Department of Mathematics

Sub Title: PROBABILITY AND QUEUING THEORY

Sub Code: 18MAB 204T

Unit -I - Probability and Random Variables

1. The Random variable which can take infinite number of values in an interval is called ……..
   (a) Random Variable (b) Continuous R.V (c) Discrete R.V (d) Range space

   Ans: (b)

2. A random variable is said to be discrete if

   a) its possible values can be counted or listed b) it can assume any real number within an interval

   c) the rules of probability apply d) it can be represented graphically **Ans: (a)**

3. The conditions satisfied by the pdf are ……

   (a) $p(x) \geq 0 \,\&\, \sum p(x) = 1$ (b) $f(x) \geq 0 \,\&\, \int_{-\infty}^{\infty} f(x)dx = 1$

   (c) $p(x) \leq 0 \,\&\, \sum p(x) = 0$ (d) $f(x) \leq 0 \,\&\, \int_{-\infty}^{\infty} f(x)dx = 1$ **Ans: (b)**

4. A ________ describes the number of occurrences of an event over a specified interval of time or space.

   a) Binomial random variable b) Poisson random variable

   c) Discrete random variable d) Continuous random variable **Ans: (d)**

5. The cumulative distribution function F(x) is a ……… function of X.
   (a) Increasing (b) non- increasing (c) non- decreasing (d) decreasing **Ans: (c)**

6. After a coin is tossed 2 times. If X is the number of heads, what is the value of $P(x = 0)$?

   a) 0 b) 1 c) $\frac{1}{4}$ d) $\frac{1}{2}$ **Ans: (c)**

7. If X is a continuous R.V, then $\frac{d}{dx}F(x) = f(x)$ at all points here F(x) is—

   (a) integrable (b) Constant (c) 1 (d) Differentiable

   Ans: (d)

8. The value of 'k' from the following table is------

x	-2	-1	0	1	2	3
p(x)	0.1	k	0.2	2k	0.3	3k

(a) 1 (b) $\frac{1}{10}$ (c) $\frac{1}{15}$ (d) $\frac{2}{3}$ **Ans: (c)**

9. The value of $P(1/2\, X < 2/3)$ from the above table is ----

(a) 0.5 (b) ∞ (c) undefined (d) 1 **Ans: (c)**

10.From the above table the value of P(X<2) is

(a) 1 (b) $\frac{1}{15}$ (c) $\frac{1}{2}$ (d) $\frac{1}{30}$ **Ans: (c)**

11. Consider a discrete random variable X that can assume the values 0,1, and 2 with probabilities 0.2,0.5, and 0.3 respectively. What is the expected value of X? a) 1.1 b) 1 c) 0.9 d) 1.3 **Ans: (a)**

12. The Relation between Variance and Standard deviation is ------

(a) $\text{var} = S.D^2$ (b) $\text{var} = \sqrt{S.D}$

(c) $\text{var} - S.D = 0$ (d) $\text{var} = \sqrt[2]{S.D}$ **Ans: (a)**

13. The Relation between Covariance and Mean is ------

(a) $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$ (b) $\text{cov}(X,Y) = E(XY) + E(X)E(Y)$

(c) $\text{cov}(X,Y) = E(XY) - (E(X)E(Y))^2$ (d) $\text{cov}(X,Y) = E(XY)^2 - (E(X)E(Y))^2$ **Ans: (a)**

14. The value of k if the pdf $f(x) = kx^2e^{-x}$, $x \geq 0$ is --------

(a) 0.5 (b) ∞ (c) 0 (d) 1 **Ans: (a)**

15. Given that the p.d.f of a R.V. X is $f(x) = kx, 0 < x < 1$. What is the value of k ?

a) 0 b) 1 c) 2 d) 3 **Ans: (c)**

16. If x is random variable then $E(ax+b) =$_____

a) $aE(x)$ b) $aE(x)+b$ c) $a^2E(x)$ d) $a^2E(x)+b$ **Ans: (b)**

17. The Relation between Variance and Mean is ------

(a) $VarX = E(x) - (E(x))^2$ (b) $VarX = E(x^2) - (E(x))^2$

(c) $VarX = (E(x))^2 - E(x^2)$ (d) $VarX = (E(x))^2 - E(x)$ **Ans: (b)**

18. If $Var(x) = 4$ find $Var(3x+8)$, where X is a random variable.

a) 11 b) 12 c) 24 d)36. **Ans: (d)**

19. The functions of R.V $Y = g(x)$ is given by

(a) $h(y) = f(x)\left|\frac{dx}{dy}\right|$ (b) $h(x) = f(y)\left|\frac{dx}{dy}\right|$

(c) $h(y) = f(x)\left|\frac{dy}{dx}\right|$ (d) $h(x) = f(y)\left|\frac{dy}{dx}\right|$ **Ans: (a)**

