UNIT-V Z- TRANSFORMS

INTRODUCTION

Transform techniques are very important tool in the analysis of signals .The Laplace transforms are popularly used for analysis of continuous time signals.

Similarly Z-transform plays an important role in analysis of liner discrete time signals.

DEFINITION OF Z- TRANSFORM:

CONSIDER A SEQUENCE :{f(n)}: {f (0),f(1),f(2) ....}

Which is defined for all positive integers n=0,1,2…….. ∞ then the Z –transform of{f(n) is defined asZ{f(n)}= $\sum_{n=0}^{\infty} f(n)z^{-n}$, where Z is an arbitrary---------1

Complex variable.

The R.H.S of eqn (1) is a function of z after putting n=0,1,2…..

Hence we denote z[f(n)= F(2) (or) Z [file}= F(2)

Here F(2) and f(n) are called a Z-transform pair and is denoted by f(n) $\xleftrightarrow{Z}$ f(2)

Note

The above transformation is called one sided (or) Z-transform of a sequence If {f(n)}is a sequence defined for n=0,1+2….. then Z{f(n)}= $\sum_{n=\infty}^{\infty} f(n)z^{-n}$ is called two sided or bilateral Z-transform of{f(n)}

PROBLEMS

1. **Determine the Z-transform and their region of convergence of the following discrete sequences:**

(i) f(n) = {3,2,5,7} (ii) f(n)= {1,2,3,4,5,6,7}

Solution:

Given f(n)= {3,2,5,7}

F(0)=3 f(1) =2 f(2)=5 f(3)=7

Are know that, F(Z)= Z[f(n)= $\sum_{n=0}^{3} f(n)z^{-n}$

$$= f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3}$$

$$F(z) = 3 + \frac{2}{z} + \frac{5}{z^2} + \frac{7}{z^3}$$

Here F (z) has finite values except at Z=0.At Z=0, F(Z) becomes infinite.

Hence, F (z) is convergence is the entire complex plane except Z=0.

(ii) Given f(n) ={1,2,3,4,5,6,7}

(i.e.) f(0)=1,f(1)=2 ,f(2)= 3, f(3)=4 , f(4)=5 , f(5)=6, f(6)7

we know that F(Z)= Z[f(n)= $\sum_{n=0}^{6} f(n)z^{-n}$

$$= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots f(6)z^{6}$$

$$= 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4} + \frac{6}{z^5} + \frac{7}{z^6}$$

Hence F(Z) has finite values except at Z=0

At Z=0, F(z) becomes infinite.

Hence F(z) is convergent for all values of Z except at Z=0. hence region of convergence is the entire complex plane Z=0.

2. Prove that Z(1) = $\left(\frac{z}{z-1}\right)$**. Find also the region of convergence.**

Soln: Here f(n)=1

$$Z(1) = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\ldots$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\ldots$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\ldots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

3. Prove that $Z\left[(-1)^n\right] = \frac{z}{z+1}$ **find also the region of convergence.**

Solution:

$$Z\left[(-1)^n\right] = \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots$$

$$= \left(1 + \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z+1}{z}\right)^{-1}$$

$$Z\left[(-1)^n\right] = \frac{z}{z+1}$$

Hence the region of convergence is $\left|\frac{1}{z}\right| < 1$ or $|z| > 1$.

4. **Prove that** $Z\left[a^n\right] = \frac{z}{z-a}$ **.** **(NOV 2012)**

Solution:

$$Z\left[a^n\right] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left(\frac{z-a}{z}\right)^{-1}$$

$$Z\left[a^n\right] = \frac{z}{z-a}$$

Hence, the region of convergence is $\left|\frac{a}{z}\right| < 1$ **(or)** $|z| > |a|$.

<u>COROLLARY:</u>

1. $Z\left[(-1)^n\right] = \frac{z}{z+1}$

2. $Z\left[e^{an}\right] = Z\left[\left(e^a\right)^n\right] = \frac{z}{z-e^a}$

3. $Z\left[e^{-an}\right] = Z\left[\left(e^{-a}\right)^n\right] = \frac{z}{z-e^{-a}}$

5. **Prove that** $Z(n) = \frac{z}{(z-1)^2}$ **or** $Z(k) = \frac{z}{(z-1)^2}$. **(June 2013)**

Solution:

We know that $Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z(n) = \sum_{n=0}^{\infty} n z^{-n}$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \dots$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \dots\right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left(\frac{z^2}{(z-1)^2}\right)$$

$$Z(n) = \frac{z}{(z-1)^2}$$

6. Prove that $Z\left[na^n\right] = \frac{az}{(z-a)^2}$ **(or)** $Z\left[ka^k\right] = \frac{az}{(z-a)^2}$

Solution:

$$Z\left(na^n\right) = \sum_{n=0}^{\infty} na^n z^{-n}$$

$$= \frac{a}{z} + 2\frac{a^2}{z^2} + 3\frac{a^3}{z^3} + \dots$$

$$= \frac{a}{z}\left[1 + 2\frac{a}{z} + 3\frac{a^2}{z^2} + \dots\right]$$

$$= \frac{a}{z}\left[1 - \frac{a}{z}\right]^{-2}$$

$$= \frac{a}{z}\left[\frac{z-a}{z}\right]^{-2}$$

$$= \frac{a}{z}\left[\frac{z^2}{(z-a)^2}\right]$$

$$Z\left[na^n\right] = \frac{az}{(z-a)^2}$$

7. Prove that $Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$.

Solution:.

$$Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty}\frac{1}{n+1}z^{-n}$$

$$= 1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} + \frac{1}{4}z^{-3} + \ldots$$

$$= 1 + \frac{1/z}{2} + \frac{1/2^2}{3+}\ldots\ldots$$

$$= \mathbf{Z}\left[\frac{1}{z} + \frac{1/z^2}{2} + \frac{1/z^3}{3} + \ldots\ldots\ldots\right]$$

$$= \mathbf{Z}\left[-\log(1-\frac{1}{z})\right]\left[\because -\log(1-x) = x + \frac{x^2}{2} + \frac{x3}{3} + \ldots\ldots\ldots\right]$$

$$= \mathbf{Z}\left[-\log\left(\frac{z-1}{z}\right)\right]$$

$$= \mathbf{Z}\left[\log\left(\frac{z-1}{z}\right)^{-1}\right]$$

$$Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right).$$

8. Find $Z(\cos n\theta)$ and hence deduce $Z\,(\cos\frac{n\pi}{2})$ (June 2013)

Solution:

Let $a = e^{i\theta}$

$$a^n = (e^{i\theta})^n = e^{in\theta} \quad = \cos n\theta + i \sin n\theta$$

we know that $Z(a^n) = \dfrac{z}{z-a}$

$$Z((e^{i\theta})^n) = \frac{z}{z - e^{i\theta}}$$

$$Z(e^{in\theta}) = \frac{z}{z - (\cos\theta + i\sin\theta)}$$

$$Z(\cos n\theta \_ i \sin n\theta) = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$Z(\cos n\theta) + iZ(\sin n\theta)$$

$$= \left[\frac{z}{(z-\cos\theta) - i\sin\theta}\right]\left[\frac{(z-\cos\theta) + i\sin\theta}{(z-\cos\theta) + i\sin\theta}\right]$$

$$= \left[\left[\frac{z(z-\cos\theta) + iz\sin\theta}{(z-\cos\theta)^2 + \sin^2\theta}\right]\right]$$

$$= \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + i\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating the real and imaginary parts, we

$$Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}, \quad |z| > 1 \text{ ....(1)}$$

$$Z(\sin n\theta) = \frac{\square z\sin\theta}{z^2 - 2z\cos\theta + 1}, \quad |z| > 1 \text{.................................. (2)}$$

Putting $\theta = \dfrac{\pi}{2}$ in (1), we get

$$Z(\cos\frac{n\pi}{2}) = \frac{z(z-\cos\frac{\pi}{2})}{z^2 - 2z\cos\frac{\pi}{2} + 1}$$

$$= \frac{z(z-0)}{z^2 - 2z(0) + 1}$$

$$= \frac{z^2}{z^2+1}, \qquad |z| > 1.$$

9. Find the Z-transform of $a^n \cos n\theta$ and $e^{-at} \sin bt$

solution:

