Math204 Calc III Final SP21

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Do any FIVE of the following SEVEN problems. Clearly indicate which TWO problems you do not want me to grade.

It is OK to use resources for this final, for example our textbook or your course notes are fine, as are graphing utilities like Desmos or Geogebra. However, all of your work must be your own, and you may not communicated with other humans (besides your instructor) about the content of this exam: no tutors, no peers, etc. You also must show all work for full credit. Your work can be typed into Overleaf or you may upload scans of handwritten work into the document, but if you do so, it must be placed under the corresponding question using the includegraphics command.

1. Take one group project posted to our D2L Discussion Board other than your own, and write a summary of it! For full credit, your summary must contain the following:

   - What were the key questions being asked, or what were the key topics being explored?
   - What were the techniques used in the project? In particular, how exactly did optimization figure into it, and why?
   - Besides optimization, what other methods were used, and why?
   - What were the conclusions of the project? What were the end results? Why was the work important or interesting?

   For this problem, your response must be typed in LaTeX following all QSG principles!

Exercise 0.0.1. Question 1

- I choose the minimum distance SP and the main topic was find the minimum distance first between two point (using the mid-point formula), then for three symmetric points, and finally three non-symmetric points.
- The main technique used throughout the project was using a function $D(x, y)$ that represented the total distance from a set of given points and then using partial derivatives (of x and of y) to find the location that minimized distance between the set of points. Lagrange multipliers were also used to help find the minimum distance between points. Optimization could be seen in find the optimal location for Mama prairie dog to dig her hole so that she was in the best location to be as close to all three of her pups as possible.
- Gradient descent was also used as more of a guess and check method by taking the gradient of the function $D(x, y)$. This method has benefits as it involves less complicated partial derivative and a relatively simple, plug and chug, strategy to find the optimal point.

- The conclusion of the project was that you can use partial derivatives and Lagrange multiplier as well as geometry as shown by the work of Fermat and Torricelli. The end result was the discovery of the optimal location for Mama prairie dog for the non symmetric location of her pup's holes. This work has very important implementation as mentioned at the beginning of the project. It help shipping company know where to place new delivery locations. This method could also be used to help a power company know where they should put generators to limit the amount of wire used to get to each household.

2. In his 225BCE treatise On Spirals, Archimedes successfully calculated the area inside a spiral with linear growth on the radius. This is believed to be the first area calculations for any figure that was not bounded by straight lines or conics!

