SRM INSTITUTE OF SCIENCE AND TECHNOLOGY

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Department of Mathematics

Sub Title: PROBABILITY AND QUEUING THEORY

Sub Code: 18MAB 204 T

Unit -V - MARKOV CHAINS

11. A discrete parameter markov process is called a

(a)Markov process (b) stationary process (c) random process (d) Markov chain **Ans: (d)**

2. A square matrix , in which the sum of all the elements of each row is one is called a

(a)unitary matrix (b) diagonal matrix (c) stochastic matrix (d) skew matrix **Ans: (c)**

3. A stochastic matrix P is said to be regular if all the entries of P^m are

(a)negative (b) positive (c) semi positive (d) either positive or negative **Ans: (b)**

4. If $\pi = (\pi_1, \pi_{2,...,}\pi_n)$ is the steady state distribution of the chain whose tpm is the n^{th} order square matrix P, then

(a) $\pi P = \pi$ (b) $\pi\mu = \pi$ (c) $\pi A = n$ (d) $\pi P = P$. **Ans: (a)**

5.The conditional probability $P[X_n = a_j/X_{n-1} = a_i]$ is called

(a) second tpm (b)one-step transition probability (c) homogeneous (d) n-step tpm **Ans:(b)**

6. If the one-step tpm does not depend on the step ie. $p_{ij}(n-1,n) = p_{ij}(m-1,m)$ the markov chain is called

(a) stationary chain (b) discrete chain (c) homogeneous markov chain

(d) regular markov chain **Ans: (c)**

7. The conditional probability $P[X_n = a_j/X_0 = a_i]$ is called

(a) second tpm (b)one-step tpm (c) homogeneous (d) n-step transition probability **Ans:(d)**

8. If P is the tpm of a homogeneous Markov chain , then the n-step tpm $P^{(n)} = P^n$ is known as

(a) probability theorem (b) Chapman- Kolmogorov Theorem

(c) Markov theorem (d) Chapman theorem **Ans: (b)**

9. State i of a Markov chain is said to be ----- with period d_i if $d_i > 1$

(a) periodic (b) not periodic (c) aperiodic (d) biperiodic **Ans: (a)**

10. State i of a Markov chain is said to be ----- with period d_i if $d_i = 1$

(a) periodic (b) not periodic (c) aperiodic (d) biperiodic **Ans: (c)**

11. Every state can be reached from every other state , the Markov chain is said to be

(a) homogeneous (b) reducible (c) irreducible (d) recurrent **Ans: (c)**

12. A non null persistent and aperiodic state is called

(a) markov (b) irreducible (c) recurrence (d) ergodic **Ans: (d)**

13. A state i is said to be ------ if the return to state i is certain.

(a) persistent (b) non persistent (c) ergodic (d) periodic **Ans:(a)**

14. A state i is said to be ------ if the return to state i is uncertain.

(a) persistent (b) non persistent (c) transient (d) periodic **Ans: (c)**

15. A state i is said to be ------ if the mean recurrence time μ_{ii} is finite.

(a) persistent (b) non persistent (c) transient (d) non null persistent **Ans:(d)**

16. A state i is said to be ------ if the mean recurrence time $\mu_{ii} = \infty$.

(a) persistent (b) non persistent (c) null persistent (d) non null persistent **Ans:(c)**

17. If a markov chain is finite irreducible , all its states are

(a) persistent (b) null persistent (c) non null persistent (d) recurrent **Ans: (c)**

18. A Markov chain is completely specified when

(a) intial probability distribution (b) tpm (c) absorbing state (d) both a & b are given **Ans:(d)**

19.If $\pi P = \pi$, where P = $\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$ then values of (π_1, π_2) is

(a) (1/3,2/3) (b) (1/2,1/2) (c) (2/3,1/3) (d) (0,1) **Ans:(a)**

20.If the tpm of a markov chain is P = $\begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ then $P[X_1 = 3/X_0 = 2]$.=

(a) 0.1 (b) 0.2 (c) 0.4 (d) 0.6 **Ans:(b)**

UNIT V - MARKOV CHAINS

PART-B

(1) Define Markov chain. When can you say that a Markov chain is homogeneous.

If for all n,

$$P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \dots X_0 = a_0\}$$
$$= P(X_n = a_n / X_{n-1} = a_{n-1})$$

Then the process $\{X_n : n = 0, 1, 2 \dots\}$ is called a Markov chain. (ie) The discrete parameter Markov process is called a Markov chain.