(i) we know that $Z(a^n f(n)) = F\left[\frac{z}{a}\right]$

$$Z(a^n \cos n\theta) = \left[Z(\cos n\theta)\right]_{z\to z/a}$$

$$= \left[\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}\right]_{z\to z/a}$$

$$= \frac{\frac{z}{a}\left(\frac{z}{a} - \cos\theta\right)}{\left[\frac{z}{a}\right]^2 - 2\frac{z}{a}\cos\theta + 1}$$

$$= \left[\frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}\right]$$

(ii) we know that , $Z(e^{-at} f(t)) = Z\left[f(t)\right]_{z\to ze^{at}}$

$$Z\left[e^{-at}\sin bt\right] = \left[Z[\sin bt]\right]_{z\to ze^{aT}}$$

$$= \left[\frac{z\sin bT}{z^2 - 2z\cos bT + 1}\right]_{z\to ze^{a}}$$

$$= \frac{ze^{aT}\sin bT}{z^2 e^{2aT} - 2ze^{aT}\cos bT + 1}$$

10. Find $Z[n(n-1)(n-2)]$. (June 2012)

Solution:

$$[n(n-1)(n-2)] = n\left[n^2 - 2n - n + 2\right]$$

$$= n^3 - 3n^2 + 2n$$

$$Z[n(n-1)(n-2)] = Z\left[n^3 - 3n^2 + 2n\right]$$

$$= Z[n^3] - 3Z[n^2] + 2Z[n]$$

$$= \frac{z\left[z^2 + 4z + 1\right]}{(z-1)^4} - 3\frac{z(z+1)}{(z-1)^3} + 2\frac{z}{(z-1)^2}$$

$$= \frac{6z}{(z-1)^4}.$$

PROPERTIES OF Z-TRANSFORM:

1. LINER PROPERTY:

Z[a f(n)+bg(n)]=a F(2)+BG when Z[f(n)=F(z)

Z[g(n)=G(z),a,b , are constants.

SOLUTION:

$$\mathbf{Z}\left[af(n)+bg(n)\right]=\sum_{n=0}^{\infty}\left[af(n)+bg(n)\right]z^{-n}$$

$$=\sum_{n=0}^{\infty} f(n)z^{n+b}+b\sum_{n=0}^{\infty} g(n)z^{-n}$$

$$= aF(z)+bG(z)$$

2. FIRST SHIFTNG [FREQUENCY SHIFTING];

If Z{f(t)=F(z),then,

$$Z\left[e^{-at}f(t)\right]=F\left[ze^{at}\right]$$

Proof :

We know that $= z\left[f(t)=\right]\sum_{n=0}^{\infty} f(nt)z^{-n}$

$$\left[\because assumethatf(t)=f(nt)\right]$$

$$\therefore z\left[e^{-at}f(t)\right]=\sum_{n=0}^{\infty}e^{-anT}f(nT)z^{z-n}$$

$$=\sum_{n=0}^{\infty} f(nT)\left(ze^{aT}\right)^{-n}$$

$$z\left[e^{-at}f(t)\right]=F\left[ze^{aT}\right]$$

NOTE:

The above result can also be written as

$$z\left[e^{-at}f(t)\right]=\{f(z)\}=\longrightarrow ze^{aT}$$

$$z\left[e^{at} f(t)\right] = \{f(z)\} = \longrightarrow ze^{-aT}$$

COROLLARY;

$$Z\left[a^n f(n)\right] = F(z/a) where$$ F(z)=z [f(n)].

Proof;

$$\mathbf{Z}\,[a^n f(n)] = \sum_{n=0}^{\infty} a^n f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n)\left(\frac{z}{a}\right)^{-n}$$

$$Z[a^n f(n)] = F\left(\frac{z}{a}\right), \text{ where } F(z) = Z[f(n)]$$

SECOND SHIFTING PROPERTY

If Z{f(t)}=F(z) then, $z\left[f(t+T)\right] = zF(z) - zf(0)$ **(June 2013)**

Proof:

$$z\left[f(t+T)\right] = \sum_{n=0}^{\infty} f(nT+T) z^{-n}$$

$$= \sum_{n=0}^{\infty} f\left[(n+1)T\right] z^{-n}$$

$$= z\sum_{n=0}^{\infty} f\left[(n+1)T\right] z^{-(n+1)}$$

$$= z\sum_{m=1}^{\infty} f\left[mT\right] z^{-m}$$, where $m = n+1$

$$= z\left[\sum_{m=0}^{\infty} f\left[mT\right] z^{-m} - f(0)\right]$$

$$= zF(z) - zf(0)$$

PROBLEMS:

1. **Find** $z\left[t^2 e^{-t}\right]$

Solution;

We know that $Z\left[e^{-at} f(t)\right] = \left[F(t)\right]_{z \to ze^{at}}$

Where ,F(z) = $z\left[f(t)\right]$

Here f(t) = $\left[t^2\right]$

Here f(t)= $\left[t^2\right]$ **and a=1**

$$Z\left[f(t)\right] = Z(t^2)$$

$$Z\left[(nT)^2\right] = T^2 Z(n^2) = T^2 \frac{z(z+1)}{(z-1)^3}$$

$$\therefore Z\left[e^{-t} t^2\right] = T^2 \left\{\frac{z(z+1)}{(z-1)^3}\right\}_{z \to ze^T}$$

$$= T^2 \left\{\frac{ze^T\left(ze^T + 1\right)}{\left(ze^T - 1\right)^3}\right\}$$

2. **Find** $Z\left(e^{-iat}\right)$

Solution:

$$Z\left[e^{-iat}\right] = Z\left[e^{-iat}..1\right]$$

Here f(t)=1 and a=+ia

$$\therefore Z\left[f(t)\right] = Z(1) = \frac{z}{z-1}$$

$$\Rightarrow Z\left[e^{-iat}\right] = \left\{\frac{z}{z-1}\right\}_{z \to ze^{iaT}}$$

$$= \frac{ze^{iaT}}{ze^{iaT} - 1}$$

3 Find $Z\left(\sin\frac{n\pi}{2}\right)$ and $Z\left(\cos\frac{n\pi}{2}\right)$

Solution:

We know that $Z\left(a^n\right)=\frac{z}{z-a}$

$$\therefore Z\left(e^{\frac{in\pi}{2}}\right)=Z\left[\left(e^{\frac{i\pi}{2}}\right)^n\right]=\frac{z}{z-e^{\frac{i\pi}{2}}}$$

$$=\frac{z}{z-\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right]}$$

$$=\frac{z}{z-i}$$

$$=\frac{z}{(z-i)(z+i)}(z+i)$$

$$Z\left[e^{\frac{in\pi}{2}}\right]=\frac{z^2+zi}{z^2+1}$$

$$\Rightarrow Z\left[\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}\right]=\frac{z^2+zi}{z^2+1}$$

Equating the real part imaginary part on both sides.

$$\Rightarrow Z\left[\cos\frac{n\pi}{2}\right]=\frac{z^2}{z^2+1}$$

$$\& Z\left[\sin\frac{n\pi}{2}\right]=\frac{z}{z^2+1}$$

4 Find $Z[\sin at]$ *and* $Z[\cos at]$

Solution:

We know that

$$Z\left[e^{-iat}\right] = \frac{ze^{iaT}}{ze^{iaT}-1}$$

$$\Rightarrow Z\left[\cos at - i\sin at\right] = \frac{ze^{iaT}}{ze^{iaT}-1}$$

$$= \frac{ze^{iaT}}{\frac{e^{iaT}}{e^{iat}}\left(ze^{iaT}-1\right)}$$

$$= \frac{z}{z-\frac{1}{e^{iaT}}}$$

$$= \frac{z}{z-e^{-iaT}}$$

$$= \frac{(z)}{z-\cos aT - i\sin aT}$$

$$= \frac{z}{(z-\cos aT)+i\sin aT}$$

$$= \frac{z\left[(z-\cos aT)-i\sin aT\right]}{(z-\cos aT)^2+\sin^2 aT}$$

$$= \frac{z(z-\cos aT)-iz\sin aT}{z^2-2z\cos aT+\cos^2 aT+\sin^2 aT}$$

$$Z\left[\cos aT - i\sin aT\right] = \frac{z(z-\cos aT)-iz\sin aT}{1+z^2-2z\cos aT}$$

Equating the real and imaginary terms on both sides.

$$Z[\cos aT] = \frac{z(z-\cos aT)}{1+z^2-2z\cos aT}$$

$$Z[\sin aT] = \frac{z\sin aT}{1+z^2-2z\cos aT}$$

5 find $Z[\cos^2 t]$ *and* $Z[\cos^3 t]$

Solution:

We know that,

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$\therefore Z[\cos^2 t] = Z\left[\frac{1+\cos 2t}{2}\right]$$

$$= \frac{1}{2}[z(1)+z(\cos 2t)]$$

$$= \frac{1}{2}\left[\frac{z}{z-1} + \frac{z(z-\cos 2t)}{z^2-2z\cos 2t+1}\right]$$

also we know that,

$$\cos^3 t = \frac{1}{4}[3\cos t + \cos 3t]$$

$$Z[\cos^3 t] = z\left[\frac{1}{4}(3\cos t+\cos 3t)\right]$$

$$= \frac{3}{4}z(\cos t) + \frac{1}{4}z(\cos 3t)$$

$$= \frac{3}{4}\left[\frac{z(z-\cos T)}{(z^2-2z\cos T+1)}\right] \frac{1}{4}\left[\frac{z(z-\cos 3T)}{z^2-2z\cos 3T+1}\right]$$

