11/3/19

UNIT - IV

MARKOV PROCESS

Markov Process

Random variable => Random Process

X

The conditional probability

$P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots\ldots X_1 = a_1, X_0 = a_0]$

$= P[X_n = a_n / X_{n-1} = a_{n-1}]$

is called Markov Process (Markov chain)

i.e The prob. of future event depends only on present but not on the past event.

MARKOV CHAIN: A discrete parameter Markov process is called Markov chain, where time is discrete or continuous.

One - step transition probability

$P[X_n = a_i / X_{n-1} = a_j] = P_{ij}$

where $a_0, a_1, a_2 \ldots a_n$ are states of Markov chain.

Homogenous Markov chain:

If $P[X_n = a_i / X_{n-1} = a_j] = P[X_n = a_i / X_{n-1} = a_j]$

i.e The probability do not depend on step, then it is Called Homgeneous Markov chain.

Transition probability Matrix (TPM)

The matrix formed by the one - step probabilities P_{ij} denoted by P is called TPM.

STOCHASTIC (OR) REGULAR MATRIX

The tpm in which every entry is positive and the row sum is equal to '1' is called standar stochastic matrix.

The markov chain with the stochastic (or) regular matrix called Regular Markov chain.

FINITE MARKOV CHAIN

It is a Markov chain with finite number of states.

STEADY-STATE (OR) LONG RUN (OR) INVORIENT PROBABILITY

In the long run,

$$\lim_{n \to \infty} P^{n} = P$$

The steady state path ,

where $\pi = (\pi_1, \pi_2 \dots \pi_n)$

and $\pi_1 + \pi_2 + \dots + \pi_n = 1$

⊛ Chapman - Kolmogorov Theorem

⊛ $P_{ij}^{(n)} = [P_{ij}]^{n}$

where $P_{ij}^{(n)}$ _ nth ~~ext~~ step prob.

P_{ij}^{n} - nth TPM

Also $P^{(n)} = P^{(0)} . P^{n}$

and

$P^{(n)} = P^{(n-1)} P$

(n) - step probability

n - power

M

8) A man either drives a car or catches a train to go to office everyday. He never goes two days successively by train but if he drives one day then the day he is just has likely drive again. Has his to travel by train suppose that one the first day of week, the man tossed a fair ~~day~~ dice and drove to work. if and only if the ~~no~~ number 6 appears. Find i) Probability that he takes a train on 3rd day .ii) Probability that he drove to work in the long run

sn State space $S = \{Train, Car\}$

Initial probability $p^{(0)} = \left(\frac{5}{6}, \frac{1}{6}\right)$

Transition prob. matrix

TPM $$P = \begin{array}{c|cc} & T & C \\ \hline T & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} \end{array} = 1$$

X_n, X_{n+1}

i) P(goes by train on 3rd day)

$$p^{(1)} = p^{(0)} P$$

$$= \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \left(\frac{1}{12} \quad \frac{5}{6} + \frac{1}{12}\right) = \left(\frac{1}{12}, \frac{11}{12}\right)$$

$$p^{(2)} = p^{(1)} P$$

$$p^{(2)} = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

$$= \left(\frac{11}{24}, \frac{1}{12} + \frac{11}{24}\right)$$

$$= \left(\frac{11}{24}, \frac{13}{24}\right)$$

P(goes by train on 3rd day) = $11/24$

(or)

$$p^{(2)} = p^{(0)} P^2$$

$$P^2 = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$

$$p^{(2)} = p^{(0)} P^2$$

$$= \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$

$$= \left(\frac{5}{12} + \frac{1}{24}, \frac{5}{12} + \frac{3}{24}\right)$$

$$= \left(\frac{11}{24}, \frac{13}{24}\right)$$

ii) Long run (steady, state or Invarriant) Prob

$$\boxed{\pi P = \pi}$$

where $\pi = (\pi_1, \pi_2)$

$$\boxed{\pi_1 + \pi_2 = 1} \quad - ①$$

$\pi P = \pi \Rightarrow$

$$(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1, \pi_2)$$

$$0\pi_1 + \frac{\pi_2}{2} = \pi_1 \Rightarrow 2\pi_1 - \pi_2 = 0 \quad - ②$$

$$\pi_1 + \frac{\pi_2}{2} = \pi_2 \Rightarrow 2\pi_1 - \pi_2 = 0$$

Solving ① & ②

$$\pi_1 + \pi_2 = 1$$
$$2\pi_1 + \pi_2 = 0$$
$$3\pi_1 = 1$$
$$\pi_1 = 1/3$$
$$\pi_2 = 2/3$$