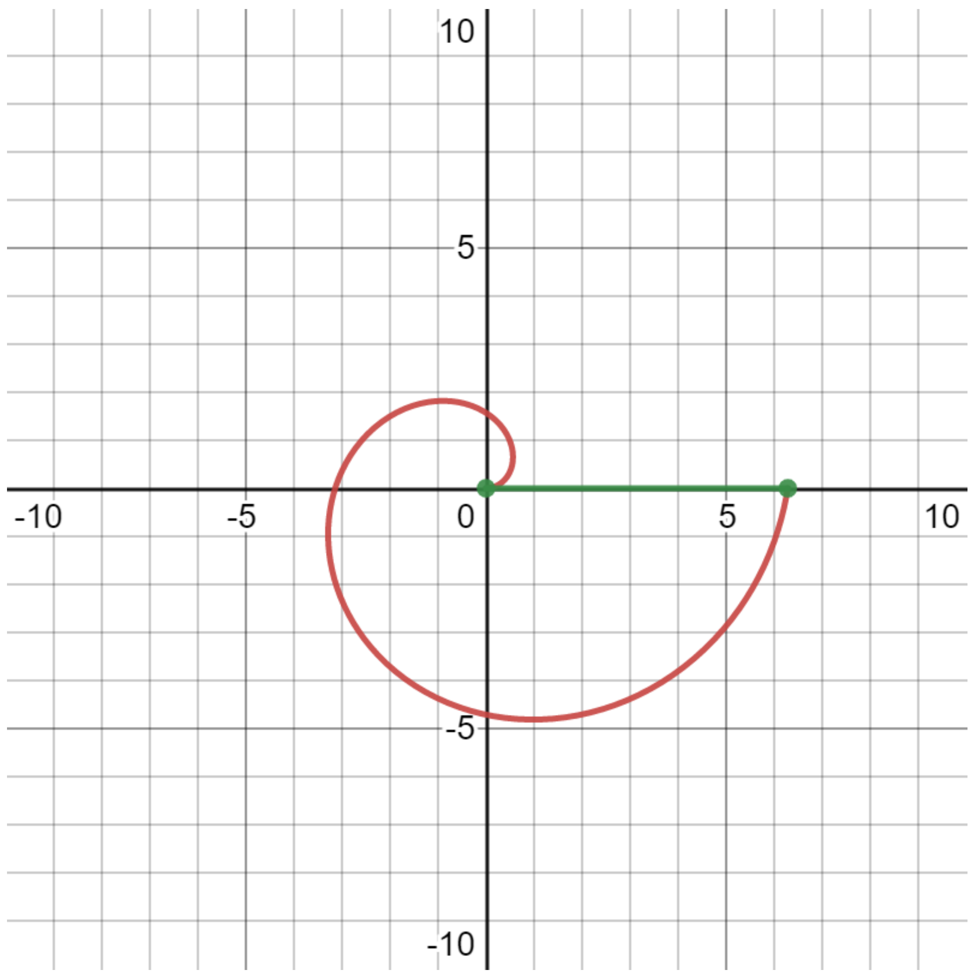
   Let us here recapture his result using much more modern machinery: Green's Theorem. In particular, consider the parametric curve $C = C_1 \cup C_2$ where

$$C_1 = \{(t\cos(t), t\sin(t)) : t \in [0, 2\pi]\}$$

   and C_2 is the line segment connecting the origin to $(2\pi, 0)$. Graph the region, and then use Green's Theorem to find the area enclosed by C. For the line integrals that arise in this calculation, can you evaluate them using the Fundamental Theorem of Line Integrals? Why or why not?

Exercise 0.0.2. Question 2

Below is the graph of C.



Now we can use the fact that the area integral of an object is $\iint_D 1\,dA$ and therefore, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. Knowing this, we can pick any vector field that satisfies this criteria where $Q(x, y)$ is the x-coordinate of the vector field and $P(x, y)$ is the y-coordinate of the vector field. One such equation is $F(Q, P) = \frac{1}{2}\begin{bmatrix} -y \\ x \end{bmatrix}$. Now that we have a vector field we can set up the line integral of Green's Theorem.

$$\begin{aligned}
\iint_C 1\,dA = \int_C \vec{f}\,ds &= \int_{C_1} \vec{f}\,ds + \int_{C_2} \vec{f}\,ds \\
&= \frac{1}{2}\int_{t=0}^{t=2\pi} \begin{bmatrix} -y \\ x \end{bmatrix} \cdot \vec{r_1}'(t)\,dt + \frac{1}{2}\int_{t=0}^{t=2\pi} \begin{bmatrix} -y \\ x \end{bmatrix} \cdot \vec{r_2}'(t)\,dt \\
&= \frac{1}{2}\int_{t=0}^{t=2\pi} \begin{bmatrix} -t\sin t \\ t\cos t \end{bmatrix} \cdot \begin{bmatrix} -t\sin t \\ t\cos t \end{bmatrix}\,dt + \frac{1}{2}\int_{t=0}^{t=2\pi} \begin{bmatrix} 0 \\ t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\,dt \\
&= \frac{1}{2}\int_{t=0}^{t=2\pi} t^2(\sin^2 t + \cos^2 t)\,dt + \frac{1}{2}\int_{t=0}^{t=2\pi} 0\,dt \\
&= \frac{1}{2}\left(\left.\frac{t^3}{3}\right|_{t=0}^{t=2\pi} \right) + 0 \\
&= \frac{4}{3}\pi^3
\end{aligned}$$

We could not have evaluated this line integral using FTLI because this vector field is not conservative and FTLI only applies to conservative vector fields. It is not conservative because $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \neq 0$ which is the definition of a conservative vector field.

