Proof Hn is order ln

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Statement 0

$$H_n = \sum_{x=1}^{n} \frac{1}{x}$$

Statement 1

$$\text{Suppose } 0 < n \text{ and } 0 \le x \le 1$$

$$= 0 \le x$$
$$= n \le n + x$$
$$= 1 \le \frac{n+x}{n}$$
$$= \frac{1}{n+x} \le \frac{1}{n}$$

Statement 2

$$\text{Suppose } n \in \mathbb{Z}_{>0}$$

$$\int_{n}^{n+1} \frac{1}{x} = \lim_{\delta x \to 0} \sum_{x=n}^{n+1} \frac{1}{x} \delta x$$

Using statement 1: $\frac{1}{n+x} \le \frac{1}{n} \forall x, n : 0 < n, 0 \le x \le 1$

$$\lim_{\delta x \to 0} \sum_{x=n}^{n+1} \frac{1}{x} \delta x \le \frac{1}{n}$$

$$\int_{n}^{n+1} \frac{1}{x} \le \frac{1}{n}$$

Statement 3

Using statement 2: $\int_{n}^{n+1} \frac{1}{x} \le \frac{1}{n} \forall n \in \mathbb{Z}_{>0}$ (n is replaced by n' in the following equations)

$$\sum_{n'=1}^{n} \int_{n'}^{n'+1} \frac{1}{x} \le \sum_{n'=1}^{n} \frac{1}{n'}$$

$$\int_{1}^{n+1} \frac{1}{x} \le \sum_{n'=1}^{n} \frac{1}{n'} = H_n$$

$$\int_{1}^{n+1} \frac{1}{x} = \ln(x)|_{1}^{n+1} = \ln(n+1) - \ln(1) = \ln(n+1) \in \Theta(ln(n))$$

$$\ln(n+1) \leq H_n$$
$$H_n \in \Omega(ln(n))$$

Statement 4

$$\text{Suppose } 1 < n \text{ and } 0 \leq x \leq 1$$
$$= 0 \geq -x$$
$$= n \geq n - x$$
$$= 1 \geq \frac{n-x}{n}$$
$$= \frac{1}{n-x} \geq \frac{1}{n}$$

Statement 5

$$\text{Suppose } n \in \mathbb{Z}_{>1}$$
$$\int_{n}^{n+1} \frac{1}{x} = \lim_{\delta x \to 0} \sum_{x=n}^{n+1} \frac{1}{x} \delta x$$

Using statement 4: $\frac{1}{n-x} \geq \frac{1}{n} \forall x, n : 1 < n, 0 \leq x \leq 1$

$$\lim_{\delta x \to 0} \sum_{x=n-1}^{n} \frac{1}{x} \delta x \geq \frac{1}{n}$$
$$\int_{n-1}^{n} \frac{1}{x} \geq \frac{1}{n}$$

Statement 6

Using statement 5: $\int_{n-1}^{n} \frac{1}{x} \geq \frac{1}{n} \forall n \in \mathbb{Z}_{>1}$ (n is replaced by n' in the following equations)

$$\sum_{n'=2}^{n} \int_{n'-1}^{n'} \frac{1}{x} \geq \sum_{n'=2}^{n} \frac{1}{n'}$$
$$\int_{1}^{n} \frac{1}{x} \geq \sum_{n'=1}^{n} \frac{1}{n'} = H_n - 1$$
$$\int_{1}^{n} \frac{1}{x} = \ln(x)|_1^n = \ln(n) - \ln(1) = \ln(n)$$
$$\ln(n) \geq H_n - 1$$
$$\ln(n) + 1 \geq H_n$$

$$\ln(n) + 1 \in \Theta(ln(n))$$

$$H_n \in O(ln(n))$$

Statement 7

Using Statement 3: $H_n \in Omega(ln(n))$ and Statement 6: $H_n \in O(ln(n))$, by defenition $H_n \in \Theta(ln(n))$