6 Find the Z-transforms of $\sin^2\left(\frac{n\pi}{4}\right)$ and $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$ (NOV 2012)

Solution:

(i) $Z\left[\sin^2\left(\frac{n\pi}{4}\right)\right]$

We know that , $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\sin^2\frac{n\pi}{4} = \frac{1-\cos\frac{n\pi}{4}}{2}$$

$$= \frac{1}{2} - \frac{1}{2}\cos\frac{n\pi}{2}$$

$$Z\left[\sin^2\left(\frac{n\pi}{4}\right)\right] = \frac{1}{2}Z(1) - \frac{1}{2}Z\left[\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-1}\right] - \frac{1}{2}\left[\frac{z^2}{z^2+1}\right]$$

(ii) $Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right]$

$$\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) = \cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}$$

$$= \cos\frac{n\pi}{2}\frac{1}{\sqrt{2}} - \sin\frac{n\pi}{2}\sin\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right]$$

$$Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] = \frac{1}{\sqrt{2}}Z\left[\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{\sqrt{2}}Z\left[\cos\frac{n\pi}{2}\right] - \frac{1}{\sqrt{2}}Z\left[\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1}\right] - \frac{1}{\sqrt{2}}\left[\frac{z}{z^2+1}\right]$$

$$= \frac{z}{\sqrt{2}}\left[\frac{z}{z^2+1} - \frac{1}{z^2+1}\right]$$

$$= \frac{z(z-1)}{\sqrt{2}\left(z^2+1\right)}$$

7. Find $Z\left[a^k \cosh \alpha k\right]$

solution:

$$Z\left[a^k \cosh \alpha k\right] = Z\left[a^k \frac{\left(e^{\alpha k}+e^{-\alpha k}\right)}{2}\right]$$

$$= \frac{1}{2} Z\left[a^k e^{\alpha k}\right] + \frac{1}{2} Z\left[a^k e^{-\alpha k}\right]$$

$$= \frac{1}{2}\left[\left(\frac{z}{z-a}\right)_{z \to ze^{-\alpha}} + \left(\frac{z}{z-a}\right)_{z \to ze^{\alpha}}\right]$$

$$\left[\because \textit{By shifting property}\right]$$

$$= \frac{1}{2}\left[\frac{ze^{-\alpha}}{ze^{-\alpha}-a} + \frac{ze^{\alpha}}{ze^{\alpha}-a}\right]$$

$$= \frac{1}{2}\left[\frac{z^{\alpha} - aze^{-\alpha} + z^2 - zae^{\alpha}}{\left(ze^{-\alpha}-a\right)\left(ze^{\alpha}-a\right)}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - az\left(e^{\alpha}+e^{-\alpha}\right)}{z^2 - aze^{-\alpha} - aze^{\alpha} + a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - 2az\cosh\alpha}{z^2 + a^2 - 2az\cosh\alpha}\right]$$

$$Z\left[a^k \cosh \alpha k\right] = \frac{z^2 - az\cosh\alpha}{z^2 + a^2 - 2az\cosh\alpha}$$

TRANSFORM OF UNIT STEP FUNCTION:-

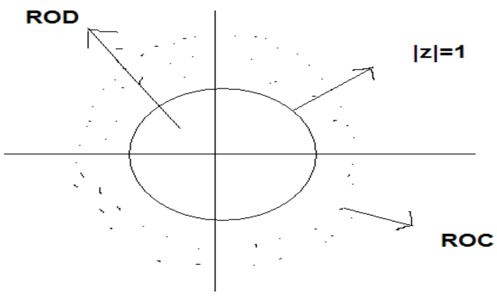
A discrete unit step function is defined as,

$$u(k):\{1,1,1,......\} = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

Hence
$$z\{u(k)\} = \sum_{k=0}^{\infty} u(k) z^{-k}$$
$$= \sum_{k=0}^{\infty} z^{-k}$$
$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\ldots$$
$$= \left(1 - \frac{1}{z}\right)^{-1}$$
$$z[u(k)] = \left(\frac{z-1}{z}\right)^{-1}$$

$$\therefore \quad z[u(k)] = \frac{z}{z-1}, \ \textit{where } u(k) \textit{ is unit step function.}$$

The region of convergence is |z| > 1



UNIT IMPULSE FUNCTION:-

A discrete unit impulse function is defined by,

$$\delta(k) : \{1,0,0,0\} = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

1. Find $z[\delta(k)], \delta(k)$ is unit impulse function (or) Determine the Z-transform of the unit sample sequence $\delta(n)$.

solution:-

$$z[\delta(k)] = \sum_{k=0}^{\infty} \delta(k) z^{-k}$$

$$z[\delta(k)] = 1$$

Since Z[$\delta(k)$] is constant and independent of values of Z the ROC is the entire complex plane.

2. Find $z[f(k)]$ when (i) f(k)= (1/2)

Solution:-

$$z[f(k)] = \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k u(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-k} \quad [\because u(k) = 1, k \geq 0]$$

$$= \sum_{k=0}^{\infty} (2z)^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2z}\right)^k$$

$$= 1 + \frac{1}{2z} + \left(\frac{1}{2z}\right)^2 + \ldots\ldots$$

$$= \left(1 - \frac{1}{2z}\right)^{-1}$$

$$= \left(2z - \frac{1}{2z}\right)^{-1}$$

$$z[f(k)] = \frac{2z}{2z - 1}$$

3. Find (i) $z[n+2]$ (ii) $z\left[4\,3^n + 2(-1)^n\right]$ (iii) $z\left[\frac{1}{(n+1)(n+2)}\right]$

Solution:-

(i) $z[n+2] = z(n)+z(2)$

$$= \frac{z}{(z-1)^2} + 2z(1)$$

$$= \frac{z}{(z-1)^2} + 2 \cdot \frac{z}{z-1}$$

$$= \frac{z+2z(z-1)}{(z-1)^2}$$

$$= z+2z^2-2z$$

$$z[n+2] = \frac{2z^2-z}{(z-1)^2}$$

(ii) $z\left[4 3^n + 2(-1)^n\right] = 4z(3^n) + 2z\left[(-1)^n\right]$

$$= 4.\frac{z}{z-3} + 2.\frac{z}{z+1}$$

(iii) $z\left[\frac{1}{(n+1)(n+2)}\right]$

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$\frac{1}{(n+1)(n+2)} = \frac{A(n+2)+B(n+1)}{(n+1)(n+2)}$$

$$1 = A(n+2) + B(n+1)$$

Put n= -2 $\Rightarrow B = -1$

Put n= -1 $\Rightarrow A = 1$

$$\therefore \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\therefore Z\left[\frac{1}{(n+1)(n+2)}\right] = z\left[\frac{1}{n+1}\right] - z\left[\frac{1}{n+2}\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} - \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= [1 + \frac{1/z}{2} + \frac{(1/z)^2}{3} + ...] - \left[\frac{1}{2} + \frac{1/z}{3} + \frac{1/z^2}{4} + ...\right]$$

$$= z\left[\frac{1}{z} + \frac{1/z^2}{2} + \frac{1/z^3}{3} + ...\right] - z^2\left[\frac{1/z^2}{2} + \frac{1/z^3}{3} + \frac{1/z^4}{4} + ...\right]$$

$$= z\left[-\log\left(1 - \frac{1}{z}\right)\right] - z^2\left[-\log\left(1 - \frac{1}{z}\right) - \frac{1}{z}\right]$$

$$= z\left[\log\left(\frac{z-1}{z}\right)^{-1}\right] - z^2\left[\log\left(\frac{z-1}{z}\right)^{-1} - \frac{1}{z}\right]$$

$$= z\log\left(\frac{z}{z-1}\right) - z^2\left[\log\left(\frac{z}{z-1}\right) - \frac{1}{z}\right]$$

$$Z\left[\frac{1}{(n+1)(n+2)}\right] = \log\left(\frac{z}{z-1}\right)[z - z^2] + z$$

PROPERTY:DIFFERENTIATION IN Z-DOMAIN

Prove that $Z[nf(n)] = -z\frac{d}{dz}\{F(z)\}$ *where* $F(z) = Z[f(n)]$

Solution: $Z[nf(n)] = \sum_{n=0}^{\infty} nf(n)z^{-n}$

We know that $F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$

$$\frac{d}{dz}F(z) = \frac{d}{dz}\sum_{n=0}^{\infty} f(n)z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n)\frac{d}{dz}(z^{-n})$$

$$= \sum_{n=0}^{\infty} f(n)(-nz^{-n-1})$$

$$\frac{d}{dz}\left[F(z)\right] = \sum_{n=0}^{\infty} nf(n)z^{-n}\left(-\frac{1}{z}\right)$$

$$\sum_{n=0}^{\infty} nf(n)z^{-n} = -z\frac{d}{dz}\left[F(z)\right]$$

$$Z\left[nf(n)\right] = -z\frac{d}{dz}\left[F(z)\right]$$

$$Z\left[kf(k)\right] = -z\frac{d}{dz}\left[F(z)\right]$$

$$In\ general,\ Z\left[k^{m} f(k)\right] = \left[-z\frac{d}{dz}\right]^{m} F(z)$$

