Math204 Quadratic Approximation - Group Project Template

March 15, 2024

Just as much of Calculus 1 and 2 can be framed in terms of power series of a single variable, much of Calculus 3 can be framed in terms of multi variable power series!

- State the ordinary Second Derivative Test from Calculus 1. Explain how this coincides with the computation of a degree two power series centered at the corresponding critical point.

Solution Author: Graham Kroll

The Second Degree Derivative test is as follows

- If $f'(c) = 0$ and $f''(c) > 0$, then there is a local minimum at $x = c$.
- If $f'(c) = 0$ and $f''(c) < 0$, then there is a local maximum at $x = c$.
- If $f'(c) = 0$ and $f''(c) = 0$, or if $f''(c)$ doesn't exist, then the test is inconclusive. There might be a local maximum or minimum, or there might be a point of inflection.

This corresponds with a degree two power series centered at c as shown below since the term $f''(c)(x-c)^2$ would correspond with a parabola centered at c, which is what we are looking for here. We would use these terms to find the critical points, or more specifically the vertex of the parabola. If $f''(c)$ was negative we would have a local maximum and is it was positive we would have a local minimum. This is the relation between second derivative test (stated above) and a degree two power series.

$$\sum_{n=1}^{2} f(x) = f(c) + f'(c)(x-c) + f''(c)(x-c)^2$$

Comments on the Above

Feedback from Ken. What you stated for the Second Derivative Test was correct, but your corresponding analysis regarding what the power series tells you graphically isn't really. We're not asking about higher and higher derivatives; we're specifically asking about degree two. Here is the key: the degree two power series will always have a parabola for its graph because it's a degree two polynomial equation, right? So now think in each of those three cases that you listed above, what would the graph of that parabola look like? When would it be upward facing? When would it be downward facing? When would it be hard to tell how it faces given that info? etc

- If F has continuous second-order partial derivatives at (x_0, y_0), then the **second-degree power series** of F at (x_0, y_0) is

$$\begin{aligned} Q(x,y) =& F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0) + \frac{1}{2}F_{xx}(x_0,y_0)(x-x_0)^2 \\ &+ F_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}F_{yy}(x_0,y_0)(y-y_0)^2 \end{aligned}$$

and the approximation $F(x,y) \approx Q(x,y)$ is called the **quadratic approximation** to F at (x_0, y_0). Verify that Q has the same first- and second-order partial derivatives as F at (x_0, y_0).

Solution Author: Brian Ortiz

$$\begin{aligned}
Q_x(x,y) &= 0 + F_x(x_0,y_0) + 0 + \frac{1}{2}F_{xx}(x_0,y_0)\cdot 2\cdot(x-x_0)\cdot 1 + F_{xy}(x_0,y_0)(y-y_0)\\
Q_x(x_0,y_0) &= F_x(x_0,y_0) + F_{xx}(x_0,y_0)(x_0-x_0) + F_{xy}(x_0,y_0)(y_0-y_0)\\
Q_x(x_0,y_0) &= F_x(x_0,y_0)\\
Q_y(x,y) &= 0 + 0 + F_y(x_0,y_0) + 0 + F_{xy}(x_0,y_0)(x-x_0) + \frac{1}{2}F_{yy}(x_0,y_0)\cdot 2\cdot(y-y_0)\cdot 1\\
Q_y(x_0,y_0) &= F_y(x_0,y_0) + F_{xy}(x_0,y_0)(x_0-x_0) + F_{yy}(x_0,y_0)(y_0-y_0)\\
Q_y(x_0,y_0) &= F_y(x_0,y_0)\\
Q_{xy}(x_0,y_0) &= F_{xy}(x_0,Y_0)(y-y_0)(x-x_0)
\end{aligned}$$

After confirming these calculations we can take the partial derivatives of both Q_x and Q_y to get,

$$Q_{xx}(x,y) = 0 + F_{xx}(x_0,y_0) + 0$$
$$Q_{xx}(x_0,y_0) = F_{xx}(x_0,y_0)$$

and

$$Q_{yy}(x,y) = 0 + 0 + F_{yy}(x_0,y_0)$$
$$Q_{yy}(x_0,y_0) = F_{yy}(x_0,y_0)$$

Comments on the Above

Feedback from Ken. This is good calculation-wise, but you still need to get everything in complete sentence form and not just have calculations floating on the page there.

– Find the first- and second-degree Taylor polynomials L and Q of $f(x,y) = e^{-x^2-y^2}$ at $(0,0)$. Graph f, L, and Q. Comment on how well L and Q approximate f.

Solution Author: Hannah Gray

$f(x,y) = e^{-x^2-y^2}$ at $(0,0)$ partial derivatives are:

* $f(x) = -2x * e^{(-x^2)(-y^2)}$
* $f(y) = -2y * e^{(-x^2)(-y^2}$
* $f(xx) = -2e^((-x^2)(-y^2)) + 4(x^2)(e^{(-x^2)(-y^2)}$
* $f(yy) = -2(e^((-x^2)(-y^2)) + 4(y^2)(e^{(-x^2)(-y^2)}$
* $f(xy) = 4x * y * e^{(-x^2)(-y^2)}$

The partial derivatives at the point (0,0).

