heat equation $\boldsymbol{u_{t=c^2 u_{xx}}}$, what is c^2?**(May/June 2013)**

Solution : $c^2 = \frac{k}{\rho.s}$ where k = thermal conductivity, ρ = density and s = Specific heat

Problem 2 An insulated rod of length 60 cm has its end at A and B maintained at 20° C and 80° C respectively. Find the steady state solution of the rod. **(Nov/Dec 2012)**

Solution : $l = 60\ cm$

When steady state condition prevail the heat flow equation is $\frac{\partial^2 u}{\partial x^2}$= o

$u(x) = ax + b \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \to (i)$

When steady state conditions exists the boundary conditions are

$u(0) = 20\ ; \quad u(l) = 80 \cdots \cdots \cdots \cdots \cdots \cdots \to (ii)$

Therefore solution can be got using the formula

$u(x) = \left(\frac{B-A}{l}\right) x + A$ where A = 20 and B = 80

$u(x) = \left(\frac{80-20}{l}\right) x + 20 \quad \because l = 60$

$u(x) = \left(\frac{80-20}{60}\right) x + 20$

$$\Longrightarrow u(x) = \left(\frac{60}{60}\right) x + 20$$

$$\boldsymbol{Ans}: u(x) = x + 20$$

Problem 3: What is the basic difference between the solutions of one dimensional wave equation and one dimensional heat equation with respect to the time? **(May/ June 2012)**

Solution :

Solution of the one dimensional wave equation is of periodic in nature. But solution of the one dimensional heat equation is not of periodic in nature.

Problem 4 State the governing equation for one dimensional heat equation and necessary to solve the problem**. (Nov/Dec 2011)**

Solution :

$\boldsymbol{u_{t= c^2 u_{xx}}}$

$c^2 = \frac{k}{\rho .s}$ where k = thermal conductivity, ρ = density and s = Specific heat

Boundary conditions

(i) $u(0, t) = 0 \quad \forall t$

(ii) $u(l, t) = 0 \quad \forall t$

Initial conditions

(iii) $u(x, 0) = 0 \quad \forall x$

Possible solutions

(i) $u(x, t) = (A\, e^{px} + B\, e^{-px})\ e^{c^2p^2t}$

(ii) $u(x, t) = (A \cos px + B\, sinpx)\ e^{-c^2p^2t}$

(iii) $u(x, t) = (A\, x + B)$

Correct solution

$$\Rightarrow u(x, t) = (A \cos px + B\, sinpx)\, e^{-c^2p^2t}$$

Problem 5 A rod 40 cm long with insulated sides has its ends A and B kept at $20°\ C$ and $60°\ C$ respectively. Find the steady state temperature at a location 15 cm from A. **(April/ May 2011)**

Solution :

$u(x) = (\frac{B-A}{l})\ x + A$ where A = 20 and B = 60

$u(x) = (\frac{60-20}{l})\, x + 20 \because l = 40$

$u(x) = (\frac{60-20}{40})\ x + 20$

$u(x) = x + 20$

At x = 15 steady temperature $u = 15 + 20 = 35°\ C$

Problem 6 : Write down the three possible solutions of one dimensional heat equation **(Nov/Dec 2010 , May/ June 2009)**

Solution :

$$u_{t} = c^2 u_{xx}$$

possible solutions are

(i) $u(x, t) = (A\, e^{px} + B\, e^{-px})\; e^{c^2 p^2 t}$

(ii) $u(x, t\,) = (A \cos px + B \sin px)\; e^{-c^2 p^2 t}$

(iii) $u(x, y\,) = (A\,x + B)$

Problem 7: A tightly stretched string with fixed end points x = 0 and x = l is initially in a position given by $y(x,0) = y_0\ sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest in this position, write the boundary conditions. **(April/ May 2010)**

Solution :

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

- **(i)** $y(0,t) = 0$, $t > 0$
- **(ii)** $y(l,t) = 0$, $t > 0$
- **(iii)** $\frac{\partial y(x,0)}{\partial t} = 0$, $0 < x < l$
- **(iv)** $y(x,0) = y_0\ sin^3\left(\frac{\pi x}{l}\right)$, $0 < x < l$

Problem 8: Classify the partial differential equation $4\frac{\partial^2 y}{\partial x^2} = \frac{\partial u}{\partial t}$

Solution :

