Markov Process

Markov model is a model of a random process where the value of the random process depends only upon the most recent previous value and independent of all values in the more distant past

Markov process defines that a future value is independent of the past values, given the present value.

Markov Chain

If, for all n,

$$P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots X_0 = a_0\} = P\{X_n = a_n / X_{n-1} = a_{n-1}\}$$

then the process $\{X_n\}, n = 0, 1, \ldots$ is called a Markov chain

$$\text{States} = (a_1, a_2, a_3 \ldots a_n, \ldots)$$

The conditional probability $P\{X_n = a_j / X_{n-1} = a_i\}$ is called a one step transition probability from state a_i to state a_j at nth step (trial) and is denoted by $p_{ij}(n-1, n)$.

If the one-step transition probability does not depend on the step

$$P_{ij}(n-1, n) = p(m-1, m)$$

then the Markov chain is called a homogenous Markov chain.

This chain is said to have stationary transition probability. Stationary → ~~no random~~ Stationary random sequence

This transition probability is put in a matrix called transition probability matrix ~~called~~ or tpm. [P]

The tpm of a Markov chain is stochastic matrix

(i) $p_{ij} > 0$

(ii) $\Sigma p_{ij} = 1$

Transition from state a_i to any one of the states is a certain event.

The conditional probability that the process is in state a_j at step n, given that it was in ~~at~~ state a_i at step 0

$$P\{X_n = a_j / X_0 = a_i\}$$

is called the nth step transition probability and denoted by $P_{ij}(n)$.

$\boxed{p_{ij} = p_{ij}^{(1)}}$ → just to know

TPM Basics

$$\text{States of } X_{n-1} = \begin{array}{c|cccc} & \text{col1} & \text{col2} & \text{col3} & \text{col4} \\ & 0 & 1 & 2 & 3 \\ \hline \text{row1 } 0 & 0 & 1 & 0 & 0 \\ \text{row2 } 1 & 0.3 & 0 & 0.7 & 0 \\ \text{row3 } 2 & 0 & 0.3 & 0 & 0.7 \\ \text{row4 } 3 & 0 & 0 & 1 & 0 \end{array}$$

one step

$1 \leftarrow 0$

Now $p_{23} = 0.7 = P\{X_1 = 2 / X_0 = 1\}$

$\underbrace{p_{2}}_{2^{nd}\text{ row}}\underbrace{_{3}}_{3^{rd}\text{ column}}$ state 2 ← to ← state 1

p_{23} means that, if the process is at state 2 at step $(n-1)$, the probability that it moves to state 3 at step n $= 0.7$ (where n = +ve integer).

Probability distribution.

If the probability that the process is in state a_i is p_i $(i = 1, 2, \dots k)$ at any arbitary step, then the row vector $p = (p_1, p_2, \dots p_k)$ is called probability distribution of the process at any time.

In particular,

$$p^{(0)} = \{p_1^{(0)}, p_2^{(0)}, \dots p_k^{(0)}\}$$

is the initial probability distribution

Chapman-kolmogorov Theorem

If P is the tpm of a homogenous Markov chain, then the n^{th} step tpm $p^{(n)}$ is equal to P^n.

$$P_{ij}^{(n)} = (P_{ij})^n$$

$$\Rightarrow P^{(n)} = P^n$$

Stochastic Matrix

A stochastic matrix is said to be a regular matrix, if all the entries of P^m (m=+ve) are positive.

A homogenous Markov Chain is said to be regular if its tpm is regular.

Some Theorems

① If $p=\{p_i\}$ is the state probability distribution of the process at an arbitary time, then that after one step pP, where P

is the tpm of the chain and that after n steps in $p\mathcal{P}^n$.

② If a homogenous Markov chain is regular, then every sequence of state probability distributions approaches a unique fixed probability distribution called the stationary (state) distribution or steady state distribution of the Markov chain

That is,

$$\lim_{n \to \infty} \{p^{(n)}\} = \pi$$

where the state probability distribution at step n,

$$p^{(n)} = (p_1^{(n)}, p_2^{(n)}, \ldots, p_k^{(n)})$$ and the stationary distribution

$$\pi = (\pi_1, \pi_2, \ldots, \pi_n)$$ are row vectors.

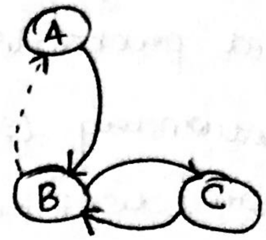
Moreover, if $\mathcal{P}$ is the tpm of the regular chain then

$$\pi P = \pi$$

Classification of States in Markov Chain

If $p_{ij}^{(n)} > 0$ for some n and for all i and j, then every state can be reached from every other state, then the Markov chain is $\boxed{\text{irreducible}}$ 1.

The tpm of an irreducible chain is an irreducible matrix.



Otherwise, the chain ~~is~~ ~~used~~ is said to be non-irreducible or $\boxed{\text{reducible}}$ 2.

State i of a Markov chain is called a $\boxed{\text{return}}$ 3 state, if $p_{ij}^{(n)} > 0$ for some $n > 1$.

The period d_i of a return state i is defined as the greatest common divisor of all m such that $p_{ii}^{(m)} > 0$

i.e) $d_i = \text{GCD}\{m : p_{ii}^{(m)} > 0\}$

State i said to be

- $\boxed{\text{periodic}}$ 4 with period d_i if $d_i > 1$ &
- $\boxed{\text{aperiodic}}$ 5 if $d_i = 1$ and $p_{ii} \neq 0$ is implicit.

The probability that the chain returns to state i, having started from state i, for the first time at the n^{th} step (or after n transitions) is denoted by $f_{ii}^{(n)}$ and called first return time probability (6) or $\boxed{\text{recurrence time probability}}$.

$\{n, f_{ii}^{(n)}\}$, $n = 1, 2, 3, \dots$ is the distribution of recurrence times of state i.

If

$$F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1 \longrightarrow \text{return to state } i \text{ is certain}$$

$$\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$$ is called mean reccurrence time of state i.

(7)

A state i is said to be $\boxed{\text{persistant}}$ or recurrent if the return to state i is certain

i.e) $F_{ii} = 1$

(8)

A state i is said to be $\boxed{\text{transient}}$ if the return to state i is uncertain

i.e) $F_{ii} < 1$.

The state i is said to be $\boxed{\text{non null persistant}}$ (9) if ~~the~~ its mean recurrence time μ_{ii} is finite.

The state i is said to be $\boxed{\text{null persistant}}$ (10) if the mean recurrence time μ_{ii} is infinite.

A non null and aperiodic state is called $\boxed{\text{ergodic}}$ (11).

<u>Theorems</u> (to classify states)

(1) If Markov chain is irreducible, then all the states are of same type

→ transient
(or)
→ null persistant
(or)
→ non-null persistant

→ periodic
(or)
→ aperiodic

(2) If a Markov chain is finite irreducible, all of its states are non-null persistant