20. Let x be a random variable with density function $f_X(x) = 2x, 0 < x < 1$. If $y = 3x + 6$, then the range of y is

a) $(0,1)$ b) $(1,3)$ c) $(3,6)$ d) $(6,9)$ **Ans: (d)**

21. The generalized form of Tchebycheff's inequality is ------------

(a) $P[|X-\mu| < k\sigma] = 1 - \frac{1}{k^2}$ (b) $P[|X-\mu| > k\sigma] = 1 - \frac{1}{k^2}$

(c) $P[|X-\mu| < k\sigma] = \frac{1}{k^2}$ (d) $P[|X-\mu| > k\sigma] = \frac{1}{k^2}$ **Ans: (a)**

22. The conditions satisfied by the pmf is ……

(a) $p(x) \geq 0 \;\&\; \sum p(x) = 1$ (b) $f(x) \geq 0 \;\&\; \int_{-\infty}^{\infty} f(x)dx = 1$

(c) $p(x) \leq 0 \;\&\; \sum p(x) = 0$ (d) $f(x) \leq 0 \;\&\; \int_{-\infty}^{\infty} f(x)dx = 1$ **Ans: (a)**

23. If Var(X) = 4, then Var(4X+5) is

(a)89 (b) 69 (c) 64 (d) 9 **Ans(c)**

24. If X and Y are independent random variables with Var 2 and 3 respectively, Then

Var(3X+4Y) is (a) 66 (b) 7 (c) 25 (d) 18 **Ans: (a)**

25. If E(X) =3 ,then E(3X+4) is

(a) 15 (b) 13 (c) 9 (d) 10 **Ans: (b)**

26. Var(aX) is

(a) aVar(X) (b) a^2 Var(X) (c) Var(X) (d) 0 **Ans: (b)**

27. Var (aX+b) =

(a) aVar(X)+b (b) a^2 Var(X) (c)aVar(X) (d)Var(X) **Ans: (b)**

28. Suppose X is a continuous random variable with probability density function $f(x)$. The moment generating

function (MGF) of X is

a) $M_X(t) = E(e^{tX}) = \sum_{x=-\infty}^{\infty} e^{tX} f(x)$ b) $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} f(x)\,dx$

c) $M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tX} f(x)$ d) $M_X(t) = E(e^{tX}) = \int_{0}^{\infty} e^{tX} f(x)\,dx$ **Ans: (b)**

PART-B

1. If a random variable X takes the values 1,2,3,4 such that P(X=1)=3P(X=2)=P(X=3)=5P(X=4). Find the probability distribution of X 2

Solution: Assume P(X=3) = α By the given equation $P(X=1) = \frac{\alpha}{2}$ $P(X=2) = \frac{\alpha}{3}$ $P(X=4) = \frac{\alpha}{5}$

For a probability distribution (and mass function) $\sum P(x) = 1$ P(1)+P(2)+P(3)+P(4) =1

$$\frac{\alpha}{2} + \frac{\alpha}{3} + \alpha + \frac{\alpha}{5} = 1 \quad \Rightarrow \quad \frac{61}{30}\alpha = 1 \quad \Rightarrow \alpha = \frac{30}{61}$$

$$P(X=1) = \frac{15}{61}\ ; P(X=2) = \frac{10}{61};\ P(X=3) = \frac{30}{61};\ P(X=4) = \frac{6}{61}$$

The probability distribution is given by

X	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

2. Let X be a continuous random variable having the probability density function

$$f(x) = \begin{cases} \frac{2}{x^3}, & x \ge 1 \\ 0, & otherwise \end{cases}$$ Find the distribution function of x.

Solution:

$$F(x) = \int_1^x f(x)\ dx = \int_1^x \frac{2}{x^3} dx = \left[-\frac{1}{x^2}\right]_1^x = 1 - \frac{1}{x^2}$$

3. A random variable X has the probability density function *f(x)* given by

$$f(x) = \begin{cases} cx\,e^{-x}, & x \ge 0 \\ 0, & otherwise \end{cases}$$. Find the value of c and CDF of X.

Solution:

$$\int_0^{\infty} f(x)\, dx = 1$$

$$\int_0^{\infty} cx e^{-x}\, dx = 1$$

$$c\left[-x e^{-x} - e^{-x}\right]_0^{\infty} = 1$$

$$c(1) = 1$$

$$c = 1$$

$$F(x) = \int_0^{x} f(x)\, dx$$

$$= \int_0^{x} cx e^{-x}\, dx$$

$$= \int_0^{x} x e^{-x}\, dx$$

$$= \left[-x e^{-x} - e^{-x}\right]_0^{x}$$

$$= 1 - x e^{-x} - e^{-x}$$

4. A continuous random variable X has the probability density function *f(x)* given by $f(x) = ce^{-|x|}, -\infty < x < \infty$. Find the value of c and CDF of X.