$4\frac{\partial^2 y}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

A = 4 , B = 0 and C = 0

$$B^2 - 4\,AC = (0)^2 - 4(4)(0)$$

$$= 0$$

Therefore, Parabolic

Problem 9: Write sown all possible solutions of one dimensional wave equation **(Nov/Dec 2009)**

Solution :

Possible solutions $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

- **(i)** $y(x,t) = (A\,e^{px} + B\,e^{-px})(C\,e^{pat} + D\,e^{-pat})$
- **(ii)** $y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$
- **(iii)** $y(x,t) = (A\,x + B)(C\,t + D)$

Problem 10: Verify that y = cos h $(\lambda x) cos\ h\ (-\lambda at)$ is a solution of $\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$ **(May/ June 2009)**

Solution :

y = cos h $(\lambda x) cos\ h\ (-\lambda at) \rightarrow (1)$

Differentiating (i) partially w.r.t x

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \lambda \sin h\ (\lambda\ x) \cosh(-\lambda\ at)$$

$$\frac{\partial^2 \boldsymbol{y}}{\partial \boldsymbol{x}^2} = \lambda^2 \cosh(\lambda\ x) \cosh(-\lambda\ at) \rightarrow (2)$$

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{t}} = -\boldsymbol{a}\lambda \cos h\ (\lambda\ x) \sinh(-\lambda\ at)$$

$$\frac{\partial^2 \boldsymbol{y}}{\partial \boldsymbol{t}^2} = a^2\lambda^2 \cosh(\lambda\ x) \cosh(-\lambda\ at)$$

$$= a^2 \frac{\partial^2 \boldsymbol{y}}{\partial \boldsymbol{x}^2} \text{ using } (2)$$

Here y = cos h $(\lambda x) cos\ h\ (-\lambda at)$ is a solution of $\frac{\partial^2 \boldsymbol{y}}{\partial \boldsymbol{t}^2} = a^2 \frac{\partial^2 \boldsymbol{y}}{\partial \boldsymbol{x}^2}$

Problem 16 A rod 50 cm long with insulated sides has its ends A and B kept at 20° C and 70° C respectively. Find the steady state temperature distribution of the rod. **(Nov/ Dec 2008)**

Solution :

$u(x) = (\frac{B-A}{l})\ x + A$ where A = 20 and B = 70

$u(x) = (\frac{70-20}{l})\ x + 20 \because l = 50\ cm$

$\text{u}(x) = (\frac{70-20}{50})\ x + 20$

$\text{u}(x) = x + 20$

Problem 11. Classify the differential equation **(Nov/ Dec 2008)**

$$3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 6\frac{\partial^2 u}{\partial y} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u = 0$$

Solution :

A = 3 , B = 4 and C = 6

B^2 - 4 A C = 16 – 72 = - 56 < 0

B^2 - 4 A C < 0 Therefore the given differential equation is elliptic

Problem 12. The ends A and B of a rod of length 10cm long have their temperature kept at 20° C and 70° C. Find the steady state temperature distribution on the rod. **(April/ May 2008)**

Solution :

$u(x) = (\frac{B-A}{l})\, x + A$ where A = 20 and B = 70

$u(x) = (\frac{70-20}{l})\, x + 20 \because l = 10\ cm$

$\mathrm{u}(x) = (\frac{70-20}{10})\, x + 20$

$\mathrm{u}(x) = 5x + 20$

Problem 13. Classify the differential equation **(April/ May 2008)**

$$3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0$$

Solution :

A = 3 , B = 4 and C = 0

B^2 - 4 A C = 16 > 0 . Therefore the equation is hyperbolic

PART B

One Dimensional Wave Equation

Problems of Vibrating string with initial velocity:

Problem 1 A tightly stretched string of length l has its ends fastened at x=0 and x= l. The midpoint of the string is then taken to height h and then released from rest in that position. Find the lateral displacement of a point of the string at time t from the instant of release**.(April/ May 2010)**

Solution:

The displacement of the string is given by $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$ (1)

Take one end as the origin. The midpoint of the string has the height 'h'.

Equation of OA is y-0 = $\left(\dfrac{h-0}{l/2-0}\right)(x-0)$

i.e. y = $\dfrac{2h}{l}.x$, when $0 \le x \le l/2$

Equation of AB is y = $\dfrac{-2h}{l}.(x-l)$, when $l/2 \le x \le l$

Hence the boundary conditions are

(i) y(0,t) = 0, for all t≥0

(ii) y(l,t) = 0, for all t≥0

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_{at t=0} = 0$, for all $0 \le x \le l$

(iv) y(x,0) = f(x) = $\begin{cases} \dfrac{2h}{l}.x, 0 \le x \le l/2 \\ \dfrac{-2h}{l}(x-l), l/2 \le x \le l \end{cases}$

Solve the equation (1) using the conditions (i), (ii), (iii) and (iv)

The most suitable solution of equation (1) is

$$Y(x,t) = (A \cos px + B \sin px)\,(C \cos pat + D \sin pat) \qquad (2)$$

Where A, B, C and D are arbitrary constants.