3. Consider the solid Q bounded by the following planes:

$$x + y + z = 1,$$

$$x + y + z = -1,$$

$$x - y + z = 1,$$

$$x - y + z = -1,$$

$$z = 1,$$

and

$$z = -1.$$

(a) What shape is the region? What are the coordinates of the vertices? Draw a graph of both, either by hand or in a 3D plotting utility, but be sure to clearly label the vertices.

(b) Take differences of pairs of vertices to get three vectors $\vec{u}, \vec{v}, \vec{w}$ that generate the figure. Use the magnitude of the triple scalar product,

$$|\vec{u} \cdot (\vec{v} \times \vec{w})|,$$

to calculate the volume of the solid.

(c) Use a triple integral along with the change of coordinates

$$u = x + y + z,$$

$$v = x - y + z,$$

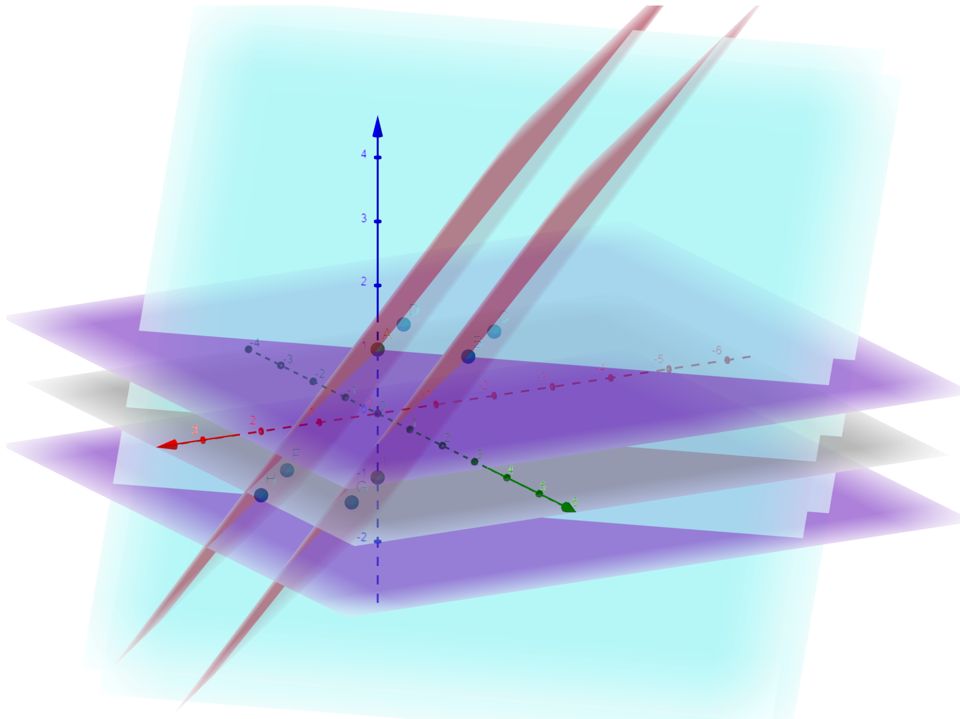
and

$$w = z$$

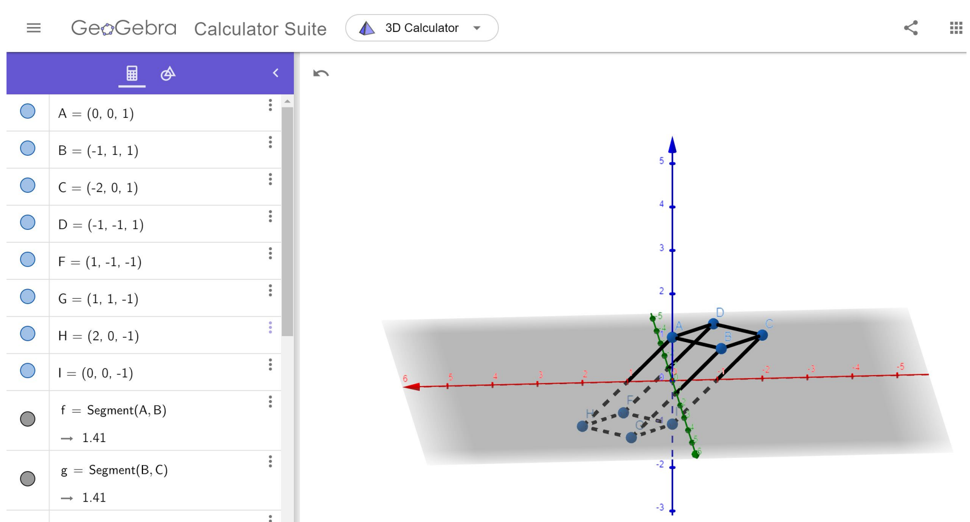
to find the volume a second time. Confirm your answers match! (**Hint.** To set up the Jacobian, you'll need to solve for $x, y,$ and z as functions of $u, v,$ and w in the change of coordinates given above.)

Exercise 0.0.3. ~~**Question 3**~~

(a) Below is the parallelepiped Q that is created by the six planes.



Here is another view with only points created my the planes and connecting side of the parallelepiped.



(b) Using $A = (0,0,1)$ as the starting point, we can say $\vec{u} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$, and

$\vec{w} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$. we can then do the triple scalar product.

$$\left\| \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right) \right\| = 4$$

Therefore the volume of the parallelepiped is 4.