TIME REVERSAL PROPERTY FOR BILATERAL Z- TRANSFORM:

If $\quad Z\left[f(k)\right] = \left[F(z)\right], then Z\left[f(-k)\right] = F\left[\frac{1}{z}\right]$

Proof: $Z\left[f(-k)\right] = \sum_{n=-\infty}^{\infty} f(-k)z^{-k}$

Put –k = n   k = -n   when k = -∞   n = ∞

K = ∞   n = - ∞

$$= \sum_{n=\infty}^{-\infty} f(n)z^{n} = \sum_{n=-\infty}^{\infty} f(n)\left(\frac{1}{z}\right)^{-n} = F\left[\frac{1}{z}\right]$$

Problem:

1. Find $Z\left[n^{2}\right]$

   Solution: $Z\left[n^{2}\right] = Z\left[n \times n\right]$

$$= -z\frac{d}{dz}\left[Z(n)\right]$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2.1-2z(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$= -z\left[\frac{-z-1}{(z-1)^3}\right]$$

$$Z\left[n^2\right] = \frac{z(z+1)}{(z-1)^3}$$

INITIAL VALUE THEOREM:

If $Z\left[f(t)\right] = \left[F(z)\right], then \lim_{z\to\infty} F(z) = f(0) = \lim_{t\to 0} f(t)$

FINAL VALUE THEOREM:

If $Z\left[f(t)\right] = \left[F(z)\right], then \lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)F(z)$

1. Find the initial and the final values of the function $F(z) = \frac{1+z^{-1}}{1-0.25z^{-2}}$

Solution: By initial value theorem , we have

$$f(0) = \lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{1+z^{-1}}{1-0.25z^{-2}}$$

f(0) = 1

By final value theorem, $f(\infty) = \lim_{z\to 1} (z-1)F(z)$

$$= \lim_{z\to 1} (z-1)\frac{1+z^{-1}}{1-0.25z^{-2}}$$

$$= \lim_{z\to 1} \frac{z^2-1}{z^2-0.25}$$

$$= 0$$

2. Find the initial and the final values of the function $F(z)=\dfrac{z}{2z^2-3z+1}$, $|z|>1$

Solution:

By initial value theorem , we have

$$f(0)=\lim_{z\to\infty}F(z)=\lim_{z\to\infty}\frac{z}{2z^2-3z+1}$$

$$=\lim_{z\to\infty}\frac{z}{z(2z-3+1/z)}$$

$$=\lim_{z\to\infty}\frac{1}{(2z-3+1/z)}$$

$$f(0)=0$$

By final value theorem, we have

$$f(\infty)=\lim_{z\to1}(z-1)F(z)$$

$$=\lim_{z\to1}(z-1)\frac{z}{2z^2-3z+1}$$

$$=\lim_{z\to1}\frac{z^2-z}{2z^2-3z+1}$$

$$=\lim_{z\to1}\frac{z^2(1-\frac{1}{z})}{z^2\left(2-\frac{3}{z}+\frac{1}{z^2}\right)}=\frac{0}{0}(I.F)$$

$$=\lim_{z\to1}\frac{2z-1}{4z-3}=1$$

$$\therefore f(\infty)=1$$

THE INVERSE Z- TRANSFORM:

If $Z[f(k)]=[F(z)]$, *thentheinverse Z – transformis defined by* $Z^{-1}[F(z)]=f(k)$

(or) *If* $Z[f(n)]=F(z)$ *then* $Z^{-1}[F(z)]=f(n)$

S.No	Z-transform	Inverse Z- transform
1.	$Z(a^n)=\dfrac{z}{z-a}$	$Z^{-1}\left[\dfrac{z}{z-a}\right]=a^n$
2.	$Z(a^{n-1})=\dfrac{1}{z-a}$	$Z^{-1}\left[\dfrac{1}{z-a}\right]=a^{n-1}$

3.	$Z[n]=\frac{z}{(z-1)^2}$	$Z^{-1}\left[\frac{z}{(z-1)^2}\right]=n$
4.	$Z\left[na^n\right]=\frac{az}{(z-a)^2}$	$Z^{-1}\left[\frac{az}{(z-a)^2}\right]=na^n$
5.	$Z\left[na^{n-1}\right]=\frac{z}{(z-a)^2}$	$Z^{-1}\left[\frac{z}{(z-a)^2}\right]=na^{n-1}$
6.	$Z\left[(n-1)a^{n-1}\right]=\frac{a}{(z-a)^2}$	$Z^{-1}\left[\frac{a}{(z-a)^2}\right]=(n-1)a^{n-1}$

Though we have many methods to find inverse Z-transform the following are very familiar methods.

1. Method of partial fraction
2. Method of residues (Using Cauchy's Residue theorem)
3. Power series expansion method or Long division method

NOTE: Different methods **of** finding inverse Z-transform of a function F(z) will give different types of mathematical expression. But after evaluating the expression for each value of k, we may get the same sequence.

PROBLEMS:

1 Find the inverse Z-transform of $F(z)=z^2\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)\left(1+2z^{-1}\right), 0<|z|<\infty$

Solution:

$$F(z)=z^2\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)\left(1+2z^{-1}\right), 0<|z|<\infty$$

$$=\left(z^2-\frac{z}{2}\right)\left(1+2z^{-1}-z^{-1}-2z^{-2}\right)$$

$$=z^2+2z-z-2-\frac{z}{2}-1+\frac{1}{2}+z^{-1}$$

$$F(z)=z^2\frac{z}{2}-\frac{5}{2}+z^{-1}$$

We know that the bilateral Z-transform is,

$$F(z) = Z[f(n)] = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

$$= \dots + f(-2)z^{2} + f(-1)z + f(0) + f(1)z^{-1} + \dots$$

Comparing like coefficients, we get

$$f(-2)=1,\ f(-1)=\frac{1}{2},\quad f(0)=-\frac{5}{2},\quad f(1)=1$$

$$\therefore f(n)=\left\{\dots 0,1,\frac{1}{2},-\frac{5}{2},1,0,\dots\right\}$$

2 Find the inverse Z-transform of $F(z)=\log\left[\frac{1}{1-az^{-1}}\right],\ |z|>|a|$

Solution: $F(z)=\log\left[\frac{1}{1-az^{-1}}\right],\ |z|>|a| = -\log\left[1-az^{-1}\right] \dots\dots (1),\ |a|>|a|$

We know that,

$$\log(1-r) = -\sum_{n=1}^{\infty}\frac{1}{n} r^{n},\ |r|<1 \dots\dots\dots\dots (2)$$

Comparing (1) *and* (2) *we get*

$$F(z)=\sum_{n=1}^{\infty}\frac{1}{n}\left(az^{-1}\right)^{n} = \sum_{n=1}^{\infty}\frac{1}{n} a^{n} z^{-n}$$

$$\therefore\ f(n)=\begin{cases}\frac{1}{n}a^{n}, & n\geq 1\\ 0, & n\leq 0\end{cases}$$

$$\therefore\ f(n)=\frac{1}{n}a^{n}u(n-1)$$

METHOD OF PARTIAL FRACTION TO FIND INVERSE Z-TRANSFORM:

1.Find $Z^{-1}\left[\frac{z-4}{(z-1)(z-2)^{2}}\right]$

Solution:

$$\left[\frac{z-4}{(z-1)(z-2)^2}\right] = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$\left[\frac{z-4}{(z-1)(z-2)^2}\right] = \frac{A(z-2)^2 + B(z-1)(z-2) + C(z-1)}{(z-1)(z-2)^2}$$

put z=2 -2 = C

C = -2

Put z=1 A=-3

Equating the coefficient of z^2 on both sides

0 = A+B -3+B = 0 B = 3

$$\left[\frac{z-4}{(z-1)(z-2)^2}\right] = \frac{-3}{z-1} + \frac{3}{z-2} - \frac{2}{(z-2)^2}$$

$$Z^{-1}\left[\frac{z-4}{(z-1)(z-2)^2}\right] = -3Z^{-1}\left[\frac{1}{z-1}\right] + 3Z^{-1}\left[\frac{1}{z-2}\right] - 2\left[\frac{1}{(z-2)^2}\right]$$

$$= -3(1)^{k-1} + 3(2)^{k-1} - 2\frac{(k-1)2^{k-1}}{2}$$

$$= -3u(k-1) - (k-1)2^{k-1}u(k-1) + 3(2)^{k-1}u(k-1)$$

(the term 2^{k-1} is valid only for k>1)