Using the boundary conditions (i), (ii) and (iii), we get

$$A = 0,\ p = \frac{n\pi}{l} \text{ and } D=0$$

Equation (2) becomes,

$$y(x,t) = B\,C \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \qquad (3)$$

The most general solution of equation of (3) is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \qquad (4)$$

Using boundary condition (iv) in (4), we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x) = \begin{cases} \frac{2h}{l}.x, & 0 \le x \le l/2 \\ \frac{-2h}{l}(x-l), & l/2 \le x \le l \end{cases}$$

L.H.S is a half-range sine series.

$$\therefore B_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

Substituting f(x) value and integrating we get,

$$B_n = \frac{8h}{n^2\pi^2} \sin\frac{n\pi}{2}$$

Substituting this in equation (4) , we get,

$$y(x,t) = \frac{8h}{\pi^2}\left[\frac{1}{l^2} \sin\frac{\pi x}{l} \cos\frac{\pi at}{l} - \frac{1}{3^2} \sin\frac{3\pi x}{l} \cos\frac{3\pi at}{l} + \cdots\right]$$

which is the required displacement.

Problem 2. A tightly stretched string with fixed end points x= 0 and x = 2l is initially in a position given by y(x,0) = $\begin{cases} \frac{kx}{l}, 0 < x < l \\ \frac{k(2l-x)}{l}, l < x < 2l \end{cases}$. If it is released from rest from this position, find the displacement function y(x,t) at any point of the string**. (Nov/ Dec 2006, 2010, May/June 2008)**

Solution:

The displacement of the string is given by $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ (1)

The boundary conditions are

(i) y(0,t) = 0, for all t≥0

(ii) y(2l,t) = 0, for all t≥0

(iii) $\left(\frac{\partial y}{\partial t}\right)_{att=0} = 0$, for all 0≤x≤ l

(iv) y(x,0) = $\begin{cases} \frac{kx}{l}, 0 < x < l \\ \frac{k(2l-x)}{l}, l < x < 2l \end{cases}$

The most suitable solution of equation (1) is

y(x,t) = (Acospx+Bsinpx)(Ccospat+Dsinpat) (2)

Where A, B, C and D are arbitrary constants.

Using the boundary conditions (i), (ii) and (iii), we get A=0, p= $\frac{n\pi}{l}$ and D=0

∴ (2) becomes, y(x,t) =BC sin $\frac{n\pi x}{2l}$ $cos\frac{n\pi at}{2l}$

The most general equation is y(x,t) = $\sum_{n=1}^{\infty} B_n sin \frac{n\pi x}{2l} cos \frac{n\pi at}{2l}$

By applying the boundary condition (iv), y(x,0) = $\sum_{n=1}^{\infty} B_n \, sin\frac{n\pi x}{2l} = \begin{cases} \frac{kx}{l}, 0 < x < l \\ \frac{k(2l-x)}{l}, l < x < 2l \end{cases}$

$B_n = \frac{1}{l}\int_0^{2l} f(x)\, sin\frac{n\pi x}{2l} dx = \frac{k}{l^2}\int_0^{l} x\, sin\frac{n\pi x}{2l} dx + \frac{k}{l^2}\int_l^{2l} (2l - x)\, sin\frac{n\pi x}{2l} dx$

$$= \frac{k}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right)+\left(\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right]_0^l + \frac{k}{l^2}\left[(2l-x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right)-\left(\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right]_l^{2l}$$

$$= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi a t}{2l}$$

Problems with non-zero initial velocity:

Problem 3 A string of length l is initially at rest in its equilibrium position and each of its points is given a velocity $\frac{\partial y}{\partial t}(x,0) = v_0 \sin^3\frac{\pi x}{l}, 0 \prec x \prec l$. Determine the displacement y(x,t).**(Nov/ Dec 2012, Nov/Dec 2004)**

Solution:

The wave equation is $\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2}$

The solution is y = (A cos px + B sin px) (C cos pat + D sin pat) (1)