Hence in Long run,

$$P = (\pi_1, \pi_2) = (1/3, 2/3)$$

P(goes by car in long run) = 2/3

Q) A gambler has ₹2, he bets ₹1 at a time and wins ₹1 with probability 1/2. He stops playing the game when he loses ₹2 and wins ₹4.

i) what is the TPM of related Markov chain

ii) what is the Prob that he has lost his money at the end of 4th the play

iii) what is the prob that he ~~teon~~ Continued the game after 5 ~~plays~~ cases.

State space $S = (0, 1, 2, 3, 4, 5, 6)$

Initial prob $p^{(0)} = (0\ 0\ 1\ 0\ 0\ 0\ 0)$

i) TPM

$$P = X_n \begin{array}{c|ccccccc} & \multicolumn{7}{c}{X_{n+1}} \\ & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 2 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 3 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 4 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 5 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

ii) $P[X_4 = 0]$

$$= p^{(4)} = p^{(3)} P$$

$$p^{(3)} = p^{(2)} P$$

$$p^{(2)} = p^{(1)} P$$

$$p^{(1)} = p^{(0)} P$$

$$p^{(1)} = (0\ 0\ 1\ 0\ 0\ 0\ 0) P$$

$$= (0\ 1/2\ 0\ 1/2\ 0\ 0\ 0)$$

$$p^{(2)} = p^{(1)} P$$

$$= (0\ 1/2\ 0\ 1/2\ 0\ 0\ 0)(\ P\ )$$

$$p^{(2)} = (1/4\ 0\ 2/4\ 0\ 1/4\ 0\ 0)$$

$p^{(3)} = p^{(2)} P = (1/4 \quad 0 \quad 2/4 \quad 0 \quad 1/4 \quad 0 0)(P)$

$= (1/4 \quad 2/8 \quad 0 \quad 3/8 \quad 0 \quad 1/8 \quad 0)$

$p^{(4)} = p^{(3)} P = (3/8 \quad 0 \quad 5/16 \quad 0 \quad 4/16 \quad 0 \quad 1/16)$

$P[X_4 = 0] = 3/8$

iii) P(continue the game after 5 plays)

$= P[X_5 = 1] + P[X_5 = 2] + P[X_5 = 3] + P[X_5 = 4] + P[X_5 = 6]$ - ①

$P[X_5 = x]$

$p^{(5)} = p^{(4)} P = (3/8 \quad 0 \quad 5/16 \quad 0 \quad 4/16 \quad 0 \quad 1/6) P$

$= (3/6 \quad 0 \quad 5/32 \quad 0 \quad 9/32 \quad 0 \quad 4/32 \quad 1/16)$ - ②

Sub in ①

P(continued the game after 5th play) $= 0 + 5/32 + 0 + 9/32 + 0 + 4/32 = 18/32$

8) A fair die is tossed repeatedly and if $\{X_n\}$ represents the maximum of the number occurring in 1^{st} n ~~times~~ trials. Find 'TPM' of the Markov chain, also find P^2 and $P[X_2 = 6]$

X_n - Max. of no occuring in first 'n' trials.

$= \{1, 2, 3, 4, 5, 6\}$

TPM

$$P = \begin{array}{cc|cccccc} & & & & X_{n+1} & & & \\ & & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & 2 & 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ X_n & 3 & 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & 4 & 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ & 5 & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ & 6 & 0 & 0 & 0 & 0 & 0 & \frac{6}{6}\ 1 \end{array}$$

$$P^2 = P \cdot P$$

$$= \frac{1}{36}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$= \frac{1}{36}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

$$P[X_2 = 6] = p^{(2)}$$

$$p^{(2)} = p^{(0)} P^2$$

$$= \frac{1}{36}\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

$$= \frac{1}{216}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\Bigg(\quad '' \quad \Bigg)$$

$$= \frac{1}{216}\begin{pmatrix} 1 & 7 & 19 & 37 & 61 & 91 \end{pmatrix} \Rightarrow P[X_2 = 6] = \frac{91}{216} \;//$$

Q) A house wife buys 3 types of vegetable A, B, C She never buys the same vegetable in successive weeks. If she buys A then the next week she buys B. However if she buys B or C the next week she buys 3 types as likely to A as the other vegetable. In this long run how often does she buy each of the 3 vegetable.