Solution:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} c e^{-|x|}\, dx = 1$$

$$2\int_0^{\infty} c e^{-|x|}\, dx = 1$$

$$2\int_0^{\infty} c e^{-x}\, dx = 1$$

$$2c\left[-e^{-x}\right]_0^{\infty} = 1$$

$$2c(1) = 1$$

$$c = \frac{1}{2}$$

$Case(i)\ x < 0$

$$F(x) = \int_{-\infty}^{x} f(x)\ dx$$

$$= \int_{-\infty}^{x} c\, e^{-|x|}\ dx$$

$$= c \int_{-\infty}^{x} e^{x}\ dx$$

$$= c\left[\ e^{x}\ \right]_{-\infty}^{x}$$

$$= \frac{1}{2} e^{x}$$

$Case(ii)\ x > 0$

$$F(x) = \int_{-\infty}^{x} f(x)\ dx$$

$$= \int_{-\infty}^{x} c\, e^{-|x|}\ dx$$

$$= c \int_{-\infty}^{0} e^{x}\ dx + c\int_{0}^{x} e^{-x} dx$$

$$= c\left[\ e^{x}\ \right]_{-\infty}^{0} + c\left[-e^{-x}\right]_{0}^{x}$$

$$= c - ce^{-x} + c$$

$$= c\left(2 - e^{-x}\right)$$

$$= \frac{1}{2}\left(2 - e^{-x}\right)$$

$$F(x) = \begin{cases} \frac{1}{2} e^{x}, & x > 0 \\ \frac{1}{2}\left(2 - e^{-x}\right), & x < 0 \end{cases}$$

5. If a random variable has the probability density $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & otherwise \end{cases}$. Find the probability that it will take on a value between 1 and 3. Also, find the probability that it will take on value greater than 0.5.

Solution:

$$P(1 < X < 3) = \int_1^3 f(x)\, dx = \int_1^3 2e^{-2x} dx = \left[-e^{-2x}\right]_1^3 = e^{-2} - e^{-6}$$

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\, dx = \int_{0.5}^{\infty} 2e^{-2x} dx = \left[-e^{-2x}\right]_{0.5}^{\infty} = e^{-1}$$

6. Is the function defined as follows a density function?

$$f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3+2x), & 2 \leq x \leq 4 \\ 0, & x > 4 \end{cases}$$

Solution:

$$\int_2^4 f(x)\ dx = \int_2^4 \frac{1}{18}(3+2x)\ dx = \left[\frac{(3+2x)^2}{72}\right]_2^4 = 1$$

Hence it is density function.

7. The cumulative distribution function (CDF) of a random variable X is $F(X) = 1-(1+x)e^{-x}$, $x > 0$. Find the probability density function of X.

Solution:

$$\begin{aligned} f(x) &= F'(x) \\ &= 0 - \left[(1+x)\left(-e^{-x}\right) + (1)\left(e^{-x}\right)\right] \\ &= xe^{-x}, \quad x > 0 \end{aligned}$$

8. The number of hardware failures of a computer system in a week of operations has the following probability mass function:

No of failures:	0	1	2	3	4	5	6
Probability :	0.18	0.28	0.25	0.18	0.06	0.04	0.01

Find the mean of the number of failures in a week.

Solution:

$$\begin{aligned} E(X) = \sum x\ P(x) &= (0)(0.18) + (1)(0.28) + (2)(0.25) + (3)(0.18) + \\ &\quad (4)(0.06) + (5)(0.04) + (6)(0.01) \\ &= 1.92 \end{aligned}$$

9. Given the p.d.f of a continuous r.v X as follows: $f(x) = \begin{cases} 6x(1-x), & 0 \leq x \leq 1 \\ 0, & elsewhere \end{cases}$ Find the CDF of X.

Solution:

$$F(x) = \int_0^x f(x)\, dx = \int_0^x 6x(1 - x)\, dx = \int_0^x 6x - 6x^2\, dx = \left[3x^2 - 2x^3\right]_0^x = 3x^2 - 2x^3$$

10. A continuous random variable X has the probability function $f(x) = k(1+x), \quad 2 \le x \le 5$. Find P(X<4).

Solution:

$$\int_2^4 f(x)dx = 1 \qquad \Rightarrow k\int_2^5 (1+x)dx = 1$$

$$\Rightarrow k\left[\frac{(1+x)^2}{2}\right]_2^5 = 1$$

$$\Rightarrow k\,\frac{27}{2} = 1$$

$$\Rightarrow k = \frac{2}{27}$$

$$P(X < 4) = \int_2^4 f(x)\, dx = \frac{2}{27}\int_2^4 (1+x)\, dx = \frac{2}{27}\left[\frac{(1+x)^2}{2}\right]_2^4 = \frac{1}{25}(25-9) = \frac{16}{27}$$