We have the following boundary conditions

(i) y = 0 when x=0

(ii) y = 0 when x=l

(iii) y = 0 when t=0

(iv)(iv) $\frac{\partial y}{\partial t}(x,0) = v_0 \sin^3\frac{\pi x}{l}$ when t=0

Applying condition (i) , (ii) in (1), we get,

A=0, p = $\frac{n\pi}{l}$ and y = Bsin $\frac{n\pi x}{l}\left(C\cos\frac{n\pi a t}{l} + D\sin\frac{n\pi a t}{l}\right)$ (2)

Using the condition (iii) in (2) we get C=0

Equation (2) becomes,

$$Y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \qquad (3)$$

Differentiating (4) w.r.t ‘t’ and using the boundary condition (iv), we get,

$$v_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

To find the values of b_1, b_2 …we have to apply sin3θ formula and compare the like terms we get, $b_1 = \frac{3v_0 l}{4\pi a}, b_2 = 0, b_3 = \frac{-v_0 l}{12\pi a}, b_4 = b_5 = 0......$

The solution is

$$Y = \frac{3v_0 l}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} - \frac{v_0 l}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l}$$

Problem 4. A tightly stretched string with fixed end points x=0 and x=*l* is initially at rest in its equilibrium position. If it is set vibrating by giving each point a velocity 3x(*l*-x). Find the displacement of the string at any time. **(Nov’03,May’08,N/Dec 2009, May/June 2013 if 3= k)**

Solution:

The displacement of the string is given by $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ (1)

The boundary conditions are

(i) y(0,t) = 0, for all t ≥ 0

(ii) y(*l* ,t) = 0, for all t ≥ 0

(iii) y(*x* ,0) = 0, for all *x* in (0,*l*)

(iv) $\frac{\partial y}{\partial t}(x,0) = 3x(l-x)$ *for all x in* $(0,l)$

The most suitable solution of equation (1) is

The solution is y = (A cos px + B sin px) (C cos pat + D sin pat) (2)

Applying the first three boundary conditions in (2) we get A=0, $p = \frac{n\pi}{l}$, C=0 and

$$y = B \sin\frac{n\pi x}{l}\left(D \sin\frac{n\pi at}{l}\right) \quad (3)$$

The most general solution of equation of (3) is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \quad (4)$$

Differentiating (4) partially w.r.t 't' we get

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \quad (5)$$

Putting t = 0 in (5), we get

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = 3x(l-x) \text{ where } b_n = B_n \frac{n\pi a}{l}$$

By expanding $3x(l-x)$ in a half range Fourier sine series, we get

$$b_n = \frac{2}{l}\int_0^l 3x(l-x) \sin\frac{n\pi x}{l} dx = \frac{6}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]$$

$$= \frac{12l^2}{n^3\pi^3}\left[1-(-1)^n\right] = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{24l^2}{n^3\pi^3}, & \text{when } n \text{ is odd} \end{cases}$$

Therefore, $B_n = b_n \frac{l}{n\pi a} = \frac{24l^3}{an^4\pi^4}$ and hence $y(x,t) = \sum_{n=1,3,5,..} \frac{24l^3}{an^4\pi^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$

One Dimensional Heat Flow Equation

Problem 5. A rod of 30 cm long has its ends A and B kept at 20^o c and 80^oc respectively, until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0^o c and kept so. Find the resulting temperature function u(x,t) taking x = 0 at A . **(N/D 2009) ,(N/D 2008)**

Solution:

The P.D.E satisfied by u(x, t) is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

$u(x, t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t}$ (2)

In steady-state, this equation reduces to $\frac{d^2 u}{dx^2} = 0$

Here $l=30$

Solving this, u = ax+b, where a and b are arbitrary constants.

u=20 at x=0 and u=80 at x=30.

$\therefore$ b=20 and 80= la+20

Solving we get a= $\frac{60}{l}$

Thus the temperature function in steady-state is

$u(x) = \frac{60}{l}x+20$ (3)

This temperature distribution reached at the steady state becomes initial temperature distribution for the unsteady state. Then the temperature function u(x, t) satisfies (1).

The new boundary conditions are

(i) $u(0, t) = 0 \ \forall\ t>0$

(ii) $u(l, t) = 0 \ \forall\ t>0$

(iii) $u(x, 0) = \frac{60}{l} x+20$ for $0<x<l$

Using the boundary condition (i) in (2),

$u(0, t) = Ae^{-\alpha^2 p^2 t}=0 \quad \forall\ t>0$

$\therefore A = 0$

Using (ii) in (2),

B sin l p $e^{-\alpha^2 p^2 t}$=0, ∀ t>0

Since B≠0, sin lp=0.