State space = $\{A, B, C\}$

TPM

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 3/4 & 0 & 1/4 \\ C & 3/4 & 1/4 & 0 \end{array}$$

Long run prob

$\pi P = \pi$

$\pi = (\pi_1, \pi_2, \pi_3)$

$\boxed{\pi_1 + \pi_2 + \pi_3 = 1}$ — ①

$\pi P = \pi$

$$(\pi_1, \pi_2, \pi_3)\begin{pmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix} = (\pi_1, \pi_2 \pi_3)$$

$0\pi_1 + \frac{3\pi_2}{4} + \frac{3\pi_3}{4} = \pi_1$

$\pi_1 + 0\pi_2 + \frac{\pi_3}{4} = \pi_2$

$0\pi_1 + \frac{\pi_2}{4} + 0 = \pi_3$

$4\pi_1 - 3\pi_2 - 3\pi_3 = 0$ — ②

$4\pi_1 - 4\pi_2 + \pi_3 = 0$ — ③

$\pi_2 - 4\pi_3 = 0$ — ④

Solving ① ② ③ , ④

$\pi_1 = \frac{15}{35}$ $\pi_2 = \frac{16}{35}$, $\pi_3 = \frac{4}{35}$

The invariant prob is

$\left(15/35 , 16/35 , 4/35 \right)$

Q.) A salesman territory consists of 3 cities A, B, & C

He never sells in same city on successive days. If he sells in city A then, the next day he sell in B. However if he sells neither B or C, then the next day he is twice as likely to sell in A as in other city. How often does he sell in each city in steady state.

State space = {A, B, C}

TPM

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 2/3 & 0 & 1/3 \\ C & 2/3 & 1/3 & 0 \end{array}$$

$A = 2C$

$A + C = 1$

$A - 2C = 0$

$3C = 1$

$C = 1/3$

$A = 2/3$

To find steady state ~~path~~ prob

$$\pi P = \pi \quad , \quad \pi_1 + \pi_2 + \pi_3 = 1$$

$$(\pi_1, \pi_2, \pi_3) \begin{pmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix} = (\pi_1, \pi_2, \pi_3)$$

$$\frac{2\pi_2}{3} + \frac{2\pi_3}{3} = \pi_1 \quad - ①$$

$$\pi_1 + \frac{\pi_3}{3} = \pi_2 \quad - ②$$

$$\frac{\pi_2}{3} = \pi_3 \quad - ③$$

By Total Prob.

$$\pi_1 + \pi_2 + \pi_3 = 1 \quad - ④$$

$$3\pi_1 - 2\pi_2 - 2\pi_3 = 0$$

$$3\pi_1 - 3\pi_2 + \pi_3 = 0$$

$$\pi_2 - 3\pi_3 = 0$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

solving ① ② ③ ④

$(\pi_1, \pi_2, \pi_3) = (1/4, 2/3, 1/12)$

Irreducible (Non - null persistent)

If $P_{ij}^{(n)} > 0$ for some n, for all i, j

i.e if every state can be accessible (reached) from every other state, then the Markov chain is called Irreducible (Non-null persistent) Markov chain.

~~Retat~~ Return state

If $P_{ii}^{(n)} > 0$ for $n \geq 1$ then the state 'i' is called as return state.

Regular Markov chain

$P_{ij}^{(n)} \geq 0$ for some n

Transcient or Recurrent

If $F_{ii} = \sum_{i} P_{ii}^{(n)} = 1$ then the state 'i' is called transcient state.

NOTE : For non - null persistant,

Mean $= \mu_{ii} = \sum_{i} n P_{ii}^{(n)}$ is finite ~~periodic~~

Periodic or (Period of order n)

Let $d_i = \text{Gcd} \{ n, P_{ii}^{(n)} \}$

If $d_i = 1$ then the state 'i' is aperiodic or (period of order 1)

If $d_i = n$, then the state 'i' is periodic of order n.