11. Given the p.d.f of a continuous R.V X as follows: .

$$f(x) = \begin{cases} 12.5x - 1.25 & 0.1 \le x \le 0.5 \\ 0, & elsewhere \end{cases}$$

Find P(0.2 < X < 0.3)

Solution:

$$P(0.2 < X < 0.3) = \int_{0.2}^{0.3} (12.5x - 1.25)\, dx$$

$$= \left[12.5\frac{x^2}{2} - 1.25x\right]_{0.2}^{0.3}$$

$$= 1.25\left[5(0.3)^2 - 0.3 - 5(0.2)^2 + 0.2\right]$$

$$= 0.1875$$

12. If the MGF of a continuous R.V X is given by $M_X(t) = \dfrac{3}{3-t}$. Find the mean and variance of X.

Solution:

$$M_X(t)=\frac{3}{3-t}=\frac{1}{1-\frac{t}{3}}=\left(1-\frac{t}{3}\right)^{-1}=1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots$$

$$E(X)=(coefficient\ \ of\ \ t)\ 1!=\frac{1}{3}\ \ is\ \ the\ \ mean$$

$$E(X^2)=\left(coefficient\ of\ t^2\right)2!=\frac{1}{9}2!=\frac{2}{9}$$

$$Variance\ \ =E(X^2)-E(X)^2=\frac{2}{9}-\frac{1}{9}=\frac{1}{9}$$

13. If the MGF of a discrete R.V X is given by $M_X(t)=\frac{1}{81}\left(1+2e^t\right)^4$,find the distribution of X.

Solution:

$$M_X(t)=\frac{1}{81}\left(1+2e^t\right)^4=\frac{1}{81}\left(1+4C_1\left(2e^t\right)+4C_2\left(2e^t\right)^2+4C_3\left(2e^t\right)^3+4C_4\left(2e^t\right)^4\right)$$

$$=\frac{1}{81}+\frac{8}{81}e^t+\frac{24}{81}e^{2t}+\frac{32}{81}e^{3t}+\frac{16}{81}e^{4t}$$

By definition of MGF ,

$$M_X(t)=\sum e^{tx}p(x)=p(0)+p(1)e^t+p(2)e^{2t}+p(3)e^{3t}+p(4)e^{4t}$$

On comparison with above expansion the probability distribution is

X	0	1	2	3	4
$p(x)$	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$

14. Find the MGF of the R.V X whose p.d.f is $f(x)=\begin{cases}\frac{1}{10}, & 0<x<10\\ 0, & elsewhere\end{cases}$ .Hence its mean.

Solution:

$$M_X(t)=\int_0^{10}\frac{1}{10}e^{tx}\,dx$$

$$=\frac{1}{10}\left(\frac{e^{tx}}{t}\right)_0^{10}$$

$$=\frac{1}{10}\left(\frac{e^{10t}-1}{t}\right)$$

$$=\frac{1}{10t}\left(1+10t+\frac{100t^2}{2!}+\frac{1000t^3}{3!}+\ldots.-1\right)$$

$$=1+5t+\frac{1000}{31}t^2+\ldots..$$

$$Mean = coefficient\ \ of\ \ t = 5$$

15. . Given the r.v X with density function $f(x)=\begin{cases}2x & , 0<x<1\\ 0 & , elsewhere\end{cases}$

Find the pdf of $y=8x^3$

Soln: The pdf of y is given by

$$f_Y(y)=f_X(x)\left|\frac{dx}{dy}\right| \qquad \text{Where } y=8x^3$$

$$x^3=\frac{y}{8}\Rightarrow x=\left(\frac{y}{8}\right)^{\frac{1}{3}}$$

$$\frac{dx}{dy}=\frac{1}{3}\left(\frac{y}{8}\right)^{\frac{1}{3}-1}\frac{1}{8}=\frac{1}{24}\left(\frac{y}{8}\right)^{\frac{-2}{3}}$$

$$f_Y(y)=2x\,\frac{1}{24}\left(\frac{y}{8}\right)^{\frac{-2}{3}}$$

$$=\frac{2}{24}\left(\frac{y}{8}\right)^{\frac{1}{3}}\left(\frac{y}{8}\right)^{\frac{-2}{3}}=\frac{2}{24}\left(\frac{y}{8}\right)^{\frac{-1}{3}} \qquad 0<x<1\Rightarrow 0<y<8$$

$$=\frac{1}{12}\left(\frac{8}{y}\right)^{\frac{1}{3}}=\frac{1}{6}(y)^{\frac{-1}{3}}\quad ,$$

$$f_Y(y) = \frac{1}{6}(y)^{\frac{-1}{3}} \ , 0 < y < 8$$

17. If the pdf of X is $f_X(x) = 2x \ , 0 < x < 1$, then find the pdf of Y = 3 x + 1

Soln: The pdf of Y is given by $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$, where Y = 3 x + 1

$$Y = 3x + 1 \Rightarrow x = \frac{y-1}{3}$$

$$\frac{dx}{dy} = \frac{1}{3} \Rightarrow \left|\frac{dx}{dy}\right| = \frac{1}{3}$$

$$\therefore \ f_Y(y) = 2x\,\frac{1}{3} = \frac{2}{3}\left(\frac{y-1}{3}\right) = \frac{2}{9}(y-1) \ , 1 < y < 4$$

