Unit - IV : Small Sample Testing & Queuing Theory

☆ F- Test (Snecdor's F Test)

Test on Variance / SD
(Difference between Population Variance)

$$F = \frac{\sigma_1^2 \rightarrow \text{Horizontal}}{\sigma_2^2 \rightarrow \text{Vertical}} \quad \text{with } \gamma_1 = n_1 - 1 \text{ and } \gamma_2 = n_2 - 1 \text{ as DOF}$$

$$\sigma_1^2 = \frac{n_1 s_1^2}{n_1 - 1} \qquad \sigma_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$$

$$s^2 = \frac{1}{n}\sum (x_i - \bar{x})^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2$$

NOTE : i) The value of F should always be greater than 1.

$$\boxed{F > 1}$$

If $F < 1$ then,

$$F = \frac{\sigma_2^2 \rightarrow \text{Horizontal}}{\sigma_1^2 \rightarrow \text{Vertical}}$$

☆ Interpolation (OR) Reading b/w the data
— To take out values not given in table

ii) F is not a 2 tail test, it's always a right-tailed test.

↓	4
20	2.87
25	2.76

$$\frac{2.87 - 2.76}{20 - 25} = -0.022$$

$20 \rightarrow 2.87$

$21 \rightarrow 2.87 - 0.022 = 2.848$

$22 \rightarrow 2.848 - 0.022 = 2.826$

$23 \rightarrow 2.826 - 0.022 = 2.804$

$24 \rightarrow 2.804 - 0.022 = 2.782$

$25 \rightarrow 2.782 - 0.022 = 2.76$

∴ If accepted, there is no significant difference.

If rejected, there is significant difference.

iii) To test if the two small samples have been drawn from the same normal population, test for significance of mean by 't' and test for significance of variance using 'F'.

Problems

1) A sample of size 13 gave an estimated population variance of 3.0 and another sample

of size 15 gave the same as 2.5. Could both the samples be from same population with same variance?

Sol: We perform F-test to test for variance

(1) $H_0 : \sigma_1^2 = \sigma_2^2$

$H_1 : \sigma_1^2 \neq \sigma_2^2$

(2) Calculated value of F

$$F_{cal} = \frac{\sigma_1^2}{\sigma_2^2} = \frac{3}{2.5} = 1.2$$

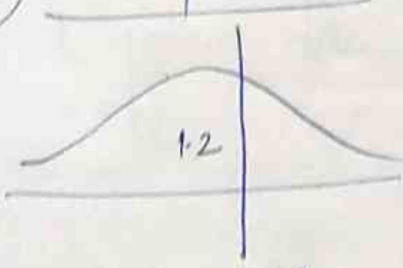
(3) $\gamma_1 = n_1 - 1 = 13 - 1 = 12$

$\gamma_2 = n_2 - 1 = 15 - 1 = 14$

$\gamma_1 \to$ horizontal, $\gamma_2 \to$ vertical

$F_{Tab} = (12, 14)$ @ 5% LOS $= 2.53$

(4) Comparison



so accept H_0

(5) Conclusion:

The 2 samples come from same population with same variance.

2) A sample of size 5 gave an estimated population variance as 5.3 and another sample of size 6 gave the same as 21.6. Could both the samples be from population with same variance?

Sol: (1) $H_0 : \sigma_1^2 = \sigma_2^2$ $\qquad H_1 : \sigma_1^2 \neq \sigma_2^2$

(2) calculated value of F

$$F_{cal} = \frac{\sigma_1^2}{\sigma_2^2} = \frac{5.3}{21.6} < 1$$

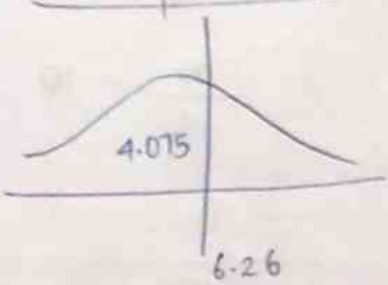
Since $F > 1$ we apply $F_{cal} = \frac{\sigma_2^2}{\sigma_1^2} = \frac{21.6}{5.3} = 4.075$

(3) $\gamma_2 = n_2 - 1 = 6 - 1 = 5$

$\gamma_1 = n_1 - 1 = 5 - 1 = 4$

$F_{Tab} = (5, 4)$ @ 5% LOS $= 6.26$

(4) Comparison



so accept H_0

(5) Conclusion

They come from population with same variance.