* $fx(0,0) = 0$
* $fy(0,0) = 0$
* $fxx(0,0) = -2$

* $fyy(0,0) = -2$
* $fxy(0,0) =$
* $f(0,0) = 1$

The first degree Taylor polynomial of $f(x,y) = e^{(-x^2)(-y^2)}$ at (0,0) is

* $L(x,y) = f(0,0) + (f(x))(0,0) * (x-0) + (f(y))(0,0)(y-0)$
* $= 1 + 0(x) + 0(y) = 1$

The second degree Taylor polynomial of $f(x) = e^{(-x^2)(-y^2)}$ at $(0,0)$ is

* $Q(x,y) = f(0,0) + (f(x))(0,0)(x-0) + (f(y))(0,0)(y-0) + (1/2)(f(xx)(0,0)((x-0)^2) + (f(xy))(0,0)(x-0)(y-0) + (1/2) * (f(yy))(0,0)((y-0)^2)$
* $= 1 - (x^2) - (y^2)$

$Q(x)$ is more closer because $L(x)$ is a constant

Comments on the Above

Feedback from Ken. Hi Hannah! One repeated issue is that you basically never want to use an asterisk for multiplication in math typesetting (even though I understand why you would since most computer algebra systems do). Instead "backslash cdot" is going to give you the correct symbol, or simple juxtaposition is always valid too. Then the other main thing is just give your output in the pdf a thorough top-to-bottom look, and you'll find many instances where the output of the code isn't matching what it shoudl look like. For example, on the very first line, you have the function

$$f(x,y) = e^{-x^2-y^2}$$

which looks great, looks just like it should. But then throughout your writeup after that, where you have similar expressions, sometimes you have

$$(-x^2)(-y^2)$$

in the power, which would mean that those quantities are multiplied rather than subtracted, and sometimes you have them spilling out of the superscript, where they are on the same line as the e instead of up in the power. The fact that it keeps changing throughout your calculation seems to indicate to me that you were typing this by hand each time. Don't do that! Instead, save yourself a ton of time by using copy-paste on the code for that expression that formatted correctly the very first time, and then it will display correctly every single time. Or if you do want to retype it that is fine of course but you'll need to make sure that it matches every single character of that first place where it displays correctly, which it certainly doesn't at the moment.
The next thing is your first round of derivative calculations has just function notation for what should be subscripts (like $f(x)$ instead of f_x). You did it correctly down below when you plugged in the point $(0,0)$, so follow that notation instead.
The other big thing I would say is on messy multi-line calculations, you don't want to use an itemize environment, which is really meant moreso for lists. Instead use the align environment as described in the transitive chain section of our QSG, and it will format much more nicely. :)

– Find the first- and second-degree Taylor polynomials L and Q for $f(x,y) = xe^y$ at $(1,0)$.

Compare the values of L, Q, and f at $(0.9, 0.1)$. Graph f, L, and Q. Comment on how well L and Q approximate f.

Solution Author: Graham Kroll

First let's find all the first a second derivatives of $f(x, y) = xe^y$ at $(x_0, y_0) = (1, 0)$.

* $F(x_0, y_0) = 1$
* $F_x(x_0, y_0) = 1$
* $F_y(x_0, y_0) = xye^y = 0$
* $F_{xx}(x_0, y_0) = 0$
* $F_{xy}(x_0, y_0) = ye^y = 0$
* $F_{yy}(x_0, y_0) = x(1) \cdot e^y + xy \cdot ye^y = xe^y + xy^2e^y$

Now that we have found all the partial derivatives, we can find the first- and second-degree polynomials. For L, a first degree polynomial...

$$\begin{aligned}
L(x, y) &= F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) \\
&= 1 + (x - x_0) + 0(y - y_0) \\
&= 1 + 1(x - 1)
\end{aligned}$$

And for a second-degree polynomial Q...

$$\begin{aligned}
Q(x, y) =& F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + \frac{1}{2}F_{xx}(x_0, y_0)(x - x_0)^2 \\
&+ F_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{1}{2}F_{yy}(x_0, y_0)(y - y_0)^2 \\
&= 1 + (x - x_0) + 0(y - y_0) + \frac{1}{2}(0)(x - x_0)^2 + (0)(x - x_0)(y - y_0) \\
&+ \frac{1}{2}xe^y + (xy^2e^y)(y - y_0)^2 \\
&= 1 + (x - 1) + 0(y) + \frac{1}{2}(0)(x - 1)^2 + (0)(x - 1)(y) + \frac{1}{2}(xe^y + xy^2e^y)(y)^2
\end{aligned}$$

