# Countability Axiom in Topology

jiang Zao*

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**Abstract**

Topology lemmas in measure theory

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## 1 Introduction

Some results we found and trusted.

## 2 First Countable Space

**Definition 2.1.** (**Herenditarily Lindelöf Space**)
If every subspace of $X$ is Lindelöf, then we say it's Herenditarily Lindelöf.

**Theorem 2.1.** ***(Separable and First Countable Space )***
*Let $X$ be a Hausdorff, locally compact and Herenditarily Lindelöf space, then it is first countable.*

**Proof.** Given a point $x$ in $X$, then the class of open sets $\{X/K|\ K$ is a paracompact neighborhood of $x$ $\}$ covers $X/\{x\}$. Since $X/\{x\}$ is oepn and $X$ is Herenditarily Lindelöf, $X/\{x\}$ is Lindelöf. There are countablely number of $K_n$, such that $\{X/K_n\}$ covers $X/\{x\}$. Without loss of generality, we assume $\{K_n\}$ is decreasing(nested). We claim that $\{X/K_n\}$ is a neighborhood basis of x. Given U is a neighborhood of x, if U doesn't contain any $K_n$, we can choose $x_n \in K_n/U$. Actually a compact space is sequential compact. So $\{x_n\} \in K_0/U$ have at least one limit point y. For any compact neighborhood $V$ of y, $V \cap K_{n_i} \neq \emptyset$ for infinite $n_i$, then $V \cap \bigcap_{n=1}^{\infty} K_n \neq \emptyset$. Since $\bigcap_{n=1}^{\infty} K_n$ is $\{x\}$, then $x \in V$. Then $x = y$, which controdicts U is a neighborhood of x. $\square$

*Sun Yat-sen University, Guang Zhou, China. Email: jiangzmath@outlook.com.

**Theorem 2.2.** *Space $R_l$ is a Lindelöf, Separable, first countable space, but it's not second countable.*

**Proof.** Open sets in $R_l$ is generated by $\{[a, b) : a, b \in R\}$ $\square$

**Theorem 2.3.** *Let $X$ is a second countable space, then every cover of it admits a countable subcover.*

**Proof.** Given an open cover $\{U_\alpha\}$ of $X$, and countable basis $\mathcal{B}$ of X. For every $B_n \in \mathcal{B}$, if there is at least one $U_\alpha$ contain it, we choose this $U_\alpha$ and denote it by $U_{\alpha_n}$. Then we have a countable class of open set $\{U_{\alpha_n}\}$. It's the subcover we need.

For any $x$ in $X$, there is $U \in \{U_\alpha\}$ such that $x \in U$, then there is a base $x \in B_k \subseteq U$. Then $x \in B_k \subseteq U_{\alpha_k}$, which completes the proof. $\square$

**Theorem 2.4.** ***(Secondly Countable)***
*Let $X$ be a separable metric space, then $X$ is second countable.*
*Let $X$ be a Lindelöf metric space, then $X$ is second countable.*
*let $X$ be a first countable metric space, then $X$ is second countable.*

**Proof.** $\square$

**Remark 2.1.** The discrete metric $d(x, y) = 1$ iff. $x = y$ on $\mathbb{R}$ is a metric space, first countable but not $C_2$.

**Corollary 2.1.** *Let $X$ is separable, Hausdorff, metric space (can be substituded by herenditarily Lindelöf), then every compact set of $X$ is $G_\delta$*

**Remark 2.2.** The space $I^I$, where $I = [0, 1]$ endowed with product topology. It's compact housdorff. From Hewitt-Marczewski-Pondiczery theorem, it's also separable, but it's not $C_1$, and the sigle point set on it is not countablely intercept of basis sets, which means that it shouldn't be in the countablely intercept of open sets.

**Remark 2.3.** (**Classical Results**)

(a) First countability axiom guarantees that $x \in \bar{X}$ iff. there is a sequance $x_n$ in X convergent to x. And f continuous iff. for every $x_n \to x$, $f(x_n) \to f(x)$.

(b) Every basis of $C_2$ space contains a countable basis.

(c) Subspaces of $C_1$ or $C_2$ spaces are $C_1$, or $C_2$.

(d) Subspaces of Lindelöf, separable space are not necessary Lindelöf or separable. But closed subspaces of Lindelöf space are Lindelöf.

(e) Eg. $I_o^2$ endowed with order topology, it's quotion space which sticks $[0,1] \times \{0,1\}$ together as one point is Lindelöf but $I \times (0,1)$ is not Lindelöf.

(f) Products of Lindelöf space is not necessary Lindelöf, but countbalely products separable space is separable.

(g) Eg. Sergenfry Space $R_l^2$ is not Lindelöf.

(h) Image of Continuous map on Lindelöf, separable is Lindelöf, separable. If a continuous map is open, then image from $C_1$, $C_2$ space is also $C_1$, $C_2$.

(i) A class of disjoint open sets in separable space is countable.

(j) X is Lindelöf, Y is compact, then $X \times Y$ is Lindelöf.

(k) If X is a $C_1$ topology group, if X is Lindelöf or separable, then X is $C_2$.[Hint: $B_n$ is a countable basis at e, D is a countable dense subset, then $\{dB_n | d \in D\}$ is a basis. If X is Lindelöf, then for every n, there is a countable set $C_n \in X$, such that $\mathcal{K}_n = \{cB_n | c \in C_n\}$ is an open cover of X, then $\bigcup\limits_{n=1}^{\infty} \mathcal{K}_n$ is a basis.]