3) 2 random samples gave the following results-

Sample	Size	Mean	Sum of the squares of deviation from mean
1	10 (n_1)	15 $(\bar{x}_1)$	90
2	12 (n_2)	14 $(\bar{x}_2)$	108

Test whether the samples come from same normal population.

sol :
i) Test on Variance — 'F'
ii) Test on Mean — 't'

i) Test on Variance $\left[s^2 = \dfrac{\sum (x_i - \bar{x})^2}{n} \right]$

$$\sum (x_{i1} - \bar{x}_1)^2 = n_1 s_1^2 = 90$$

$$\sum (x_{i2} - \bar{x}_2)^2 = n_2 s_2^2 = 108$$

$$\sigma_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{90}{10-1} = \frac{90}{9} = 10$$

$$\sigma_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{108}{12-1} = \frac{108}{11} = 9.818$$

$$F = \frac{\sigma_1^2}{\sigma_2^2} = \frac{10}{9.818} = 1.018$$

① $H_0 : \sigma_1^2 = \sigma_2^2$ $H_1 : \sigma_1^2 \neq \sigma_2^2$

② $$F_{cal} = \frac{\sigma_1^2}{\sigma_2^2} = \frac{10}{9.818} = 1.018$$

③ $\gamma_1 = 9$, $\gamma_2 = 11$

$\gamma = (9, 11)$ — To find the value we interpolate

	$8 \longleftarrow 9 \longrightarrow$		12
10	3.07		2.91
11	$\boxed{2.96}$	$\boxed{2.92}$	$\boxed{2.8}$
12	2.85		2.69

$(10 \longleftrightarrow 12)$

$$\frac{3.07 - 2.85}{10 - 12} = -0.11$$

$11 \longrightarrow 3.07 - 0.11 = 2.96$ (8)

$2.91 - 2.69 = 2.8$ (12)

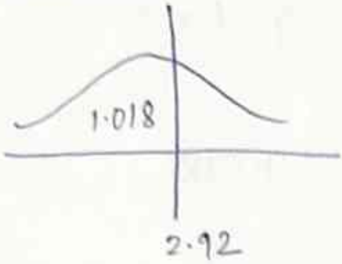
$(8 \longleftrightarrow 12)$

$$\frac{2.96 - 2.8}{8 - 12} = -0.04$$

$9 \longrightarrow 2.96 - 0.04 = 2.92$ (11)

$F_{Tab} = \gamma(9, 11) = 2.92$

④ **Comparison**



so accept H_0 for variance

ii) **t- Test on Mean**

① $H_0 : \overline{x_1} = \overline{x_2}$

$H_1 : \overline{x_1} \neq \overline{x_2}$

$$\left\{ \begin{aligned} s^2 &= \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} \\ &= \frac{90 + 108}{20} = 9.9 \\ s &= 3.14 \end{aligned} \right\}$$

② 2 Tail Test @ 5% LOS

$\gamma = n_1 + n_2 - 2 = 10 + 12 - 2 = 20$

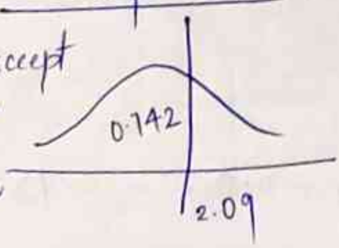
$t_\alpha = 20$ @ 5% LOS $= 2.09$

③ $$|t| = \left| \frac{\overline{x_1} - \overline{x_2}}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \right| = \left| \frac{15 - 14}{3.14\sqrt{\frac{1}{10} + \frac{1}{12}}} \right|$$

$$= 0.742$$

④ **Comparison**

so accept for mean



⑤ **Final Conclusion :**

The 2 samples come from same normal population.

✫ Application of χ^2 - Test

1) Test goodness of Fit (check if sample comes from a population required).

2) Test independence of attributes (if they are associated or not)

$$\chi^2 = \sum_{i=1}^{n} \left[\frac{(O_i - E_i)^2}{E_i} \right] \qquad \boxed{\gamma = n-1}$$

O_i = Observed (Practical) frequency

E_i = Expected (Theoretical) frequency

✫ χ^2 distribution by Karl Pearson.

— Conditions

1) $N \geq 50$ 2) No. of classes : $4 \leq n \leq 16$

3) $O_i \geq 10$ (Individual frequencies)

If $O_i < 10$ then, combine with the neighbouring frequencies such that combined freq. ≥ 10

Problems

1) The no. of aircraft accidents that occured during various days of the week are given below.

Test whether the accidents are uniformly distributed over the week.

Days :	Mon	Tue	Wed	Thu	Fri	Sat
No. of accidents :	15	19	13	12	16	15

Sol :

$$N = \sum f_i = 90 > 50$$

$$\text{Classes } (n) = 6 \in [4, 16]$$

$$\text{and } \quad O_i > 10$$

Hence, χ^2 Test conditions are satisfied.