Hence lp = nπ

i.e. p = $\frac{n\pi}{l}$, where n is any integer.

Therefore, (2) becomes,

$$u(x, t) = B_n sin \frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

The most general solution of (1) is,

$$u(x, t) = \sum_{n=1}^{\infty} B_n sin \frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \quad (4)$$

Using the initial condition (iii) in (4),

$$u(x, 0) = \sum_{n=1}^{\infty} B_n sin \frac{n\pi x}{l} = \frac{60}{l}x+20, \text{ for } 0<x< l.$$

Thus is itself Fourier half-range sine series for $\frac{60}{l}$x+20 in 0<x<l if

$$B_n = \frac{2}{l}\int_0^l \left(\frac{60}{l}x + 20\right) sin \frac{n\pi x}{l} dx$$

$$= \frac{40}{n\pi}[1 + 4(-1)^{n+1}]$$

Substituting this value of B_n in (4).

$$u(x, t) = \sum_{n=1}^{\infty} \left[\frac{40}{n\pi}[1 + 4(-1)^{n+1}]\right] sin \frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

Problem 6. A rod of length 20 cm has its ends A and B kept at temperature 30^oc and 90^oc respectively until steady state conditions prevail. If the temperature at each end is then suddenly reduced to 0^o c and maintained so, find the temperature distribution at a distance from A at time 't'(N/D 2005)

Solution:

The P.D.E satisfied by u(x, t) is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

u(x, t) = (Acospx+Bsinpx) $e^{-\alpha^2 p^2 t}$ (2)

In steady-state, this equation reduces to $\frac{d^2u}{dx^2} = 0$

Here *l=20*

Solving this, u = ax+b, where a and b are arbitrary constants.

u=30 at x=0 and u=90 at x=20.

$\therefore$ b=30 and 90= *l*a+30

Solving we get a= $\frac{60}{l}$

Thus the temperature function in steady-state is

u(x) = $\frac{60}{l}$x+30 (3)

This temperature distribution reached at the steady state becomes initial temperature distribution for the unsteady state. Then the temperature function u(x, t) satisfies (1).

The new boundary conditions are

(i) u(0, t) = 0 $\forall$ t>0

(ii) u(*l*, t) = 0 $\forall$ t>0

(iii) u(x, 0) = $\frac{60}{l}$ x +30for 0<x< *l*

Using the boundary condition (i) in (2),

u(0, t) = A$e^{-\alpha^2 p^2 t}$=0 $\forall$ t>0

$\therefore$A = 0

Using (ii) in (2),

B sin *l* p $e^{-\alpha^2 p^2 t}$=0, $\forall$ t>0

Since B≠0, sin *l*p=0.

Hence *l*p = nπ

i.e. p=$\frac{n\pi}{l}$, where n is any integer.

Therefore, (2) becomes,

$$u(x, t) = B_n sin \frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

The most general solution of (1) is,

$$u(x, t) = \sum_{n=1}^{\infty} B_n sin \frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \qquad (4)$$

Using the initial condition (3) in (4),

$$u(x, 0) = \sum_{n=1}^{\infty} B_n\, sin \frac{n\pi x}{l} = \frac{60}{l}x+30, \text{ for } 0<x< l.$$

Thus is itself Fourier half-range sine series for $\frac{60}{l}x+30$ in $0<x<l$ if

$$B_n = \frac{2}{l}\int_0^l \left(\frac{60}{l}x + 30\right) sin \frac{n\pi x}{l} dx = 0 \text{ , n is odd}$$

$$= \frac{-120}{n\pi}\text{, n is even}$$

Substituting this value of B_n in (4).

$$u(x, t) = \sum_{n=2,4,6\cdots}^{\infty} \frac{-120}{n\pi} sin \frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

Problem 7. The ends A and B of a rod l cm long have their temperatures kept at 30^0C and 80^0C until steady state conditions prevails. The temperature of the end B is suddenly reduced to 60^0C and that of A is increased to 40^0C. Find the temperature distribution in the rod after time'‘t'. **(May'07)**

Solution:

The P.D.E satisfied by u(x, t) is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

In steady-state, this equation reduces to $\frac{d^2u}{dx^2} = 0$

Solving this, u = ax+b, where a and b are arbitrary constants.

u=30 at x=0 and u=80 at x= l.

