IE531: Algorithms for Data Analytics
Spring, 2018
Homework 1: Review of Linear Algebra, Probability & Statistics and Computing
Due Date: February 16, 2018

Instructions

1. You can modify any of the C++ code on Compass to solve these problems, if you want. It might help you with honing your programming skills. If these attempts (at using C++ code) is turning out to to be intense, you can use MATLAB just this once.

2. You will submit a PDF-version of your answers on Compass on-or-before midnight of the due date.

Instructions

1. **Linear Algebra**: (*30 points*) Consider the following set of linear equations involving six equations in five unknowns $x_1, x_2, \ldots, x_5$.

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & 6 & 2 \\ 1 & -1 & -1 & 4 & 0 \\ 2 & -2 & 2 & 4 & 0 \\ 2 & -2 & 0 & 6 & 0 \\ -4 & 4 & -8 & -4 & 0 \\ 4 & 0 & -2 & 14 & 4 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 2 \\ 7 \\ 18 \\ 16 \\ -40 \\ 2 \end{pmatrix}}_{\mathbf{y}}$$

   (a) (*2 points*) Show that there a solution to these equations.

   Look at what was done in Linear_Algebra_1.cpp, or figure 4 of my review of Linear Algebra.

   (b) (*10 points*) Present a general formula for the set of all solutions that satisfy these equations.

   Look at what was done in Linear_Algebra_2.cpp, or figure 5 of my review of Linear Algebra. Once you have the required basic solutions, take their *affine-combination* to obtain the general solution. If you need additional help, look at the example in section 3.1 of the Linear Algebra review.

   (c) (*3 points*) Verify the general formula using MATLAB.

   You can also do this by hand, if you want. MATLAB's symbolic toolkit will let you do the symbolic multiplications if you are not up to doing it by hand. You can always cut-n-paste MATLAB's output on to your solutions (as you have to generate a PDF to upload on Compass; hard if you are hand-verifying it).

(d) (*5 points*) Show that

$$\begin{pmatrix} -7\lambda_1 - 4\lambda_2 + 8 \\ -7\lambda_1 - 7\lambda_2 \\ 1 - \lambda_2 \\ -\lambda_2 \\ -7 + 7\lambda_1 + 7\lambda_2 \end{pmatrix} \tag{1}$$

and

$$\begin{pmatrix} a \\ -7a - 7b \\ -\frac{8}{3}a - \frac{7}{3}b + \frac{11}{3} \\ -\frac{8}{3}a - \frac{7}{3}b + \frac{8}{3} \\ -7 + 7a + 7b \end{pmatrix}. \tag{2}$$

are solutions to the equation $\mathbf{Ax} = \mathbf{y}$ identified earlier.

Trust you can work this one out on your own.

(e) (*5 points*) Present a cogent argument in support of the claim that even though the solution identified by equations 1 and 2 look different, they are in effect the same.

Trust you can work this one out on your own. See 16:00 marker of Lecture 3 on Echo360, if you are lost.

(f) (*5 points*) Show that if add the constraints

$$\begin{aligned} 7x_4 + x_5 &= 7 \\ x_2 + 2x_5 &= 7, \end{aligned}$$

in addition to those already in the requirement that $\mathbf{Ax} = \mathbf{y}$ as described in problem 1, there can be only one solution to the combined set of equations/constraints.

Trust you can work this one out on your own.

2. (*30 points*) **Generating RVs using the Inverse Transform Technique**: You may want to use the various C++ code samples on Compass (along with some graphing/plotting software) to verify/experiment-with things before you put your answers down for this problem.

(a) (*10 points*) The *triangular distribution* has a *probability density function*

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 < x \le 2 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

In three/four sentences describe how you can generate i.i.d. r.v's that are distributed according to equation 3.

You have the PDF, figure out what the CDF should be, then take its inverse, and work out the nitty-gritty as per the *Inverse Transform Method*.

(b) (*10 points*) What is the i.i.d. distribution generated by repeated calls to the code shown below?

```
double blah1( )
{
  (double) x = get_uniform();
  if (x <= 0.5)
                    return (1 + $\sqrt{2x}$);
  else
        return (1 − $\sqrt{2 - 2x}$);
}
```

This is an exercise in converting "what-you-see-in-code" to the generated CDF/PDF. You take an empirical approach and experiment with the code, and then justify it with some cogent reasoning (as any good experimental scientist would). Take your pick!

(c) (*10 points*) What is the i.i.d. distribution generated by repeated calls to the code shown below?

```
double blah2( )
{
  (double) x = get_uniform();
  if (x <= 0.5)
                    return (2 − $\sqrt{1 - 2x}$);
  else
        return ( $\sqrt{2x - 1}$);
}
```

This is an exercise in converting "what-you-see-in-code" to the generated CDF/PDF. You take an empirical approach and experiment with the code, and then justify it with some cogent reasoning (as any good experimental scientist would). Take your pick!

3. (*40 points*) **General Programming Concepts**:

   (a) (*10 points*) Consider the two versions of the code shown in figure 1. Suppose I type the string "kablooey" as input, give me the reasoning behind the output each of the programs generate[1].