Ergodic

. A finite, Irreducible and aperiodic Markov chain is called Ergodic.

NOTE : A non null persistent, finite and aperiodic Markov chain is also Ergodic Markov chain.

Q) Three boys A, B and C throwing a ball to each other. A always throw the ball to B and B always throws the ball to C but C is just as likely to throw the ball to B as to A. Show that the process is Markovian and find the TPM and classify the state.

12m

State space = {A, B, C}

TPM

$$P = X_n \begin{array}{c|ccc} & \multicolumn{3}{c}{X_{n+1}} \\ & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/2 & 1/2 & 0 \end{array}$$

Since X_{n+1} depends on X_n, the process is Markovian.

$$P^2 = P.P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

$$P^3 = P^2 . P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

$$P^4 = P^3 . P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$$

$$P^5 = P^4 . P = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{pmatrix}$$

Irreducible : since $P_{ij}^{(n)} > 0$ for some n the given Markov chain is irreducible. Since the states {A, B, C} is finite, it is a finite, non null persistant Markov chain.

To find Period of states

For State A,

$P_{AA}^{(1)} = 0,\ P_{AA}^{(2)} = 0,\ P_{AA}^{(3)} = 1/2 > 0,\ P_{AA}^{(4)} = 0,\ P_{AA}^{(5)} = 1/4 > 0$

period $= \text{Gcd}\{3, 5, \ldots\} = 1$

State A is aperiodic

State B

$P_{BB}^{(1)} = [illegible] > 0,\ P_{BB}^{(2)} = 1/2,\ P_{BB}^{(3)} = 1/2 > 0,\ P_{BB}^{(4)} = 1/2 > 0,$

$P_{BB}^{(5)} = 1/4 > 0 \ldots.$

Period $= \text{Gcd}\{2, 3, 4, 5 \ldots\} = 1$

state B is aperiodic.

statec

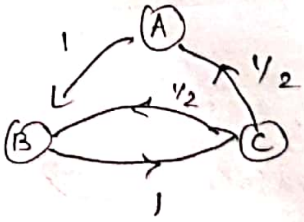
$P_{cc}^{(1)} = 0,\ P_{cc}^{(2)} = 1 > 0,\ P_{cc}^{(3)} = 0,\ P_{cc}^{(4)} = 1/2 > 0,\ P_{cc}^{(5)} = 1/4 > 0$

Period $= \text{Gcd}\{2, 4, 5, \ldots\} = 1$

State c is aperiodic.

Since each state is irreducible, finite, non null the given Markov chain is persistant and aperiodic Ergodic.

Transition diagram



9) Find the nature of the states of M C

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 0 & 1 & 0 \end{array}$$

$$P^2 = P.P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} = P$$

$$P^4 = P^3 . P = P . P = P^2$$

$$P^5 = P^4 . P = P^2 . P = P^3 = P$$

$$P = P^3 = P^5 = P^7 \dots, \quad P^2 = P^4 = P^6 = \dots\dots$$

Since $P_{ij}^{(n)} > 0$ for some n, the given MC is irreducible. Also it has only three state, it is finite and non null persistant.

Period of states

For state '0'

$$P_{00}^{(1)} = 0, \; P_{00}^{(2)} = 1/2 > 0, \; P_{00}^{(3)} = 0, \; P_{00}^{(4)} = 1/2 > 0 \dots$$

Period = Gcd $\{2, 4, 6 \dots\} = 2$

Period of state '0' is 2.

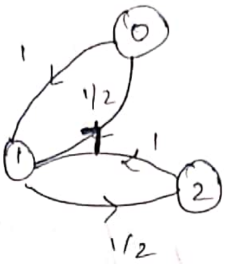
For state '1'

$$P_{11}^{(1)} = 0, \; P_{11}^{(2)} = 1/2 > 0, \; P_{11}^{(3)} = 0 \dots\dots$$

Period = Gcd $\{2, 4, 6 \dots\} = 2$

Similarly. the state '2' is of period 2. Since the states are not aperiodic, the given MC is not Ergodic.

