Homework 3: Review of Linear Algebra, Probability Statistics and Computing

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1. **Exercise 3.7** Let A be a square $n \times n$ matrix whose rows are orthonormal. Prove that the columns of A are orthonormal.
   Solution:
   Since the row vectors of A are orthonormal, we have that $AA^{\intercal} = I$. For square matrix A, this implies that $A^{\intercal} = A^{-1}$. Since $A^{-1}A = I$, we have $A^{\intercal}A = I$, which implies that the column vectors of A are also orthonormal.

2. **Exercise 3.13** Let $\sum\limits_{i} \sigma_i u_i v_i^{\intercal}$ be the singular value decomposition of a rank r matrix A. Let $A_k = \sum\limits_{i=1}^{k} \sigma_i u_i v_i^{\intercal}$ be a rank k approximation to A for some $k < r$. Express the following quantities in terms of the singular values $\{\sigma_i, 1 \leq i \leq r\}$.

   - $||A_k||_F^2$
     Solution: $\sum\limits_{i=1}^{k} \sigma_i^2$
   - $||A_k||_2^2$
     Solution: σ_1^2
   - $||A - A_k||_F^2$
     Solution: $\sum\limits_{i=k+1}^{r} \sigma_i^2$
   - $||A - A_k||_2^2$
     Solution: σ_{k+1}^2

3. **Exercise 3.14** If A is a symmetric matrix with distinct singular values, show that the left and right singular vectors are the same and that $A = VDV^{\intercal}$.
   Solution:
   A is a symmetric matrix, so that $A^{\intercal}A = AA^{\intercal}$. Suppose $A = UDV^{\intercal}$ is the Singular Value Decomposition of A. Then

$$
\begin{aligned}
A^\intercal A &= (UDV^\intercal)^\intercal UDV^\intercal \\
&= V^\intercal D^\intercal U^\intercal UDV^\intercal \\
&= VD^2V^\intercal
\end{aligned}
$$

$$
\begin{aligned}
AA^\intercal &= UDV^\intercal(UDV^\intercal)^\intercal \\
&= UD^2U^\intercal
\end{aligned}
$$

So the left and right singular vectors are the same and that $A = VDV^\intercal$.

4. **Exercise 3.16** Use the power method to compute the singular value decomposition of the matrix.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Solution: Please see attachment.