18..If X has an exponential distribution with parameter α, find the pdf of y = log x

Soln: The pdf of exponential distribution is $f(x) = \alpha \ e^{-\alpha x}$,

The pdf of Y is given by $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$ where y = log x

y = log x $\Rightarrow x = e^y$, $\frac{dx}{dy} = e^y \Rightarrow \left|\frac{dx}{dy}\right| = e^y$

$$\therefore \ f_Y(y) = \alpha \ e^{-\alpha x} e^y ,$$
$$\Rightarrow f_Y(y) = \alpha \ e^{-\alpha e^y} e^y , -\infty < y < \infty$$

19. If $Y = x^2$, where x is a Gaussian r.v. with zero mean and variance σ^2 ,find the

Pdf of the variable Y

Soln: $F_Y(y) = p(Y \le y) = p\left[x^2 \le y\right]$

$$= p\left[-\sqrt{y} \le x \le \sqrt{y}\right], \ if \ y \ge 0$$
$$= F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right) \quad ----------(1)$$

and $F_Y(y) = 0 \quad if \quad y < 0$

Differentiating (1) with respect to y , we have

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{y}}\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right] & if \quad y \geq 0 \\ 0 & if \quad y < 0 \end{cases} \qquad ---------(2)$$

It is given that X follows $N(0,\sigma)$ $\quad \therefore \quad f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \quad , -\infty < x < \infty$

Using this value in (2) , we have

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y}{2\sigma^2}} + \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y}{2\sigma^2}}\right]$$

$$= \frac{1}{2\sqrt{y}} \; \frac{2}{\sigma\sqrt{2\pi}} e^{-\frac{y}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi y}} e^{-\frac{y}{2\sigma^2}} ,$$

$$\therefore \qquad f_Y(y) = \frac{1}{\sigma\sqrt{2\pi y}} e^{-\frac{y}{2\sigma^2}} , y > 0$$

20. If X is a Gaussian r.v. with zero mean and variance σ^2 ,find the Pdf of $Y = |x|$

Soln: $F_Y(y) = p[Y \leq y] = p\;[|x| \leq y]$

$$p[-y \leq x \leq y]$$

$$F_Y(y) = F_X(y) - F_X(-y) \quad --------(1)$$

Differentiating (1) both sides w.r.t. y , we have

$$f_Y(y) = f_X(y) + f_X(-y) \quad , y > 0 \quad ----(2)$$

Since $X \to N(0 \;, \sigma)$, the density function is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \quad , -\infty < x < \infty$$

$$(2) \Rightarrow f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} + \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}}$$

$$\Rightarrow f_Y(y) = \frac{2}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \quad , \quad y > 0.$$

21.If X is a r.v. with cdf as F(x) , show that the r.v.Y = F(x) is uniformly distributed in (0,1)

Soln: The pdf of Y is given by $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$ where Y = F(x)

Y = F(x) $\frac{dy}{dx} = \frac{d}{dx}F(x) = f(x)$

Then $f_Y(y) = f(x).\frac{1}{f(x)} = 1$

So , Y is uniformly distributed in (0,1).

PART-C

1. The density function of a random variable X is given by $f(x) = kx(2-x), \quad 0 \le x \le 2$. Find k, mean , variance and r^{th} moment.

$$\int_0^2 f(x)\,dx = 1 \qquad \int_0^2 kx(2-x)\,dx = 1$$

Solution: $k\int_0^2 \left(2x - x^2\right) dx = 1 \qquad k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$

$$k\left(4 - \frac{8}{3}\right) = 1 \qquad k = \frac{3}{4}$$

$$\mu_r' = \int_0^2 x^r \frac{3}{4}x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 \left(2x^{r+1} - x^{r+2}\right) dx$$

$$= \frac{3}{4}\left[2\frac{x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2 \qquad = \frac{3}{4}\left[\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$= \frac{3}{4}2^{r+3}\left[\frac{1}{r+2} - \frac{1}{r+3}\right] \qquad = 6\left(2^r\right)\frac{1}{(r+2)(r+3)}$$

put $r = 1,2$ $\quad \mu_{1'} = \frac{12}{(3)(4)} = 1 \quad \mu_2{}' = \frac{24}{(4)(5)} = \frac{6}{5}$

Mean = 1 and variance = $\mu_{2'} - \mu_{1'^2} = \frac{6}{5} - 1 = \frac{1}{5}$

2. The monthly demand for Allwyn watches is known to have the following probability distribution.

Demand:	1	2	3	4	5	6	7	8
Probability:	0.08	0.3k	0.19	0.24	k^2	0.1	0.07	0.04

Determine the expected demand for watches. Also, compute the variance.