(1) H_0 : Accidents are uniformly distributed over the week

H_1 : Accidents are non-uniformly distributed over the week

$$E_i = \frac{\sum f_i}{n} = \frac{90}{6} = 15$$

(2) $\gamma = n-1 = 6-1 = 5$ @ 5% LOS

$$\chi^2_{Tab} = 11.07$$

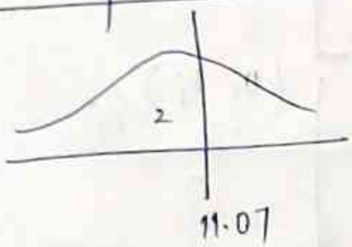
(3) $$\chi^2 = \sum_{i=1}^{n} \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

O_i	E_i	$(O_i - E_i)^2$	$(O_i - E_i)^2 / E_i$
15	15	0	0
19	15	16	$16/15 = 1.067$

13	15	4	$4/15 = 0.267$
12	15	9	$9/15 = 0.6$
16	15	1	$1/15 = 0.067$
15	15	0	0
			$\Sigma = 2.001$

$$\chi^2_{cal} = 2$$

(4) Comparison



so accept H_0.

(5) Conclusion :

The accidents are uniformly distributed over the week

2) A sample analysis of exam results of 500 students was made. It was found that 200 failed, 170 secured 3rd class, 90 secured 2nd class and rest secured 1st class. Do the data indicate the general belief that the above categories are in the ratios 4:3:2:1.

Sol :

500	200	170	90	40
N	F	III	II	I

$$N = \sum f_i = 500$$

classes $(n) = 4$ $\quad O_i > 10$

χ^2 Test conditions are satisfied.

1. H_0 : Data supports the general belief of the ratio 4:3:2:1

H_1 : Data doesn't support the belief

2. $\gamma = n-1 = 4-1 = 3$

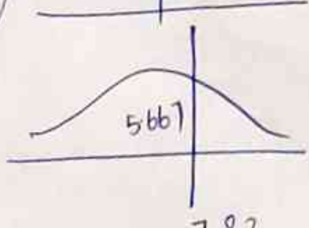
$$\chi^2_{Tab} = 3 \text{ @ } 5\% \text{ LOS} = 7.82$$

3. $$\chi^2 = \sum_{i=1}^{n} \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

O_i	E_i	$(O_i - E_i)^2$	$(O_i - E_i)^2 / E_i$
200	$\frac{4}{10} \times 500 = 200$	0	0
170	$\frac{3}{10} \times 500 = 150$	400	$400/150 = 2.667$
90	$\frac{2}{10} \times 500 = 100$	100	$100/100 = 1$
40	$\frac{1}{10} \times 500 = 50$	100	$100/50 = 2$
			$\Sigma = 5.667$

$$\chi^2_{cal} = 5.667$$

4. Comparison.



so accept H_0

5. Conclusion :

Data supports the general belief of the ratio 4:3:2:1.

☆ Fit Binomial (OR) Poisson (Fit?)

(Checking Goodness using χ^2)

$$\boxed{\gamma = n - 2}$$

1) Fit a Poisson distribution for the following data and test the goodness of fit using χ^2

X :	0	1	2	3	4	5
f :	142	156	69	27	5	1

Sol :

$$N = \Sigma f = 400$$

$$\text{Mean} = \lambda = \frac{\Sigma xf}{\Sigma f} \Rightarrow \lambda = 1$$

$$\text{Theoretical frequencies} = N_i \cdot P(x) = 400\left[\frac{e^{-1}\, 1^{x}}{x!}\right]$$

Ef :	147	147	74	25 $\boxed{32}$	6 (7)	1
Of :	142	156	69	27 $\boxed{33}$	5 (6)	1

Testing the goodness of fit using χ^2

$N = 400 > 50$ $\qquad n = 4 \in [4, 16]$

$O_i \not> 10$ so merge neighbouring frequencies

O_i	E_i	$(O_i-E_i)^2$	$(O_i-E_i)^2/E_i$
142	147	25	$25/147 = 0.170$
156	147	81	$81/147 = 0.551$
69	74	25	$25/74 = 0.337$
33	32	1	$1/32 = 0.0312$
			$\sum = 1.0872$

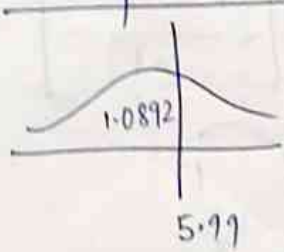
(1) H_0 : Fit is good
H_1 : Fit is not good

(2) $\gamma = n-2 = 4-2 = 2$

$\chi^2_{Tab} = 2$ @ 5% LOS $= 5.99$

(3) $\chi^2_{cal} = 1.0892$

(4) Comparison



so accept H_0

(5) Conclusion :
Fit is good.

