A Framework for Equivalence of Cast Calculi

by Lu Kuang-Chen

1 Definitions

Definition 1. $s \approx^* s'$ if and only if there exists a state t such that $s \longrightarrow^* t$ and $t \approx s'$. Intuitively, $s \approx^* s'$ means s will be related to s'.

Notation 2. $[\![c]\!]$ means $\lambda V.\mathrm{applyCast}(V, c)$.

Lemma 3. If $U \approx U' : A$ then $[\![A \Rightarrow^l B]\!](U) \approx [\![\lceil A \Rightarrow^l B \rceil]\!](U') : B$.

Definition 4. $\mathrm{extCont}(c, [\Box\langle d\rangle]k) = [\Box\langle \mathrm{seq}(c,d)\rangle]k$

Lemma 5. $\mathrm{extCont}(\mathrm{seq}(c,d), \kappa) = \mathrm{extCont}(c, \mathrm{extCont}(d, \kappa))$

Lemma 6. If $\kappa \approx \kappa'$ then $[\Box\langle A \Rightarrow^l B\rangle]\kappa \approx \mathrm{extCont}(\lceil A \Rightarrow^l B\rceil, \kappa')$

Lemma 7. For all $U \approx U'$ and $V \approx V'$ and $k \approx k'$ then $\mathrm{doApp}(U, V, \kappa) \approx^* \mathrm{doApp}'(U', V', \kappa')$.

Proof. By induction on $U \approx U'$. There are two cases.

Case $\langle \lambda x : A.\, e : B, E\rangle \approx \langle \lambda x : \mathrm{id}(A).\, e : \mathrm{id}(B), E'\rangle$

In this case

$$
\begin{aligned}
& \mathrm{doApp}(U, V, k) \\
={} & \mathrm{doApp}(\langle \lambda x : A.\, e : B, E\rangle, V, \kappa) \\
={} & \langle e, E[x \leftarrow V], \kappa\rangle
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{doApp}'(U', V', \kappa') \\
={} & \mathrm{doApp}'(\langle \lambda x : \mathrm{id}(A).\, e : \mathrm{id}(B), E'\rangle, V', \kappa') \\
={} & [\![\mathrm{id}(A)]\!](V') \ggcurly \lambda V'.\langle e, E'[x \leftarrow V'], \kappa'\rangle \\
(\star)\quad ={} & \underline{\mathrm{Just}\, V'} \ggcurly \lambda V'.\langle e, E'[x \leftarrow V'], \kappa'\rangle \\
={} & \langle e, E'[x \leftarrow V'], \kappa'\rangle
\end{aligned}
$$

The step marked with $(\star)$ follows from the basic property of $\mathrm{id}(\cdot)$, that is

$$[\![\mathrm{id}(A)]\!](V) = \mathrm{Just}\, V$$

By definition, $\langle e, E[x \leftarrow V], \kappa\rangle \approx \langle e, E'[x \leftarrow V'], \kappa'\rangle$.

Thus $\langle e, E[x \leftarrow V], \kappa\rangle \approx^* \langle e, E'[x \leftarrow V'], \kappa'\rangle$.

Case $\langle U_1\langle A \to B \Rightarrow^l C \to D\rangle, E\rangle \approx \langle \lambda x : \mathrm{seq}(\lceil C \Rightarrow^l A\rceil, c).\, e : \mathrm{seq}(d, \lceil B \Rightarrow^l D\rceil), E'\rangle$

Let $U_1' = \langle \lambda x^{c,d}.e, E'\rangle$, we also have $U_1 \approx U_1'$.

In this case,

$$
\begin{aligned}
&\mathrm{doApp}(U, V, k) \\
={}&\mathrm{doApp}(\langle U_1 \langle A \to B \Rightarrow^l C \to D\rangle, E\rangle, V, k) \\
\longrightarrow{}& [\![C \Rightarrow^l A]\!](V) \gg \lambda V_1.\langle V_1, [(U_1 \,\Box)][B \Rightarrow^l D]\kappa\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{doApp}'(U', V', k') \\
={}&\mathrm{doApp}'(\langle \lambda x : \mathrm{seq}(\lceil C \Rightarrow^l A\rceil, c).e : \mathrm{seq}(d, \lceil B \Rightarrow^l D\rceil), E'\rangle, V', k') \\
={}&[\![\mathrm{seq}(\lceil C \Rightarrow^l A\rceil, c)]\!](V') \gg \lambda V_1'.\langle e, E'[x \leftarrow V_1'], \mathrm{extCont}(\mathrm{seq}(d, \lceil B \Rightarrow^l D\rceil), k')\rangle \\
(\star)\quad ={}&\underline{[\![\lceil C \Rightarrow^l A\rceil]\!](V') \gg [\![c]\!]} \gg \lambda V_1'.\langle e, E'[x \leftarrow V_1'], \mathrm{extCont}(\mathrm{seq}(d, \lceil B \Rightarrow^l D\rceil), k')\rangle
\end{aligned}
$$

The step marked with $(\star)$ follows from the basic property of $\mathrm{seq}(\cdot, \cdot)$, that is

$$
[\![\mathrm{seq}(c, d)]\!](V) = [\![c]\!](V) \gg [\![d]\!]
$$

By Lemma 3, we can case-split $[\![C \Rightarrow^l A]\!](V)$ and $[\![\lceil C \Rightarrow^l A\rceil]\!](V')$ at the same time.