3. Find $Z^{-1}\left[\frac{z-4}{(z+2)(z+3)}\right]$

Solution:

$$\frac{z-4}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$$

$$\frac{z-4}{(z+2)(z+3)}=\frac{A(z+3)+B(z+2)}{(z+2)(z+3)}$$

$$\Rightarrow z-4=A(z+3)+B(z+2)$$

$$\begin{array}{ll} Put\ z=-3 & Put\ z=-2 \\ -7=-B & -6=A \\ B=7 & A=-6 \end{array}$$

$$\Rightarrow \frac{z-4}{(z+2)(z+3)}=-\frac{6}{z+2}+\frac{7}{z+3}$$

$$Z^{-1}\left[\frac{z-4}{(z+2)(z+3)}\right]=-6\,Z^{-1}\left[\frac{1}{z+2}\right]+7Z^{-1}\left[\frac{1}{z+3}\right]$$

$$=-6(-2)^{k-1}+7(-3)^{k-1}$$

$$\therefore Z^{-1}\left[\frac{z-4}{(z+2)(z+3)}\right]=-6(-2)^{k-1}u(k-1)+7(-3)^{k-1}u(k-1)$$

Since the term a^{k-1} is valid only for k>1 and hence multiplied by u(k-1).

NOTE:

1. To find inverse transform of a function F(z) by using partial fraction method , it is convenient to write F(z) as $\frac{F(z)}{z}$ and split the function by partial fraction.
2. Here, in F(z) the degree of both numerator and denominator are equal. Hence to split F(z) into partial fraction, we should divide F(z) till we get the degree of numerator is less than the degree of the denominator.

PROBLEM:

1. Find $Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right]$

Solution:

$$Let \ F(z) = \frac{z^3}{(z-1)^2(z-2)}$$

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)^2(z-2)}$$

$$\frac{z^2}{(z-1)^2(z-2)} = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$= \frac{A(z-1)^2 + B(z-1)(z-2) + C(z-2)}{(z-1)^2(z-2)}$$

$$\Rightarrow z^2 = A(z-1)^2 + B(z-1)(z-2) + C(z-2)$$

Put $z = 1$ *Put* $z = 2$

$C = -1$ $A = 4$

Equating thecoefficient of z^2 *onbothsides*, $1 = A + B \Rightarrow B = -3$

$$\frac{F(z)}{z} = \frac{4}{z-2} - \frac{3}{z-1} - \frac{1}{(z-1)^2}$$

$$\Rightarrow F(z) = \frac{4z}{z-2} - \frac{3z}{z-1} - \frac{z}{(z-1)^2}$$

$$\therefore Z^{-1}[F(z)] = 4Z^{-1}\left[\frac{z}{z-2}\right] - 3Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$= 4(2)^n - 3(1)^n - n$$

2 .Find $Z^{-1}\left[\frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}\right]$

Solution:

$$Given\ F(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$= \frac{1}{1-\frac{1.5}{z}+\frac{0.5}{z^2}}$$

$$F(z) = \frac{z^2}{z^2-1.5z+0.5}$$

$$\frac{F(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

$$\frac{z}{(z-1)(z-0.5)} = \frac{A}{z-1} + \frac{B}{z-0.5}$$

$$= \frac{A(z-0.5)+B(z-1)}{(z-1)(z-0.5)}$$

$$\Rightarrow \quad z = A(z-0.5)+B(z-1)$$

$Put\ z = 0.5$ $\quad\quad$ $Put\ z = 1$

$B = -1$ $\quad\quad$ $A = 2$

$$\therefore \frac{z}{(z-1)(z-0.5)} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$F(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

$$Z^{-1}[F(z)] = 2Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{z-0.5}\right]$$

$$= 2(1)^n - (0.5)^n$$

$$= 2u(n) - (0.5)^n \quad \left[\because Z[u(n)] = \frac{z}{z-1}\right.$$

TO FIND INVERSE Z-TRANSFORM USING RESIDUE THEOREM:

If Z[f(n)] = F(z), then f(n) which gives the inverse Z-transform of F(z) is obtained from the following result.

(i.e) $f(n) = \frac{1}{2\pi i}\int_C z^{n-1}F(z)dz$, where C is the closed contour which encloses all the poles of the integrand. To evaluate $\int_C z^{n-1}F(z)dz$ we can use the Residue theorem.

1. Find the inverse Z-transform of $\frac{z}{(z-1)(z-2)}$ using Residue theorem.

Solution:

$$Let\ \ Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]=f(n)$$

$$then\ \ f(n)=\frac{1}{2\pi i}\int_C z^{n-1}F(z)dz$$

$$=\frac{1}{2\pi i}\int_C z^{n-1}\frac{z}{(z-1)(z-2)}dz$$

$$=\frac{1}{2\pi i}\int_C \frac{z^n}{(z-1)(z-2)}dz.....................(1)$$

$$To evaluate \int_C \frac{z^n}{(z-1)(z-2)}dz$$

$$\Rightarrow \int_C \frac{z^n}{(z-1)(z-2)}dz = 2\pi i\,[Sum\ of\ the residues\ of\ \phi(z)\ at\ each\ of\ its\ poles].......(2)$$

$$Where\ \ \phi(z)=\frac{z^n}{(z-1)(z-2)}$$

The poles are z = 1, z =2

$$\operatorname{Res}_{z=1}\{\phi(z)\}=\lim_{z\to 1}(z-1)\phi(z)$$

$$=\lim_{z\to 1}(z-1)\frac{z^n}{(z-1)(z-2)}$$

$$=-(1)^n.................(3)$$

$$\operatorname{Res}_{z=2}\{\phi(z)\}=\lim_{z\to 2}(z-2)\frac{z^n}{(z-1)(z-2)}$$

$$=2^n..................(4)$$

$$Sub\ (3)\ and\ (4)\ in\ eqn\ (2)$$

$$\therefore \int_C \frac{z^n}{(z-1)(z-2)}dz = 2\pi i\,[2^n-(1)^n]...........(5)$$

$$Sub\ (5)\ in\ eqn\ (1),\qquad f(n)=\frac{1}{2\pi i}2\pi i[2^n-(1)^n]$$

$$=2^n-(1)^n$$

Checking:

$$Z\left[2^n-(1)^n\right]=Z\left[2^n\right]-Z\left[(1)^n\right]$$

$$=\frac{z}{z-2}-\frac{z}{z-1}=\frac{z}{(z-1)(z-2)}$$

2. Find $Z^{-1}\left[\frac{z}{z^2-2z+2}\right]$ using residue theorem.

Solution:

$$Let\ \ Z^{-1}\left[\frac{z}{z^2-2z+2}\right]=f(n)$$

$$then\quad f(n)=\frac{1}{2\pi i}\int_C z^{n-1}F(z)dz$$

$$=\frac{1}{2\pi i}\int_C z^{n-1}\frac{z}{z^2-2z+2}dz$$

$$=\frac{1}{2\pi i}\int_C\frac{z^n}{z^2-2z+2}dz\ ....(1)$$

$$To\ evaluate\int_C\frac{z^n}{(z-1)(z-2)}\ dz$$

$$\Rightarrow\int_C\frac{z^n}{z^2-2z+2}dz=2\pi i\left[Sum\ of\ the\ residues\ of\ \phi(z)\ at\ each\ of\ its\ poles\right]........(2)$$

$$Where\quad\phi(z)=\frac{z^n}{z^2-2z+2}$$

In $\phi(z)$ equate the denominator to zero

$$z^2-2z+2=0$$

$$z=\frac{2\pm\sqrt{4-4(1)(2)}}{2}$$

$$=\frac{2\pm\sqrt{-4}}{2}=1\pm i$$

$$\therefore The\ poles\ are\ z=1+i\ ,1-i$$

$$\underset{z=1+i}{\mathrm{Res}}\{\phi(z)\}=\lim_{z\to1+i}[z-(1+i)]\frac{z^n}{[z-(1+i)][z-(1-i)]}$$

$$=\frac{(1+i)^n}{2i}$$

$$\underset{z=1-i}{\mathrm{Res}}\{\phi(z)\}=\lim_{z\to1-i}[z-(1-i)]\frac{z^n}{[z-(1+i)][z-(1-i)]}$$

$$=\frac{(1-i)^n}{-2i}$$

$$\therefore\int_C\frac{z^n}{z^2-2z+2}dz=2\pi i\left[\frac{(1+i)^n}{2i}-\frac{(1-i)^n}{2i}\right]$$

$$=\pi\left[(1+i)^n-(1-i)^n\right]$$

$$The\ \mathrm{mod}ulus\ and\ amplitude\ form\ of\quad(1+i)^n=2^{n/2}\left[\cos n\frac{\pi}{4}+i\sin n\frac{\pi}{4}\right]$$

$$(1-i)^n=2^{n/2}\left[\cos n\frac{\pi}{4}-i\sin n\frac{\pi}{4}\right]$$

$$\therefore(1+i)^n-(1-i)^n=2i\,2^{n/2}\sin n\frac{\pi}{4}$$

$$\Rightarrow\int_C\frac{z^n}{z^2-2z+2}dz=\pi\left[2i\,2^{n/2}\sin n\frac{\pi}{4}\right]$$

$$\therefore f(n)=\frac{1}{2\pi i}2\pi i2^{n/2}\sin n\frac{\pi}{4}=2^{n/2}\sin n\frac{\pi}{4}$$

POWER SERIES METHOD (LONG DIVISION METHOD)

a. Find the inverse Z-transform of F(z) = $\frac{1}{1-az^{-1}}, |z| > |a|$ using power series method.

