Solution.

CS 3200: Introduction to Scientific Computing

In-class Activity: Cubic Interpolation and positivity

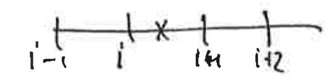
Consider using Lagrange Interpolation for which a polynomial interpolant of degree N with value U_i at x_i, where the points x_i are equally spaced with spacing $h = x_i - x_{i-1}$ is given by

$$U_n(x) = \sum_{i=0}^{n} U_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \frac{(x-x_0)(x-x_1)....(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_0)(x_i-x_1)....(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}$$

Consider the case in the resilience example in the lectures in which the solution at fine mesh points given by $x_{i+1/2} = 1/2(x_i + x_{i+1})$ is required.

1. Write down the four parts of the of the cubic polynomial that uses the values at $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ to create a cubic polynomial and simplify the expression by writing them in terms of the mesh spacing using the fact that $x_{i+1/2} = (1/2)(x_i + x_{i+1})$



$$L_{i-1} = \frac{(x-x_i)(x-x_{i+1})(x-x_{i+2})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})(x_{i-1}-x_{i+2})} = \frac{\frac{h}{2}\left(-\frac{h}{2}\right)\left(-\frac{3h}{2}\right)}{(-h)(-2h)(-3h)} = -\frac{1}{16}$$

$$L_i = \frac{(x-x_{i-1})(x-x_{i+1})(x-x_{i+2})}{(x_i-x_{i-1})(x_i-x_{i+1})(x_i-x_{i+2})} = \frac{\frac{3h}{2}\left(-\frac{h}{2}\right)\frac{3h}{2}}{h(-h)(-2h)} = \frac{9}{16}$$

$$L_{i+1} = \frac{(x-x_{i-1})(x-x_i)(x-x_{i+2})}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)(x_{i+1}-x_{i+2})} = \frac{\frac{3h}{2}\,\frac{h}{2}\,\frac{-3h}{2}}{2h\; h(-h)} = \frac{9}{16}$$

$$L_{i+2} = \frac{(x-x_{i-1})(x-x_i)(x-x_{i+1})}{(x_{i+2}-x_{i-1})(x_{i+2}-x_i)(x_{i+2}-x_{i+1})} = \frac{\frac{3h}{2}\,\frac{h}{2}\,\frac{-h}{2}}{3h\;2h\;h} = \frac{-1}{16}$$

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2. Use these simplified expressions to express the interpolation procedure in the form

$$U_{i+1/2} = aU_{i-1} + bU_i + cU_{i+1} + dU_{i+2}$$

$$= -\frac{1}{16} U_{i-1} + \frac{9}{16} U_i + \frac{9}{16} U_{i+1} - \frac{1}{16} U_{i+2}$$

What are *a,b,c,d, (hint a and d are negative)* ?

$a =$ $-1/16$

$b =$ $4/16$

$c =$ $9/16$

$d =$ $-1/16$

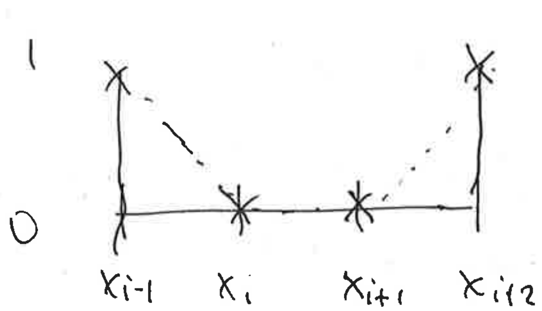
3. If the values U_{i-1}, U_i, U_{i+1} and U_{i+2} are positive write down a condition for $U_{i+1/2}$ to be negative

it is necessary for

$$(U_{i-1} + U_{i+2}) < 9(U_i + U_{i+1})$$

4. Draw an example of a function which would give rise to a negative value of $U_{i+1/2}$ (hint let some *U* values be zero)

A simple example is



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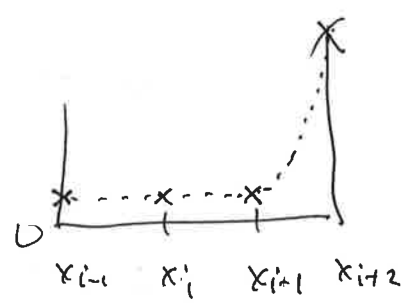

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