2) Fit a Binomial distribution for the following data and test the goodness of fit using χ^2

x :	0	1	2	3	4	5	6
of :	5	18	28	12	7	6	4

Sol: $N = \sum f = 80$, $n = 6$

$$\text{Mean} = np = \frac{\sum xf}{\sum f}$$

$$6p = \frac{0\times5 + 1\times18 + 2\times28 + 3\times12 + 4\times7 + 5\times6 + 6\times4}{80}$$

$$6p = \frac{192}{80} \Rightarrow p = 0.4,\ q = 0.6$$

Theoretical frequencies $= N\cdot P(x) = N\left[{}^nC_x\, p^x q^{n-x}\right]$

$x=0$: $N\cdot P(X=0) = 80\left[{}^6C_0\,(0.4)^0(0.6)^6\right] = 3.732 \simeq 4$

$x=1$: $N\cdot P(X=1) = 80\left[{}^6C_1\,(0.4)(0.6)^5\right] = 14.929 \simeq 15$

$x=2$: $N\cdot P(X=2) = 80\left[{}^6C_2\,(0.4)^2(0.6)^4\right] = 24.88 \simeq 25$

$x=3$: $N\cdot P(X=3) = 80\left[{}^6C_3\,(0.4)^3(0.6)^3\right] = 22.11 \simeq 22$

$x=4$: $N\cdot P(X=4) = 80\left[{}^6C_4\,(0.4)^4(0.6)^2\right] = 11.05 \simeq 11$

$x=5$: $N\cdot P(X=5) = 80\left[{}^6C_5\,(0.4)^5(0.6)\right] = 2.949 \simeq 3$

$x=6$: $N\cdot P(X=6) = 80\left[{}^6C_6\,(0.4)^6(0.6)^0\right] = 0.327 \simeq 0$

Ef :	4	15 $\boxed{19}$	25	22	$\boxed{14}$ 11	3 (3)	0
Of :	5	18 $\boxed{23}$	28	12	$\boxed{17}$ 7	6 (10)	4

Testing the goodness of fit using χ^2

$N = 80 > 50$ $\quad n = 4 \in [4, 16]$

$O_i \ngtr 10$ so merge neighbouring frequencies

O_i	E_i	$(O_i - E_i)^2$	$(O_i - E_i)^2 / E_i$
23	19	16	$16/19 = 0.842$
28	25	9	$9/25 = 0.36$
12	22	100	$100/22 = 4.545$
17	14	9	$14/9 = 1.555$
			$\Sigma = 7.302$

1. H_0 : Fit is good

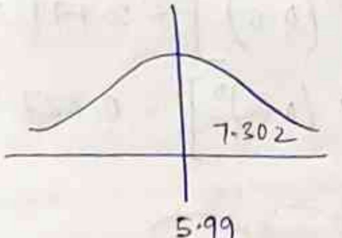
   H_1 : Fit is not good

2. $\gamma = n - 2 = 4 - 2 = 2$

   $\chi^2_{Tab} = 2$ @ 5% LOS $= 5.99$

3. $\chi^2_{cal} = 7.302$

4. Comparison



so reject H_0

5. Conclusion :

   Fit is not good.

☆ Testing Independence of Attributes

Income (A)

m:
- Below poverty
- Low
- Medium
- High
- Extremely high

Education (B)

n:
- Primary
- secondary
- Higher secondary
- College UG
- College PG

Education $\longrightarrow$

$$
\text{Income} \uparrow \quad
\begin{array}{c|cccc|l}
 & B_1 & B_2 & B_3 & \cdots\ B_n & \\
\hline
A_1 & & & & & \longrightarrow \boxed{}\ RT \\
A_2 & & & & & \\
A_3 & & & & & \\
\vdots & & & & & \\
A_n & & & & & \\
\hline
 & \boxed{}\ CT & & & &
\end{array}
$$

$m \times n$ matrix

$$E_i = \frac{\text{Row Total} \times \text{Column Total}}{\text{Grand Total}}$$

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

$$\gamma = (R-1)(C-1) = (m-1)(n-1)$$

Problems

1) The following table gives the matrix for the level of education and the marriage adjustment score.

Marriage Adjustment

Level of education

	Very low	low	high	Very high	RT
College	24	97	62	58	241
High School	22	28	30	41	121
Middle school	32	10	11	20	73
CT	78	135	103	119	435

Can you conclude from the above data, higher the level of education, greater is the adjustment in marriage?

Sol: (1) H_0 : The level of education is dependent on adjustment in marriage.

H_1 : The level of education is independent of adjustment in marriage.

(2)

O_i	$E_i = \frac{RT \times CT}{GT}$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
24	$\frac{241 \times 78}{435} = 43.2 \simeq 43$	361	8.395
97	$\frac{241 \times 135}{435} = 74.7 \simeq 75$	484	6.453
62	$\frac{241 \times 103}{435} = 57.06 \simeq 57$	25	0.438
58	$\frac{241 \times 119}{435} = 65.92 \simeq 66$	64	0.969