$\therefore$ b=20 and 40=10a+b

Solving we get a=2

Thus the temperature function in steady-state is

$u(x) = 2x+20$ (2)

When the temperatures at A and B are changed, the state is no longer steady. Then the temperature function u(x, t) satisfies (1).

The boundary conditions in the second state are

(iv) $u(0, t) = 50 \ \forall \ t>0$

(v) $u(10, t) = 10 \ \forall \ t>0$

(vi) $u(x, 0) = 2x+20$ for $0<x<10$

Consider, $u(x, t) = u_s(x)+u_t(x, t)$ (3)

where $u_s(x)$ is a solution of (1) involving x only and satisfying the boundary conditions (i) and (ii), $u_t(x, t)$ is a function defined by (3) and satisfying the equation (1).

Thus, $u_s(x)$ is a steady-state solution of (1) and $u_t(x, t)$ may then be regarded as a transient solution which decreases with increase of t; $u_s(x)$ satisfies (1).

$\frac{d^2u_s}{dx^2} = 0$, where $u_s(0) = 50$ and $u_s(10) = 10$.

Solving, $u_s(x) = ax+b$.

$u_s(0) = b = 50$, using(i)

and $u_s(10) = 10a+50 = 10$, using (ii)

Hence a = -4.

Thus $u_s(x) = 50-4x$ (4)

Consequently,

$u_t(0, t) = u(0, t)- u_s(0)=50-50=0$ (iv)

$u_t(10, t) = u(10, t)- u_s(10)=10-10=0$ (V) and $u_t(x, 0) = u(x, 0) - u_s(x) = (2x+20)-(50-4x)$

i.e, $u_t(x, 0) = 6x-30$ (vi)

Now, $u_t(x, t)$ also satisfies (1) and (iv), (v), (vi).

Solving (1) and selecting a suitable solution

$$u_t(x, t) = (A\cos\lambda x + B\sin\lambda x)\, Ce^{-\alpha^2\lambda^2 t} \quad (5)$$

Using the boundary condition (iv) in (5),

$u_t(0, t) = Ae^{-\alpha^2\lambda^2 t} = 0 \ \forall\ t>0$

$\therefore A = 0$

Using (v) in (5),

$B\sin 10\lambda e^{-\alpha^2\lambda^2 t} = 0, \ \forall\ t>0$

Since $B\neq 0$, $\sin 10\lambda = 0$.

Hence $10\lambda = n\pi$

i.e. $\lambda = \frac{n\pi}{10}$, where n is any integer.

Therefore, (5) becomes,

$$u_t(x, t) = B_n \sin\frac{n\pi x}{10}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}}$$

The most general solution of (1) is,

$$u_t(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}} \quad (6)$$

Using the initial condition (vi) in (6),

$$u_t(x, 0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = 6x-30, \text{ for } 0<x<10.$$

Thus is itself Fourier half-range sine series for $6x-30$ in $0<x<10$ if

$$B_n = \frac{2}{10}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}dx$$

$$= \frac{-60}{n\pi}[1+(-1)^n] = \frac{-120}{n\pi}, \text{ for n even}$$

$= 0$, for n odd.

Substituting this value of B_n in (6).

$u_t(x, t) = \sum_{n=2,4,6\cdots}^{\infty} \frac{-120}{n\pi} sin \frac{n\pi x}{10} e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}}$

$= \frac{-60}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} sin \frac{n\pi x}{5} e^{\frac{-\alpha^2 n^2 \pi^2 t}{25}}$

$\therefore$ u(x, t) = $u_s(x)+u_t(x, t)$

$= 50-4x-\frac{-60}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} sin \frac{n\pi x}{5} e^{\frac{-\alpha^2 n^2 \pi^2 t}{25}}$

Problem 8. The ends A and B of a rod *l* cm long have the temperature 40^0C and 90^0C until steady state prevails. The temperature at A is suddenly raised to 90^0C and at the same time that at B is lowered to 40^0C. Find the temperature distribution in the rod at time t. Also show that the temperature at the mid point of the rod remains unaltered for all time, regardless, of the material of the rod.

(Apr'03)

solution

The P.D.E satisfied by u(x, t) is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

In steady-state, this equation reduces to $\frac{d^2u}{dx^2} = 0$

Solving this, u = ax+b, where a and b are arbitrary constants.

u=40 at x=0 and u=90 at x= *l*.

