IE531: Algorithms for Data Analytics
Midterm Exam
Spring, 2018

Instructions

1. **A (correct) answer that is not accompanied by an appropriate justification will not be given any credit. Make sure you answer all questions.**

2. **Try to finish on time.**

Instructions

1. (*25 points*) **(Rank (and Determinant) of a Matrix)** Consider the matrix **A** shown below, where $x \in \mathcal{R}$ is a real-valued variable.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 & 7 & 0 \\ 2 & 4 & 6 & 8 & 0 \\ 0 & 0 & 0 & 0 & x \end{pmatrix}$$

   (a) (*2 points*) What is the value of x that makes *Rank*(**A**) as small as possible?

   (b) (*2 points*) What is the smallest possible value for *Rank*(**A**)?

   (c) (*2 points*) What is the value of x that makes *Rank*(**A**) as large as possible?

   (d) (*2 points*) What is the largest possible value for *Rank*(**A**)?

   (e) (*2 points*) What is the value of x that ensures there is a solution to the equation $\mathbf{Ax} = \mathbf{y}$, where $\mathbf{y} = (1\ 1\ 0)^T$?

   (f) (*2 points*) What is the value of x that ensures there is a solution to the equation $\mathbf{Ax} = \mathbf{y}$, where $\mathbf{y} = (1\ 0\ 1)^T$?

   (g) (*2 points*) Suppose **X** and **Y** are arbitrary matrices, is it possible for $rank(\mathbf{X}+\mathbf{Y}) > rank(\mathbf{X}) + rank(\mathbf{Y})$?

   (h) (*2 points*) Suppose **z** is a $n \times 1$ column. What are the number of rows and number of columns of $\mathbf{z}\mathbf{z}^T$? What is $rank(\mathbf{z}\mathbf{z}^T)$?

   (i) (*2 points*) If **X** is an $n \times n$, square matrix with $rank(\mathbf{X}) < n$, what is $det(\mathbf{X})$?

   (j) (*7 points*) If **X** is an $n \times n$, square matrix, where the (i, j)-th entry, $\mathbf{X}_{i,j} = i + j$, what is $rank(\mathbf{X})$?

   To receive full-credit for this problem, make sure you give me an explanation for your answers.

2. (*25 points*) **(Generating Constrained, High-Dimensional RVs)** The *d-dimensional Unit-Cube* is defined as

$$\mathbf{C} = \{\mathbf{x} \in \mathcal{R}^d \mid 0 \leq \mathbf{x}_i \leq 1, i = 1, 2, \ldots, d\}$$

   That is, **C** is a collection of d-dimensional vectors where each entry/component of the vector is in the closed-interval $[0, 1]$.

(a) (*10 points*) How would you generate points Uniformly at Random in **C**? I am looking to see some pseudo-code, along with an informal justification for why your code will pick any point in **C** with equal probability.

(b) (*15 points*) How would you generate points Uniformly at Random on the ($(d-1)$-dimensional) *surface* of **C**? I am looking to see some pseudo-code, along with an informal justification for why your code will pick any point on the ($d-1$-dimensional) surface of **C** with equal probability.

Some Observations (Start)

If you understand what is meant by the term "($(d-1)$-dimensional) *surface of* **C**," you can skip this factoid. You may want to think about solving problem 4b by trying your hand for the special-cases of $d = 1, 2, 3, 4$ first (this in itself will get you no points!), and then generalize it to arbitrary values of d.

A *1-dimensional Unit-Cube* is the Unit-Interval [0, 1]. There are 2-many 0-dimensional surfaces (i.e. points) on this Unit-cube, and they are the points 0 and 1.

A *2-dimensional Unit-Cube* is a square with unit-length sides. There are 4-many 1-dimensional surfaces (i.e. lines) on this Unit-cube, and they are the four lines that define the square.

A *3-dimensional Unit-Cube* is a cube with unit-length sides. There are 6-many 2-dimensional surfaces (i.e. areas) on this Unit-cube, and they are the six square-areas that define the cube.

There are 8-many 3-dimensional surfaces (i.e. volumes) in a *4-dimensional Unit-Cube*.

