Homework 4: Markov Chains

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1. **(Random Walks on Graphs)** Show that in steady-state you d v will spend 2/card(E) % of time in vertex v, where card( ) is the cardinality (i.e. size) of the set-argument.
   Solution: Let π to be:

$$\pi = \left[\frac{d_{v1}}{2card(E)}, \frac{d_{v2}}{2card(E)}, \frac{d_{v3}}{2card(E)}, \frac{d_{v4}}{2card(E)}, \frac{d_{v5}}{2card(E)}, \cdots\right]$$

.
   To prove this we only need prove $\pi \times P = \pi$
   Let P be:

$$\begin{bmatrix} 1/d_{v1} & 0 & 1/d_{v1} & \dots & x_{1n} \\ 0 & 0 & 1/d_{v2} & \dots & 1/d_{v2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1/d_{vn} & 1/d_{vn} & \dots & 0 \end{bmatrix}$$

   Then we compute $\pi \times P$, we can get π back again.

2. **(Random Knights Tour)** In this problem we start the knight at one of the four corner squares in an otherwise empty chessboard. The knight selects one of the next positions at random independently of the past moves.

   (a) Interpret the Knights tour as a Markov Chain, where the discrete-states represent the 64 squares of the chessboard. (PS: I am just looking for an implicit definition of the 64 64 probability-matrix here; if you want, give me a piece of pseudo-code that constructs the probability- matrix).
   Solution: the next position X_{n+1} is a function of the current position X_n and a random choice ξ_n of a neighbor. Hence the problem is in the same form as the one above. Hence (X_n) is a Markov chain. The state space is the set of the squares of the chess board. There are $8 \times 8 = 64$ squares. We can label them by a pair of integers. Hence the state space is:

$$S = \{(i_1, i_2) : 1 \le i_1 \le 8, 1 \le i_2 \le 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\} \times \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

   (b) As a follow-on to problem 2a, is the Markov Chain irreducible? Is it aperiodic?
   Solution: For irreducibility, we can find a path that takes the knight from any

square to any other square. Hence every state communicates with every other state, i.e. **it is irreducible**.
For periodicity, we can find the period for a specific state, e.g. from (1, 1). You can see that, if you start the knight from (1, 1) you can return it to (1, 1) only in even number of steps. Hence the period is 2. So **it is not aperiodic**.

(c) Find the stationary distribution of this Markov Chain.
Solution: First, there is a lot of symmetry. So squares (states) that are symmetric with respect to the center of the chess board must have the probability under the stationary distribution. So, for example, states (1, 1), (8, 1), (1, 8), (8, 8) have the same probability. And so on. Second, you should realize that (1, 1) must be less likely than a square closer to the center, e.g. (4, 4). The reason is that (1, 1) has fewer next states (exactly 2) than (4, 4) (which has 8 next states). So let us make the guess that if $x = (i_1, i_2)$, then (x) is proportional to the number N(x) of the possible next states of the square x:

$$\pi(x) = CN(x)$$

. So we must show that such a π satisfies the balance equations:

$$\pi(x) = \sum \pi(y)p(y, x)$$

. The sum on the right is zero unless x is a NEIGHBOUR of y:

$$N(x) = \sum_{x \ neighbour \ of \ y} N(y)_{p(y,x)}$$

Therefore, all we have to do is count the neighbors of each square x which equals to 336. So:

$$\pi(1, 1) = 2/336, \pi(1, 2) = 3/336, \pi(1, 3) = 4/336, ..., \pi(4, 4) = 8/336$$

(d) Where is the Knight most likely to be spotted in steady-state? Where is the Knight least likely to be spotted in steady-state?
Solution: The 16 middle ones are the most likely: 8/336; The corner ones are the least likely: 2/336.