$\therefore$ b = 40 and 90 = *l* a + b

Solving we get a = $\frac{50}{l}$

Thus the temperature function in steady-state is

$$u(x) = \frac{50}{l}x+40 \qquad (2)$$

When the temperatures at A and B are changed, the state is no longer steady. Then the temperature function u(x, t) satisfies (1).

The boundary conditions in the second state are

(vii) u(0, t) = $\forall$ t>0

(viii) $u(10, t) = 10 \ \forall \ t>0$

(ix) $u(x, 0) = 2x+20$ for $0<x<10$

Consider, $u(x, t) = u_s(x)+u_t(x, t)$ (3)

where $u_s(x)$ is a solution of (1) involving x only and satisfying the boundary conditions (i) and (ii), $u_t(x, t)$ is a function defined by (3) and satisfying the equation (1).

Thus, $u_s(x)$ is a steady-state solution of (1) and $u_t(x, t)$ may then be regarded as a transient solution which decreases with increase of t; $u_s(x)$ satisfies (1).

$\frac{d^2u_s}{dx^2} = 0$, where $u_s(0) = 50$ and $u_s(10) = 10$.

Solving, $u_s(x) = ax+b$.

$u_s(0) = b = 50$, using(i)

and $u_s(10) = 10a+50 = 10$, using (ii)

Hence $a = -4$.

Thus $u_s(x) = 50-4x$ (4)

Consequently,

$u_t(0, t) = u(0, t)- u_s(0)=50-50=0$ (iv)

$u_t(10, t) = u(10, t)- u_s(10)=10-10=0$ (v)

and $u_t(x, 0) = u(x, 0)- u_s(x)= (2x+20)-(50-4x)$

i.e, $u_t(x, 0) = 6x-30$ (vi)

Now, $u_t(x, t)$ also satisfies (1) and (iv), (v), (vi).

Solving (1) and selecting a suitable solution

$u_t(x, t) = (A\cos px+B\sin px)\, Ce^{-\alpha^2p^2t}$ (5)

Using the boundary condition (iv) in (5),

$u_t(0, t) = Ae^{-\alpha^2p^2t}=0 \ \forall \ t>0$

$\therefore A = 0$

Using (v) in (5),

$B\sin 10p e^{-\alpha^2p^2t}=0, \ \forall \ t>0$

Since $B\neq 0$, $\sin 10p= 0$

Hence $10p=n\pi$

i.e. $p=\frac{n\pi}{10}$, where n is any integer.

Therefore, (5) becomes,

$u_t(x, t) = B_n sin\frac{n\pi x}{10}e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}}$

The most general solution of (1) is,

$$u_t(x, t) = \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi x}{10}e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}} \qquad (6)$$

Using the initial condition (vi) in (6),

$u_t(x, 0) = \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{10} = 6x-30$, for $0<x<10$.

Thus is itself Fourier half-range sine series for 6x-30 in $0<x<10$ if

$B_n = \frac{2}{10}\int_0^{10}(6x-30)\, sin\frac{n\pi x}{10}dx$

$= \frac{-60}{n\pi}[1+(-1)^n] = \frac{-120}{n\pi}$, for n even

$= 0$, for n odd.

Substituting this value of B_n in (6).

$u_t(x, t) = \sum_{n=2,4,6\cdots}^{\infty} \frac{-120}{n\pi}\, sin\frac{n\pi x}{10}e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}}$

$= \frac{-60}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}sin\frac{n\pi x}{5}e^{\frac{-\alpha^2 n^2 \pi^2 t}{25}}$

$\therefore\ u(x, t) = u_s(x)+u_t(x, t)$

$= 50-4x-\frac{60}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}sin\frac{n\pi x}{5}e^{\frac{-\alpha^2 n^2 \pi^2 t}{25}}$

Problem 9. A metal bar 10cm long with insulated sides has its ends A and B kept at 20^0C and 40^0C respectively, until steady state conditions prevail. The temperature at A is then suddenly raised to 50^0C and at the same instant that at B is lowered to 10^0C. Find the subsequent temperature at any point at the bar at any time. **(Nov/Dec 2005)**

Solution:

The P.D.E satisfied by u(x, t) is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

In steady-state, this equation reduces to $\frac{d^2u}{dx^2} = 0$

Solving this, u = ax+b, where a and b are arbitrary constants.

u=20 at x=0 and u=40 at x=10.