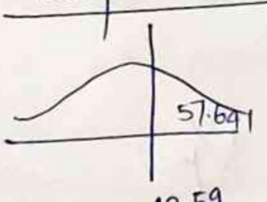
22	$\frac{121 \times 78}{435} = 21.69 \simeq 22$	0	0
28	$\frac{121 \times 135}{435} = 37.551 \simeq 37$	81	2.187
30	$\frac{121 \times 103}{435} = 28.65 \simeq 29$	1	0.034
41	$\frac{121 \times 119}{435} = 33.101 \simeq 33$	64	1.939
32	$\frac{73 \times 78}{435} = 13.08 \simeq 13$	361	27.76
10	$\frac{73 \times 135}{435} = 22.655 \simeq 23$	169	7.347
11	$\frac{73 \times 103}{435} = 17.28 \simeq 17$	36	2.117
20	$\frac{73 \times 119}{435} = 19.97 \simeq 20$	0	0
	435		$\sum = 57.641$

$$\chi^2_{cal} = 57.641$$

(3) $\gamma = (R-1)(c-1) = (3-1)(4-1) = 6$

$$\chi^2_{Tab} = 6 \text{ @ } 5\% \text{ LOS} = 12.59$$

(4) Comparison



so reject Ho

(5) Conclusion :

The level of education is independent of adjustment in marriage.

☆ Queuing Theory

→ Characteristics of Queuing Theory

- Input (OR) Arrival Pattern (λ = Mean arrival rate)
- Service Pattern (OR) Service Discipline
- Queue Discipline
  - → LIFO
  - → FIFO
  - → SIRO (Selection In Random Order)
  - → Priority

→ Arrival pattern follows Poisson distribution.

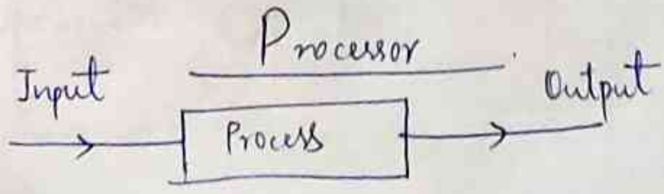
<u>Inter Arrival Time</u> : Time duration b/w first arrival and second arrival.
(will follow exponential distribution).

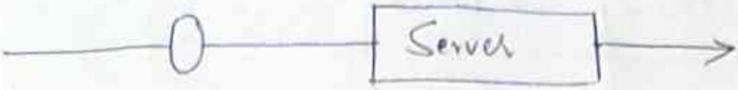
The mean inter arrival rate $= \dfrac{1}{\lambda}$

→ <u>Service Pattern</u> (μ) follows Poisson distribution

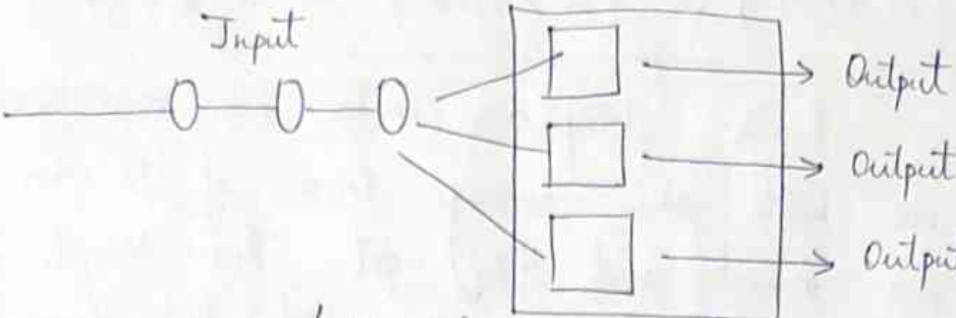
Service rate $= \mu$

Mean Inter Service rate $= 1/\mu$
(exponential distribution)

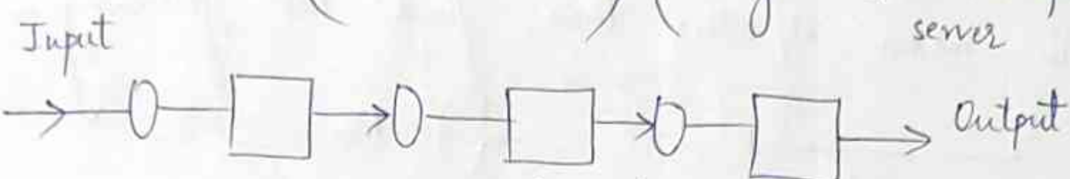




(Single Q single server)



(Parallel server) (Single Q Multiple server)



(Series path) (single Q Multiple server)

☆ Kendell's Notation

(a/b/c) (d/e/f)

a → Type of distribution of the no. of arrivals per unit time

b → Type of distribution of service time

c → no. of servers available for service

d → capacity of the system (maximum queue length)

e → Queue discipline

f → The size of the calling source (finite or infinite)

(a/b/c) (d/e)

① (M/M/1) (∞/FIFO)

② (M/M/1) (K/FIFO)

→ Markovian Arrival

→ Markovian Service

☆ Model I $(M/M/1):(\infty/FIFO)$ $(\lambda < \mu)$

$P_n(t)$ – Probability that there are n customers in the system at time t.