Some Observations (End)

3. (*25 points*) Consider the following SVD of a matrix $\mathbf{A} = \mathbf{UDV}^T$, where

$$\underbrace{\begin{pmatrix} 4.7511 & 1.9078 & 0.9344 & 3.2316 \\ 0.1722 & 3.8276 & 2.4488 & 3.5468 \\ 2.1937 & 3.9760 & 2.2279 & 3.7734 \end{pmatrix}}_{=\mathbf{A}} = \underbrace{\begin{pmatrix} -0.5493 & -0.7744 & -0.3139 \\ -0.5379 & 0.6152 & -0.5763 \\ -0.6394 & 0.1477 & 0.7545 \end{pmatrix}}_{=\mathbf{U}} \underbrace{\begin{pmatrix} 9.8247 & 0 & 0 \\ 0 & 3.7411 & 0 \\ 0 & 0 & 0.2958 \end{pmatrix}}_{=\mathbf{D}} \underbrace{\begin{pmatrix} -0.4179 & -0.5750 & -0.3313 & -0.6205 \\ -0.8685 & 0.3915 & 0.2973 & 0.0633 \\ 0.2178 & 0.6594 & -0.0800 & -0.7151 \end{pmatrix}}_{=\mathbf{V}^T}$$

(a) (*5 points*) What is the best-possible rank 1 approximation to **A**? (**PS**: No need to carry out the multiplication; just give me the form/structure)

(b) (*5 points*) What is the best-possible rank 2 approximation to **A**? (**PS**: No need to carry out the multiplication; just give me the form/structure)

(c) (*5 points*) What is the best-possible rank 3 approximation to **A**? (**PS**: No need to carry out the multiplication; just give me the form/structure)

(d) (*10 points*) What is the best-possible rank 4 approximation to **A**? (**PS**: No need to carry out the multiplication; just give me the form/structure)

4. (*25 points*) **MCMC-MH**: We need to generate identically-distributed samples $X \sim f(x)$ according to some (complicated) PDF $f(\bullet)$ that can be computed upto a proportionality constant.

   Assume that for right now, you have a present value $x^{(j)}$ for $X \sim f(x)$. You have access to a known/well-understood *Proposal Distribution* $q(\bullet)$ that is conditioned on $x^{(j)}$. You get a sample $\widetilde{x} \sim q(x \mid x^{(j)})$, you set $x^{(j+1)} = \widetilde{x}$ with probability

$$p(\widetilde{x}, x^{(j)}) = \min\left\{1, \frac{f(\widetilde{x})}{f(x^{(j)})} \times \frac{q(x^{(j)} \mid \widetilde{x})}{q(\widetilde{x} \mid x^{(j)})}\right\}. \tag{1}$$

   The sequence of (not necessarily independent) samples $\{x^{(j)}\}_{j=1}^{\infty}$ will be distributed according $f(x)$. In many cases, we will pick *Proposal Distributions* that are symmetric, that is, $q(x^{(j)} \mid \widetilde{x}) = q(\widetilde{x} \mid x^{(j)})$, in which case the above expression simplifies to

$$p(\widetilde{x}, x^{(j)}) = \min\left\{1, \frac{f(\widetilde{x})}{f(x^{(j)})}\right\}. \tag{2}$$

   (a) (*7.5 points*) Show that $q(\widetilde{x} \mid x^{(j)}) = N(x^{(j)}, 1)$ (i.e. it returns r.v.'s that are Normally-distributed with *mean* $x^{(j)}$ with *variance* of 1) is a *symmetric* distribution (cf. the definition/discussion above).

   (b) (*7.5 points*) Suppose $q(\widetilde{x} \mid x^{(j)}) = \text{Uniform}(x^{(j)}, x^{(j)} + 10)$ (i.e. it returns r.v.'s that are Uniformly-distributed between $x^{(j)}$ and $x^{(j)} + 10$). Is this *Proposal Distribution* symmetric? Explain your answer (or, you get no points!)

   (c) (*10 points*) As a follow-on to problem 4b, which version of the acceptance probability would you use – Equation 1? Equation 2? Explain your answer (**ps.** This is a trick-question, just saying!).