If the one step transition probability does not depend on the step, then the Markov chain is called a homogeneous Markov chain.

(2) Define transition probability matrix (tpm) of a Markov chain. What is Stochastic matrix? when it is said to be regular.

If the Markov chain is homogeneous, the one step transition probability is denoted by P_{ij} and the matrix $P = (P_{ij})$ is called the one step transition probability matrix.

The tpm is a stochastic matrix if

(i) $P_{ij} \geq 0 \quad \forall i, j$

(ii) $\sum_j P_{ij} = 1$ for all i (ie, sum of elements of any row is 1).

A stochastic matrix P is said to be regular if all the entries of $P^{(m)}$ are positive (for some positive integer m).

(3) Define n-step transition probability in a Markov chain and state Chapman-Kolmogorov equation. (2)

The conditional probability

$$P_{ij}^{(n)} = P(X_n = a_j | X_0 = a_i)$$ is called the n-step transition probability.

If P is the tpm of a Markov chain, then the n-step tpm $P^{(n)}$ is equal to P^n

ie) $[P_{ij}^{(n)}] = [P_{ij}]^n$.

(4) If the transition probability matrix of a Markov chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, find the steady state distribution of the chain.

Given tpm $P = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Let $\pi = (\pi_1 \;\; \pi_2)$ be the steady state distribution.

Now, $\pi P = \pi$ with $\pi_1 + \pi_2 = 1$

$$(\pi_1 \;\; \pi_2) \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1 \;\; \pi_2)$$

$$\left(\frac{\pi_2}{2} \;\;\; \pi_1 + \frac{\pi_2}{2} \right) = (\pi_1 \;\; \pi_2)$$

$\frac{\pi_2}{2} = \pi_1$ & $\pi_1 + \frac{\pi_2}{2} = \pi_2$

$\boxed{\pi_2 = 2\pi_1}$ We have $\pi_1 + \pi_2 = 1$

$\pi_1 + 2\pi_1 = 1 \Rightarrow 3\pi_1 = 1$

$\boxed{\pi_1 = \frac{1}{3}}$ $\therefore \boxed{\pi_2 = \frac{2}{3}}$

$\therefore \pi = \left(\frac{1}{3} \;\; \frac{2}{3} \right)$.

(5) The tpm of a Markov chain $(X_n : n = 1, 2, 3 \ldots)$ with three states $0, 1$ & 2 is (3)

$$P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$ with initial distribution

$P(0) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Find $P(X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 2)$.

Given
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix} \end{array} = \begin{pmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{pmatrix}.$$

and $P(0) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$P(X_0 = 0) = P(X_0 = 1) = P(X_0 = 2) = \frac{1}{3}$$

$$P(X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 2)$$
$$= P(X_3 = 1 / X_2 = 1)\, P(X_2 = 1 / X_1 = 1)\, P(X_1 = 1 / X_0 = 2)\, P(X_0 = 2)$$
$$= P_{11}^{(1)} P_{11}^{(1)} P_{21}^{(1)} P(X_0 = 2)$$
$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{16}.$$

(6) If the initial state distribution of a Markov chain is $P^{(0)} = \left(\frac{5}{6} \;\; \frac{1}{6}\right)$. and the tpm of the chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, find the probability distribution of the chain after 2 steps.

Given, $P^{(0)} = \left(\frac{5}{6} \;\; \frac{1}{6}\right)$

and $P = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

$$P^{(1)} = P^{(0)} P$$

$$= \left(\tfrac{5}{6} \quad \tfrac{1}{6}\right) \begin{pmatrix} 0 & 1 \\ \tfrac{1}{2} & \tfrac{1}{2} \end{pmatrix}$$

$$= \left(0 + \tfrac{1}{12} \quad \tfrac{5}{6} + \tfrac{1}{12}\right)$$

$$P^{(1)} = \left(\tfrac{1}{12} \quad \tfrac{11}{12}\right)$$

$$P^{(2)} = P^{(1)} P = \left(\tfrac{1}{12} \quad \tfrac{11}{12}\right) \begin{pmatrix} 0 & 1 \\ \tfrac{1}{2} & \tfrac{1}{2} \end{pmatrix}$$

$$= \left(0 + \tfrac{11}{24} \quad \tfrac{1}{12} + \tfrac{11}{24}\right)$$

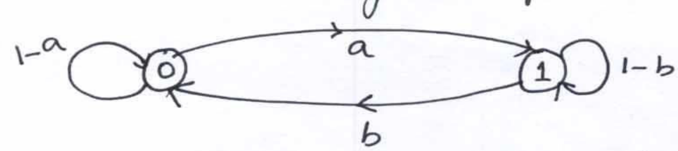
$$P^{(2)} = \left(\tfrac{11}{24} \quad \tfrac{13}{24}\right).$$

