IE531: Algorithms for Data Analytics
Spring, 2018
Homework 3: Review of Linear Algebra, Probability & Statistics and Computing
Due Date: March 16, 2018

Instructions

1. You can modify any of the C++ code on Compass to solve these problems, if you want. It might help you with honing your programming skills. If these attempts (at using C++ code) is turning out to to be intense, you can use MATLAB just this once.
2. You will submit a PDF-version of your answers on Compass on-or-before midnight of the due date.

Instructions

1. (*25 points*) Exercise 3.7 (Text).

   Since the rows of $\mathbf{A}$ are orthonormal, we know $\mathbf{AA}^T = \mathbf{I}$. Since $\mathbf{A}$ is square, it follows that $\mathbf{A}^{-1} = \mathbf{A}^T$. Since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, it follows that $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, as well. That is, the column vectors of $\mathbf{A}$ are orthonormal, as well.

   Some Observations (Start)

   This claim is not true if $\mathbf{A}$ is not a square matrix. The following matrix is a counter-example,

   $$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

   Some Observations (End)

2. (*25 points*) Exercise 3.13 (Text)

   $||\mathbf{A}_k||_F^2 = \sum_{i=1}^{k} \sigma_i^2$, $||\mathbf{A} - \mathbf{A}_k||_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$, $||\mathbf{A}||_2^2 = \sigma_1^2$ $||\mathbf{A} - \mathbf{A}_k||_2^2 = \sigma_{k+1}^2$

3. (*25 points*) Exercise 3.14 (Text)

   There are many ways to showing this. I am presenting one of them here. If $\mathbf{A}$ is symmetric, it is square (why?). Any symmetric square matrix can be orthogonally diagonalized. That is,

   $$(\mathbf{A} = \mathbf{PDP}^T) \Leftrightarrow (\mathbf{D} = \mathbf{P}^T\mathbf{AP})$$

   where $\mathbf{PP}^T = \mathbf{P}^T\mathbf{P} = \mathbf{I}$. This is an SVD-decomposition of $\mathbf{A}$ (= $\mathbf{U\Sigma V}^T$) with $\mathbf{U}$ = $\mathbf{V}$ = $\mathbf{P}$.

4. (*25 points*) Exercise 3.16 (Text)

Straightforward. But, if you need help –

$$\mathbf{B} = \mathbf{A}^T\mathbf{A} = \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix} \Rightarrow \mathbf{B}^5 = \begin{pmatrix} 7885120 & 11189024 \\ 11189024 & 15877280 \end{pmatrix}$$

Normalizing the first column of $\mathbf{B}^5$ to a unit-vector, we get

$$\mathbf{v}_1 = \begin{pmatrix} 0.5760 \\ 0.8174 \end{pmatrix} \Rightarrow \mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 2.2109 \\ 4.9978 \end{pmatrix} = \overbrace{5.4650}^{\sigma_1} \times \underbrace{\begin{pmatrix} 0.4046 \\ 0.9145 \end{pmatrix}}_{=\mathbf{u}_1 \text{(unit-vector)}}$$

To get the second singular value and its associated vectors, we proceed as follows.

$$\begin{aligned} \mathbf{C} = (\mathbf{A} - \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T) &= \begin{pmatrix} -0.2736 & 0.1928 \\ 0.1210 & -0.0853 \end{pmatrix} \\ \Rightarrow \mathbf{D} = \mathbf{C}^T\mathbf{C} &= \begin{pmatrix} 0.0895 & -0.0631 \\ -0.0631 & 0.0444 \end{pmatrix} \\ \mathbf{D}^5 &= 10^{-4} \times \begin{pmatrix} 0.2879 & -0.2029 \\ -0.2029 & 0.1430 \end{pmatrix} \end{aligned}$$

Normalizing the first column of $\mathbf{D}^5$ to a unit-vector, we get

$$\mathbf{v}_2 = \begin{pmatrix} 0.8174 \\ -0.5760 \end{pmatrix} \Rightarrow \mathbf{A}\mathbf{v}_2 = \begin{pmatrix} -0.3347 \\ 0.1481 \end{pmatrix} = \overbrace{0.3660}^{\sigma_2} \times \underbrace{\begin{pmatrix} -0.9145 \\ 0.4046 \end{pmatrix}}_{=\mathbf{u}_2 \text{(unit-vector)}}$$

Therefore

$$\begin{aligned} \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{=\mathbf{A}} &= \left\{ \overbrace{5.4650}^{\sigma_1} \times \underbrace{\begin{pmatrix} 0.4046 \\ 0.9145 \end{pmatrix}}_{=\mathbf{u}_1} \overbrace{\begin{pmatrix} 0.5760 & 0.8174 \end{pmatrix}}^{=\mathbf{v}_1^T} \right\} + \\ & \quad \left\{ \overbrace{0.3660}^{\sigma_2} \times \underbrace{\begin{pmatrix} -0.9145 \\ 0.4046 \end{pmatrix}}_{=\mathbf{u}_2} \overbrace{\begin{pmatrix} 0.8174 & -0.5760 \end{pmatrix}}^{=\mathbf{v}_2^T} \right\} \end{aligned}$$