1) Avg no. of customers in system (length)

$$L_s = \frac{\lambda}{\mu - \lambda}$$

(OR) Expected no. of person in the system $E(N_s)$

2) Avg no. of customers in the queue $L_q = \dfrac{\lambda^2}{\mu(\mu-\lambda)}$

(OR) Expected no. of customers in queue $E(N_q)$

3) P_0 = Probability of zero (no) customers in system $= 1 - \left(\dfrac{\lambda}{\mu}\right)$

P_n = Prob of 'n' customers in the system $= \left(\dfrac{\lambda}{\mu}\right)^n \left(1 - \dfrac{\lambda}{\mu}\right)$

4) Probability that the no. of customers in the system exceeds K

$$p(N > K) = \left(\frac{\lambda}{\mu}\right)^{K+1}$$

5) Average waiting time of a customer in the system:

$$E(W_s) = \frac{1}{\mu - \lambda}$$

6) Average waiting time of a customer in the queue:

$$E(W_q) = \frac{\lambda}{\mu(\mu - \lambda)}$$

7) Average waiting time of a customer in the queue, if he has to wait

$$E(W_q / W_q > 0) = \frac{1}{\mu - \lambda}$$

8) Probability that the waiting time of a customer in the system exceeds t:

$$p(W_s > t) = e^{-(\mu - \lambda)t}$$

9) Average length of the queue that forms from time to time:

$$E(N_q / N_s > 1) = \frac{\mu}{\mu - \lambda}$$

— Little's Formula

1) $E(N_s) = \lambda E(W_s)$

2) $E(N_q) = \lambda E(W_q)$

3) $E(W_s) = E(W_q) + \frac{1}{\mu}$

4) $E(N_s) = E(N_q) + \frac{\lambda}{\mu}$

☆ Model II $[(M/M/1) : (K/FIFO)]$

K — System's capacity

so only a max of K people can be accomodated in the system

1) Values of P_0 & P_n

$$P_0 = \begin{cases} \dfrac{1 - \left(\frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} & \text{if } \lambda \neq \mu \\[2ex] \dfrac{1}{K+1} & \text{if } \lambda = \mu \end{cases}$$

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \left[\dfrac{1 - \left(\frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}}\right] & \text{if } \lambda \neq \mu \\[2ex] \dfrac{1}{K+1} & \text{if } \lambda = \mu \end{cases}$$

2) Average no. of customers in the system:

$$E(N_s) = \begin{cases} \left(\frac{\lambda}{\mu-\lambda}\right) - \left[\dfrac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}\right] & \text{if } \lambda \neq \mu \\ \frac{K}{2} & \text{if } \lambda = \mu \end{cases}$$

3) Average no. of customers in the queue:

$$E(N_q) = E(N_s) - \left(\frac{\lambda'}{\mu}\right)$$

where λ' is the overall effective arrival rate and is given by

$$\boxed{\lambda' = \mu(1-P_0)}$$

4) Average waiting time in the system and in the queue: (By Little's Formula)

$$E(W_s) = \frac{1}{\lambda'} E(N_s)$$

where λ' = overall effective arrival rate

$\lambda' = \mu(1-P_0)$

$$E(W_q) = \frac{1}{\lambda'} E(N_q)$$

☆ Queuing Theory Model

$(M/M/1) : (\infty / FIFO)$ where $(\lambda < \mu)$

1) Arrivals at a telephone booth are considered to be Poisson with an average time of 12 min between one arrival and the next. The length of the phone call is assumed to be distributed exponentially with mean 4 min. then, find the following —

12 min

$\frac{1}{\lambda} = 12$

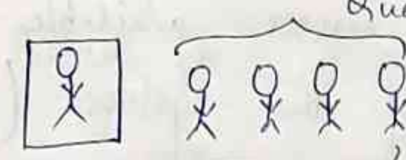
Inter-arrival time (exponential distribution) : $\boxed{\lambda = 1/12 \text{ /min}}$

Service $\longrightarrow \mu$ (exponential) : $\frac{1}{\mu} = 4$ min

$\boxed{\mu = 1/4 \text{ /min}}$

i) Find the average number of persons waiting in the system.

Queue (Q)



system

System : Telephone booth

Q : No. of people waiting to make a call

$$E(N_s) = \frac{\lambda}{\mu - \lambda} = \frac{1/12}{1/4 - 1/12} = \frac{1/12}{2/12}$$

$= 1/2 = 0.5$

$\therefore E(N_s) = 0.5 \simeq 1$ person

ii) What is the probability that a person arriving at the booth will have to wait in the queue?

Sol: $P(\text{person has to wait}) = 1 - P(\text{system is empty})$

$= 1 - P_0$

$= 1 - \left[1 - \frac{\lambda}{\mu}\right] = \frac{\lambda}{\mu} = \frac{1/12}{1/4} = 1/3$

iii) What is the probability that it will take him more than 10 min altogether to wait for the phone & complete the call?

Sol: $P(W_s > 10) = e^{-\left(\frac{1}{4} - \frac{1}{12}\right)10} = e^{-5/3} = 0.188$

$\therefore \left[P(W_s > t) = e^{-(\mu-\lambda)t}\right]$

iv) Estimate the fraction of the day when the phone will be in use.