7. State tpm of a two state Markov chain and draw the transition diagram and also give the steady state probability.

The tpm of the Markov chain is

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

The transition diagram of a two state Markov chain is



The steady state probability of this Markov chain is

$$\pi = \left(\frac{b}{a+b} \quad \frac{a}{a+b}\right).$$

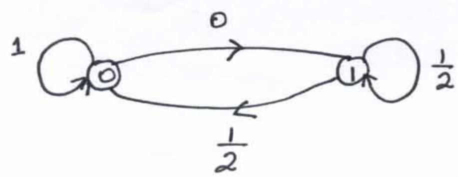
(8) Consider a Markov chain with state space $\{0, 1\}$ and the tpm

$$P = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(i) Draw the transition diagram

(ii) Is the chain irreducible ?

$$\text{Given} \quad P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$



The chain is not irreducible since $P_{ii}^{(n)} = 0$, $n = 1, 2, 3 \ldots$

(9) Prove that the matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ is the tpm of an irreducible Markov chain.

$$\text{Given,} \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad P^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

$P_{11}^{(3)} > 0$, $P_{13}^{(2)} > 0$, $P_{21}^{(2)} > 0$, $P_{22}^{(2)} > 0$, $P_{33}^{(2)} > 0$

and all other $P_{ij}^{(n)} > 0$.

$\therefore$ The chain is irreducible.

PART-C

6

① The one step tpm of a Markov chain $(X_n : n=0,1,2\ldots)$ having state space $S=(1,2,3)$ is $\begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $\pi_0 = (0.7 \ \ 0.2 \ \ 0.1)$. Find

(i) $P(X_2=3 / X_0=1)$ (ii) $P(X_3=2, X_2=3, X_1=3, X_0=2)$ and (iii) $P(X_2=3)$.

$$\text{Given} \quad P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

and initial state distribution is $P(X_0=1)=0.7$, $P(X_0=2)=0.2$, $P(X_0=3)=0.1$

$$P^2 = P \cdot P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}\begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

(i) $P(X_2=3 / X_0=1) = P_{13}^{(2)} = 0.26$

(ii) $P(X_3=2, X_2=3, X_1=3, X_0=2)$

$$= P(X_3=2 / X_2=3)\, P(X_2=3 / X_1=3)\, P(X_1=3 / X_0=2)\, P(X_0=2)$$

$$= P_{32}^{(1)}\, P_{33}^{(1)}\, P_{23}^{(1)}\, P(X_0=2)$$

$$= (0.4)\,(0.3)\,(0.2)\,(0.2)$$

$$= 0.0048$$

(iii) $P(X_2=3) = \sum_{i=1}^{3} P(X_2=3 / X_0=i)\, P(X_0=i)$

$= P(X_2=3 \mid X_0=1)\, P(X_0=1) + P(X_2=3 / X_0=2)\, P(X_0=2)$ 7

$+ P(X_2=3 / X_0=3)\, P(X_0=3)$

$= P_{13}^{(2)}\, P(X_0=1) + P_{23}^{(2)}\, P(X_0=2) + P_{33}^{(2)}\, P(X_0=3)$

$= (0.26)(0.7) + (0.34)(0.2) + (0.29)(0.1)$

$= 0.182 + 0.068 + 0.029$

$= 0.279.$

Problem 2: The tpm of a Markov chain with three states 0,1,2 is $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$ and the initial state distribution of the chain is $P(X_0=i) = \frac{1}{3}$, $i = 0,1,2$. Find (i) $P(X_2=2)$ (ii) $P(X_3=1, X_2=2, X_1=1, X_0=2)$ (iii) $P(X_2=1, X_0=0)$.