(c) Solving for x, y, and z in the equations above we get that

$$x = \frac{1}{2}(u - v - 2w) = f(u, v, w),$$

$$y = \frac{u - v}{2} = g(u, v, w),$$

and

$$z = w = h(u, v, w)$$

In order to find the volume using a triple intergral, we first need to find the Jacobin for our change in coordinate by taking the determinate of x, y, and z.

$$J = \det \begin{bmatrix} f(u, v, w) \\ g(u, v, w) \\ h(u, v, w) \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

4. Consider the surface S given by the parameterization

$$S = \left\{ \vec{r}(u,v) = \begin{bmatrix} \cos(u) \\ \sin(u) \\ v \end{bmatrix} : u \in [0, 2\pi], v \in [0,1] \right\}$$

with boundary curve given by

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

where

$$C_1 = \left\{ \vec{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 1 \end{bmatrix} : t \in [0, 2\pi] \right\},$$

$$C_2 = \left\{ \vec{r}(t) = \begin{bmatrix} 1 \\ 0 \\ 1-t \end{bmatrix} : t \in [0, 1] \right\},$$

$$C_3 = \left\{ \vec{r}(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{bmatrix} : t \in [0, 2\pi] \right\},$$

and

$$C_4 = \left\{ \vec{r}(t) = \begin{bmatrix} 1 \\ 0 \\ t \end{bmatrix} : t \in [0, 1] \right\},$$

(a) Just in words, describe each of the shapes given above:

- S: S is a cylinder without a top and bottom or no end caps (a circle stretch from 0 to 1 in the z dimension) that is one unit tall.

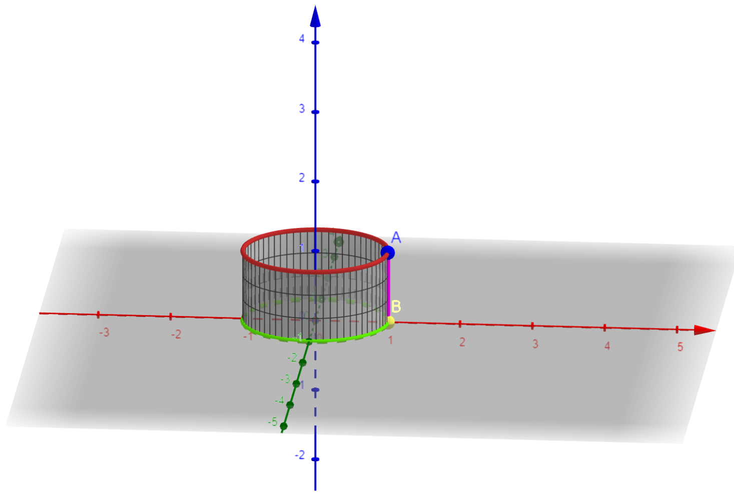
- C_1: A circle at the top of the cap-less cylinder that travels in the CCW direction making an normal vector pointing in the positive z direction.

