Burnside's Lemma

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Outline

Motivation

- How many colorings does a graph have?

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- How many colorings does a graph have? k^n where we have k colors and n objects (vertices or edges)
- How do we account for equivalence classes of colorings?

Group

4.2.3. Definition. A **group** is a set G together with a composition law satisfying the four properties below. Here we write composition as multiplication.

(1) *identity*: there exists $e \in G$ such that $e\pi = \pi = \pi e$ for all $\pi \in G$.

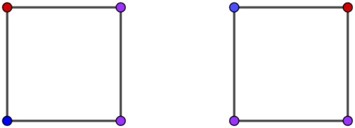
(2) *inverse*: for $\pi \in G$, there is a unique $\pi^{-1} \in G$ with $\pi\pi^{-1} = e = \pi^{-1}\pi$, where e is the identity element.

(3) *closure*: $\pi\sigma \in G$ for all $\pi, \sigma \in G$.

(4) *associativity*: $(\pi\sigma)\tau = \pi(\sigma\tau)$ for all $\pi, \sigma, \tau \in G$.

Equivalence Classes

"Equivalence classes of colorings result from 'symmetry operators' that permute the things being colored"

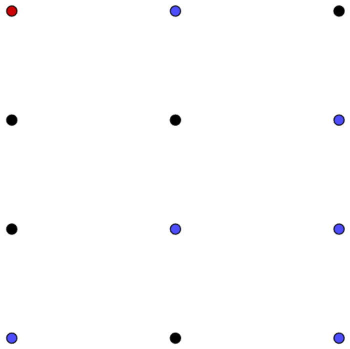


Two colorings are equivalent if we can permute the objects to turn one coloring into another.

Two colorings f and f' are **indistinguishable** or **equivalent** if there exists $\pi \in G$ such that $f'(\pi(x)) = f(x)$ for all $x \in X$. ("π turns f into f'")

Orbits

When G is a group of permutations of a set C and $u \in C$, the **orbit** of u is $\{\pi(u) : \pi \in G\}$



Counting Manually

Every permutation in G is mapping some coloring f to an element in it's orbit.
Summing the number of permutations taking f to each element of its orbit counts the orbit $|G|$ times so need to also divide the total by $|G|$ to count everything once.

4.2.6 Lemma

Lemma

If G is a group of permutations of C, and $u, v \in C$ are in the same orbit, then $|\{\pi \in G : \pi(u) = v\}| = |\{\pi \in G : \pi(v) = v\}|$.

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Proof Idea:

Let $A = |\{\pi \in G : \pi(u) = v\}|$ and $B = |\{\pi \in G : \pi(v) = v\}|$.
Since u, v are in the same orbit $\exists \sigma \in G$ st $\sigma(u) = v$.
Compose σ with an element from A and an element from B separately and you would have functions from A to B and B to A.
Prove injective.

Burnside's Lemma

Lemma

Under the action of a group G of permutations of C, the number of orbits of C is $\frac{1}{|G|}\sum_{\pi \in G} \psi(\pi)$, where $\psi(\pi)$ is the number of elements of C left fixed by the action of π on C.

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Lemma

Under the action of a group G of permutations of C, the number of orbits of C is $\frac{1}{|G|}\sum_{\pi\in G}\psi(\pi)$, where $\psi(\pi)$ is the number of elements of C left fixed by the action of π on C.

Proof Idea:

$$\sum_{E} 1 = \frac{1}{|G|}\sum_{E}|G| = \frac{1}{|G|}\sum_{E}\sum_{v\in E}|\{\pi : \pi(u_E) = v\}|$$

$$= \frac{1}{|G|}\sum_{v\in C}\phi(v) = \frac{1}{|G|}\sum_{\pi\in G}\psi(\pi).$$