$= 1 - P(\text{phone is idle i.e system is empty})$

$= 1 - P_0 = 1 - \left[1 - \frac{\lambda}{\mu}\right] = \frac{\lambda}{\mu} = \frac{1/12}{1/4} = 1/3$

v) Find the average length of the queue that forms from time to time.

Sol: $E(N_q / \text{queue is formed time to time}) = E(N_q / N_s > 1)$

$$= \frac{\mu}{\mu - \lambda} = \frac{1/4}{1/4 - 1/12} = \frac{1/4}{2/12} = 3/2 = 1.5$$

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vi) The telephone dept. will install the 2nd booth when convinced that an arrival has to wait on the average for atleast 3 min for the phone? By how much the flow of the arrivals should increase inorder to justify a second booth?

Sol: $E(W_q) > 3$ (for 2nd both to be installed)

$\lambda_R \rightarrow$ required arrival rate

$$\frac{\lambda_R}{\mu(\mu - \lambda_R)} > 3$$

$$\frac{\lambda_R}{\frac{1}{4}\left(\frac{1}{4} - \lambda_R\right)} > 3$$

$$\lambda_R > 3\left(\frac{1}{4}\right)\left(\frac{1}{4} - \lambda_R\right)$$

$$\lambda_R > \frac{3}{16} - \frac{3}{4}\lambda_R$$

$$\lambda_R + \frac{3}{4}\lambda_R > 3/16$$

$$\frac{7}{4}\lambda_R > 3/16$$

$$\boxed{\lambda_R > 3/28}$$

Arrival rate should increase by $= \lambda_R - \lambda$

$$= \frac{3}{28} - \frac{1}{12} = \frac{1}{42}$$

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2) Customers arrive at a 1-man barber shop according to a Poisson's process with a mean inter arrival time of 12 min. Customers spend on an average of 10 min in the barber's chair. Find the following —

i) Expected no. of customers in the barber's shop and in the queue.

ii) Calculate the percentage of time an arrival can walk straight into the barber's chair without having to wait.

iii) How much time can a customer expect to spend in the barber's shop?

iv) What is the average time the customer spends in the queue?

v) What is the probability that the waiting time in the system is greater than 30 min?

vi) Calculate the percentage of customers who have to wait prior to getting into the barber's chair?

vii) The management will provide another chair & hire another barber when a customer's waiting time in the shop exceeds 1.25 hours? How much must the average rate of arrivals increase to warrant a second barber?

Sol: Barber's shop $\rightarrow$ system

People waiting for haircut $\rightarrow$ queue (not on barber's chair)

$$\lambda = \frac{1}{12}/\text{min} \quad , \quad \mu = 1/10 \,/\text{min}$$

i) $$E(N_s) = \frac{\lambda}{\mu-\lambda} = \frac{1/12}{\frac{1}{10} - 1/12} = \frac{1/12}{1/60} = 5$$

(Expected no. of customers in barber's shop)

Expected no. of customers in queue

$$E(N_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(1/12)^2}{1/10\,(1/10 - 1/12)} = \frac{1/144}{1/10\,(1/60)}$$

$$= 600/144 = 4.166 \simeq 5$$

ii) $P(\text{customer doesn't wait}) = P(\text{system is empty})$

$$= P_0 = 1 - \left(\frac{\lambda}{\mu}\right) = 1 - \left(\frac{1/12}{1/10}\right)$$

$$= 1 - \frac{5}{6} = 1/6$$

Percentage of time $= \frac{1}{6} \times 100 = 16.667\%$

iii) $E(\text{customer's time in shop}) = E(W_s)$

$$= \frac{1}{\mu-\lambda} = \frac{1}{1/10 - 1/12} = \frac{1}{1/60} = 60 \text{ min}$$

iv) $E(W_q) = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1/12}{1/10\left(1/10 - 1/12\right)} = \frac{1/12}{1/10\left(1/60\right)}$

$= 600/12 = 50$ min

v) $P(W_s > t) = e^{-(\mu-\lambda)t}$

$P(W_s > 30) = e^{-\left(\frac{1}{10}-\frac{1}{12}\right)30} = e^{-30/60} = e^{-0.5}$

$= 0.6065$

vi) P (customer wait prior to get into barber's chair) = 1 − $P\Big(\underbrace{\text{system is empty}}_{\text{when customer doesn't wait}}\Big)$

$= 1 - P_0 = 1 - \left[1 - \left(\frac{\lambda}{\mu}\right)\right]$

$= \frac{\lambda}{\mu} = \frac{1/12}{1/10} = 5/6$

percentage of customers $= 5/6 \times 100 = 83.33\%$

vii) $E(W_q) > 1.25$ hours $= 75$ min (for another barber to be hired)

