

Countability Axiom in Topology

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Abstract

Topology lemmas in measure theory

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1 Introduction

Some results we found and trusted.

2 First Countable Space

Definition 2.1. (Herenditarily Lindelöf Space)

If every subspace of X is Lindelöf, then we say it's Herenditarily Lindelöf.

Theorem 2.1. (*Separable and First Countable Space*)

Let X be a Hausdorff, locally compact and Herenditarily Lindelöf space, then it is first countable.

Proof. Given a point x in X , then the class of open sets $\{X/K \mid K \text{ is a paracompact neighborhood of } x\}$ covers $X/\{x\}$. Since $X/\{x\}$ is open and X is Herenditarily Lindelöf, $X/\{x\}$ is Lindelöf. There are countably number of K_n , such that $\{X/K_n\}$ covers $X/\{x\}$. Without loss of generality, we assume $\{K_n\}$ is decreasing(nested). We claim that $\{X/K_n\}$ is a neighborhood basis of x . Given U is a neighborhood of x , if U doesn't contain any K_n , we can choose $x_n \in K_n/U$. Actually a compact space is sequential compact. So $\{x_n\} \in K_0/U$ have at least one limit point y . For any compact neighborhood V of y , $V \cap K_{n_i} \neq \emptyset$ for infinite n_i , then $V \cap \bigcap_{n=1}^{\infty} K_n \neq \emptyset$. Since $\bigcap_{n=1}^{\infty} K_n$ is $\{x\}$, then $x \in V$. Then $x = y$, which contradicts U is a neighborhood of x . \square

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Theorem 2.2. *The space R_l is a Lindelöf, Separable, first countable space, but it's not second countable.*

Proof. Open sets in R_l is generated by $\{[a, b) : a, b \in \mathbb{R}\}$ □

Theorem 2.3. *Let X be a second countable space, then every cover of it admits a countable subcover.*

Proof. Given an open cover $\{U_\alpha\}$ of X , and countable basis \mathcal{B} of X . For every $B_n \in \mathcal{B}$, if there is at least one U_α contain it, we choose this U_α and denote it by U_{α_n} . Then we have a countable class of open set $\{U_{\alpha_n}\}$. It's the subcover we need.

For any x in X , there is $U \in \{U_\alpha\}$ such that $x \in U$, then there is a base $x \in B_k \subseteq U$. Then $x \in B_k \subseteq U_{\alpha_k}$, which completes the proof. □

Theorem 2.4. (Secondly Countable)

Let X be a separable metric space, then X is second countable.

Let X be a Lindelöf metric space, then X is second countable.

let X be a first countable metric space, then X is second countable.

Proof. □

Remark 2.1. The discrete metric $d(x, y) = 1$ iff. $x \neq y$ on \mathbb{R} is a metric space, first countable but not C_2 .

Corollary 2.1. *Let X be separable, Hausdorff, metric space (can be substituted by hereditarily Lindelöf), then every compact set of X is G_δ*

Remark 2.2. The space I^I , where $I = [0, 1]$ endowed with product topology. It's compact housdorff. From Hewitt-Marczewski-Pondiczery theorem, it's also separable, but it's not C_1 , and the sigle point set on it is not countably intercept of basis sets, which means that it shouldn't be in the countably intercept of open sets.

Remark 2.3. (Classical Results)

(a) First countability axiom guarantees that $x \in \bar{X}$ iff. there is a sequence x_n in X convergent to x . And f continuous iff. for every $x_n \rightarrow x$, $f(x_n) \rightarrow f(x)$.

(b) Every basis of C_2 space contains a countable basis.

(c) Subspaces of C_1 or C_2 spaces are C_1 , or C_2 .

(d) Subspaces of Lindelöf, separable space are not necessary Lindelöf or separable. But closed subspaces of Lindelöf space are Lindelöf.

- 54 (e) Eg. I_o^2 endowed with order topology, it's quotient space which sticks $[0, 1] \times \{0, 1\}$
 55 together as one point is Lindelöf but $I \times (0, 1)$ is not Lindelöf.
- 56 (f) Products of Lindelöf space is not necessary Lindelöf, but countably products separable
 57 space is separable.
- 58 (g) Eg. Sorgenfrey Space R_l^2 is not Lindelöf.
- 59 (h) Image of Continuous map on Lindelöf, separable is Lindelöf, separable. If a continuous
 60 map is open, then image from C_1 , C_2 space is also C_1 , C_2 .
- 61 (i) A class of disjoint open sets in separable space is countable.
- 62 (j) X is Lindelöf, Y is compact, then $X \times Y$ is Lindelöf.
- 63 (k) If X is a C_1 topology group, if X is Lindelöf or separable, then X is C_2 . [Hint: B_n
 64 is a countable basis at e , D is a countable dense subset, then $\{dB_n | d \in D\}$ is a
 65 basis. If X is Lindelöf, then for every n , there is a countable set $C_n \in X$, such that
 66 $\mathcal{K}_n = \{cB_n | c \in C_n\}$ is an open cover of X , then $\bigcup_{n=1}^{\infty} \mathcal{K}_n$ is a basis.]