- C_2:A line beginning at the point $(1,0,1)$ and traveling straight down in the $-z$ direction to the point $(1,0,0)$.

- C_3: A circle at the bottom of the cap-less cylinder that travels in the CW (when looked at from above) direction making an normal vector pointing in the positive $-z$ direction.
- C_4: A line beginning at the point $(1,0,0)$ and traveling straight up in the z direction to the point $(1,0,1)$.

(b) Draw a picture of the surface S along with the path C. Explain why C is a valid boundary curve for S.

- **Answer:**



- We know that this is a valid boundary since we can in vision unwinding the cylinder to make it into a rectangle where the top edge is red (C_1). The bottom is green (C_3). the left side is a line that goes from point-A to point-B (C_2), and the right side is a line that goes from point-B to point-A (C_4).

(c) Verify Stokes' Theorem on S (that is, directly compute both the left and right hand sides and verify they match) using the vector field

$$\vec{F}(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

Exercise 0.0.4. Question 4c

First we will state Stokes Theorem:

$$\iint_S \text{curl}\vec{f} \cdot dS = \int_C \vec{f} \cdot ds.$$

In order to use Stokes's Theorem, we first need to calculate the normal vector and curl of $\vec{f}$.

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{bmatrix} -\sin u \\ \cos u \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos u \\ \sin u \\ 0 \end{bmatrix}$$

$$\text{curl}\vec{f} = \nabla \times \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now that we have the curl and normal vector we can calculate the LHS and RHS starting with the LHS.

$$\iint_S \text{curl}\vec{f} \cdot dS = \int_{u=0}^{u=2\pi} \int_{v=0}^{v=1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \cos u \\ \sin u \\ 0 \end{bmatrix} \, dv \, du = 0$$

To calculate the RHS, we need to break the line integral into four different sections (C_1,C_2,C_3, and C_4).

$$\int_C \vec{f} \cdot ds = \int_{C_1} \vec{f} \cdot ds + \int_{C_2} \vec{f} \cdot ds + \int_{C_3} \vec{f} \cdot ds + \int_{C_4} \vec{f} \cdot ds$$

We are going to calculate each integral separately and then add

$$\int_{C_1} \vec{f} \cdot ds = \int_{t=0}^{t=2\pi} \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix} \, dt = 0$$

$$\int_{C_2} \vec{f} \cdot ds = \int_{t=0}^{t=2\pi} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \, dt = 0$$

$$\int_{C_3} \vec{f} \cdot ds = -\int_{t=0}^{t=2\pi} \begin{bmatrix} \cos t \\ -\sin t \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ -\cos t \\ 0 \end{bmatrix} \, dt = 0$$

$$\int_{C_2} \vec{f} \cdot ds = -\int_{t=0}^{t=2\pi} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \, dt = 0$$

Therefore the RHS equals $0+0+0+0=0$ which equals the LHS and we have verified Stokes's Theorem.

5. Recall our claim that a Möbius strip is a non-orientable surface. Let us make this a little more explicit!

   Consider the following parameterization of the Möbius strip:

$$\vec{r}(u,v) = \begin{bmatrix} \left(1 + \frac{v}{2}\cos\left(\frac{u}{2}\right)\right)\cos(u) \\ \left(1 + \frac{v}{2}\cos\left(\frac{u}{2}\right)\right)\sin(u) \\ \frac{v}{2}\sin\left(\frac{u}{2}\right) \end{bmatrix}.$$

   Calculate the normal vector $\vec{r}_u \times \vec{r}_v$. Where does this calculation break down? Specifically, find two ordered pairs (u, v) which correspond to the same point $\vec{r}(u, v)$ but have different normal vectors (hence implying the surface cannot have a well-defined orientation). Illustrate your answer with a diagram.

