Synge Theorem and Corollaries

JiangZao*

November 16, 2024 9:34pm

Abstract

Introduced the basic concepts concerning Synge Theorem and the prove of main theorems.

Keywords: Riemannian Geometry, Differential manifold, Space form, Synge Theorem

1 Introduction

When I was doing exercises in Riemannian Geometry, some well known results are found, which are helpful to have an inner sight into Riemannian Geometry, but hard to remember. Hence, I have to write them down for reviewings in the future. Every covering map in this paper, we automatically admit that the covering spaces and base spaces are connected and locally path-connected, and thus path connected(Why?).

2 Space Form

Definition 2.1. act in a totally discontinuous manner
Let (M, g) be a Riemannian manifold. We say group G (some homeomorphisms of M) **acts in a totally discontinuous manner** on M, if for every x in M, there is an neighbourhood of x denoted by U, such that for all $g \neq e \in G$, $g(U) \cap U = \emptyset$.

Remark 2.1. Since G is a group, for any $g_1, g_2 \in G$, $g_1^{-1} \circ g_2(U) \cap U = \emptyset$. Thus $g_1(U) \cap g_2(U) = \emptyset$.

*Sun Yat-Sen University, Guang Zhou, China. Email: Jiangzmath@outlook.com.

Definition 2.2. Covering transformation, deck transformation
Let (Y_1, p_1) and (Y_2, p_2) are covering spaces of topology space Y, $F : Y_1 \to Y_2$ is called a covering transformation if $p_2 \circ F = p_1$ and F is continous. Directly from the definition we have F is a covering map. So if F is an injection, then the it's a homeomorphism when we call F a covering isomorphism.

If F is a automorphism, which means $Y_1 = Y_2 = Y, p_1 = p_2 = p$, then we call it a deck transformation of (Y, p). And $Deck(Y, p)$ is all the deck transformations on (Y, p) which forms a group.

Definition 2.3. Regular covering map
A map π is a covering map, and in addition for every $x \in M$, any two elements $x_1 \neq x_2$ in $\pi^{-1}(\{x\})$, there is a deck transformation f such that $f(x_1) = x_2$, which means $Deck(\widetilde{M})$ acts transitively on $\pi^{-1}(\{x\})$.

The group action is discrete and free means that trajectories of G are discrete and there is no fix points for every $g \in G$. G acts transitively means that $Gx = M$. Then for any G acts freely and transitively, there is a natural projection$\pi : M \to M/G$.

Theorem 2.1. *Let M be a connected and locally path-connected topology space, and G be the set of homeomorphisms that acts in a totally discontinuous manner on M, then the quotion map $\pi : M \to M/G$ is a regular covering map. And homeomorphisms in G is consisted of deck transformations.*

Proof. Clearly that π is continuous and surjective. For any given $x \in M$, there is an open set U in M, such that $g(U) \cap U = \emptyset$ for any $g \in G$. Then $\widetilde{U} = \pi^{-1}(\pi(U))$ is a open set in M, decomposite $\widetilde{U}$ by the union of it's componets. That is $\widetilde{U} = \bigcup\limits_{\alpha \in I} U_\alpha$.

Since G acts in a totally discontinuous manner on U, U_α doesn't intersect each other. And because elements in M/G are trajectories of G's action on U, we have $U_\alpha = \alpha(U)$ and $U_e = U$, where e is the identity of G. Thus U_α is homeomorphic to each other, π is a identity from U to $[U]$. So π is a regular covering map. And every $\alpha \in G$ is a deck transformation.

Given a deck transformation σ such that it maps a given element $a \in U$ to $\sigma(a) \in U_\alpha$, and $\pi(x) = \pi\sigma(x)$ for any $x \in M$. Since $a, \sigma(a) \in \pi^{-1}(a)$, there is a g in G that maps a to $\sigma(a)$. Then $\sigma \circ g$ and $g \circ \sigma$ are deck transformations on M with at least one fix point each, but every deck transformation with one fix point is a identity[Why?]. Hence $\sigma \in G$. The prove is completed. $\square$

- Universal covering map is a covering map such that $\pi_1(\widetilde{M}) = e$.