Solution:

Given state space $S = (0, 1, 2)$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} \end{array} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix}$$

and initial state distribution is

$$P(X_0=i) = \frac{1}{3}, \quad i = 0,1,2$$

$$P^{(2)} = P^2 = P \cdot P = \begin{bmatrix} \frac{5}{8} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{8}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{4}{16} \end{bmatrix}$$

(i) $P(X_2=2) = \sum_{i=0}^{2} P(X_2=2 \mid X_0=i)\, P(X_0=i)$ (8)

$= P(X_2=2 \mid X_0=0)\, P(X_0=0) + P(X_2=2 \mid X_0=1)\, P(X_0=1)$

$+ P(X_2=2 \mid X_0=2)\, P(X_0=2)$

$= P_{02}^{(2)}\, P(X_0=0) + P_{12}^{(2)}\, P(X_0=1) + P_{22}^{(0)}\, P(X_0=2)$

$= \left(\frac{1}{16} \times \frac{1}{3}\right) + \left(\frac{3}{16} \times \frac{1}{3}\right) + \left(\frac{4}{16} \times \frac{1}{3}\right)$

$= \frac{1}{48} + \frac{3}{48} + \frac{4}{48} = \frac{8}{48} = \frac{1}{6}$

(ii) $P(X_3=1, X_2=2, X_1=1, X_0=2)$

$= P(X_3=1 \mid X_2=2)\, P(X_2=2 \mid X_1=1)\, P(X_1=1 \mid X_0=2)\, P(X_0=2)$

$= P_{21}^{(1)}\, P_{12}^{(1)}\, P_{21}^{(1)}\, P(X_0=2)$

$= \cancel{\frac{9}{16} \times \frac{3}{16} \times \frac{9}{16}}\ \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} \times \frac{1}{3} = \frac{3}{64}$

(iii) $P(X_2=1, X_0=0) = P(X_2=1 \mid X_0=0)\, P(X_0=0)$

$= P_{01}^{(2)}\, P(X_0=0) = \frac{5}{16} \times \frac{1}{3} = \frac{5}{48}$

Problem ③ : A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair dice and drove to work iff a 6 appeared. Find (i) The probability that he takes a train on the third day. and

(ii) The probability that he drives to work in the long run.

The State Space of the Markov chain is {Train, Car} (9)

The transition probability matrix is

$$P = \begin{array}{c} \\ T \\ C \end{array} \begin{array}{c} \begin{array}{cc} T & C \end{array} \\ \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Now on the first day,

$P(\text{driving to work}) = P(\text{getting } 6) = \frac{1}{6}$

$P(\text{taking a train}) = \frac{5}{6}$.

$\therefore$ The initial probability distribution is

$$p^{(1)} = \left(\frac{5}{6} \quad \frac{1}{6} \right)$$

Now $$P^2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

The probability distribution for the third day

$$p^{(3)} = p^{(1)} P^2 = \left(\frac{5}{6} \quad \frac{1}{6} \right) \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$= \left(\frac{11}{24} \quad \frac{13}{24} \right)$$

Probability that he takes a train on the third day $= \frac{11}{24}$

Let the steady state distribution be

$$\pi = (\pi_1 \quad \pi_2)$$

Then $\pi P = \pi$

$$[\pi_1 \quad \pi_2] \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [\pi_1 \quad \pi_2]$$

$$\left[\frac{\pi_2}{2} \quad \pi_1 + \frac{\pi_2}{2} \right] = [\pi_1 \quad \pi_2]$$

(10)

$$\frac{\pi_2}{2} = \pi_1 \quad \Rightarrow \pi_2 = 2\pi_1$$

Since π is a probability distribution,

$$\pi_1 + \pi_2 = 1$$
$$2\pi_1 + \pi_1 = 1$$
$$3\pi_1 = 1$$
$$\boxed{\pi_1 = \frac{1}{3}} \qquad \boxed{\pi_2 = \frac{2}{3}}$$

$$\therefore \pi = \left(\frac{1}{3} \quad \frac{2}{3}\right)$$

$\therefore$ The probability that the man drives to work in the long run $= \frac{2}{3}$.

Problem ④: A fair die is tossed repeatedly. If $\{X_n\}$ denotes the maximum of the numbers occurring in the first n trials. find the transition probability matrix P of the Markov chain $\{X_n\}$. Also find P^2 and $P(X_2 = 6)$.

Solution:

The state space is $\{1, 2, 3, 4, 5, 6\}$

X_n = maximum of the numbers occurring in the first n trials.

X_{n+1} = Maximum of the numbers occurring in the first $n+1$ trials

$= \max\{X_n$, number in the $(n+1)^{th}$ trial$)$

Let us see how the 1^{st} row of the tpm is filled.