Solution: Since the Region Of Convergence is $|z| > |a|$, which is outside the circle whose radius is ‘a’ unit, the sequence is unilateral or causal. In the unilateral sequence, since all powers of ‘z’ are negative, we perform the long division such that the powers of ‘z’ are negative. The long division is performed as follows to get power series expansion.

Thus, we the series,

```
                1 + a z  + a   z  + a   z
1- az      |  1
           |  1 - a z
           |  az
           |  az  -  a   z
           |         a   z
           |         a   z  - a   z
           |                  a   z
           |                  a   z  - a   z
           |                           a   z  ...
```

$$F(z) = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + ...$$

$$= \frac{1}{1-az^{-1}}, |z| > |a| \quad (or)\ F(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

Comparing this sequence with the basic definition we have, $f(n) = a^n u(n)$

2. Using power series technique, find the inverse Z-transform of

F(z) = $\dfrac{z}{2z^2 - 3z + 1}, |z| > 1$

Solution:

Here the ROC is $|z| > 1$

$\therefore$ The sequence should be unilateral or causal sequence.

Thus, we must divide so as to obtain a series in powers of z^{-1} as follows.

```
                         1/2 z    + 3/4 z
                     ------------------------------
2z  - 3z + 1        | z
                    | z -  3/2 +  1/2 z
                    |------------------------------
                    |      3/2 -  1/2 z
                    |      3/2 -   9/4 z  + 3/4 z
                    |------------------------------
                    |              7/4 z  - 3/4 z  ...
```

$$\therefore F(z) = \frac{1}{2} z^{-1} + \frac{3}{4} z^{-2} + \frac{7}{8} z^{-3} + ...$$

$$By\ definition\ we\ get, \quad f(0) = 0, \quad f(1) = \frac{1}{2}, \quad f(2) = \frac{3}{4}, \quad f(3) = \frac{7}{8} \ .....$$

$$\therefore f(n) = \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8} ....\right\}$$

$$= \frac{2^n - 1}{2^n}, \quad n = 0, 1, ...\infty$$

3. Find the inverse Z-transform of F(z) = $\dfrac{4z}{(z-1)^3}$ by long division method.

Solution:

Here, the ROC is not given. Since we are dealing only unilateral sequence, we may assume that the ROC is $|z| > 1$and the powers of 'z' in the sequence should be negative.

Thus, we rewrite the given sequence as

$$F(z) = \frac{4z}{(z-1)^3} = \frac{4z^{-2}}{(1-z^{-1})^3} = \frac{4z^{-2}}{1-3z^{-1}+3z^{-2}-z^{-3}}$$

```
                  4z   + 12z   + 24z
             ------------------------------------
1-3z  +3z  -z |  4z
              |  4z   - 12z   +12z   - 4z
              ------------------------------------
              |       12z  - 12z  + 4z
              |       12z  -  36z  + 36z   - 12z
              ------------------------------------
              |            24z  - 32z  + 12z
              |            24z  - 72z  +  72z   - 24z
              ------------------------------------
              |                 40z  - 60z  + 24z   ...
```

$$\therefore F(z) = 4z^{-2} + 12z^{-3} + 24z^{-4} + \ldots.$$

Comparing this with the definition of Z-transform, we get

f(0) = 0, f(1) = 0, f(2) = 4, f(3) = 12, f(4) = 24….

$$\therefore\ f(n) = \{0,0,4,12,24,\ldots\}$$

$$(or)\ f(n) = \begin{cases} 0, & for\ n = 0,1 \\ 2n(n-1)u(n), & for\ n \geq 1 \end{cases}$$

SOLVING DIFFERENCE EQUATION USING Z-TRANSFORM:

To solve difference equations we need the following results.

$$Z[y(k+1)] = zF(z) - zy(0) \ (or) \ z[F(z) - y(0)]$$
$$Z[y(k+2)] = z^2F(z) - z^2y(0) - zy(1) \ (or) \ z^2\left[F(z) - y(0) - \frac{y(1)}{z}\right]$$
$$Z[y(k+3)] = z^3F(z) - z^3y(0) - z^2y(1) - zy(2) \ (or) \ z^3\left[F(z) - y(0) - \frac{y(1)}{z} - \frac{y(2)}{z^2}\right]$$

Where, Z[y(k)] = F(z).

1. Solve the difference equation y(k+2) – 4y(k+1) +4y(k) = 0 where y(0) = 1,y(1) =0

   Solution:

Given y(k+2) – 4y(k+1) +4y(k) = 0

Taking Z-transform on both sides, we get Z[y(k+2)]-4Z[y(k+1)]+4Z[y(k)] = 0

$z^2\left[F(z) - y(0) - \frac{y(1)}{z}\right]$ -4 $z[F(z) - y(0)]$+4F(z) = 0, Where, Z[y(k)] = F(z).

$$z^2[F(z) - 1] - 4z[F(z) - 1] + 4F(z) = 0$$
$$F(z)[z^2 - 4z + 4] = z^2 - 4z$$
$$F(z) = \frac{z^2 - 4z}{z^2 - 4z + 4}$$
$$Z[y(k)] = \frac{z^2 - 4z}{(z-2)^2} \Rightarrow y(k) = Z^{-1}\left[\frac{z^2 - 4z}{(z-2)^2}\right]$$
$$Let \ \ F(z) = \frac{z^2 - 4z}{(z-2)^2} \quad \Rightarrow \frac{F(z)}{z} = \frac{z-4}{(z-2)^2} = \frac{A}{z-2} + \frac{B}{(z-2)^2}$$
$$\frac{z-4}{(z-2)^2} = \frac{A(z-2) + B}{(z-2)^2}$$
$$z - 4 = A(z-2) + B$$
$$Put \ z = 2 \qquad\qquad Put \ z = 0$$
$$B = -2 \qquad\qquad A = 1$$
$$\therefore \frac{F(z)}{z} = \frac{1}{z-2} - \frac{2}{(z-2)^2} \Rightarrow F(z) = \frac{z}{z-2} - \frac{2z}{(z-2)^2}$$
$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{z-2}\right] - Z^{-1}\left[\frac{2z}{(z-2)^2}\right] = 2^k - k2^k$$
$$\therefore \quad y(k) = 2^k(1-k).$$

2. Solve the difference equation y(k+2) +y(k) = 1, y(0) = y(1) = 0 (June 2012)

Solution:

Given y(k+2) +y(k) = 1

Taking Z-transform on both sides, we get Z[y(k+2)]+Z[y(k)] = Z(1)

$$z^2\left[F(z) - y(0) - \frac{y(1)}{z}\right] + F(z) = \frac{z}{z-1}$$

$$z^2F(z) + F(z) = \frac{z}{z-1}$$

$$F(z) = \frac{z}{(z-1)(z^2+1)}$$

$$Z[y(k)] = \frac{z}{(z-1)(z^2+1)}$$

$$y(k) = Z^{-1}\left[\frac{z}{(z-1)(z^2+1)}\right]$$

$$Let \quad F(z) = \frac{z}{(z-1)(z^2+1)} \Rightarrow \frac{F(z)}{z} = \frac{1}{(z-1)(z^2+1)}$$

$$\frac{1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$$

$$1 = A(z^2+1) + (Bz+C)(z-1)$$

$$Put \quad z = 1 \Rightarrow A = \frac{1}{2}$$

$$Equating\ the coefficient\ of\ z^2, \quad A + B = 0 \Rightarrow B = -\frac{1}{2}$$

$$Equating\ the coefficient\ of\ cons\tan ts, \quad 1 = A - C \Rightarrow C = -\frac{1}{2}$$

$$\therefore \frac{F(z)}{z} = \frac{1/2}{z-1} - \frac{1/2(z+1)}{z^2+} \Rightarrow F(z) = \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\frac{z^2+z}{z^2+1}$$