$\therefore$ b=20 and 40=10a+b

Solving we get a=2

Thus the temperature function in steady-state is

$u(x) = 2x+20 \qquad (2)$

When the temperatures at A and B are changed, the state is no longer steady. Then the temperature function u(x, t) satisfies (1).

The boundary conditions in the second state are

(x) $u(0, t) = 50 \; \forall \; t>0$

(xi) $u(10, t) = 10 \; \forall \; t>0$

(xii) $u(x, 0) = 2x+20$ for $0<x<10$

Consider, $u(x, t) = u_s(x)+u_t(x, t) \qquad (3)$

where $u_s(x)$ is a solution of (1) involving x only and satisfying the boundary conditions (i) and (ii), $u_t(x, t)$ is a function defined by (3) and satisfying the equation (1).

Thus, $u_s(x)$ is a steady-state solution of (1) and $u_t(x, t)$ may then be regarded as a transient solution which decreases with increase of t; $u_s(x)$ satisfies (1).

$\frac{d^2u_s}{dx^2} = 0$, where $u_s(0) = 50$ and $u_s(10) = 10$.

Solving, $u_s(x) = ax+b$.

$u_s(0) = b = 50$, using(i)

and $u_s(10) = 10a+50 = 10$, using (ii)

Hence a = -4.

Thus $u_s(x) = 50-4x \qquad (4)$

Consequently,

$u_t(0, t) = u(0, t)- u_s(0)=50-50=0 \qquad$ (iv)

$u_t(10, t) = u(10, t)- u_s(10)=10-10=0 \qquad$ (V)

and $u_t(x, 0) = u(x, 0)- u_s(x)= (2x+20)-(50-4x)$

i.e, $u_t(x, 0) = 6x-30 \qquad$ (vi)

Now, $u_t(x, t)$ also satisfies (1) and (iv), (v), (vi).

Solving (1) and selecting a suitable solution

$$u_t(x, t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \quad (5)$$

Using the boundary condition (iv) in (5),

$u_t(0, t) = Ae^{-\alpha^2\lambda^2 t} = 0 \ \forall\ t>0$

$\therefore A = 0$

Using (v) in (5),

$B\sin 10\lambda\, e^{-\alpha^2\lambda^2 t} = 0, \ \forall\ t>0$

Since $B\neq 0$, $\sin 10\lambda = 0$.

Hence $10\lambda = n\pi$

i.e. $\lambda = \frac{n\pi}{10}$, where n is any integer.

Therefore, (5) becomes,

$$u_t(x, t) = B_n \sin\frac{n\pi x}{10}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{100}}$$

The most general solution of (1) is,

$$u_t(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{100}} \quad (6)$$

Using the initial condition (vi) in (6),

$$u_t(x, 0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = 6x-30, \text{ for } 0<x<10.$$

Thus is itself Fourier half-range sine series for $6x-30$ in $0<x<10$ if

$$B_n = \frac{2}{10}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}dx$$

$$= \frac{-60}{n\pi}[1+(-1)^n] = \frac{-120}{n\pi}, \text{ for n even}$$

$= 0$, for n odd.

Substituting this value of B_n in (6).

$$u_t(x, t) = \sum_{n=2,4,6\cdots}^{\infty} \frac{-120}{n\pi} \sin\frac{n\pi x}{10}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{100}}$$

$$= \frac{-60}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{5}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{25}}$$

$\therefore\ u(x, t) = u_s(x) + u_t(x, t)$

$$= 50-4x-\frac{60}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{5}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{25}}$$

Problem 10. Solve the heat flow equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the following boundary conditions

(i) $u(0,t) = 0$

(ii) $u(l,t) = 0$

(iii) $u(x,0) = x$

Solution:

The solution of one dimensional heat equation is given by

$$u = (A\cos px + B\sin px)Ce^{-\alpha^2 p^2 t} \qquad (1)$$

Using boundary conditions (i) and (ii) in (1), we get,

$A = 0,\ p = n\pi/l$

Substituting these values in (1) we get

The most general solution of (1) is

$$u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{\frac{-\alpha^2 p^2 \pi^2 t}{l^2}} \qquad (2)$$

Applying the condition (iii) in (2), we get

$$x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

The R.H.S is Half range sine series

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{-2}{n\pi}\left[l(-1)^n\right]$$

The solution is u = $\sum_{n=1}^{\infty} \frac{-2l}{n\pi} (-1)^n \sin \frac{n\pi x}{l} \, e^{\frac{-\alpha^2 p^2 \pi^2 t}{l^2}}$