Transition diagram



Q) The transition probability matrix of a MC $\{X_n\}$, $n = 1, 2, 3 \ldots$

Ex) having three states 1, 2, 3 is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

and the initial probability is

$$p^{(0)} = [0.7, 0.2, 0.1]$$

i) Find $P[X_2 = 3]$ ii) $P[X_2 = 3 / X_0 = 1]$

iii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

i) $$P^2 = P.P = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$P[X_2 = 3] = P^2$

$= p^{(0)} \cdot$ $\quad P^2 = (0.7 \quad 0.2 \quad 0.1) \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$

$= (0.38 \quad 0.33 \quad 0.28)$

$\therefore P[X_2 = 3] = 0.28$

ii) $P[X_2 = 3 / X_0 = 1] = p_{13}^{(2)} = 0.26$

iii) $P[\underbrace{X_3 = 2}_{A}, \underbrace{X_2 = 3, X_1 = 3, X_0 = 2}_{B}]$

$= P[X_3 = 2 / X_2 = 3, X_1 = 3, X_0 = 2] \, P(X_2 = 3, X_1 = 3, X_0 = 2)$

$= P[x_3 = 2 | x_2 = 3] \; P[x_2 = 3 / x_1 = 3] \; P[x_1 = 3 / x_0 = 2] \; P[x_0 = 2]$

$= p_{32}^{(1)} \; p_{33}^{(1)} \; p_{23}^{(1)} (x_0 = 2) = (0.4)(0.3)(0.2)(0.2)$

$= 0.00p48$ //

Q) Suppose that the probability of a die dry day (state 0) following a rainy day (State 1) is 1/3, and probability of a rainy day following a dry day is 1/2. Given that May 1 is a dry day. find probability that i) May 3 is a dry day ii) May 5 is also a dry day.

States = {D, R}

TPM

$$P = \begin{array}{c c} & \begin{array}{cc} D & R \end{array} \\ \begin{array}{c} D \\ R \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

$p^{(0)} = (1, 0)$

$$P^2 = P.P = \begin{pmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} \frac{27}{72} & \frac{43}{72} \\ \frac{43}{108} & \frac{65}{108} \end{pmatrix}$$

$$P^4 = \begin{pmatrix} \frac{173}{432} & \frac{259}{432} \\ \frac{259}{648} & \frac{384}{648} \end{pmatrix}$$

i) May 3 is a dry day

$$p^{(2)} = p^{(0)} P^2 = (1 \; 0) \begin{pmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{pmatrix}$$

$$= (5/12 \quad 7/12)$$

P(May 3 is a dry day) = 5/12

(ii) May 5 is a dry day

$$p^{(4)} = p^{(0)} p^4 = (1\ 0) \begin{pmatrix} \frac{173}{432} & \frac{259}{432} \\ \frac{259}{648} & \frac{384}{648} \end{pmatrix}$$

$$= \left(\frac{173}{432} , \frac{259}{432} \right)$$

$$P(\text{May 5 is a dry day}) = \frac{173}{432}$$

Part B

1) Find invariant (Steady state) probability of the MC $\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$

The steady state probability is

$$\pi P = \pi , \quad \pi_1 + \pi_2 = 1 \quad - ③$$

$$(\pi_1 , \pi_2) \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix} = (\pi_1 , \pi_2)$$

$$\frac{\pi_1}{2} + \frac{\pi_2}{3} = \pi_1 \Rightarrow 3\pi_1 + 2\pi_2 = 6\pi_1 \quad ①$$

$$\frac{\pi_1}{2} + \frac{2\pi_2}{3} = \pi_2 \Rightarrow 3\pi_1 + 4\pi_2 = 6\pi_2 \quad - ②$$

① $\Rightarrow 3\pi_1 - 2\pi_2 = 0$

② $\Rightarrow 3\pi_1 - 2\pi_2 = 0$

③ $\Rightarrow \pi_1 + \pi_2 = 1$

$$3\pi_1 - \cancel{2\pi_2} = 0$$
$$2\pi_1 + \cancel{2\pi_2} = 2$$
$$5\pi_1 = 2$$
$$\pi_1 = 2/5$$
$$\pi_2 = 3/5$$

Solving $\pi_1 = 2/5$, $\pi_2 = 3/5$

The steady state prob is

$$(\pi_1 , \pi_2) = (2/5 , 3/5)$$