Case $[\![C \Rightarrow^l A]\!](V) = \mathrm{Err}\, l$ and $[\![\lceil C \Rightarrow^l A\rceil]\!](V') = \mathrm{Err}\, l$

In this case

$$
\begin{aligned}
&\mathrm{doApp}(U, V, \kappa) \\
\longrightarrow{}& \mathrm{Err}\, l \gg \lambda V_1.\langle V_1, [(U_1 \,\Box)][B \Rightarrow^l D]\kappa\rangle \\
\longrightarrow{}& \mathrm{Err}\, l
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{doApp}'(U', V', \kappa') \\
={}& \mathrm{Err}\, l \gg [\![c]\!] \gg \lambda V_1'.\langle e, E'[x \leftarrow V_1'], \mathrm{extCont}(\mathrm{seq}(d, \lceil B \Rightarrow^l D\rceil), k')\rangle \\
={}& \mathrm{Err}\, l
\end{aligned}
$$

By definition, $\mathrm{Err}\, l \approx \mathrm{Err}\, l$.

Thus $\mathrm{doApp}(U, V, \kappa) \approx^* \mathrm{doApp}'(U', V', \kappa')$.

Case $[\![C \Rightarrow^l A]\!](V) = \mathrm{Just}\, V_1$ and $[\![\lceil C \Rightarrow^l A\rceil]\!](V') = \mathrm{Just}\, V_1'$ and $V_1 \approx V_1'$

In this case

$$
\begin{aligned}
&\mathrm{doApp}(U, V, \kappa) \\
\longrightarrow{}& \mathrm{Just}\, V_1 \gg \lambda V_1.\langle V_1, [(U_1 \,\Box)][B \Rightarrow^l D]\kappa\rangle \\
\longrightarrow{}& \langle V_1, [(U_1 \,\Box)][B \Rightarrow^l D]\kappa\rangle \\
\longrightarrow{}& \mathrm{doApp}(U_1, V_1, [\Box\langle B \Rightarrow^l D\rangle]\kappa)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{doApp}'(U', V', \kappa') \\
={}& \mathrm{Just}\, V_1' \gg [\![c]\!] \gg \lambda V_2'.\langle e, E'[x \leftarrow V_2'], \mathrm{extCont}(\mathrm{seq}(d, \lceil B \Rightarrow^l D\rceil), k')\rangle \\
={}& [\![c]\!](V_1') \gg \lambda V_2'.\langle e, E'[x \leftarrow V_2'], \mathrm{extCont}(\mathrm{seq}(d, \lceil B \Rightarrow^l D\rceil), k')\rangle \\
(\star)\quad ={}& [\![c]\!](V_1') \gg \lambda V_2'.\langle e, E'[x \leftarrow V_2'], \mathrm{extCont}(d, \mathrm{extCont}(\lceil B \Rightarrow^l D\rceil, \kappa'))\rangle \\
={}& \mathrm{doApp}'(\langle \lambda x^{c,d}.e, E'\rangle, V_1', \mathrm{extCont}(\lceil B \Rightarrow^l D\rceil, \kappa')) \\
={}& \mathrm{doApp}'(U_1', V_1', \mathrm{extCont}(\lceil B \Rightarrow^l D\rceil, \kappa'))
\end{aligned}
$$

The step marked with $(\star)$ follows from Lemma 5.

Now we can use the induction hypothesis two show that

$$
\mathrm{doApp}(U_1, V_1, [\Box\langle B \Rightarrow^l D\rangle]\kappa) \approx^* \mathrm{doApp}'(U_1', V_1', \mathrm{extCont}(\lceil B \Rightarrow^l D\rceil, \kappa'))
$$

because $U_1 \approx U_1'$ and $V_1 \approx V_1'$ and, by Lemma 6, $[\square \langle B \Rightarrow^l D \rangle] \kappa \approx \mathrm{extCont}(\lceil B \Rightarrow^l D \rceil, \kappa')$.

Thus $\mathrm{doApp}(U, V, \kappa) \approx^* \mathrm{doApp}'(U', V', \kappa')$.

$\square$