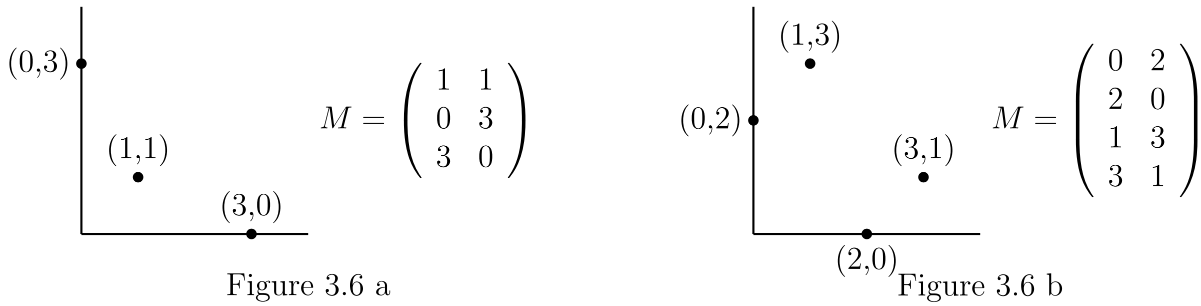

Figure 3.6: SVD problem

1. *Run the power method starting from* $x = \binom{1}{1}$ *for* $k = 3$ *steps. What does this give as an estimate of* v_1*?*

2. *What actually are the* v_i*'s,* σ_i*'s, and* u_i*'s? It may be easiest to do this by computing the eigenvectors of* $B = A^T A$*.*

3. *Suppose matrix* A *is a database of restaurant ratings: each row is a person, each column is a restaurant, and* a_{ij} *represents how much person* i *likes restaurant* j*. What might* v_1 *represent? What about* u_1*? How about the gap* $\sigma_1 - \sigma_2$*?*

Exercise 3.7 *Let* A *be a square* $n \times n$ *matrix whose rows are orthonormal. Prove that the columns of* A *are orthonormal.*

Exercise 3.8 *Suppose* A *is a* $n \times n$ *matrix with block diagonal structure with* k *equal size blocks where all entries of the* i^{th} *block are* a_i *with* $a_1 > a_2 > \cdots > a_k > 0$*. Show that* A *has exactly* k *nonzero singular vectors* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ *where* $\mathbf{v_i}$ *has the value* $(\frac{k}{n})^{1/2}$ *in the coordinates corresponding to the* i^{th} *block and 0 elsewhere. In other words, the singular vectors exactly identify the blocks of the diagonal. What happens if* $a_1 = a_2 = \cdots = a_k$*? In the case where the* a_i *are equal, what is the structure of the set of all possible singular vectors?*
Hint: *By symmetry, the top singular vector's components must be constant in each block.*

Exercise 3.9 *Interpret the first right and left-singular vectors for the document term matrix.*

Exercise 3.10 *Verify that the sum of* r*-rank one matrices* $\sum_{i=1}^{r} c_i \mathbf{x_i} \mathbf{y_i}^T$ *can be written as* XCY^T*, where the* $\mathbf{x_i}$ *are the columns of* X*, the* $\mathbf{y_i}$ *are the columns of* Y*, and* C *is a diagonal matrix with the constants* c_i *on the diagonal.*

Exercise 3.11 *Let* $\sum_{i=1}^{r} \sigma_i \mathbf{u_i} \mathbf{v_i}^T$ *be the SVD of* A*. Show that* $\left|\mathbf{u_1}^T A\right| = \sigma_1$ *and that* $\left|\mathbf{u_1}^T A\right| = \max_{|\mathbf{u}|=1} \left|\mathbf{u^T} A\right|$*.*

Exercise 3.12 *If $\sigma_1, \sigma_2, \ldots, \sigma_r$ are the singular values of A and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are the corresponding right-singular vectors, show that*

1. $A^T A = \sum_{i=1}^{r} \sigma_i^2 \mathbf{v_i} \mathbf{v_i}^T$
2. $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_r$ *are eigenvectors of* $A^T A$.
3. *Assuming that the eigenvectors of $A^T A$ are unique up to multiplicative constants, conclude that the singular vectors of A (which by definition must be unit length) are unique up to sign.*

Exercise 3.13 *Let $\sum_i \sigma_i u_i v_i^T$ be the singular value decomposition of a rank r matrix A. Let $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ be a rank k approximation to A for some $k < r$. Express the following quantities in terms of the singular values $\{\sigma_i, 1 \le i \le r\}$.*

1. $||A_k||_F^2$
2. $||A_k||_2^2$
3. $||A - A_k||_F^2$
4. $||A - A_k||_2^2$

Exercise 3.14 *If A is a symmetric matrix with distinct singular values, show that the left and right singular vectors are the same and that $A = VDV^T$.*

Exercise 3.15 *Let A be a matrix. How would you compute*

$$\mathbf{v_1} = \arg\max_{|\mathbf{v}|=1} |A\mathbf{v}|?$$

How would you use or modify your algorithm for finding $\mathbf{v_1}$ to compute the first few singular vectors of A.

Exercise 3.16 *Use the power method to compute the singular value decomposition of the matrix*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Exercise 3.17

1. *Write a program to implement the power method for computing the first singular vector of a matrix. Apply your program to the matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & 9 & 10 \\ 2 & 3 & 4 & \cdots & 10 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 9 & 10 & 0 & \cdots & 0 & 0 \\ 10 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$