Solution:

$$\sum P(x) = 1$$

$$(0.08) + (0.3k) + (0.19) + (0.24) + (k^2) + (0.1) + (0.07) +)(0.04) = 1$$

$$k^2 + 0.3k - 0.28 = 0 \quad \Rightarrow k = 0.4$$

$$E(X) = \sum x\ P(x) = (1)(0.18) + (2)(0.12) + (3)(0.19) +$$
$$(4)(0.24) + (5)(0.16) + (6)(0.1) + (7)(0.07) + (8)(0.04)$$
$$= 4.02 \ \ is \ \ the \ \ mean$$

$$E(X^2) = \sum x^2\ P(x) = (1)(0.18) + (4)(0.12) + (9)(0.19) +$$
$$(16)(0.24) + (25)(0.16) + (36)(0.1) + (49)(0.07) + (64)(0.04)$$
$$= 19.7$$

$$Variance \ = E(X^2) - E(X)^2 = 19.07 - 4.02^2 = 3.54$$

3. The distribution of a random variable X is given by $F(X) = 1 - (1+x)e^{-x}$, $x > 0$.Find the r^{th} moment, mean and variance.

Solution:

$$f(x) = F'(x)$$

(i) $$= 0 - \left[(1+x)\left(-e^{-x}\right) + (1)\left(e^{-x}\right)\right]$$

$$= xe^{-x}, \quad x > 0$$

(ii)

$$E\left(X^r\right) = \mu_r{}' = \int_0^\infty x^r\ f(x)\ dx$$
$$= \int_0^\infty x^r\ xe^{-x}\ dx$$
$$= \int_0^\infty x^{r+1}\ e^{-x}\ dx$$
$$= (r+1)!$$

(iii) $E(X) = \mu_1' = (1+1)! = 2$

(iv) $E\left(X^2\right) = \mu_2' = (2+1)! = 6$ Variance = $E\left(X^2\right) - E(X)^2$ =2

4. Suppose that the duration 'X' in minutes of long distance calls from your home, follows exponential law with p.d.f $f(x) = \frac{1}{5} e^{-\frac{x}{5}}, \ x > 0$. Find p(X > 5), p(3≤X≤6),mean and variance.

Solution:

(i) $p(X > 5) = \int_5^\infty f(x)\, dx = \int_5^\infty \frac{1}{5} e^{-\frac{x}{5}} dx = \left[-e^{-\frac{x}{5}} \right]_5^\infty = e^{-1}$

(ii) $p(3 < X < 6) = \int_3^6 f(x)\, dx = \int_3^6 \frac{1}{5} e^{-\frac{x}{5}} dx = \left[-e^{-\frac{x}{5}} \right]_3^6 = -e^{-1.2} + e^{-0.5}$

(iii)

$$E(X) = \int_0^\infty x f(x)\, dx = \int_0^\infty \frac{1}{5} x e^{-\frac{x}{5}} dx$$

$$= \frac{1}{5}\left[-x e^{-\frac{x}{5}} 5 - e^{-\frac{x}{5}} 25 \right]_0^\infty$$

$$= \frac{1}{5}(0 + 25) = 5$$

(iv) $E(X^2) = \int_0^\infty x^2 f(x)\, dx = \int_0^\infty \frac{1}{5} x^2 e^{-\frac{x}{5}} dx$

$$= \frac{1}{5}\left[-x^2 e^{-\frac{x}{5}} 5 - 2x e^{-\frac{x}{5}} 25 + 2e^{-\frac{x}{5}} 125 \right]_0^\infty$$

$$= \frac{1}{5}(0 + 250) = 50$$

Variance = $E\left(X^2\right) - E(X)^2 = 50 - 25 = 25$

5. A random variable X has the following probability distribution.

X:	0	1	2	3	4	5	6	7
P(X=x):	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) the value of k (ii) $p(1.5 < X < 4.5 \mid X > 2)$

(iii) the smallest value of λ such that $p(X \le \lambda) > \frac{1}{2}$.