Exercise 0.0.5. ~~**Question 5**~~

6. Let the region R be the part of the unit circle that lies in the first quadrant. Then consider the following density functions:

$$d_1(x, y) = 1$$

and

$$d_2(x, y) = \sqrt{x^2 + y^2}.$$

(a) Explain why for both d_1 and d_2, the center of mass of R must lie along the line $y = x$.

(b) Just using geometry/physical intuition, which center of mass would you expect to be closer to the origin, the one for d_1 or for d_2?

(c) Use moment integrals to calculate both centers of mass and verify your prediction from the previous part.

Exercise 0.0.6. Question 6

(a) The reason that both of the these functions are on the line $y = x$ is that x and y are weighted the same. We could consider the quarter circle and prove this fact for $d_1(x, y) = 1$ by imagining folding it in half. It would have a line of symmetry on the half way point or where it crosses the line $y = x$. Since every point is weighted equally, we now that the center of mass has to be on the $y = x$ line.

(b) I would expect the $d_1(x, y) = 1$ density function to be closer to the origin since $d_2(x, y) = \sqrt{x^2 + y^2}$ is more massive as we get further away from the origin in the radial direction.

(c) Since we know that both function lie on the $y = x$ line, we only have to calculate $\overline{x}$ or $\overline{y}$ and we know our coordinate will be the same for both. Below is the formula for the center of mass:

$$\text{C.O.M.}_y = \overline{y} = \frac{M_x}{m} = \frac{\iint_R y \cdot d_1(x, y)}{\iint_R d_1(x, y)\, dA}$$

where M_x is the moment integral with respect to x and m is the mass. This calculation can be made easier if we change to cylindrical coordinates.

$$R = \left\{(\theta, r) = \mathbb{R}^2 : \theta \in [0, \frac{\pi}{2}], r \in [0, 1]\right\},$$

We will chose to calculate $\overline{y}$.

$$\begin{aligned}\overline{y} &= \frac{\iint_R y \cdot d_1(x, y)}{\iint_R d_1(x, y)\, dA} \\ &= \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1} y \cdot (1)\, dr\, d\theta}{\iint_R 1\, dA}\end{aligned}$$

