Module-2

MODEL-II $(M/M/1):(K/FIFO)$

1. Probability of zero customers in the system

$$p_0 = \begin{cases} \dfrac{1-\dfrac{\lambda}{\mu}}{1-\left(\dfrac{\lambda}{\mu}\right)^{k+1}} & if \;\; \lambda \neq \mu \\ \dfrac{1}{k+1} & if \;\; \lambda = \mu \end{cases}$$

Probability of n customers in the system

$$p_n = \begin{cases} \left(\dfrac{\lambda}{\mu}\right)^n \left[\dfrac{1-\dfrac{\lambda}{\mu}}{1-\left(\dfrac{\lambda}{\mu}\right)^{k+1}}\right] & , if \;\; \lambda \neq \mu \\ \dfrac{1}{k+1} & , if \;\; \lambda = \mu \end{cases}$$

2. $L_s = The \;\; average \;\; no.of \;\; customers \;\; in \;\; the \;\; system$

$$L_s = \begin{cases} \left(\dfrac{\lambda}{\mu-\lambda}\right) - \left[\dfrac{(k+1)\left(\dfrac{\lambda}{\mu}\right)^{k+1}}{1-\left(\dfrac{\lambda}{\mu}\right)^{k+1}}\right] & , if \;\; \lambda \neq \mu \\ \dfrac{k}{2} & , if \;\; \lambda = \mu \end{cases}$$

3. Probability of k customer turned away = $P_k = \left(\dfrac{\lambda}{\mu}\right)^k P_0$

4. Effective arrival rate $\lambda' = \mu(1-p_0)$

5. $L_q = L_s - \dfrac{\lambda'}{\mu}$ where λ' is the effective arrival rate $\lambda' = \mu(1-p_0)$

6. $W_s = \dfrac{L_s}{\lambda'}$

7. $W_q = \dfrac{L_q}{\lambda'}$

8. Traffic intensity or utilization factor $\rho = \dfrac{\lambda}{\mu}$

Problem – 1

Patients arrive at a clinic according to Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients Investigation time per patient in exponential with mean rate of 20 per hour.

a) Determine the effective arrival rate at the clinic.

b) What is the probability that an arriving patient will not wait?

c) What is the expected time (waiting) until the patient is discharged from the Clinic?

Solution:

This is $(M/M/I):(K/FIFO)$ model hence $\lambda =$ 30 patients/ hr,

and $\mu =$ 20/ $hr.$, $K =$ 14 + 1 = 15

(a) Effective arrival rate $\lambda' = \mu(1 - p_0)$

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1`}} = = \frac{1-1.5}{1-(1.5)^{14+1`}} = 0.00076$$

λ'= 20 (1- 00076) = 19.98/ Hr

(b) The probability that an arriving patient will not wait P_0 = 0.00076

(c) The expected time (waiting) until the patient is discharged from the Clinic

$$L_s = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1`}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1`}} \text{ if } \lambda \neq \mu$$

$$L_s = \frac{1.5}{1-1.5} - \frac{(15+1)(1.5)^{15+1`}}{1-(1.5)^{15+1`}}$$

= 13 patients onl

Problem-2 :

A one person barber shop has 6 chairs to accommodate people waiting for haircut .Assume that the customers who arrive when all the 6 chairs are full leave without entering the barbers shop. Customers arrive at the shop one after another at an average of every 20 min and spend in average of 15 min in the barber's chair.

Find (i) the probability that a customer can get directly in to the barber's chair.

(ii) Probability that there are at least 3 customers in the shop.

(iii) Expected number of customers in the shop.

(iv) Average number of customers waiting for a hair cut

(v) Average time a customer expect to spend in the barber shop and in the queue.

(vi) What fraction of a potential customers turned away?

Solution:

Here Arrival rate is Poisson.

Service rate is Exponential.

Number of server = 1

Capacity = (6 + 1) = 7

$\therefore$ This problem comes under the model (M/M/1) : (k/FIFO)

Given Mean arrival rate λ = 20 min/customer

$$= \frac{1}{20} \times 60 = 3/hour$$

Mean service rate μ = 15 min/ customer

$$= \frac{1}{15} \times 60 = 4/hour$$

Capacity of the system k = 7(6 customers can wat + 1 customer is in service)

(i) A customer can get directly in to the barber's chair if ther are no customers in the system.