Solution:

$$\sum P(x) = 1$$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0 \quad \Rightarrow k = -1, \frac{1}{10}$$

$$k = \frac{1}{10} = 0.1$$

$$A = 1.5 < X < 4.5 = \{2,3,4\}$$

$$B = X > 2 = \{3,4,5,6,7\}$$

(ii) $A \cap B = \{3,4\}$

$$p(1.5 < X < 4.5 \mid X > 2) = p(A \mid B) = \frac{p(A \cap B)}{p(B)} = \frac{p(3,4)}{p(3,4,5,6,7)}$$

$$= \frac{2k + 3k}{2k + 3k + k^2 + 2k^2 + 7k^2 + k} = \frac{5k}{10k^2 + 6k} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

(iii)

X	0	1	2	3	4	5	6	7
p(X)	0	0.1	2k = 0.2	2k = 0.2	3k=0.3	k^2=0.01	$2k^2$ = 0.02	$7k^2$+k = 0.17
F(X)	0	0.1	0.3	0.5	0.8	0.81	0.83	1

From the table for X = 4,5,6,7 F(X) > $\frac{1}{2}$ and the smallest value is 4

Therefore $\lambda = 4$.

6. Find the MGF of triangular distribution whose density function is given by $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & elsehwere \end{cases}$ .Hence its mean and variance.

Solution:

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^1 e^{tx}\, x\, dx + \int_1^2 e^{tx}(2-x)\, dx$$

$$= \left[x\frac{e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$M_X(t) = \frac{e^{2t} - 2e^t + 1}{t^2}$$

expanding the above in powers of t, we get

$$M_X(t) = \frac{e^{2t} - 2e^t + 1}{t^2} = \frac{1}{t^2}\left[\left(1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \dots\right) - 2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + .\right) - 1\right]$$

$$= \frac{1}{t^2}\left(\frac{2t^2}{2!} + \frac{6t^3}{3!} + \frac{14t^4}{4!} + \dots\right)$$

$$= 1 + t + \frac{7t^2}{12} + \frac{t^3}{4} + \dots.$$

Mean = E(X) = (coefficient of t) 1! = 1

$E(X^2)$ = (coefficient of t^2)2! = $\frac{7}{6}$

Variance = $E(X^2) - E(X)^2$ = $\frac{1}{6}$

7. Find the MGF of the RV X, whose pdf is given by $f(x) = \frac{1}{2}e^{-|x|}, \ -\infty < x < \infty$. Hence its mean and variance.

Solution:

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{0} e^{tx} e^{x}\,dx + \int_{0}^{\infty} e^{tx} e^{-x}\,dx$$

$$= \int_{-\infty}^{0} e^{(t+1)x}\,dx + \int_{0}^{\infty} e^{-(1-t)x}\,dx$$

$$= \left[\frac{e^{(t+1)x}}{(t+1)}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-t)x}}{-(1-t)}\right]_{0}^{\infty}$$

$$M_X(t) = \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right) = \frac{1}{1-t^2} = 1 + t^2 + t^4 + \ldots.$$

Mean = E(X) = (coefficient of t) 1! = 0

$E(X^2)$ = (coefficient of t^2)2! =2

Variance = $E(X^2) - E(X)^2 =$ 2

8. The p.m.f of a RV X, is given by $p(X = j) = \dfrac{1}{2^j}$, $j = 1,2,3\ldots$ Find MGF, mean and variance.

Solution:

$$M_X(t) = E\left(e^{tX}\right) = \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{1}{2^x}$$

$$= \sum_{x=0}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \left(\frac{e^t}{2}\right)^4 + \ldots.$$

$$= \frac{e^t}{2}\left(1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \left(\frac{e^t}{2}\right)^4 + ..\right)$$

$$= \frac{e^t}{2} \frac{1}{1 - \frac{e^t}{2}} = \frac{e^t}{2 - e^t}$$

Differentiating twice with respect to t

$$M'_X(t) = \frac{\left(2-e^t\right)\left(e^t\right)-e^t\left(-e^t\right)}{\left(2-e^t\right)^2} = \frac{2e^t}{\left(2-e^t\right)^2}$$

$$M''_X(t) = \frac{\left(2-e^t\right)^2\left(2e^t\right)-2e^t\,2\left(2-e^t\right)\left(-e^t\right)}{\left(2-e^t\right)^4} = \frac{4e^t+2e^{2t}}{\left(2-e^t\right)^3}$$

put t = 0 above $E(X) = M'_X(0) = 2$

$$E\left(X^2\right) = M''_X(0) = 6$$

$$Variance = E\left(X^2\right) - E\left(X\right)^2 = 6-4=2$$

10..If X has the probability density function $f(x) = k\,e^{-3x},\ x>0$

Find (i) k (ii) p(0.5 ≤X≤1) (iii) Mean of X.