- A regular covering map also means that $\pi_*(\pi_1(\widetilde{M}))$ is a normal subgroup of $\pi_1(M)$

Theorem 2.2. *Let $\widetilde{M}$ be the universal covering of M. If $\pi_1(M)$ is not $\{e\}$, then $\pi^{-1}(x)$ have cardinality at least 2. More generaly, cardinality of $\pi^{-1}(x)$ is equal to the cardinality of $\pi_1(M)$, since number of sheets of a covering map is equal to the lagrange index $[\pi_*(\pi_1(\widetilde{M}))\pi_1(M)]$(Why? prove it by the remark below).*

And any given $x_1 \neq x_2 \in \pi^{-1}(x)$, there exists a unique deck transformation σ on $\widetilde{M}$ such that $\sigma(x_1) = x_2$.

- $\pi_* : \pi_1(\widetilde{M}, \widetilde{x}_0) \to \pi_1(M, x_0)$ is injective.

Theorem 2.3. ***(Differential Geometry Xu senlin Page.395)*** *M has a universal covering space, then the covering space is unique up to the equivalence of covering isomorphism. M is semi locally simply connected, then M has a universal covering space.*

Theorem 2.4. ***(Xu Senlin Page.395)*** *Let $(\widetilde{M}, p)$ is a universal covering of M. And given $x_1, x_2 \in \widetilde{M}, p(x_i) = x, i = 1, 2$. Then there is a unique deck transformation $\widetilde{h}$ such that $\widetilde{h}(x_1) = x_2$*

Theorem 2.5. ***(Do Carmo Riemannian Geometry page.165)*** *Let (M, g) be a complete Riemannian manifold with constant sectional curvature K(i.e. a space form), without loss of generality K is chosen to be -1, 0 or 1. Then M is isometric to $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is the universal covering space of M, Γ is a subgroup of isometries which acts in a totally discontinuous manner on $\widetilde{M}$, actually it's the group of deck transformations on $\widetilde{M}$ and the metric on $\widetilde{M}$ is induced from the covering map $\pi : \widetilde{M} \to \widetilde{M}/\Gamma$.*

Remark 2.2.

- Now the covering map is regular. [prove it]
- If the fiber of an universal covering map of M has finite cardinality, then the fundamental group of M is finite.

3 Fix Point

Definition 3.1. (orientation)

Theorem 3.1. ***(Kobayashi Transformation groups in Differential Geometry page.63)*** *Let M be a compact orientable manifold with positive sectional curvature. Let f be an isometry of M.*

- *If $dim(M)$ is even and f is orientation preserving, then f has a fix point.*
- *If $dim(M)$ is odd and f is orientation-inversing, then f has a fix point.*

Proof. $\square$

Theorem 3.2. ***(Kobayashi page.65) Synge Theorem*** *let (M, g) be a compact Riemannian manifold positive sectional curvature, worth noting that every compact Riemannian manifold is complete by Hopf-Rinow theorem.*

- *if $dim(M)$ is even and orientable, then M is simply connected.*
- *if $dim(M)$ is odd, then M is orientable*

Proof. If $dim(M)$ is even and orientable, then we consider it's universal covering $\widetilde{M}$, with $\pi : \widetilde{M} \to M$ is the covering map, we choose the orientation of $\widetilde{M}$ which forces π is orientation preserving. And the metric on $\widetilde{M}$ is $\pi^* g$ which is locally isometric to M.

We now have every deck transformation(i.e. covering transformation) f on $\widetilde{M}$ is orientation preserving. Since for every $x \in \widetilde{M} = \pi \circ f = \pi$, then $(d\pi)_{f(x)} \circ (df)_x = (d\pi)_x$. Then they must have fix points, but every covering transformation must have no fix points, and the proof is at following.

If $f(x_1) = x_2$,$x_1 \neq x_2$, $\pi(x_1) = \pi(x_2)$ and there is a fix point x of f, then we lift the paths from x to x_1, and x to x_2. By the uniqueness of lifting, the two lifting curves should be identical, which contradicts $x_1 \neq x_2$. $\square$