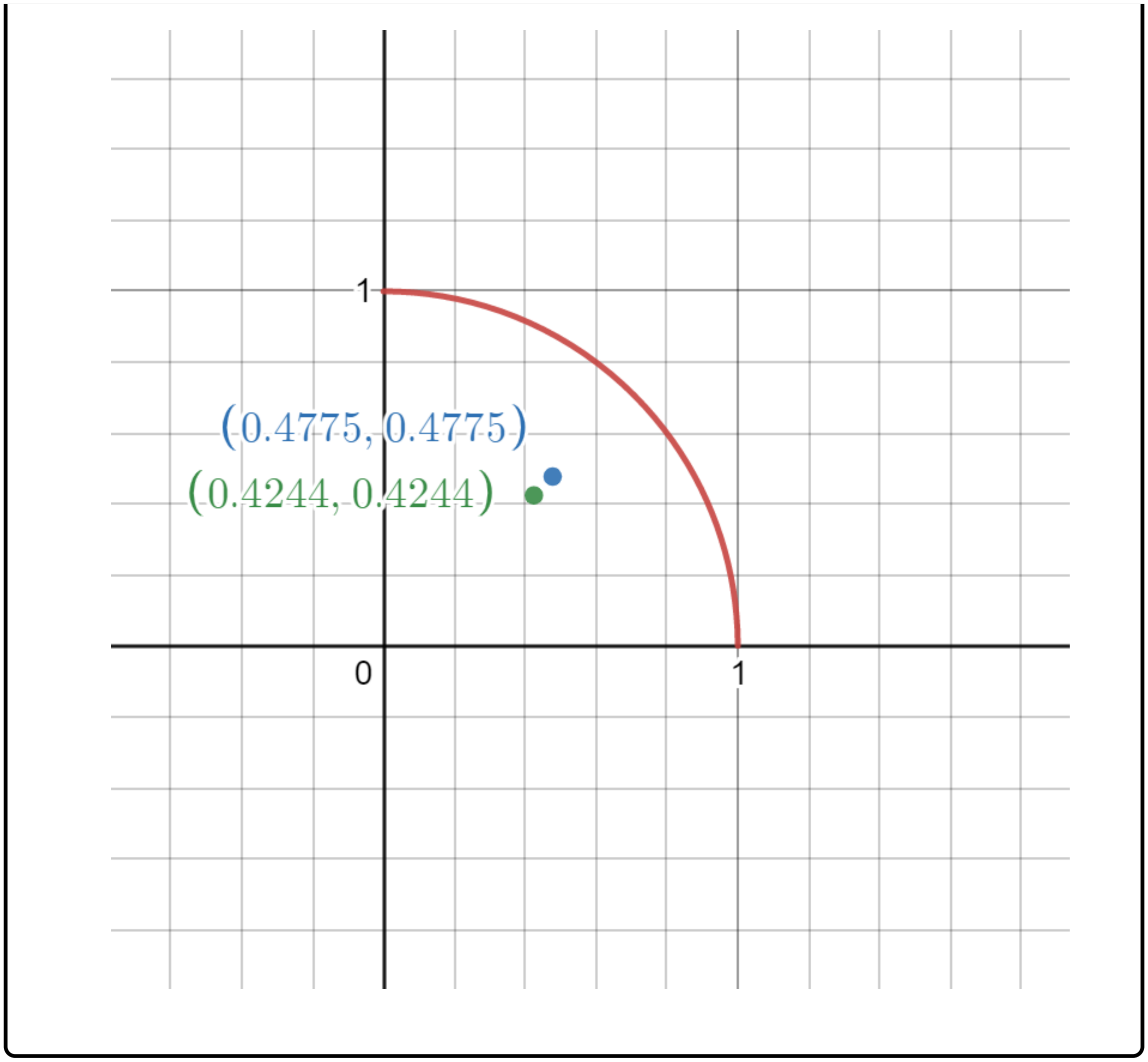
Since the density function is equal to one, we know mass equals area. The area of a quarter unit circle is $\frac{\pi}{4}$ so we can say this is the mass, and put its reciprocal in front of our moment integral

$$
\begin{aligned}
&= \frac{4}{\pi}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1} r^2 \sin\theta \, dr \, d\theta \\
&= \frac{4}{\pi}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{r^3}{3}\sin\theta\Big|_{r=0}^{r=1} d\theta \\
&= \frac{4}{\pi}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{3}\sin\theta \, d\theta \\
&= \frac{4}{\pi}\left(-\frac{1}{3}\cos\theta\right)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} \\
&= \frac{4}{3\pi}
\end{aligned}
$$

Therefore, the C.O.M. for $d_1(x, y)$ is $(x, y) = \left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$. Now we can use the same process for $d_2(x, y) = \sqrt{x^2 + y^2}$, but since we substituted for cylindrical coordinate we can say $\sqrt{x^2 + y^2} = r$. This time, we will calculate $\overline{x}$.

$$
\begin{aligned}
\overline{x} &= \frac{\iint_R y \cdot d_1(x, y)}{\iint_R d_1(x, y)\, dA} \\
&= \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1} r^3 \cos\theta \, dr \, d\theta}{\int_{\theta=0}^{\theta=\frac{\pi}{2}} r^2 \, dr \, d\theta} \\
&= \frac{\frac{1}{4}}{\frac{\pi}{6}} \\
&= \frac{3}{2\pi}
\end{aligned}
$$

Therefore, the C.O.M. for $d_2(x, y)$ is $(x, y) = \left(\frac{3}{2\pi}, \frac{3}{2\pi}\right)$. This agrees our hypothesis as can be seen in the graph below where the green point is the values of d_1 and the blue point is the values for d_2.



