Zao Jiang *

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Abstract

We detail some proofs in Hörmander's book concerning Fredholm theory.

Keywords: Functional analysis, Fredholm operator.

1 Introduction

To learn functional analysis through pde theory, my functional analysis courses imported some topics from Hörmander's book (*The Analysis of Linear Partial Differential Operators III Pseudo-Differential Operators*), and Serge Lang's book, (*Real and Functional Analysis*). But sometimes it's really hard to follow Hörmander's proof. So I'd like to choose some important theorems to figure out the details of their proofs.

2 Basic Knowledges

Definition 2.1. Banach Algebra
An algebra is a vector space A (over $\mathbb{R}$ or $\mathbb{C}$) with bilinear map $A \times A \to A$. Then we can define commutitive, associative and unit of the bilinear map in a natural way.

An normed algebra is an associative algebra whose vector space is normed satisfy the condition $\|uv\| < \|u\|\|v\|$. If the norm is complete, then it's a Banach algebra.

Remark 2.1. since $\|e\| \neq 0$(Why ?), we can always assume that if there is an unit element on Banach algebra A, we assume that $\|e\| = 1$.

Theorem 2.1. ***(The set of invertable elements is an Open Set)***
Let A be a Banach algebra with unit element e. Then the set of invertable elements is open in A. If $v \in A$ and $\|v\| < 1$, then $e + v$ is invertable.

*Sun Yat-Sen University, Guang dong, China. Email: jiangzmath@outlook.com.

Proof. Since $(e-v)(e+v+v^2+...+v^n) = e - v^{n+1}$, $e-v$ is invertable. If $u \in A$, for any $\|w-u\| < \frac{1}{\|u^{-1}\|}$ we have $\|wu^{-1} - e\| < 1$. Then wu^{-1} is invertable, whence w is invertable. □

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Corollary 2.1. *Let E, F be Banach spaces. Then the set of toplinear isomorphisms of E onto F is open in $\mathcal{L}(E, F)$.*

Remark 2.2. Toplinear isomorphism is a terminology from Serge Lang, which means a linear operator, and it's also a homeomorphism.

Definition 2.2. (Index of Fredholm operator)
X, Y are banach space, $T \in \mathcal{L}(X, Y)$ is Fredholm operator if $dimKer(T) < +\infty$, with closed range and $dim(coker(T))$ is finite (without this we call it a general Fredholm operator), where $coker(T) = Y/Range(X)$.

Theorem 2.2. *Let E, F be Banach spaces. Then $\mathcal{F}(E, F)$ is open in $\mathcal{L}(E, F)$, and the function $T \to ind(T)$ is continuous on $\mathcal{F}(E, F)$, hence constant on connected components.*

Proof. Let W be the kernel of T, and let G be a closed complement for $T(E)$, that is $E = N \oplus G$. Then T induces a toplinear isomorphism of G on its image $T(G)$ (by the open mapping theorem), and we can write $F = T(G) \oplus H$ for some finite dimensional subspace H. The map

$$G \times H \to T(G) \oplus H = F$$

$$(x, y) \to Tx + y$$

is a toplinear operator. By Corollary(2.1), if $\|S\|$ is small enough, then the map

$$G \times H \to (T + S)(G) \oplus H = F$$

$$(x, y) \to (T + S)x + y$$

is also a toplinear operator.

Hence the kernel of $T + S$ is finite dimensional. The image of $T + S$ has finite codimension, and is consequently closed. This proves that T is Fredholm, and proves our first assertion. The second assertion comes from the observation

$$E = G \oplus Ker(T) \oplus M$$

□

Corollary 2.2. ***(Copied from Hörmander)***
Let X, Y are banach space, with $T \in \mathcal{F}(X,Y)$ (permitted be $+\infty$), if $\|S\|$ is small enough, $dimKer(T+S) \le dimKerT$ and $T+S$ has close range. And $ind(T+S) = ind(T)$.

Remark 2.3. Actually we can present three different ways to prove it. They come from books of Serge Lang, Hörmander, and Martin Schechter respectively. But in this paper we aim only to explain some omitted details in their books.