Here $X_n = 1$.

$X_{n+1} = 1$ if 1 appear in $(n+1)^{th}$ trial.

$= 2$ if 2 appears in $(n+1)^{th}$ trial.

(11)

$X_{n+1} = 3$ if 3 appear in $(n+1)^{th}$ trial.

$\vdots$

6 if 6 appear in $(n+1)^{th}$ trial.

Now in the $n+1^{th}$ trial, each of the numbers 1, 2, 3, 4, 5, 6 occurs with probability $\frac{1}{6}$.

If $X_n = 2$, $P(X_{n+1} = 2) = \frac{2}{6}$

and $P(X_{n+1} = k) = \frac{1}{6}$ for $k = 3, 4, 5, 6$.

Proceeding similarly, the tpm is

$$P = \begin{array}{cc} & X_{n+1} \\ X_n & \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{6}{6} \end{bmatrix} \end{array}$$

$$P^2 = \begin{bmatrix} \frac{1}{36} & \frac{3}{36} & \frac{5}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & \frac{4}{36} & \frac{5}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & \frac{9}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & 0 & \frac{16}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & \frac{25}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & 0 & \frac{36}{36} \end{bmatrix}$$

Now since all values (1, 2, 3, 4, 5, 6) are equally likely, the initial probability distribution is

$$\left[\frac{1}{6} \ \frac{1}{6} \ \frac{1}{6} \ \frac{1}{6} \ \frac{1}{6} \ \frac{1}{6} \right]$$

(12)

$$P(X_2=6) = \sum_{i=1}^{6} P(X_2=6 / X_0=i)\; P(X_0=i)$$

$$= P(X_2=6/X_0=1)\,P(X_0=1) + P(X_2=6/X_0=2)\,P(X_0=2) + P(X_2=6/X_0=3)\,P(X_0=3) + P(X_2=6/X_0=4)\,P(X_0=4) + P(X_2=6/X_0=5)\,P(X_0=5) + P(X_2=6/X_0=6)\,P(X_0=6)$$

$$= P_{16}^{(2)} P(X_0=1) + P_{26}^{(2)} P(X_0=2) + P_{36}^{(2)} P(X_0=3) + P_{46}^{(2)} P(X_0=4) + P_{56}^{(2)} P(X_0=5) + P_{66}^{(2)} P(X_0=6)$$

$$= \left(\frac{11}{36}\times\frac{1}{6}\right) + \left(\frac{11}{36}\times\frac{1}{6}\right) + \left(\frac{11}{36}\times\frac{1}{6}\right) + \left(\frac{11}{36}\times\frac{1}{6}\right) + \left(\frac{11}{36}\times\frac{1}{6}\right) + \left(\frac{36}{36}\times\frac{1}{6}\right)$$

$$= \frac{11}{216} + \frac{11}{216} + \frac{11}{216} + \frac{11}{216} + \frac{11}{216} + \frac{36}{216} = \frac{91}{216}.$$

Problem (5): A salesman territory consists of three cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day, he sells in city B. However, if he sells in either B or C, the next day he is twice as likely to sell in city A as in the other city. In the long run, how often does he sell in each of the cities?

The tpm of the given problem is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

The given problem describes a Markov chain with three states as three cities A, B, C.

we require the steady state distribution of the Markov chain for finding the probabilities in the long run. (13)

Let $\pi = (\pi_1 \; \pi_2 \; \pi_3)$ be the steady state probability distribution.

Then $\pi P = \pi$ and $\pi_1 + \pi_2 + \pi_3 = 1$.

$$\pi P = \pi \implies (\pi_1 \; \pi_2 \; \pi_3) \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} = (\pi_1 \; \pi_2 \; \pi_3)$$

$$\left(\frac{2\pi_2}{3} + \frac{2\pi_3}{3} \quad \pi_1 + \frac{\pi_3}{3} \quad \frac{\pi_2}{3} \right) = (\pi_1 \; \pi_2 \; \pi_3)$$

$$\frac{2\pi_2}{3} + \frac{2\pi_3}{3} = \pi_1, \quad \pi_1 + \frac{\pi_3}{3} = \pi_2, \quad \frac{\pi_2}{3} = \pi_3$$

$$3\pi_1 - 2\pi_2 - 2\pi_3 = 0, \; \text{—(1)} \quad 3\pi_1 - 3\pi_2 + \pi_3 = 0, \; \text{—(2)} \quad \pi_2 = 3\pi_3 \; \text{—(3)}$$