1
(0.4775, 0.4775)
(0.4244, 0.4244)
0
1

7. Verify the Divergence Theorem (i.e., compute the left- and right-hand sides independently and verify they match) for the following:

   - The solid Q is the cube with vertices $(0,0,0)$, $(0,0,1)$,$(0,1,0)$,$(1,0,0)$,$(0,1,1)$,$(1,0,1)$,$(1,1,0)$, and $(1,1,1)$.
   - The vector field is

$$\vec{f}(x,y,z) = \begin{bmatrix} x+y \\ y \\ xz \end{bmatrix}.$$

Exercise 0.0.7. Question 7

First we will write out the divergence theorem:

$$\iiint_Q \text{DIV}\vec{f}\,dV = \iint_S \vec{f}\,dS.$$

Now we will calculate the divergence theorem starting with the LHS.

$$\begin{aligned}
\iiint_Q \text{DIV}\vec{f}\,dV &= \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} x+y \\ y \\ xz \end{bmatrix} dz\,dy\,dx \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} 2+x\,dz\,dy\,dx \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=1} 2+x\,dy\,dx \\
&= \int_{x=0}^{x=1} 2+x\,dx \\
&= 2x+\frac{x^2}{2}\Big|_{x=0}^{x=1} \\
&= \frac{5}{2}
\end{aligned}$$

In order to calculate the RHS of this equation we need to break this surface integral into the six sides ($S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$).
The parameters of the six sides of the are given below:

$$S_1 = \left\{ \vec{r}(u,v) = \begin{bmatrix} u \\ -v \\ 0 \end{bmatrix} : u \in [0,1], v \in [0,1] \right\},$$

$$S_2 = \left\{ \vec{r}(u,v) = \begin{bmatrix} u \\ 0 \\ -v \end{bmatrix} : u \in [0,1], v \in [0,1] \right\},$$

$$S_3 = \left\{ \vec{r}(u,v) = \begin{bmatrix} 1 \\ u \\ v \end{bmatrix} : u \in [0,1], v \in [0,1] \right\},$$

$$S_4 = \left\{ \vec{r}(u,v) = \begin{bmatrix} u \\ 1 \\ v \end{bmatrix} : u \in [0,1], v \in [0,1] \right\},$$

$$S_5 = \left\{ \vec{r}(u,v) = \begin{bmatrix} 0 \\ u \\ v \end{bmatrix} : u \in [0,1], v \in [0,1] \right\},$$

and

$$S_6 = \left\{ \vec{r}(u,v) = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} : u \in [0,1], v \in [0,1] \right\}.$$

Where S_1 is the surface when $z = 0$, S_2 is the surface when $y = 0$, S_3 is the surface when $x = 1$, S_4 is the surface when $y = 1$, S_5 is the surface when $x = 0$, and S_6 is the surface when $z = 1$. We also need the normal vector for each face of the cube.

$$\vec{n}_1 = \vec{r}_u \times \vec{r}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{n}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{n}_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{n}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{n}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can now calculate the double integral for each surface and sum the results.

$$\iint_{s_1} \vec{f}\, ds = \int_{u=0}^{u=1} \int_{v=0}^{v=1} \begin{bmatrix} u+v \\ -v \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \, dv\, du = 0$$

$$\iint_{s_2} \vec{f}\, ds = 0$$

$$\iint_{s_3} \vec{f}\, ds = -\frac{3}{2}$$

$$\iint_{s_4} \vec{f}\, ds = 1$$

$$\iint_{s_5} \vec{f}\, ds = \frac{1}{2}$$

$$\iint_{s_6} \vec{f}\, ds = 1$$

Summing all of our integrals we get a result of we get a value of 1.