CS 3200: Introduction to Scientific Computing

In-class Activity: Numeric Integration

Problem: You need to perform integration on a function and have completely forgotten your calculus training. Use a various numeric integration methods to determine the integral of the function. (For all methods, $N = 3$)

$$\int_a^b f(x)\,dx\text{, where } a = 1, b = 4, f(x) = 2x^2 + x + 1 \text{ (sln: 52.5)}$$

Some Hints:

$f(1) = 4$	$f\left(-\frac{3}{2}\sqrt{\frac{3}{5}}+\frac{5}{2}\right) = f(1.34) = 5.9$	$f\left(\frac{3}{2}\right) = 7$	$f(2) = 11$	$f\left(\frac{5}{2}\right) = 16$
$f(3) = 22$	$f\left(\frac{3}{2}\sqrt{\frac{3}{5}}+\frac{5}{2}\right) =$ [redacted] $= f(3.66) = 31.5$	$f\left(\frac{7}{2}\right) = 29$	$f(4) = 37$	

1. **Solve** the problem using the Composite Trapezoid Method

Remember:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i)$$

$\Delta x = \frac{b-a}{N-1}$

$x_i = a + (i-1)\Delta x$

$w_i = \begin{cases} \frac{\Delta x}{2}, & i = 1, N \\ \Delta x, & i = 2, \ldots, N-1 \end{cases}$

$$\Delta x = \frac{4-1}{3-1} = \frac{3}{2} \qquad x_i = 1 + (i-1)\frac{3}{2}$$

$$w_i = \begin{cases} 3/4, & i = 1, 3 \\ 3/2, & i = 2 \end{cases}$$

$$\int \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$= \frac{3}{4} f\left(1 + (1-1)\frac{3}{2}\right) + \frac{3}{2} f\left(1 + (2-1)\frac{3}{2}\right) + \frac{3}{4} f\left(1 + (3-1)\frac{3}{2}\right)$$

$$= \frac{3}{4} f(1) + \frac{3}{2} f\left(\frac{5}{2}\right) + \frac{3}{4} f(4)$$

$$= \frac{3}{4} \cdot 4 + \frac{3}{2} \cdot 16 + \frac{3}{4} \cdot 37 = \boxed{54.75}$$

Name: ______________ uNID: ______________

2. **Solve** the problem using the Composite Simpson Rule

> **Remember:**
>
> $$\int_a^b f(x)dx \approx \sum_{i=1}^{N} \int_a^b g(x)dx = \sum_{i=1}^{2N+1} w_i f(x_i)$$
>
> $$\Delta x = \frac{b-a}{2N}$$
>
> $$w_i = \begin{cases} \frac{\Delta x}{3} & : \quad i = 1, 2N+1 \\ \frac{4\Delta x}{3} & : \quad i = 2, \ldots, 2N \quad (i \text{ even}) \\ \frac{2\Delta x}{3} & : \quad i = 3, \ldots, 2N-1 \quad (i \text{ odd}) \end{cases}$$ $1/6$, $2/3$, $1/3$
>
> $$x_i = a + (i-1)\Delta x = 1+(i-1)\tfrac{1}{2} = \tfrac{1}{2}+\tfrac{i}{2}$$

$$\Delta x = \frac{4-1}{2\cdot 3} = \frac{1}{2}$$

$x_1 = \frac{1}{2}+\frac{1}{2} = 1$ $\quad w_1 = 1/6$

$x_2 = \frac{1}{2}+\frac{2}{2} = 3/2$ $\quad w_2 = 2/3$

$x_3 = \frac{1}{2}+\frac{3}{2} = 2$ $\quad w_3 = 1/3$

$x_4 = \frac{1}{2}+\frac{4}{2} = 5/2$ $\quad w_4 = 2/3$

$x_5 = \frac{1}{2}+\frac{5}{2} = 3$ $\quad w_5 = 1/3$

$x_6 = \frac{1}{2}+\frac{6}{2} = 7/2$ $\quad w_6 = 2/3$

$x_7 = \frac{1}{2}+\frac{7}{2} = 4$ $\quad w_7 = 1/6$

$$\int \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + w_4 f(x_4) + w_5 f(x_5) + w_6 f(x_6) + w_7 f(x_7)$$

$$= \frac{f(1)}{6} + \frac{2f(3/2)}{3} + \frac{f(2)}{3} + \frac{2f(5/2)}{3} + \frac{f(3)}{3} + \frac{2f(7/2)}{3} + \frac{f(4)}{6}$$

$$= \frac{4}{6} + \frac{2\cdot 7}{3} + \frac{11}{3} + \frac{2\cdot 16}{3} + \frac{22}{3} + \frac{2\cdot 29}{3} + \frac{37}{6} = \frac{315}{6} = \boxed{52.5}$$

3. **Solve** the problem using the Gauss-Legendre Rule

> **Remember:**
>
> $$\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{i=1}^{N} w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right)$$
>
> | N | x_i | w_i |
> |---|---|---|
> | 1 | 0 | 2 |
> | 2 | $\pm\sqrt{\frac{1}{3}}$ | 1 |
> | 3 | 0 | $\frac{8}{9}$ |
> | | $\pm\sqrt{\frac{3}{5}}$ | $\frac{5}{9}$ |

$$\int \approx \frac{4-1}{2}\sum_{i=1}^{3} w_i f\left(\frac{4-1}{2}x_i + \frac{4+1}{2}\right)$$

$$= \frac{3}{2}\sum_{i=1}^{3} w_i f\left(\frac{3}{2}x_i + \frac{5}{2}\right)$$

$$= \frac{3}{2}\left(\frac{8}{9}f\left(\frac{3}{2}\cdot 0+\frac{5}{2}\right) + \frac{5}{9}f\left(\frac{3}{2}\cdot\sqrt{\frac{3}{5}}+\frac{5}{2}\right) + \frac{5}{9}f\left(-\frac{3}{2}\sqrt{\frac{3}{5}}+\frac{5}{2}\right)\right)$$

$$= \frac{3}{2}\left(\frac{8}{9}\cdot 16 + \frac{5}{9}\,31.5 + \frac{5}{9}\,5.9\right)$$

$$\approx 21.333 + 26.25 + 4.9166 \approx \boxed{52.5}$$

Name: ____________________ uNID: ____________________