   This is a straightforward exercise in programming.

   (b) (*10 points*) The *greatest common divisor* (GCD) of two positive numbers m and n is the largest number k that divides m and n. That is, $\frac{m}{k}$ and $\frac{n}{k}$ leaves no remainder. Consider the C++ code shown in figure 2. Using the fact that the GCD of two numbers does not change if the smaller number is subtracted from the larger number, show that the recursive funtion gcd(m, n) in the code shown in figure 2 implements the GCD computation.

   (**Note**: Please do not give me an illustrative example as a "proof" of that the code does what it is supposed to do. *I am looking for a rigorous attempt at showing the correctness of the code for* $\underline{\textit{any}}$ *pair of numbers*.)

[1] I am looking for an answer that justifies why we get to see the output generated by each of these programs.

```cpp
#include <iostream>
#include <cmath>
using namespace std ;
void f()
{
    char c;
    cin.get(c);
    // this line reads character "c"
    if ('\n' == c)
        // '\n' is "return/newline"
        return;
    cout << c;
    f();
}
int main()
{
    f();
    cout << endl;
}
```

(a) Problem 3a(I)

```cpp
#include <iostream>
#include <cmath>
using namespace std ;
void f()
{
    char c;
    cin.get(c);
    // this line reads character "c"
    if ('\n' == c)
        // '\n' is "return/newline"
        return;
    f();
    cout << c;
}
int main()
{
    f();
    cout << endl;
}
```

(b) Problem 3a(II)

Figure 1: (Two versions of the) Code for problem 3a.

```
#include <iostream>
using namespace std;
long gcd(long m, long n)
{
        if (m == n)
                return n;
        else if (m < n)
                return gcd(m,n-m);
        else
                return gcd(m-n, n);
}
int main()
{
        cout << gcd(259, 111) << endl;
}
```

Figure 2: Problem 3b.

Think through the recursion, and provide a formal proof of its correctness. This is standard-fare in many texts, but you should be able to figure this one out with little assistance from these texts.

(c) (*10 points*) What is the output of the C++ code shown in figure 3. Make sure you give me your reasoning for your answer.

```
#include <iostream>
using namespace std;
void print_integer (int num)
{
    putchar (num % 10 + 'A');
        if (num / 10)
                print_integer(num / 10);
}
int main()
{
        print_integer (1234);cout << endl;
}
```

Figure 3: Problem 3c.

What is the output of the above program? What is the reasoning behind the output that this program generates?

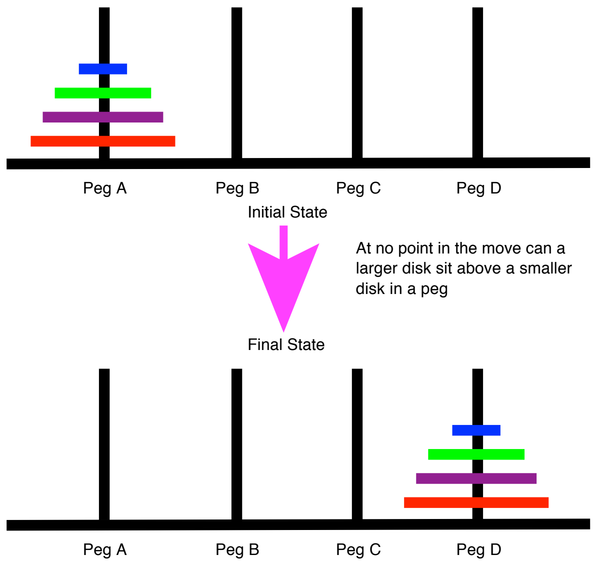
Straightforward.

(d) (*10 points*) My review of computing contains the *Frame-Stewart* solution to the k-peg version of the *Tower of Hanoi* problem. Consider the following solution to the 4-peg version of the problem.

Alternate Solution to 4-Peg Tower of Hanoi Problem

```
1: $int\ k = \lfloor \sqrt{2 * n} \rfloor$;
2: Move_Using_Four_Pegs ($n - k$, source, intermediate_1, intermediate_2, destination);
3: Move_Using_Four_Pegs ($k - 1$, source, intermediate_2, destination, intermediate_1);
4: cout << "Move the top disk from peg " << source << " to peg " << destination << endl;
5: Move_Using_Four_Pegs ($k - 1$, intermediate_2, destination, intermediate_1, source);
6: Move_Using_Four_Pegs ($n - k$, intermediate_1, destination, intermediate_2, source);
```

Comment on the correctness of this algorithm. Will this work? (**You might want to spend a little time thinking about this problem before you wrote down your answer**)


Peg A
Peg B
Peg C
Peg D
Initial State
At no point in the move can a larger disk sit above a smaller disk in a peg
Final State
Peg A
Peg B
Peg C
Peg D

This one could be tricky – you have to think about this some. If you took an empirical approach, make sure you try cases where n ranges from 3 to 10. Check if any of the moves have been illegal (you can write code to check for legality, just saying!).