Countability Axiom in Topology

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December 14, 2024 3:29pm

Abstract

Topology lemmas in measure theory

Keywords: topology, measure theory

1 Introduction

Some results we found and trusted.

2 First Contable Space

Definition 2.1. (**Herenditarily Lindelöf Space**)
If every subspace of X is Lindelöf, then we say it's Herenditarily Lindelöf.

Theorem 2.1. ***(Separable and First Countable Space)***
Let X be a Hausdorff, locally compact and Herenditarily Lindelöf space, then it is first countable.

Proof. Given a point x in X, then the class of open sets $\{X/K|\ K$ is a paracompact neighborhood of x $\}$ covers $X/\{x\}$. Since $X/\{x\}$ is oepn and X is Herenditarily Lindelöf, $X/\{x\}$ is Lindelöf. There are contablely number of K_n, such that $\{X/K_n\}$ covers $X/\{x\}$. Without loss of generality, we assume $\{K_n\}$ is decreasing(nested). We claim that $\{X/K_n\}$ is a neighborhood basis of x. Given U is a neighborhood of x, if U doesn't contain any K_n, we can choose $x_n \in K_n/U$. Actually a compact space is sequential compact. So $\{x_n\} \in K_0/U$ have at least one limit point y. For any compact neighborhood V of y, $V \cap K_{n_i} \neq \emptyset$ for infinite n_i, then $V \cap \bigcap\limits_{n=1}^{\infty} K_n \neq \emptyset$. Since $\bigcap\limits_{n=1}^{\infty} K_n$ is $\{x\}$, then $x \in V$. Then $x = y$, which controdicts U is a neighborhood of x. $\square$

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Theorem 2.2. *The $R_{\updownarrow}$ is a Lindelöf, Separable, first countable space, but it's not second countable.*

Proof. $\square$

Theorem 2.3. *Let X is a second countable space, then every cover of it admits a countable subcover.*

Proof. Given an open cover $\{U_\alpha\}$ of X, and countable basis $\mathcal{B}$ of X. For every $B_n \in \mathcal{B}$, if there is at least one U_α contain it, we choose this U_α and denote it by U_{α_n}. Then we have a countable class of open set $\{U_{\alpha_n}\}$. It's the subcover we need.

For any x in X, there is $U \in \{U_\alpha\}$ such that $x \in U$, then there is a base $x \in B_k \subseteq U$. Then $x \in B_k \subseteq U_{\alpha_k}$, which completes the proof. $\square$

Theorem 2.4. ***(Secondly Countable)***
Let X be a separable metric space, then X is second countable.
Let X be a Lindelöf metric space, then X is second countable.
let X be a first countable metric space, then X is second countable.

Proof. $\square$

Corollary 2.1. *Let X is separable, Hausdorff, locally compact, metric space, then every cover of it admits a countable subcover. Then every compact set of X is G_δ*

Remark 2.1. (**Classical Results**)

(a) First countablility axiom guarantees that $x \in \bar{X}$ iff. there is a sequance x_n in X convergent to x. And f continuous iff. for every $x_n \to x$, $f(x_n) \to f(x)$.

(b) Every basis of C_2 space contains a countable basis.

(c) Subspaces of C_1 or C_2 spaces are C_1, or C_2.

(d) Subspaces of Lindelöf, separable space are not necessary Lindelöf or separable. But closed subspaces of Lindelöf space are Lindelöf.

(e) Eg. I^2 endowed with order topology is Lindelöf but $I \times (0, 1)$ is not Lindelöf.

(f) Products of Lindelöf space is not necessary Lindelöf, but countbalely products separable space is separable.

(g) Eg. Sergenfry Space R_l^2 is not Lindelöf.

(h) Image of Continuous map on Lindelöf, separable is Lindelöf, separable. If a continuous map is open, then image from C_1, C_2 space is also C_1, C_2.

(i) A class of disjoint open sets in separable space is countable.

(j) X is Lindelöf, Y is compact, then $X \times Y$ is Lindelöf.

(k) If X is a C_1 topology group, if X is Lindelöf or separable, then X is C_2.[Hint: B_n is a countable basis at e, D is a countable dense subset, then $\{dB_n | d \in D\}$ is a basis. If X is Lindelöf, then for every n, there is a countable set $C_n \in X$, such that $\mathcal{K}_n = cB_n | c \in C_n$ is an open cover of X, then $\bigcup\limits_{n=1}^{\infty} \mathcal{K}_n$ is a basis.]