$$Z^{-1}[F(z)] = \frac{1}{2}Z^{-1}\left[\frac{1}{z-1}\right] - \frac{1}{2}Z^{-1}\left[\frac{z^2}{z^2+1}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z^2+1}\right]$$

$$= \frac{1}{2}\left[1 - \cos k\frac{\pi}{2} - \sin k\frac{\pi}{2}\right]$$

$$\therefore y(k) = \frac{1}{2}\left[1 - \cos k\frac{\pi}{2} - \sin k\frac{\pi}{2}\right] \quad \left[\because Z\left[\cos k\frac{\pi}{2}\right] = \frac{z^2}{z^2+1}\right.$$

$$Z\left[\sin k\frac{\pi}{2}\right] = \frac{z}{z^2+1}$$

3. Solve the difference equation y(n+3)-3y(n+1)+2y(n) = 0 given that y(0) = 4, y(1) = 0, y(2) = 8. (May 2011, Nov 2012)

Solution: Given y(n+3)-3y(n+1)+2y(n) = 0

Taking Z-transform on both sides, we get

Z[y(n+3)]-3Z[y(n+1)]+2Z[y(n)] = 0

$$z^3\left[F(z) - y(0) - \frac{y(1)}{z} - \frac{y(2)}{z^2}\right] - 3z\left[F(z) - y(0)\right] + 2F(z) = 0$$

$$F(z)\left[z^3 - 3z + 2\right] = 4z^3 - 4z$$

$$F(z) = \frac{4z^3 - 4z}{z^3 - 3z + 2} \Rightarrow \frac{F(z)}{z} = \frac{4z^2 - 4}{z^3 - 3z + 2}$$

$$\frac{4z^2 - 4}{(z+2)(z-1)^2} = \frac{A}{z+2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$4z^2 - 4 = A(z-1)^2 + B(z-1)(z+2) + C(z+2)$$

Put $z = 1$ *Put* $z = -2$

$C = 0$ $A = \frac{4}{3}$

Equating the coefficient of z^2, *we get* $A + B = 4 \Rightarrow B = \frac{8}{3}$

$$\therefore \frac{F(z)}{z} = \frac{4/3}{z+2} + \frac{8/3}{z-1} \Rightarrow F(z) = \frac{4}{3}\frac{z}{z+2} + \frac{8}{3}\frac{z}{z-1}$$

$$Z^{-1}\left[F(z)\right] = \frac{4}{3}Z^{-1}\left[\frac{z}{z+2}\right] + \frac{8}{3}Z^{-1}\left[\frac{z}{z-1}\right]$$

$$\therefore y(n) = \frac{4}{3}(-2)^n + \frac{8}{3}(1)^n$$

4. solve by Z-transform $y_{n+2} - 2y_{n+1} + y_n = 2^n$ with $y_0 = 2$ *and* $y_1 - 1$.

(May 2010 , Nov 2010)

solution:

Given $y_{n+2} - 2y_{n+1} + y_n = 2^n$

$$Z(y_{n+2}) - 2Z(y_{n+1}) + Z(y_n) = Z(2^n)$$

$$z^2Y(z) - z^2 y(0) - zy(1) - 2[zY(z) - zy(0)] + Y(z) = \frac{z}{z-2}$$

$$(z^2 - 2z + 1)Y(z) - 2z^2 - z + 4z = \frac{z}{z-2}$$

$$(z-1)^2 Y(z) - 2z^2 + 3z = \frac{z}{z-2}$$

$$(z-1)^2 Y(z) = \frac{z}{z-2} + 2z^2 - 3z$$

$$= \frac{z + 2z^3 - 4z^2 - 3z^2 + 6z}{z-2}$$

$$= \frac{2z^3 - 7z^2 + 7z}{z-2}$$

$$Y(z) = \frac{z(2z^2 - 7z + 7)}{(z-2)(z-1)^2}$$

$$\frac{Y(z)}{z} = \frac{(2z^2 - 7z + 7)}{(z-2)(z-1)^2}$$

using the method of partial fraction,

$$\frac{Y(z)}{z} = \frac{z(2z^2 - 7z + 7)}{(z-2)(z-1)^2} = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

……………………………………(1)

$$(2z^2 - 7z + 7) = A(z-1)^2 + B(z-1)(z-2) + C(z-2)$$

putting $z = 2$, *we get* $A = 1.$

putting $z = 1$, *we get* $C = -2.$

putting $z = 0$, *we get* $B = 1.$

$$\frac{Y(z)}{z} = \frac{1}{z-2} + \frac{1}{z-1} - \frac{2}{(z-1)^2}$$

$$Y(z) = \frac{z}{z-2} + \frac{z}{z-1} - \frac{2z}{(z-1)^2}$$

$$Z(y(n)) = \frac{z}{z-2} + \frac{z}{z-1} - \frac{2z}{(z-1)^2}$$

$$y(n) = Z^{-1}\left[\frac{z}{z-2}\right] + Z^{-1}\left[\frac{z}{z-1}\right] - z^{-1}\left[\frac{2z}{(z-1)^2}\right]$$

$$= 2^n + 1^n - 2n$$

5. Solve $y_{(n+2)} + 6y_{(n+1)} + 9y_{(n)} = 2^n$ given that $y_0 = y_1 = 0$. (Nov 2009,Nov 2012)

Solution:

Given $y_{(n+2)} + 6y_{(n+1)} + 9y_{(n)} = 2^n$

$$Z[y_{(n+2)}] + 6Z[y_{(n+1)}] + 9Z[y_{(n)}] = Z[2^n]$$

$$[z^2 y(z) - z^2 y(0) - zy(1)] + 6[zY(z) - zy(0)] + 9Y(z) = \frac{z}{z-2}$$

$$[z^2 Y(z) + 6zY(z) + 9Y(z)] = \frac{z}{z-2}$$

$$[z^2 + 6z + 9]Y(z) = \frac{z}{z-2}$$

$$[(z+3)^2]Y(z) = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

$$= \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2} \quad \text{..................................(1)}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Comparing the coefficients, we get

$$A = \frac{1}{25}, \quad B = \frac{-1}{25} \quad and \quad C = \frac{-1}{5}$$

Therefore , (1)$\Rightarrow$ $\frac{Y(z)}{z} = \frac{1}{25}\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\frac{1}{(z+3)^2}$

$$Y(z) = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$y(n) = \frac{1}{25}Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25}Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5}Z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$y(n) = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n + \frac{1}{15}(-3)^n n$$

6. solve $u_{(n+2)} + 4u_{(n+1)} + 3u_{(n)} = 3^n$ given that $u_0 = 0, u_1 = 1.$ (NOV 2011)

solution: Rewriting the above equation is $y_{(n+2)} + 4y_{(n+1)} + 3y_{(n)} = 3^n$

$$Z\left[y_{(n+2)}\right] + 4Z\left[y_{(n+1)}\right] + 3Z\left[y_{(n)}\right] = Z\left[3^n\right]$$

$$\left[z^2 y(z) - z^2 y(0) - zy(1)\right] + 4\left[zY(z) - zy(0)\right] + 3Y(z) = \frac{z}{z-3}$$

$$\left[z^2 + 4z + 3\right]Y(z) - z = \frac{z}{z-3}$$

$$(z+1)(z+3)Y(z) = \frac{z}{z-3} + z$$

$$= \frac{z + z(z-3)}{z-3}$$

$$= \frac{z^2 - 2z}{z-3}$$

$$Y(z) = \frac{z(z-2)}{(z+1)(z+3)(z-3)}$$

$$\frac{Y(z)}{z} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{(z-3)} \text{ ....(1)}$$

$$z-2=A(z+3)(z-3)+B(z+1)(z-3)+C(z+1)(z+3)$$

Comparing the coefficients, we get

$$A=\frac{3}{8},\quad B=\frac{-5}{12}\quad and\quad C=\frac{1}{24}$$

$$(1)\Rightarrow\frac{Y(z)}{z}=\frac{3/8}{z+1}+\frac{(-5/12)}{z+3}+\frac{1/24}{(z-3)}$$

$$Y(z)=\frac{3}{8}\frac{z}{z+1}-\frac{5}{12}\frac{z}{z+3}+\frac{1}{24}\frac{z}{(z-3)}$$

$$y(n)=\frac{3}{8}(-1)^n-\frac{5}{25}(-3)^n+\frac{1}{24}(3)^n$$

7. Form the difference equation of second order by eliminating the arbitrary constants A and B from solve $y_n = A(-2)^n + Bn$. (NOV 2011)

solution:

$$y_n = A(-2)^n + Bn\text{.......................................}(1)$$

$$y_{n+1} = A(-2)^{n+1} + B(n+1)$$

$$= A(-2)^n(-2) + B(n+1)\text{................}(2)$$

$$y_{n+2} = A(-2)^{n+2} + B(n+2)$$

$$= A(-2)^n(-2)^2 + B(n+2)\text{....................}(3)$$

Eliminating $A(-2)^n$ *and* B, we get

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & -2 & n+1 \\ y_{n+2} & 4 & n+2 \end{vmatrix}=0$$

$$\Rightarrow (3n+1)\,y_{n+2}+(3n-2)\,y_{n+1}-(6n+8)\,y_n=0$$

CONVOLUTION OF TWO SEQUENCES:

The convolution of two sequences $\{f(n)\}\, and\, \{g(n)\}$ is defined as

$$\left[f(n)*g(n)\right]=\sum_{r=0}^{n} f(r)g(n-r) \quad \left[\textit{For right sided sequence}\right.$$

$$(or)\quad \left[f(n)*g(n)\right]=\sum_{r=-\infty}^{\infty} f(r)g(n-r) \quad \left[\textit{For twosided or bilateral sequence}\right.$$

The convolution of two functions f(t) and g(t) is defined as

$$f(t)*g(t)=\sum_{r=0}^{n} f(rT)g(n-r)T$$, where T is the sampling period.