Simplifying all term we get a final answers of

$$L(x, y) = 1 + (x - 1)$$

and,

$$Q(x, y) = 1 + (x - 1) + \frac{1}{2}(xe^y + xy^2e^y)y^2.$$

Now we can find the values for L, Q, and f at $(0.9, 0.1)$.

$$L(0.9, 0.1) = 1 + 1(0.9 - 1) = 0.9,$$

$$Q(x, y) = 1 + (0.9 - 1) + \frac{1}{2}((0.9)e^{0.1} + (0.9)(0.1)^2e^{0.1})(0.1)^2 = 0.905023...,$$

and

$$f(x, y) = (0.9)e^{0.1} = 0.99465....$$

Both L and Q were only accurate to the first term which makes it a fairly inaccurate power series. Also, this power series looks like it converges slowly so we would need many more terms (like a tenth degree polynomial) to get a good approximation of $f(x, y)$.

Comments on the Above

Feedback from Ken. The phrase "we would not many more terms" I think has a missing word. Good otherwise!

- In this problem we analyze the behavior of the polynomial $Q(x,y) = ax^2 + bxy + cy^2$ (without using the Second Derivatives Test) by identifying the graph as a paraboloid. Note that specializing the critical point to be at (0,0,0) carries no loss of generality, since extrema are invariant under translation.

  – By completing the square with respect to the variable x (treating y as a constant), show that if $a \neq 0$, then

$$Q(x,y) = ax^2 + bxy + cy^2 = a\left(\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac-b^2}{4a^2}\right)y^2\right).$$

Solution Author: Hannah Gray

$$= a\left(\left(x + \frac{b}{2a}y^2\right) + \left(\frac{4ac-b^2}{4a^2}\right)y^2\right).$$

$$= a(-a * \frac{by^2}{2a} + \frac{4ac-b^2}{4a^2} * y^2 + x))$$

a

Comments on the Above

Feedback from Ken. Incomplete. Let me give a hint to get you going! It's quite tricky to start with the LHS shown above and end up with the RHS. But if instead you start with the RHS and just expand everything out, you will see lots of nice cancellation, and you'll end up with the LHS.

  – Let $D = 4ac - b^2$. Show that if $D > 0$ and $a > 0$, then Q has a local minimum at $(0,0)$.

Solution Author: Graham Kroll

Using the complete square form of $Q(x,y)$, we can see that the formula looks very similar to the generic vetex formula of a parabola for algebra given below.

$$f(x) = a(x-h)^2 + k$$

We also know that D and a are positive and all other values are squared so they must be zero or positive as well. Therefore, since we have

$$Q(x,y) = (+\text{ number})\left[\left(x + \frac{b}{2a}y\right)^2 + \frac{+\text{ number}}{(\text{number })^2}y^2\right].$$

which mean that the lowest possible value we could get is when x and y are equal to zero, since any other numbers would give us a result greater than $(0,0)$. In other words, since the equation must equal zero or some positive number, the minimum value would be when both x and y are zero.

Comments on the Above

– Show that if $D > 0$ and $a < 0$, then Q has a local maximum at $(0, 0)$.

Solution Author: Brian Ortiz

As Graham said, above the equation used to complete the square of $Q(x, y)$, is in fact quite similar to

$$f(x) = a(x - h)^2 + k$$

and since D is positive and a is negative, we know that the equation $Q(x, y)$ will always be a down facing parabola

$$Q(x, y) = (-number)\left[\left(x + \frac{b}{2(-a)}y\right)^2 + \frac{+number}{(number)^2}y^2\right]$$

Comments on the Above

Feedback from Ken. Incomplete. Here is a hint: follow the same sort of analysis as the above two problems! Specifically, look for two directions, one in which the parabola will be upward facing, and one in which the parabola will be downward facing.

– Show that if $D < 0$, then $(0, 0)$ is a saddle point.

Solution Author:

Comments on the Above

- Explain why between the two previous problems, the translation $a = \frac{1}{2}F_{xx}$, $b = F_{xy}$, and $c = \frac{1}{2}F_{yy}$ is valid. Then, plug these values in for the D-test described above, and conclude the Second Derivative Test as any textbook states it!

Solution Author: Graham Kroll

The translations shown above are valid in the equation for $Q(x, y)$ because they are being multiplied by their respective squares, in the case of $a = \frac{1}{2}F_{xx}$ and $c = \frac{1}{2}F_{yy}$. For instance, F_{xx} is being multiplied by $(x - x_0)^2$. Since we want the constant to cancel out when we take the second derivative. Therefore, we have to and the half in front so that it cancels out when we do the second derivative. However, for the $b = F_{xy}$ term, neither term is squared so we don't need the half in front. Now using the D-test. We know that $D = 4ac - b^2$ so we can plug in our values for a, b, and c.

$$D = 4ac - b^2 = 4\left(\frac{1}{2}F_{xx} \cdot \frac{1}{2}F_{yy}\right) - (F_{xy})^2$$

Simplifying the factions we get the final answer below

$$D = (F_{xx} \cdot F_{yy}) - (F_{xy})^2.$$

Comments on the Above