$\lambda_R \rightarrow$ required arrival rate

$$\frac{\lambda_R}{\mu(\mu-\lambda_R)} > 75$$

$$\frac{\lambda_R}{\frac{1}{10}\left(\frac{1}{10}-\lambda_R\right)} > 75$$

$$\lambda_R > \frac{75}{10}\left(\frac{1}{10} - \lambda_R\right)$$

$$\lambda_R > \frac{75}{100} - \frac{75}{10}\lambda_R$$

$$\lambda_R + \frac{75}{100}\lambda_R > \frac{75}{100}$$

$$\frac{7}{4}\lambda_R > \frac{3}{4}$$

$$\boxed{\lambda_R > 3/7}$$

Arrival rate should increase by $= \lambda_R - \lambda$

$$= \frac{3}{7} - \frac{1}{12} = \frac{29}{84}$$

☆ Model II $(M/M/1) : (K/FIFO)$.

1) Patients arrive at a clinic according to Poisson distribution at the rate of 30 patients per hour. The waiting room doesn't accomodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour. Find the following —

i) Effective arrival rate at the clinic

ii) What is the probability that an arriving patient will not wait?

iii) What is the expected waiting time until a patient is discharged from the clinic?

Sol : K = System's capacity = 14 (waiting room) + 1 (being examined)

$$K = 15$$

$$\lambda = 30/\text{hour}$$

$$\mu = 20/\text{hour} \qquad \lambda \neq \mu$$

i) Effective arrival rate, $\lambda' = \mu(1-P_0)$

$$P_0 = \frac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} = \frac{\left(1-\frac{3}{2}\right)}{1-\left(\frac{3}{2}\right)^{15+1}}$$

$$= \frac{-0.5}{-655.84} = 0.0007624$$

$$\lambda' = 20\left(1-0.0007624\right) = 19.985 \text{ patients per hour}$$

ii) A patient need not have to wait if system is empty

$$P_0 = 0.0007624$$

iii) Expected waiting time in clinic $= E(W_s)$

$$= \frac{1}{\lambda'} E(N_s)$$

$$E(N_s) = \left(\frac{\lambda}{\mu-\lambda}\right) - \left[\frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}\right]$$

$$= \left(\frac{30}{20-30}\right) - \left[\frac{(15+1)\left(\frac{3}{2}\right)^{15+1}}{1-\left(\frac{3}{2}\right)^{16}}\right]$$

$$= -3 - \left[\frac{16\left(3/2\right)^{16}}{1-\left(3/2\right)^{16}}\right] = 13.024 \text{ persons/hr}$$

$$E(W_s) = \frac{13.024}{19.985} = 0.652 \text{ hours}$$

$$\simeq 39.12 \text{ min}$$

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2) At a railway station, only 1 train is handled at a time. The railway yard is sufficient only for 2 trains to wait while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6/hr and the railway station can handle them on an average of 6/hr. Assuming Poisson arrivals and exponential service distributions, find the probability for the no. of trains in the system. Also find the average waiting time of a new train coming into the yard. If the handling rate is doubled, how will the above results get modified?

Sol: K = System's capacity = 2 (trains in yard) + 1 (train given signal)

$$K = 3 \quad , \quad \mu = \lambda = 6/hr$$

$$P(\text{no of trains in system}) = P_n = \frac{1}{K+1}$$

$$= \frac{1}{3+1} = 1/4 = 0.25$$

$$E(W_s) = \frac{1}{\lambda'} E(N_s)$$

(Average waiting time of train in yard)

$$\lambda' = \mu(1 - P_0) = 6\left(1 - \frac{1}{K+1}\right) = 6(1 - 0.25)$$

$$= 4.5$$

$$E(W_s) = \frac{K/2}{\lambda'} = \frac{3/2}{4.5} = 1/3 \text{ hr} = 20 \text{ min}$$

If handling (service) rate, $\mu = 12/hr$ $\quad (\lambda \neq \mu)$

$\lambda = 6/hr$, $K = 3$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left[\frac{1 - \left(\frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}}\right]$$

$$= \left(\frac{6}{12}\right)^n \left[\frac{1 - (6/12)}{1 - (6/12)^{3+1}}\right]$$

$$= (0.5)^n (0.5333) \quad \text{for } n = 0, 1, 2, 3 \ldots$$

$$E(W_s) = \frac{1}{\lambda'} E(N_s)$$

$$\lambda' = \mu(1 - P_0) = 12(1 - 0.5333) = 5.604$$

$$E(N_s) = \left(\frac{\lambda}{\mu-\lambda}\right) - \left[\frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}\right]$$

$$= \left(\frac{6}{12-6}\right) - \left[\frac{(3+1)\left(\frac{6}{12}\right)^{3+1}}{1-\left(6/12\right)^{3+1}}\right]$$

$$= 11/15 = 0.73$$

$$E(W_s) = \frac{0.73}{5.604} = 0.1302 \text{ hrs} \simeq 7.812 \text{ min}$$