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{1-\frac{3}{4}}{1-\left(\frac{3}{4}\right)^{7+1}} = 0.2778$$

(ii) Probability that there are at least 3 customers in the shop = P (n≥ 3)

= 1 – P (n < 3)

= 1 – [$P_0 + P_1 + P_2$]

$$= 1 - [P_0 + \frac{\lambda}{\mu}P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0]$$

= 0.3576

(iii) Expected number of customers in the shop = Ls

$$= \frac{3}{4-3} - \frac{(7+1)\left(\frac{3}{4}\right)^{7+1}}{1-\left(\frac{3}{4}\right)^{7+1}} \quad \text{if } \lambda \neq \mu$$

= 2.111 ≈ 2 customers

(iv) Average number of customers waiting for haircut = Lq

$$= L_s - \frac{\lambda'}{\mu}$$

$$= 2.111 - \frac{\mu(1-P_0)}{\mu}$$

= 2.111 – (1-P_0)

= 1.389 ≈ 1 customer

(v) Expected time a customer spend in the shop $W_s = \frac{L_s}{\lambda'}$

$$= \frac{2.111}{2.89}$$

= 0.7304 hr = 0.7304 x 60 = 43.82 min

(vi) Expected time a customer spend in the queue $W_q = \frac{L_q}{\lambda'}$

$$= \frac{1.389}{2.89} = 0.4806\,hr = 28.84\min$$

(vii) Probability of a customer turned away , if ther are 7 or more customers are in the system = P (k > 7) we have $P_k = \left(\frac{\lambda}{\mu}\right)^k P_0$

$$= \left(\frac{3}{4}\right)^7 (0.2778)$$

= 0.037

Problem – 3

A local one man barber shop can accommodate a maximum of 5 peoples at a time (4waiting and one getting hair cut) the customers arrive following poisson process at an average rate of 5 per hour and the barber cuts hair at an average rate of 4 per hour, according to exponential distribution.

Find (i) What percentage of the time the barber is idle?

(ii) What fraction of a potential customers turned away?

(iii) What is the expected number of customers waiting for a hair cut?

(iv) How much time a customer expect to spend in the barber shop?

This is $(M/M/I):(K/FIFO)$ model hence $\lambda = 5$ customers/ hr,

and $\mu = 4/hr.$, $K = 5$

$$P_0 = \frac{1-1.25}{1-(1.25)^{5+1}} .= 0.888$$

(i) P(the barber is idle) = P(n=0)

$= P_0 = 0.888$

Percentage of the time the barber is idle = 8.8% = 9 %

(ii) P(a customers turned away) = P (n >5)

$$= \left(\frac{\lambda}{\mu}\right)^k - \left\{\frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right\}$$

$$= (1.25)^5 - \left\{\frac{(5+1)(1.25)^{5+1}}{1-(1.25)^{5+1}}\right\} = 0.2771$$

(iii) The expected number of customers waiting for a hair cut = Lq

$$= Ls - \frac{\lambda'}{\mu} = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} - (1 - P_0)$$

= 2.2 Customers

(iv) A customer expect to spend in the barber shop $Ws = \frac{Ls}{\lambda'}$

$$= \frac{L_s}{\mu(1-P0)} = \frac{3.137}{3.6448} = 0.8592 \text{ hour} = 51.5 \text{ min}$$

Problem – 4

At a railway station only one train is handled at a time. The railway yard is sufficient only for 2 trains to wait, while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 6 per hour. Assuming Poisson arrivals and Exponential service distribution find the probabilities for the numbers of trains in the system. Also find the average waiting time of a new train coming into the yard. If the handling rate is doubled, how will the above results are modified?

Solution:

Given: A railway station → Single server

Train → Finite capacity

∴ The given problem is (M|M|l) : (K|FIFO) model .

Given : Arrival rate = λ = 6 per hour

Service rate = μ = 6 per hour

Finite capacity = k = 3 ($\because$ 2 trains + 1 train = 3 trains)

Since $\lambda = \mu$,

$$P_0 = \frac{1}{k+1} = \frac{1}{4}$$

$$P_n = \frac{1}{k+1} = \frac{1}{4} \quad \text{for } n = 1,2,3$$

(i) To find the average waiting time in the station of a new train coming in to the yard:

$$W_s = \frac{L_s}{\lambda'}$$

$$\lambda' = \mu(1 - p_0)$$

$$= 6(1\text{-}0.25) = 4.5$$

$$Ls = \frac{k}{2} = \frac{3}{2} = 1.5$$

$$W_s = \frac{L_s}{\lambda'} = \frac{1.5}{4.5}$$

$$= \frac{1}{3} \text{ hour} = \frac{1}{3} \times 60 \text{ min} = 20 \text{ min}$$

(ii) To find if the handling rate is doubled:

i.e., Service rate is doubled = $\mu = 2 \times 6 = 12$

To find P_0 :

$$p_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$p_0 = \frac{1 - \frac{6}{12}}{1 - \left(\frac{6}{12}\right)^{3+1}} \quad \Rightarrow \quad p_0 = \frac{8}{15}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$P_n = \left(\frac{6}{12}\right)^n \times \frac{8}{15} \quad , n = 1,2,3$$

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$$L_s = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} \quad \text{if } \lambda \neq \mu$$

$$= \frac{6}{12-6} - \frac{(3+1)\left(\frac{6}{12}\right)^{3+1}}{1 - \left(\frac{6}{12}\right)^{3+1}} = 1 - 0.2667$$

$$= 0.7333 \text{ trains}$$

To find Ws :

$$W_s = \frac{0.7333}{5.6} = 0.131 \text{ hour}$$

$$= 0.131 \times 60 = 7.86 \text{ min}$$