Synge Theorem and Corollaries

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Abstract

Introduced the basic concepts concerning Synge Theorem and the prove of main theorems.

Keywords: Riemannian Geometry, Differential manifold, Space form, Synge Theorem

1 Introduction

When I was doing exercises in Riemannian Geometry, some well known results are found, which are helpful to have an inner sight into Riemannian Geometry, but hard to remember. Hence, I have to write them down for reviewings in the future.

2 Space Form

Definition 2.1. act in a totally discontinuous manner
Let (M, g) be a Riemannian manifold. We say group G (some homeomorphisms of M) **acts in a totally discontinuous manner** on M, if for every x in M, there is an neighbourhood of x denoted by U, such that for all $g \neq e \in G$, $g(U) \cap U = \emptyset$.

Remark 2.1. Since G is a group, for any $g_1, g_2 \in G$, $g_1^{-1} \circ g_2(U) \cap U = \emptyset$. Thus $g_1(U) \cap g_2(U) = \emptyset$.

Definition 2.2. Covering transformation, deck transformation
Let (Y_1, p_1) and (Y_2, p_2) are covering spaces of topology space Y, $F : Y_1 \to Y_2$ is called a

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covering transformation if $p_2 \circ F = p_1$ and F is continous. Directly from the definition we have F is a covering map. So if F is an injection, then the it's a homeomorphism when we call F a covering isomorphism.

If F is a automorphism, which means $Y_1 = Y_2 = Y, p_1 = p_2 = p$, then we call it a deck transformation of (Y, p). And $Deck(Y, p)$ is all the deck transformations on (Y, p) which forms a group.

Definition 2.3. Regular covering map
A map π is a covering map, and in addition for every $x \in M$, any two elements $x_1 \neq x_2$ in $\pi^{-1}(\{x\})$, there is a deck transformation f such that $f(x_1) = x_2$, which means $Deck(\widetilde{M})$ acts transitively on $\pi^{-1}(\{x\})$.

The group action is discrete and free means that trajectories of G are discrete and there is no fix points for every $g \in G$. G acts transitively means that $Gx = M$. Then for any G acts freely and transitively, there is a natural projection$\pi : M \to M/G$.

If we let M be a connected and locally path-connected topology space, and G be the sets of homeomorphisms that acts in a totally discontinuous manner, then the quotion map $\pi : M \to M/G$ is a regular covering map. And homeomorphisms in G is consisted of deck transformations.

Proof. Apparently π is a continuous and surjective. Given ant open set U of M/G, then $\widetilde{U} = \pi^{-1}(U)$ is a open set in M, take componets of $\widetilde{U} = \bigcup\limits_{\alpha \in I} U_\alpha$. Since G acts in a totally discontinuous manner, U_α doesn't intersect each other, thus $\square$

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- Universal covering map is a covering map such that $\pi_1(\widetilde{M}) = e$.

- A regular covering map also means that $\pi_*(\pi_1(\widetilde{M}))$ is a normal subgroup of $\pi_1(M)$

Theorem 2.1. ***(Differential Geometry Xu senlin Page.395)*** *M has a universal covering space, then the covering space is unique up to the equivalence of covering isomorphism. M is semi locally simply connected, then M has a universal covering space.*

Theorem 2.2. ***(Xu Senlin Page.395)*** *Let $(\widetilde{M}, p)$ is a universal covering of M. $x_1, x_2 \in \widetilde{M}, p(x_i) = x,$ Then there is a unique deck transformation $\widetilde{h}$ such that $\widetilde{h}(x_1) = x_2$*

Theorem 2.3. ***(Do Carmo Riemannian Geometry page.165)*** *Let (M, g) be a complete Riemannian manifold with constant sectional curvature K(i.e. a space form), without loss of generality K is chosen to be -1, 0 or 1. Then M is isometric to $\widetilde{M}/\Gamma$,* *where $\widetilde{M}$ is the universal covering space of M, Γ is a subgroup of isometries which acts in a totally discontinuous manner on $\widetilde{M}$, actually it's the group of deck transformations on $\widetilde{M}$ and the metric on $\widetilde{M}$ is induced from the covering map $\pi : \widetilde{M} \to \widetilde{M}/\Gamma$.*

Proof. □

Remark 2.2.

- Now the covering map is regular. [prove it]
- If the fiber of an universal covering map of M has finite cardinality, then the fundamental group of M is finite.

3 Fix Point

Definition 3.1. (**orientation**)

Theorem 3.1. ***(Kobayashi Transformation groups in Differential Geometry page.63)*** *Let M be a compact orientable manifold with positive sectional curvature. Let f be an isometry of M.*

- *If dim(M) is even and f is orientation preserving, then f has a fix point.*
- *If dim(M) is odd and f is orientation-inversing, then f has a fix point.*

Proof. □

Theorem 3.2. ***(Kobayashi page.65) Synge Theorem*** *let (M, g) be a compact Riemannian manifold positive sectional curvature, worth noting that every compact Riemannian manifold is complete by Hopf-Rinow theorem.*

- *if dim(M) is even and orientable, then M is simply connected.*
- *if dim(M) is odd, then M is orientable*

Proof. If $dim(M)$ is even and orientable, then we consider it's universal covering $\widetilde{(}M)$, with $\pi : \widetilde{M} \to M$ is the covering map, we choose the orientation of $\widetilde{M}$ which forces π is orientation preserving. And the metric on $\widetilde{M}$ is $\pi^* g$ which is locally isometric to M.

We now have every deck transformation(i.e. covering transformation) f on $\widetilde{M}$ is orientation preserving. Since for every $x \in \widetilde{M} = \pi \circ f = \pi$, then $(d\pi)_{f(x)} \circ (df)_x = (d\pi)_x$. Then they must have fix points, but every covering transformation must have no fix points, and the proof is at following.

If $f(x_1) = x_2$,$x_1 \neq x_2$, $\pi(x_1) = \pi(x_2)$ and there is a fix point x of f, then we lift the paths from x to x_1, and x to x_2. By the uniqueness of lifting, the two lifting curves should be identical, which contradicts $x_1 \neq x_2$. □

4 Citation

I am citing [?] as an example.