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### Abstract

We details some proves in Hörmander's book concerning Fredholm theory.

**Keywords:** Functional analysis, Fredholm operator.

## 1 Introduction

To learn functional analysis through the pde, my functional analysis courses used some learning materials from Hörmander's book *The Analysis of Linear Partial Differential Operators III Pseudo-Differential Operators*, and Serge Lang, *Real and Functional Analysis*. But Sometimes it's really hard to follow Hörmander's book. So i chose some important points to detail the prove, with references from Serge Lang.

## 2 Basic Knowledges

### Definition 2.1. Banach Algebra

An algebra is a vector space  $A$  (over  $\mathbb{R}$  or  $\mathbb{C}$ ) with bilinear map  $A \times A \rightarrow A$ . Then we can define commutitive, associative and unit of the bilinear map in a natural way.

An normed algebra is an associative algebra whose vector space is normed satisfy the condition  $\|uv\| < \|u\|\|v\|$ . If the norm is complete, then it's a Banach algebra.

**Remark 2.1.** since  $\|e\| \neq 0$ (Why ?), we can always assume that if there is an unit element on Banach algebra  $A$ , we assume that  $\|e\| = 1$ .

### Theorem 2.1. (*The set of invertable elements is an Open Set*)

Let  $A$  be a Banach algebra with unit element  $e$ . Then the set of invertable elements is open in  $A$ . If  $v \in A$  and  $\|v\| < 1$ , then  $e + v$  is invertable.

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**Proof.** Since  $(e - v)(e + v + v^2 + \dots + v^n) = e - v^{n+1}$ ,  $e - v$  is invertable. If  $u \in A$ , for any  $\|w - u\| < \frac{1}{\|u^{-1}\|}$  we have  $\|wu^{-1} - e\| < 1$ . Then  $wu^{-1}$  is invertable, whence  $w$  is invertable.  $\square$

**Corollary 2.1.** *Let  $E, F$  be Banach spaces. Then the set of toplinear isomorphisms of  $E$  onto  $F$  is open in  $\mathcal{L}(E, F)$ .*

**Remark 2.2.** Toplinear isomorphism is a terminology from Serge Lang, which means a linear operator, and it's also a homeomorphism.

**Definition 2.2. (Index of Fredholm operator)**

$X, Y$  are Banach space,  $T \in \mathcal{L}(X, Y)$  is Fredholm operator if  $\dim \text{Ker}(T) < +\infty$ , with closed range and  $\dim(\text{coker}(T))$  is finite (without this we call it a general Fredholm operator), where  $\text{coker}(T) = Y/\text{Range}(T)$ .

**Theorem 2.2.** *Let  $E, F$  be Banach spaces. Then  $\mathcal{F}(E, F)$  is open in  $\mathcal{L}(E, F)$ , and the function  $T \rightarrow \text{ind}(T)$  is continuous on  $\mathcal{F}(E, F)$ , hence constant on connected components.*

**Proof.** Let  $W$  be the kernel of  $T$ , and let  $G$  be a closed complement for  $T(E)$ , that is  $E = W \oplus G$ . Then  $T$  induces a toplinear isomorphism of  $G$  on its image  $T(G)$  (by the open mapping theorem), and we can write  $F = T(G) \oplus H$  for some finite dimensional subspace  $H$ . The map

$$\begin{aligned} G \times H &\rightarrow T(G) \oplus H = F \\ (x, y) &\rightarrow Tx + y \end{aligned}$$

is a toplinear operator. By Corollary(2.1), if  $\|S\|$  is small enough, then the map

$$\begin{aligned} G \times H &\rightarrow (T + S)(G) \oplus H = F \\ (x, y) &\rightarrow (T + S)x + y \end{aligned}$$

is also a toplinear operator.

Hence the kernel of  $T + S$  is finite dimensional. The image of  $T + S$  has finite codimension, and is consequently closed. This proves that  $T$  is Fredholm, and proves our first assertion. The second assertion comes from the observation

$$E = W \oplus \text{Ker}(T) \oplus M$$

$\square$

53 **Corollary 2.2.** (*Copied from Hörmander*)

54 *Let  $X, Y$  be Banach spaces, with  $T \in \mathcal{F}(X, Y)$  (permitted be  $+\infty$ ), if  $\|S\|$  is small enough,*  
55  *$\dim \text{Ker}(T + S) \leq \dim \text{Ker} T$  and  $T + S$  has closed range. And  $\text{ind}(T + S) = \text{ind}(T)$ .*

56 **Remark 2.3.** Actually we can present three different ways to prove it. They come from  
57 books of Serge Lang, Hörmander, and Martin Schechter respectively. But in this paper  
58 we aim only to explain some omitted details in their books.