Appendix

Theorem 1. *Assume the RLST-ZNN model operates in a noise-free environment, i.e., $d(t) = 0$, and the error function $E(\mathbf{x}(t), t)$ is continuously differentiable. Then, as $t \to \infty$, the error function $E(\mathbf{x}(t), t)$ converges to zero:*

$$\lim_{t \to \infty} E(\mathbf{x}(t), t) = 0.$$

Proof. The argument starts by analyzing the RLST-ZNN model under noise-free conditions ($d(t) = 0$), where the error dynamics are given by:

$$\dot{\mathbf{x}}(t) = -\left(\frac{\partial E}{\partial \mathbf{x}(t)}\right)^{\dagger} \left(\gamma\phi(E) + \frac{\partial E}{\partial t} - \rho(t)\right).$$

For the error function $E(\mathbf{x}(t), t)$, the time derivative is:

$$\dot{E}(\mathbf{x}(t), t) = -\gamma\phi(E) - \rho(t).$$

We consider the general case where the activation function is $\phi(x) = \phi_2(x) = x$, taking the Laplace transform of the differential equation for $E(\mathbf{x}(t), t)$ yields:

$$\mathcal{L}\left\{\dot{E}(\mathbf{x}(t), t)\right\} = s \cdot E(\mathbf{x}(s), s) - E(\mathbf{x}(0), 0),$$

which leads to:

$$sE(\mathbf{x}(s), s) - E(\mathbf{x}(0), 0) = -\gamma E(\mathbf{x}(s), s) - \mathcal{L}\left\{\rho(t)\right\}.$$

Rearranging gives:

$$(s + \gamma)E(\mathbf{x}(s), s) = E(\mathbf{x}(0), 0) - \mathcal{L}\left\{\rho(t)\right\}.$$

Hence, the Laplace transform of $E(\mathbf{x}(t), t)$ is:

$$E(\mathbf{x}(s), s) = \frac{E(\mathbf{x}(0), 0) - \mathcal{L}\left\{\rho(t)\right\}}{s + \gamma}.$$

Next, the Laplace transform of $\rho(t)$ is examined. Two cases arise:

- For $t < T$, $\rho(t) = 0$, implying:

$$\mathcal{L}\left\{\rho(t)\right\} = 0 \quad \text{for } t < T.$$

- For $t \geq T$, computing the Laplace transform of $\rho(t)$ proceeds as follows:

$$\rho(t) = \rho(t - T) + \vartheta_1 + \vartheta_2 \int_0^t \operatorname{sign}(E(\mathbf{x}(s), s))\, \mathrm{d}s.$$

Then:

$$\mathcal{L}\left\{\rho(t)\right\} = \mathcal{L}\left\{\rho(t - T)\right\} + \frac{\vartheta_1}{s} + \frac{\mathcal{L}\left\{\operatorname{sign}(E(\mathbf{x}(s), s))\right\}}{s}.$$

Because $\rho(t - T)$ is a delayed function,

$$\mathcal{L}\left\{\rho(t - T)\right\} = e^{-Ts}\mathcal{L}\left\{\rho(t)\right\}.$$

Substituting into the expression for $\mathcal{L}\left\{\rho(t)\right\}$ gives:

$$\mathcal{L}\left\{\rho(t)\right\} = e^{-Ts}\mathcal{L}\left\{\rho(t)\right\} + \frac{\vartheta_1}{s} + \frac{\mathcal{L}\left\{\operatorname{sign}(E(\mathbf{x}(s), s))\right\}}{s}.$$

Isolating $\mathcal{L}\left\{\rho(t)\right\}$:

$$\mathcal{L}\left\{\rho(t)\right\} - e^{-Ts}\mathcal{L}\left\{\rho(t)\right\} = \frac{\vartheta_1}{s} + \frac{\mathcal{L}\left\{\operatorname{sign}(E(\mathbf{x}(s), s))\right\}}{s},$$

and factoring out $\mathcal{L}\left\{\rho(t)\right\}$ leads to:

$$\mathcal{L}\left\{\rho(t)\right\}\left(1 - e^{-Ts}\right) = \frac{\vartheta_1}{s} + \frac{\mathcal{L}\left\{\operatorname{sign}(E(\mathbf{x}(s), s))\right\}}{s}.$$

Solving for $\mathcal{L}\left\{\rho(t)\right\}$:

$$\mathcal{L}\left\{\rho(t)\right\} = \frac{\vartheta_1 + \mathcal{L}\left\{\operatorname{sign}(E(\mathbf{x}(s), s))\right\}}{s(1 - e^{-Ts})}.$$

Substituting this Laplace transform of $\rho(t)$ into the expression for $E(\mathbf{x}(s), s)$ results in:

$$E(\mathbf{x}(s), s) = \frac{E(\mathbf{x}(0), 0)}{s + \gamma} - \frac{\mathcal{L}\left\{\rho(t)\right\}}{s + \gamma}.$$

The location of the poles of the transfer function determines system stability. Poles occur when $s + \gamma = 0$, yielding a pole at $s = -\gamma$.

For $t < T$, $\mathcal{L}\left\{\rho(t)\right\} = 0$, so the system simplifies to a first-order system with a pole at $s = -\gamma$, ensuring stability for $\gamma > 0$.

For $t \geq T$, the feedback term $\rho(t)$ introduces a delay, but the factor $1 - e^{-Ts}$ in the denominator ensures continued stability. The transfer function poles remain in the left half of the complex plane, causing the response to decay as time progresses.

Consequently, the system stays stable for all $t > 0$. □

Theorem 2. *Assume the RLST-ZNN model is subjected to periodic noise interference $d(t)$, where $d(t - T) = d(t)$, and the error function $E(\mathbf{x}(t), t)$ is continuously differentiable. Then, as $t \to \infty$, the error function $E(\mathbf{x}(t), t)$ converges to zero:*

$$\lim_{t \to \infty} E(\mathbf{x}(t), t) = 0.$$

Proof. Since $d(t) = d(t - T)$ for all t, we define

$$m(t) = d(t) - \rho(t).$$

Under this substitution, the perturbed RLST-ZNN model can be rewritten as

$$\dot{E}(\mathbf{x}(t), t) \;=\; -\gamma\, \phi\big(E(\mathbf{x}(t), t)\big) \;-\; m(t).$$

this system falls into the same framework as in Theorem 1, where the overall stability analysis applies directly. Therefore, we do not expand the details further here. $\square$