Solution:

(i)

$$p(0.5 \le X \le 1) = \int_{0.5}^{1} f(x)\ dx$$

$$= \int_{0.5}^{1} 3\,e^{-3x}\ dx$$

$$= 3\left[\frac{-e^{-3x}}{3}\right]_{0.5}^{1}$$

$$= -e^{-3}+e^{-1.5}$$

(iii)

$$Mean = E\left(X\right) = \int_{0}^{\infty} x\,f(x)\ dx$$

$$= \int_{0}^{\infty} 3x\,e^{-3x}\ dx$$

$$= 3\left[-x\frac{e^{-3x}}{3} - \frac{e^{-3x}}{9}\right]_{0}^{\infty}$$

$$= 3\left(\frac{1}{9}\right) = \frac{1}{3}$$

11. If a RV X has the pdf $f(x) = \begin{cases} \frac{1}{4}, & |x| < 2 \\ 0, & otherwise \end{cases}$.

Obtain (i) p(X <1) (ii) p($|X|$ >1) (iii) p(2X+3 > 5) (iv) p ($|X|$ < 0.5 | X < 1

Solution:

(i) p(X < 1) = $\int_{-2}^{1} \frac{1}{4} dx = \frac{1}{4}[x]_{-2}^{1} = \frac{3}{4}$

(ii) p($|X| \le 1$) = $\int_{-1}^{1} \frac{1}{4} dx = \frac{1}{4}[x]_{-1}^{1} = \frac{1}{2}$

Hence p($|X|$ >1)= 1 - p($|X| \le 1$) = $\frac{1}{2}$

(iii) p(2X+3 > 5) = p(X > 1) = 1- p(X $\le$1) = 1- $\frac{3}{4} = \frac{1}{4}$

(iv) p ($|X|$ < 0.5 | X < 1) = $\frac{p(|X| < 0.5 \cap X < 1)}{p(X < 1)}$

$$= \frac{p([-0.5 < X < 0.5] \cap X < 1)}{p(X < 1)}$$

$$= \frac{p([-0.5 < X < 0.5])}{p(X < 1)} =$$

$$\frac{\int_{-1}^{1} \frac{1}{4} dx}{\frac{3}{4}} = \frac{[x]_{-0.5}^{0.5}}{3} = \frac{1}{3}$$

12. .If X has the distribution function

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{3}, & 1 \le x < 4 \\ \frac{1}{2}, & 4 \le x < 6 \\ \frac{5}{6}, & 6 \le x < 10 \\ 1, & x > 10 \end{cases}$$

(1) Probability distribution of X (2) p(2<X<6) (3) Mean (4) variance

Solution:

(1)As there is no x terms in the distribution function given is a discrete random variable. Hence the probability distribution is given by

X	1	4	6	10
$p(X)$	$\frac{1}{3}$	$\frac{1}{2}-\frac{1}{3}$	$\frac{5}{6}-\frac{1}{2}$	$1-\frac{5}{6}$
	$\frac{1}{3}$	$=\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

(2) p(2<X<6) = p(4) = $\frac{1}{6}$

(3) Mean = E(X) =

(4) E(X^2) = $\sum x^2 \; p(x) = (1)\left(\frac{1}{3}\right)+(16)\left(\frac{1}{6}\right)+(36)\left(\frac{1}{3}\right)+(100)\left(\frac{1}{6}\right) = \frac{95}{3}$

Variance = E(X^2) – E(X)^2 = $\frac{95}{3}$ - $\frac{196}{9}$ = $\frac{89}{9}$

13. A continuous random variable X has the distribution function

$$F(x) = \begin{cases} 0 & : \quad x \le 1 \\ k(1-x)^4 & : \quad 1 < x \le 3 \\ 0 & : \quad x > 3 \end{cases}$$

Find *k*, the probability density function *f(x)* and *P(X <2)*.

Solution:

Since it is a distribution function

$F(\infty) = F(3) = 1$

$k(3-1)^4 = 1$

$k = \frac{1}{16}$

The density function is $f(x) = F'(x) = \frac{1}{16}4(1-x)^3 = \frac{(1-x)^3}{4}, \; 1 \le x \le 3$

p(X < 2) = F(2) = $\frac{1}{16}(2-1)^4 = \frac{1}{16}$

14. If the cumulative distribution function of a R.V X is given by $F(x) = \begin{cases} 1-\frac{4}{x^2}, & x > 2 \\ 0, & x \le 2 \end{cases}$

find (i) P(X < 3) (ii) P(4 < X < 5) (iii) P (X ≥3).

Solution:

(i) P(X < 3) = F(3) = $1-\frac{4}{3^2} = \frac{5}{9}$

(ii) P(4 < X < 5) = F(5) - F(4) = $\left(1-\dfrac{4}{5^2}\right)$ - $\left(1-\dfrac{4}{4^2}\right) = \dfrac{21}{25}-\dfrac{3}{4} = \dfrac{9}{100}$

(iii) P (X ≥3) = 1- F(3) = 1 - $\left(1-\dfrac{4}{3^2}\right)$ = 1- $\dfrac{5}{9} = \dfrac{4}{9}$