Solving (1), (2) & (3) with $\pi_1 + \pi_2 + \pi_3 = 1$,

$$\pi_1 = \frac{8}{20}, \quad \pi_2 = \frac{9}{20}, \quad \pi_3 = \frac{3}{20}$$

$\therefore$ The steady state distribution is

$$\pi = \left(\frac{8}{20} \; \frac{9}{20} \; \frac{3}{20} \right) = (0.40 \; 0.45 \; 0.15)$$

Thus in the long run, he sells 40% of the time in city A, 45% of the time in the city B and 15% of the time in city C.

Problem (6): Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states. 14

Solution: The State space is [A B C].

$$\text{The tpm is } P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

The state in the $(n+1)^{th}$ step depends only on the n^{th} step and not on the previous steps. Hence the process is Markovian.

$$\text{Now} \quad P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^3 = P^2 . P = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^4 = P^3 . P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$P^5 = P^4 . P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

Classification of states (15)

$P_{11}^{(3)} > 0, \quad P_{11}^{(5)} > 0 \cdots$

Period = GCD of $\{3, 5, \ldots\} = 1$

$P_{22}^{(2)} > 0, \quad P_{22}^{(3)} > 0, \quad P_{22}^{(4)} > 0 \cdots$

Period = G.C.D of $\{2, 3, 4 \cdots\} = 1$

$P_{33}^{(2)} > 0, \quad P_{33}^{(3)} > 0, \quad P_{33}^{(4)} > 0 \cdots$

Period = G.C.D of $\{2, 3, 4, \cdots\} = 1$

$\therefore$ All the states A, B, C have period 1

i.e) they are aperiodic.

Now, $P_{11}^{(3)} > 0, \quad P_{12}^{(1)} > 0, \quad P_{13}^{(2)} > 0$

$P_{21}^{(2)} > 0, \quad P_{22}^{(2)} > 0, \quad P_{23}^{(1)} > 0$

$P_{31}^{(1)} > 0, \quad P_{32}^{(1)} > 0, \quad P_{33}^{(2)} > 0$

$\therefore$ The chain is irreducible. Also since there are only 3 states, the chain is finite.

i.e) the chain is finite and irreducible.

$\therefore$ All the states are non-null persistent. Since all the states are aperiodic and non-null persistent, they are ergodic.

Problem (7):

Let $\{X_n : n = 1, 2, 3, \cdots\}$ be a Markov chain on the space $S = \{1, 2, 3\}$ with one step transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

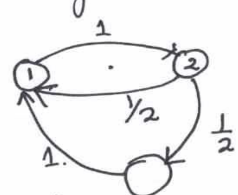
(i) Sketch the transition diagram

(ii) Is the Chain irreducible? Explain.

(iii) Is the chain Ergodic? Explain.

Solution:

(i) Transition diagram:



(ii) $P^2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

$P^4 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$, $P^5 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

$\therefore P_{11}^{(2)} > 0,\ P_{12}^{(1)} > 0,\ P_{13}^{(2)} > 0$

$P_{21}^{(1)} > 0,\ P_{22}^{(2)} > 0,\ P_{23}^{(1)} > 0$

$P_{31}^{(1)} > 0,\ P_{32}^{(2)} > 0,\ P_{33}^{(3)} > 0.$

$\therefore$ The chain is irreducible.

(iii) Period of state 1 = G.C.D of $\{2,3,4,5\} = 1$ 17

$P_{11}^{(2)} > 0$, $P_{11}^{(3)} > 0$, $P_{11}^{(4)} > 0$, $P_{11}^{(5)} > 0$ ....

$P_{22}^{(2)} > 0$, $P_{22}^{(3)} > 0$, $P_{22}^{(4)} > 0$, $P_{22}^{(5)} > 0$ ....

Period of state 2 = G.C.D of $\{2,3,4,5,\ldots\}$ $= 1$.

$P_{33}^{(3)} > 0$, $P_{33}^{(5)} > 0$, ...

Period of state 3 = G.C.D of $\{3,5,\ldots\} = 1$.

∴ All the states are aperiodic.

There are only 3 states, hence the chain is finite.

∴ All these states are non-null persistent.

Since all the states are aperiodic and non-null persistent, they are ergodic.