CONVOLUTION THEOREM:

(i) $Z\left[f(n)*g(n)\right]=F(z)G(z)\ where\ Z\left[f(n)\right]=F(z)\ and\ Z\left[g(n)\right]=G(z)$

(ii) $Z\left[f(t)*g(t)\right]=F(z)G(z)\ where\ Z\left[f(t)\right]=F(z)\ and\ Z\left[g(t)\right]=G(z)$

Proof:

$$Let\ Z[f(n)] = F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$G(z) = Z[g(n)] = \sum_{n=0}^{\infty} g(n)z^{-n}$$

$$\therefore F(z)G(z) = \sum_{n=0}^{\infty} f(n)z^{-n} \sum_{n=0}^{\infty} g(n)z^{-n}$$

$$= \left[f(0) + f(1)z^{-1} + f(2)z^{-2} + ...\right]\left[g(0) + g(1)z^{-1} + g(2)z^{-2} + ...\right]$$

$$= f(0)g(0) + [f(0)g(1) + f(1)g(0)]z^{-1} + \ .. + \left[\sum_{r=0}^{n} f(r)g(n-r)\right]z^{-n} + ...$$

$$= \sum_{n=0}^{\infty} \left\{\sum_{r=0}^{n} f(r)g(n-r)\right\} z^{-n}$$

$$= \sum_{n=0}^{\infty}[f(n) * g(n)]z^{-n} = Z[f(n) * g(n)]$$

$$\therefore Z[f(n) * g(n)] = F(z)G(z) \Rightarrow Z^{-1}[F(z)G(z)] = f(n) * g(n)$$

(ii) (ii) $Let\ F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n}$ and $G(z) = Z[g(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n}$

$$Now\ F(z)G(z) = \sum_{n=0}^{\infty} f(nT)z^{-n} \sum_{n=0}^{\infty} f(nT)z^{-n}$$

$$= \left[f(0T) + f(T)z^{-1} + f(2T)z^{-2} + ...\right]\left[g(0T) + g(T)z^{-1} + g(2T)z^{-2} + \square\right]$$

$$= \sum_{n=0}^{\infty} \left\{\sum_{r=0}^{n} f(rT)g(n-r)T\right\} z^{-n}$$

$$= \sum_{n=0}^{\infty}[f(t) * g(t)]z^{-n}$$

$$\therefore Z[f(t) * g(t)] = F(z)G(z) \Rightarrow Z^{-1}\{F(z)G(z)\} = f(t) * g(t)$$

Problems:

1. Find the Z –transform of f(n)*g(n) if

   $(i)\ f(n) = 2^n u(n)\ and\ \ g(n) = 2^n u(n)$ (ii) $f(n) = 2^n u(n)\ \ g(n) = 3^n u(n)$

Solution:

$(i)\ \ Z[f(n)] = Z[2^n u(n)]$

$$= \frac{z}{z-2} = F(z)$$

$Z[g(n)] = Z[2^n u(n)]$

$$= \frac{z}{z-2} = G(z)$$

$$\therefore Z[f(n)*g(n)] = F(z)G(z) = \left(\frac{z}{z-2}\right)^2$$

$$(ii) \quad Z[f(n)] = Z[2^n u(n)]$$

$$= \frac{z}{z-2} = F(z)$$

$$Z[g(n)] = Z[3^n u(n)]$$

$$= \frac{z}{z-3} = G(z)$$

$$\therefore Z[f(n)*g(n)] = F(z)G(z) = \frac{z^2}{(z-2)(z-3)}$$

2. If f(n) = U(n) and g(n) = $\delta(n) + \left(\frac{1}{2}\right)^n U(n)$, then find Z[f(n)*g(n)].

Solution:

$$Z[f(n)] = Z[U(n)] = \frac{z}{z-1}$$

$$Z[g(n)] = Z\left[\delta(n) + \left(\frac{1}{2}\right)^n U(n)\right] = Z[\delta(n)] + Z\left[\left(\frac{1}{2}\right)^n U(n)\right]$$

$$= 1 + \frac{z}{z-1/2} = \frac{4z-1}{2z-1}$$

$$\therefore Z[f(n)*g(n)] = F(z)G(z) = \frac{z}{z-1}\frac{4z-1}{2z-1}$$

$$Z[f(n)*g(n)] = \frac{z(4z-1)}{(z-1)(2z-1)}$$

3. Using convolution theorem evaluate $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$ (May 2011)

Solution:

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]=Z^{-1}\left[\frac{z}{z-1}\,\frac{z}{z-3}\right]$$

$$=(1)^n*(3)^n \qquad \left[\because Z\,(1)^n=\frac{z}{z-1},\ Z\,(3)^n=\frac{z}{z-3}\right]$$

$$=\sum_{r=0}^{n}(1)^r(3)^{n-r}=3^n+3^{n-1}+3^{n-2}+\cdots+3^1+1$$

$$=1+3+3^2+\cdots+3^n$$

$$\therefore Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]=\frac{3^{n+1}-1}{2}$$

4. Using convolution theorem, find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$ (June 2013)

Solution:

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]=Z^{-1}\left[\frac{z}{z-a}\,\frac{z}{z-b}\right]$$

$$=(a)^n*(b)^n \qquad \left[\because Z\,(a)^n=\frac{z}{z-a},\ Z\,(b)^n=\frac{z}{z-b}\right]$$

$$=\sum_{m=0}^{n}a^m b^{n-m}=b^n\sum_{m=0}^{\infty}\left(\frac{a}{b}\right)^m$$

$$=b^n\frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}}$$

$$\therefore Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]=\frac{b^{n+1}-a^{n+1}}{b-a}$$

5. Find $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+3)}\right]$ using convolution theorem. (June 2012)

Solution:

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+3)}\right] = Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{3}{4}\right)}\right]$$

$$= Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\cdot\frac{z}{z+\frac{3}{4}}\right] = Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] Z^{-1}\left[\frac{z}{z+\frac{3}{4}}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(-\frac{3}{4}\right)^n$$

$$= \sum_{r=0}^{n}\left(-\frac{3}{4}\right)^r\left(\frac{1}{2}\right)^{n-r} = \left(\frac{1}{2}\right)^n \sum_{r=0}^{n}\left(-\frac{3}{4}\right)^r\left(\frac{1}{2}\right)^{-r}$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n}\left(-\frac{3}{4}\right)^r 2^r = \left(\frac{1}{2}\right)^n \sum_{r=0}^{n}\left(-\frac{3}{2}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \left\{\frac{1-\left(-\frac{3}{2}\right)^{n+1}}{1-\left(-\frac{3}{2}\right)}\right\} = \left(\frac{1}{2}\right)^n \left\{\frac{1-\left(-\frac{3}{2}\right)^{n+1}}{\frac{5}{2}}\right\}$$

$$= \frac{2}{5}\left\{\left(\frac{1}{2}\right)^n + \frac{3}{2}\left(-\frac{3}{4}\right)^n\right\}$$

$$\therefore Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+3)}\right] = \frac{2}{5}\left(\frac{1}{2}\right)^n + \frac{3}{5}\left(-\frac{3}{4}\right)^n$$

6. Using convolution theorem, find the inverse Z-transform of $\frac{Z^2}{(z+a)^2}$.(NOV 2012)

Solution:

$$Z^{-1}\left[\frac{Z^2}{(z+a)^2}\right] = Z^{-1}\left[\frac{z}{z+a}\cdot\frac{z}{z+a}\right]$$

$$= Z^{-1}\left[\frac{z}{z+a}\right] * Z^{-1}\left[\frac{z}{z+a}\right]$$

$$= (-a)^n * (-a)^n$$

$$= \sum_{K=0}^{n}(-a)^K(-a)^{n-K}$$

$$= \sum_{K=0}^{n}(-a)^n = (n+1)(-a)^n .$$